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Abstract approved:

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In this study, I seek to examine undergraduate STEM majors' beliefs about and attitude towards mistakes in the context of counting. This is a particularly fruitful setting for such an investigation both because combinatorics is widely applicable to various fields such as physics, biology, chemistry, and computer science (Kapur, 1970), and because it is acknowledged by many in the field of math education research that students struggle with learning to count (Hadar & Hadass, 1981; Lockwood, 2014; Batanero, Navarro-Pelayo, & Godino, 1997). Specifically, given that students tend to display negative affect towards mistakes (Turner, Thorpe, & Meyer, 1998), despite the beneficial nature of mistakes (Borasi, 1987, p. 2), and that affective factors like attitudes and beliefs have a significant impact on students' problem-solving activity (Carlson and Bloom, 2005), enumerative combinatorics is an ideal setting to study individuals' mindsets (as in Dweck, 2006 and Boaler & Dweck, 2016). I helped to interview five students, asking them to engage in combinatorial problem solving, and reflect on their prior experiences with counting. I found that students' self-reported mindsets and beliefs towards mistakes affected their counting activity. Furthermore, I also found evidence to support that the concept of mindset is a spectrum, rather than a dichotomy. These results serve to inform the existing literature, provide implications for the teaching and learning of enumerative combinatorics, and offer opportunities for future research.

Student Affect Towards Mistakes in the Context of Counting

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Major Professor, representing Mathematics

Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Samantha McGee, Author

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Chapter 1: Introduction

1.1 Introductory Remarks

As today's society expands to become more industrialized and technological, the study of mathematics, as well as its applications, becomes increasingly vital. However, previous studies indicate that affective variables, such as beliefs, attitudes, and emotions, have a significant influence on the problem solving process and behavior of problem solvers (Carlson & Bloom, 2005). This is significant because previous empirical studies have found that the affective domain could stand to negatively impact an individual's mathematical endeavors. For instance, researchers found that certain beliefs about mathematics, like the belief that mathematics is all about memorization, undermine pupils' problem-solving performance (Schoenfeld 1989). Furthermore, Blum-Anderson (1992) also found that negative affect has an impact on enrollment in higher-level mathematics classes. Thus, there seems to be evidence that students struggle to be successful in the imperative subject of mathematics, and unfortunately students can be deterred from engaging with and studying mathematics. As a result, many studies have been dedicated to ascertaining a better understanding the role of affect and beliefs in various mathematical contexts (e.g. Goldin, 2000; Ioannou & Nardi 2009; Liljedahl, 2005; Martínez-Sierra & García-González, 2015; Weber, 2008)

In particular, a great deal of research has been done around student errors in mathematics as well as affect towards mistakes in various mathematical domains. For instance, Borasi (1987) noted that mistakes are not only a valuable avenue toward discerning pupils' learning difficulties (and thus, to directing remediation), but they can also serve as a gateway to mathematical exploration. In other words, Borasi (1987) argues that students' mistakes can lead to explorations that lend a better understanding of the mathematics. Yet, despite the potential benefits mistakes can yield, much of the mathematics education research surrounding students' attitudes towards and beliefs about errors in mathematics indicate that students often experience negative affect as a result of making a mistake (e.g. Elliot & Dweck, 1988; Kloosterman, 1988; Turner, Thorpe, & Meyer, 1998). While there are students who experience positive affect after making the mistakes, these authors argue that individuals who are focused on performing well, rather than understanding the content, often experience negative affect towards mistakes in mathematics.

A particularly fruitful context in which to study affect towards mistakes is enumerative combinatorics, or solving counting problems. This is true for several reasons, which include, but are not limited to the applicability of combinatorics as a field, the accessibility of counting problems, and the notorious difficulties students face while engaged in combinatorial problem solving (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Hadar & Hadass, 1981; Kapur, 1970). In fact, it has been noted that combinatorics is applicable in other mathematical areas like number theory and probability theory, as well as in many other scientific areas such as physics, biology, chemistry, and computer science (Kapur, 1970; Lockwood & Gibson, 2015). Furthermore, because counting requires only a "modest" mathematical background (Allenby & Slomson, 2011, p. 14) and permits multiple solution approaches (English, 1991), it acts as an accessible setting for mathematical problem solving (Lockwood, 2015).

However, despite the applicable and accessible nature of counting, much of the mathematics education research focused on enumerative combinatorics emphasize the immense struggle students encounter as they attempt to solve counting problems (Batanero et al., 1997; Hadar & Hadass, 1981; Lockwood, 2014;). In fact, it has even been called one of the most difficult mathematical topics to teach and learn (Eizenberg & Zaslavsky, 2004). In particular, students experience troubles with issues of permutations, combinations, and repetition (Mellinger, 2004), verifying their solutions (Eizenberg & Zaslavsky, 2004), and constructing a systematic method of counting (Hadar & Hadass, 1981), among other things. While some of these troubles have been attributed to the complex nature of the field, as exemplified by a lack of algorithms (Martin, 2001) and need for clever insights and logical reasoning (Tucker, 2002), there is still a need to continue investigating this field. Specifically, there is a need to study students' struggles, particularly their affective struggles in the context of counting.

1.2 Research Questions

Given the importance of better understanding students' attitudes and beliefs in mathematics in general, and in combinatorics in particular, I pose the following research questions:

1. How does self-reported information about mindset and mistakes manifest itself during undergraduate STEM majors' combinatorial activity?
2. What evidence can be found to support that the concept of mindset applies to a spectrum, and not a dichotomy?
3. What role do mindsets and mistakes play in undergraduates' learning of enumerative combinatorics?

In the next chapter, I will demonstrate how these questions address significant gaps in previous literature surrounding mistakes, student affect, the teaching and learning of combinatorics, and the intersections of these fields.

After examining earlier studies in Chapter 2, Chapter 3 will describe the constructs that I implemented as a theoretical perspective while designing this study and analyzing the data. In addition, I will also use Chapter 3 to briefly expound upon how I related mindset and affect. In particular, I will describe how I am interpreting mindset to be one aspect of affect. In Chapter 4, I will describe the methods used to gather and analyze data. In particular, I will describe how students were recruited, the participants, the data collection process, and how the data was analyzed. Furthermore, Chapter 5 provides a representative report of each student’s interview as well as some brief discussion, and Chapter 6 seeks to relate the data to the research questions and provide some interesting points of discussion. Lastly, in Chapter 7, I provide implications for educators as well as potential avenues for future research.

Chapter 2: Literature Review

In this chapter, I discuss previous empirical studies that pertain to combinatorics and mathematical affect to situate my study in existing literature. More specifically, these studies focus on the influence of student affect in mathematics, students' attitudes and beliefs towards mathematical mistakes, as well as student affect and mistakes in the context of counting. In Section 2.1, I begin by considering literature related to student affect about making mistakes in mathematics. I first examine studies centered around the affective domain and mistakes, separately, and in the theoretical perspective I discuss empirical research that unites these topics. In Section 2.2, I synthesize previous empirical studies that examine student thinking about counting problems. Specifically, I discuss the value of counting, students' struggles with solving counting problems, and remedies that have been proposed to ease these struggles. Finally, in Section 2.3, I discuss previous findings concerning student mistakes in the context of counting. I also provide several examples of student affect that appear in existing literature, in an attempt not only to demonstrate the existence of affect in the domain of counting, but also to argue for the importance of studying the topic of affect. More precisely, I will make the case for studying student affect about mistakes in the context of counting specifically.

2.1: Student Affect Pertaining to Mistakes in Mathematics

2.1.1 The affective domain in mathematics.

Interest in studying affective variables, such as student beliefs, emotions, and attitudes, has been increasing in the math education community (Zan et al., 2006) because of such variables' significant influence on the problem solving process and behavior of problem solvers (Carlson & Bloom, 2005). Early research on the affective domain was highly focused on attitudes towards mathematics and mathematical anxiety, and it later progressed to consider other dimensions of affect like enjoyment of mathematics and value of mathematics (Mcleod, 1994). One of the most well-known affect scales that contributed to research on mathematical affect was the Fennema-Sherman Mathematics Attitude Scales. This scale included nine Likert-type scales that measure attitudes related to the learning of mathematics. The Attitude toward Success in Mathematics Sale, The Mathematics as a Male Domain Scale, and The Confidence in Learning Mathematics Scale, among others, were a few of these scales (Fennema and Sherman, 1976). I did not use these scales to measure affect in my study, but I mention them here to trace the development of measuring affect in mathematics education.

Apart from expanding the domain itself, early investigations surrounding affect also made other contributions to the field of math education. For instance, researchers found that certain beliefs about mathematics, like the belief that mathematics was all about memorization, undermine pupils' problem solving performance (Schoenfeld 1989). Furthermore, they also found that negative affect has an impact on enrollment in higher-level mathematics classes (Blum-Anderson, 1992). In fact, Blum-Anderson (1992) argued,

> teachers who emphasize affective issues during the teaching and learning of mathematical concepts and procedures increase the likelihood that more students from all levels of the achievement range will choose to enroll in higher-level mathematics classes when mathematics becomes an elective (p. 433).

Thus, attending to the affective domain in mathematics can help pupils in their current mathematical endeavors and, furthermore, lead them to pursue higher level mathematics classes.

Other studies have sought to establish theoretical constructs for studying affect. Among the theoretical constructs is Goldin's (2000) affective pathways model in which he describes two affective pathways. While both pathways begin with curiosity, puzzlement, and bewilderment, it is what follows bewilderment, or the feeling of being "lost" in a math problem, that distinguishes the pathways. In one pathway, encouragement, pleasure, and satisfaction follow bewilderment, and in the opposing pathway, frustration, anxiety, and despair follow (Goldin, 2000). Liljedahl (2005) would argue that changing from the latter pathway to the former requires "long periods of sustained and successive success" (p.231). This is evidenced by the results of his study, in which Liljedahl (2005) witnessed the positive effect of an "aha!" moment on students. Liljedahl noted that these moments of sudden, complete comprehension produced a sense of confidence and elation in otherwise mathematically anxious students (Liljedahl, 2005). The study of affect in mathematics, however, has also extended beyond the context of general problem solving.

Researchers like Weber (2008), Ioannou and Nardi (2009), and others (e.g. Martínez-Sierra & García-González, 2015; Selden, McKee, & Selden, 2009) sought to study affect in particular mathematical contexts. More specifically, Weber (2008) and Ioannou and Nardi (2009) examined student affect in an undergraduate real analysis and undergraduate abstract algebra courses, respectively. Despite the different mathematical settings, the studies appear to have yielded similar results. In particular, the authors in both studies noted marked decrease in engagement and increased feelings of frustration as the courses progressed. Referring to Real Analysis, one student reported, "…I don't get it and it frustrates me to do it when I don't get it so I would rather not do it…" (Weber, 2008, p. 77). Weber's findings also indicated that frustration, as well as despair and anxiety, leads to rote problem solving techniques, such as memorization. Furthermore, both studies indicated that the abstract nature and less intuitive structure of the respective mathematical domains contributed to negative affect in students.

In a more recent study, Martínez-Sierra and García-González (2015) also examined student affect in a particular area of mathematics. Their work explored students' emotional experiences in an undergraduate linear algebra course via focus group interviews. Analysis of the data was based on the theory of cognitive structure of emotions, which "specifies eliciting conditions for each emotion and the variables that affect intensity of emotions" (p. 87). Findings indicated that students experienced emotions such as satisfaction, fear, and self-criticism. Furthermore, these emotions are triggered by their assessment of situations such as going to the black board, failure in a course, and difficulty attributed to Linear Algebra courses, among other things. This analysis appears to align with previous findings pertaining to student affect in specific domains. In fact, in all three of the studies, students experienced frustration, and in both Weber (2008) and Martínez-Sierra and García-González (2015), there were explicit references to emotions related to fear.

As evidenced in the preceding discussions, there is much reason to study student affect in mathematics. McLeod (1992) asserted that to maximize the impact of research on instruction and learning on students and teachers, "affective issues need to occupy a more central position in the minds of researchers" (p. 575). Moreover, the existence of studies that examine affect in specific mathematical domains, particularly at the undergraduate level, and their findings, indicates that there is value in striving to understand students' affect in other, specific domains as well. In particular, there is value in studying affect in the context of undergraduate combinatorics. In light of the powerful nature of mistakes, as discussed in the subsequent section, I will make a case that there is value in studying affect related to mistakes in combinatorics.

2.1.2 Research surrounding student mistakes in mathematics.

It has been acknowledged by the math education community that student mistakes can be a valuable method of discerning pupils' learning difficulties, which can help inform remediation (Borasi, 1987, p. 2). Specifically, Borasi noted an increased awareness of the value of probing the causes of errors as they can inform educators of the difficulties a student encountered in learning a specific concept. Furthermore, because effective remediation involves educators hypothesizing on and verifying the cause of a given mistake, such errors can also inform future instruction

(Borasi, 1987). Borasi's (1987) offers an example of a contribution of mathematical mistakes, and he shows evidence of student errors serving as a gateway to mathematical exploration. In this analysis, Borasi's findings suggest a number of ways in which educators can utilize mistakes. First, she indicated that mistakes can be thought of as a motivational tool and starting point for "creative mathematical explorations involving valuable problem solving and problem posing activities" (p. 7). Furthermore, she also indicated that mistakes can foster a deeper understanding of mathematics. These findings are further supported by the results of a teaching experiment conducted in the interest of treating errors, both impromptu student errors and those carefully selected by the researcher, as learning opportunities (Borasi, 1994). In a carefully controlled setting, students were able to utilize mistakes as a "springboard for inquiry" (p. 166) and develop a more profound understanding of mathematics.

The aforementioned studies in this section exemplify the potential utility of mistakes in the teaching and learning of mathematics. In fact, mistakes possess such promising potential that learners' brains develop more connections each time they makes a mistake (Boaler and Dweck, 2016). Furthermore, according to a recent psychological study, this growth occurs whether or not they are conscious that a mistake has been made (Moser et al., 2011). However, in spite of these studies and their findings, other research suggests that students do not always perceive mistakes as beneficial (e.g. Boaler & Dweck, 2016; Turner et al., 1998), as discussed in the theoretical perspective.

