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ON THE GENERATING FUNCTIONS OF THE NEWLY DEFINED GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, we have defined new generalizations of some hypergeometric functions and fractional operators with the help of Fox-Wright function. Then, using each of the generalized fractional operators, we derived linear and bilinear generating function relations for these functions. Finally, we have shown that the newly defined hypergeometric functions and fractional operators can be reduced to functions and operators presented in many studies in the literature by giving special values for their parameters.

Keywords: Beta function, Fox-Wright function, hypergeometric functions, fractional operators, generating functions.

AMS Subject Classification: 33B15, 33C05, 33C65, 26A33, 05A15.

1. Introduction

In many studies conducted in recent years, it has been observed that researchers have defined new hypergeometric functions and fractional operators using various generalizations of the beta function [1, 2, 7, 8, 9, 12, 13, 19, 22, 23, 24, 28, 29]. However, in some publications it can be seen that generalized fractional operators are used to obtain generating function relations of generalized hypergeometric functions [3, 4, 5, 10, 16, 17, 20, 21, 27]. In these studies, a single generalized fractional operator was used to obtain the generating function relations of the defined generalized hypergeomeric functions. In 2018, Çetinkaya et al. [11] defined generalized hypergeometric functions in a different way and were able to obtain generating function relations using several generalized fractional operators.

Our motivation is to define new generalized hypergeometric functions which are more general in a similar way to Çetinkaya et al. and to obtain the generating function relations using various fractional operators. In particular, we have defined generalized Gauss F, Appell F_1, Appell F_2, Lauricella F_D^3 hypergeometric functions and generalized Riemann-Liouville fractional derivative and integral, Caputo fractional derivative and Kober-Erdelyi fractional integral operators using a beta function defined in [7].

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2. Preliminaries

The definitions of some special functions needed in this paper are given below.
Gamma function [26]:

$$\Gamma(\xi)=\int_0^\infty \Delta^{\xi-1}\exp(-\Delta)d\Delta,\quad \big(\Re(\xi)>0\big).$$

Pochhammer symbol [26]:

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)},\quad \big(\lambda\neq 0,-1,-2,\dots\big).$$

Gauss hypergeometric function [26]:

$$F(w_1,w_2;w_3;\mu)=\sum_{n=0}^{\infty}\frac{(w_1)_n(w_2)_n}{(w_3)_n}\frac{\mu^n}{n!},\quad \big(|\mu|<1\big).$$

Appell F_1 hypergeometric function [26]:

$$F_1(w_1,w_2,w_3;w_4;\xi,\eta)=\sum_{n,m=0}^{\infty}\frac{(w_1)_{n+m}(w_2)_n(w_3)_m}{(w_4)_{n+m}}\frac{\xi^n}{n!}\frac{\eta^m}{m!},$$

$$\big(\max\{|\xi|,|\eta|\}<1\big).$$

Appell F_2 hypergeometric function [26]:

$$F_2(w_1,w_2,w_3;w_4,w_5;\xi,\eta)=\sum_{n,m=0}^{\infty}\frac{(w_1)_{n+m}(w_2)_n(w_3)_m}{(w_4)_n(w_5)_m}\frac{\xi^n}{n!}\frac{\eta^m}{m!},$$

$$\big(|\xi|+|\eta|<1\big).$$

Lauricella F_D^3 hypergeometric function [26]:

$$F_D^3(w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu)=\sum_{n,m,r=0}^{\infty}\frac{(w_1)_{n+m+r}(w_2)_n(w_3)_m(w_4)_r}{(w_5)_{n+m+r}}\frac{\xi^n}{n!}\frac{\eta^m}{m!}\frac{\mu^r}{r!},$$

$$\big(\max\{|\xi|,|\eta|,|\mu|\}<1\big).$$

${}_u\Psi_v$-beta function [7]:

$${}^{\Psi}\hat{B}_\delta(\xi,\eta)={}^{\Psi}B_\delta\left[\begin{matrix}(B_i,A_i)_{1,u}\\(D_j,C_j)_{1,v}\end{matrix}\,\middle|\,\xi,\eta\right]=\int_0^1\Delta^{\xi-1}(1-\Delta)^{\eta-1}{}_u\Psi_v\left(\frac{-\delta}{\Delta(1-\Delta)}\right)d\Delta,\qquad (1)$$

$$\big(\Re(\delta)>0,\Re(\xi)>0,\Re(\eta)>0\big).$$

Note that by choosing $u=0$ and $v=1$ in (1), it is reduced to the Ψ-beta function in [8].
Fox-Wright function [18]:

$${}_u\Psi_v(\mu)={}_u\Psi_v\left[\begin{matrix}(B_i,A_i)_{1,u}\\(D_j,C_j)_{1,v}\end{matrix}\,\middle|\,\mu\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{u}\Gamma(A_in+B_i)}{\prod_{j=1}^{v}\Gamma(C_jn+D_j)}\frac{\mu^n}{n!},$$

where $\mu,B_i,D_j\in\mathbb{C}$ and $A_i,C_j\in\mathbb{R}$ $(i=1,\dots,u;j=1,\dots,v)$. Fox-Wright (or, general Wright) function was investigated by Fox ([14, 15]) and Wright ([30, 31, 32]), who presented its asymptotic expansion for large values of the argument μ under the condition

$$\sum_{j=1}^{v}C_j-\sum_{i=1}^{u}A_i>-1.$$

Remark 2.1. *In particular, the function* ${}_u\Psi_v(\mu)$ *is immediately reduced to the generalized hypergeometric function* ${}_uF_v(\mu)$ *when* $A_i = C_j = 1$*, and multiplied by* $\frac{\Gamma(D_1)...\Gamma(D_v)}{\Gamma(B_1)...\Gamma(B_u)}$.

Also, the definitions of classical fractional operators are as follows.
Riemann-Liouville fractional integral [18]:

$$\left[I_a^{\epsilon}\kappa\right](\xi) = \frac{1}{\Gamma(\epsilon)}\int_a^{\xi}(\xi-\Delta)^{\epsilon-1}\kappa(\Delta)d\Delta, \quad \big(\Re(\epsilon)>0\big). \tag{2}$$

