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Title: Construction and Numerical Simulation of A Two-Dimensional Analogue to the KdV Equation

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Abstract approved: ______________________________

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Arising from an investigation in Hydrodynamics, the Korteweg-de Vries equation demonstrates existence of nonlinear waves that resume their profile after interaction. In this thesis, the classical equations governing wave motion are the starting point for the development of an analogue of the KdV that describes the evolution of a wave surface. The resulting partial differential equation is non-linear and third order in two spatial variables. The linear and and non-linear parts of this equation are analyzed separately. A variant of the method of stationary phase is used to study the linear third order terms, and it is found that the non-linear part equates to the non-viscous Burger's equation. Numerical methods are also used to investigate behavior of wave shapes. We find initial conditions that behave in a manner similar to those of the KdV in that the waves are nonlinear but retain their shape after interaction. These include all solutions of the KdV, but also some "lump" initial conditions.

Construction and Numerical Simulation of A Two-dimensional Analogue to the KdV Equation

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Wendy Black, Author

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1. INTRODUCTION

1.1. Background on Solitary waves

In 1895, Korteweg and de Vries [22] presented a mathematical model to study the "solitary wave" described by Scott Russell in 1844 [2],[10],[22]. The "solitary wave" was a disturbance consisting of a single elevated wave that had been observed to travel an extensive distance without noticeable change of shape or speed. In fact, the profile of such a wave was distinct from the more familiar oscillatory waves in that it tended to be tall and thin with long, shallow troughs. Through experimentation, Russell demonstrated existence of such waves in shallow water. Furthermore, given an equilibrium depth l and wave amplitude α, he established an estimate of its velocity as $\sqrt{g(l+\alpha)}$. Airy also published work in 1845 which gave a different estimate of the speed $\sqrt{gl}(1+\frac{\alpha}{2l})$, and concluded that such waves could not exist in a permanent form. To resolve the dispute, mathematical theory needed to be constructed. Boussinesq in 1871 obtained an equation for long waves with solitary waves solutions. The partial differential equation developed in [22] was nonlinear and was found to have a solitary traveling wave solution which agreed with observation.

The partial differential equation (PDE) was derived in a formal manner as a shallow water wave with a factor to account for surface tension and has the general form

$$\zeta_t + (c_0 + c_1\zeta)\zeta_x + \delta\zeta_{xxx} = 0. \tag{1.1}$$

Although equation (1.1) (known as the Korteweg - de Vries or KdV equation) was originally developed in addressing a problem in Hydrodynamics, ζ may be taken to represent observable properties other than surface height. It may be also be interpreted as a small variation in velocity or pressure, leading to applications in plasma physics, anharmonic lattices and other problems in fluid dynamics and quantum mechanics. It has solutions strongly tied to another equation derived in 1838 by Airy while doing research in optics, and has been derived independently in the field of optics. The interaction between waves evolving in time according to (1.1) has been found to be of great interest. However the model is restricted to one spatial dimension so the usefulness of the equation is limited. It cannot describe transverse interactions (as seen in interference patterns), nor can it describe waves with variable cross-sections.

In this thesis, we will develop a generalization of (1.1) having independent spatial dimensions x and y and including the effects of surface tension. Both non-linear and dispersive terms are found in (1.1) and we will see such terms in our higher-dimensional results. Separately these properties create challenges for interpretation of solutions and for numerical modeling. The interaction between these properties produces some interesting and unexpected behavior. This introduction is used to present definitions and discuss general properties of wave-like systems. We first review the derivation of equations from classical mechanics that govern the behavior of water waves. The phenomenon of dispersion will be discussed with some examples of one-dimensional, linear, dispersive PDE's, along with a look at the behavior encountered in non-linear PDE's. Non-linear dispersion theory only partially follows from the linear case, so requires the development of further techniques. The introduction ends with a discussion of some basics of the numerical methods needed to model our equation.

In the second section, we focus on literature in which the KdV has been studied and other relevant background material. The original development of the KdV equation and its solution is summarized. The linearized form is related to a function known as Airy's integral. The Airy integral is, in fact, important in understanding the behavior of the non-linear KdV solution. We continue by presenting more recent work in which numerical experiments were used to study the development of initial conditions evolving according to (1.1). These experiments prompted new research into methods of solving the KdV, and closed-form multiple-wave solutions were found. One well-known modification of (1.1) allows for "weak" distortion in the y-direction in an effort to study this type of wave over two spatial variables. The physical reasoning leading to the resulting equation is also discussed in the second section. On the other hand, the derivation in this thesis will begin from the same basic concepts from which the KdV was developed.

Section 3 consists of the development of the desired partial differential equation in a manner parallel to the original construction of the KdV. In the theory of shallow water waves, many of the well-known results are one-dimensional. That is, the wave propagates in one horizontal direction while the cross section is constant with respect to a second horizontal dimension. However, we have been able to omit this restriction and develop a partial differential equation for a wave surface dependent on two horizontal coordinates. With exception for the increase in number of spatial dimensions, the assumptions imposed on the KdV are retained. We follow the original method of development of (1.1), making an approximation by means of asymptotic expansions. The resulting equation has a linear dispersive component as well as nonlinear terms.

As there are few methods available for studying such an equation, our analysis proceeds by viewing different parts of the problem independently. The next sections are then devoted to gaining information about the non-linear and linear parts separately and then visualizing the behavior of various initial conditions. To address the linear part, the Fourier transform will be applied. The resulting integral solution will be estimated through series expansion techniques described in the literature review. In the numerical modeling of equation (1.1), we must deal with the nonlinearity as well as the third-order term. A method that had been designed for the equation (1.1) will be modified to accomodate the extra spatial variable.

1.2. Classical Mechanics Framework

The equations governing surface waves in shallow water are developed for example in the classic texts of Batchelor [4], Witham [34] and Lamb [23] for example. Using vector notation, we begin with three rectilinear coordinates $\mathbf{x} =< x_1, x_2, x_3 >$ where the horizontal coordinates are $x_1,\ x_2$, where $(x_1, x_2) \in \mathbb{R}^2$ lie in some domain

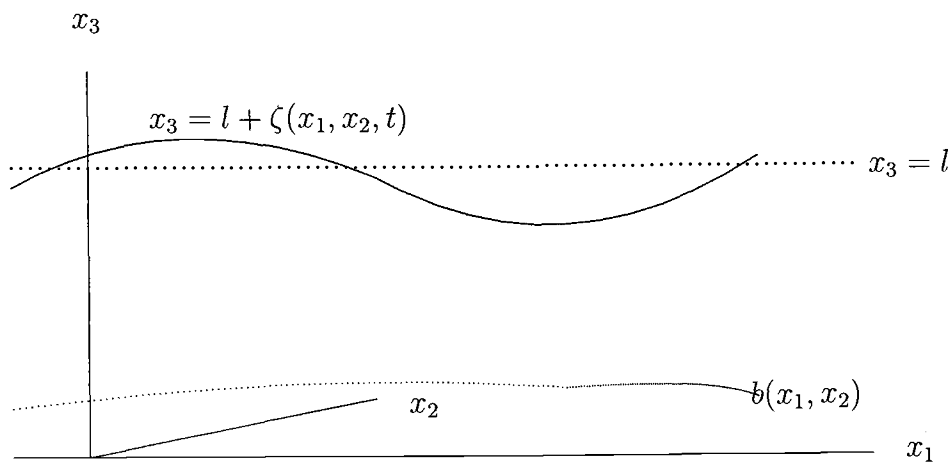

FIGURE 1.1: A wave surface.

determined by the problem. In some applications it may extend to infinity, whereas in other scenarios it may be bounded in one or both horizontal coordinates. Under certain circumstances the domain may be taken to be one dimensional. The vertical coordinate is x_3, and it points upward. Where it is helpful to make the notation more compact, we may interchange $x_1 = x,\ x_2 = y,\ x_3 = z$ and $\mathbf{x} =< x, y, z >$.

A wave surface is thus a constraint on the vertical coordinate z, with the system generally described as follows:

$$\begin{aligned} z &= l \text{ is the equilibrium position of the surface above the reference plane} \\ z &= l + \zeta(x, y, t) \text{ is the free surface} \\ z &= b(x, y)\text{, the fixed bottom relative to the reference plane .} \end{aligned}$$

The velocity field will be

$$\mathbf{u} \;=\; < u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t) >^T,$$

where these components may also be designated by $\mathbf{u} =< u_1, u_2, u_3 >$ as with the spatial variables.

The gradient, divergence, Laplacian and curl of vector quantities will be given in the following notation (where the summation convention is applied):

$$\begin{aligned}
\nabla \mathbf{u} &= < \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z} > \\
\nabla \cdot \mathbf{u} &= \frac{\partial u_i}{\partial x_i} \\
\nabla^2 \mathbf{u} &= \frac{\partial^2 u_i}{\partial x_i^2} \\
\nabla \times \mathbf{u} &= < \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} > .
\end{aligned}$$

1.2.1. Physical laws throughout the body of fluid

Let a fixed set of particles constitute a material volume element $V(t)$, with boundary $\partial V(t)$ and exterior unit normal $\mathbf{n} =< n_1, n_2, n_3 >$. The fluid is considered to be homogeneous, with parameters specific to the fluid in question:

$$\begin{aligned}
\mu &= \text{viscosity coefficient} \\
\rho &= \text{density} \\
\tau &= \text{coefficient of surface tension.}
\end{aligned}$$

In general, transport of some quantifiable property is described by an integral expression in the following manner. Suppose $\Theta(\mathbf{x}(t), t)$ is an arbitrary local observable or property per unit mass. Then

$$\int_{V(t)} \rho \Theta dV$$

is the total amount of Θ associated with the material volume V. Changes over time in this amount are then expressed by taking the derivative with respect to time. We have, by Reynolds' transport theorem

$$\frac{d}{dt}\int_{V(t)} \rho\Theta dV = \int_V (\rho\Theta)_t \, dV + \int_{\partial V} \rho\Theta(\mathbf{u}\cdot\mathbf{n}) dA, \tag{1.2}$$

where V is a control volume coinciding with $V(t)$ (see, for example, [24]). The flux of the quantity outward along the boundary due to fluid motion is the rightmost term and applying the divergence theorem,

$$\int_{\partial V} \rho\Theta(\mathbf{u}\cdot\mathbf{n}) dA = \int_V \nabla\cdot(\rho\Theta\mathbf{u}) \, dV. \tag{1.3}$$

Within V there may be sources or sinks contributing to the net changes. These are represented by

$$\textstyle\int_V Q dV. \tag{1.4}$$

A conservation law is a statement that (1.2) and (1.4) balance. In general form it is given as

$$\int_V (\rho\Theta)_t \, dV + \int_V \nabla\cdot(\rho\Theta\mathbf{u}) \, dV = \int_V Q \, dV. \tag{1.5}$$

The first case we consider results from setting $\Theta = 1$ and $Q = 0$ in (1.5). The result is the usual expression of conservation of mass when no sources or sinks are present

$$\int_V \rho_t + \nabla\cdot(\rho\mathbf{u}) \, dV = 0. \tag{1.6}$$

That is, within each control volume V, the rate of change of the mass in the volume equals the net rate of mass flux across the boundary. Noting (1.6) holds for arbitrary volumes,

$$\rho_t + \nabla\cdot\rho\mathbf{u} = 0.$$

When the fluid is incompressible ($\rho =$ const.) this reduces to

$$\nabla\cdot\mathbf{u} = 0. \tag{1.7}$$

To set up our second application of (1.5), let the vector quantity $\mathbf{F}$ denote the forces acting "at a distance" on particles of V. Then the total of body forces acting on V is represented by

$$\int_V \rho \mathbf{F} dV.$$

Water is usually considered to be a "Newtonian fluid", that is "linearly viscous", and isotropic (description of mechanical properties is independent of direction). The development of the equations of motion proceeds under these assumptions [24]. On ∂V, normal and shearing stresses take effect. These include pressure and the effects of viscosity. The *first coefficient of viscosity* is our parameter μ and the *coefficient of bulk viscosity* is the quantity $\lambda + \frac{2}{3}\mu$. Suppose the internal pressure acting on ∂V is generated by the scalar function

$$P(\mathbf{x}, t) \quad = \quad \text{scalar pressure.}$$

The stress tensor describing the surface forces on a surface element has components

$$\sigma_{ij} \quad = \quad -P\delta_{ij} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \lambda(\nabla \cdot \mathbf{u})\delta_{ij},$$

δ_{ij} indicating the Kronecker delta. We will be employing the assumption that the fluid is incompressible, so may apply the continuity equation (1.7) to get

$$\sigma_{ij} \quad = \quad -P\delta_{ij} + \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]. \tag{1.8}$$

The sum of the surface forces on the boundary is

$$\int_{\partial V} \sigma_{ij} n_j dA$$

where the summation convention on multiple indices is assumed to hold. By applying the divergence theorem, this may be converted to a volume integral with integrand $\frac{\partial \sigma_{ij}}{\partial x_j}$, otherwise seen as

$$-\nabla P + \mu\big[\ (\nabla \cdot \nabla)\mathbf{u} + \nabla(\nabla \cdot \mathbf{u})\big],$$

which may again be simplified by eliminating the divergence of $\mathbf{u}$.

We are now in a position to describe the principle of conservation of momentum. Momentum is the product of mass and velocity $\rho\mathbf{u}$, so a momentum flux is expressed by assigning $\Theta = \mathbf{u}$ in (1.3). Momentum transport is governed by Newton's second law, which states that the rate of change of momentum equals the sum of forces acting on the volume, so we take $Q = \rho\mathbf{F} + (\frac{\partial \sigma_{ij}}{\partial x_j})$. Then the general conservation law (1.5) becomes

$$\int_V (\rho\mathbf{u})_t \, dV + \int_{\partial V} \mathbf{u}\rho\mathbf{u} \cdot \mathbf{n} dA = \int_V \rho\mathbf{F} - \nabla P + \mu \ \nabla^2 \mathbf{u} \, dV,$$

or written in vector notation as

$$\rho\mathbf{u}_t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \rho\mathbf{F} - \nabla P + \mu\nabla^2\mathbf{u}. \tag{1.9}$$

These are also well-known as the Navier-Stokes equations for an incompressible Newtonian fluid.

In order to work with these equations, it is often assumed the fluid is irrotational

$$\nabla \times \mathbf{u} = 0 \tag{1.10}$$

as well as incompressible. Then there is a potential Φ with the property

$$\nabla\Phi = \mathbf{u} \tag{1.11}$$

and Laplace's equation follows from (1.7), that is,

$$\nabla^2\Phi = 0. \tag{1.12}$$

It may be further assumed the fluid is inviscid ($\mu = 0$) and the only body force acting on the fluid is gravity,

$$\mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

(where g is the gravitational constant). In this case, equation (1.9) is integrated from, say $\mathbf{x}$ within the fluid to $\mathbf{x_0}$ on the boundary, to find the Bernoulli equation,

$$\Phi_t + \frac{1}{2}||\nabla\Phi||^2 = -gz - \frac{1}{\rho}(P_0 - P) + C(t), \tag{1.13}$$

where C(t) may be adjusted at will by reassigning

$$\tilde{\Phi} = \Phi - \int C(t)dt$$

1.2.2. Boundary condition at the bottom

When it is supposed that the bottom is fixed at $z = b(x, y)$, the unit normal to the bottom pointing outward from the fluid will be

$$\mathbf{n}_b = \frac{< b_x, b_y, -1 >}{\sqrt{b_x^2 + b_y^2 + 1}}$$

so if the bed is impermeable,

$$\mathbf{n}_b \cdot (\nabla\Phi) = 0 \text{ on } z = b(x, y).$$

When the bottom is flat, i.e. $b(x, y)$ is a constant, the result is

$$\Phi_z = 0 \text{ on } z = b. \tag{1.14}$$

1.2.3. Boundary condition at the free surface

When a volume element is entirely within the fluid , the pressure P on the boundary is taken to be constant (and $P_0 - P = 0$), but there will be a change in pressure across the air-water interface, denoted by

$$[P] = \lim_{\mathbf{x} \longrightarrow \text{atmosphere}} P(\mathbf{x}, t) - \lim_{\mathbf{x} \longrightarrow \text{surface}} P(\mathbf{x}, t).$$

When the surface is at rest, this change can be measured giving the surface tension coefficient, τ. Then, as the surface is moved from equilibrium, the surface tension acts as a restoring force, dependent upon the mean curvature at each point. $[P]$ is then proportional to the mean curvature of the surface (see Batchelor [4] for instance). To obtain an expression for the surface tension, we view the wave surface at any fixed time as a manifold in $\mathbb{R}^3$, expressed by

$$z - (l + \zeta(x, y, t)) = 0 \tag{1.15}$$

where t is treated as a parameter. This manifold has unit normal

$$\mathbf{n}_s = \frac{< -\zeta_x, -\zeta_y, 1 >}{\sqrt{\zeta_x^2 + \zeta_y^2 + 1}}.$$

The mean curvature of (1.15) is computed as

$$\begin{aligned} H(x, y) &= \left(\frac{\partial n_1}{\partial x} + \frac{\partial n_2}{\partial y}\right) \\ &= \frac{-\zeta_{xx}(1 + \zeta_y^2) - \zeta_{yy}(1 + \zeta_x^2) + 2\zeta_x\zeta_y\zeta_{xy}}{(\zeta_x^2 + \zeta_y^2 + 1)^{3/2}}. \end{aligned} \tag{1.16}$$

A discussion of the differential geometry involved may be found in Boothby [6] or Gray [13].

