TWMS J. App. and Eng. Math. V.14, N.1, 2024, pp. 154-172

STABILITY OF DUAL CONTROLLED G-FUSION FRAMES IN HILBERT SPACES

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ABSTRACT. Some properties of controlled K-g-fusion frame have been discussed. Characterizations of controlled K-g-fusion frame are being presented. We also establish a relationship between quotient operator and controlled K-g-fusion frame. Some algebric properties of controlled K-g-fusion frame have been described. Finally, we shall discuss the stability of dual controlled g-fusion frame.

Keywords: g-fusion frame, K-g-fusion frame, quotient operator, controlled g-fusion frame, controlled K-g-fusion frame.

AMS Subject Classification: 42C15; 94A12; 46C07.

1. INTRODUCTION

In 1952, Duffin and Schaeffer [9] introduced frame for Hilbert space to study some fundamental problems in non-harmonic Fourier series. At present, frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, system modeling and so on. Several generalization of frames namely, K-frame [11], fusion frame [5], g-frame [22], g-fusion frame [14, 21] and K-g-fusion frame [1] etc had been proposed in recent times. P. Ghosh and T. K. Samanta [12] studied the stability of dual g-fusion frame in Hilbert space and they also discussed generalized atomic subspace for operator in Hilbert space and presented the frame operator for a pair of g-fusion Bessel sequences [13].

One of the newest generalization of frame is controlled frame. I. Bogdanova et al. [4] introduced controlled frame for spherical wavelets to get numerically more efficient approximation algorithm. Thereafter, weighted and controlled frame in Hilbert space was devepolded by P. Balaz [3]. In recent times, several generalizations of controlled frame

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§ Manuscript received: December 05, 2021; accepted: February 14, 2022.

namely, controlled K-frame [18], controlled g-frame [19], controlled fusion frame [16], controlled g-fusion frame [15], controlled K-g-fusion frame [20] etc. have been appeared.

In this paper, we develop some results in K-g-fusion frame to the controlled K-g-fusion frame. We construct new type of controlled g-fusion frame from a given controlled K-g-fusion frame by using a invertible bounded linear operator. A necessary and sufficient condition for controlled g-fusion Bessel sequence to be a controlled K-g-fusion frame is established. Finally, stability of controlled g-fusion frame and its dual have been presented.

Throughout this paper, H is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and $\{H_j\}_{j \in J}$ are the collection of Hilbert spaces, where J is subset of integers $\mathbb{Z}$. I_H is the identity operator on H. $\mathcal{B}(H_1, H_2)$ is a collection of all bounded linear operators from H_1 to H_2. In particular $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H. For $S \in \mathcal{B}(H)$, we denote $\mathcal{N}(S)$ and $\mathcal{R}(S)$ for null space and range of S, respectively. Also, $P_M \in \mathcal{B}(H)$ is the orthonormal projection onto a closed subspace $M \subset H$. $\mathcal{G}\mathcal{B}(H)$ denotes the set of all bounded linear operators which have bounded inverse. If $S, R \in \mathcal{G}\mathcal{B}(H)$, then R^*, R^{-1} and SR are also belongs to $\mathcal{G}\mathcal{B}(H)$. $\mathcal{G}\mathcal{B}^+(H)$ is the set of all positive operators in $\mathcal{G}\mathcal{B}(H)$.

In a complex Hilbert space, every bounded positive operator is self-adjoint and any two bounded positive operators can be commute with each other.

2. Preliminaries

In this section, we recall some necessary definitions and theorems.

Theorem 2.1. *(Douglas' factorization theorem)* [8] *Let $S, V \in \mathcal{B}(H)$. Then the following conditions are equivalent:*

(i) $\mathcal{R}(S) \subseteq \mathcal{R}(V)$.
(ii) $SS^* \leq \lambda^2 VV^*$ for some $\lambda > 0$.
(iii) $S = VW$ for some bounded linear operator W on H.

Theorem 2.2. [7] *The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$ which is defined as for $R, S \in \mathcal{S}(H)$*

$$R \leq S \Leftrightarrow \langle Rf, f \rangle \leq \langle Sf, f \rangle \;\; \forall f \in H.$$

Definition 2.1. [17] *A self-adjoint operator $U : H_1 \to H_1$ is called positive if $\langle Ux, x \rangle \geq 0$ for all $x \in H_1$. In notation, we can write $U \geq 0$. A self-adjoint operator $V : H_1 \to H_1$ is called a square root of U if $V^2 = U$. If, in addition $V \geq 0$, then V is called positive square root of U and is denoted by $V = U^{1/2}$.*

Theorem 2.3. [17] *The positive square root $V : H_1 \to H_1$ of an arbitrary positive self-adjoint operator $U : H_1 \to H_1$ exists and is unique. Further, the operator V commutes with every bounded linear operator on H_1 which commutes with U.*

Theorem 2.4. [10] *Let $M \subset H$ be a closed subspace and $T \in \mathcal{B}(H)$. Then $P_M T^* = P_M T^* P_{\overline{TM}}$. If T is an unitary operator (i. e $T^*T = I_H$), then $P_{\overline{TM}} T = T P_M$.*

Theorem 2.5. [7] *Let H_1, H_2 be two Hilbert spaces and $V : H_1 \to H_2$ be a bounded linear operator with closed range $\mathcal{R}_V$. Then there exists a bounded linear operator $V^\dagger : H_2 \to H_1$ such that $VV^\dagger x = x \;\forall x \in \mathcal{R}_V$.*

The operator $V^\dagger$ defined in Theorem 2.5, is called the pseudo-inverse of V.

Definition 2.2. [21] *Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights and let $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then the* *family* $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ *is called a generalized fusion frame or a g-fusion frame for* H *respect to* $\{H_j\}_{j \in J}$ *if there exist constants* $0 < A \leq B < \infty$ *such that*

$$A \, \|f\|^2 \leq \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j}(f) \right\|^2 \leq B \, \|f\|^2 \quad \forall f \in H. \tag{1}$$

The constants A *and* B *are called the lower and upper bounds of g-fusion frame, respectively. If* $A = B$ *then* Λ *is called tight g-fusion frame and if* $A = B = 1$ *then we say* Λ *is a Parseval g-fusion frame. If* Λ *satisfies only the right inequality of (1) it is called a g-fusion Bessel sequence with bound* B *in* H.

Define the space

$$l^2\left(\{H_j\}_{j \in J}\right) = \left\{ \{f_j\}_{j \in J} : f_j \in H_j, \; \sum_{j \in J} \|f_j\|^2 < \infty \right\}$$

with inner product is given by

$$\langle \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \rangle = \sum_{j \in J} \langle f_j, g_j \rangle_{H_j}.$$

Clearly $l^2\left(\{H_j\}_{j \in J}\right)$ is a Hilbert space with the pointwise operations [1].