Riemann-Liouville fractional derivative [18]:

$$\left[D_a^{\epsilon}\kappa\right](\xi) = \frac{1}{\Gamma(m-\epsilon)}\left(\frac{d}{d\xi}\right)^m\int_a^{\xi}(\xi-\Delta)^{m-\epsilon-1}\kappa(\Delta)d\Delta, \tag{3}$$

$$\big(\Re(\epsilon)>0, m-1<\Re(\epsilon)<m, m\in\mathbb{N}\big).$$

Caputo fractional derivative [18]:

$$\left[{}^cD_a^{\epsilon}\kappa\right](\xi) = \frac{1}{\Gamma(m-\epsilon)}\int_a^{\xi}(\xi-\Delta)^{m-\epsilon-1}\kappa^{(m)}(\Delta)d\Delta, \tag{4}$$

$$\big(\Re(\epsilon)>0, m-1<\Re(\epsilon)<m, m\in\mathbb{N}\big).$$

Kober-Erdelyi fractional integral [18]:

$$\left[I_{a;\sigma,\eta}^{\epsilon}\kappa\right](\xi) = \frac{\sigma\xi^{-\sigma(\epsilon+\eta)}}{\Gamma(\epsilon)}\int_a^{\xi}\Delta^{\sigma\eta+\sigma-1}(\xi^{\sigma}-\Delta^{\sigma})^{\epsilon-1}\kappa(\Delta)d\Delta, \tag{5}$$

$$\big(\Re(\epsilon)>0, \sigma>0, \eta\in\mathbb{C}\big).$$

3. New Generalized Hypergeometric Functions and Fractional Operators

We will begin by giving definitions of the new generalized hypergeometric functions.

Definition 3.1. *Let* χ_1 *and* χ_2 *be arbitrary parameters and* $|\mu|<1$*. The new generalized Gauss hypergeometric function is defined as:*

$$\begin{aligned}{}^{\Psi}F_{\delta}(w_1,w_2;w_3;\mu;\chi_1,\chi_2) &= {}^{\Psi}F_{\delta}\left[\begin{matrix}(B_i,A_i)_{1,u}\\(D_j,C_j)_{1,v}\end{matrix}\,\middle|w_1,w_2;w_3;\mu;\chi_1,\chi_2\right]\\ &:= \sum_{n=0}^{\infty}\frac{(w_1)_n(w_2)_n}{(w_3)_n}\frac{{}^{\Psi}\hat{B}_{\delta}(w_2-\chi_1+n, w_3-w_2+\chi_2)}{B(w_2-\chi_1+n, w_3-w_2+\chi_2)}\frac{\mu^n}{n!}.\end{aligned}$$

Definition 3.2. *Let* χ_1 *and* χ_2 *be arbitrary parameters and* $|\xi|<1$*,* $|\eta|<1$*. The new generalized Appell* F_1 *hypergeometric function is defined as:*

$$\begin{aligned}{}^{\Psi}F_{1,\delta}(w_1,w_2,w_3;w_4;\xi,\eta;\chi_1,\chi_2) &= {}^{\Psi}F_{1,\delta}\left[\begin{matrix}(B_i,A_i)_{1,u}\\(D_j,C_j)_{1,v}\end{matrix}\,\middle|w_1,w_2,w_3;w_4;\xi,\eta;\chi_1,\chi_2\right]\\ &:= \sum_{n,m=0}^{\infty}\frac{(w_1)_{n+m}(w_2)_n(w_3)_m}{(w_4)_{n+m}}\frac{{}^{\Psi}\hat{B}_{\delta}(w_1-\chi_1+n+m, w_4-w_1+\chi_2)}{B(w_1-\chi_1+n+m, w_4-w_1+\chi_2)}\frac{\xi^n}{n!}\frac{\eta^m}{m!}.\end{aligned}$$

Definition 3.3. *Let χ_1 and χ_2 be arbitrary parameters and $|\xi| + |\eta| < 1$. The new generalized Appell F_2 hypergeometric function is defined as:*

$$
\begin{aligned}
{}^{\Psi}F_{2,\delta}(w_1, w_2, w_3; w_4, w_5; \xi, \eta; \chi_1, \chi_2) &= {}^{\Psi}F_{2,\delta}\left[\begin{matrix}(B_i, A_i)_{1,u}\\(D_j, C_j)_{1,v}\end{matrix}\,\middle|\, w_1, w_2, w_3; w_4, w_5; \xi, \eta; \chi_1, \chi_2\right]\\
&:= \sum_{n,m=0}^{\infty} \frac{(w_1)_{n+m}(w_2)_n(w_3)_m}{(w_4)_n(w_5)_m} \frac{{}^{\Psi}\hat{B}_\delta(w_2-\chi_1+n, w_4-w_2+\chi_2)}{B(w_2-\chi_1+n, w_4-w_2+\chi_2)}\\
&\quad\times \frac{{}^{\Psi}\hat{B}_\delta(w_3-\chi_1+m, w_5-w_3+\chi_2)}{B(w_3-\chi_1+m, w_5-w_3+\chi_2)} \frac{\xi^n}{n!} \frac{\eta^m}{m!}.
\end{aligned}
$$

Definition 3.4. *Let χ_1 and χ_2 be arbitrary parameters and $|\xi| < 1, |\eta| < 1, |\mu| < 1$. The new generalized Lauricella F_D^3 hypergeometric function is defined as:*

$$
{}^{\Psi}F^3_{D,\delta}(w_1, w_2, w_3, w_4; w_5; \xi, \eta, \mu; \chi_1, \chi_2) = {}^{\Psi}F^3_{D,\delta}\left[\begin{matrix}(B_i, A_i)_{1,u}\\(D_j, C_j)_{1,v}\end{matrix}\,\middle|\, w_1, w_2, w_3, w_4; w_5; \xi, \eta, \mu; \chi_1, \chi_2\right]
$$

$$
:= \sum_{n,m,r=0}^{\infty} \frac{(w_1)_{n+m+r}(w_2)_n(w_3)_m(w_4)_r}{(w_5)_{n+m+r}} \frac{{}^{\Psi}\hat{B}_\delta(w_1-\chi_1+n+m+r, w_5-w_1+\chi_2)}{B(w_1-\chi_1+n+m+r, w_5-w_1+\chi_2)} \frac{\xi^n}{n!} \frac{\eta^m}{m!} \frac{\mu^r}{r!}.
$$

Remark 3.1. *If we choose $\delta = u = v = 0$ in Definition 3.1-3.4, we get the classical hypergeometric functions F, F_1, F_2 and F_D^3, respectively. Let us also state that the convergence regions of ${}^{\Psi}F_\delta$, ${}^{\Psi}F_{1,\delta}$, ${}^{\Psi}F_{2,\delta}$ and ${}^{\Psi}F^3_{D,\delta}$ are same as the classical ones.*

Let us now give the definitions of the new generalized fractional operators.