Replacing $[P]$ by $\tau H(x,y)$ in the momentum equation (1.13) we obtain

$$\Phi_t + \frac{1}{2}\|\nabla\Phi\|^2 + g\zeta = \frac{\tau}{\rho}\frac{\zeta_{xx}(1+\zeta_y^2) + \zeta_{yy}(1+\zeta_x^2) - 2\zeta_x\zeta_y\zeta_{xy}}{(\zeta_x^2+\zeta_y^2+1)^{3/2}}. \tag{1.17}$$

This restriction of the Bernoulli equation (1.13) to the surface is called the dynamic free surface condition.

A second equation applies to the free surface. Consider that the free surface (1.15) has normal velocity

$$-\frac{\zeta_t}{\sqrt{\zeta_x^2+\zeta_y^2+1}},$$

and on the interior, the normal velocity of the fluid near the surface is

$$\mathbf{n}_s \cdot \mathbf{u} = \frac{u\zeta_x + v\zeta_y - w}{\sqrt{\zeta_x^2+\zeta_y^2+1}}.$$

These must be equal since the fluid does not cross the interface. Therefore,

$$\zeta_t + u\zeta_x + v\zeta_y = w. \tag{1.18}$$

This constraint on the interface is called the kinematic free surface boundary condition.

1.3. Some Characterizing Properties of Partial Differential Equations

The study of equations expressing mass or momentum transport is commonly approached by making approximations of various types. In particular the velocities, the free surface or the forces acting on a volume may be considered as perturbations of a constant value. The approximation used provides insight to different properties, such as dispersion or nonlinearity.

1.3.1. Dispersion

Dispersion is the phenomenon of individual waves in a train moving apart. In the linear theory of arbitrary dimensions, dispersive PDE's are those which admit elementary solutions of the form

$$\zeta(x, y, t) = A \exp(\mathbf{i}\mathbf{k} \cdot \mathbf{x} - \mathbf{i}\omega(\mathbf{k})\mathbf{t}). \tag{1.19}$$

Here, $\mathbf{i} = \sqrt{-1}$, and $\mathbf{k} =< k_1, \ldots, k_n >$. $\mathbf{k}$ and ω are related by a *dispersion relation* $\omega = \omega(\mathbf{k})$, obtained by substituting (1.19) into the PDE and simplifying. In Whitham [34], a dispersive linear PDE is defined by the existence of solutions of the form (1.19), a real dispersion relation, and

$$\det \left[\frac{\partial^2 \omega(\mathbf{k})}{\partial k_i \partial k_j} \right] \neq 0.$$

Examples of dispersive, linear equations describing the propagation of long waves in one dimension include:

$$\text{the linear KdV} \quad \zeta_t + c_0\zeta_x + \delta\zeta_{xxx} = 0 \tag{1.20}$$

$$\text{the linear Boussinesq} \quad \zeta_{tt} - \alpha^2\zeta_{xx} - \beta^2\zeta_{xxtt} = 0 \tag{1.21}$$

There is much information on the behavior of the solution contained in the dispersion relation. The *modes* $(\mathbf{k}, \omega(\mathbf{k}))$ allow us to identify *phase* $\theta = \mathbf{k} \cdot \mathbf{x} - \omega\mathbf{t}$, *wave number* $\nabla\theta = \mathbf{k}$, *phase velocity* $\mathbf{c} = \frac{\omega\mathbf{k}}{|\mathbf{k}|^2}$, *angular frequency* $-\theta_t = \omega$, *wave length* $\lambda = \frac{2\pi}{|\mathbf{k}|}$ and *frequency* $\tau = \frac{2\pi}{\omega}$. In particular, different wave numbers are identified with different speeds, and the waves of different numbers separate or disperse.

Since these waves are linear, solutions may be superimposed so it is clear that cosines and sines are included, as well as Fourier integrals such as

$$\int_{-\infty}^{\infty} F(\mathbf{k}) \exp(\mathbf{ik} \cdot \mathbf{x} - \mathbf{i}\omega\mathbf{t})\, \mathbf{dk}.$$

From this expression, we see that the requirement of a real dispersion relation prevents each fundamental solution from either growing or decaying since the exponent is purely imaginary. However, the solutions described are generally a train of oscillatory waves. As an example, consider the linearization of the equations governing water waves (1.12),(1.17) and (1.18) when surface tension is absent. This example is given because it is commonly used as a reference for modeling wave behavior (see [34]). It is the dispersion relation for the surface $\zeta(x, y, t)$ that is sought, and the potential equation is considered to be dispersive in the horizontal coordinates, but not the vertical. If we denote the horizontal coordinates as $\mathbf{x} = (x, y)$ and treat z separately, we have

$$\Phi = Z(z) \exp(\mathbf{ik} \cdot \mathbf{x} - \mathbf{i}\omega\mathbf{t}).$$

Throughout the body, $0 < z < l + \zeta$, the potential satisfies

$$\nabla^2\Phi = 0 \tag{1.22}$$

and if we assign $k = \sqrt{k_1^2 + k_2^2}$ or equivalently $\|\mathbf{k}\|$, Laplace's equation (1.22) becomes the ODE

$$Z'' = k^2 Z. \tag{1.23}$$

Assuming a horizontal bed, say at $z = 0$, the lower boundary condition (1.14) will translate into

$$Z'(0) = 0. \tag{1.24}$$

Now the solution to this BVP (1.23), (1.24) is

$$Z(z) = A\cosh(kz)$$

for some constant A.

The surface is linearized to $z = l$, and equations (1.17), (1.18) are, to first order,

$$\Phi_t + g\zeta = 0$$

$$\zeta_t - \Phi_z = 0$$

which combine to give

$$\Phi_{tt} + g\Phi_z = 0,$$

from which the resulting equation for ω is

$$-\omega^2\cosh(kl) + gk\sinh kl = 0.$$

We now identify a real dispersion relation for the surface approximated by $z = l$ as

$$\omega^2 = gk\text{tanh }(kl).$$

Φ is now known (up to the amplitude, A) and ζ is found through

$$\zeta = -\frac{1}{g}\Phi_t.$$

Solutions and dispersion relations of nonlinear systems are expected to be natural extensions of the linear case, where amplitude may be included in the dispersion relation.

1.3.2. Nonlinearities

Two of the effects attributed to nonlinearities are the development of shocks and the existence of solitary wave solutions. The solitary wave is distinct from a train of oscillatory waves (such as in the sine or cosine solutions) in that it consists of a single elevation. Nonlinear dispersive systems may demonstrate both periodic wave trains and solitary waves, but latter has no counterpart in linear theory [34]. A standard example used to study the difficulties encountered in non-linear PDEs (without dispersion) is Burgers' equation

$$\zeta_t + \zeta\zeta_x = 0. \tag{1.25}$$

This is the same nonlinearity encountered in (1.1). One uses the method of characteristics to solve the equation, and finds that paths on which ζ remains constant will intersect if the initial data is decreasing. When this happens, the solution becomes multi-valued even if the initial profile was smooth. This phenomenon is called a *shock*. Special techniques are required to solve or model a shock.

However, with inclusion of a viscosity term

$$\zeta_t + \zeta\zeta_x = \mu\zeta_{xx} \tag{1.26}$$

the shocks in (1.25) do not occur. In fact, smooth solutions exist and can be computed by transforming (1.26) into the heat equation. This is done by means of the Hopf-Cole transformation [9], [17], as follows: Begin by introducing the change of variable

$$\zeta = \psi_x.$$

(1.26) is transformed into

$$\begin{aligned} \psi_{xt} + \psi_x \psi_{xx} - \mu \psi_{xxx} &= 0 \quad \text{or} \\ \frac{\partial}{\partial x}(\psi_t + \frac{1}{2}(\psi_x)^2 - \mu \psi_{xx}) &= 0. \end{aligned}$$

Integration and then further substitution of

$$\psi = -2\mu \log \phi \tag{1.27}$$

produces the linear heat equation

$$\phi_t = \mu \phi_{xx}.$$

Solutions of the heat equation are well known, and may be found in texts on partial differential equations such as [14]. It is instructional to note that separation of variables produces the two ordinary differential equations

$$\begin{aligned} T'(t) &= \kappa T(t) \\ X''(x) &= \kappa X(x). \end{aligned}$$

The first provides growth or decay in time, and the second is a Sturm-Liouville eigenvalue problem. The form of the solution obtained clearly relies on the initial and boundary conditions of the problem.

In its general form, the KdV equation

$$\zeta_t + c_1 \zeta \zeta_x + \delta \zeta_{xxx} = 0 \tag{1.28}$$

embodies both nonlinearity and dispersion. Physically, surface tension provides a stabilizing influence for short waves. As with the viscous Burger's equation, (1.26), smooth solutions to (1.28) also exist. If we consider a wave traveling along a characteristic line, $\zeta(x,t) = \zeta(x - ct)$ then the t-derivative in (1.28) is eliminated and the solution

$$\zeta \;=\; A\text{sech}^2(B(x-ct)) \tag{1.29}$$

is found, provided

$$B^2 = \frac{c_1 A}{12\delta} \quad \text{and} \quad c = \frac{A}{3}. \tag{1.30}$$

This is the solitary wave solution originally found in [22]. Because of the relationship between the parameters, taller waves travel faster and are steeper.

A procedure similar to the Hopf-Cole transformation above (1.27) allowed discovery of multiple-wave solutions of the KdV when in the form

$$\zeta_t - 6\zeta\zeta_x + \zeta_{xxx} = 0. \tag{1.31}$$

The procedure is known as the *Inverse Scattering Transform*, and will be discussed in the next section.

1.4. Numerical Challenges

When modeling linear dispersive equations, the dispersion relation is used to study stability and determine if there are restrictions on the time step relative to the spatial step. The dispersion relation allows us to isolate a mode (or wave), and to compute wave speed, amplitude and phase for that mode. Thus, for a linear problem, we have at hand a known solution and computational errors can be quantified. Fourier stability analysis identifies whether growth or decay is occurring in the numerical method when they do not occur in the true solution. Fourier analysis is also applicable to multiple spatial dimensions as long as the system is linear. In order to evaluate a method for a non-linear PDE by Fourier analysis, we must linearize and approximate. In practice, it is often found that the nonlinear term is not offset by the third derivative in the numerical method (as opposed to the true solution), limiting the usefulness of such methods over long time periods.

Implicit methods generally provide more stability than explicit methods. However, in two spatial dimensions the matrices required quickly raise issues of storage and computation time. We also risk smoothing out a solution by using a coarser grid. There are some methods for specific PDE's in which the difference operators can be split (somewhat like factoring) into two steps, each evolving one spatial dimension for instance. One such example was reviewed by this author in [5] regarding the use of a Hamiltonian structure to discretize the Non-linear Schrödinger equation. It was found that the wave speed varied from known results, producing an unexpected error unless severe restrictions were placed on the grid. That is, errors introduced by instability and meeting storage and efficiency requirements are only some of the issues that arise in modeling a PDE. Thorough analysis of a numerical method will ideally address its effect on velocity and phase changes as well, comparing results to known solutions.

2. LITERATURE

Although the KdV (1.1) was originally constructed in 1895, little progress into finding solutions other than those given by Korteweg and de Vries was made until 1965. In that year, numerical experiments run by Zabusky and Kruskal were published [35]. They observed and reported some interesting interactions between waves evolving under a variant of (1.1). After their paper, further research was inspired and resulted in the discovery of new solutions. In this section we present the historical development of the PDE and material needed to analyze its behavior, along with a brief look at some of the methods of solution.

Later in this thesis, we will be developing a PDE by generalizing the method of Korteweg and de Vries to two, rather than one, spatial dimensions. Therefore, the details of the original paper [22] are not reproduced here as they are a special case of our later work. Instead at this point, we give a conceptual outline of the expansion method by which (1.1) was developed.

2.1. Asymptotic Expansions

In the introduction, equations for conservation of mass and momentum were derived in terms of a potential $\nabla\Phi = \mathbf{u}$. The system was assumed to have no sources or sinks and gravity was the only force $\mathbf{F}$ acting on the body. The pressure remained constant except when crossing the fluid interface where surface tension is accounted for by the curvature function $H(x, y; t)$. Coefficients of surface tension and density (respectively τ, ρ) have been taken to be constant and the fluid is assumed to be inviscid ($\mu = 0$). Under these assumptions, the equations for an incompressible and irrotatinal fluid (1.7) and (1.9) are now restated in terms of the velocity components.

$$\nabla \times \mathbf{u} = \mathbf{0} \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F} - \nabla(\frac{\tau H}{\rho}). \tag{2.3}$$

On an impermeable flat bottom at $z = 0$ the vertical velocity component vanishes,

$$w(x, y, 0, t) = 0. \tag{2.4}$$

On the wave surface $z = l + \zeta(x, y, t)$, we obtained the kinematic (1.18) and dynamic (1.17) conditions

$$\zeta_t + u\zeta_x + v\zeta_y = w \tag{2.5}$$

$$u_t + (uu_x + vv_x + ww_x) + g\zeta_x = -\frac{\tau}{\rho}H_x \tag{2.6}$$

$$v_t + (uu_y + vv_y + ww_y) + g\zeta_y = -\frac{\tau}{\rho}H_y \tag{2.7}$$

Progress toward understanding this system of differential equations is made by making approximations to the velocities, u, v, w (or to the potential in some references) and to the surface ζ. Power series of successive orders in z^n when z is small are used for the approximations, terminating at the order of interest. The power series is one type of asymptotic expansion of a function. Although the power series is convergent for small enough z, there also exist asymptotic series which do not converge but are still useful in giving good approximations of functions after only a few terms. Thus, asymptotic sequences provide a tool for analysis. The following definition of an asymptotic expansion of a function $f(z)$ is attributed to Poincare and found in [7], stated in terms of complex variables for generality. Here we restate the definition in terms of real variables.

Definition 1 *Suppose we are given a domain $\Omega \subset \mathbb{R}$ with limit point ν_0. Let there be a sequence $\{\phi_n(\nu)\}$ for which $\phi_n(\nu) \neq 0$ on a punctured neighborhood of ν_0 and $\phi_{n+1} = o(\phi_n)$ as $\nu \longrightarrow \nu_0$. Then given coefficients a_n independent of ν, $\sum_{n=0}^{\infty} a_n \phi_n(\nu)$ is an* **asymptotic expansion** *of a function $f(\nu)$ if for all n,*

$$f(\nu) - \sum_{n=0}^{m} a_n \phi_n(\nu) = o(\phi_m(\nu)).$$

The coefficients are determined by

$$a_m = \lim_{z \longrightarrow \nu_0} \frac{f(z) - \sum_{n=0}^{m-1} a_n \phi_n(\nu)}{\phi_m(\nu)}.$$

The coefficients then depend on the sequences $\{\phi_n(\nu)\}$, which might not be unique for a given f.

Beyond expanding the equations (2.1)-(2.7) in terms of power series, asymptotic expansions are useful in interpreting formal integral solutions. One result cited in Copson [7] regarding asymptotic behavior of the Fourier integral over a finite interval is:

Theorem 1 *Let $h(x)$ be N times continuously differentiable on the interval $\alpha \leq x \leq \beta$. Then, as $\nu \longrightarrow \infty$*

$$\int_{\alpha}^{\beta} \exp(\mathrm{i}\nu x) h(x)\, dx \;=\; \sum_{n=0}^{N-1} \frac{\mathrm{i}^{n+1}}{\nu^{n+1}} (\exp(\mathrm{i}\nu\alpha) \frac{d^n h}{dx^n}(\alpha) - \exp(\mathrm{i}\nu\beta) \frac{d^n h}{dx^n}(\beta)) \;+\; o(\nu^{-N}).$$

The proof follows from integration by parts. Often, only the dominant term, when $n = 0$, is needed to find information about the behavior of the integral.

More general integrals of the form

$$\int_{\alpha}^{\beta} \exp(\mathbf{i}\nu f(x))h(x)dx$$

where $f(x)$ is real may be treated using *the method of stationary phase*, also found in [7]. In this procedure, one needs to first determine the stationary points of the phase $\nu f(x)$, i.e. where $f'(x) = 0$ (note that in this case, the term phase would not be the same as when used in describing a dispersive wave). These are used to partition the interval $[a, b]$ into closed subintervals, each closed subinterval containing no more than one stationary point, and that stationary point occurring at an endpoint of the subinterval. Under this construction, it is found that as ν becomes large, the dominant term of the asymptotic expansion is dependent only on the stationary points and endpoints of the interval.