Now that I have made a case for the importance of studying affect and, in particular, affect towards mistakes in mathematics, I will focus on the value of studying student affect towards mistakes in enumerative combinatorics. I begin by examining the utility of counting as a mathematical domain.

2.2: Enumerative Combinatorics

2.2.1 Counting in mathematics and other scientific domains.

Combinatorics has sundry significant applications in other mathematical domains such as probability and number theory, as well as many scientific fields (Kapur, 1970; Lockwood & Gibson, 2015). Furthermore, learning enumerative combinatorics enhances student understanding of particular mathematical concepts and aids in the development of "deep and profound reasoning" as well as "rich mathematical thinking" (Lockwood, 2015, p. 340), which not only enriches students as mathematicians, but also proves useful in circumstances outside of their mathematics classes.

Counting provides an accessible context for students to engage in constructive problem solving (Lockwood, 2015) because it necessitates only a "modest" amount of mathematical background (Allenby & Slomson, 2011, p. 14) and permits multiple approaches to finding solutions (English, 1991). Such problem solving is constructive in the sense that it has the potential to foster an understanding of mathematical concepts such as rigorous proof writing, isomorphisms, generalization, making conjectures, and systematic thinking (Kapur, 1970). All of these consequences of combinatorial problem solving are skills that will not only help students in their mathematics classes, but also in other extracurricular experiences. Yet, in spite of the value of counting problems, an examination of previous literature reveals that students struggle to solve counting problems.

2.2.2 Student difficulties with counting problems.

Despite the aforementioned applicability and accessibility of combinatorics, as well as its ability to aid in the development of students as mathematicians, it is widely acknowledged that students often struggle with counting (Batanero et al., 1997; Hadar & Hadass, 1981; Lockwood, 2014;). In fact, Eizenberg and Zaslavsky (2004) described it as "one of the most difficult mathematical topics to teach and to learn" (p. 16). In an attempt to study students' verification strategies, they supported this claim by noting that out of 108 problems, only 43 initial solutions were correct (p. 31). In their study of undergraduates, Godino et al. (2005) also found that students struggled with solving counting problems reporting that, on average, pupils could only solve six out 13 "simple" combinatorial problems. Aside from simply acknowledging that difficulties occur, however, many researchers have also conjectured the reasons behind this struggle (which I discuss in the remainder of the section).

Much of the existing combinatorial literature has attributed a portion of this difficulty to the complex nature of counting problems. For instance, Annin and Lai (2010) observed, "most problems do not fall cleanly into one and only one standard category of counting problems" (p. 404). Thus, there are few algorithms and each problems seems different from previous ones (Martin, 2001). Furthermore, successfully solving counting problems requires clever insights and logical reasoning just as much as it requires an "inventory of special techniques" (Tucker, 2002, p. 169). In other words, previous findings suggest that many novice counters struggle because unlike the mathematics they may be accustomed to, combinatorics lacks standard algorithms and a rigid structure. Moreover, the type of insights and reasoning deemed as necessary to solving counting problems tends to come primarily through experience (Martin, 2001).

While this complex nature of enumerative combinatorics certainly contributes to student difficulties, there are other causes as well. Several researchers have put forth various obstacles that students must overcome to successfully solve counting problems (e.g. Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981; Mellinger, 2004). For example, Hadar and Hadass (1981) recognized that students often labor to construct a systematic method of counting, which can lead to doubt as to if each outcome has been counted exactly once. They also indicated that students struggle to understand the "compact nature of the mathematical language" (p. 436), break a problem into sub-problems, and generalize, among others. Lockwood (2014) added that students' difficulties can be exacerbated by a lack of attention to sets of outcomes during the combinatorial problem solving process. In another study, Mellinger (2004) noted that students also experience difficulties understanding and coordinating the ideas of permutation, combination, and repetition. In addition, these difficulties are accentuated by the fact that students also struggle to develop efficient verification strategies (Eizenberg & Zaslavsky, 2004). Considering typical student errors, many empirical studies have examined potentially beneficial strategies and productive ways of thinking for solving counting problems, as discussed in the next section.

2.2.3 Proposed solutions to student difficulties.

There have been several studies dedicated to alleviating students' difficulties with solving counting problems. In a more critical analysis of these studies, two prominent themes can be identified. Some studies, as it will be discussed in Section 2.3.1, consider the nature of student difficulties and possible contributing factors. Other studies pose possible remedies for student difficulties, which I discuss in Section 2.2.3. My examination of potential remedies revealed two general types of remediation. In Section 2.2.3.1, I discuss studies like Lockwood and Gibson (2015), English (1993), as well as others that presented particular strategies that could prove useful for students as they solve counting problems. In Section 2.2.3.2, I examine analyses such as Maher et al. (2010), Lockwood (2014), and Halani (2012), which examined productive ways of thinking as one way of easing students' struggles.

2.2.3.1 Useful strategies for solving counting problems.

Researchers have explored various valuable strategies to aid students in their endeavors to successfully solve counting problems. For instance, some authors have recommended utilizing listing as a tool for solving counting problems. In particular, studies like English (1991, 1993) and Lockwood and Gibson (2015) examined the effects of listing on student success in combinatorial problem solving. Lockwood and Gibson's (2015) findings suggested that creating a list of possible outcomes, or even partial lists of the sets of outcomes "led to significant improvements in performance in students' success on problems" (p. 247). Furthermore, they argued that constructing such a "systematic, organized list" (p. 251) can enable students to justify why they have counted all of the outcomes. English (1991, 1993) had similar results in her examination of children's strategies for solving two and three dimensional counting problems. English (1991, 1993) found that the children's approaches ranged from an odometer strategy to a guess and check methodology. As it pertains to the former, there were instances in which both a two dimensional odometer strategy, involving the fixation of one variable, as well as a three dimensional *odometer strategy*, involving the fixation of two variables, were used. By fixing one or two variables, the participants were then able to cycle through items that were to be placed in other positions. In effect, they were able to create a systematic, organized list and according to English, "it was the more sophisticated strategies that facilitate goal attainment" (English, 1993, p. 265).

Aside from listing, researchers have also proposed other strategies as a means of easing students' difficulties. For example, the common problem solving heuristic of solving smaller, similar problems has been identified as a potentially useful strategy for students. According to Lockwood (2015), this particular tool proved beneficial to students' problem solving process due to its tendency to facilitate systematic listing. In turn, this listing aided in the detection of patterns as well as identification of over-counts. The benefits of using smaller, similar cases is not only evidenced by Lockwood's (2015) study, but also by Maher et al.'s (2010) longitudinal study, in which the evolution students' combinatorial reasoning was traced over a 12 year period. In one of many studies by Maher and her colleagues, third and fourth grade students were asked to determine the number of towers that could be built using four blocks of two different colors (Maher et al., 2010, pp. 35-37). In particular, one fourth grader was able to recognize the recursive nature of the problem due to his consideration of simpler cases. However, Lockwood (2015) advocated with caution because of some observed potential downfalls in utilizing this strategy. For instance, the author warned that if not done carefully, reducing a parameter of the original problem to create a smaller problem may cause mathematical properties between the two problems to differ. Furthermore, even after the problem has been reduced carefully, the smaller problem must be worked with "precision and attention to detail" (p. 359). Again, this heuristic is another strategy that can aid students in their efforts to solve counting problems and furthermore, it supports the aforementioned potential benefits of listing.

Another productive approach observed by multiple authors is relying on patterning and sets of outcomes. In particular, Lockwood et al., (2015) guided two students' reinvention of counting formulas and witnessed how two students' reliance on patterning and sets of outcomes enabled them to successfully reconstruct counting formulas. Furthermore, Maher et al. (2010) also suggested pattern recognition as one of many productive strategies they observed students using. However, like solving smaller, similar problems, this strategy of patterning and writing sets of outcomes is accompanied by a warning. Lockwood et al. (2015) advised that an overreliance on patterning may lead to the preclusion of the development of the multiplication principle, a fundamental aspect of counting which enables students to understand and justify the basic formulas.

Apart from using smaller cases and pattern recognition, Maher and her colleagues (2010) also observed several other advantageous strategies that students utilized while engaged in combinatorial problem solving. For instance, among the most prominent strategies, students were often cited justifying their answers with proof by cases. In earlier years, this took the form of what English (1991) might describe as the less sophisticated odometer strategy, "guess and check," to find as many outcomes as they could. This can be witnessed in two third grade students' attempts to determine how many towers could be built from four blocks of two colors (Maher et al., 2010, pp. 28-29). In as early as grade 5, researchers noted that the implementation of proof by cases became more systematic and thus, effective. One significant way in which the proof by cases methodology evolved was in pupils' incorporation of recursion (Maher et al., 2010, pp. 69-72). While all of these strategies have proven to be potentially effective in alleviating the severity of students' struggles, there are some that have been further developed to become thought of as a productive way of thinking. They are discussed in the next section.

2.2.3.2 Productive ways of thinking.

As an alternative to proposing specific strategies, other researchers have suggested adopting ways of thinking that are potentially productive to solving counting problems. For instance, Halani (2012) expounded upon English's (1991, 1993) categorization of children's approaches to combinatorial problems to make the case that the odometer strategy can be thought of as a way of thinking (in the sense of Harel, 2008). However, as Halani indicated,

> It can be difficult to distinguish between whether students are using the odometer strategy, as described by English (1991), or engaging in the Odometer way of thinking. The most important distinction is that students are able to anticipate the result of their mental acts when engaging in a form of Odometer thinking. It is only through probing

> the students' utterances and actions that the researcher is able to determine if the students have simply stumbled upon a plan of action that is currently fruitful, or if the students are truly engaging in a way of thinking (p. 234).

This distinction is significant because those who have adopted an Odometer way of thinking will also be able to recognize why it will generate all possible outcomes. Halani went on to describe the "forms" of Odometer thinking as follows (See Table 1):

Form	Description
Standard Odometer	"One would first hold an item constant in a given position and then systematically (and possibly recursively) vary the other items. Following this, the item in the given position is changed and the process repeats until all possible items for the given position are exhausted" (p.236).
Wacky Odometer	"Sometimes students would hold a single item constant and vary the other items. However, following that, the students would change the position of this item and repeat until all possible elements of the solution set had been generated" (p. 238).
Generalized Odometer	"It is an extension of the Wacky Odometer way of thinking in the sense that although things are being held constant, they are not in the same position. However, in contrast to the Wacky Odometer way of thinking, an array of items is being held constant instead of just one item. In this way, it is a more sophisticated way of thinking than either Standard or Wacky Odometer thinking" (p. 241-242).

Table 1: Halani's (2012) forms of Odometer thinking

Lockwood (2014) also proposed a way of thinking related to one of the aforementioned strategies and emphasized a set-oriented perspective, which she described as "a way of thinking about counting that involved viewing an explicit focus on sets of outcomes as a fundamental aspect of solving counting problems" (p. 32). She argued that this perspective may be an "effective factor in helping students correctly solve counting problems" as it can help students avoid many of the common pitfalls of counting (p. 31). In fact, this perspective is grounded in earlier research in which Lockwood (2013) (later revised in Lockwood et al., 2015) proposed a model of students' combinatorial thinking, illustrated below.

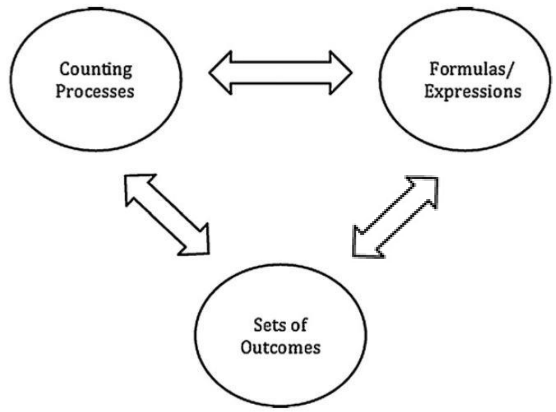

Figure 1: Lockwood et al.'s (2015) model of students' combinatorial thinking

In her model of students' thinking, Lockwood (2013) focused on three components of students' thinking as they contemplate counting problems. That is, her model established reflexive relationships between counting processes and sets of outcomes, counting processes and formulas/expressions, and between formulas/expressions and sets of outcomes (See Figure 1). Lockwood defined formulas/expressions as mathematical expressions that result in some numerical answer (p. 252), counting processes as the enumeration processes that a counter engages in as they solve counting problems (p. 253), and set of outcomes as the collection of objects that are being counted (p. 253). In particular, she emphasized the connection between counting processes and sets of outcomes because "sets of outcomes can provide a way for students to ground their combinatorial activity and can ultimately help to determine whether a counting process is correct" (p. 258). As it pertains to the strategies mentioned in Section 2.2.3.1, Lockwood's (2013) model of students' combinatorial thinking, as well as its associated set-oriented thinking (Lockwood, 2014), informed and helped motivate Lockwood and Gibson's (2015) examination of undergraduates' listing strategies.

As it was previously mentioned, Maher et al (2010) sought to trace the evolution of students' combinatorial thinking in their 12-year study. They aimed to determine what students thought of as convincing, what made sense to students, and how students developed their answers. As it pertains to the latter motivating factors, the authors were also able to observe another productive way of thinking for students as they reasoned about counting problems. More specifically, Maher et al. (2010) observed how generalizing, recognizing isomorphic problem structures, and employing properties of transitivity became a particularly effective way of thinking. An example of this can be witnessed in Chapter 12 of their book. In this study, the authors took note of four students who were able to solve a problem by making the generalization that Pascal's triangle underlies the mathematical structure of a particular problem and that particular problem was isomorphic to a different problem. So, by transitivity, the students were able to reason that Pascal's triangle underlies that second problem as well (p. 147). Ultimately this enabled the four students to adequately justify their tentative conclusions.