Definition 3.5. *The new Riemann-Liouville fractional integral for $\Re(\epsilon) > 0$ is defined as:*

$$
\begin{aligned}
{}^{\Psi}I^{\epsilon,\delta}_{RL}\left[\kappa(\mu)\right] &= {}^{\Psi}I^{\epsilon,\delta}_{RL}\left[\begin{matrix}(B_i, A_i)_{1,u}\\(D_j, C_j)_{1,v}\end{matrix}\,\middle|\,\kappa(\mu)\right]\\
&:= \frac{1}{\Gamma(\epsilon)} \int_0^\mu (\mu-\Delta)^{\epsilon-1}\, {}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right)\kappa(\Delta)d\Delta. \qquad (6)
\end{aligned}
$$

Definition 3.6. *The new Riemann-Liouville fractional derivative for $m-1 < \Re(\epsilon) < m$, $m \in \mathbb{N}$ is defined as:*

$$
\begin{aligned}
{}^{\Psi}D^{\epsilon,\delta}_{RL}\left[\kappa(\mu)\right] &= {}^{\Psi}D^{\epsilon,\delta}_{RL}\left[\begin{matrix}(B_i, A_i)_{1,u}\\(D_j, C_j)_{1,v}\end{matrix}\,\middle|\,\kappa(\mu)\right]\\
&:= \frac{1}{\Gamma(m-\epsilon)} \frac{d^m}{d\mu^m} \int_0^\mu (\mu-\Delta)^{m-\epsilon-1}\, {}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right)\kappa(\Delta)d\Delta. \qquad (7)
\end{aligned}
$$

Definition 3.7. *The new Caputo fractional derivative for $m-1 < \Re(\epsilon) < m, m \in \mathbb{N}$ is defined as:*

$$
\begin{aligned}
{}^{\Psi}D^{\epsilon,\delta}_{C}\left[\kappa(\mu)\right] &= {}^{\Psi}D^{\epsilon,\delta}_{C}\left[\begin{matrix}(B_i, A_i)_{1,u}\\(D_j, C_j)_{1,v}\end{matrix}\,\middle|\,\kappa(\mu)\right]\\
&:= \frac{1}{\Gamma(m-\epsilon)} \int_0^\mu (\mu-\Delta)^{m-\epsilon-1}\, {}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right)\kappa^{(m)}(\Delta)d\Delta. \qquad (8)
\end{aligned}
$$

Definition 3.8. *The new Kober-Erdelyi fractional integral for $\Re(\epsilon) > 0$ and $\gamma \in \mathbb{C}$ is defined as:*

$$
\begin{aligned}
{}^{\Psi}I^{\gamma,\epsilon,\delta}_{KE}\left[\kappa(\mu)\right] &= {}^{\Psi}I^{\gamma,\epsilon,\delta}_{KE}\left[\begin{matrix}(B_i, A_i)_{1,u}\\(D_j, C_j)_{1,v}\end{matrix}\,\middle|\,\kappa(\mu)\right]\\
&:= \frac{\mu^{-\epsilon-\gamma}}{\Gamma(\epsilon)} \int_0^\mu \Delta^\gamma (\mu-\Delta)^{\epsilon-1}\, {}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right)\kappa(\Delta)d\Delta. \qquad (9)
\end{aligned}
$$

Remark 3.2. *If we choose $\delta = u = v = 0$ in Definition 3.5-3.8, we get the classical fractional operators* (2)*,* (3)*,* (4) *and* (5)*.*

For the sake of shortness, we use the notations $\hat{F}$, $\hat{F}_1$, $\hat{F}_2$, $\hat{F}_D^3$, $\hat{I}_{RL}^{\epsilon}$, $\hat{D}_{RL}^{\epsilon}$, $\hat{D}_C^{\epsilon}$ and $\hat{I}_{KE}^{\gamma,\epsilon}$ instead of ${}^{\Psi}F_{\delta}$, ${}^{\Psi}F_{1,\delta}$, ${}^{\Psi}F_{2,\delta}$, ${}^{\Psi}F_{D,\delta}^3$, ${}^{\Psi}I_{RL}^{\epsilon,\delta}$, ${}^{\Psi}D_{RL}^{\epsilon,\delta}$, ${}^{\Psi}D_C^{\epsilon,\delta}$ and ${}^{\Psi}I_{KE}^{\gamma,\epsilon,\delta}$ respectively.

4. Generalized Fractional Derivative and Integrals of Some Functions

In this section we will compute the generalized fractional derivatives and integrals of some functions and use them to obtain the generating function relations of the generalized hypergeometric functions in the following topics. In some of the following theorems, only the equations for the generalized Caputo fractional derivative have been given and proved. Other equations given as corollaries can be proved in a similar way. However, proofs of similar equations in the literature can also be examined (see, e.g. [3, 4, 5, 10, 16, 17, 19, 20, 21, 27]).

Theorem 4.1. *The following equations holds true:*

$$\begin{aligned}
\hat{I}_{RL}^{\epsilon}\left[\mu^{\Lambda_1}\right] &= \frac{\Gamma(\Lambda_1+1)}{\Gamma(\Lambda_1+1+\epsilon)}\frac{{}^{\Psi}\hat{B}_{\delta}(\Lambda_1+1,\epsilon)}{B(\Lambda_1+1,\epsilon)}\mu^{\Lambda_1+\epsilon}, \qquad (10)\\
\hat{D}_{RL}^{\epsilon}\left[\mu^{\Lambda_1}\right] &= \frac{\Gamma(\Lambda_1+1)}{\Gamma(\Lambda_1+1-\epsilon)}\frac{{}^{\Psi}\hat{B}_{\delta}(\Lambda_1+1,m-\epsilon)}{B(\Lambda_1+1,m-\epsilon)}\mu^{\Lambda_1-\epsilon},\\
\hat{D}_{C}^{\epsilon}\left[\mu^{\Lambda_1}\right] &= \frac{\Gamma(\Lambda_1+1)}{\Gamma(\Lambda_1+1-\epsilon)}\frac{{}^{\Psi}\hat{B}_{\delta}(\Lambda_1-m+1,m-\epsilon)}{B(\Lambda_1-m+1,m-\epsilon)}\mu^{\Lambda_1-\epsilon},\\
\hat{I}_{KE}^{\gamma,\epsilon}\left[\mu^{\Lambda_1}\right] &= \frac{\Gamma(\Lambda_1+\gamma+1)}{\Gamma(\Lambda_1+\gamma+1+\epsilon)}\frac{{}^{\Psi}\hat{B}_{\delta}(\Lambda_1+\gamma+1,\epsilon)}{B(\Lambda_1+\gamma+1,\epsilon)}\mu^{\Lambda_1}.
\end{aligned}$$

Proof. For $\Re(\epsilon) > 0$ and $\Re(\Lambda_1) > -1$, with direct calculations, we have

$$\begin{aligned}
\hat{I}_{RL}^{\epsilon}\left[\mu^{\Lambda_1}\right] &= \frac{1}{\Gamma(\epsilon)}\int_0^{\mu}\Delta^{\Lambda_1}(\mu-\Delta)^{\epsilon-1}{}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right)d\Delta\\
&= \frac{\Gamma(\Lambda_1+1)}{\Gamma(\Lambda_1+1+\epsilon)}\frac{{}^{\Psi}\hat{B}_{\delta}(\Lambda_1+1,\epsilon)}{B(\Lambda_1+1,\epsilon)}\mu^{\Lambda_1+\epsilon}.
\end{aligned}$$