Definition 2.3. [15] *Let* $\{W_j\}_{j \in J}$ *be a collection of closed subspaces of* H *and* $\{v_j\}_{j \in J}$ *be a collection of positive weights. Let* $\{H_j\}_{j \in J}$ *be a sequence of Hilbert spaces,* $T, U \in \mathcal{GB}(H)$ *and* $\Lambda_j \in \mathcal{B}(H, H_j)$ *for each* $j \in J$. *Then* $\Lambda_{TU} = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ *is a* (T, U)*-controlled g-fusion frame for* H *if there exist constants* $0 < A \leq B < \infty$ *such that*

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \|f\|^2 \quad \forall f \in H. \tag{2}$$

If $A = B$ *then* Λ_{TU} *is called* (T, U)*-controlled tight g-fusion frame and if* $A = B = 1$ *then we say* Λ_{TU} *is a* (T, U)*-controlled Parseval g-fusion frame. If* Λ_{TU} *satisfies only the right inequality of (2) it is called a* (T, U)*-controlled g-fusion Bessel sequence in* H.

Definition 2.4. [15] *Let* Λ_{TU} *be a* (T, U)*-controlled g-fusion Bessel sequence in* H *with a bound* B. *Suppose, for each* $j \in J$, *the operator* $T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U$ *is positive. The synthesis operator* $T_C : \mathcal{K}_{\Lambda_j} \to H$ *is defined as*

$$\begin{aligned} &T_C\left(\left\{ v_j \left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U\right)^{1/2} f \right\}_{j \in J}\right) \\ &= \sum_{j \in J} v_j^2 \, T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, \end{aligned}$$

for all $f \in H$ *and the analysis operator* $T_C^* : H \to \mathcal{K}_{\Lambda_j}$ *is given by*

$$T_C^* f = \left\{ v_j \left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U\right)^{1/2} f \right\}_{j \in J} \quad \forall f \in H,$$

where

$$\begin{aligned} \mathcal{K}_{\Lambda_j} &= \left\{ \left\{ v_j \left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U\right)^{1/2} f \right\}_{j \in J} : f \in H \right\} \\ &\subset l^2\left(\{H_j\}_{j \in J}\right). \end{aligned}$$

The frame operator $S_C : H \to H$ *is defined as follows:*

$$S_C f = T_C T_C^* f = \sum_{j \in J} v_j^2 T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f \;\; \forall f \in H$$

and it is easy to verify that

$$\langle S_C f, f \rangle = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \;\; \forall f \in H.$$

Furthermore, if Λ_{TU} *is a* (T, U)*-controlled g-fusion frame with bounds* A *and* B *then* $A I_H \leq S_C \leq B I_H$*. Hence,* S_C *is bounded, invertible, self-adjoint and positive linear operator. It is easy to verify that* $B^{-1} I_H \leq S_C^{-1} \leq A^{-1} I_H$*.*

Definition 2.5. [20] *Let* $K \in \mathcal{B}(H)$ *and* $\{W_j\}_{j \in J}$ *be a collection of closed subspaces of* H *and* $\{v_j\}_{j \in J}$ *be a collection of positive weights. Let* $\{H_j\}_{j \in J}$ *be a sequence of Hilbert spaces,* $T, U \in \mathcal{G}\mathcal{B}(H)$ *and* $\Lambda_j \in \mathcal{B}(H, H_j)$ *for each* $j \in J$*. Then the family* $\Lambda_{TU} = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ *is a* (T, U)*-controlled* K*-g-fusion frame for* H *if there exist constants* $0 < A \leq B < \infty$ *such that*

$$A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \| f \|^2 \;\; \forall f \in H.$$

If $A = B$ *then* Λ_{TU} *is called* (T, U)*-controlled tight* K*-g-fusion frame and if* $A = B = 1$ *then we say* Λ_{TU} *is a* (T, U)*-controlled Parseval* K*-g-fusion frame.*

Throughout this paper, Λ_{TU} denotes the family $\{(W_j, \Lambda_j, v_j)\}_{j \in J}$.

3. Some properties of controlled K-g-fusion frame

In this section, we describe few properties of controlled K-g-fusion frame. Constructions of controlled g-fusion frame from a given controlled K-g-fusion frame have been discussed. We also give some characterizations of controlled K-g-fusion frame. Relationship between controlled K-g-fusion frame and quotient operator is established. We also establish some algebric properties of controlled K-g-fusion frame. We start with this section by give an example of a controlled K-g-fusion frame.

3.1. **Example.** Let $H = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for H. Define $K : H \to H$ by $K f = \sum\limits_{j=1}^{3} \langle f, e_j \rangle e_j, \;\; f \in H$. Suppose $W_1 = \overline{span}\{e_2, e_3\}, \;\; W_2 = \overline{span}\{e_1, e_3\}$ and $W_3 = \overline{span}\{e_1, e_2\}$. Define $\Lambda_1 f = \langle f, e_2 \rangle e_3, \;\; \Lambda_2 f = \langle f, e_3 \rangle e_1$ and $\Lambda_3 f = 2 \langle f, e_1 \rangle e_2$. We show that $\{(W_j, \Lambda_j, 1)\}_{j=1}^{3}$ is a K-g-fusion frame for H. It is easy to verify that $K^* e_1 = e_1, \;\; K^* e_2 = e_2, \; K^* e_3 = e_3$. Now, for any $f \in H$, we have

$$\| K^* f \|^2 = \left\| \sum_{j=1}^{3} \langle f, e_j \rangle K^* e_j \right\|^2 = \left\| \sum_{j=1}^{3} \langle f, e_j \rangle e_j \right\|^2 = \| f \|^2,$$

$$\begin{aligned} \sum_{j=1}^{3} \left\| \Lambda_j P_{W_j} f \right\|^2 &= |\langle f, e_2 \rangle|^2 + |\langle f, e_3 \rangle|^2 + 4 \, |\langle f, e_1 \rangle|^2 \\ &= \| f \|^2 + 3 \, |\langle f, e_1 \rangle|^2 . \end{aligned}$$

Thus,

$$\| K^* f \|^2 \le \sum_{j=1}^{3} \left\| \Lambda_j P_{W_j} f \right\|^2 \le 4 \| f \|^2, \ \forall f \in H.$$

Let $T(f_1, f_2, f_3) = (2f_1, 3f_2, 5f_3)$ and $U(f_1, f_2, f_3) = \left(\frac{f_1}{2}, \frac{f_2}{3}, \frac{f_3}{4} \right)$ be two operators on H. Then it is easy to verify that $T, U \in \mathcal{GB}^+(H)$, $TU = UT$. Now, for any $f = (f_1, f_2, f_3) \in H$,

$$\sum_{j=1}^{3} \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle = 4 f_1^2 + f_2^2 + \frac{5}{4} f_3^2.$$

Thus,

$$\| K^* f \|^2 \le \sum_{j=1}^{3} \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \le 4 \| f \|^2, \ \forall f \in H.$$

Hence, $\{(W_j, \Lambda_j, 1)\}_{j=1}^{3}$ is a (T, U)-controlled K-g-fusion frame for H.