In conclusion, combinatorics is an important and useful domain of mathematics as it has many applications in various scientific fields. Furthermore, its accessibility can foster meaningful and engaging problem solving experiences for students and enhance their understanding of other mathematical concepts. Yet, whether it is because of the complex nature of counting or the numerous barriers students must transcend to find success, existing literature indicates that it is also a domain in which many students struggle. Fortunately, the math education community has focused on identifying these difficulties as well as ways in which to alleviate the severity of the struggle. The remedies not only include short term solutions like strategies that can be utilized problem by problem, but the remedies also include the adoption of particular ways of thinking, which can be a more enduring, transformative solution for novice counters. They are transformative in the sense that they require students to develop a significantly deep understanding and alter the way they look at counting problems. As it was mentioned in Section 2.2.3, another approach to easing student difficulties is to study the nature of student struggles and possible contributing factors. This, as well as the possible influence of student affect on students' combinatorial problem solving process, is discussed in the following section.

2.3: Student Affect and Mistakes in the Context of Counting

2.3.1 Student errors in counting.

Aside from simply identifying common errors that students make while solving counting problems, as described above, attempts have also been made to better understand the nature of these mistakes as well as contributing factors. For instance, while studying the effect of the implicit combinatorial model on student reasoning before and after instruction, Batanero et al. (1997) also categorized typical student errors in counting. Among the most common mistakes were misinterpreting the problem statement, errors relating to order, errors related to repetition, confusing the type of object being counted, using or remembering formulas incorrectly,. Furthermore, the study also revealed that the types of errors students commonly make depended on several factors. Examples of such factors include the complexity of the problem statement, the combinatorial model (selection, distribution, or partition), and additional conditions on a permutation problem, such as repetition, among others. The authors argue that understanding these errors as well as the variables that influence them are "two fundamental steps for making the learning of this subject easier" (p. 182).

Eizenberg and Zaslavsky (2004), on the other hand, did not explicitly study the nature of student errors, but rather, difficulties with verification as a contributing factor. Furthermore, they also advocated for the use of verification as a means of attaining more success in solving counting problems. In this particular study, the researchers examined students' tendencies to verify their solution, the verification strategies that students utilized, and the efficiency of these strategies.

There were attempts made by students, out of their own initiative, for 66 out of 108 combinatorial problems. Findings also indicated that the most common verification strategies were reworking the solution, evaluation of the reasonability of the answer, adding justification to the solution, modifying some component of the answer, and using a different solution method and comparing results. Moreover, Eizenberg and Zaslavsky also highlighted the need to explicitly teach verification strategies because although 34 of those aforementioned solutions remained incorrect after an attempted verification, indicating that students struggle to efficiently check their solutions, the study also demonstrated that there are students who were able to find efficient ways of checking their answers. This is significant because as they explained,

> Verification is considered part of a “looking back” strategy and plays a critical role in problem solving (Poly, 1957; Schoenfeld, 1984, 1985; Silver, 1987; Wilson, Fernandez, & Hadaway, 1993). First and most obvious, by checking solutions one can catch careless errors, at a local level, and find support for a solution as well as alternative solutions at a more global level (Schoenfeld, 1985, p.17)

While this particular statement pertains to general problem solving, it can certainly be extended to combinatorial problem solving.

The aforementioned articles raise an important point pertaining to the analysis of students' mistakes in the context of counting. It is essential that mistakes not only be identified, as many have already done (e.g. Annin & Lai, 2010, Batanero, et al., 1997, and Hadar & Hadass, 1981), but there is also value in understanding the nature of these mistakes, the variables that influence these mistakes, and ways in which educators can help students navigate through, or perhaps, even around them. Both Batenero et al. (1997) and Eizenberg and Zaslavsky (2004) indicated that in doing so, we are one step closer to addressing the difficulties students' face in solving counting problems.

2.3.2 Affect in counting.

As noted previously, affective variables such as beliefs, attitudes, and emotions, have a significant influence on one's problem solving process in mathematics (Carlson & Bloom, 2005; Liljedahl, 2005; Weber, 2008). Given that combinatorics is simply a subclass of mathematics, it follows that combinatorial problem solving is also subject susceptible to the influence of the affective domain. Yet, my search through combinatorial literature did not yield any existing studies that explicitly examined affect in the context of counting. In fact, Goldin (2016) noted that much of affective research focused on traditional school mathematics, which does not include discrete mathematics, and specifically identified the "need for research on the affective and conative dimensions of discrete mathematics" (p. 7). My exploration of the literature, on the other hand, did produce several previous empirical studies in which one can witness several instances of affect in students as they work with counting problems, even if that was not the main focus of the paper. This further emphasizes the potential importance of studying of students' beliefs and emotions in the context of counting. While some of these aforementioned instances of affect are indicative of the students' beliefs, others reveal emotions they experience while solving counting problems.

An example of a statement that divulges a student's belief, is, "…I usually do mess up on whether to use a combination or permutation, because I am lazy…" (Lockwood, 2014, p. 34). This student holds the belief that she commonly makes errors when determining if order matters, which granted, may be true, but it is interesting for a couple of reasons. First, it indicates a lack of confidence in her counting abilities, and second, she attributes the mistake to the fact that she is "lazy." Interestingly, this student was not the only student to accredit their struggles to laziness. Lockwood (2015) quotes a student, Anderson, as saying, "…since my brain's not all that math oriented, I guess I'll just write it out…well, my brain's too lazy to come up with a specific example…I guess it's that step my brain kept skipping due to laziness that made me overlook that one problem" (p. 355). Again, this student not only identifies "laziness" as the cause of his troubles, but also generalizes to convey a belief about mathematics. This statement clearly conveys the student's belief that his brain is not "math-oriented," but it also could indicate that he believes mathematics is a gift that some people have and others do not. Later, Lockwood (2015), went on to quote a different student as saying, "…oh man, I am bad at counting…" (p. 355). Like the first example, this is another excerpt that seems to communicate a lack of confidence in their counting abilities. Similar to Anderson's statement, it also suggests the belief that mathematics, or in this case, counting, is something a person is either good at or bad at.

On the other hand, there is also an example of affect in the literature that relates to the emotional experiences of students as they engage in counting. Lockwood et al. (2015) quoted a student, Thomas, as saying, "We're getting really frustrated trying to write out all of the possibilities, because we're just noticing it's just going to keep growing and growing" (p. 39) as he and another student, Robin, were attempting to solve counting problems. As evidenced by his statement, their efforts to solve this problem were temporarily hindered by their frustration with the magnitude of the number of outcomes. While Thomas and Robin were able to continue solving that problem as well as several others, this instance acts as compelling evidence for the existence of student affect in the context of counting. More specifically, it demonstrates the existence of student affect related to emotions, contrasted with affect pertaining to beliefs, which was discussed in the preceding paragraph.

The instances of affect described above not only serve as an existence proof for affect in the combinatorial domain, but the noted commonalities also suggest that students might be experiencing the affective domain in the context of counting in similar ways. In this case, we had students who appeared to lack confidence, believe that mathematical abilities, counting or otherwise, were primarily fixed attributes, and even more particularly, a couple of students that attributed their struggles to laziness. This is important because such affect possesses a potentially significant influence on pupils and if we are to begin easing student difficulties with counting, we must first understand this powerful aspect of combinatorial problem solving.

2.4 Conclusion

In sum, there is an abundance of literature that speaks to the importance of studying counting problems as well as various others that contest to the worth of studying student affect in mathematics, student mistakes in mathematics, and student affect towards mistakes in mathematics. Furthermore, it has been shown that although combinatorics is an important aspect of mathematics, it is also a "complex topic" (Eizenberg & Zaslavsky, 2004, p. 31) that students struggle to learn. As Batanero et al. (1997) pointed out, "two fundamental steps for making the learning of this subject [combinatorics] easier are understanding the nature of pupils' mistakes when solving combinatorial problems and identifying the variables that might influence this difficulty" (p. 182). While the "variables" they were referring to were factors such as the complexity of the problem statement, the combinatorial model (selection, distribution, or partition), and additional conditions on a permutation problem, Turner et al. (1998) indicated that affect could be one of those factors as well. I have provided evidence for the existence of affect in students as they try to solve counting problems. Yet, despite this evidence and value of studying student affect towards mistakes in mathematics, there appears to exist a gap in current combinatorial literature. To my knowledge, there are not any existing studies that examine student affect towards mistakes in the context of counting, a mathematical domain Hadar and Hadass

(1981) described as “strewn with pitfalls” (p. 435). Therefore, it is my intention to attend to this gap in the literature by investigating how students regard mistakes in the context of counting.

Chapter 3: Theoretical Perspective

In the following section, I describe the theoretical perspective I used while gathering and analyzing data for my research. The theoretical perspective is rooted in Jo Boaler's (2016) recent mathematical perspective on Carol Dweck's (2006) psychological concept of mindsets, known as mathematical mindset. Because the concept of mindsets also has roots in previous research, such as Turner, et al. (1998) and Elliot and Dweck (1988), I incorporate these contributions into the theoretical perspective as well.

3.1: Performance-Focused Students

A number of previous empirical studies have found that students generally experience negative affect towards mistakes in mathematics (Elliott & Dweck, 1988; Kloosterman, 1988; Turner, et al., 1998). In particular, Elliott and Dweck (1988) described two very different reactions to errors in mathematics, which appear to be typical among students. On one hand, some students exhibit a "helpless" (p. 5) reaction, they blame their failures on a lack of mathematical ability, they experience negative affect, and they show a dramatic deterioration in their performance. In fact, Turner et al. (1998) indicate that even "threat appraisals (such as the possibility of failing) evoke negative emotions like anger and anxiety and lead to coping actions to bolster self-worth or to restore well-being" (p. 759). These coping strategies include withdrawing effort or trying to reduce difficulty of the given task. Turner et al. (1998) describes this group of students as those who possess "ability-focused or performance goals" (p. 759).

In general, students who are performance-oriented strive to demonstrate competence or achieve at high levels of normative ability. In fact, Turner et al. (1998) argue that these students view learning as a medium for public recognition rather than as a desirable result in and of itself. Pupils with performance goals also perceive effort as a sign of low ability and believe that if effort does not yield success, it is a threat to their perceived ability. Thus, they tend to value success with little effort, and moreover, "report using fewer effortful cognitive and metacognitive self-regulatory behaviors like planning, organizing, asking questions, seeking help when needed, and reviewing mistakes" (Turner et al., 1998, p. 759). Turner et al. (1998) also note that while these students experience positive affect after achieving success with little effort, they often experience negative affect as a result of failure, accompanied by the belief that they lack ability, as discussed above.

These insights by Turner, et al. (1998) offer potential reasons for why students with ability-focused goals experience negative affect due to mistakes. Specifically, because their goal is to demonstrate competence, mistakes are viewed as a threat to their perceived ability. As a result, they also tend to shy away from challenges and work problems they know they are more likely to do correctly. This, in part, could also be explained by the moderate correlation between students' self-confidence and achievement (Kloosterman, 1988) as well as Turner et al.'s (1998) claim that these students "interpret success as a reflection of their scholastic abilities and a comment on their self-worth (Covington, 1992)." In other words, making a mistake detracts from these students' success and, consequently, their self-worth and self-confidence.

3.2: Learning-Focused Students

In contrast to performance-oriented students, there are students who, after making a mistake, exhibit "solution-oriented self-instructions" (p. 5), positive affect, and sustained, if not improved, performance. Turner et al. (1998) described this group of pupils as those who possess "learning-focused or mastery goals" (p. 759) and note that, unlike their ability-focused counterparts, they are often more willing to attempt challenging problems, as they are not afraid to make mistakes. In fact, they perceive errors as constructive, rather than debilitating, and "exploit mistakes as information about the need to reevaluate themselves, the task, or their strategies" (Turner et al., 1998, p. 759). In other words, they prefer to take on more difficult problems because they believe the challenge as well as any mistakes that might occur provide more opportunities to learn.

In general, students who pursue mastery goals define success as developing new skills, gaining an understanding of content, and making individual progress. Furthermore, they believe that increased effort is the means through which one ascertains this understanding and achieves academic success. In fact, when met with a challenge, learning-focused individuals are more likely to increase effort and demonstrate persistence. Thus, they also tend to "use more effortful self-regulatory behavior such as cognitive and metacognitive strategies" (Turner et al., 1998, p.759) and often experience positive affect as a result of such effortful involvement.

As in the case of performance-oriented students, we are able to make inferences as to why these students react to mistakes the way that they do. In particular, we may surmise why pupils with mastery goals experience positive affect after making mistakes. For instance, one reason is that they strive to understand content and acquire new skills, and perceive mistakes as learning opportunities. In addition, their persistence in the face of challenges could be explained by their belief that effortful involvement ultimately leads to understanding and academic success.

3.3: A Note on Terminology

These classifications of learning-focused and ability-focused students support and align with Dweck's description of students' learning mentalities (as discussed in Boaler & Dweck, 2016). There are two types of students: those with a *fixed mindset* and those with a *growth mindset*. Students who believe intelligence can increase with hard work are said to possess a *growth mindset,* and, like Turner et al.'s (1998) learning-focused pupils, they perceive mistakes as "a challenge and motivation to do more" (Boaler & Dweck, 2016, p. 7). Furthermore, Boaler and

Dweck (2016) also add that students with a growth mindset possess a greater awareness of errors and as a result, are more likely to review and correct those mistakes. On the other hand, those with a *fixed mindset* believe one is either smart or they are not and are more likely to give up when working with a difficult problem (Boaler & Dweck, 2016). Moreover, like Turner et al.'s (1998) ability-focused students, learners with a fixed mindset are "less willing to try more challenging work or subjects because they are afraid of slipping up and no longer being seen as smart" (Boaler & Dweck, 2016, p. 7).

As it relates to affect, I am considering mindset to be one component of affect. This is because I interpret mindset to encompass, primarily, beliefs about and attitudes towards ability and mistakes. In turn, how one perceives ability and mistakes can lead to other affective reactions, including anger, frustration, or excitement. In addition, they might also influence an individual's beliefs about and attitude towards a particular subject like counting, or more generally, mathematics. Although I considered emotions as well as beliefs about and attitudes towards a particular subject to be a part of the affective domain, I did not consider them to be an aspect of mindset. So, I would describe mindset as a subdomain of affect.