Also, taking $\kappa(\mu) = \mu^{\Lambda_1}$ in equations (7), (8) and (9), we have

- for $m-1 < \Re(\epsilon) < m$ and $\Re(\Lambda_1) > -1$,
$$\hat{D}_{RL}^{\epsilon}\left[\mu^{\Lambda_1}\right] = \frac{d^m}{d\mu^m}\left(\hat{I}_{RL}^{m-\epsilon}\left[\mu^{\Lambda_1}\right]\right), \qquad (11)$$
- for $m-1 < \Re(\epsilon) < m$ and $\Re(\Lambda_1) > m-1$,
$$\hat{D}_{C}^{\epsilon}\left[\mu^{\Lambda_1}\right] = \frac{\Gamma(\Lambda_1+1)}{\Gamma(\Lambda_1-m+1)}\hat{I}_{RL}^{m-\epsilon}\left[\mu^{\Lambda_1-m}\right], \qquad (12)$$
- for $\Re(\epsilon) > 0$ and $\Re(\Lambda_1+\gamma) > -1$,
$$\hat{I}_{KE}^{\gamma,\epsilon}\left[\mu^{\Lambda_1}\right] = \mu^{-\epsilon-\gamma}\hat{I}_{RL}^{\epsilon}\left[\mu^{\Lambda_1+\gamma}\right]. \qquad (13)$$

As a result, the proofs of the other cases (11), (12) and (13) are completed by considering the equation (10). □

Theorem 4.2. *The analytical function $\kappa(\mu)=\sum_{n=0}^{\infty} w^n\mu^n$ provides the following equations for $|\mu|<r$:*

$$\hat{I}_{RL}^{\epsilon}\big[\kappa(\mu)\big]=\sum_{n=0}^{\infty} w^n \hat{I}_{RL}^{\epsilon}\big[\mu^n\big],$$
$$\hat{D}_{RL}^{\epsilon}\big[\kappa(\mu)\big]=\sum_{n=0}^{\infty} w^n \hat{D}_{RL}^{\epsilon}\big[\mu^n\big],$$
$$\hat{D}_{C}^{\epsilon}\big[\kappa(\mu)\big]=\sum_{n=0}^{\infty} w^n \hat{D}_{C}^{\epsilon}\big[\mu^n\big],$$
$$\hat{I}_{KE}^{\gamma,\epsilon}\big[\kappa(\mu)\big]=\sum_{n=0}^{\infty} w^n \hat{I}_{KE}^{\gamma,\epsilon}\big[\mu^n\big].$$

Proof. The desired results are obtained by using the analytical function $\kappa(\mu)$ in equations (6), (7), (8) and (9). □

Theorem 4.3. *For $m-1<\Re(\Lambda_1-\epsilon)<m<\Re(\Lambda_1)$ and $|w\mu|<1$ holds true:*

$$\hat{D}_{C}^{\Lambda_1-\epsilon}[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}]=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}(\Lambda_2,\Lambda_1;\epsilon;w\mu;\chi,\chi). \tag{14}$$

Proof. The power series expansion [6] is as follows:

$$(1-w\mu)^{-\Lambda_2}=\sum_{n=0}^{\infty}(\Lambda_2)_n\frac{(w\mu)^n}{n!},\quad \big(|w\mu|<1\big). \tag{15}$$

Considering the power series and making the necessary calculations, we have

$$\begin{aligned}&\hat{D}_{C}^{\Lambda_1-\epsilon}[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}]\\ &\quad=\frac{1}{\Gamma(m-\Lambda_1+\epsilon)}\int_0^{\mu}(\mu-\Delta)^{m-\Lambda_1+\epsilon-1}\ {}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right)\\ &\quad\times\left(\frac{d^m}{d\Delta^m}\left\{\Delta^{\Lambda_1-1}(1-w\Delta)^{-\Lambda_2}\right\}\right)d\Delta\\ &\quad=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}(\Lambda_2,\Lambda_1;\epsilon;w\mu;\chi,\chi).\end{aligned}$$
□

Corollary 4.1. *Similarly, for other generalized fractional operators, the following results are obtained as:*

$$\hat{I}_{RL}^{-(\Lambda_1-\epsilon)}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}\right]=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}(\Lambda_2,\Lambda_1;\epsilon;w\mu;0,0),$$
$$\hat{D}_{RL}^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}\right]=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}(\Lambda_2,\Lambda_1;\epsilon;w\mu;0,\chi),$$
$$\hat{I}_{KE}^{\gamma,\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}\right]=\frac{\Gamma(\Lambda_1+\gamma)}{\Gamma(2\Lambda_1+\gamma-\epsilon)}\mu^{\Lambda_1-1}\hat{F}(\Lambda_2,\Lambda_1+\gamma;2\Lambda_1+\gamma-\epsilon;w\mu;0,0).$$

Theorem 4.4. *For $|w\mu|<1$, $|r\mu|<1$, and $m-1<\Re(\Lambda_1-\epsilon)<m<\Re(\Lambda_1)$ holds true:*

$$\hat{D}_{C}^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}\right]=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}_1(\Lambda_1,\Lambda_2,\Lambda_3;\epsilon;w\mu,r\mu;\chi,\chi). \tag{16}$$

Proof. Considering the power series (15) and making the necessary calculations, we have

$$\begin{aligned}
&\hat{D}_C^{\Lambda_1-\epsilon}[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}] \\
&\quad= \frac{1}{\Gamma(m-\Lambda_1+\epsilon)}\int_0^{\mu}(\mu-\Delta)^{m-\Lambda_1+\epsilon-1}{}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right) \\
&\quad\times\left(\frac{d^m}{d\Delta^m}\left\{\Delta^{\Lambda_1-1}(1-w\Delta)^{-\Lambda_2}(1-r\Delta)^{-\Lambda_3}\right\}\right)d\Delta \\
&\quad= \frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}_1(\Lambda_1,\Lambda_2,\Lambda_3;\epsilon;w\mu,r\mu;\chi,\chi).
\end{aligned}$$