Theorem 3.1. *Let $K \in \mathcal{B}(H)$ and Λ_{TU} be a (T, U)-controlled g-fusion Bessel sequence in H with synthesis operator T_C. Then the following statements hold:*

- (i) *If Λ_{TU} is a (T, U)-controlled tight K-g-fusion frame for H, then $\mathcal{R}(T_C) = \mathcal{R}(K)$.*
- (ii) *$\mathcal{R}(T_C) = \mathcal{R}(K)$ if and only if there exist constants positive A, B such that*

$$A \| K^* f \|^2 \le \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \le B \| K^* f \|^2 \tag{3}$$

for all $f \in H$.

Proof. (i) Suppose Λ_{TU} is a (T, U)-controlled tight K-g-fusion frame for H. Then for each $f \in H$, there exists constant $A > 0$ such that

$$A \| K^* f \|^2 = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle = \| T_C^* f \|^2.$$

This shows that $A K K^* = T_C T_C^*$ and hence by Theorem 2.1, $\mathcal{R}(T_C) = \mathcal{R}(K)$.

(ii) First we suppose that $\mathcal{R}(T_C) = \mathcal{R}(K)$. Then by Theorem 2.1, there exist positive constants A and B such that $A K K^* \le T_C T_C^* \le B K K^*$ and therefore for each $f \in H$, we have

$$A \| K^* f \|^2 \le \| T_C^* f \|^2 = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \le B \| K^* f \|^2.$$

Conversely, suppose that (3) holds. Using synthesis operator T_C, the inequality (3) can be written as $A \| K^* f \|^2 \le \| T_C^* f \|^2 \le B \| K^* f \|^2$ and this implies that $A K K^* \le T_C T_C^* \le B K K^*$ and hence by Theorem 2.1, $\mathcal{R}(T_C) = \mathcal{R}(K)$. □

In the following theorem, we will see that every controlled g-fusion frame is a controlled K-g-fusion frame and the converse is also true under some condition.

Theorem 3.2. *Let $K \in \mathcal{B}(H)$. Then*

- (i) *Every (T, U)-controlled g-fusion frame is a (T, U)-controlled K-g-fusion frame.*
- (ii) *If $\mathcal{R}(K)$ is closed, every (T, U)-controlled K-g-fusion frame is a (T, U)-controlled g-fusion frame for $\mathcal{R}(K)$.*

Proof. (i) Let Λ_{TU} be a (T, U)-controlled g-fusion frame for H with bounds A and B. Then for each $f \in H$, we have

$$\frac{A}{\|K\|^2}\|K^* f\|^2 \leq A\|f\|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B\|f\|^2.$$

Hence, Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H with bounds $\dfrac{A}{\|K\|^2}$ and B.

(ii) Let Λ_{TU} be a (T, U)-controlled K-g-fusion frame for H with bounds A and B. Since $\mathcal{R}(K)$ is closed, by Theorem 2.5, there exists an operator $K^\dagger \in \mathcal{B}(H)$ such that $K K^\dagger f = f \ \forall f \in \mathcal{R}(K)$. Then for each $f \in \mathcal{R}(K)$,

$$\frac{A}{\|K^\dagger\|^2}\|f\|^2 \leq A\|K^* f\|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B\|f\|^2.$$

Thus, Λ_{TU} is a (T, U)-controlled g-fusion frame for $\mathcal{R}(K)$ with bounds $\dfrac{A}{\|K^\dagger\|^2}$ and B. □

In the next two theorems, we will construct new type of controlled g-fusion frame from a given controlled K-g-fusion frame by using a invertible bounded linear operator.

Theorem 3.3. *Let $V \in \mathcal{B}(H)$ be an invertible operator on H and V^* commutes with T and U. Let Λ_{TU} be a (T, U)-controlled K-g-fusion frame for H for some $K \in \mathcal{B}(H)$. Then $\Gamma_{TU} = \left\{\left(V W_j, \Lambda_j P_{W_j} V^*, v_j\right)\right\}_{j \in J}$ is a (T, U)-controlled $V K V^*$-g-fusion frame for H.*

Proof. Since Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H, there exist positive constants A and B such that

$$A\|K^* f\|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B\|f\|^2.$$

Now, for each $f \in H$, using Theorem 2.4, we have

$$\begin{aligned}
&\sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U f, \Lambda_j P_{W_j} V^* P_{V W_j} T f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* U f, \Lambda_j P_{W_j} V^* T f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U V^* f, \Lambda_j P_{W_j} T V^* f \right\rangle \\
&\leq B\|V^* f\|^2 \leq B\|V\|^2\|f\|^2.
\end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned}
\frac{A}{\|V\|^2}\|(V K V^*)^* f\|^2 &= \frac{A}{\|V\|^2}\|V K^* V^* f\|^2 \leq A\|K^* V^* f\|^2 \\
&\leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U V^* f, \Lambda_j P_{W_j} T V^* f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* U f, \Lambda_j P_{W_j} V^* T f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U f, \Lambda_j P_{W_j} V^* P_{V W_j} T f \right\rangle.
\end{aligned}$$

Thus, Γ_{TU} is a (T, U)-controlled $V K V^*$-g-fusion frame for H. $\square$

Theorem 3.4. *Let $V \in \mathcal{B}(H)$ be an invertible operator on H and V^*, $(V^{-1})^*$ commutes with T and U. Let $\Gamma_{TU} = \left\{ \left(V W_j, \Lambda_j P_{W_j} V^*, v_j \right) \right\}_{j \in J}$ is a (T, U)-controlled K-g-fusion frame for H, for some $K \in \mathcal{B}(H)$. Then Λ_{TU} is a (T, U)-controlled $V^{-1} K V$-g-fusion frame for H.*

Proof. Since Γ_{TU} is a (T, U)-controlled K-g-fusion frame for H, for each $f \in H$, there exist constants $A, B > 0$ such that

$$A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U f, \Lambda_j P_{W_j} V^* P_{V W_j} T f \right\rangle \leq B \| f \|^2.$$