Turner et al. (1998) summarized this key point aptly in saying, "Thus it appears that the relationship between performance goals [learning-focused and ability-focused goals] and resulting academic behaviors is negative affect about failure" (p. 760). They later went on to confirm, via results from a cluster analysis of student surveys, that affect not only relates performance goals and academic behaviors, but the role it plays is important (p. 768).

It is important to note that because the aforementioned terms refer to similar concepts, I will be using the terms interchangeably. More specifically, I will treat the terms, "fixed mindset," "ability focused," and "performance oriented" as synonymous, and, "growth mindset," "learning focused," and "mastery oriented" as synonymous. Also note that the similarities are not only observed in my studies of the literature, but Turner et al. (1998, p. 759) note them as well, and as a result, used the terms synonymously, too. Furthermore, Turner et al. (1998) acknowledge that most research treats these concepts as a dichotomy; however, I argue that it is a spectrum and that students do not always align exclusively with one mindset or the other. As Turner et al., (1998) indicates, this would necessitate students' simultaneous pursuit of multiple goals. This is not only supported by this study, but also in other previous empirical research as well. In particular, Turner et al. (1998) discuss several studies in which students were determined, via a cluster analysis, to be pursuing multiple goals simultaneously (e.g. Meece & Holt, 1993; Pintrich, 1989) In addition, Dweck (2006) indicates that "many people have elements of both [mindsets]" (p.47), and Boaler and Dweck (2016) indicate that approximately 40% of students possess a fixed mindset, 40% hold a growth mindset, and the other 20% waver between the two mindsets, and they also argue that one's mindset can change.

Chapter 4: Methods

In this chapter, I discuss the methodology used to gather and analyze data for this study. In particular, I will describe the participants that were involved in the study, the data collection procedure (including a discussion of the mathematical tasks), and how the data was analyzed.

4.1 Participants

The participants of this study were five undergraduate students enrolled in a discrete mathematics course at a large university in the Western United States. Table 2, seen below, contains information on each of the participants, including a gender-preserving pseudonym, the course they were enrolled in, if they were repeating the class, and their major. Only one of the students was in another math class at the time of the interview (Linear Algebra II), but all of them had taken other math classes prior to the interviews. The highest-level math class they had all taken was Calculus I, but altogether, they had taken classes that included Calculus II, Calculus III, Sequences and Series, Linear Algebra, Differential Equations, and Matrices and Power Series Methods.

The students were enrolled in either Math 355 or Math 231, two discrete mathematics courses that are taught within the mathematics department. Math 355 is a discrete mathematics course intended for mathematics majors. It is considered a transition to proof course, and its course catalog reads:

> Proof analysis and development in the context of discrete mathematics for math majors transitioning to upper-division course work. Topics include elementary logic and set theory, quantifiers, basic counting principles, elementary combinatorics, equivalence relations, the binomial theorem, and mathematical induction. Additional topics may include recurrence relations, generating functions, and introductory graph theory.

Math 231 is a discrete mathematics course intended for computer science majors. The course catalog describes Math 231 as covering “elementary logic and set theory, functions, direct proof techniques, contradiction and contraposition, mathematical induction and recursion, elementary combinatorics, basic graph theory, [and] minimal spanning trees."

Student (Pseudonym)	Course Enrolled	Repeating	Major
Carl	Math 355	No	Math and Economics
Clarice	Math 231	Yes	Computer Science
Sofia	Math 231	Yes	Computer Science
Damien	Math 231	No	Electrical Engineer
Nathan	Math 231	No	Computer Science

Table 2 Participant information

I desired to interview students who were currently taking a discrete mathematics course so that they would have studied the concepts involved in counting recently. In particular, I wanted them to be able to reflect on their feelings and beliefs about counting, and so I wanted them to have some recent prior experience with counting problems. Moreover, I also wanted to make sure they would have seen counting and had some exposure to the concepts that would be asked of them in the interview. Below, I describe the processes involved in the data collection and data analysis.

4.2 Data Collection

Each of the students were recruited from discrete mathematics courses that were being offered in the Fall and Winter terms at the academic institution. I went into the classes and distributed informational flyers asking for volunteers to participate in one hour-long interview aimed at ascertaining their attitude towards mistakes in the context of counting in exchange for monetary compensation. Five students responded that they were interested in participating, and I included all five of these students in the study.

The interviews were structured so that the first 40-45 minutes of the hour were designed to be task-based, in which we gave students counting problems to work through, while the remaining 15-20 minutes were used to conduct a semi-structured interview aimed at understanding their beliefs about and attitude towards counting problems in general, as well as mistakes in the context of counting. Another researcher with experience studying students' counting conducted the interviews, while I filmed them. During the task-based portion of the interview, students were given questions that were designed to elicit common mistakes, as discussed in Section 4.2.1.

A portion of the interview was chosen to involve counting tasks that elicit mistakes for several reasons. First, I wanted to be able to observe how students would react to solving counting problems that might pose a challenge. In particular, I desired to determine whether the students were more likely to exhibit the helpless reaction typical of a fixed mindset (Boaler & Dweck, 2016; Elliot & Dweck, 1988), or self-regulatory behaviors and determination typical of a growth mindset (Boaler & Dweck, 2016; Turner et al., 1998). Second, I also wanted to provide students with a recent encounter with counting problems during the interview so that they could more readily recall prior experiences with counting, and, potentially, experiences with making mistakes while solving counting problems. This would enable them to answer the questions asked of them in the second portion of the interview confidently and comprehensively. Finally, I wanted to see how students demonstrated their mindsets while they actually solved combinatorics problems (and not only when they reflected and self-reported on their mindsets). Thus, I wanted to be able to observe students solving counting problems.

Like the task-based portion of the interview, the questions in the latter portion of the interview were designed to explore students' beliefs about and attitudes towards mistakes in the context of counting. Specifically, I desired to determine not only whether their verbal statements pertaining to their prior experiences with counting problems were indicative of a growth mindset or fixed mindset, but also if their self-reported beliefs and attitudes were reflected in their counting activity. We asked questions pertaining to their confidence in their ability to improve their counting skills, what role mistakes played in one's learning, how they think mistakes are perceived my teachers and peers, and so on. One example of a question that we asked is similar to a question that Dweck (2006) poses, among many, to determine one's mindset is the following:

Of the two statements below, which one best reflects your own experience with learning to solve counting problems:

a. I can significantly improve my ability to count.

b. I can learn new things, but I can't really change my ability to count.

Dweck argues that individuals with a growth mindset would be more likely to choose the first option because it aligns with their belief that abilities and knowledge are not static. In other words, they would believe that with hard work, their abilities can increase significantly. People who possess a fixed mindset, however, would be more likely to choose the second option because they identify with the idea that people's abilities are fixed. They believe that subjects like mathematics are talents that people are either born with or will inherently lack.

4.2.1 Common tasks.

We asked the students the following three common counting problems (Table 3). In the remainder of this section I will discuss the solutions to these tasks in order to clarify subsequent presentation of the students' work. Many of these tasks were chosen because they involve some common mistakes that we thought could arise for the students, and I will also justify why we chose the tasks for the interviews.

Task	Problem Statement
Domino Problem	A domino is a small, thin, rectangular tile that has dots on one of its broad faces. That face is split into two halves and there can be zero through six dots on each of those halves. Suppose you want to make a complete set of dominoes, how many distinguishable dominoes would you have to make for a complete set?
Round Table Problem	How many ways are there to seat 10 people around a round table?
Sequence Problem	How many ways are there to form a three-letter sequence using the letters *a, b, c, d, e, f*: a. With repetition of letters allowed? b. Without repetition of any letter? c. Without repetition and containing the letter *e*? d. With repetition and containing *e*?

Table 3 Interview tasks

4.2.1.1 The domino problem.

For the sake of simplicity, we will establish notation so that (a,b) is the domino with a dots on one side and b dots on the other side, where $a,b\in[0,6]$, and (x,y) is equivalent to (y,x). For instance, (0,1) represents the domino that is blank on one side and has one dot on the other side, and it is considered the same as (1,0). To answer this question, we could break the problem down into cases. The first case counts the dominos that are blank on one side: (0,0), (0,1), (0,2), … (0,6). We would count seven dominos in this first case. The second case would count all the dominos with one dot on one side: (1,1), (1,2), (1,3), … (1,6). Note that we do not count (1,0) because it was already counted in the first case as (0,1). Thus, there are six dominos. In the third case, we would count all the dominos that have a two on one side; however, since (0,2) and (1,2) have already been counted, there would be five dominos in the third case. In a similar manner, we can determine that there are four dominos in the fourth case, three dominos in the fifth case, two dominos in the sixth case, and one domino in the seventh case. So, altogether, there are

$$7+6+5+4+3+2+1=28$$

dominos.

A common mistake for students to make is to overcount by treating outcomes such as (0,1) and (1,0) as different, which results in an answer of 49 dominos. Students often attempt to employ the multiplication principle[1] and determine that since there are seven choices for the first half of the domino and seven choices for the second half of the domino, there must be

$$7 \times 7 = 49$$

dominos in a complete set. This process implies that, say, (0,1) and (1,0) are different. The other way students arrive at 49 as their final answer is explicitly counting outcomes like (0,1) and (1,0) as different while listing the outcomes.

Lockwood and Swinyard, (2016) have also discussed students' tendencies to solve this problem incorrectly and have shared anecdotal experiences of students insisting on using familiar combinatorial operations and formulas (such as factorials and permutations), even when they do not apply. Given this prior work, I felt that this Domino problem could put students in situations where they might make an error, which might elicit conversations about their mindsets.

4.2.1.2 The round table problem.

To solve this problem, we first consider the number of ways to sit ten people in a line.

____ ____ ____ ____ ____ ____ ____ ____ ____ ____

There are ten people to choose from for the first position and if someone sits in that first spot, that leaves nine people to choose from for the second position. Likewise, after someone sits in the second position, there are eight choices for the third position, seven for the fourth, and so on. Thus, there are 10! ways to seat ten people in a line. This leads us to the mistake most students make

[1] The multiplication principle states: "Suppose a procedure can be broken into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage,…, and r_m different outcome in the m^{th} stage. If the number of outcomes at each stage is independent of the choices in previous stages and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \ldots \times r_m$ different composite outcomes" (Tucker, 2002, p. 170).

while solving the round table problem. Students often think that the number of outcomes for people sitting in a line is the same as for people sitting at a round table. So, they leave their final answer as 10!.

However, the situations are not the same, and the number of outcomes differ. In particular, outcomes that are simply a rotation of the table, like, for example, those in Figure 2, are considered equivalent. So, 10! would be over count the actual number of ways to arrange ten people around a round table.

Figure 2: Equivalent outcomes for the Round Table problem

Thus, we must divide 10! by the number of times the table can be rotated, which, since there are ten seats at this table, is ten times. So, our final answer is $\frac{10!}{10}$, which is 9!. Again, because I have had prior anecdotal experience with students thinking the answer was 10!, I thought this problem could raise issues that might elicit discussion of mistakes.

4.2.1.3 The sequence problem.

This problem is found in Tucker (2002), and he highlights a common error in part d) of the sequence problem (p. 172). To determine how many three letter sequences can be formed using the letters *a*, *b*, *c*, *d*, *e*, and *f*, permitting repetition, we will look at the number of possibilities for each position. Specifically, since we are allowed to repeat letters, there are six possibilities for each of the three positions. So, employing the multiplication principle, there are 6 x 6 x 6, or 6^3, number of three letter sequences.

To determine how many three letter sequences can be formed using the letters *a, b, c, d, e,* and *f*, without repetition, we will, again, look at the number of possibilities for each position. Unlike last time, we cannot repeat letters. So there are six possibilities for the first letter, five possibilities for the second letter, and four possibilities for the third letter. This yields 6 x 5 x 4, or 120, three letter sequences if repetition is not permitted.

Now we want to determine how many three letter sequences can be formed using the letters *a, b, c, d, e,* and *f* if it must contain *e* and repetition is not allowed. Note that since repetition is not allowed, any three letter sequence will only have one *e* and furthermore, there are three places the *e* could be placed:

__e__ _____ _____   _____ __e__ _____   _____ _____ __e__

In each of these cases, there are five choices for one of the remaining spots and four choices for the other spot. Thus, for each of the above scenarios, there are 5 x 4, or 20, three letter sequences for that particular placement of *e*. Since there are three placements, there are 3 x 20, or 60, total three letter outcomes.

Finally, to determine the number of three letter sequences that can be formed using the letters *a, b, c, d, e,* and *f*, permitting repetition, and containing the letter *e*, we will consider cases. Specifically, we will consider cases based on the number of *e's* contained in the three-letter sequence. The first case, which will be the case containing all *e*'s, has one outcome:

__e__ __e__ __e__

In the second case, we will count the number three letter sequences that contain exactly two *e*'s:

__e__ _____ __e__

Since this case specifies that we are to have exactly two *e'*s, there are five choices for the middle spot. Moreover, since there are three ways we can arrange those two *e'*s, there are 3 x 5, or 15, outcomes in this case.

In the final case, we will determine the number of three letter sequences that can be formed if there is exactly one *e*.

_____ _____ __e__

Recall that we can repeat letters, but we want to make sure this three letter sequence contains exactly one *e*. This means there are five choice for each of the remaining slots. So with this particular placement of the *e*, there are 5 x 5, or 25, three letter sequences. However, since there are three ways to place the *e*, this final case actually has 75 total outcomes. Knowing the total number of outcomes for each case, we can add them together to get the total number of outcomes. So, there are 1+15+75, or 91, three letter sequences that repeat letters and contain *e*.