□

Corollary 4.2. *Similarly, for other generalized fractional operators, the following results are obtained as:*

$$\begin{aligned}
\hat{I}_{RL}^{-(\Lambda_1-\epsilon)}\left[z^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}\right] &= \frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}_1(\Lambda_1,\Lambda_2,\Lambda_3;\epsilon;w\mu,r\mu;0,0), \\
\hat{D}_{RL}^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}\right] &= \frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}_1(\Lambda_1,\Lambda_2,\Lambda_3;\epsilon;w\mu,r\mu;0,\chi), \\
\hat{I}_{KE}^{\gamma,\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}\right] &= \frac{\Gamma(\Lambda_1+\gamma)}{\Gamma(2\Lambda_1+\gamma-\epsilon)}z^{\Lambda_1-1} \\
&\quad\times\hat{F}_1(\Lambda_1+\gamma,\Lambda_2,\Lambda_3;2\Lambda_1+\gamma-\epsilon;w\mu,r\mu;0,0).
\end{aligned}$$

Theorem 4.5. *For $|w\mu|<1$, $|r\mu|<1$, $|t\mu|<1$ and $m-1<\Re(\Lambda_1-\epsilon)<m<\Re(\Lambda_1)$ holds true:*

$$\begin{aligned}
&\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}(1-t\mu)^{-\Lambda_4}\right] \\
&\qquad= \frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}_D^3(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4;\epsilon;w\mu,r\mu,t\mu;\chi,\chi). \qquad (17)
\end{aligned}$$

Proof. Considering the power series (15) and making the necessary calculations, we have

$$\begin{aligned}
&\hat{D}_C^{\Lambda_1-\epsilon}[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}(1-t\mu)^{-\Lambda_4}] \\
&\quad= \frac{1}{\Gamma(m-\Lambda_1+\epsilon)}\int_0^{\mu}(\mu-\Delta)^{m-\Lambda_1+\epsilon-1}{}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right)d\Delta \\
&\quad\times\left(\frac{d^m}{d\Delta^m}\left\{\Delta^{\Lambda_1-1}(1-w\Delta)^{-\Lambda_2}(1-r\Delta)^{-\Lambda_3}(1-t\Delta)^{-\Lambda_4}\right\}\right) \\
&\quad= \frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}_D^3(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4;\epsilon;w\mu,r\mu,t\mu;\chi,\chi).
\end{aligned}$$

□

Corollary 4.3. *Similarly, for other generalized fractional operators, the following results are obtained as:*

$$\begin{aligned}
\hat{I}_{RL}^{-(\Lambda_1-\epsilon)}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}(1-t\mu)^{-\Lambda_4}\right] &= \frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1} \\
&\quad\times\hat{F}_D^3(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4;\epsilon;w\mu,r\mu,t\mu;0,0), \\
\hat{D}_{RL}^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}(1-t\mu)^{-\Lambda_4}\right] &= \frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1} \\
&\quad\times\hat{F}_D^3(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4;\epsilon;w\mu,r\mu,t\mu;0,\chi),
\end{aligned}$$

$$\hat{I}_{KE}^{\gamma,\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}(1-t\mu)^{-\Lambda_4}\right]=\frac{\Gamma(\Lambda_1+\gamma)}{\Gamma(2\Lambda_1+\gamma-\epsilon)}\mu^{\Lambda_1-1}$$
$$\times\hat{F}_D^3(\Lambda_1+\gamma,\Lambda_2,\Lambda_3,\Lambda_4;2\Lambda_1+\gamma-\epsilon;w\mu,r\mu,t\mu;0,0).$$

Theorem 4.6. *For* $|\xi|+|w\mu|<1$ *and* $m-1<\Re(\Lambda_1-\epsilon)<m<\Re(\Lambda_1)$ *holds true:*

$$\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_3;\Lambda_4;\frac{\xi}{1-w\mu};\chi,\chi\right)\right]=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}$$
$$\times\hat{F}_2(\Lambda_2,\Lambda_3,\Lambda_1;\Lambda_4,\epsilon;\xi,w\mu;\chi,\chi). \tag{18}$$

Proof. Considering the power series (15) and making the necessary calculations, we have

$$\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_3;\Lambda_4;\frac{\xi}{1-w\mu};\chi,\chi\right)\right]$$
$$=\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n(\Lambda_3)_n}{(\Lambda_4)_n}\frac{\xi^n}{n!}\frac{1}{\Gamma(m-\Lambda_1+\epsilon)}\int_0^{\mu}(\mu-\Delta)^{m-\Lambda_1+\epsilon-1}{}_u\Psi_v\left(\frac{-\delta\mu^2}{\Delta(\mu-\Delta)}\right)$$
$$\times\frac{{}^{\Psi}\hat{B}_\delta(\Lambda_3+n-\chi,\Lambda_4-\Lambda_3+\chi)}{B(\Lambda_3+n-\chi,\Lambda_4-\Lambda_3+\chi)}\left(\frac{d^m}{d\Delta^m}\left\{\Delta^{\Lambda_1-1}(1-w\Delta)^{-(\Lambda_2+n)}\right\}\right)d\Delta$$
$$=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}\hat{F}_2(\Lambda_2,\Lambda_3,\Lambda_1;\Lambda_4,\epsilon;\xi,w\mu;\chi,\chi).$$
□

Corollary 4.4. *Similarly, for other generalized fractional operators, the following results are obtained as:*

$$\hat{I}_{RL}^{-(\Lambda_1-\epsilon)}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_3;\Lambda_4;\frac{\xi}{1-w\mu};0,0\right)\right]=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}$$
$$\times\hat{F}_2(\Lambda_2,\Lambda_3,\Lambda_1;\Lambda_4,\epsilon;\xi,w\mu;0,0),$$
$$\hat{D}_{RL}^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_3;\Lambda_4;\frac{\xi}{1-w\mu};0,\chi\right)\right]=\frac{\Gamma(\Lambda_1)}{\Gamma(\epsilon)}\mu^{\epsilon-1}$$
$$\times\hat{F}_2(\Lambda_2,\Lambda_3,\Lambda_1;\Lambda_4,\epsilon;\xi,w\mu;0,\chi),$$
$$\hat{I}_{KE}^{\gamma,\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_3;\Lambda_4;\frac{\xi}{1-w\mu};0,0\right)\right]=\frac{\Gamma(\Lambda_1+\gamma)}{\Gamma(2\Lambda_1+\gamma-\epsilon)}\mu^{\Lambda_1-1}$$
$$\times\hat{F}_2(\Lambda_2,\Lambda_3,\Lambda_1+\gamma;\Lambda_4,2\Lambda_1+\gamma-\epsilon;\xi,w\mu;0,0).$$

5. Generating Function Relations

We should point out that we omit the proofs of the following corollaries, since the proofs of similar theorems are contained in the articles [3, 4, 5, 10, 16, 17, 20, 21, 27] and in the book [25] (Sects. 5.2 and 5.3).