Now, for each $f \in H$, using Theorem 2.4, we have

$$\begin{aligned}
\frac{A}{\| V \|^2} \left\| \left(V^{-1} K V \right)^* f \right\|^2 &= \frac{A}{\| V \|^2} \left\| V^* K^* (V^{-1})^* f \right\|^2 \\
&\leq A \left\| K^* \left(V^{-1} \right)^* f \right\|^2 \\
&\leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U \left(V^{-1} \right)^* f, \Lambda_j P_{W_j} V^* P_{V W_j} T \left(V^{-1} \right)^* f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* U \left(V^{-1} \right)^* f, \Lambda_j P_{W_j} V^* T \left(V^{-1} \right)^* f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* \left(V^{-1} \right)^* U f, \Lambda_j P_{W_j} V^* \left(V^{-1} \right)^* T f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle.
\end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned}
&\sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U V^* \left(V^{-1} \right)^* f, \Lambda_j P_{W_j} T V^* \left(V^{-1} \right)^* f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* U \left(V^{-1} \right)^* f, \Lambda_j P_{W_j} V^* T \left(V^{-1} \right)^* f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U \left(V^{-1} \right)^* f, \Lambda_j P_{W_j} V^* P_{V W_j} T \left(V^{-1} \right)^* f \right\rangle \\
&\leq B \left\| \left(V^{-1} \right)^* f \right\|^2 \leq B \left\| V^{-1} \right\|^2 \| f \|^2.
\end{aligned}$$

Thus, Λ_{TU} is a (T, U)-controlled $V^{-1} K V$-g-fusion frame for H. $\square$

In the following theorem, we will construct a controlled K-g-fusion frame by using a controlled g-fusion frame under some sufficient conditions.

Theorem 3.5. *Let $K \in \mathcal{B}(H)$ be an invertible operator on H and Λ_{TU} be a (T, U)-controlled g-fusion frame for H with frame bounds A, B and S_C be the associated frame operator. Suppose $S_C^{-1} K^*$ commutes with T and U. Then the family $\Gamma_{TU} = \left\{ \left(K S_C^{-1} W_j, \Lambda_j P_{W_j} S_C^{-1} K^*, v_j \right) \right\}_{j \in J}$ is a (T, U)-controlled K-g-fusion frame for H with the corresponding frame operator $K S_C^{-1} K^*$.*

Proof. Let $V = K\,S_C^{-1}$. Then V is invertible on H and $V^* = S_C^{-1}\,K^*$. Now, it is easy to verify that

$$\|\,K^*\,f\,\|^2 \,\leq\, B^2\,\left\|\,S_C^{-1}\,K^*\,f\,\right\|^2 \quad \forall\, f \,\in\, H.$$

Now, for each $f \in H$, using Theorem 2.4, we have

$$\begin{aligned}
&\sum_{j \in J} v_j^2\,\left\langle\,\Lambda_j\,P_{W_j}\,V^*\,P_{V\,W_j}\,U\,f,\,\Lambda_j\,P_{W_j}\,V^*\,P_{V\,W_j}\,T\,f\,\right\rangle \\
&\quad= \sum_{j \in J} v_j^2\,\left\langle\,\Lambda_j\,P_{W_j}\,V^*\,U\,f,\,\Lambda_j\,P_{W_j}\,V^*\,T\,f\,\right\rangle \\
&\quad= \sum_{j \in J} v_j^2\,\left\langle\,\Lambda_j\,P_{W_j}\,U\,S_C^{-1}\,K^*\,f,\,\Lambda_j\,P_{W_j}\,T\,S_C^{-1}\,K^*\,f\,\right\rangle \\
&\quad\leq B\,\|\,S_C^{-1}\,\|^2\,\,\|\,K^*\,f\,\|^2 \\
&\quad\leq \frac{B}{A^2}\,\|\,K\,\|^2\,\|\,f\,\|^2\;\;[\text{ using } B^{-1}\,I_H \,\leq\, S_C^{-1} \,\leq\, A^{-1}\,I_H\,].
\end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned}
&\sum_{j \in J} v_j^2\,\left\langle\,\Lambda_j\,P_{W_j}\,V^*\,P_{V\,W_j}\,U\,f,\,\Lambda_j\,P_{W_j}\,V^*\,P_{V\,W_j}\,T\,f\,\right\rangle \\
&\quad= \sum_{j \in J} v_j^2\,\left\langle\,\Lambda_j\,P_{W_j}\,U\,S_C^{-1}\,K^*\,f,\,\Lambda_j\,P_{W_j}\,T\,S_C^{-1}\,K^*\,f\,\right\rangle \\
&\quad\geq A\,\left\|\,S_C^{-1}\,K^*\,f\,\right\|^2 \,\geq\, \frac{A}{B^2}\,\|\,K^*\,f\,\|^2.
\end{aligned}$$

Thus, $\Gamma_{T\,U}$ is a $(T,\,U)$-controlled K-g-fusion frame for H.
Furthermore, for each $f \in H$, we have

$$\begin{aligned}
&\sum_{j \in J} v_j^2\,T^*P_{V\,W_j}\,\left(\Lambda_j\,P_{W_j}\,V^*\right)^*\,\left(\Lambda_j\,P_{W_j}\,V^*\right)\,P_{V\,W_j}\,U\,f \\
&= \sum_{j \in J} v_j^2\,T^*\,\left(P_{V\,W_j}\,V\,P_{W_j}\right)\,\Lambda_j^*\,\Lambda_j\,\left(P_{W_j}\,V^*\,P_{V\,W_j}\right)\,U\,f \\
&= \sum_{j \in J} v_j^2\,T^*\,\left(P_{W_j}\,V^*\,P_{V\,W_j}\right)^*\,\Lambda_j^*\,\Lambda_j\,(\,P_{W_j}\,V^*\,P_{V\,W_j}\,)\,U\,f \\
&= \sum_{j \in J} v_j^2\,T^*\,\left(P_{W_j}\,V^*\right)^*\,\Lambda_j^*\,\Lambda_j\,P_{W_j}\,V^*\,U\,f\;\;\;[\text{ using Theorem 2.4 }] \\
&= \sum_{j \in J} v_j^2\,T^*\,V\,P_{W_j}\,\Lambda_j^*\,\Lambda_j\,P_{W_j}\,V^*\,U\,f \\
&= \sum_{j \in J} v_j^2\,V\,T^*\,P_{W_j}\,\Lambda_j^*\,\Lambda_j\,P_{W_j}\,U\,V^*\,f \,=\, V\,S_C\,V^*\,f \\
&= \left(K\,S_C^{-1}\right)\,S_C\,\left(S_C^{-1}\,K^*\,f\right) \,=\, K\,S_C^{-1}\,K^*\,f.
\end{aligned}$$