To approximate double integrals over a domain D of the form

$$\int\int_{D} \exp(\mathbf{i}\nu f(\mathbf{x}))\phi(\mathbf{x})d\mathbf{x},$$

Jones and Kline [20] provide an extension of the method of stationary phase. By utilizing the δ-function, they begin by re-writing the integral as

$$\int_{\alpha}^{\beta} \exp(\mathbf{i}\nu s)\ h(s)\ ds \tag{2.8}$$

where the limits of integration α and β are the minimum and maximum values of $f(\mathbf{x})$ over the domain, and

$$h(s) \quad = \quad \int\int_{D} \phi(\mathbf{x})\delta(s - f(\mathbf{x}))\ d\mathbf{x}. \tag{2.9}$$

Once this is done, the differentiability of $h(s)$ must be assessed, and the integral (2.8) is determined by critical values. The term *critical value* is used here to describe points $\mathbf{x}$ at which the corresponding s-value is not contained in an open interval. Such points include local extrema, cusps and endpoints of the s-domain. After a several transformations are applied, a series expansion of $h(s)$ is found in the new variables and the integral is evaluated using Erdelyi's Theorem [7],[20]:

Theorem 2 *If $\phi(s)$ is N times continuously differentiable for $\alpha \leq s \leq \beta$, and $0 < \lambda \leq 1$, $0 < \mu \leq 1$, then as $\nu \longrightarrow \infty$*

$$\int_{\alpha}^{\beta} \exp(\mathbf{i}\nu s)(s-\alpha)^{\lambda-1}(\beta-s)^{\mu-1}\phi(s)\, ds \quad = \quad B_N(\nu) + A_N(\nu)$$

where

$$A_N(\nu) \quad = \quad \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n!\nu^{n+\lambda}} \exp(\mathbf{i}\frac{\pi}{2}(n+\lambda)) \exp(\mathbf{i}\nu\alpha) \frac{d^n}{ds^n}[(\beta-s)^{\mu-1}\phi(s)]_{s=\alpha}$$

$$B_N(\nu) \quad = \quad \sum_{n=0}^{N-1} \frac{\Gamma(n+\mu)}{n!\nu^{n+\mu}} \exp(\mathbf{i}\frac{\pi}{2}(n-\mu)) \exp(\mathbf{i}\nu\beta) \frac{d^n}{ds^n}[(s-\alpha)^{\mu-1}\phi(s)]_{s=\beta}$$

2.2. The Equation of Korteweg and de Vries

In [22], Korteweg and de Vries investigated a model to describe a solitary wave in a long channel. Their method proceeded in a formal manner utilizing rapidly convergent series. The resulting equation had a solution which agreed with observation. By considering the wave to have a constant cross section, the horizontal coordinates reduce to $\mathbf{x} = x$. It was assumed the vertical scale was small compared with the wave length, so the horizontal velocity component u and vertical component

w could be expanded by power series in z (the vertical coordinate). That is,

$$u(x,z,t) = \sum_{n=0}^{\infty} u_n(x,t)z^n$$
$$w(x,z,t) = \sum_{n=0}^{\infty} w_n(x,t)z^n.$$

Applying the classical assumptions of conservation of mass and momentum for an irrotational, incompressible fluid on a flat, horizontal, impermeable bed (2.1)-(2.4), the velocity expansions were simplified. In fact, they found they could write the coefficients of each of these expansions in terms of only the first horizontal coefficient u_0.

The surface elevation was taken as a perturbation from equilibrium depth,

$$z = l + \zeta(x,t)$$

and the initial term of u was taken to be a perturbation from a value scaled to be $\sqrt{gl}$

$$u_0(x,t) = \sqrt{gl} + \beta(x,t).$$

The first estimations of u and w were inserted into the dynamic and kinematic surface conditions (2.5), (2.6). Retaining only linear first order terms, they solved for the velocity perturbation β in terms of the surface elevation ζ. The velocity coefficient u_0 was then approximated in terms of ζ plus a smaller perturbation,

$$u_0 = \sqrt{gl} - \sqrt{\frac{g}{l}}(\zeta + \tilde{\beta}).$$

When the surface equations (2.5), (2.6) were expanded to second order, with scalings on x, t, the result is the KdV equation in its original form,

$$\zeta_t = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}(\frac{1}{2}\zeta^2 + \frac{2}{3}\alpha\zeta + \frac{1}{3}\sigma\zeta_{xx}). \tag{2.10}$$

The parameter α is an arbitrary constant linked to the Bernoulli equation (1.13). Although not mentioned in [22], α may be taken as zero by taking a suitably adjusted potential function, as was noted in the introduction. The other parameter is $\sigma = \frac{1}{3}l^3 - \frac{\tau l}{\rho g}$. For a specified medium, surface tension τ and density ρ are fixed, and the equilibrium depth l determines the sign of σ. In [22], the value at which σ changes sign is approximately $l = .47$ cm.

Upon assumption that $\frac{\partial \zeta}{\partial t} = 0$, the equation (2.10) was integrated to find the stationary solution

$$\zeta = A\text{sech}^2\left(x\sqrt{\frac{A}{4\sigma}}\right). \tag{2.11}$$

This result is equivalent to the solution (1.29)-(1.30) which has constant profile translated along a line. More general solutions were also found by Korteweg and/ de Vries in terms of Jacobi elliptic functions.

There are several methods for producing this PDE. A similar derivation of the KdV equation is often cited, beginning from the same system of equations stated in terms of the potential Φ and its power series representation (rather than expanding the velocities), as seen in [2] or [10]. This type of approach is essentially the same in that the water depth is assumed to be small in comparison with the wavelength (x-scale) and the time derivatives are of higher order than the spatial derivatives. Whitham [34] gives a derivation beginning with a variation of the dynamic surface equation. In this, he does not use the surface tension term and the time derivative is of the same order as the other variables. He determines a relation that will eliminate first order terms, and then determines that if the velocity is of the form

$$u(x,t) = C_1\zeta + C_2\zeta^2 + C_3\zeta_{xx}$$

with constraints on the coefficients C_i, then the Kinematic and Dynamic surface equations will be consistent. In the next section, we will see how the equation was re-derived in other fields using Taylor approximations to difference equations.

2.3. Numerical Experiments

2.3.1. The Fermi-Pasta-Ulam problem

In 1955 Fermi, Pasta and Ulam undertook numerical experiments to study a one-dimensional anharmonic lattice [11]. According to Ablowitz and Segur [2], this had been motivated in part by a question of Debye in 1914 on the role of a nonlinear coupling on thermal conductivity. The particles on the lattice were of equal mass m and connected by strings obeying a non-linear spring law

$$F(\Delta) = -K(\Delta + \alpha\Delta^2).$$

Here Δ represented the difference between consecutive masses (and Δ^2 represented the difference of squares of distances). α was a parameter used to determine the degree of nonlinearity of the force between points. In their work, the continuum had fixed ends and was discretized into 64 points. The displacement of a point from its equilibrium was denoted by U_i and the model became

$$\begin{aligned} \frac{m}{K}(U_i)_{tt} &= (U_{i+1} - 2U_i + U_{i-1}) + \alpha((U_{i+1} - U_i)^2 - (U_i - U_{i-1})^2) \\ i &= 1, 2, \ldots, N-1 \\ U_0 = U_N &= 0. \end{aligned}$$

Energy was taken as the sum of kinetic and potential energies. They expected to find that the long-term behavior of the system would demonstrate an equipartition of energy between the modes given smooth initial conditions. Instead their results indicated little tendency towards mixing but rather a pattern of "almost-recurrence".

2.3.2. Zabusky and Kruskal

Zabusky and Kruskal followed up in 1963 by performing numerical studies based on the above system. They set Δx to be the grid spacing $(x_{k+1} - x_k)$ and transformed

$$t \longrightarrow \sqrt{\frac{K}{m}}t.$$

After applying Taylor expansions of u_{k+1}, u_{k-1} about u_k, the result is the model

$$u_{tt} = \Delta x^2 u_{xx} + \frac{\Delta x^4}{12} u_{xxxx} + 2\alpha \Delta x^3 u_x u_{xx}.$$

A transformation to an asymptotic traveling wave solution

$$\begin{aligned} X &= x - t \\ T &= \frac{\epsilon t}{2} \\ u &\sim \Phi(X, T) \end{aligned}$$

results in the form

$$\Phi_{XT} + \Phi_X \Phi_{XX} + \delta^2 \Phi_{XXXX} + \mathcal{O}(h^2, \frac{h^4}{\epsilon}) \;=\; 0$$

which upon setting $\zeta = \Phi_X$ yields

$$\zeta_T + \zeta \zeta_X + \delta^2 \zeta_{XXX} \;=\; 0. \tag{2.12}$$

The transformation $t \longrightarrow a\frac{3}{2}\sqrt{\frac{g}{l}}t$, $x \longrightarrow -ax$, provides a connection between the equation (2.10) and that used by Zabusky and Kruskal (2.12) where

$$\delta = \frac{a^2}{3}(\frac{l^3}{3} - \frac{\tau l}{\rho g}). \tag{2.13}$$

They used a leap-frog method in time, and center differences to approximate the spatial derivative. Here, we introduce the notation for the center difference approximations

$$\begin{aligned} D_x^1(U_k) &= U_{k+1} - U_{k-1} \\ D_x^3(U_k) &= U_{k+2} - 2U_{k+1} + 2U_{k-1} + U_{k-2}. \end{aligned}$$

The non-linear factor was discretized as a spatial average over three grid points. Specifically, the method is

$$U_k^{n+1} = U_k^{n-1} + \frac{\Delta t}{3\Delta x}(U_{k+1}^n + U_k^n + U_{k-1}^n)D_x^1(U_k^n) + \frac{\Delta t}{(\Delta x)^3}D_x^3(U_k^n) \tag{2.14}$$

using fixed grid steps

$$\begin{aligned} \Delta x &= x_{k+1} - x_k \\ \Delta t &= t_{n+1} - t_n. \end{aligned}$$

Using the periodic initial conditions $\cos(\pi x)$, several crests developed. These were observed to travel at different speeds (depending on height) and pass through

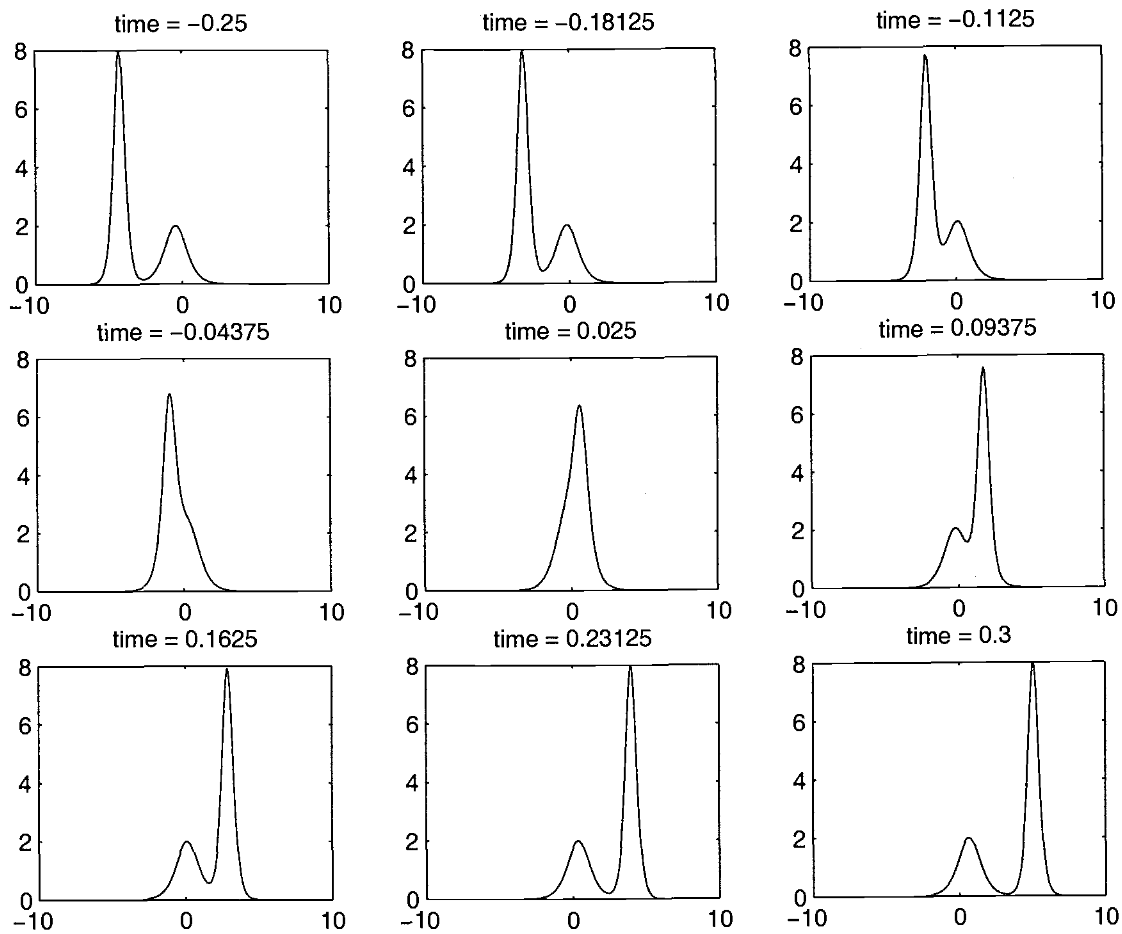

FIGURE 2.1: Two-soliton interaction computed with the Zabusky-Kruskal method.

each other. The unexpected result was that after an interaction, the waves resumed their shape with a phase shift, rather than evening out (refer to figure 2.1) The authors named a wave that demonstrates this kind of behavior *soliton*, a term that has come to be applied to a non-linear wave that recovers its profile and speed after a non-linear interaction with another wave of the same type.

Although numerical solutions did eventually become discontinuous, the scheme was more stable than might be expected considering the nonlinearity. It happens that the spatial averaging allowed for conserved momentum and "almost-conserved" energy. To see what is meant, consider the discrete analog of the conservation law (1.5). Momentum conservation is represented by the sum of the values of U^n over the spatial grid. In [35], periodic boundary conditions were used, and one needs only check

$$\begin{aligned}\sum_k (U^n_{k+1} + U^n_k + U^n_{k-1})(U^n_{k+1} - U^n_{k-1}) &= 0 \\ \sum_k U^n_{k+2} - 2U^n_{k+1} + 2U^n_{k-1} - U^n_{k-2} &= 0\end{aligned}$$

in order to establish

$$\sum_k U^{n+1}_k = \sum_k U^{n-1}_k.$$

Energy at step n would be conserved if

$$\sum_k \frac{1}{2}(U^{n+1}_k)^2 = \sum_k \frac{1}{2}(U^{n-1}_k)^2,$$

but terms only cancel to $\mathcal{O}(\Delta t^2)$. However, they had noted their periodic initial condition tended to become discontinuous around a final time of about $1/\pi$, as the nonlinear instabilities limited the time intervals over which a solution could be computed.

2.3.3. Evaluations of the Zabusky-Kruskal scheme

A generalization of the numerical method of Zabusky and Kruskal was reviewed by Pen-Yu and Sanz-Serna in 1980 [30]. The spatial discretation of the nonlinear term was treated as a weighted combination of approximations to $\zeta D^1_x \zeta)$ and $\frac{1}{2}D^1_x(\zeta^2)$. Given smooth solutions of the KdV with periodic initial conditions and numerical solution U, they established

$$||\zeta(t) - U(t)|| = \mathcal{O}(\Delta x^2)$$

when $\Delta x \longrightarrow 0$ and $t > 0$. It is to be noted that this result was obtained by a perturbation method rather than linearizing the KdV.

In 1981, Sanz-Serna [31] investigated some explicit methods that might improve the grid restriction. With $\Delta x = .02$, $\Delta t = .005$ they noted a blow up at about 3000 time steps (a final time of about 15) when the smooth initial condition

$$0.9\text{sech}^2(\sqrt{\frac{.3}{4\delta}}(x-1)).$$

was advanced according to the Zabusky-Kruskal scheme. A necessary grid restriction $\Delta t \leq \frac{2}{3\delta\sqrt{3}}(\Delta x)^3$ was established. In order to improve this result, Sanz-Serna developed alternate explicit methods designed to conserve energy exactly. The first method was a variation of the Z-K scheme, using a fixed step. It provided comparable accuracy and efficiency with more stability at $\Delta x = .02$, $\Delta t = .005$ and $\Delta x = .01$, $\Delta t = .0005$. However it required more storage and produced larger phase errors. As an improved method, the time step was modified to create a "self-adaptive" conservative scheme. This provided a grid restriction that was less than Zabusky-Kruskal's by a multiplicative factor (i.e. still $\mathcal{O}(\Delta x^3)$.