Theorem 5.1. *For* $|\mu|<\min\{1,|1-\Delta|\}$, $|\Delta|<|1-\mu|$ *and* $m-1<\Re(\Lambda_1-\epsilon)<m<\Re(\Lambda_1)$ *the following equality holds true:*

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;\chi,\chi)\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_1;\epsilon;\frac{\mu}{1-\Delta};\chi,\chi\right). \tag{19}$$

Proof. Let's use the following equation given in [25]:

$$\big((1-\mu)-\Delta\big)^{-\Lambda_2}=(1-\Delta)^{-\Lambda_2}\left(1-\frac{\mu}{1-\Delta}\right)^{-\Lambda_2}. \tag{20}$$

Considering the binomial series (15) in equation (20) we get

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}(1-\mu)^{-\Lambda_2-n}\Delta^n=(1-\Delta)^{-\Lambda_2}\left(1-\frac{\mu}{1-\Delta}\right)^{-\Lambda_2}. \tag{21}$$

Multiplying the equation (21) by μ^{Λ_1-1} and applying the $\hat{D}_C^{\Lambda_1-\epsilon}$ operator, we have

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-\mu)^{-\Lambda_2-n}\right]\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}\left(1-\frac{\mu}{1-\Delta}\right)^{-\Lambda_2}\right].$$

Using equation (14), we obtain

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}\left(\Lambda_2+n,\Lambda_1;\epsilon;\mu;\chi,\chi\right)\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_1;\epsilon;\frac{\mu}{1-\Delta};\chi,\chi\right). \qquad \square$$

Corollary 5.1. *Similarly, the following results are obtained as:*

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;0,0)\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_1;\epsilon;\frac{\mu}{1-\Delta};0,0\right),$$

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1+\gamma;2\Lambda_1+\gamma-\epsilon;\mu;0,0)\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_1+\gamma;2\Lambda_1+\gamma-\epsilon;\frac{\mu}{1-\Delta};0,0\right),$$

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;0,\chi)\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_1;\epsilon;\frac{\mu}{1-\Delta};0,\chi\right).$$

Theorem 5.2. *For* $|\mu|<\min\left\{1,\left|\frac{1-\Delta}{\Delta}\right|\right\}$, $|\Delta|<|1-\mu|^{-1}$ *and* $m-1<\Re(\Lambda_1-\epsilon)<m<\Re(\Lambda_1)$ *the following equality holds true:*

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_3-n,\Lambda_1;\epsilon;\mu;\chi,\chi)\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{F}_1\left(\Lambda_1,\Lambda_2,\Lambda_3;\epsilon;\frac{-\mu\Delta}{1-\Delta},\mu;\chi,\chi\right).$$

Proof. Let's use the following equation also given in [25]:

$$\big(1-(1-\mu)\Delta\big)^{-\Lambda_2}=(1-\Delta)^{-\Lambda_2}\left(1+\frac{\mu\Delta}{1-\Delta}\right)^{-\rho}. \tag{22}$$

Considering the binomial series (15) in equation (22) we have

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\big((1-\mu)\Delta\big)^n=(1-\Delta)^{-\Lambda_2}\left(1+\frac{\mu\Delta}{1-\Delta}\right)^{-\Lambda_2}. \tag{23}$$

Multiplying the equation (23) by $\mu^{\Lambda_1-1}(1-\mu)^{-\Lambda_3}$ and applying the $\hat{D}_C^{\Lambda_1-\epsilon}$ operator, we obtain

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-\mu)^{n-\Lambda_3}\right]\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-\mu)^{-\Lambda_3}\left(1+\frac{\mu\Delta}{1-\Delta}\right)^{-\Lambda_2}\right].$$

Using equation (14) and equation (16), we get

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_3-n,\Lambda_1;\epsilon;\mu;\chi,\chi)\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{F}_1\left(\Lambda_1,\Lambda_2,\Lambda_3;\epsilon;\frac{-\mu\Delta}{1-\Delta},\mu;\chi,\chi\right). \quad \square$$

Corollary 5.2. *Similarly, the following results are obtained as:*

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_3-n,\Lambda_1;\epsilon;\mu;0,0)\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{F}_1\left(\Lambda_1,\Lambda_2,\Lambda_3;\epsilon;\frac{-\mu\Delta}{1-\Delta},\mu;0,0\right),$$

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_3-n,\Lambda_1+\gamma;2\Lambda_1+\gamma-\epsilon;\mu;0,0)\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{F}_1\left(\Lambda_1+\gamma,\Lambda_2,\Lambda_3;2\Lambda_1+\gamma-\epsilon;\frac{-\mu\Delta}{1-\Delta},\mu;0,0\right),$$

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_3-n,\Lambda_1;\epsilon;\mu;0,\chi)\Delta^n=(1-\Delta)^{-\Lambda_2}\hat{F}_1\left(\Lambda_1,\Lambda_2,\Lambda_3;\epsilon;\frac{-\mu\Delta}{1-\Delta},\mu;0,\chi\right).$$

Theorem 5.3. *For* $|\mu|<1$, $\left|\frac{(1-u)\Delta}{1-\mu}\right|<1$, $\left|\frac{\mu}{1-\Delta}\right|+\left|\frac{u\Delta}{1-\Delta}\right|<1$, $m-1<\Re(\Lambda_1-\epsilon)<m<\Re(\Lambda_1)$ *and* $m-1<\Re(\Lambda_3-\Lambda_4)<m<\Re(\Lambda_3)$ *the following equality holds true:*

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;\chi,\chi)\hat{F}(-n,\Lambda_3;\Lambda_4;u;\chi,\chi)\Delta^n=(1-\Delta)^{-\Lambda_2}$$
$$\times\hat{F}_2\left(\Lambda_2,\Lambda_1,\Lambda_3;\epsilon,\Lambda_4;\frac{\mu}{1-\Delta},\frac{-u\Delta}{1-\Delta};\chi,\chi\right).$$

Proof. Substituting $(1-u)\Delta$ for Δ in the equation (19), we obtain

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;\chi,\chi)\big((1-u)\Delta\big)^n$$
$$=\big(1-(1-u)\Delta\big)^{-\Lambda_2}\hat{F}\left(\Lambda_2,\Lambda_1;\epsilon;\frac{\mu}{1-(1-u)\Delta};\chi,\chi\right).$$

Multiplying the above equation by u^{Λ_3-1} and applying the $\hat{D}_C^{\Lambda_3-\Lambda_4}$ operator, we get

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;\chi,\chi)\hat{D}_C^{\Lambda_3-\Lambda_4}\left[u^{\Lambda_3-1}(1-u)^n\right]\Delta^n$$
$$=\hat{D}_C^{\Lambda_3-\Lambda_4}\left[u^{\Lambda_3-1}\big(1-(1-u)\Delta\big)^{-\Lambda_2}\hat{F}(\Lambda_2,\Lambda_1;\epsilon;\frac{\mu}{1-(1-u)\Delta};\chi,\chi)\right].$$