This implies that $K\,S_C^{-1}\,K^*$ is the corresponding frame operator of $\Gamma_{T\,U}$. □

Corollary 3.1. *Let $\Lambda_{T\,U}$ be a $(T,\,U)$-controlled g-fusion frame for H with frame operator S_C. If P_V is the orthogonal projection onto closed subspace $V \subset H$ and $S_C^{-1}\,P_V$ commutes with $T,\,U$ then $\left\{\,\left(\,P_V\,S_\Lambda^{-1}\,W_j,\;\Lambda_j\,P_{W_j}\,S_\Lambda^{-1}\,P_V,\;v_j\,\right)\,\right\}_{j \in J}$ is a $(T,\,U)$-controlled K-g-fusion frame for H with the corresponding frame operator $P_V\,S_C^{-1}\,P_V$.*

Proof. Proof of this corollary directly follows from the Theorem 3.5, by putting $K = P_V$. □

Theorem 3.6. *Let $K \in \mathcal{B}(H)$ and Λ_{TU} be a (T, U)-controlled K-g-fusion frame for H with frame bounds A, B. If $V \in \mathcal{B}(H)$ with $\mathcal{R}(V) \subset \mathcal{R}(K)$, then Λ_{TU} is a (T, U)-controlled V-g-fusion frame for H.*

Proof. Since Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H, for each $f \in H$,

$$A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \| f \|^2.$$

Since $\mathcal{R}(V) \subset \mathcal{R}(K)$, by Theorem 2.1, there exists some $\lambda > 0$ such that $V V^* \leq \lambda K K^*$. Thus, for each $f \in H$, we have

$$\frac{A}{\lambda} \| V^* f \|^2 \leq A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \| f \|^2.$$

Hence, Λ_{TU} is a (T, U)-controlled V-g-fusion frame for H. □

The following theorem shows that any controlled K-g-fusion frame is a K-g-fusion frame and conversely any K-g-fusion frame is a controlled K-g-fusion frame under some conditions.

Theorem 3.7. *Let $K \in \mathcal{B}(H)$, $T, U \in \mathcal{G}\mathcal{B}^+(H)$ and K commutes with T and U and $S_\Lambda T = T S_\Lambda$. Then Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H if and only if Λ_{TU} is a K-g-fusion frame for H, where S_Λ is the K-g-fusion frame operator defined by*

$$S_\Lambda f = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} f, \quad f \in H.$$

Proof. First we suppose that Λ_{TU} is a K-g-fusion frame for H with bounds A and B. Then for each $f \in H$, we have

$$A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} f \right\|^2 \leq B \| f \|^2.$$

Now according to the Lemma 3.10 of [2], we can deduced that

$$m m' A K K^* \leq T S_\Lambda U \leq M M' B I_H,$$

where m, m' and M, M' are positive constants. Then for each $f \in H$, we have

$$m m' A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle T P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle \leq M M' B \| f \|^2$$

$$\Rightarrow m m' A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq M M' B \| f \|^2.$$

Hence, Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H.

Conversely, suppose that Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H. Then for each $f \in H$, there exist constants $A, B > 0$ such that

$$A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \| f \|^2.$$

Now, for each $f \in H$, we have

$$A \| K^* f \|^2 = A \left\| (TU)^{1/2} (TU)^{-1/2} K^* f \right\|^2$$
$$= A \left\| (TU)^{1/2} K^* (TU)^{-1/2} f \right\|^2$$
$$\leq \left\| (TU)^{1/2} \right\|^2 \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U (TU)^{-1/2} f, \Lambda_j P_{W_j} T (TU)^{-1/2} f \right\rangle$$
$$= \left\| (TU)^{1/2} \right\|^2 \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U^{1/2} T^{-1/2} f, \Lambda_j P_{W_j} T^{1/2} U^{-1/2} f \right\rangle$$
$$= \left\| (TU)^{1/2} \right\|^2 \left\langle \sum_{j \in J} v_j^2 U^{-1/2} T^{1/2} P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U^{1/2} T^{-1/2} f, f \right\rangle$$
$$= \left\| (TU)^{1/2} \right\|^2 \left\langle U^{-1/2} T^{1/2} S_\Lambda U^{1/2} T^{-1/2} f, f \right\rangle = \left\| (TU)^{1/2} \right\|^2 \langle S_\Lambda f, f \rangle$$
$$= \left\| (TU)^{1/2} \right\|^2 \sum_{j \in J} v_j^2 \langle P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} f, f \rangle$$
$$\Rightarrow \frac{A}{\| (TU)^{1/2} \|^2} \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (f) \right\|^2.$$

On the other hand, it is easy to verify that

$$\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (f) \right\|^2 = \langle S_\Lambda f, f \rangle \leq B \left\| (TU)^{-1/2} \right\|^2 \| f \|^2.$$

Thus, Λ_{TU} is a K-g-fusion frame for H. This completes the proof. □

A characterization of a controlled K-g-fusion frame is given by in the next theorem.

Theorem 3.8. *Let $K \in \mathcal{B}(H)$, $T, U \in \mathcal{GB}^+(H)$ and K commutes with T and U and $S_\Lambda T = T S_\Lambda$. Then Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H if and only if Λ_{TU} is a (TU, I_H)-controlled K-g-fusion frame for H, where S_Λ is the K-g-fusion frame operator defined by*

$$S_\Lambda f = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} f, \quad f \in H.$$

.

Proof. For each $f \in H$, we have

$$\sum_{j \in J} v_j^2 \langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \rangle = \left\langle \sum_{j \in J} v_j^2 T P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle$$
$$= \langle T S_\Lambda U f, f \rangle = \langle S_\Lambda T U f, f \rangle$$
$$= \left\langle \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} T U f, f \right\rangle$$
$$= \sum_{j \in J} v_j^2 \langle \Lambda_j P_{W_j} T U f, \Lambda_j P_{W_j} f \rangle.$$

Hence, Λ_{TU} is (T, U)-controlled K-g-fusion frame for H with bounds A and B is equivalent to:

$$A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} T U f, \Lambda_j P_{W_j} f \right\rangle \leq B \| f \|^2 \ \forall f \in H.$$

Thus, Λ_{TU} is a (TU, I_H)-controlled K-g-fusion frame for H with bounds A and B. This completes the proof. □

Corollary 3.2. *Let $K \in \mathcal{B}(H)$, $T, U \in \mathcal{G}\mathcal{B}^+(H)$ and K commutes with T and U and $S_\Lambda T = T S_\Lambda$. Then Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H if and only if Λ_{TU} is a $\left((TU)^{1/2}, (TU)^{1/2}\right)$-controlled K-g-fusion frame for H.*

Proof. According to the proof of the Theorem 3.8, for each $f \in H$, we have

$$\begin{aligned}
\sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle &= \left\langle S_\Lambda (TU)^{1/2} f, (TU)^{1/2} f \right\rangle \\
&= \left\langle \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} (TU)^{1/2} f, (TU)^{1/2} f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} (TU)^{1/2} f, \Lambda_j P_{W_j} (TU)^{1/2} f \right\rangle.
\end{aligned}$$

Thus, Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H if and only if Λ_{TU} is a $\left((TU)^{1/2}, (TU)^{1/2}\right)$-controlled K-g-fusion frame for H. □

In the following theorem, we give a necessary and sufficient condition for controlled g-fusion Bessel sequence to be a controlled K-g-fusion frame with the help of quotient operator.