Operator splitting methods have also been investigated. One example is provided by [16]. This paper, written in 1990, considers treating the linear and nonlinear parts separately. The linear part was dealt with by either the (explicit) third center difference or by the fast Fourier transform. The ODE that results from the transform was solved using the (implicit) Crank-Nicholson method as explicit methods became unstable. The nonlinear part represents a conservation law, so the authors considered the Gudonov and ENO methods as well as a "spectral viscosity" method which is a variation of a Fourier transform method, again using the Crank-Nicholson to solve the ODE. Results were compared for different combinations of these methods, and the most accurate was the combination of FFT (with Crank-Nicholson) and the spectral viscosity methods.

2.4. Behavior and Known Solutions

The traveling wave solution (2.11) of the KdV not only propagates without change of form, but it also has been shown that waves with the sech^2 profile interact elastically with each other. That is, after interaction, the wave profile remained as before, with only a change in phase [35]. The findings that periodic initial conditions evolved into a series of such waves is in part explained by examining the linearized equation.

2.4.1. The dispersive effect

The linear part of the KdV Equation with initial condition

$$\zeta_t + \zeta_{xxx} = 0 \tag{2.15}$$

$$\zeta(x,0) = \zeta_0(x) \tag{2.16}$$

may be solved by Fourier Transform. That is,

$$\zeta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta_0(\eta) \int_{-\infty}^{\infty} \exp(\mathrm{i}k(x-\eta)+k^3 t))\; dk\; d\eta.$$

If the initial condition is taken to be a delta, then $\zeta(x,t)$ reduces to the Airy integral (a type of Bessel function)

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos(sz + \frac{1}{3}s^3)ds. \tag{2.17}$$

having substituted $z = \frac{x}{(3t)^{1/3}}$ and $s = (3t)^{1/3}k$. This integral is a solution of the differential equation studied by Airy (1838) in the context of optics

$$Ai''(z) - z\ Ai(z) = 0. \tag{2.18}$$

Analysis of (2.17) is an aid in studying the KdV itself [8]. The use of asymptotic expansions to identify the dominant term is discussed in [7] [15], yielding for large z and $|\arg z| < \frac{\pi}{3}$,

$$Ai(z) \sim \frac{z^{-1/4}}{2\sqrt{\pi}}\exp(-\frac{2z^{-3/2}}{3}), \tag{2.19}$$

and also

$$Ai(-z) \sim \frac{z^{-1/4}}{\sqrt{\pi}}\sin(\frac{2}{3}z^{3/2} + \frac{\pi}{4}) \tag{2.20}$$

Referring to figure (2.2), we see that waves traveling to the right decay and those traveling to the left oscillate.

Asymptotic methods also provide the approximation for large $x > 0$

$$\int_0^x Ai(z)dz \sim \frac{1}{3} - \frac{x^{-3/4}}{2\pi^{1/2}}\exp(-\frac{2x^{3/2}}{3}) \tag{2.21}$$

and for large $x < 0$

$$\int_x^0 Ai(z)dz \sim \frac{2}{3} - \frac{x^{-3/4}}{\pi^{1/2}}\cos(-\frac{2x^{3/2}}{3} + \frac{\pi}{4}). \tag{2.22}$$

In particular, the if the wave travels leftward it develops oscillatory motion, while solitons are generated with the rightward motion.

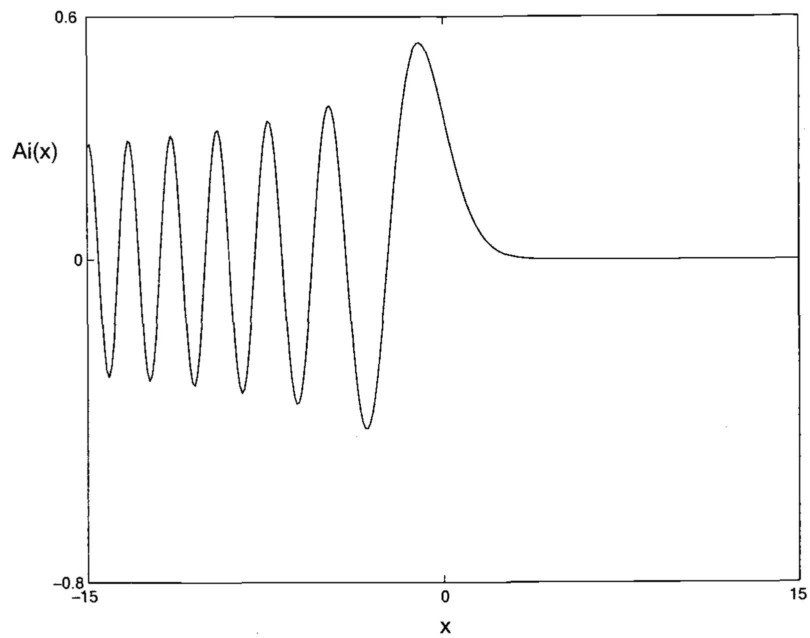

FIGURE 2.2: The Airy function

2.4.2. The Inverse Scattering Transform

It was found in [26] that the assignment

$$\zeta = \psi^2 + \psi_x \tag{2.23}$$

brings teh KdV equation (1.31) into the form

$$(2\psi + \frac{\partial}{\partial x})(\psi_t - 6\psi^2\psi_x + \psi_{xxx}) = 0. \tag{2.24}$$

Progress is made by working with the relation (2.23) rather than the resulting PDE (2.24). Note that under the change of variables

$$\begin{aligned} \zeta &\longrightarrow \zeta - \lambda \\ x &\longrightarrow x - 6\lambda t \end{aligned}$$

equation (1.31) is left invariant. If the additional assignment

$$\psi = \frac{\partial}{\partial x} \log \phi$$

is made in the Riccati equation (2.23), we obtain the linear, time-independent Schrödinger equation

$$\phi_{xx} + (\lambda - u)\phi = 0. \tag{2.25}$$

The method of solution of (1.31) through the use of (2.25) was developed by Miura, Gardner, et. al. in [12] and is known as the Inverse Scattering Transform. This method has been used to solve many problems in one spatial dimension, but the theory has not been satisfactorily extended to a higher number of spatial dimensions.

The solution of (2.25) is comprised of the discrete spectrum

$$\sum_{m=1}^{N} c_m^2(0) \exp(8\kappa_m^3 t - \kappa_m \xi)$$

and the continuous spectrum

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} b(\kappa, 0) \exp(\mathbf{i}(8\kappa^3 t + \kappa\xi)) d\kappa.$$

One should take a moment to compare these terms to the Airy integral (2.17). The solution of the linear equation (2.25), $\phi_{(t)}(\xi)$, is the sum of these two expressions. To translate back to the original equation (1.31), one solves for $K_{(t)}(x, y)$ in the integral equation

$$K_{(t)}(x, y) + \phi_{(t)}(x + y) + \int_{x}^{\infty} \phi_{(t)}(y + z) K_{(t)}(x, z) dz \;=\; 0.$$

Once K is known, the solution to (1.31) is

$$\zeta(x, t) \;=\; -2 \frac{d}{dx} K(x, x).$$

In the case $b(\kappa, 0) = 0$ (termed *reflectionless*) there is no contribution from the continuous spectrum. Then the inverse scattering method produces multiple wave solutions that develop a sech^2 profile upon interaction, but with relationships between amplitude phase and speed dependent upon the number of waves interacting. In particular, N soliton solutions develop from initial profile

$$-N(N+1)\text{sech}^2(x). \tag{2.26}$$

2.4.3. Restrictions on the initial condition

In response to the discovery of the Inverse Scattering Transform as a means to solve the KdV, Cohen [8] studied the effect of rate of decay of initial conditions on the resulting solutions. She demonstrated the initial value problem for (2.15) is solved by the Fourier solution, and established decay rates of this solution. These solutions were related back to the KdV, finding the initial datum $\zeta(x, 0) = \zeta_0(x)$ determines the smoothness of the solution. She placed the following requirements on the initial data:

$$\zeta_0 \in C^3(\mathbb{R})$$

$$\zeta_0 \text{ is piecewise of class } C^4(\mathbb{R})$$

$$\zeta_0^{(j)} \text{ decays at an algebraic rate for } j \leq 4$$

2.5. Two-Dimensional Extensions

2.5.1. The Sjöstrand problem

The time-independent problem

$$\frac{\partial}{\partial x}[\nabla^2 \psi] = 0$$

was studied by Sjöstrand in 1936 [32] and by von Wolfersdorf in 1970 [33]. Von Wolfersdorf looked at the more general form

$$\frac{\partial}{\partial x}[\nabla^2 \psi] = F(\psi, \psi_x, \psi_y, \psi_{xx}, \psi_{xy}, \psi_{yy})$$

where F is linear in ψ and its derivatives, with coefficients dependent on x, y. The problem was approached by working in the complex plane, and does not generalize to the time-dependent case.

2.5.2. The K-P equation

There are a few equations which extend the KdV to two horizontal dimensions. One such was developed by Kadomtsev and Petviashvili in [21] and known as the K-P equation:

$$(\zeta_t + \zeta\zeta_x + \zeta_{xxx})_x + \zeta_{yy} = 0. \tag{2.27}$$

This equation is derived through physical reasoning under the assumption of "weak" transverse motion. Specifically, one begins with the KdV equation and adds a correction term to account for weak y-coordinate dependence, such as a bending distortion. Although the equation was originally developed to model waves with a weak phase and amplitude variations in the y-direction, it has subsequently been used to fit data from fully two dimensional, periodic shallow water waves [18].

Like the KdV, the K-P equation has been studied by means of the Inverse Scattering Transform [2]. This method yields soliton solutions through an analytical process. Solutions to (2.27) include the essentially one-dimensional solution traveling at an angle to the y-axis

$$\zeta(x, y, t) = 2\kappa^2 \mathrm{sech}^2[\kappa(x + \lambda y - (4\kappa^2 + 3\lambda^2)t + \delta_0)]$$

and the "two line-soliton" solution

$$\begin{aligned}
\zeta(x,y,t) &= 2\frac{\partial^2}{\partial x^2}\ln F(x,y,t) \\
F(x,y,t) &= 1+\exp(\eta_1)+\exp(\eta_2)+\exp(\eta_1+\eta_2+A_{12}) \\
\eta_i &= 2\kappa_i(x+\lambda_i y-(4\kappa_i^2+3\lambda_i^2)t+\delta_i) \\
\exp(A_{12}) &= \frac{4(\kappa_1-\kappa_2)^2-(\lambda_1-\lambda_2)^2}{4(\kappa_1+\kappa_2)^2-(\lambda_1-\lambda_2)^2}
\end{aligned}$$

Neither of these solutions decay as $(x^2+y^2)^{1/2} \longrightarrow \infty$. It is clear the first takes the form of KdV solution, and away from the interaction in the latter example the waves tend toward a sech^2 profile (see [1]).

It is possible to derive a PDE of this form using series expansion methods as with the KdV (see, for example [1]). To do this, one starts with the equations governing water waves under the assumption that the y-scale is small compared to the x-scale.

3. DEVELOPMENT

We proceed to derive an equation for a two-dimensional wave in a manner analogous to the approximation method presented in [22] and outlined in the introduction.

3.1. Power Series Expansions of the Velocities

The fluid is incompressible (1.7), irrotational (1.10) and homogeneous. Recall that throughout the body, the velocities u, v, w are governed by the following relations:

$$w_y - v_z = 0 \tag{3.1}$$

$$u_z - w_x = 0 \tag{3.2}$$

$$u_y - v_x = 0 \tag{3.3}$$

$$u_x + v_y + w_z = 0. \tag{3.4}$$

The bottom is a horizontal plane taken as $z = 0$ and there is no vertical flow out of the bed, so (1.14) is applied to get

$$w(x, y, 0, t) = 0. \tag{3.5}$$

Assuming all functions to be "smooth enough", and the vertical scale in z to be small, there are power series expansions of the velocities

$$u(x,y,z,t) = \sum_{n=0}^{\infty} z^n u_n(x,y,t) \tag{3.6}$$

$$v(x,y,z,t) = \sum_{n=0}^{\infty} z^n v_n(x,y,t) \tag{3.7}$$

$$w(x,y,z,t) = \sum_{n=0}^{\infty} z^n w_n(x,y,t). \tag{3.8}$$

To comply with the condition that the bed is impermeable (3.5), $w_0 = 0$ in (3.8). By equating coefficients of powers of z when these power series are substituted into the relations described by (3.1) and (3.2) we see

$$u_{n+1} = \frac{1}{n+1}(w_n)_x \tag{3.9}$$

$$v_{n+1} = \frac{1}{n+1}(w_n)_y. \tag{3.10}$$

Next w may be written in terms of u_n, v_n by using the relation determined by conservation of mass. Computing the derivatives of expansions (3.6)-(3.8), combining according to (3.4) and equating coefficients of powers of z yields

$$w_n = -\frac{1}{n}[(u_{n-1})_x + (v_{n-1})_y] \tag{3.11}$$

for $n \geq 1$. Therefore,

$$w = \sum_{n=1}^{\infty}(-\frac{1}{n})[(u_{n-1})_x + (v_{n-1})_x]z^n.$$

Since $w_0 = 0$, we see from (3.9) and (3.10) that $u_1 = v_1 = 0$. Moreover, by repeatedly applying (3.9)-(3.11), $w_{2n} = u_{2n+1} = v_{2n+1} = 0$ for all n. That is, the odd terms drop out of the horizontal velocity components. Additionally, all even terms

w_{2n} are zero. This leaves u_0 and v_0 undetermined, but non-zero if the system is to have two horizontal components of velocity.

Expressions for the remaining u_n and v_n can be obtained in terms of in terms of v_0 and u_0 by combining the equations (3.9)-(3.10) with (3.11):

$$\begin{aligned} v_{n+1} &= -\frac{1}{n(n+1)}\left[(u_{n-1})_x + (v_{n-1})_y\right]_y \\ u_{n+1} &= -\frac{1}{n(n+1)}\left[(u_{n-1})_x + (v_{n-1})_y\right]_x. \end{aligned}$$

The irrotational assumption (3.3) implies

$$(u_n)_y = (v_n)_x$$

for all n, and substitution of this to the prior expressions leaves

$$\begin{aligned} u_{n+1} &= -\frac{1}{n(n+1)}\left[(u_{n-1})_{xx} + (u_{n-1})_{yy}\right] \\ v_{n+1} &= -\frac{1}{n(n+1)}\left[(v_{n-1})_{xx} + (v_{n-1})_{yy}\right]. \end{aligned}$$

Now v_n and u_n can be recursively expressed in terms of $\tilde{u} = u_0$ and $\tilde{v} = v_0$ and the equations (3.6)-(3.8) for u, v and w are reduced to

$$u = \tilde{u} - \frac{z^2}{2}(\tilde{u}_{xx} + \tilde{u}_{yy}) + \frac{z^4}{24}(\tilde{u}_{xxxx} + 2\tilde{u}_{xxyy} + \tilde{u}_{yyyy}) + ... \tag{3.12}$$

$$v = \tilde{v} - \frac{z^2}{2}(\tilde{v}_{xx} + \tilde{v}_{yy}) + \frac{z^4}{24}(\tilde{v}_{xxxx} + 2\tilde{v}_{xxyy} + \tilde{v}_{yyyy}) + ... \tag{3.13}$$

$$w = -z(\tilde{u}_x + \tilde{v}_y) + \frac{z^3}{6}(\tilde{u}_{xxx} + \tilde{u}_{yyx} + \tilde{v}_{xxy} + \tilde{v}_{yyy}) - ... \tag{3.14}$$

3.2. Expansion of the Surface Equations

Constraints on the surface, $z = l + \tilde{\zeta}$ were developed in the introduction. Restated, these boundary conditions are

$$0 = u_t + (uu_x + vv_x + ww_x) + g\tilde{\zeta}_x + \frac{\tau}{\rho}H_x \quad (3.15)$$

$$0 = v_t + (uu_y + vv_y + ww_y) + g\tilde{\zeta}_y + \frac{\tau}{\rho}H_y, \quad (3.16)$$

recalling $H(x, y, t)$ is the curvature function given in (1.16), and

$$0 = \tilde{\zeta}_t + u\tilde{\zeta}_x + v\tilde{\zeta}_y - w. \quad (3.17)$$

At this point, it will simplify notation to use the Laplacian in 2 variables,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

since all equations will now restricted to the surface. Substitution of the velocity expansions (3.12)-(3.14) into the dynamic surface equations (3.15), (3.16) gives

$$0 = \tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{v}\tilde{v}_x + g\tilde{\zeta}_x - \frac{\tau}{\rho}\nabla^2\tilde{\zeta}_x - \frac{(l+\tilde{\zeta})^2}{2}(\tilde{u}\nabla^2\tilde{u}_x + \tilde{v}\nabla^2\tilde{v}_x) + ... \quad (3.18)$$

$$0 = \tilde{v}_t + \tilde{u}\tilde{u}_y + \tilde{v}\tilde{v}_y + g\tilde{\zeta}_y - \frac{\tau}{\rho}\nabla^2\tilde{\zeta}_y - \frac{(l+\tilde{\zeta})^2}{2}(\tilde{u}\nabla^2\tilde{u}_y + \tilde{v}\nabla^2\tilde{v}_y) + ... \quad (3.19)$$

and (3.17) is expressed as

$$0 = \tilde{\zeta}_t + \tilde{\zeta}_x\tilde{u} + \tilde{\zeta}_y\tilde{v} + (l+\tilde{\zeta})(\tilde{u}_x + \tilde{v}_y) - \frac{(l+\tilde{\zeta})^3}{6}(\nabla^2\tilde{u}_x + \nabla^2\tilde{v}_y) + ... \quad (3.20)$$

3.2.1. Linear first approximation

The wave surface and velocities are expanded to first order by setting

$$z = l + \epsilon\zeta(x, y, t) \quad (3.21)$$

$$\tilde{u}(x, y, t) = q + \epsilon\beta(x, y, t) \quad (3.22)$$

$$\tilde{v}(x, y, t) = r + \epsilon\gamma(x, y, t). \quad (3.23)$$

The constants q and r are the horizontal components of the mean current. These are scaled to satisfy $q^2+r^2=gl$. In our first approximation, we will find expressions for β and γ, which will then be used to form a second approximation.