Using equation (14) and equation (18), we have

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;\chi,\chi)\hat{F}(-n,\Lambda_3;\Lambda_4;u;\chi,\chi)\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{F}_2\left(\Lambda_2,\Lambda_1,\Lambda_3;\epsilon,\Lambda_4;\frac{\mu}{1-\Delta},\frac{-u\Delta}{1-\Delta};\chi,\chi\right). \quad \square$$

Corollary 5.3. *Similarly, the following results are obtained as:*

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;0,0)\hat{F}(-n,\Lambda_3;\Lambda_4;u;0,0)\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{F}_2\left(\Lambda_2,\Lambda_1,\Lambda_3;\epsilon,\Lambda_4;\frac{\mu}{1-\Delta},\frac{-u\Delta}{1-\Delta};0,0\right),$$
$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1+\gamma;2\Lambda_1+\gamma-\epsilon;\mu;0,0)\hat{F}(-n,\Lambda_3+\theta;2\Lambda_3+\theta-\Lambda_4;u;0,0)\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{F}_2\left(\Lambda_2,\Lambda_1+\gamma,\Lambda_3+\theta;2\Lambda_1+\gamma-\epsilon,2\Lambda_3+\theta-\Lambda_4;\frac{\mu}{1-\Delta},\frac{-u\Delta}{1-\Delta};0,0\right),$$
$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}(\Lambda_2+n,\Lambda_1;\epsilon;\mu;0,\chi)\hat{F}(-n,\Lambda_3;\Lambda_4;u;0,\chi)\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{F}_2\left(\Lambda_2,\Lambda_1,\Lambda_3;\epsilon,\Lambda_4;\frac{\mu}{1-\Delta},\frac{-u\Delta}{1-\Delta};0,\chi\right).$$

Theorem 5.4. *For* $|w\mu|<\min\{1,|1-\Delta|\}$, $|\Delta|<|1-w\mu|$, $|r\mu|<1$, $|t\mu|<1$ *and* $m-1<\Re(\Lambda_1-\epsilon)<m<\Re(\Lambda_1)$ *the following equality holds true:*

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}_D^3(\Lambda_1,\Lambda_2+n,\Lambda_3,\Lambda_4;\epsilon;w\mu,r\mu,t\mu;\chi,\chi)\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{F}_D^3\left(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4;\epsilon;\frac{w\mu}{1-\Delta},r\mu,t\mu;\chi,\chi\right).$$

Proof. Using $w\mu$ instead of μ in (20) we have

$$\left((1-w\mu)-\Delta\right)^{-\Lambda_2}=(1-\Delta)^{-\Lambda_2}\left(1-\frac{w\mu}{1-\Delta}\right)^{-\Lambda_2}. \tag{24}$$

Considering the binomial series (15) in equation (24) we get

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}(1-w\mu)^{-\Lambda_2-n}\Delta^n=(1-\Delta)^{-\Lambda_2}\left(1-\frac{w\mu}{1-\Delta}\right)^{-\Lambda_2}. \tag{25}$$

Multiplying the equation (25) by $\mu^{\Lambda_1-1}(1-r\mu)^{-\Lambda_3}(1-t\mu)^{-\Lambda_4}$ and applying the $\hat{D}_C^{\Lambda_1-\epsilon}$ fractional operator, we obtain

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}(1-w\mu)^{-\Lambda_2-n}(1-r\mu)^{-\Lambda_3}(1-t\mu)^{-\Lambda_4}\right]\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{D}_C^{\Lambda_1-\epsilon}\left[\mu^{\Lambda_1-1}\left(1-\frac{w\mu}{1-\Delta}\right)^{-\Lambda_2}(1-r\mu)^{-\Lambda_3}(1-t\mu)^{-\Lambda_4}\right].$$

Using equation (17), we get

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}_D^3(\Lambda_1,\Lambda_2+n,\Lambda_3,\Lambda_4;\epsilon;w\mu,r\mu,t\mu;\chi,\chi)\Delta^n$$
$$=(1-\Delta)^{-\Lambda_2}\hat{F}_D^3\left(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4;\epsilon;\frac{w\mu}{1-\Delta},r\mu,t\mu;\chi,\chi\right).$$ □

Corollary 5.4. *Similarly, the following results are obtained as:*

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}_D^3(\Lambda_1,\Lambda_2+n,\Lambda_3,\Lambda_4;\epsilon;w\mu,r\mu,t\mu;0,0)\Delta^n = (1-\Delta)^{-\Lambda_2}\hat{F}_D^3\left(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4;\epsilon;\frac{w\mu}{1-\Delta},r\mu,t\mu;0,0\right),$$

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}_D^3(\Lambda_1+\gamma,\Lambda_2+n,\Lambda_3,\Lambda_4;2\Lambda_1+\gamma-\epsilon;w\mu,r\mu,t\mu;0,0)\Delta^n = (1-\Delta)^{-\Lambda_2}\hat{F}_D^3\left(\Lambda_1+\gamma,\Lambda_2,\Lambda_3,\Lambda_4;2\Lambda_1+\gamma-\epsilon;\frac{w\mu}{1-\Delta},r\mu,t\mu;0,0\right),$$

$$\sum_{n=0}^{\infty}\frac{(\Lambda_2)_n}{n!}\hat{F}_D^3(\Lambda_1,\Lambda_2+n,\Lambda_3,\Lambda_4;\epsilon;w\mu,r\mu,t\mu;0,\chi)\Delta^n = (1-\Delta)^{-\Lambda_2}\hat{F}_D^3\left(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4;\epsilon;\frac{w\mu}{1-\Delta},r\mu,t\mu;0,\chi\right).$$

6. Conclusions

In this paper, we defined the new generalized hypergeometric functions $\hat{F}$, $\hat{F}_1$, $\hat{F}_2$, $\hat{F}_D^3$ and new generalized fractional operators $\hat{I}_{RL}^{\epsilon}$, $\hat{D}_{RL}^{\epsilon}$, $\hat{D}_{C}^{\epsilon}$, $\hat{I}_{KE}^{\gamma,\epsilon}$ by using the beta function ${}^{\Psi}\hat{B}_{\delta}(\xi,\eta)$. Then we calculated the fractional derivatives and integrals of some functions to obtain the generating function relations. Finally, we determined the generating function relations using generalized fractional operators. The generating function relations obtained here are not likely to be found in the literature.

To the best of our knowledge, the functions and operators defined here have a more general structure than most hypergeometric functions and fractional operators we have studied in the literature. The relations between the functions defined here and the functions in the literature can be summarized as follows.