Theorem 3.9. *Let $K \in \mathcal{B}(H)$ and Λ_{TU} be a (T, U)-controlled g-fusion Bessel sequence in H with frame operator S_C. Then Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H if and only if the quotient operator $\left[K^* / S_C^{1/2}\right]$ is bounded.*

Proof. First we suppose that Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H with bounds A and B. Then for each $f \in H$, we have

$$A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \| f \|^2.$$

Thus, for each $f \in H$, we have

$$A \| K^* f \|^2 \leq \langle S_C f, f \rangle = \left\| S_C^{1/2} f \right\|^2.$$

Now, it is easy to verify that the quotient operator $T : \mathcal{R}\left(S_C^{1/2}\right) \to \mathcal{R}(K^*)$ defined by $T\left(S_C^{1/2} f\right) = K^* f \ \forall f \in H$ is well-defined and bounded.

Conversely, suppose that the quotient operator $\left[K^* / S_C^{1/2}\right]$ is bounded. Then for each $f \in H$, there exists some $B > 0$ such that

$$\begin{aligned}
&\| K^* f \|^2 \leq B \left\| S_C^{1/2} f \right\|^2 = B \langle S_C f, f \rangle \\
&\Rightarrow \| K^* f \|^2 \leq B \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle.
\end{aligned}$$

Thus, Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H. □

Now, we establish that a quotient operator will be bounded if and only if a controlled K-g-fusion frame becomes controlled $V\,K$-g-fusion frame, for some $V \in \mathcal{B}(H)$.

Theorem 3.10. *Let $K \in \mathcal{B}(H)$ and Λ_{TU} be a (T, U)-controlled K-g-fusion frame for H with frame operator S_C. Let $V \in \mathcal{B}(H)$ be an invertible operator on H and V^* commutes with T and U. Then the following statements are equivalent:*

(*i*) $\Gamma_{TU} = \left\{ \left(V W_j,\, \Lambda_j P_{W_j} V^*,\, v_j \right) \right\}_{j \in J}$ *is a (T, U)-controlled $V\,K$-g-fusion frame for H.*

(*ii*) *The quotient operator* $\left[(V K)^* \,/\, S_C^{1/2} V^* \right]$ *is bounded.*

(*iii*) *The quotient operator* $\left[(V K)^* \,/\, (V S_C V^*)^{1/2} \right]$ *is bounded.*

Proof. (*i*) $\Rightarrow$ (*ii*) Suppose Γ_{TU} is a (T, U)-controlled $V\,K$-g-fusion frame with bounds A and B. Then for each $f \in H$, we have

$$A \,\|(V K)^* f\|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U f,\; \Lambda_j P_{W_j} V^* P_{V W_j} T f \right\rangle \leq B \,\|f\|^2.$$

By Theorem 2.4, for each $f \in H$, we have

$$\begin{aligned} &\sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U f,\; \Lambda_j P_{W_j} V^* P_{V W_j} T f \right\rangle \\ &\quad = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* U f,\; \Lambda_j P_{W_j} V^* T f \right\rangle \\ &\quad = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U V^* f,\; \Lambda_j P_{W_j} T V^* f \right\rangle = \langle S_C V^* f, V^* f \rangle. \qquad (4) \end{aligned}$$

Thus, for each $f \in H$, we have

$$A \,\|(V K)^* f\|^2 \leq \langle S_C V^* f, V^* f \rangle = \left\| S_C^{1/2} V^* f \right\|^2.$$

Now, we define a operator $T : \mathcal{R}\left(S_C^{1/2} V^* \right) \to \mathcal{R}((V K)^*)$ by $T\left(S_C^{1/2} V^* f \right) = (V K)^* f \;\; \forall\, f \in H$. Then it is easy verify that the quotient operator T is well-defined and bounded.

(*ii*) $\Rightarrow$ (*iii*) It is obvious.

(*iii*) $\Rightarrow$ (*i*) Suppose that the quotient operator $\left[(V K)^* \,/\, (V S_C V^*)^{1/2} \right]$ is bounded. Then for each $f \in H$, there exists $B > 0$ such that

$$\|(V K)^* f\|^2 \leq B \left\| (V S_C V^*)^{1/2} f \right\|^2.$$

Now, by (4), for each $f \in H$, we have

$$\begin{aligned} &\sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U f,\; \Lambda_j P_{W_j} V^* P_{V W_j} T f \right\rangle \\ &\quad = \langle S_C V^* f, V^* f \rangle = \left\| (V S_C V^*)^{1/2} f \right\|^2 \geq \frac{1}{B} \,\|(V K)^* f\|^2. \end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned}&\sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} V^* P_{V W_j} U f,\ \Lambda_j P_{W_j} V^* P_{V W_j} T f \right\rangle \\ &\quad = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U V^* f,\ \Lambda_j P_{W_j} T V^* f \right\rangle \\ &\quad \leq C \, \| U^* f \|^2 \leq C \, \| U \|^2 \, \| f \|^2.\end{aligned}$$

Hence, Γ_{TU} is a (T, U)-controlled $V K$-g-fusion frame for H.
This completes the proof. □

In the following theorem, we will present some algebric properties of controlled K-g-fusion frame.

Theorem 3.11. *Let $K_i \in \mathcal{B}(H)$ and Λ_{TU} be a (T, U)-controlled K_i-g-fusion frame for H for all $i = 1, 2, \cdots, n$. Then*

- (i) *If a_i, for $i = 1, 2, \cdots, n$, are finite collection of scalars, then Λ_{TU} is a (T, U)-controlled $\sum_{i=1}^{n} a_i K_i$-g-fusion frame for H.*
- (ii) *Λ_{TU} is a (T, U)-controlled $\prod_{i=1}^{n} K_i$-g-fusion frame for H.*

Proof. (i) Since Λ_{TU} is a (T, U)-controlled K_i-g-fusion frame for H for all i, there exist constants $A, B > 0$ such that

$$A \, \| K_i^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \, \| f \|^2.$$

Then for each $f \in H$, we have

$$\begin{aligned}&\frac{A}{n \max\limits_i |a_i|^2} \left\| \left(\sum_{i=1}^{n} a_i K_i \right)^* f \right\|^2 \leq A \, \| K_i^* f \|^2 \\ &\quad \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \, \| f \|^2.\end{aligned}$$

Thus, Λ_{TU} is a (T, U)-controlled $\sum_{i=1}^{n} a_i K_i$-g-fusion frame for H.