As an aside, we take a moment to examine a first approximation to the surface equations assuming the independent variables are all of the same order. Then the first derivative terms of equations (3.18)-(3.20) are

$$\beta_t + g\zeta_x + q\beta_x + r\gamma_x = \mathcal{O}(\epsilon) \tag{3.24}$$

$$\gamma_t + g\zeta_y + q\beta_y + r\gamma_y = \mathcal{O}(\epsilon) \tag{3.25}$$

$$\zeta_t + l(\beta_x + \gamma_y) + q\zeta_x + r\zeta_y = \mathcal{O}(\epsilon). \tag{3.26}$$

Choosing a transformation

$$\tilde{x} = x - qt \tag{3.27}$$

$$\tilde{y} = y - rt \tag{3.28}$$

$$\tilde{t} = t, \tag{3.29}$$

the resulting system can be represented as a matrix operator

$$\begin{bmatrix} g\frac{\partial}{\partial\tilde{x}} & (\frac{\partial}{\partial\tilde{t}} - r\frac{\partial}{\partial\tilde{y}}) & r\frac{\partial}{\partial\tilde{x}} \\ g\frac{\partial}{\partial\tilde{y}} & q\frac{\partial}{\partial\tilde{y}} & (\frac{\partial}{\partial\tilde{t}} - q\frac{\partial}{\partial\tilde{x}}) \\ \frac{\partial}{\partial\tilde{t}} & l\frac{\partial}{\partial\tilde{x}} & l\frac{\partial}{\partial\tilde{y}} \end{bmatrix} \begin{bmatrix} \zeta \\ \beta \\ \gamma \end{bmatrix} = 0. \tag{3.30}$$

Solutions of this system are kernels of the operator

$$(\frac{\partial}{\partial\tilde{t}} - q\frac{\partial}{\partial\tilde{x}} - r\frac{\partial}{\partial\tilde{y}})(\frac{\partial^2}{\partial\tilde{t}^2} - gl(\frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{y}^2})). \tag{3.31}$$

In this form, two well-known equations appear. The first operator is a linearization of an advection equation

$$0 = \zeta_t - (u\zeta)_x - (v\zeta)_y.$$

This is equivalent to the case $\frac{\partial}{\partial t} = 0$ in the original system before the transformations (3.27) were applied. It would have a steady solution travelling in the direction of a characteristic line. The second operator is the wave equation,

$$\Phi_{tt} = gl\nabla^2\Phi$$

which also has well-known solutions, including radially symmetric waves, as seen in texts on mathematical physics such as [14].

3.2.2. Scaling of the independent variables

The KdV equation (2.10) was constructed to approximate slow variations, so instead of having the independent variables of the same order, we apply the scaling

$$x \longrightarrow \epsilon^{1/2}x \tag{3.32}$$

$$y \longrightarrow \epsilon^{1/2}y \tag{3.33}$$

$$t \longrightarrow \epsilon^{3/2}t \tag{3.34}$$

This choice weights the equations so that in the linear case, the wave will be stationary as in [22], and so that higher order derivatives will be higher orders of ϵ. Substituting the velocity approximations (3.21)-(3.23) into the series expansions of the surface equations (3.18)-(3.20) along with the scaling (3.32)-(3.34) and eliminating constants, the first approximations to the surface equations become

$$g\zeta_x + q\beta_x + r\gamma_x = \mathcal{O}(\epsilon^{3/2}) \tag{3.35}$$

$$g\zeta_y + q\beta_y + r\gamma_y = \mathcal{O}(\epsilon^{3/2}) \tag{3.36}$$

$$l(\beta_x + \gamma_y) + q\zeta_x + r\zeta_y = \mathcal{O}(\epsilon^{3/2}). \tag{3.37}$$

Immediately from (3.35) and (3.36), or from the integrated form of these equations,

$$\zeta = -\frac{1}{g}(q\beta + r\gamma). \tag{3.38}$$

Recall, any constant of integration or functions of t alone may be taken as zero by adjusting the potential in the Bernoulli equation (1.13). At this point, the relation (3.38) along with the expectation that (at first order) velocity depends on wave height suggests we may take $\beta = A_1\zeta$ and $\gamma = A_2\zeta$. This is essentially a statement that the fluid velocity components are dependent on wave height at this order. Replacement in (3.37) gives

$$(q + lA_1)\zeta_x + (r + lA_2)\zeta_y = \mathcal{O}(\epsilon^{3/2}). \tag{3.39}$$

Thus, the $\mathcal{O}(\epsilon^{3/2})$ terms of both the surface equations will vanish if

$$\begin{aligned} \beta &= -\frac{q}{l}\zeta \\ \gamma &= -\frac{r}{l}\zeta. \end{aligned}$$

3.2.3. Second approximation

Using the representations of β, γ in terms of ζ, we aim to find a quasi-linear equation in terms of ζ and its derivatives of order three or less. We continue to the next approximation assuming the coordinates have been scaled according to (3.32)-(3.34) as before. A small perturbation to the velocity components (3.22), (3.23) is added to give

$$\tilde{u} = q - \epsilon\frac{q}{l}\zeta + \epsilon^2\tilde{\beta}(x, \epsilon y, t) \tag{3.40}$$

$$\tilde{v} = r - \epsilon\frac{r}{l}\zeta + \epsilon^2\tilde{\gamma}(\epsilon x, y, t) \tag{3.41}$$

$$z = l + \epsilon\zeta(x, y, t). \tag{3.42}$$

For reference, the velocities now take on the form

$$\begin{aligned}
u &= q - \epsilon\frac{q}{l}\zeta + \epsilon^2(\tilde{\beta} + \frac{ql}{2}\nabla^2\zeta) + \ldots \\
v &= r - \epsilon\frac{r}{l}\zeta + \epsilon^2(\tilde{\gamma} + \frac{rl}{2}\nabla^2\zeta) + \ldots \\
w &= \epsilon^{3/2}(q\zeta_x + r\zeta_y) + \epsilon^{5/2}(\frac{q}{l}\zeta\zeta_x) + \frac{r}{l}\zeta\zeta_y - l\tilde{\beta}_x - l\tilde{\gamma}_y - \frac{ql^2}{6}\nabla^2\zeta_x - \frac{rl^2}{6}\nabla^2\zeta_y
\end{aligned}$$

The approximations (3.40), (3.41), (3.42) are substituted into (3.18), (3.19) and (3.20). Because of the choice of β and γ in the first approximation, the terms up to order $\mathcal{O}(\epsilon^{5/2})$ vanish leaving

$$\begin{aligned}
\mathcal{O}(\epsilon^{5/2}) &= \beta_t + \beta\beta_x + \gamma\gamma_x - \frac{\tau}{\rho}\nabla^2\zeta_x - \frac{l^2}{2}(q\nabla^2\beta_x + r\nabla^2\gamma_x) + q\tilde{\beta}_x \\
\mathcal{O}(\epsilon^{5/2}) &= \gamma_t + \beta\beta_y + \gamma\gamma_y - \frac{\tau}{\rho}\nabla^2\zeta_y - \frac{l^2}{2}(q\nabla^2\beta_y + r\nabla\gamma_y) + r\tilde{\gamma}_y \\
\mathcal{O}(\epsilon^{5/2}) &= \zeta_t + (\beta\zeta)_x + (\gamma\zeta)_y - \frac{l^3}{6}(\nabla^2\beta_x + \nabla^2\gamma_y) + l\tilde{\beta}_x + l\tilde{\gamma}_y
\end{aligned}$$

or by replacing $\beta = -\frac{q}{l}\zeta$ and $\gamma = -\frac{r}{l}\zeta$ and combining terms with $q^2 + r^2 = gl$,

$$\mathcal{O}(\epsilon^{5/2}) = -\frac{q}{l}\zeta_t + \frac{g}{l}\zeta\zeta_x + (\frac{gl^2}{2} - \frac{\tau}{\rho})\nabla^2\zeta_x + q\tilde{\beta}_x \tag{3.43}$$

$$\mathcal{O}(\epsilon^{5/2}) = -\frac{r}{l}\zeta_t + \frac{g}{l}\zeta\zeta_y + (\frac{gl^2}{2} - \frac{\tau}{\rho})\nabla^2\zeta_y + r\tilde{\gamma}_y \tag{3.44}$$

$$\mathcal{O}(\epsilon^{5/2}) = \zeta_t - \frac{2q}{l}\zeta\zeta_x - \frac{2r}{l}\zeta\zeta_y + \frac{l^2}{6}(q\nabla^2\zeta_x + r\nabla^2\zeta_y) + l\tilde{\beta}_x + l\tilde{\gamma}_y. \tag{3.45}$$

At this point, it is apparent that each of the surface equations have assumed a form similar to a KdV equation with cross-derivative terms and a perturbation. These equations can be combined so that the $\tilde{\beta}_x$ and $\tilde{\gamma}_y$ terms are eliminated. Thus, in the case neither q nor r is zero, our final equation is of the form

$$0 = 3qr\zeta_t + \alpha_1\zeta\zeta_x + \sigma_1(\zeta_{xxx} + \zeta_{yyx}) + \alpha_2\zeta\zeta_y + \sigma_2(\zeta_{xxy} + \zeta_{yyy}), \tag{3.46}$$

where the coefficients are

$$\begin{aligned}
\alpha_1 &= -\frac{r}{l}(gl + 2q^2) \\
\alpha_2 &= -\frac{q}{l}(gl + 2r^2) \\
\sigma_1 &= rl(\frac{T}{\rho} - \frac{l(2q^2 + 3r^2)}{6}) \\
\sigma_2 &= ql(\frac{T}{\rho} - \frac{l(3q^2 - 2r^2)}{6}).
\end{aligned}$$

Independent variables, as well as ζ, are easily scaled to change the coefficients if desired.

3.3. An Alternate Derivation without Surface Tension

The third derivative term will appear in the equation even when surface tension has been neglected, due to the form of the velocity expansions. Following the method of Whitham [34] but as in the prior section, incorporating two spatial variables, we give an outline of the procedure.

Variables are scaled

$$\begin{aligned}
x &\longrightarrow l\epsilon_2^{-1/2}x \\
y &\longrightarrow l\epsilon_2^{-1/2}y \\
z &\longrightarrow lz \\
t &\longrightarrow \sqrt{\frac{l}{g}\epsilon_2^{-1/2}}t \\
\zeta &\longrightarrow l\epsilon_1\zeta \\
\Phi &\longrightarrow l\sqrt{gl}\epsilon_1\epsilon_2^{-1/2}\Phi
\end{aligned}$$

so the equations for the potential read

$$\epsilon_2(\Phi_{xx} + \Phi_{yy}) + \Phi_{zz} = 0 \qquad 0 < z < 1 + \epsilon_1\zeta$$

$$\Phi_z = 0 \qquad z = 0$$

so the expansion of Φ is

$$\Phi = \tilde{\Phi} - \epsilon_2 \frac{z^2}{2} \nabla^2 \tilde{\Phi} + \epsilon_2^2 \frac{z^4}{24} \nabla^2(\nabla^2 \tilde{\Phi}).$$

The derivatives of $\tilde{\Phi}$ are taken to be dependent on ζ and its derivatives:

$$\begin{aligned} \tilde{\Phi}_x &= A_1\zeta + \epsilon_1\beta + \epsilon_2\tilde{\beta} \\ \tilde{\Phi}_y &= A_2\zeta + \epsilon_1\gamma + \epsilon_2\tilde{\gamma}. \end{aligned}$$

Now, on the surface $z = 1 + \epsilon_1\zeta$, the scaled kinematic surface condition is re-written as

$$\begin{aligned} \mathcal{O}(\epsilon_1\epsilon_2, \epsilon_2^2) &= \zeta_t + [(1+\epsilon_1\zeta)\tilde{\Phi}_x]_x + [(1+\epsilon_1\zeta)\tilde{\Phi}_y]_y - \epsilon_2[\frac{1}{6}\nabla^2(\nabla^2\tilde{\Phi})] \\ &= [\zeta_t + A_1\zeta_x + A_2\zeta_y] + \epsilon_1[\beta_x + \gamma_y + 2\zeta(A - 1\zeta_x + A_2\zeta_y)] \\ &\quad + \epsilon_2[\tilde{\beta}_x + \tilde{\gamma}_y - \frac{1}{6}(A_1\nabla^2\zeta_x + A_2\nabla^2\zeta_y)]. \end{aligned}$$

The x- and y-derivatives of the dynamic surface condition (without surface tension) become (respectively)

$$\begin{aligned} \mathcal{O}(\epsilon_2^2) &= [A_1\zeta_t + \zeta_x] + \epsilon_1[\beta_t + A_1^2\zeta\zeta_x + A_1A_2\zeta\zeta_y] - \epsilon_2[\frac{1}{2}A_1\nabla^2\zeta_t + \tilde{\beta}_t] \\ \mathcal{O}(\epsilon_2^2) &= [A_2\zeta_t + \zeta_y] + \epsilon_1[\gamma_t + A_1A_2\zeta\zeta_x + A_2^2\zeta\zeta_y] - \epsilon_2[\frac{1}{2}A_2\nabla^2\zeta_t + \tilde{\gamma}_t]. \end{aligned}$$

In order to make the $\mathcal{O}(1)$ terms drop out of the kinematic equation, let

$$\mathcal{O}(\epsilon_1) = \zeta_t + (A_1\zeta_x + A_2\zeta_y)$$

and use this for substitutions in the $\mathcal{O}(\epsilon_1)$ and $\mathcal{O}(\epsilon_2)$ terms. Combined with making the first terms in the dynamic equation $\mathcal{O}(\epsilon_1)$, we must require

$$A_1^2 + A_2^2 = 1.$$

The next step is to make the $\mathcal{O}(\epsilon_1)$ terms consistent, but not necessarily zero, between the kinematic and dynamic equations. The representations

$$\beta_t - A_1\zeta\zeta_t$$

$$\gamma_t - A_2\zeta\zeta_t$$

allow us to recognize that we need $\beta = \frac{a}{2}\zeta^2$ and $\gamma = \frac{b}{2}\zeta^2$. Now

$$(\tfrac{a}{2}\zeta^2)_x + (\tfrac{b}{2}\zeta)_y + 2\zeta(A_1\zeta_x + A_2\zeta_y)$$

$$-a\zeta(A_1\zeta_x + A_2\zeta_y) + A_1(A_1\zeta_x + A_2\zeta_y)$$

$$-b\zeta(A_1\zeta_x + A_2\zeta_y) + A_2\zeta(A_1\zeta_x + A_2\zeta_y)$$

will all represent the same expressions if

$$\beta = \frac{A_1(A_1-2)}{2(1+A_1)}\zeta^2$$

$$\gamma = \frac{A_2(A_2-2)}{2(1+A_2)}\zeta^2.$$

The same procedure is applied to the $\mathcal{O}(\epsilon_2)$ terms,

$$\tilde{\beta}_x + \tilde{\gamma}_y - \tfrac{1}{6}\nabla^2(A_1\zeta_x + A_2\zeta_y)$$

$$\tilde{\beta}_t - \tfrac{1}{2}A_1\nabla^2\zeta_t$$

$$\tilde{\gamma}_t - \tfrac{1}{2}A_2\nabla^2\zeta_t$$

to find these equations are consistent if

$$\tilde{\beta} = \frac{A_1(3A_1+1)}{6(A_1+1)}\nabla^2\zeta$$

$$\tilde{\gamma} = \frac{A_2(3A_2+1)}{6(1+A_2)}\nabla^2\zeta.$$

4. RESULTS

The differential equation derived in the preceeding section

$$\zeta_t + \alpha_1 \zeta \zeta_x + \alpha_2 \zeta \zeta_y + \sigma_1 \nabla^2 \zeta_x + \sigma_2 \nabla^2 \zeta_y = 0$$

is non-linear and third order, making it unlikely that an explicit solution can be identified. The method of inverse scattering that provided solutions for the Korteweg-de Vries equation was developed for the case of one spatial dimension, and it is not clear how it would generalize to the current case. Instead, we approach the problem by splitting the equation into linear and non-linear parts for further study. The notation follows that in prior sections, indicating vectors by bold print and vector elements by subscripts, such as $\mathbf{k} =< k_1, k_2 >$ and $\|\mathbf{k}\| = (k_1^2 + k_2^2)^{1/2}$.