This final portion of the Sequence Problem is usually where students make a mistake, and in fact, Tucker (2002) explicitly discusses this common error. In particular, students fall into a line of thinking that is similar to what is required to solve the third part and argue that there are three places to place an *e*:

__e__ _____ _____   _____ __e__ _____   _____ _____ __e__

In each of these instances, since repetition is allowed, there are six choices for the first spot and six choices for the second spot. Thus, there must be 6 x 6 three letter sequences for each placement, and with three ways to place *e*, there must be 3 x 6 x 6, or 108, three letter sequences. However, to properly employ the multiplication principle, as this solution attempts to do, it must be that all of the outcomes are unique, which is not the case, here. For instance, let the boldface *e* represent the *e* that has been fixed. Then the outcome, (e, c, e) is counted twice:

__**e**__ __c__ __e__   __e__ __c__ __**e**__

Thus, the proposed solution of 3 x 6 x 6 will over count the true number of outcomes (Tucker, 2002). Tucker highlights this as a common incorrect answer, and the interviewer has had prior experience with students making mistakes on this problem. Thus, as with the other problems, I chose this problem thinking that students might make mistakes that we could discuss with them.

4.2.2 Student specific tasks.

There are a couple of tasks that I gave to one student but not others, simply because we ended up having more time with one student than others. I briefly describe these, as I will discuss his solutions to these problems in the Results section.

Task	Problem Statement
The Jupiter Problem	How many arrangements of JUPITER are there with the vowels occurring in alphabetic order?
The Digit Problem	How many even five digit numbers have no repeated digits without leading zeros?

Table 4 Student specific tasks

4.2.2.1 The Jupiter problem.

As the problem statement indicates, we must keep the vowels, *u*, *i*, and *e*, in alphabetical order. Note, this implies that there is only one way to arrange the letters because the order cannot change. For now, though, we will focus on the remaining letters. There are several ways to solve this problem, but we highlight the solution that I had in mind when selecting this problem. Specifically, to solve this problem, we will calculate the number of ways to place the vowels, and once they are placed, they do not need to be arranged or ordered in any way. Then, we can arrange the remaining four letters in the remaining four positions.

Because there are seven letter in JUPITER, we have seven options to choose when placing the three vowels. Furthermore, we desire to place three letters. So, there are $\binom{7}{3}$ ways to place those three vowels among the seven available spaces, and, as previously discussed, the placement of the vowels has already been determined as a result of placing the consonants . Then, there are 4! ways to arrange the remaining four letters.. Thus, there are $\binom{7}{3}$ *4! ways to arrange the letters of JUPITER if the vowels are in alphabetical order.

I chose this problem because choosing the places in which to place the vowels is a subtle strategy that is sometimes difficult for students to recognize (Lockwood, Wasserman, & McGuffey, 2016). I thought that this problem might pose more of a challenge for students and could raise discussion about identifying and implementing this clever strategy.

4.2.2.2 The digit problem.

We can solve this problem by breaking the problem into cases, which are determined by the last digit. In particular, since the number must be even, it must end with a zero, two, four, six, or an eight. So, we have the following cases:

Case 1: _____ _____ _____ _____ __0__ Case 4: _____ _____ _____ _____ __6__

Case 2: _____ _____ _____ _____ __2__ Case 2: _____ _____ _____ _____ __8__

Case 2: _____ _____ _____ _____ __4__

In the first case, since zero has already been fixed as the last digit, we have nine choices for the first place, eight choices for the second place (because we cannot repeat digits), seven choices for the third place, and six choices for the fourth place. Thus, in the first case, there are 9*8*7*6 possible five digit numbers, without repeating digits, ending in zero. In the next case, we have fixed the last digit as two; however, since the problem specifies that we cannot lead with a zero, we only have eight choices for the first place, eight for the second place (since we can use zero), seven digits for the third place, and six digits for the fourth place. As a result, we have 8*8*7*6 possible five digit numbers, without repeating digits, ending in two. Note that the remaining cases follow the same argument. So, if we consider the total number of possibilities in the second, third, fourth, and fifth case, we have a total of (8*8*7*6)*4 possibilities, and if we also include the first case, we have (9*8*7*6) + [(8*8*7*6)*4] five digit even numbers, without leading zeros, where the digits do not repeat.

This is a difficult problem (Lockwood & Schaub, 2016) and one that students can tend to struggle with. The most common inclination for students on this problem is to try to do one direct implementation of the multiplication principle by moving left to right – starting with the first digit and considering options moving toward the last digit. However, the case breakdown I described above is necessary because there is dependency in the choices based on whether or not zero was included in any of the initial choices. So students tend to not recognize this dependency, or if they do, they do not know what to do about it. Solving the problem correctly requires realizing that they can break the problem into cases based on whether or not zero is the last digit, and this can be difficult to see. Thus, I chose this problem as a problem to give to a student who had generally demonstrated solid combinatorial understanding, thinking it could provide a challenge and afford opportunities for rich discussion.

4.3 Data Analysis

To begin analysis, the videos were transcribed. Then, I watched each of the videos while following the text in the transcript to code for instances that were suggestive of either a belief about or attitude towards counting or mathematics, or possession of a fixed or growth mindset. If something was said during the latter portion of the interview or if something in their counting activity reflected either one of these things, I would make note of the necessary portion of the transcript, copy it into a separate document, code it appropriately, and comment on what, specifically, stood out about the exchange. For example, one student stated, “In order to get good at physics, you need repetition, repetition, repetition. You need to get ingrained how to actually do a process to actually be a good at it. That’s basically entire math.” This was selected, coded as reflecting an attitude towards mathematics, and in particular, I commented on the student’s apparent perception of math as a rote subject, as demonstrated by a desire for “repetition, repetition, repetition” and an ingrained process. With these codes in mind, I was able to continue the data analysis process by writing short vignettes, which I discuss in more detail below, after a brief discussion about the codes.

As it pertains to coding for mindsets, I considered learning goals (Elliot & Dweck, 1988), mastery goals (Turner et al., 1998), and growth mindsets (Boaler & Dweck, 2016) to be the same and, as a result, I considered them to represent similar characteristics. Likewise, I considered each of their respective counterparts to reflect similar traits. Thus, I would code an instance as reflecting a fixed mindset if a student appeared to value correctness over understanding, perceive intelligence as a fixed entity, possess a desire to achieve at levels of normative ability, perceive academic achievement as a comment on their self-worth, value success with little effort, or any of the other characteristics typical of this mindset, as discussed in the theoretical perspective. For instance, one student reported, “It makes it hard to be successful because the fact is when you don’t know what you’re supposed to do, it’s hard to get the right answers.” This was selected, coded as reflecting a fixed mindset, and in particular, I noted that her success depended on the accuracy of her answers, rather than her understanding of the concepts.

I would code an instance as reflecting a growth mindset if they appeared to persevere when challenged, employ more self-regulatory behaviors while counting, perceive mistakes as potential learning opportunities, believe significant improvement is possible, or other traits typical of a growth mindset. For instance, one pupil was highly confident in his abilities to improve as evidenced by his statement, “I would say that I can significantly improve my ability to count.” This excerpt was selected, coded as demonstrating a growth mindset, and I noted his significant confidence in his ability to improve his counting capabilities.

After coding, I began writing short vignettes of each of the participants, using excerpts that were particularly demonstrative of their mindset. Instances were selected if it was representative of a reoccurring theme in the interview, or if it was a prominent example. The purpose of this was two-fold. The first was to provide the audience with an accurate depiction of the students, as shown in the results section. However, it was also meant as a step in the analysis so that I too could obtain a better understanding of their mindsets. In writing up each student, I was able to familiarize myself with each individual’s mindset as well as determine how they were similar or different from the others. As it relates to this, I was also able to gain an understanding of where the students lie, relative to each other on the mindset spectrum.

Chapter 5: Results

In this section, I present the findings as they relate to the research questions posed in Chapter 1. Specifically, I provide a short vignette of each participant that portrays the themes pertaining to mindsets that surfaced during their interviews. In addition, throughout the vignettes, I also draw connections and indicate differences among the ways in which the interviewees' mindsets manifested themselves. I would like to note, however, that this is not intended to measure one students against another, but rather, to note similarities and differences that can contribute to answering my research questions. To compose these vignettes, I rely heavily on the work of Elliot and Dweck (1988), Turner et al., (1998), Dweck (2006), as well as Boaler and Dweck (2016), as discussed in Chapter 2.

In the following subsections, I present each of the vignettes as well as some discussion regarding the students' mindsets. Moreover, I also note that they have been organized in a way that facilitates comparison. These vignettes and subsequent discussion serve as the results of my study; however, in the Discussion section, I also synthesize these results, connect them to my research questions, and provide additional interesting points to consider. As it pertains to the vignettes, I begin with Carl, a math and economics major enrolled in MTH 355.

5.1 Carl (A growth mindset)

In Carl, we see a prominent case of a student with a growth mindset. This was evidenced in a number of contexts, including during explicit questioning about mindset, during counting activity, and during reflection on his mathematical experiences. In this section, I provide evidence from his interview that demonstrates his persistent and well-developed growth mindset.

First, we see his growth mindset in his responses to explicit questions targeting his mindset. We began the interview by asking Carl the following question, in which we intended to explicitly target whether or not he aligned with a growth or a fixed mindset. This question is similar to one that Dweck (2006) uses to assess mindset:

Int: Of the two statements below, which one best reflects your own experience with learning to solve counting problems:

a. I can significantly improve my ability to count.

b. I can learn new things, but I can't really change my ability to count.

His response, provided below, is indicative of a growth mindset as it suggests he did not perceive ability to be a static entity.

Carl: I would say that I can significantly improve my ability to count. It's like, there's- I am sure there's methods I haven't learned yet as well. I mean, I am not- I haven't taken advanced probability yet, but I am planning to. I am sure there's something there.
Int: Sure, okay. And you feel confident that like, like yeah, that you are capable of learning more?
Carl: Oh, absolutely.

In particular, the fact that he felt he was capable of learning more suggests that he viewed himself as one that could grow in his mathematical thinking and abilities. According to Dweck, this is indicative of a growth mindset because those with a growth mindset believe that effort yields an increase in ability.

Carl's growth mindset was further reflected in his counting activity as well as in the remainder of his interview regarding his attitude towards mistakes made while solving counting problems. As it pertains to the former, one of the most prominent aspects of his counting that reflected his growth mindset was the apparent lack of impact making a mistake had on his confidence. He had been working on the Jupiter problem. Initially, he tried to apply an incorrect formula (specifically, a formula for selection with repetition problems that tends to be incorrectly applied). We see that when the interviewer asked him about it, he realized his mistake and ultimately corrected the answer.

Int: Okay, and what about that problem suggests the stars and bars method to you?
Carl: Oh, the fact that they're in alphabetical order means that those are set in a position relative to themselves, but not relative to the other letters. So, it's like it gives some information for

how the problem must be set up initially. So, you have e, i, u and then that's fixed, but what you don't know is what's in between them. So, that's why you can consider them to be like dividers, bins.

Int: *So here is a question. Um, so when you solve a problem like this, are you treating, I mean, if I'm just counting x's, am I treating the other letters as just being identical to each other?*

Carl: *Oh, wait, wait, wait. I only fig- you're right, you're right. I only figure out the number of different arrangements. I need to- yeah, so that* [referring to previous incorrect solution of seven choose 4] *times four factorial.*

Int: *Okay, and explain what that gets you.*

Carl: Because you are allowed to rearrange the other objects. What I actually figured out - I was think- the divider problems we typically had uh were for indistinguishable objects. These are distinguishable objects. So that's where so yeah I was solving a slightly different question.

In this exchange we see that making a mistake had no perceivable impact on his confidence nor his efforts to solve other counting problems as the interview progressed. Moreover, this excerpt also demonstrates how Carl was not only able to accept and effectively use the interviewer's intervention to solve the problem correctly, but he was also able to articulate the difference between the problem he was solving and that which he was intended to solve. According to Turner et al. (1998), this would be uncharacteristic of an ability-oriented student, who would not seek help. In fact, Dweck (2006) also argues that those with a growth mindset tend to be more self-regulatory, which could be why Carl was able to discern the difference between what was intended and what he actually did.

There are other examples in which Carl exhibited self-regulatory behavior and effectively utilized the interviewer's suggestion. In the following excerpt, Carl was attempting to solve the Digits problem. This was the first problem that Carl really experienced significant difficulties in solving.

Carl: How many even five digit numbers have no repeated digits without leading zeros? Okay so, that's a bit…so it's probably best to split – since it has to be even, it's like the last digit must be even and, but it also has, but the first terms also have bearings. So you can split that into cases for how many preceding even digits you have. So let's just start off. There's nine choices you can make for the first one…or maybe, maybe I could split it into an odd and even case for the first digit. You know what, I'll do cases.

As in the previous problems, Carl began this problem exhibiting self-regulatory behavior. Specifically, he employed the cognitive problem solving strategy of breaking the problem into cases; however, as the excerpt continues, we see that this was where he began to experience troubles:

Carl: So the number of even digits be 5, 4, 3, 2, 1. So if there's five even digits then you have you got four choices for the first one for the first term because you can't have zero and there's five total even digits and you can't choose zero so if you've got five you've got then you're uh. Whew. Five digits but one of them is not allowed that means there's four possible digits to the first one, but then that digit comes back in later on for the next one so you have four then times 3 x 2 x 1. Now if you have, uh, if you have four even digits then you know that you have five possibilities for the first one um, no, no four no but then you don't know which even number comes first so that method probably doesn't work very well.

Here, we see that Carl continued attempting to solve this particular problem by using cases and seemed to be talking himself through whether or not that was a reasonable approach. As previously stated, this in itself demonstrates a growth mindset as it is an example of self-regulatory behavior; however, this excerpt is indicative of a growth mindset because when he made mistakes, he used the mistake as a way to evaluate his approach. This is evidenced by his statement, "so that method probably doesn't work very well." As Turner et al. (1998) indicate, mastery-oriented students often determine that their troubles are a result of employing a less than optimal strategy and reevaluate their methods, which is exactly what we see in Carl. However, as stated earlier, this is also a problem in which one can witness Carl accepting help from the interviewer without a visible loss in confidence. As this exchange begins, Carl had already been working on the problem for approximately eight minutes, though he could not seem to find an appropriate approach:

Int: So here's a question. So if this if the last digit is the sticking point, right, could you consider cases based on what's in the final digit?