Agarwal et al. [4]:

$${}^{\Psi}F_{\delta}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|w_1,w_2;w_3;\mu;0,\chi\right] = \frac{\Gamma(A)}{\Gamma(B)}F_{\delta;1,1}(w_1,w_2;w_3;\mu;\chi),$$

$${}^{\Psi}F_{1,\delta}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|w_1,w_2,w_3;w_4;\xi,\eta;0,\chi\right] = \frac{\Gamma(A)}{\Gamma(B)}F_{1,\delta;1,1}(w_1,w_2,w_3;w_4;\xi,\eta;\chi),$$

$${}^{\Psi}F_{2,\delta}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|w_1,w_2,w_3;w_4,w_5;\xi,\eta;0,\chi\right] = \frac{\Gamma(A)}{\Gamma(B)}F_{2,\delta;1,1}(w_1,w_2,w_3;w_4,w_5;\xi,\eta;\chi),$$

$${}^{\Psi}F_{D,\delta}^{3}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;0,\chi\right] = \frac{\Gamma(A)}{\Gamma(B)}F_{D,\delta;1,1}^{3}(w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;\chi),$$

$${}^{\Psi}D_{RL}^{\epsilon,\delta}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|\kappa(\mu)\right] = \frac{\Gamma(A)}{\Gamma(B)}\frac{d^m}{d\mu^m}D_{\mu}^{\epsilon-m,\delta;1,1}\kappa(\mu).$$

Agarwal et al. [5]:

$${}^{\Psi}F_{\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\middle|w_1,w_2;w_3;\mu;\chi,\chi\right] = F_{\delta;1}(w_1,w_2;w_3;\mu;\chi),$$

$${}^{\Psi}F_{1,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\middle|w_1,w_2,w_3;w_4;\xi,\eta;\chi,\chi\right] = F_{1,\delta;1}(w_1,w_2,w_3;w_4;\xi,\eta;\chi),$$

$$
\begin{aligned}
{}^{\Psi}F_{2,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3;w_4,w_5;\xi,\eta;\chi,\chi\right]&=F_{2,\delta;1}(w_1,w_2,w_3;w_4,w_5;\xi,\eta;\chi),\\
{}^{\Psi}F^3_{D,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;\chi,\chi\right]&=F^3_{D,\delta;1}(w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;\chi),\\
{}^{\Psi}D^{\epsilon,\delta}_{C}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|\kappa(\mu)\right]&=D^{\epsilon,\delta;1}_{\mu}\kappa(\mu).
\end{aligned}
$$

Kıymaz et al. [16]:

$$
\begin{aligned}
{}^{\Psi}F_{\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2;w_3;\mu;\chi,\chi\right]&={}_2F_1(w_1,w_2;w_3;\mu;\delta),\\
{}^{\Psi}F_{1,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3;w_4;\xi,\eta;\chi,\chi\right]&=F_1(w_1,w_2,w_3;w_4;\xi,\eta;\delta),\\
{}^{\Psi}F_{2,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3;w_4,w_5;\xi,\eta;\chi,\chi\right]&=F_2(w_1,w_2,w_3;w_4,w_5;\xi,\eta;\delta),\\
{}^{\Psi}F^3_{D,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;\chi,\chi\right]&=F^3_{D,\delta}(w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;\delta),\\
{}^{\Psi}D^{\epsilon,\delta}_{C}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|\kappa(\mu)\right]&=D^{\epsilon,\delta}_{\mu}\kappa(\mu).
\end{aligned}
$$

Luo et al. [19]: (for $\rho=\lambda=1$ in [19])

$$
{}^{\Psi}I^{\epsilon,\delta}_{RL}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\Big|\kappa(\mu)\right]=\frac{\Gamma(A)}{\Gamma(B)}I^{\epsilon,\delta}_{\mu}\left\{\kappa(\mu)\right\}.
$$

Özarslan et al. [20]:

$$
\begin{aligned}
{}^{\Psi}F_{1,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3;w_4;\xi,\eta;0,0\right]&=F_1(w_1,w_2,w_3;w_4;\xi,\eta;\delta),\\
{}^{\Psi}F_{2,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3;w_4,w_5;\xi,\eta;0,0\right]&=F_2(w_1,w_2,w_3;w_4,w_5;\xi,\eta;\delta),\\
{}^{\Psi}F^3_{D,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;0,0\right]&=F^3_{D,\delta}(w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu),\\
{}^{\Psi}D^{\epsilon,\delta}_{RL}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|\kappa(\mu)\right]&=\frac{d^m}{d\mu^m}D^{\epsilon-m}_{\mu}\left\{\kappa(\mu)\right\}.
\end{aligned}
$$

Parmar [21]:

$$
\begin{aligned}
{}^{\Psi}F_{1,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3;w_4;\xi,\eta;0,0\right]&=F_1(w_1,w_2,w_3;w_4;\xi,\eta;\delta;1),\\
{}^{\Psi}F_{2,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3;w_4,w_5;\xi,\eta;0,0\right]&=F_2(w_1,w_2,w_3;w_4,w_5;\xi,\eta;\delta;1),\\
{}^{\Psi}F^3_{D,\delta}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;0,0\right]&=F^3_{D,\delta;1}(w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu),\\
{}^{\Psi}D^{\epsilon,\delta}_{RL}\left[\begin{matrix}(B_i,A_i)_{1,0}\\(D_j,C_j)_{1,0}\end{matrix}\Big|\kappa(\mu)\right]&=\frac{d^m}{d\mu^m}D^{\epsilon-m;1}_{\mu}\left\{\kappa(\mu)\right\}.
\end{aligned}
$$

Srivastava et al. [27]: $\left(\text{for } K_l = \frac{(A)_l}{(B)_l}\right)$

$$^{\Psi}F_{1,\delta}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|w_1,w_2,w_3;w_4;\xi,\eta;0,0\right]=\frac{\Gamma(A)}{\Gamma(B)}F_1^{\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}(w_1,w_2,w_3;w_4;\xi,\eta;\delta),$$

$$^{\Psi}F_{2,\delta}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|w_1,w_2,w_3;w_4,w_5;\xi,\eta;0,0\right]=\frac{\Gamma(A)}{\Gamma(B)}F_2^{\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}(w_1,w_2,w_3;w_4,w_5;\xi,\eta;\delta),$$

$$^{\Psi}F_{D,\delta}^{3}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu;0,0\right]=\frac{\Gamma(A)}{\Gamma(B)}F^{(3)}_{D,\left(\{K_l\}_{l\in\mathbf{N_0}};\delta\right)}(w_1,w_2,w_3,w_4;w_5;\xi,\eta,\mu),$$

$$^{\Psi}D_{RL}^{\epsilon,\delta}\left[\begin{matrix}(A,1)_{1,1}\\(B,1)_{1,1}\end{matrix}\middle|\kappa(\mu)\right]=\frac{\Gamma(A)}{\Gamma(B)}\frac{d^m}{d\mu^m}D^{\epsilon-m,\delta}_{\mu,\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}\{\kappa(\mu)\}.$$

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