4.1. Non-linear Part

In the case where the coefficients σ_i are negligible or the third derivatives become small compared to the α_i, the non-linear terms dominate. The method of characteristics allows us to solve the non-linear part,

$$\zeta_t + \alpha_1 \zeta \zeta_x + \alpha_2 \zeta \zeta_y = 0. \tag{4.1}$$

In the case of constant solutions along a parameterized curve $(x(\tau), y(\tau), t(\tau))$, the characteristic equations are

$$\begin{aligned}\frac{d\zeta}{d\tau} &= 0\\ \frac{dx}{d\tau} &= \alpha_1\zeta(\tau)\\ \frac{dy}{d\tau} &= \alpha_2\zeta(\tau)\\ \frac{dt}{d\tau} &= 1.\end{aligned}$$

These equations are integrated with respect to τ on $0 \leq \tau \leq s$ to find

$$\begin{aligned}\zeta(s) &= \zeta_0\\ x(s) &= x_0 + \alpha_1\zeta_0 s\\ y(s) &= y_0 + \alpha_2\zeta_0 s\\ t(s) &= t_0 + s.\end{aligned}$$

Let $t_0 = 0$, and any initial value of $\zeta_0 = \zeta(x_0, y_0)$ follows a line parameterized by

$$< x(s), y(s) > \;=\; < \alpha_1\zeta_0, \alpha_2\zeta_0 > s + < x_0, y_0 > .$$

The initial data then is always transported along a line parallel to $< \alpha_1, \alpha_2 >$, and characteristics through initial values $v_0 = (x_0, y_0)$ and $v_1 = (x_1, y_1)$ will intersect only if $v_1 - v_0$ is parallel to $< \alpha_1, \alpha_2 >$. It then relies on the linear part to create interaction between these lines.

4.2. Linear Part

The linear part consists of the third-order spatial derivatives

$$\zeta_t + \sigma_1(\zeta_{xxx} + \zeta_{yyx}) + \sigma_2(\zeta_{xxy} + \zeta_{yyy}) = 0.$$

Immediately, a fundamental solution of the form $\exp(\mathbf{ik} \cdot \mathbf{x} - \mathbf{i}\tilde{\omega}(\mathbf{k})t)$ can be found since this is a dispersive equation. We are interested in the behavior of the superposition of such waves. The analysis will be more direct if the coordinate system is rotated through an angle

$$\theta = \tan^{-1}(\frac{\sigma_1}{\sigma_2}).$$

After also scaling the time variable, the PDE is simplified to

$$\zeta_t + \nabla^2 \zeta_x = 0. \tag{4.2}$$

Let an arbitrary initial condition be

$$\zeta(\mathbf{x}, 0) = \zeta_0(\mathbf{x}). \tag{4.3}$$

Then an integral expression for the solution is obtained via Fourier Transform:

$$\hat{\zeta}(\mathbf{k}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta(\mathbf{x}, t) \exp(-\mathbf{ik} \cdot \mathbf{x}) d\mathbf{x}.$$

Thus, assuming appropriate decay rates and differentiability, the initial value problem (4.2), (4.3) is transformed to

$$\begin{aligned} \hat{\zeta}_t &= -\mathbf{i}\omega(\mathbf{k})\hat{\zeta} \\ \hat{\zeta}_0(\mathbf{k}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta_0(\eta) \exp(-\mathbf{i}\eta \cdot \mathbf{k}) d\eta \end{aligned}$$

which has solution

$$\hat{\zeta}(\mathbf{k}, t) = \hat{\zeta}_0 \exp(-\mathbf{i}\omega(\mathbf{k})t)$$

where $\omega(\mathbf{k})$ satisfies the (real) dispersion relation

$$\omega(k_1, k_2) = k_1 ||\mathbf{k}||^2. \tag{4.4}$$

It follows that the original problem has a formal integral solution

$$\zeta(\mathbf{x},t) = (\frac{1}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(\mathbf{i}(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t))(\int_{\infty}^{\infty}\int_{-\infty}^{\infty} \zeta_0(\eta)\exp(-\mathbf{i}\eta\cdot\mathbf{k})d\eta)d\mathbf{k}.$$

The order of integration may be reversed

$$\zeta(\mathbf{x},t) \;=\; (\frac{1}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \zeta_0(\eta)\ \mathbf{I}\ d\eta$$

where

$$\mathbf{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(\mathbf{i}\mathbf{k}\cdot <\mathbf{x}-\eta> -\mathbf{i}\omega(\mathbf{k})t)\ d\mathbf{k}. \tag{4.5}$$

The argument of the exponent is a cubic function in $\mathbf{k}$, and this integral can be manipulated into an Airy integral through a change of variables. Let

$$\begin{aligned} k_1 &= -\frac{1}{(12t)^{1/3}}(\xi_1+\xi_2) \\ k_2 &= -\frac{\sqrt{3}}{(12t)^{1/3}}(\xi_1-\xi_2) \end{aligned}$$

so the argument becomes

$$\mathbf{i}\Big[-(\tfrac{(x-\eta_1)+\sqrt{3}(y-\eta_2)}{(12t)^{1/3}})\xi_1 - (\tfrac{(x-\eta_1)-\sqrt{3}(y-\eta_2)}{(12t)^{1/3}})\xi_2 + \tfrac{1}{3}\xi_1^3 + \tfrac{1}{3}\xi_2^3\Big]$$

and the differential becomes

$$d\mathbf{k} \;=\; -\frac{2\sqrt{3}}{(12t)^{2/3}}d\xi.$$

We now can represent (4.5) as the product

$$\begin{aligned} \mathbf{I} &= \int_{-\infty}^{\infty} \exp(\mathbf{i}(\nu_1\xi_1 + \frac{1}{3}\xi_1^3))d\xi_1 \int_{-\infty}^{\infty} \exp(\mathbf{i}(\nu_2\xi_2 + \frac{1}{3}\xi_2^3))d\xi_2 \\ \nu_1 &= -(\frac{(x-\eta_1)+\sqrt{3}(y-\eta_2)}{(12t)^{1/3}}) \\ \nu_2 &= -(\frac{(x-\eta_1)-\sqrt{3}(y-\eta_2)}{(12t)^{1/3}}). \end{aligned} \tag{4.6}$$

The given substitution can also be used to change the variable of the outer integral

$$\eta_1 = \frac{(12t)^{1/3}}{2}(\nu_1 + \nu_2) + x$$
$$\eta_2 = \frac{(12t)^{1/3}}{2\sqrt{3}}(\nu_1 - \nu_2) + y$$
$$d\eta = -\frac{(12t)^{2/3}}{2\sqrt{3}}d\nu$$

so the general solution becomes

$$\zeta(\mathbf{x}, t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta_0 \mathrm{Ai}(\nu_1)\mathrm{Ai}(\nu_2)\ d\nu \tag{4.7}$$
$$\zeta_0 = \zeta_0(\frac{(12t)^{1/3}}{2}(\nu_1 + \nu_2) + x, \frac{(12t)^{1/3}}{2\sqrt{3}}(\nu_1 - \nu_2) + y).$$

Thus, we expect the linear part of the PDE to provide a smoothing effect analagous to the case of one spatial dimension.

5. NUMERICAL EXPLORATION

In the previous section, the analytical behavior of the equation

$$\zeta_t + \alpha_1 \zeta \zeta_x + \alpha_2 \zeta \zeta_y + \sigma_1 \nabla^2 \zeta_x + \sigma_2 \nabla^2 \zeta_y = 0$$

was discussed. We would expect that non-linear terms would produce steep wave fronts (especially in the direction of $< \alpha_1, \alpha_2 >$, while third-derivative terms produce a smoothing effect. When these balance in the one-dimensional case, solitons are formed. In this section, we first describe a numerical method and then proceed to generate some examples to study the behavior of various initial conditions.

5.1. Numerical Method

Here, an explicit method is preferred because of demands on storage for the case of two spatial dimensions. However it is acknowledged that there are limitations on such methods. We will follow the method of Zabusky and Kruskal, generalizing it to include the mixed third derivatives. The spatial grid will be

$$x_0 < x_k < x_K \qquad \text{for } k = 1, \ldots, K-1$$

$$y_0 < y_l < y_L \qquad \text{for } l = 1, \ldots, L-1,$$

where constant intervals

$$\begin{aligned} \Delta x &= x_k - x_{k-1} \\ \Delta y &= y_l - y_{l-1} \end{aligned}$$

will be assumed. In fact, due to the $x - y$ symmetry of the equation, it is appropriate to adopt a square spatial mesh so that $\Delta_x = \Delta_y$. We treat these symbols as interchangeable, but retain the separate notation to the extent that it serves as a reminder of which coordinate is being referred to. The time steps will also be held equal for

$$t_0 < t_n < t_N \qquad \text{for } n = 1, \ldots, N.$$

Assign the approximation at each grid point and time t_n as

$$U^n_{k,l} \;\approx\; \zeta(x_k, y_l, t_n).$$

When given an expression involving only one time level, the superscript may be omitted.

All space and time derivatives will be approximated by center differences. The notation to be used for these discretizations follows the pattern

$$\begin{aligned} D^1_x(U^n_{k,l}) &= U^n_{k+1,l} - U^n_{k-1,l} \\ D^2_x(U^n_{k,l}) &= U^n_{k+1,l} - 2U^n_{k,l} + U^n_{k-1,l}. \end{aligned}$$

The time derivative will also be a center difference, following the same pattern. The third derivative terms are then approximated by a composition of difference operators such as

$$D^1_x(D^2_x + D^2_y)(U^n_{k,l}) \;\approx\; \frac{\partial}{\partial x}\nabla^2\zeta(x_k, y_l, t_n).$$

Periodic boundary conditions will be adopted. In the notation for central differences laid forth above, using 'Ave' to denote the spatial average used for the nonlinear term, the model difference equation is

$$\begin{aligned} \frac{1}{2\Delta t}D^1_t(U^n_{k,l}) &= [\frac{\alpha_1}{2\Delta x}\text{Ave}\cdot D^1_x + \frac{\sigma_1}{2(\Delta x)^3}[D^1_x(D^2_x + D^2_y)](U^n_{k,l}) \\ &+ [\frac{\alpha_2}{2\Delta y}\text{Ave}\cdot D^1_y + \frac{\sigma_2}{2(\Delta y)^3}D^1_x(D^2_x + D^2_y)](U^n_{k,l}) \end{aligned}$$

However, there is still a concern regarding treatment of the non-linear term. We chose to generalize the method of Zabusky and Kruskal because the conservation properties appear to be helpful in getting reasonable numerical results, but there is more than one approach to the discretization Ave $\approx U_{k,l}$. For example, consider two possibilities–one method using a five-point average spread out in both x- and y- directions (as with the Laplacian difference approximation), and the other using distinct three-point averages–the first over x-values and the second over y-values.

$$\begin{aligned}
\text{method 1:} \quad \text{Ave} &= \frac{1}{5}(U_{k+1,l} + U_{k,l+1} + U_{k,l} + U_{k-1,l} + U_{k,l-1}) \\
\text{method 2:} \quad \text{Ave}_x &= \frac{1}{3}(U_{k+1,l} + U_{k,l} + U_{k-1,l}) \\
\text{Ave}_y &= \frac{1}{3}(U_{k,l+1} + U_{k,l} + U_{k,l-1})
\end{aligned}$$

Recall that conservation of momentum occurred in the Zabusky-Kruskal scheme because sum of the spatial discretization was zero. We begin by checking the sum of the non-linear terms of method 1 over both sets of indices, treating terms individually

$$\textstyle\sum_k \sum_l (U_{k+1,l} + U_{k,l+1} + U_{k,l} + U_{k-1,l} + U_{k,l-1})(U_{k+1,l} - U_{k-1,l}).$$

Now, it is already known from the original work by Zabusky and Kruskal that

$$\sum_k (U_{k+1,l} + U_{k,l} + U_{k-1,l})(U_{k+1,l} - U_{k-1,l}) \quad = \quad 0$$

so the sum left to compute is

$$\textstyle\sum_k \sum_l (U_{k,l+1} + U_{k,l-1})(U_{k+1,l} - U_{k-1,l}).$$

However, these terms do not cancel each other out, so method 1 does not conserve momentum.

On the other hand, it is direct to confirm that method 2 will retain the momentum conservation properties as it is the sum of one Z-K method in y and one in x along with some mixed derivative linear terms. The sum of the $D_x^1 D_y^2$ terms is

$$\sum_l \sum_k U_{k+1,l+1} - 2U_{k+1,l} + U_{k+1,l-1} - U_{k-1,l+1} + 2U_{k-1,l} - U_{k-1,l-1}$$

which clearly cancels when summed over k. Method 2 is then chosen for the spatial average term.

5.1.1. Truncation Error

Both the first and second center differences are well known to be second order accurate, as found by Taylor series expansions. It is possible to compute the truncation error of this method without linearizing, as follows:

$$\zeta(x_{k\pm1},\cdot,\cdot) = \zeta(x_k,\cdot,\cdot) \pm \zeta_x(x_k,\cdot,\cdot)\Delta x + \zeta_{xx}(x_k,\cdot,\cdot)\frac{(\Delta x)^2}{2!} \pm \zeta_{xxx}(x_k,\cdot,\cdot)\frac{(\Delta_x)^2}{3!}\cdots. \tag{5.1}$$

The center difference D_x^1 then cancels even order terms, and the sum over three horizontal grid points cancels the odd order terms, leaving

$$\begin{aligned} [\text{Ave}_x][D_x^1] &= [\frac{1}{3}(3\zeta(x_k,\cdot,\cdot) + \mathcal{O}(\Delta x)^2)][(2\Delta x)(\zeta_x(x_k,\cdot,\cdot) + \mathcal{O}(\Delta x)^2)] \\ &= 2(\Delta x)(\zeta\zeta_x + \mathcal{O}(\Delta x)^2). \end{aligned}$$

Thus, Taylor series expansion of these approximations to the non-linear terms gives a truncation error of $\mathcal{O}(\Delta x)^2 + \mathcal{O}(\Delta y)^2$ and for the time step the error is $\mathcal{O}(\Delta t)^2$. That is, the method is 2nd order accurate.

5.1.2. Linear Stability Analysis

A guideline for choosing relative sizes of the time and spatial increments is established by considering the stable Fourier modes

$$U^n_{k,l} \quad = \quad \exp(\mathbf{i}(\xi_1 x_k + \xi_2 y_l - \omega t_n)). \tag{5.2}$$

This information is used to find a necessary (but not always sufficient) condition on Δt in terms of Δx for stability. This method was used in [31] to obtain the refinement path for the one-dimensional case. As reported in Section 2, the relation could be simplified to $\Delta t = \mathcal{O}(\Delta x^3)$.

Recall that our increments are fixed, and $\Delta x = \Delta y$ When (5.2) is substituted into the linearized difference equation

$$\frac{1}{2\Delta t} D^1_t(U^n_{k,l}) \quad = \quad \left[\frac{M}{2\Delta x}(\alpha_1 D^1_x + \alpha_2 D^1_y) + \frac{1}{2(\Delta x)^3}(\sigma_1 D^1_x + \sigma_2 D^1_y)(D^2_x + D^2_y)\right](U^n_{k,l}) \tag{5.3}$$

there will immediately be a common factor of $\exp(\mathbf{i}(\xi_1 x_k + \xi_2 y_l - \omega t_n))$. Neglecting this term, the contributions from the difference operators are

$$D^1_t \quad \longrightarrow \quad \exp(-\mathbf{i}\omega\Delta t) - \exp(\mathbf{i}\omega\Delta t)$$

$$= -2\mathbf{i}\sin(\omega\Delta t) \tag{5.4}$$

$$\alpha_1 D^1_x + \alpha_2 D^1_y \quad \longrightarrow \quad 2\mathbf{i}(\alpha_1 \sin(\xi_1\Delta x) + \alpha_2 \sin(\xi_2 \Delta y)). \tag{5.5}$$

The third order terms are a bit more involved, but reduce to

$$(\sigma_1 D^1_x + \sigma_2 D^1_y)(D^2_x + D^2_y) \longrightarrow$$

$$2\mathbf{i}(\sigma_1 \sin(\xi_1\Delta x) + \sigma_2 \sin(\xi_2\Delta y))(2\cos(\xi_1\Delta x) + 2\cos(\xi_2\Delta y) - 4). \tag{5.6}$$

Substitution of (5.4)-(5.6) into the difference formula (5.3) and multiplying by $-\mathbf{i}\Delta x$ leaves

$$\begin{aligned} -\frac{\Delta x^3}{\Delta t}\sin(\omega\Delta t) &= M\Delta x\Big[\alpha_1\sin(\xi_1\Delta x)+\alpha_2\sin(\xi_2\Delta y)\Big] \\ &+ 2\Big[\sigma_1\sin(\xi_1\Delta x)+\sigma_2\sin(\xi_2\Delta y)\Big]\Big[\cos(\xi_1\Delta x)+\cos(\xi_2\Delta y)-4\Big]. \end{aligned}$$

The result given by Sanz-Serna [31] essentialy set $M = 0$ and maximized $|\sin(\xi_1\Delta x)(\cos(\xi_2)\Delta x)-1)|$. We approach the problem by considering the first and third order terms separately. The first order terms will be the least restrictive:

$$\Delta x^3 \leq \Delta t\Delta x^2|M(\alpha_1+\alpha_2)|,$$

so a refinement path would be $\Delta t = \frac{\Delta x}{|M(\alpha_1+\alpha_2)|}$. However, the third-order terms pose a more restrictive condition.

$$\begin{aligned} \Delta x^3 &\leq 2\Delta t|\sigma_1\sin(\xi_1\Delta x)(\cos(\xi_1\Delta x)-1+\cos(\xi_2\Delta y)-1)| \\ &\quad +2\Delta t|\sigma_2\sin(\xi_2\Delta y)(\cos(\xi_2\Delta y)-1+\cos(\xi_1\Delta x)-1)| \\ &\leq 2\Delta t(\frac{3\sqrt{3}}{4}+2)|\sigma_1+\sigma_2| \end{aligned}$$

or $\Delta t \leq \frac{2\Delta x^3}{(3\sqrt{3}+4)|\sigma_1+\sigma_2|}$.