Carl: Yes, um, that's a good a point. So, then what would happen is you could have 0, 2, 4, 6, 8, and then the preceding ones, it does not – so actually, that that would be a way to solve. So if I take case where 0 is the final number then we would have so 0 is the final number because *we know the final number must be an even. Then that leaves a total of nine for the first number uh eight for the second, seven, six. And now two is the final then we still have we have eight for the first one 8, times 7, times 6. And now since there's no real differ- there's no real restrictions on two versus four six or eight I could just multiply this by, so if I just like 2/ 4/ 6 /8. So I multiply this by four so then adding these two should get the total number.*

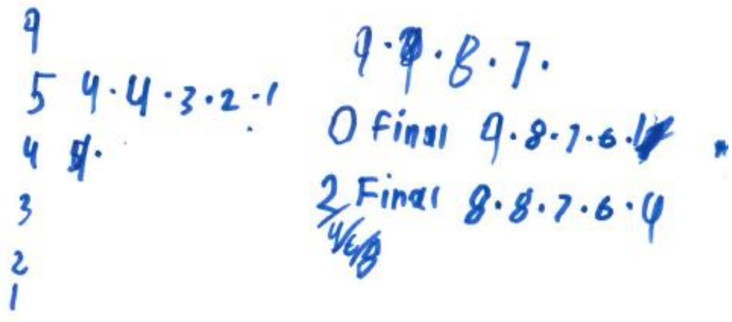

Figure 3: Carl's work for the Digit problem

Thus, we see Carl accepted the help, and, given that he spent a total of eleven minutes on the problem, we also see an example of Carl's persistence. According to Turner et al. (1998), persistence is another typical attribute of a fixed mindset. In fact, Turner and colleagues argue that when faced with a setback, students who are learning oriented are more likely to increase effort, just as Carl did, despite making several mistakes.

Furthermore, as it pertains to his attitude towards mistakes, he often made attempts to catch his own mistakes by examining his reasoning. For example, in the episode below, we see his work on the last part of the Sequence problem. He had decided to approach solving this problem by breaking it down into cases based on the number of *e*'s contained in the sequence. As he begins to work on the third case, we see him identify and self-correct a problem.

Carl: And then the third case is one e. So that's, uh, three choose one, also three, and then, um, so it would be five- yeah, five, five times four different cases and you can sum those up. Alternatively, though, you can also take the total number of cases and then remove all of those that don't have an e, which is 216 so with repetition. Oh wait, so with repetition that shouldn't be four that should just be five.

Int: Oh cool, so explain what you just fixed there.

Carl: Yeah, I forgot that this was with repetition so I accidentally wrote down four just like with the previous problem, which did not include repetition, but yeah, I'm allowed – it's like the only thing I can't add an extra of is an e because we are assuming one e.

This suggests that, unlike an ability-oriented individual, Carl utilized “more effortful self-regulatory behaviors” because he did not “seek to avoid failure” (Turner et al., 1998, p. 759). In particular, he checked his answer by employing an alternative solution approach to the problem and as a result, was able to discover a mistake he had made. This was common in his work as he did not shy away from making nor discovering mistakes as he solved counting problems.

We also gain insight into Carl’s growth mindset as we asked him reflective questions about his work, both during the interview and more broadly in his prior mathematical experiences. The fact that Carl possessed a growth mindset was further evidenced by many of the responses he provided when we inquired about mistakes in the context of counting, as demonstrated in the following excerpt.

Int: *Can you talk about, do you feel that making mistakes and learning from them is an important part of your study?*
Carl: *Yes.*
[…]
Int: *That’s cool. Um, so I guess does making a mistake when you do math make you feel discouraged or frustrated?*
Carl: *Nope.*
Int: *Okay, and how come?*
Carl: *Because that means I won’t be making the same mistake on a test. If anything, it’s helpful.*

So we can see his growth mindset presented itself not only in his claims of unwavering confidence even while making mistakes, as supported by his counting activity, but also in his perception of mistakes as potentially positive occurrences, as they could potentially help prevent him from making the same mistake again. I infer that this means he viewed mistakes as opportunities from which he could learn. We went on to have another exchange with Carl that underscores this same point.

Int: *Okay, nice and do you think it’s your own maybe do you think your attitude towards making mistakes is related to your confidence in your ability to do math, or just based in prior experiences, or what contributes to you feeling that way about making mistakes.*

Carl: I would definitely say that it has to do with my prior experiences with making mistakes and then coming back and like, like there is precedent for me making mistakes and then me understand the material better. It's like I have done that more times than I can count. But uh, and as for confidence, I can certainly see if somebody weren't confident in their math abilities that uh, making mistakes would be quite discouraging. It's like, "Oh, I keep messing up I keep making the same mistake." I would cert – I might – I would probably have that thought process if I had kept making the same mistake over and over and over, but if I don't, then I can't, I really can't see how a non-repeated mistake would be discouraging, at least for me.

The above excerpts indicate that he not only held the belief that mistakes were a critical and "helpful" part of learning, but that he also had a history of learning in this manner. Thus, he did not experience discouragement or frustration as a result of making mistakes.

In fact, Carl also reported that the times when he did experience frustration were those in which he did not know what to do with a problem and as a result, could not make a mistake at all. This could arise if he did not have anywhere to start on a problem, as he indicated in the following excerpt:

Carl: There was another linear algebra problem way earlier in the term. It was the first time in years that I'd looked at a problem and I was like, 'I don't have any idea how to solve this at all,' and the main issue was it wasn't in the textbook; it was something a bit more extra – it like, it was something extra that the professor wanted us to learn. And I couldn't go to his office hours because it conflicted with one of my other classes. And so I was really frustrated not because I was making mistakes but because there weren't any mistakes to make. It's like, I flat out didn't know what to do with the problem. I went to multiple tutors and they all like, they all said the same sort of thing, 'It works this way and that way,' and uh, and one of the tutors said, 'You know at this point, if you don't understand this at all, eventually you're just going to have to, uh, keep thinking about it and eventually, you'll wake up with the answer,' which is exactly what happened.
Int: Really?
Carl: I mean, I didn't wake up with the answer, but I woke up, went to one of my other classes, started thinking again. It's like, oh, well that method works. I guess that thought process.

Both his perception of mistakes and his reaction to mistakes are key aspects of a growth mindset, particularly as it pertains to mindset in the context of mathematics (Boaler & Dweck, 2016). Moreover, in the above excerpt we can again witness Carl's willingness to seek and accept help, which I infer to be a characteristic of a mastery-oriented student based on Turner et al.'s (1998) work, and in addition, we can see how Carl's growth mindset influenced him to put forth more effort when faced with a setback. According to Boaler and Dweck (2016), this is characteristic of a growth mindset. Furthermore, as the following excerpt demonstrates, Carl also valued understanding over correctness, which is another critical aspect of the mastery-oriented mindset (Tuner et al. 1998).

Int: Sure, well, and do you think that that has to do with also your attitude about what it means to do math. That like it's not just that you have to get the answer right away on your first try, but that there's something about sort of the process and understanding that is more important than just the right answer?

Carl: I would definitely say there is, there is uh, you should kn – there is this joke made by Tom Lehrer that's like with new math the important thing is to understand what you're doing rather than to get the right answer…even then, I think there is definitely some truth to that. It's like now long term, refer back to what I said about, it's like long term, it's like if you're making long term mistakes, long term errors, then there is definitely something wrong there. It's like it shows that you're not understanding what you're doing. But if you're getting just, if you're getting like the answer to one question wrong and then you're able to go back and then understand and actually understand and then get it right, it's like the understanding helps get to get the right answer. And referring back to the joke, I do, it's like just in general, I don't see how if you understand, if you actually understand the material I don't see how you could consistently get answers wrong, repeatedly.

This is an insightful comment from an undergraduate student, and it demonstrates the value he placed on understanding the material. He seemed to view mistakes as allowing him to increase his understanding, because he is committed to use short term mistakes to contribute to understanding in the long run.

It is my belief that Carl's growth mindset and, consequently, his perception of mistakes as an important and helpful process for learning mathematics, has enabled him to develop a positive attitude towards mathematics and towards counting in particular. For example, at one point, when asked about whether counting is difficult in general, he said, "I would say it's not really that difficult in general." He also said, "I would never say that I don't make mistakes. That's kind of, making mistakes is kind of, I feel, a given, if you're doing math for any extended period of time…"

These responses make sense for someone who feels that students (including himself) have the ability to learn counting. This positive attitude manifested itself in his confidence in himself as someone who was capable of successfully doing mathematics. In both his counting activity and responses, one is able to witness the prominent presence of a growth mindset as well as its significant influence. Carl was confident in himself as a mathematician and appears to possess a thorough understanding of the role mistakes play in his learning of mathematics. In particular, he perceived them as critical to the process of learning and consequently, did not become frustrated with making mistakes. Rather, he sought to find his mistakes so that he could exploit them as the learning opportunities he perceived them to be. As a result, he saw mistakes as a positive aspect of mathematics and did not appear to view counting in a negative light.

In summary, we see a clear instance of a growth mindset in Carl. He not only reported possessing a positive attitude towards mistakes, valuing self-regulatory behavior such as seeking and accepting help as well as proceeding with careful and effortful involvement, and having confidence in his ability to improve his counting capabilities, but he also demonstrated these attributes during his interview. As a result, we can witness positive affect towards mistakes, in general, as well as a positive attitude towards counting. It is worth noting, however, that Carl was enrolled in an upper level discrete mathematics course, and thus, is more advanced in his mathematical career. As a result, he has had more time and experience to develop his ability to count, and additionally, his growth mindset. This could, in part, explain why Carl demonstrated such a significantly striking example of a growth mindset. We acknowledge this difference between Carl and the other students, but we still feel that we can draw meaningful comparisons because we are trying to show examples of mindsets in undergraduate students. As a stark contrast with Carl's growth mindset, we now discuss a student who does not possess the same attitude. Clarice was a computer science major enrolled in MTH 231 for the second time.

5.2 Clarice (A fixed mindset)

In contrast to Carl's growth mindset, Clarice presented a striking example of a student with a fixed mindset. This mindset could clearly be witnessed in her counting activity, and it was additionally evidenced by many of her statements made during the interview. Below, I have provided several examples of such activity and statements as well as indications of how they demonstrate a fixed mindset. In fact, evidence of a fixed mindset was prominent almost immediately – before the interview began, Clarice stated that she could not understand why we would want to study such an "annoying" subject. When questioned further, she and the interviewer had the following exchange:

Int: *And how much prior experience – so you just mentioned why do we study counting and you said it's such an an – did you say annoying subject?*
Clarice: [nods]
Int: *Okay, so tell me about your prior experience with counting and why you feel like that.*
Clarice: Uh, I'm retaking this because I suck at counting.
Int: *Okay, and why do you think that you suck at counting?*
Clarice: Because counting is actually, you wouldn't think, it sounds so easy and then when you look at it you're like, 'oh, this is really annoying.' Especially since the fact that most discrete mathematics does not involve a calculator.
Int: *Okay, great. So what makes – well first of all, why does the fact that it doesn't involve a calculator make it annoying?*
Clarice: Because you had to do all the computations from your head and if you had a calculator, it would just make them way much easier.

According to Turner et al. (1998), individuals with fixed mindsets typically value success with little effort due to the fact that they perceive effort as a sign of low ability. Thus, I interpret the excerpt above as a manifestation of Clarice's desire to minimize effort, and this claim was further supported in her counting activity as well, which will be discussed later. This was not the case for Carl, who not only reported effortful involvement in his attempts to solve a linear algebra problem, but demonstrated such involvement during his work on the Digits problem. Furthermore, this excerpt is also indicative of Clarice's lack of confidence in her ability to count, as illustrated by her comment, "I'm retaking this because I suck at counting."

In the next excerpt, she went on, with an apparent increasing sense of frustration, to explain why she found counting "annoying."

Int: Okay, um, so is that the main reason why it's annoying? Why else is it annoying?
Clarice: Uh, also the different ways certain problems have to be uh thought through. And the fact that they don't generally – it's slightly harder to decide how what they actually want you to do because they never really fully explain that in a problem. You're just supposed to know what you're supposed to do.
Int: Nice. Um, I agree that those are some things that make challenging – er - counting challenging. Um, any other insights you can offer about - so you said annoying. Do you feel like that makes it difficult? Do you feel like that makes it hard to be successful?
Clarice: It makes it hard to be successful because the fact is when you don't know what you're supposed to do, it's hard to get the right answers.

Here, we see that Clarice held the belief that she "sucks" at counting due to the fact that she typically cannot understand what the problem statements are asking of her, and thus, she struggles to get the right answer. This is one of the primary characteristics of a fixed mindset. That is, those with a fixed mindset hold the belief that academic success is a comment on one's self worth and scholastic ability (Turner et al., 1998), and attribute failure to low ability (Elliot & Dweck, 1988). In this case, Clarice attributed failure, or incorrect answers, to the fact that she "sucks." Recall, however, that Carl, while solving the Digits problem, attributed his mistake to a less than optimal approach, rather than his ability. Indeed, based on the excerpt below, I would argue that Clarice interpreted incorrect answers as failure because she valued demonstrating competence over learning.

Int: Can I ask you what does it mean to be considered good at counting?
Clarice: Um, to be able to get an A on a counting test.
Int: Okay, and that, that's, okay.
Clarice: And being able to – basically, you're good at counting if you actually know what you're doing.

Int; Okay, sure, and that might be reflected on getting a A on a test.
Clarice: Yes, because generally if you know what to do you can generally get a A on the test unless you do some annoying thing with your mathematics, which is very possible in counting problems.