Proof of (ii) For each $f \in H$, we have

$$\begin{aligned}&\frac{A}{\prod_{i=1}^{n-1} \| K_i^* \|^2} \left\| \left(\prod_{i=1}^{n} K_i \right)^* f \right\|^2 \leq A \, \| K_n^* f \|^2 \\ &\quad \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \, \| f \|^2.\end{aligned}$$

Thus, Λ_{TU} is a (T, U)-controlled $\prod_{i=1}^{n} K_i$-g-fusion frame for H. □

4. Stability of dual controlled g-fusion frame

In frame theory, one of the most important problem is the stability of frame under some perturbation. P. Casazza and Chirstensen [6] have been generalized the Paley-Wiener perturbation theorem to perturbation of frame in Hilbert space. P. Ghosh and T. K. Samanta [12] discussed stability of dual g-fusion frame in Hilbert space. In this section, we give an important on stability of perturbation of controlled K-g-fusion frame and dual controlled g-fusion frame.

Following theorem provides a sufficient condition on a family Λ_{TU} to be a controlled K-g-fusion frame, in the presence of another controlled K-g-fusion frame.

Theorem 4.1. *Let Λ_{TU} be a (T, U)-controlled K-g-fusion frame for H with bounds A, B and $\Gamma_{TU} = \{(V_j, \Gamma_j, v_j)\}_{j \in J}$. If there exist constants $\lambda_1, \lambda_2, \mu$ with $0 \leq \lambda_1, \lambda_2 < 1$, $0 \leq \mu < A(1 - \lambda_1)$ such that for each $f \in H$,*

$$0 \leq \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} - P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} \right) U f, f \right\rangle \leq \mu \|K^* f\|^2 +$$

$$+ \lambda_1 \sum_{j \in J} v_j^2 \left\langle T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle + \lambda_2 \sum_{j \in J} v_j^2 \left\langle T^* P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} U f, f \right\rangle$$

then Γ_{TU} is a (T, U)-controlled K-g-fusion frame for H.

Proof. Since Λ_{TU} is a (T, U)-controlled K-g-fusion frame for H with bounds A, B, for each $f \in H$, we have

$$A \|K^* f\|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \|f\|^2.$$

Now, for each $f \in H$, we have

$$\sum_{j \in J} v_j^2 \left\langle T^* P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} U f, f \right\rangle$$

$$= \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} - P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} \right) U f, f \right\rangle +$$

$$+ \sum_{j \in J} v_j^2 \left\langle T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle.$$

$$\Rightarrow (1 - \lambda_2) \sum_{j \in J} v_j^2 \left\langle T^* P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} U f, f \right\rangle$$

$$\leq (1 + \lambda_1) \sum_{j \in J} v_j^2 \left\langle T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle + \mu \|K^* f\|^2.$$

$$\Rightarrow (1 - \lambda_2) \sum_{j \in J} v_j^2 \left\langle \Gamma_j P_{V_j} U f, \Gamma_j P_{V_j} T f \right\rangle$$

$$\leq (1 + \lambda_1) \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle + \mu \|K\|^2 \|f\|^2$$

$$\Rightarrow \sum_{j \in J} v_j^2 \left\langle \Gamma_j P_{V_j} U f, \Gamma_j P_{V_j} T f \right\rangle \leq \left[\frac{(1 + \lambda_1) B + \mu \|K\|^2}{(1 - \lambda_2)} \right] \|f\|^2.$$

On the other hand, for each $f \in H$, we have

$$\sum_{j \in J} v_j^2 \left\langle T^* P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} U f, f \right\rangle \geq \sum_{j \in J} v_j^2 \left\langle T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle -$$
$$- \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} - P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} \right) U f, f \right\rangle.$$
$$\Rightarrow (1 + \lambda_2) \sum_{j \in J} v_j^2 \left\langle T^* P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} U f, f \right\rangle$$
$$\geq (1 - \lambda_1) \sum_{j \in J} v_j^2 \left\langle T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle - \mu \, \| K^* f \|^2.$$
$$\Rightarrow \sum_{j \in J} v_j^2 \left\langle \Gamma_j P_{V_j} U f, \Gamma_j P_{V_j} T f \right\rangle \geq \left[\frac{(1 - \lambda_1) A - \mu}{(1 + \lambda_2)} \right] \| K^* f \|^2.$$

Thus, Γ_{TU} is a (T, U)-controlled K-g-fusion frame for H. □

Corollary 4.1. *Let Λ_{TU} be a (T, U)-controlled K-g-fusion frame for H with bounds A, B and $\Gamma_{TU} = \{ (V_j, \Gamma_j, v_j) \}_{j \in J}$. If there exists constant $0 < D < A$ such that for each $f \in H$,*

$$0 \leq \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} - P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} \right) U f, f \right\rangle \leq D \, \| K^* f \|^2$$

then Γ_{TU} is a (T, U)-controlled K-g-fusion frame for H.

Proof. For each $f \in H$, we have

$$\sum_{j \in J} v_j^2 \left\langle \Gamma_j P_{V_j} U f, \Gamma_j P_{V_j} T f \right\rangle = \sum_{j \in J} v_j^2 \left\langle T^* P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} U f, f \right\rangle$$
$$= \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} - P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} \right) U f, f \right\rangle +$$
$$+ \sum_{j \in J} v_j^2 \left\langle T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle$$
$$\leq \left(B + D \, \| K \|^2 \right) \| f \|^2.$$

On the other hand,

$$\sum_{j \in J} v_j^2 \left\langle T^* P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} U f, f \right\rangle \geq \sum_{j \in J} v_j^2 \left\langle T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f, f \right\rangle -$$
$$- \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} - P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} \right) U f, f \right\rangle$$
$$\geq (A - D) \, \| K^* f \|^2 \quad \forall f \in H.$$

This completes the proof. □

Theorem 4.2. *Let Λ_{TU} be a (T, U)-controlled g-fusion frame for H and S_C be corresponding frame operator. Assume that S_C^{-1} is commutes with T and U. Then $\Gamma_{TU} = \left\{ \left(S_C^{-1} W_j, \Lambda_j P_{W_j} S_C^{-1}, v_j \right) \right\}_{j \in J}$ is a (T, U)-controlled g-fusion frame for H with the corresponding frame operator S_C^{-1}.*

Proof. Proof of this theorem directly follows from the Theorem 3.5, by putting $K = I_H$. □

The family Γ_{TU} defined in the Theorem 4.2 is called the canonical dual (T, U)-controlled g-fusion frame of Λ_{TU}. We now give the stability of dual controlled g-fusion frame.