5.2. Tests and Examples

It is easily established that any solution of the KdV (such as the Jacobi elliptic function solutions) also solve the equation we have derived by virtue of y-independence. This provides one means to test a numerical method. That is, we can compare the current method to known results from cases of one spatial dimension.

Once the method is implemented, we first generate a parallel translation of the (1-dimensional) 2-soliton as a test. Here we use non-linear coefficients $\alpha_1 = 1$, $\alpha_2 = 0$, third order coefficients $\sigma_1 = .022^2$, $\sigma_2 = 0$ and initial condition $3\text{sech}^2(\sqrt{\frac{1}{12\sigma_1}}(x-1))$. The grid size is set with equal spacing $\Delta x = \Delta y = 1/40$ and time step $\Delta t = .005$. See figure (5.1). The results in this case suggest the y-derivative approximations do not introduce errors if we model a soliton solution of the KdV that is independent of y. That is, method appears to be comparable to the original Zabusky-Kruskal method.

5.2.1. Line Solitons

We first look at solutions with a single argument:
$\zeta = \zeta(\chi(x, y, t))$.

$$(\chi_t + (\alpha_1\chi_x + \alpha_2\chi_y)\zeta)\zeta' + (\sigma_1\chi_x + \sigma_2\chi_y)(\chi_x^2 + \chi_y^2)\zeta''' = 0.$$

Whenever χ is linear in the independent variables, the wave retains its shape while being translated along a line in the $x - y$ plane and the PDE reduces to

$$-c\zeta' + (\alpha_1 k_1 + \alpha_2 k_2)\zeta\zeta' + (\sigma_1 k_1 + \sigma_2 k_2)(k_1^2 + k_2^2)\zeta''' = 0$$

where $'$ indicates the derivative with respect to $\chi = k_1 x + k_2 y - ct$. Direct integration yields

$$-c\zeta + \frac{(\alpha_1 k_1 + \alpha_2 k_2)}{2}\zeta^2 + (\sigma_1 k_1 + \sigma_2 k_2)(k_1^2 + k_2^2)\zeta'' = 0.$$

The constant of integration is taken as 0 since the solution of interest and its derivatives decay rapidly as $|\chi| \longrightarrow \infty$. The next step is to multiply by ζ' and integrate

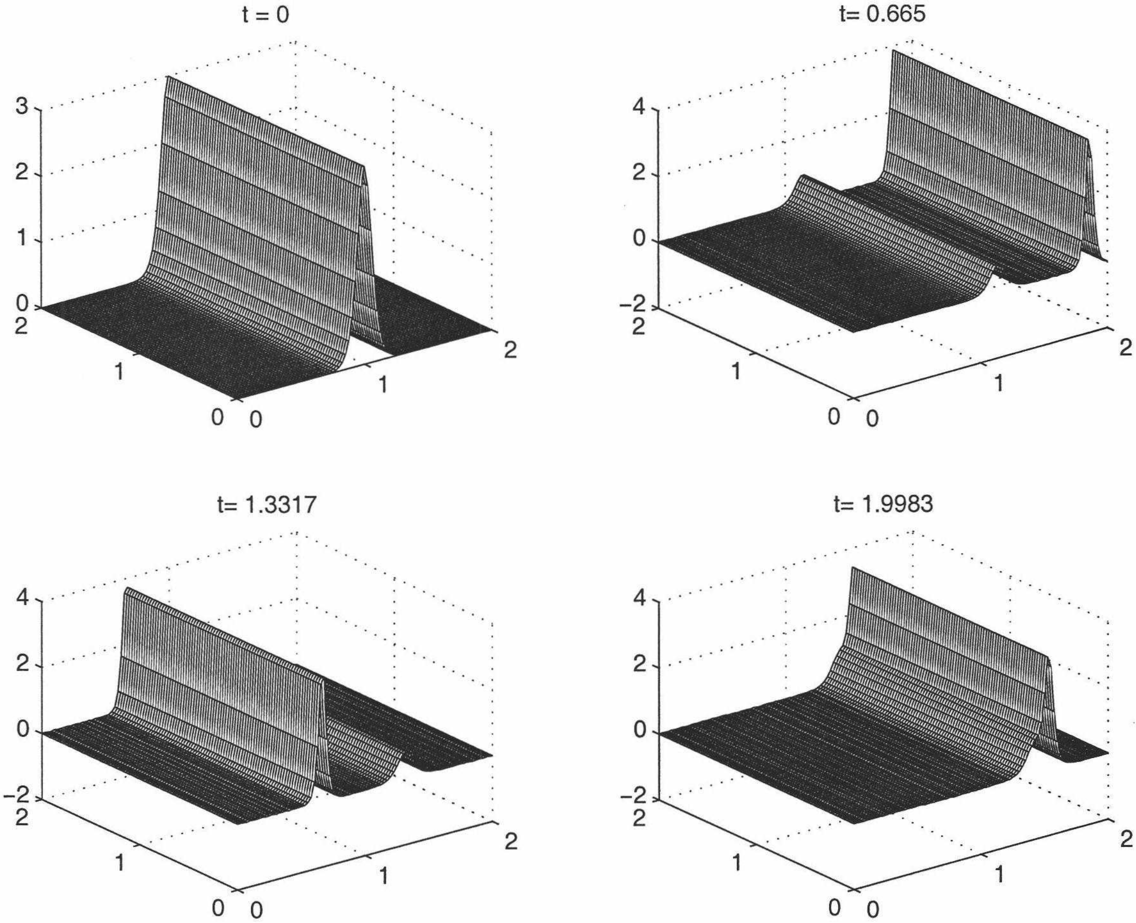

FIGURE 5.1: A one-dimensional two-soliton demonstrates that cross-derivative terms do not introduce unexpected error.

again

$$-\frac{c}{2}\zeta^2 + \frac{(\alpha_1 k_1 + \alpha_2 k_2)}{6}\zeta^3 + (\sigma_1 k_1 + \sigma_2 k_2)(k_1^2 + k_2^2)\frac{1}{2}(\zeta')^2 \;=\; 0.$$

If the gradients of the system are to be real and remain bounded, $(\zeta')^2$ must remain positive and bounded. Considering this reveals that the surface height can only vary between $\zeta = 0$ and $\zeta = \frac{3c}{\alpha_1 k_1 + \alpha_2 k_2}$. Solving for ζ' we obtain a solution analogous to that in [22]

$$\zeta(x, y, t) \;=\; A\mathrm{sech}^2(k_1 x + k_2 y - ct)$$

provided $A,\ k_1,\ k_2,\ c$ are related by

$$\begin{aligned} A &= 12\frac{(k_1^2 + k_2^2)(\sigma_1 k_1 + \sigma_2 k_2)}{\alpha_1 k_1 + \alpha_2 k_2} \\ c &= 4(\sigma_1 k_1 + \sigma_2 k_2)(k_1^2 + k_2^2). \end{aligned}$$

This type of wave will be called a *line soliton.* The distinction between this case and the prior example is that line solitons may intersect at arbitrary angles.

We give two examples of this type of interaction. In both, $\Delta x = \Delta y = 1/50$ and $t = .001$. Two line solitons of amplitude 1 that run perpendicular to each other are given as the initial condition, but the coefficients are varied. For the first case, shown in 5.2, we have set $\alpha_1 = 1$, $\sigma_1 = .022^2$ and $\alpha_2 = \sigma_2 = 0$. Note that the speed at which the higher area (from the intersection) travels faster, but in the same direction as the line-soliton.

In the second case, shown in 5.3 we have set $\alpha_1 = \alpha_2 = \sqrt{2}/2$ and $\sigma_1 = \sigma_2 = .022^2$. Here, the new coefficients cause the "lump" from the original intersection to

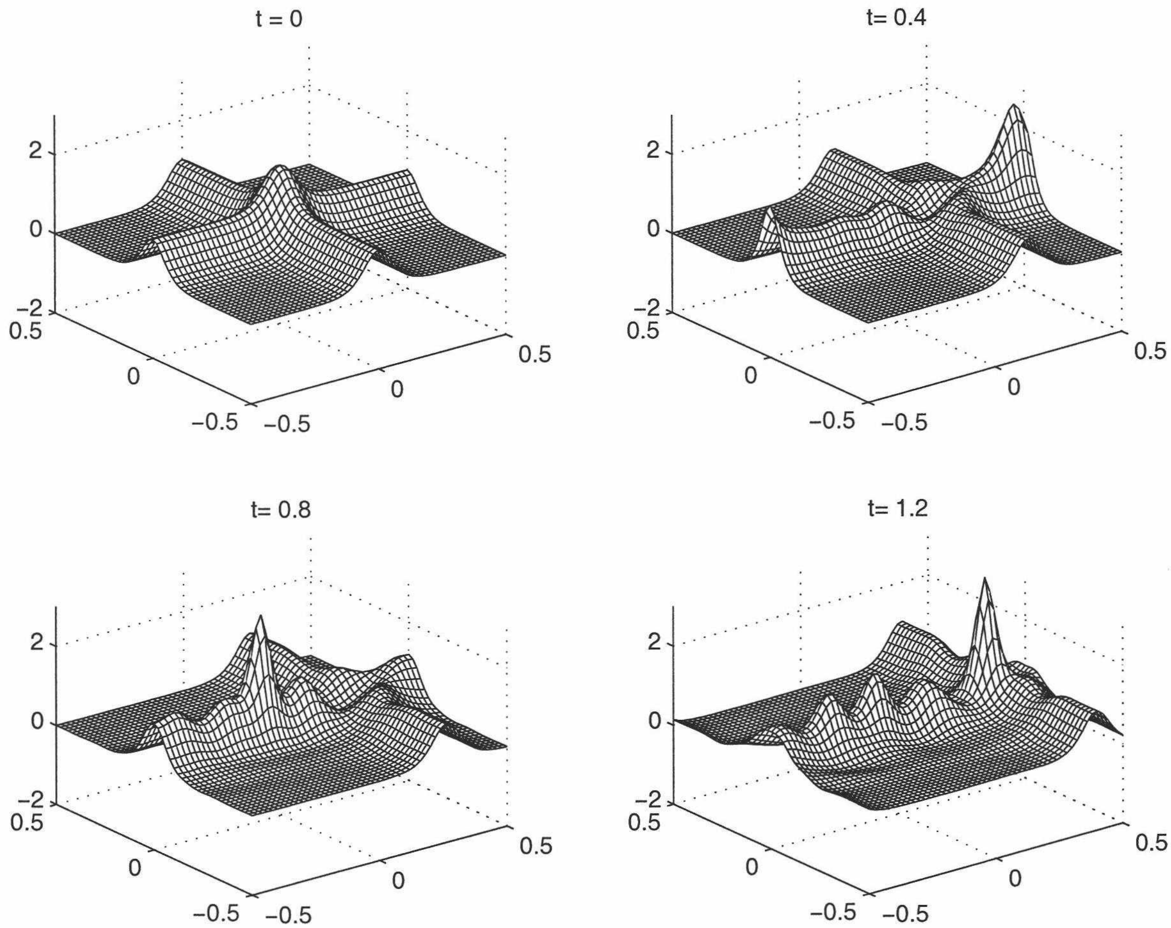

FIGURE 5.2: Intersecting line solitons of initial amplitude 1 with motion parallel to one of the initial waves.

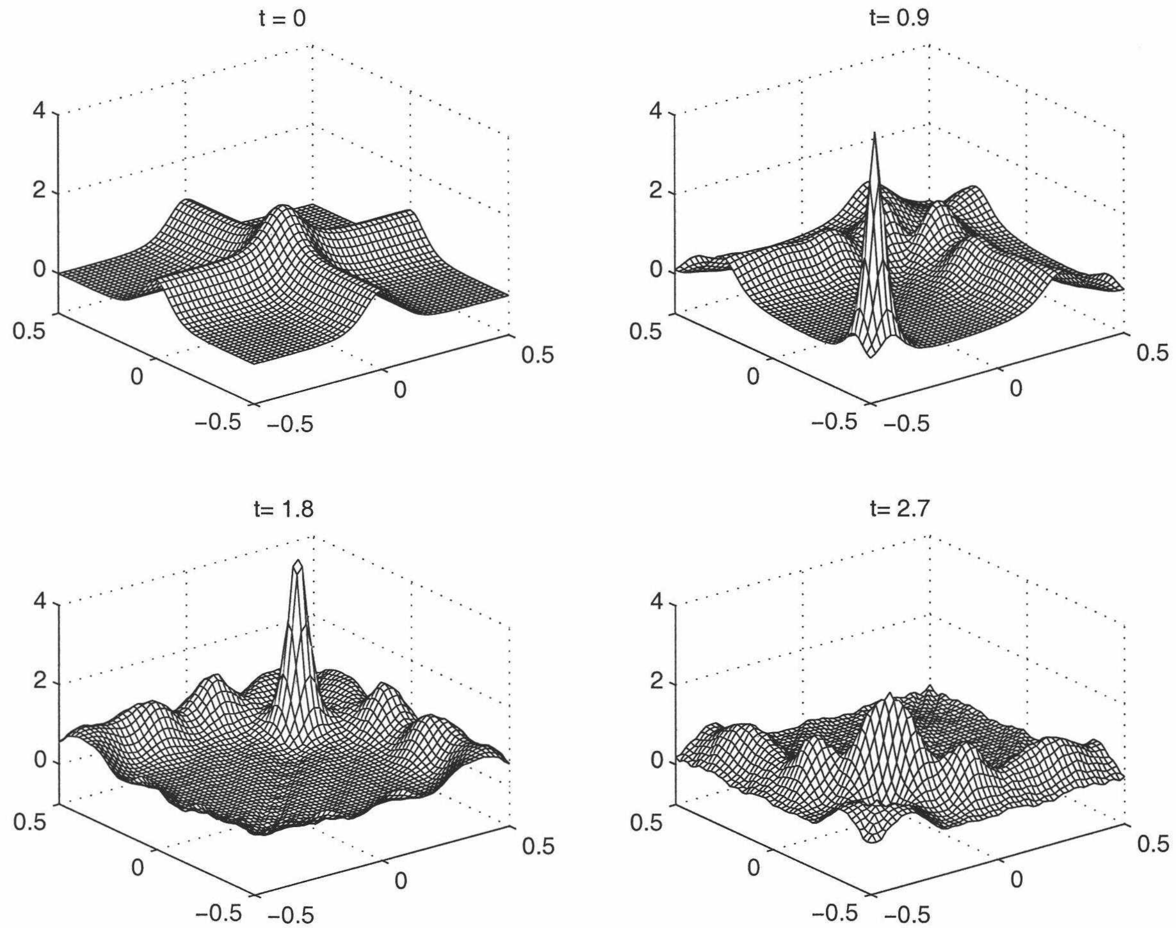

FIGURE 5.3: Intersecting line solitons of initial amplitude 1 with diagonal.

move at a diagonal faster than the intersection itself. We also see a change in the shape of the "lump", becoming taller and narrower, but reversing the change as it approaches the intersection again around $t = 2.7$.

5.2.2. Lump Initial Conditions

Now that the extension of the Zabusky-Kruskal method to two spatial dimensions has been tested on line solitons, we wish to investigate initial conditions that are radially symmetric versions of the N-solitons. That is,

$$\frac{N(N+1)}{2}\text{sech}^2(\sqrt{\frac{1}{12s}}R) \tag{5.7}$$

where we have assigned $s = (\sigma_1^2 + \sigma_2^2)^{1/2}$ and $R = (x^2 + y^2)^{1/2}$.