While Clarice acknowledged that it was important to “actually know what you’re doing,” which could suggest some understanding, it appeared to be secondary to the ability to demonstrate that understanding and receive an A on a test. Based on Turner et al. (1998)’s work examining student affect and motivation, I infer that a focus on grades, rather than on an understanding of the content, is typical of students with fixed mindsets. In fact, Turner et al. (1998) indicates that students with fixed mindsets strive to demonstrate competence and achieve at high levels of normative ability (p. 759). This desire to demonstrate competence, however, was present throughout the interview. Moreover, it was present not only in her interview, but also in her counting activity. For example, while working on the first part of the Sequence problem, the following exchange occurred:

Clarice: Hmm, a b c d e f. Hmm, should I just call them a b and c or 1 2 and 3 for these?
Int: So you can call them - so I want a three letter password or a three letter sequence that can be - and these are the letters I have to choose from. So I guess you can call them one, two, three if you want. And so you’re solving that first one. So, how about with repetition allowed?
Clarice: It would be that [3^6 3^5 3^4].
Int: Okay, and so explain what you did there.
Clarice: Well, 3^6 because I do remember this type of question from the homework. The reason why it’s 3 to the six is because there’s total of three poss- pretty much there’s six different times you can do for the first one. And then the five is like the three different possibilities. Or it’s, it’s either that, or it’s six times one and six times two and six times three. You know what, I think it’s the other way. [Crosses out previous answer and writes 6^3 6^2 6^1]

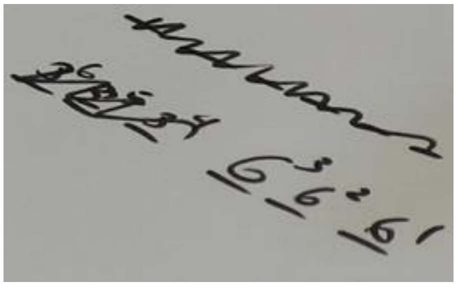

Figure 4 Clarice's work for the Sequence problem, part a

Int: *Okay, and so that means six times six times six, six times six, and six. And do you add them up or what do you do with them?*
Clarice: You're supposed to add them up, but I don't really want to add them up at the moment.
Int: *No, that's okay. But it would be like six cubed plus six squared plus six to first.*
Clarice: Yeah.
Int: *Okay, and just explain one more time how you got that.*
Clarice: Well, since like that [refers to previous counting problem that she attempted] *except for this because there's exactly six things for everyone, which probably means I did that wrong* [referring to previous problem again].

While attempting the first part of this problem, which had a correct answer of six cubed, Clarice took little time to consider the problem before providing an initial answer and even less time before she changed the answer. This was a common occurrence as she worked through the remaining three parts of the password problem. In fact, while working on the second part, Clarice changed her answer three times. While working on the third part, she changed her answer to the second part at least eight times, she changed her answer to the first part at least twice, and her answer to the third part at least twice. While working on the fourth part, she changed her answer once. The image of her work on this problem in Figure 5 reflects her tendency to quickly cross out and move on to a new answer.

To clarify, however, it is not that Clarice changed her answer several times that is noteworthy, but rather, it is how she changed her answers. Specifically, Clarice changed answers rapidly, without justification, and it did not appear as though she even attempted to understand expressions before declaring them the final solution. Interestingly, this aligns with research that has documented students' tendencies toward quickly applying counting formulas even in situations in which they are not appropriate (e.g., Batanero, et al., 1997; Lockwood, 2014). In essence, it seemed as though Clarice sought to present the correct solution quickly and effortlessly.

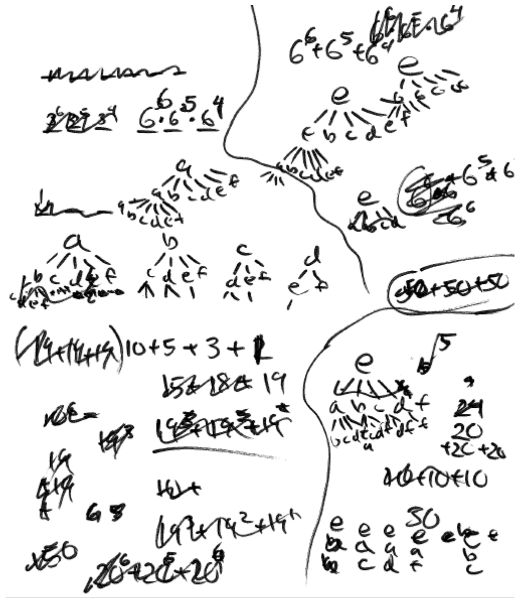

Figure 5 Clarice's work for the Sequence problem

Lastly, when prompted by the interviewer to begin listing outcomes, her behavior continued as she changed her answer to the second part at least three times. The interviewer let her change her answer and tried to keep up with her thought process. Eventually, after Clarice appeared to be done with the problem, the interviewer attempted to engage by presenting her with the correct answer with the hope of helping her to think about what might be going on in the problem. The following exchange shows what happened:

*Int: Okay, and I guess I'm going to argue, and I think this is a great list and you're right that there's twenty of them, but so, I think that the total - so if you made a list with b and a list with c, d, e, and f, I think you're just going to have 20*6 total. What do you think about that?*
Clarice: You're going to actually have 20 times-oh…
Int: Like just 20 times six. Let me actually write that.
Clarice: Um, no, I wouldn't say that. You cannot have 20 -
Int: Okay, how come?
Clarice: Because you said that the order matters.
Int: Okay, so the order does matter. So, were writing were doing with repetition, sorry, without repetition and the order does matter.
Clarice: So, the full answer is 20 times six cubed.

Nearly immediately, Clarice disagreed with the answer provided by the interviewer. This is quite unlike Carl's thoughtful evaluation, eventual acceptance, and careful implementation of the interviewer's suggestions on the Jupiter problem and the Digits problem. As she went on to explain why her answer was correct, she changed her answer at least three more times before agreeing with the solution given by the interviewer. With this solution, she was then able to change her answer to first part to arrive at the correct solution. In total, she had changed her solutions to this password problem at least 22 times. This is indicative of a fixed mindset because pupils with performance goals perceive effort as a sign of low ability and often value success with little effort (Turner et al., 1998). Clarice did not want it to seem as though she had to put forth a substantial amount of effort to arrive at the correct solution. In addition, her counting activity in this problem also suggests a fixed mindset because she did not seek or accept help (Turner et al., 1998).

In fact, when the interviewer attempted to provide help and ask questions about her thinking during a completely different problem (The Round Table problem), the following excerpt indicates that Clarice interpreted these interventions as a comment on her ability. More specifically, it appears as though she interpreted them as marks against her ability. After describing aspects of counting she often struggles with, which included understanding the problem statements, the interviewer attempted to probe her thinking further:

Int: *Okay, did you feel like any of the problems we talked about today you felt like that? Like, that it was either difficult to figure out what they were asking or difficult to figure out how to solve.*
Clarice: Uh, I would say that one because you said I sucked at that one because it seemed that I did really bad on that one.
Int: *Which one?*
Clarice: [Points to the Round Table problem]
Int: *Okay, I didn't say you sucked at that one.*
Clarice: I just did…uh, no I don't think they're that hard. It's just that their hard to do on tests.

As the interviewer indicated, it was never implied that Clarice performed poorly on any problem. In fact, Clarice is referring to is the Round Table problem, which is one that the interviewer had asked her to revisit and provided some slight guidance; however, as one can see, Clarice perceived this intervention as an indication that she had low ability, and indeed, she said that the interviewer "said I sucked" on that problem, which the interviewer certainly did not say or indicate. So, unlike Carl, the interviewer's offers of help appeared to diminish her confidence and frustrated her. Furthermore, it is interesting to note that what Clarice considered difficult was determined by an external source, rather than by her own judgment. In particular, the problem Clarice identified as difficult was one Clarice believed damaged the interviewer's perception of her ability. To reiterate, this excerpt demonstrates her fixed mindset because she did not employ self-regulatory behaviors nor did she appear to value understanding as much as she did appearing competent. Even more, she internalized the interviewer's discussion about a potentially correct solution as an attack on her ability to solve the problem.

Clarice's desire to demonstrate competence was also evident when we asked reflective questions during the interview. For instance, Clarice was prompted with the following question:

Int: Of the two statements below, which one best reflects your own experience with learning to solve counting problems:

a. *I can significantly improve my ability to count.*
b. *I can learn new things, but I can't really change my ability to count.*

Her response below suggests an attention on demonstrating competence.

Clarice: Hmmm…I know which one you want me to say, but…
Int: No, there's actually not one, you can say whatever you want.
Clarice: Uh, I would say both.
Int: Okay, so say a little more about that.
Clarice: While it's very hard for me to change it, more of its more of – I can – it's not that I can – it's not that I had to change my ability to count. I more I need to figure out what the problem is saying. So I guess I would say option C. Or yes, I can learn new things, but I can't change my ability to count. That is true. I can significantly improve my ability to count. That is true. The problem with my counting is not that I'm bad at it or cannot do it. It's that I don't know what the problem is first telling me to do until going to the massive amount of work which you can't do on tests which is why I always suck at counting tests.

Again, Clarice not only appeared to want to give the answer she thought was "right," but we also see the need to preserve her perceived intelligence in her claims that it "is not that [she is] bad at it [counting] or cannot do it." Although this initially this appears to contradict her previous statement that she "sucks" at counting, and perhaps her proclaimed belief that she "can significantly improve [her] ability to count" would appear to suggest a growth mindset, I interpret both of these statements as attempts to preserve perceived ability. Even in a low-stakes environment with only the interviewer and myself present, she desired to give the impression that her ability was not what needs to change; she claimed that she was not bad at it because she desired to appear competent. This is typical of an individual with a fixed mindset, as students who focus on competency often resort to coping mechanisms that bolster self-worth and restore appearance when they feel threatened (Turner et al., 1998). Furthermore, placing blame on the problems statements, rather than on a need for improvement, focusing on her performance on tests, and stating that she "can't change [her] ability to count" further supports this claim.

The idea that ability is a static entity is another chief aspect of a fixed mindset (Boaler & Dweck, 2016). Aside from the instance describe directly above, there were many instances in which Clarice indicated that the ability to count was something someone either possessed or did not, as shown below. Given the length of the exchange, I bolded some sections that are particularly striking.

Int: Okay, that makes sense. Do you feel that so when, if and when, you make a mistake on a counting problem, say, and we all do it, I mean, they're hard, um, do you think that helps you learn better like learn counting better when you make mistakes? Or is it demoralizing and discouraging or both?

Clarice: Uh, it's confusing.

Int: Okay, say more about that.

Clarice: Um, it's just really hard for me to – the thing is ***– in order to not completely suck at counting problems you need to actually change the way your brain thinks to realize what a counting problem is.*** *The problem is, that is very hard, especially for me. And you know that you have to do it yourself. And you actually have to change that part of you so yes, you can understand what they're saying. And also, counting is one of the things you realize* *when you finally look at the way people -* ***when people finally tell you you're wrong and then they actually tell you the answer, you're like ooh, it makes sense. But until then, you're like, 'oh, I don't know what to do.'***

Int: So do you feel like either the book or your teacher like you haven't been given the tools to succeed, or do you think you haven't learned the skill of interpreting, or do you, I don't know. Do you know what I mean? Like, you feel like there's something that you could be taught or is it just a challenge that you're always going to face.

Clarice: ***It's a challenge I am going to face until I am finally realize how to interpret them Because you can't be taught the interpretations****. It's like, it's like math. In order to get good at physics, you need repetition, repetition, repetition. You need to get ingrained how to actually do a process to actually be a good at it. That's basically entire math.* ***But in order to actually get good at it, your brain must actually change the way it actually thinks****. This is the same with physics and same with calculus. In order to actually do it you have to change the way you think which is much harder and teachers can't really teach.*

According to Clarice, counting will continue to be a challenge until she can learn how to interpret combinatorial problem statements; however, she also claimed, "you can't be taught the interpretations." I interpret this to mean that she believed that students either possess the ability to understand problems statements, or they do not, and she falls into the latter category. For this to change, her "brain must actually change the way it actually thinks." In other words, the only way she could change this is to modify her brain and its way of thinking, which again, according to Clarice, is not something that can be taught. Until then, it appears that she is left unsure of what to do. This not only demonstrates a fixed mindset because of her belief in static ability, but also because according to Elliot and Dweck (1988), students with performance goals are often more vulnerable to feeling helpless when faced with a challenge. To state it more explicitly, Clarice's self-reported lack of ability to discover and correct her mistakes indicated a fixed mindset.

The belief that her abilities are static and, furthermore, that her brain would need to change, was a common a theme in her interview. She made the following statement when asked if she felt discouraged or lost confidence when counting:

Clarice: I have gotten to the point where I have done so horribly on mathematics and physics tests that I don't really care.

Int: Okay, I am sorry to hear that, but I know what you mean.
Clarice: Basically you get disappointed enough if you take, if you're like me, who's never been completely that good at math or always had problems wrapping my head around concepts. And I remember my first years of college, I basically did math two years straight, and I do mean actually two years straight, even through the summers. So yeah, I know enough not to be disappointed; it's expected.

Thus, not only was her ability fixed, but she anticipated future failure. Again, this significantly different from Carl, who was confident in his abilities to improve as evidenced by much of his interview. Furthermore, as the following excerpt demonstrates, she appears to have held the belief that the failure is inevitable:

Int: Let's say you had, you know, a few weeks and just all you had to do, like, you were just gonna get paid to get better at counting. Right, and you didn't have any other distractions, you didn't have classes. I guess my question is, are there - do you think that there are things you can do to get better? And if so, like what would you do, what would your game plan be? Or, is it just like, I – it won't happen?
Clarice: It won't happen in six weeks, no offense. It still hasn't clicked with calculus. I am pretty sure it's not going to click with that.
Int: Okay, and how about, um, – that's fair – what if it wasn't a six week time limit, but just, I mean, do you feel like there's - is there anything you can do to improve? And if the answer is no, that's okay, I'm just curious. Does it feel kind of helpless?
Clarice: Uhhh, I don't know how to convince myself to change. It's just very confusing. I guess the only thing you can do is, I don't know, look at, uh, I guess the only thing you could do is look through all the book and figure out which counting problems match which method. And then figure out, like, I don't know, a mathematical way of figuring out of which ones to actually make. Or, pretty much, you would need to look at all the different counting problems, like how the book actually words them or how the teacher words to be able to figure out which one is which. Like, I guess you could always look for keywords.