Theorem 4.3. *Let Λ_{TU} and Γ_{TU} be two (T, U)-controlled g-fusion frames for H with bounds A_1, B_1 and A_2, B_2 having their corresponding frame operators S_C and S'_C, respectively. Consider $\Delta_{TU} = \{(X_j, \Delta_j, v_j)\}_{j \in J}$ and $\Theta_{TU} = \{(Y_j, \Theta_j, v_j)\}_{j \in J}$ as the canonical dual (T, U)-controlled g-fusion frames of Λ_{TU} and Γ_{TU}, respectively. Assume that S_C^{-1} and $(S'_C)^{-1}$ commutes with both T and U. Then the following statements hold:*

(i) *If the condition*

$$\left| \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle - \sum_{j \in J} v_j^2 \left\langle \Gamma_j P_{V_j} U f, \Gamma_j P_{V_j} T f \right\rangle \right| \leq D \|f\|^2$$

holds for each $f \in H$ and for some $D > 0$ then for all $f \in H$, we have

$$\left| \sum_{j \in J} v_j^2 \left\langle \Delta_j P_{X_j} U f, \Delta_j P_{X_j} T f \right\rangle - \sum_{j \in J} v_j^2 \left\langle \Theta_j P_{Y_j} U f, \Theta_j P_{Y_j} T f \right\rangle \right| \leq \frac{D}{A_1 A_2} \|f\|^2.$$

(ii) *If for each $f \in H$, there exists $D > 0$ such that*

$$\left| \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} - P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} \right) U f, f \right\rangle \right| \leq D \|f\|^2$$

then

$$\left| \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{X_j} \Delta_j^* \Delta_j P_{X_j} - P_{Y_j} \Theta_j^* \Theta_j P_{Y_j} \right) U f, f \right\rangle \right| \leq \frac{D}{A_1 A_2} \|f\|^2.$$

Proof. (i) Since $S_C - S'_C$ is self-adjoint so

$$\begin{aligned} \|S_C - S'_C\| &= \sup_{\|f\|=1} \left| \left\langle (S_C - S'_C) f, f \right\rangle \right| \\ &= \sup_{\|f\|=1} \left| \left\langle S_C f, f \right\rangle - \left\langle S'_C f, f \right\rangle \right| \\ &= \sup_{\|f\|=1} \left| \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle - \sum_{j \in J} v_j^2 \left\langle \Gamma_j P_{V_j} U f, \Gamma_j P_{V_j} T f \right\rangle \right| \\ &\leq \sup_{\|f\|=1} D \|f\|^2 = D. \end{aligned}$$

Then

$$\begin{aligned} \left\| S_C^{-1} - (S'_C)^{-1} \right\| &\leq \|S_C^{-1}\| \, \|S_C - S'_C\| \, \left\| (S'_C)^{-1} \right\| \\ &\leq \frac{1}{A_1} D \frac{1}{A_2} = \frac{D}{A_1 A_2}. \end{aligned} \tag{5}$$

Now, for each $f \in H$, we have

$$\begin{aligned}
&\sum_{j \in J} v_j^2 \left\langle \Delta_j P_{X_j} U f, \Delta_j P_{X_j} T f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} S_C^{-1} P_{S_C^{-1} W_j} U f, \Lambda_j P_{W_j} S_C^{-1} P_{S_C^{-1} W_j} T f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} S_C^{-1} U f, \Lambda_j P_{W_j} S_C^{-1} T f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U S_C^{-1} f, \Lambda_j P_{W_j} T S_C^{-1} f \right\rangle \\
&= \sum_{j \in J} v_j^2 \left\langle T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U S_C^{-1} f, S_C^{-1} f \right\rangle \\
&= \left\langle S_C S_C^{-1} f, S_C^{-1} f \right\rangle = \left\langle f, S_C^{-1} f \right\rangle .
\end{aligned}$$

Similarly, it can be shown that

$$\sum_{j \in J} v_j^2 \left\langle \Theta_j P_{Y_j} U f, \Theta_j P_{Y_j} T f \right\rangle = \left\langle f, \left(S'_C \right)^{-1} f \right\rangle .$$

Therefore, for each $f \in H$, we have

$$\begin{aligned}
&\left| \sum_{j \in J} v_j^2 \left\langle \Delta_j P_{X_j} U f, \Delta_j P_{X_j} T f \right\rangle - \sum_{j \in J} v_j^2 \left\langle \Theta_j P_{Y_j} U f, \Theta_j P_{Y_j} T f \right\rangle \right| \\
&= \left| \left\langle f, S_C^{-1} f \right\rangle - \left\langle f, \left(S'_C \right)^{-1} f \right\rangle \right| = \left| \left\langle f, \left(S_C^{-1} - \left(S'_C \right)^{-1} \right) f \right\rangle \right| \\
&\leq \left\| S_C^{-1} - \left(S'_C \right)^{-1} \right\| \| f \|^2 \leq \frac{D}{A_1 A_2} \| f \|^2.
\end{aligned}$$

Proof of (ii) In this case, we also find that

$$\begin{aligned}
&\left\| S_C - S'_C \right\| = \sup_{\| f \| = 1} \left| \left\langle \left(S_C - S'_C \right) f, f \right\rangle \right| \\
&= \sup_{\| f \| = 1} \left| \left\langle S_C f, f \right\rangle - \left\langle S'_C f, f \right\rangle \right| \\
&= \sup_{\| f \| = 1} \left| \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} - P_{V_j} \Gamma_j^* \Gamma_j P_{V_j} \right) U f, f \right\rangle \right| \\
&\leq \sup_{\| f \| = 1} D \| f \|^2 = D.
\end{aligned}$$

Then for each $f \in H$, we have

$$\begin{aligned}
&\left| \sum_{j \in J} v_j^2 \left\langle T^* \left(P_{X_j} \Delta_j^* \Delta_j P_{X_j} - P_{Y_j} \Theta_j^* \Theta_j P_{Y_j} \right) U f, f \right\rangle \right| \\
&= \left| \left\langle \left(S_C^{-1} - \left(S'_C \right)^{-1} \right) f, f \right\rangle \right| \leq \left\| S_C^{-1} - \left(S'_C \right)^{-1} \right\| \| f \|^2 \leq \frac{D}{A_1 A_2} \| f \|.
\end{aligned}$$

This completes the proof. □

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