Keeping the coefficients as before, the grid size is set at $\Delta x = \Delta y = 1/50$ and time step is reset to $\Delta t = .001$. The initial profile is determined by (5.7) with $N = 1$. In figure (5.4), the initial lump is seen to be moving left, with a change in the surface where the lump has passed. We make note that no disturbance was seen when using the one-dimensional method to model the 1- or 2-soliton (see figure (2.1))

Modifying the equation so $\alpha_1 = \alpha_2 = \frac{\sqrt{2}}{2}$ and $\sigma_1 = \sigma_2 = .022^2(\frac{\sqrt{2}}{2})$, we find the same single lump initial condition produces figure 5.5. In this example, $\Delta x = \frac{1}{50}$, $\Delta t = .0005$ and the final time is $t_f = 10$.

Moving on to the case $N = 2$ with $\alpha_1 = \alpha_2 = 1$ and $\sigma_1 = \sigma_2 = \frac{\sqrt{2}}{2}(.022^2)$ gives a lump which separates into 2, the taller moving faster. This behavior is similar to the one-dimensional 2-soliton, except there appears to be some growth. The initial condition for figure 5.6 is $3\text{sech}^2(\sqrt{\frac{1}{12s}}R)$ with $\Delta x = 1/40$, $\Delta t = .001$ and $t_f = 3$.

Adjusting the initial amplitude to .4 with $\alpha_1 = \alpha_2 = 1$, $\sigma_1 = \sigma_2 = .022^2$, $\Delta x = \Delta y = \frac{1}{50}$, $\Delta t = .0005$ produces the wave seen in figure 5.7. Note that this example has been run to an extended time $t = 30$ without blowup.

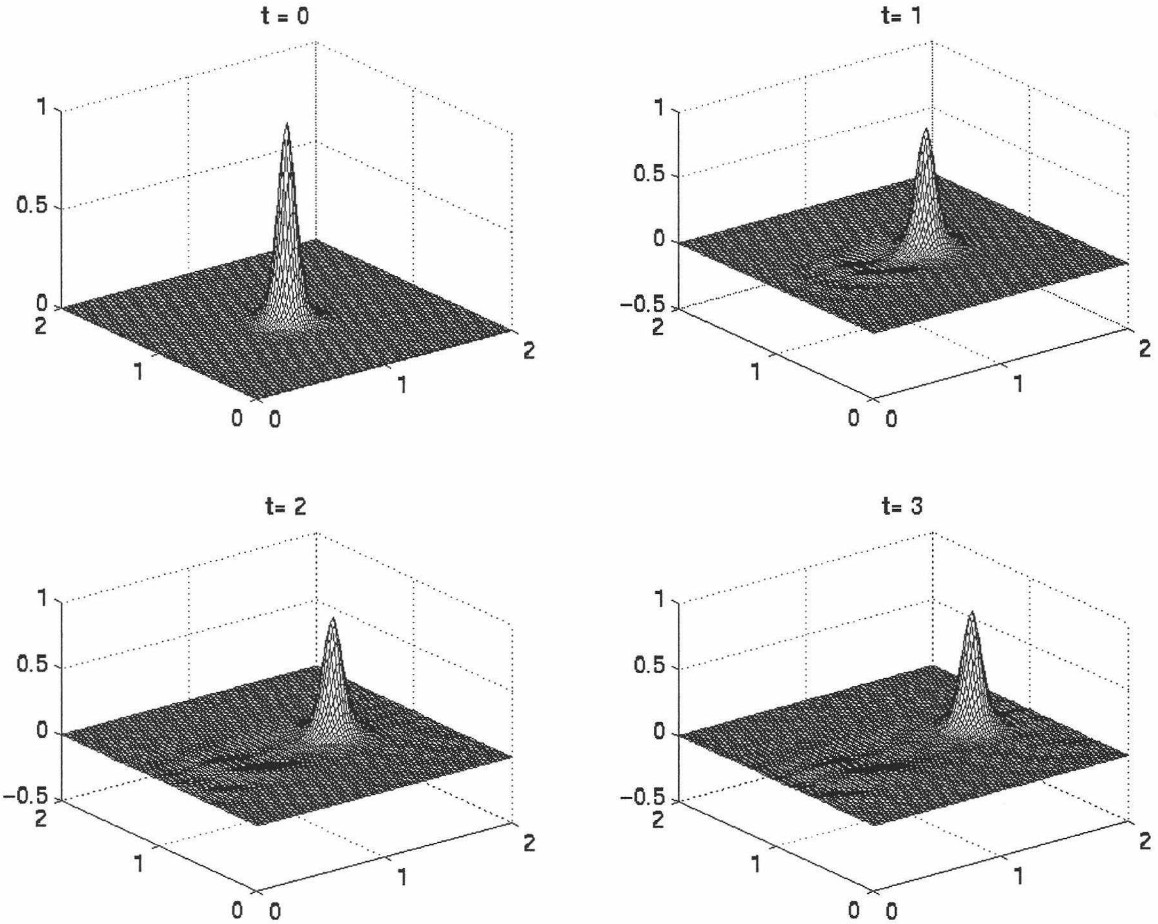

FIGURE 5.4: Radially symmetric initial condition, amplitude 1

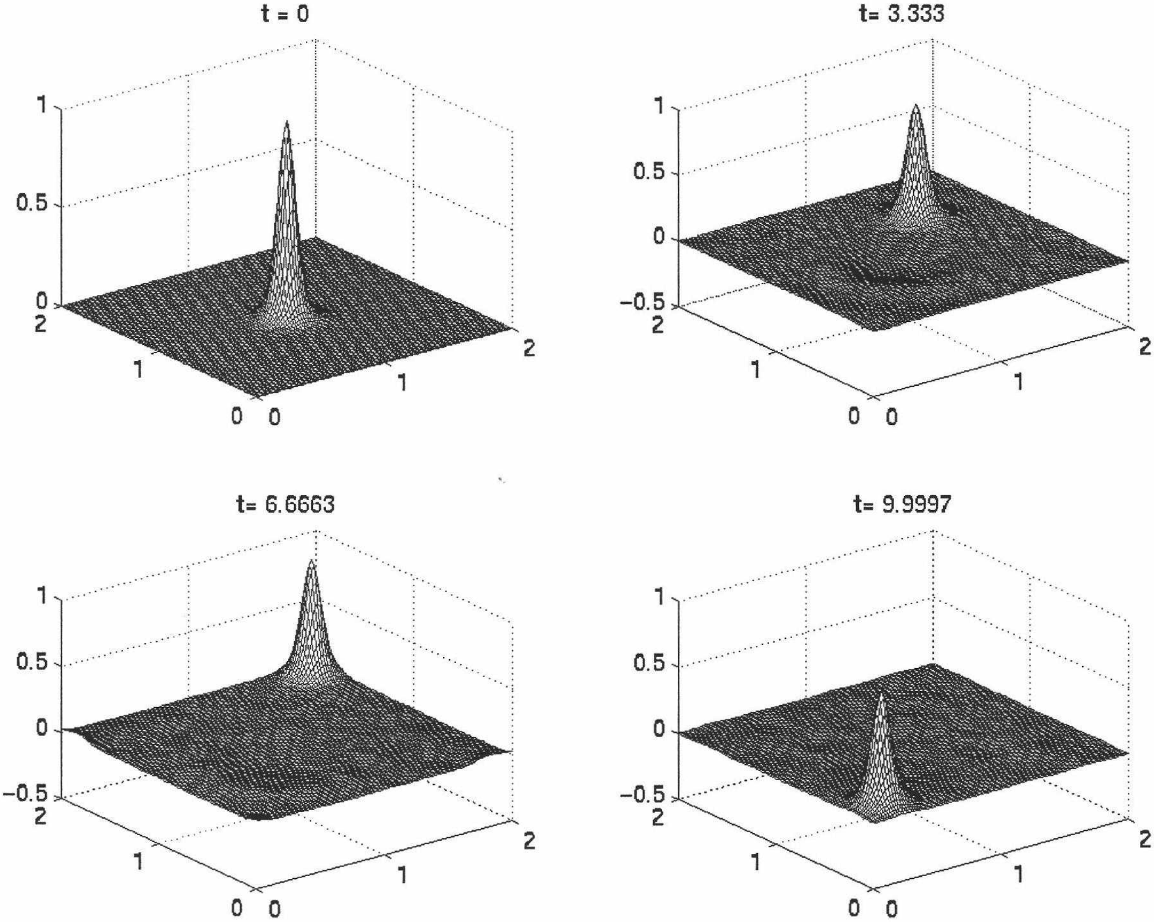

FIGURE 5.5: Radially symmetric initial condition, amplitude 1 with diagonal motion

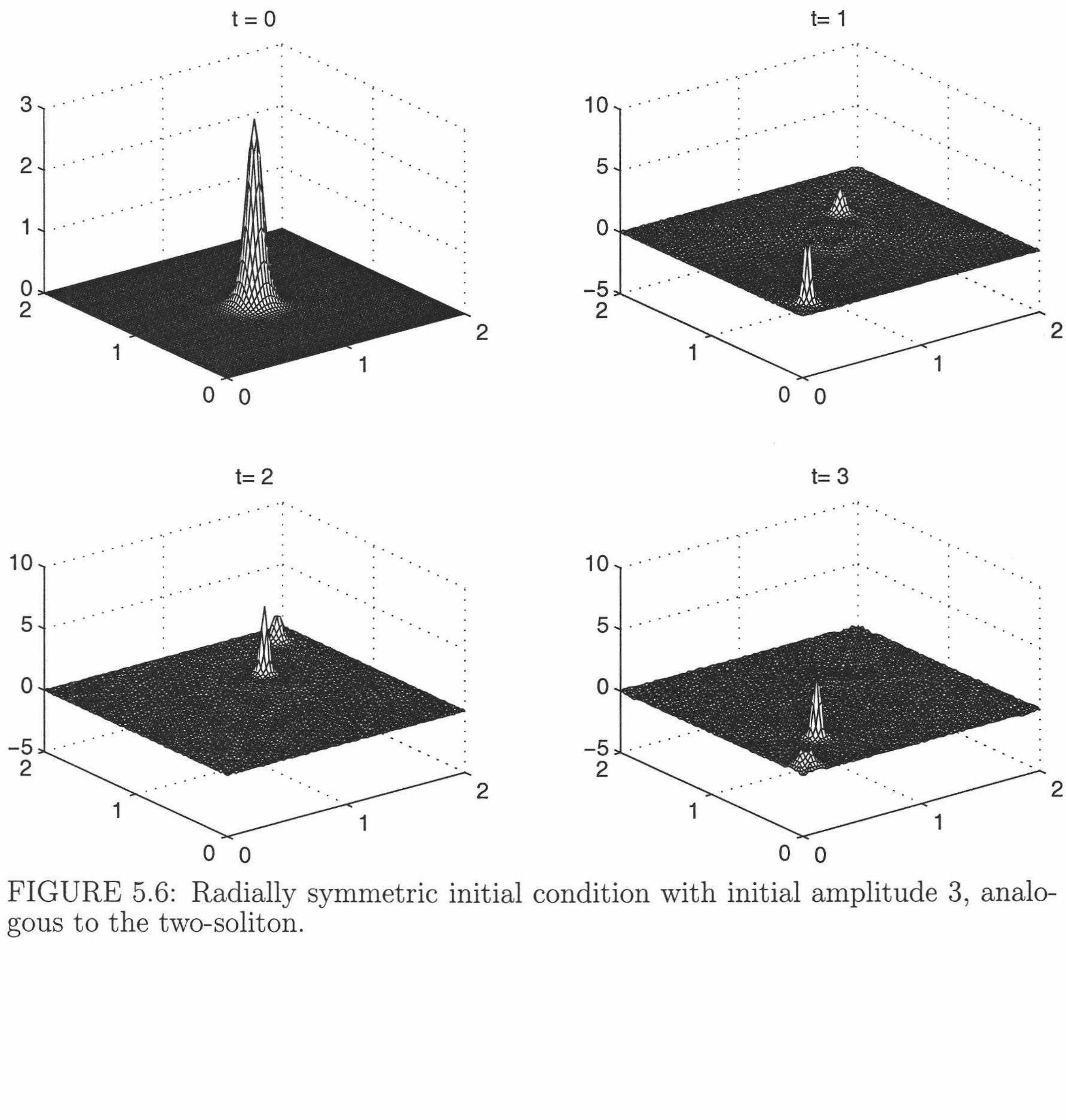

FIGURE 5.6: Radially symmetric initial condition with initial amplitude 3, analogous to the two-soliton.

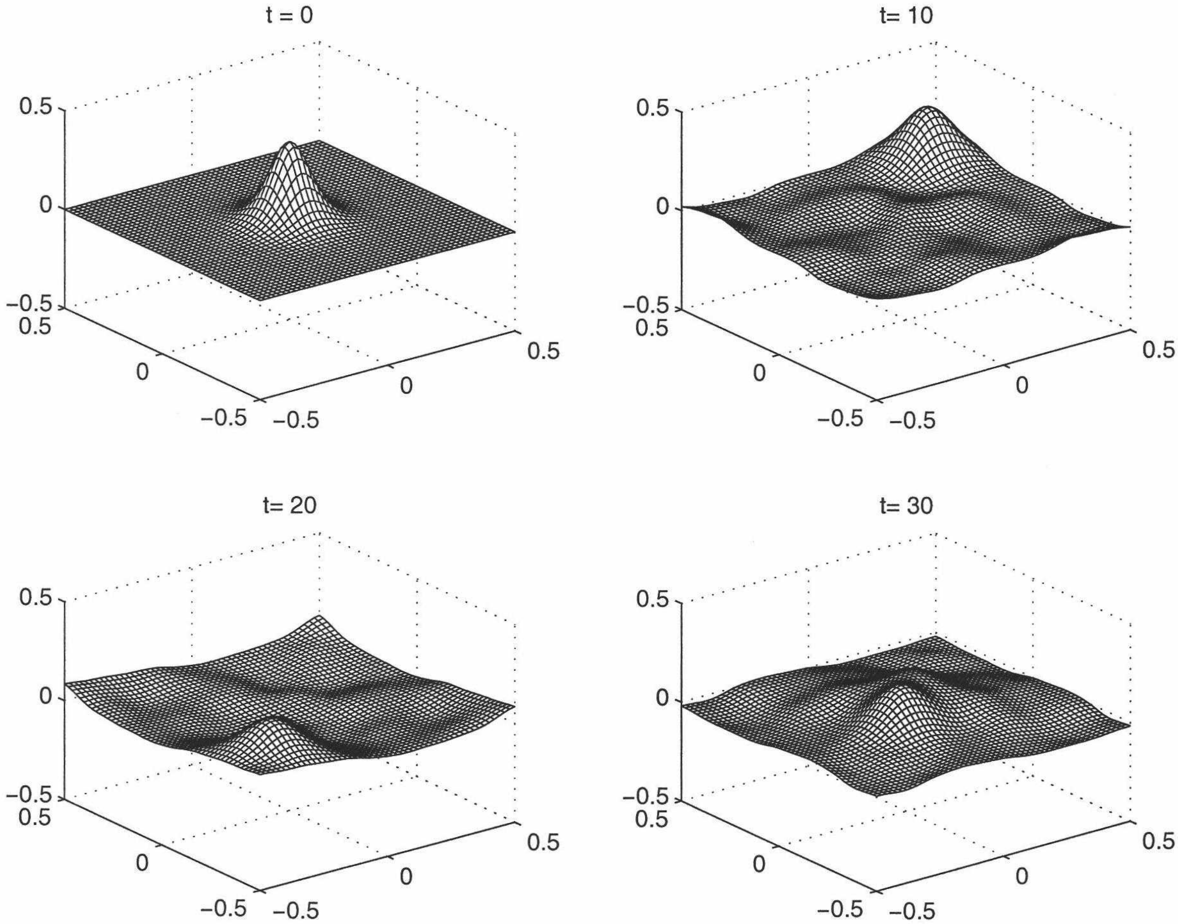

FIGURE 5.7: .4sech^2 solution decays.

6. CONCLUSIONS

A non-linear, third order differential equation describing a surface in two spatial dimensions has been constructed, beginning with the equations governing water waves. It is possible to derive this equation either with or without an accounting for surface tension. The equation resembles the sum of two KdV equations–one in x and one in y. It also resembles the KdV in that it represents long waves of small amplitude relative to the depth of the fluid. The result is distinguished from the KdV in that it carries cross-derivative terms ζ_{xxy} and ζ_{yyx} and can describe a wave surface rather than just a profile.

The non-linear terms of the equation essentially represent the inviscid Burgers equation. This would indicate a tendency to develop shocks. As with the one-dimensional KdV, the third order dispersive terms appear to compensate for steep gradients given appropriate initial conditions. Numerical runs on variety of initial conditions were performed to find that there are both line solutions and lump solutions that appear to behave as solitons.

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