Clarice's statements indicated that there was no amount of time nor any method, aside from, perhaps, matching problem statements and identifying keywords, that would enable her to improve her abilities. As it pertains to learning from mistakes and teachers' perceptions of mistakes, Clarice also stated:

Clarice: Um, I'm pretty sure when teachers they've seen it enough that while at the same time they've seen it enough and understand what you're doing wrong, they, at the same time, it's not like they can really change it. They don't have enough time in the schedule to help you change it. Unless you actually want to change, but the main problem with these problems is you actually have to change your mind. And that's something much more based

> *on you and not the teachers so it's not like the teachers can really help. And in a sense, I think they probably know that. That certain times they just need to let the students figure it out on their own.*

Again, Clarice perceived her situation as helpless because one has to be able to change their mind and teachers cannot teach this. More significantly, she acknowledged that teachers can often identify the nature of students' mistakes, but do not possess the power to change it. I interpret this to mean that Clarice did not see value in mistakes. She, unlike Carl, did not view mistakes as potential learning opportunities because of her belief that she cannot be taught the skills necessary to improving her abilities. That she did view mistakes in a positive light was further evidenced by the negative affect towards mistakes, and by her nearly immediate rejection of help while working on the Sequence problem. In fact, Turner et al. (1998) indicates that those who strive to demonstrate competence, or those with a fixed mindset, often experience negative affect towards mistakes because they are seen as a threat to one's ability. It is my belief that Clarice did not want to accept help from the interviewer as it would draw attention to her mistakes, and, ultimately, she would interpret the mistake as a comment on her self-worth, resulting in negative affect.

In summary, Clarice presented as a striking case of a fixed mindset. In both her relatively chaotic counting activity and reflection on prior counting experiences, one can clearly see many of the prominent indications of a fixed mindset. This includes the belief that ability is a static entity, the perception that effort indicates low ability, a lack of self-regulatory behaviors such as asking for or accepting help, the desire to demonstrate high normative ability or competence, and failure to demonstrate such competence is a negative comment on her self-worth, among others. Clarice completely gave up on this changing because for such an alteration to occur, she must literally change her brain and teachers cannot teach that. Moreover, it is my belief that these characteristics have had an impact on her beliefs about mathematics, as a field, counting, and herself as a student of mathematics and, as the following excerpts demonstrates, as a computer programmer.

Int: Do you, but you like, I mean you're a computer science major. Do you enjoy programming?
Clarice: I enjoy programming but while even my dad says that - he has worked in the engineering for as long as I can remember - longer than I was born. He basically said while at the same time you're supposed to know this, it doesn't necessarily come back – it is very possible that will never have to use it again.
Int: Oh, interesting. For the math?
Clarice: Yeah.
Int: For your future in programming?
Clarice: Yes. Actually, math is not that used. It really depends on what you're doing. Majority of programming does not involve a lot of math.
Int: Okay, so does that make it, maybe, hard to be motivated to do the math when just rather be programming.
Clarice: No.
Int: Okay.
Clarice: I am not good enough of a programmer to say that.

Clarice lacked confidence in her computer programming abilities as well as mathematical abilities, especially those needed for counting. As a result, she held a generally negative view of these subjects. In particular, she understood mathematics to be a rote discipline in which one requires "repetition, repetition, repetition" so as "to get ingrained how to actually do a process to actually be a good at it." Furthermore, she viewed mathematics as useless to her major, computer science. Generally, she also perceived counting to be a confusing, "annoying," structure-less subject.

As it pertains to the mindset spectrum, I argue that Carl and Clarice lie at opposite ends. In fact, they share little, if anything, in common with the way in which they approach mathematics. Carl was careful and thoughtful in his work, accepted and implemented interventions, valued learning, and demonstrated a positive attitude towards mistakes as potential learning opportunities. Clarice, on the other hand avoided even acknowledging her mistakes, rejected help and further, interpreted it as a negative mark against her ability, valued competence over learning, and desired presenting solutions quickly and effortlessly. Furthermore, Clarice appears to possess a negative perception of counting as it is an “annoying” subject, and, more generally, a negative perception of mathematics as she thinks of it as a rote subject. This is evidenced my by her belief that the only way to “get good” at mathematics is to “get ingrained how to actually do a process” through “repetition, repetition, repetition.” Despite the extreme nature of each case, neither Carl nor Clarice should be considered the case with which to measure others.

I would like to note that neither the discussion pertaining to the selected excerpts of Clarice’s interview, nor the remarks comparing her’s and Carl’s mindsets were meant to be derogatory or callous. Rather, I intended to use episodes of Clarice’s interview that would offer an accurate portrayal of her mindset as it presented itself during her interview, and provide a contrast with Carl. Both Carl and Clarice provide a striking example of their respective mindsets and thus, provide valuable insights into mindsets as well as the teaching and learning of combinatorics.

The next student, Sofia, is a computer science major enrolled in MTH 231, also for the second time.

5.3 Sofia (A growth mindset)

In Sofia, we see our second instance of a student with, primarily, a growth mindset. Although she made several statements that, if examined superficially, could demonstrate a fixed mindset, there were always stipulations attached that indicated the statement was actually suggestive of a growth mindset. Thus, I argue that, based solely on her counting activity and self-reported prior experiences with counting during this interview, the presence of a fixed mindset is barely present, if at all. Below, I will discuss examples of the aforementioned subtle statements indicating a growth mindset as well as of how the more prominent attributes of her growth mindset presented themselves. In presenting Sofia’s case I will discuss similarities and differences with her and Clarice and Carl.

Perhaps one of the most prominent ways Sofia's growth mindset revealed itself was in her counting. In fact, while attempting to provide a solution to the Domino problem, the following exchange occurred:

Int: *Okay, cool. Can I ask you get to explain what you were doing and what you were thinking about?*
Sofia: *I was thinking of having these be different dominos. You would have six on the top and six on the bottom. Then you would have five choices. Then you would have four choices. Then you would have three choices, then two, then one kind of thing. I realized that's wrong.*
Int: *What's wrong about that?*
Sofia: *Because my answer is totally off. It's in the thousands, and it shouldn't be in the thousands for a set of dominos.*
Int: *Maybe first before you fix that, explain what you were doing here.*
Sofia: *I was multiplying these out.*
Int: *The* $6 \times 5 \times 4$*?*
Sofia: *Mm-hmm.*

Here, we see Sofia exhibited self-regulatory behavior, which, as noted above, is more typical of students with a growth mindset Turner et al. (1998). In particular, we see that Sofia performed a reasonability check. This is note-worthy for two reasons. First, as it relates to self-regulatory behavior, it shows that she did not simply throw out the first numbers that came to mind as an answer, as Clarice did. Rather, like Carl, she was careful and effortful in her work, which would be atypical of an individual with a fixed mindset given that they desire success with little effort (Turner et al., 1998). Second, also similar to Carl, it demonstrates that Sofia did not shy away from making mistakes nor did she avoid finding errors in her work. Again, this is atypical of those with a fixed mindset because fixed mindsets perceive errors as a threat to their competence (Turner et al., 1998). In fact, according to Boaler and Dweck (2016), those with a growth mindset are actually more likely to be aware of and fix errors in their work. Below, we see what happened as she continued to work on the Domino problem:

Sofia: *So now I'm thinking there's seven slots, and there's two choices here, so* 7×2*. I think my answer is 24, or I think it's 24.*

Int: *Okay, you think the answer might be 7 × 2? I think that's 14. Explain what you were thinking with the 7 × 7.*
Sofia: *I was thinking because there's seven choices it would taper down.*
Int: *Why did you switch to 7 × 2?*
Sofia: *Because there are seven choices, and there's only two lines.*
Int: *Can you write down a few examples of dominos you would want to count?*
Sofia: *Mm-hmm.*
Int: *Can you explain what you just did?*
Sofia: *These are the same.*
Int: *So you wrote a 2, 1 and realized you already have the 1, 2?*
Sofia: *Mm-hmm.*
Int: *Sorry to interrupt. It looks like you're being somewhat methodical. Can you explain what you're doing as you're writing them out?*
Sofia: *Here I know there's going to be an option of it both being 0, and then I'll move on and go up to 6. Then I start over, and half the top ones starting at 1, and then go through until it hit 6. I'm also identifying when they're same to cross it out.*

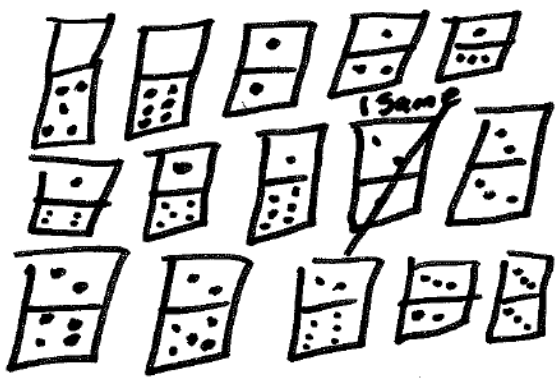

Figure 6 Sofia's work on the Domino problem

Although Sofia struggled to solve this problem correctly, there are several aspects of this last exchange that reflect her growth mindset. As in the previous excerpt, we can still see her careful, "methodical" work as she implemented the one-dimensional odometer strategy (English, 1991). Furthermore, this effortful work continued, despite several unsuccessful attempts to solve the problem. This is significant because, as Elliot and Dweck (1988) note, individuals with a fixed mindset tend to display the "helpless" response when faced with a challenge, while pupils with a growth mindset tend to persist and increase effort. It is noteworthy that both Carl and Sofia demonstrated this persistence when faced with a difficulty in their work. It is also worth noting that although each of her proposed solutions in the provided excerpts were incorrect, she could justify her reasoning behind each potential solution, and ultimately she was able to obtain the correct solution. This is a similar awareness that Carl displayed as he was able to identify the difference between the problem he was meant to solve and the problem he did solve while working on the Jupiter problem.

In this next excerpt, we see Sofia's self-regulatory behavior in her work on the Round Table problem.

Int: *Try this one* [the Round Table problem]. *Feel free to write down more if you want. Tell me what you're doing right there.*
Sofia: *I'm making a smaller example than this one, and then mentally switching them around, and trying to identify a pattern that I can add it out to or multiply it out to.*
Int: *Would you mind physically writing them down instead of just mentally?*
Sofia: *Okay.*
Int: *Is doing a smaller example something common that you do if you have to solve a problem?*
Sofia: *Mm-hmm. This is too big to really mess with. This one's a little easier and smaller to mess with, so I can try to reason it out that way and then apply it to that.*
Int: *Nice, that's great.*
Sofia: *So now I'm thinking this might be like the Domino problem where it would taper down, because eventually you would run out of choices that are the same.*

In this instance, Sofia implemented the strategy of utilizing smaller cases to solve the given problem as well as attempted to identify patterns that would permit progress. Eventually, these efforts led to a breakthrough as she was able to identify a similarity between this problem and her previous work on the Domino problem, which again, demonstrates effective problem-solving strategies.

As another example of a growth mindset, during work on the Sequence problem, Sofia not only accepted the interviewer's guidance and implemented it effectively, but she also did so without it being a noticeable detriment to her confidence. Again, this appears to demonstrate an attitude towards assistance similar to Carl's, but significantly different from Clarice's, who interpreted interventions as a criticism of her ability. To be more specific, Sofia had been working on the Sequence problem and although the first three parts of the question did not seem to hinder progress for Sofia, she struggled significantly to solve the last part of the question. The interviewer provided a few minor interventions, and after several minutes of little to no progress, Sofia accepted a hint offered by the interviewer. It is important to note that this hint was not taken as the result of a helpless attitude, but rather, she was able to acknowledge her need for assistance and accept it. While this alone demonstrates a growth mindset, I also argue that accepting a hint evidences a growth mindset because it conveys that she, like Carl, valued understanding over demonstrating competency. Specifically, it is either the case that Sofia did not perceive assistance as a threat to competency, or if she did, she clearly valued understanding over demonstrating competence because she still accepted the hint. However, the following exchange, which occurred later in the interview as we attempted to explore her beliefs about counting, makes a stronger case for the latter argument:

Int: *Okay, great. What does it mean to you to be considered good at counting?*
Sofia: *I would say knowing how to explain it well, being able to show others you method of doing something, not just putting a number on the paper and having it better right, but also being right is part of it, because maybe 8 times of 10 you should be able to get a counting problem right to be considered good.*
Int: *So being able to solve them correctly, but then also being able to explain to someone else?*
Sofia: *Mm-hmm.*

Although Sofia included correctness as a part of being good at counting, it appears to be secondary to the ability to explain concepts and, in particular, to explain them *well*. However, the ability to explain material well suggests an understanding of concepts. Thus, this exchange supports my claim that Sofia seemed to value understanding over demonstrating competence. It is important to note, though, that either of the aforementioned cases would be indicative of a growth mindset (Turner et al., 1998).

Sofia's counting activity was not the only setting in which one can discern the presence of a growth mindset. In particular, the following excerpt shows what happened when we asked her the question adapted from Dweck's (2006) book:

Int: *... First, if you can respond to this. So of the two statements below, which one best reflects your own experience with learning to solve counting problems?*
Sofia: *I think I can learn to improve my ability to count by practicing.*
Int: *Okay, so you think through practice you can improve it?*
Sofia: *Mm-hmm, yeah, like learning the concept better.*
Int: *Okay, nice. What do you mean by practice?*
Sofia: *Just doing more counting problems, watching YouTube videos on different methods, or just learning how to visualize the problems better.*

This exchange demonstrates that Sofia not only perceived ability as a dynamic entity, which is a prominent characteristic of a growth mindset (Boaler & Dweck, 2016), but we also see that she appeared to share with Carl the belief that with effort, she could improve her own abilities. In fact, as the interviewer probed her thinking further, it was revealed that Sofia believed this of all students. That is, effort or "practice" can lead to improving one's abilities and "learning the concept[s] better." Such a belief would not be typical of an individual with a fixed mindset as they often perceive effort as an indication of low ability and value success with little effort (Turner et al., 1998).

As it was previously mentioned, there were a few statements Sofia made during her interview that, if examined superficially, could indicate a slight fixed mindset. Below, one can see what happened when the interviewer began exploring Sofia's beliefs pertaining to mistakes in the context of counting.

Int: *Does making a mistake when solving counting problems make you feel discouraged or frustrated? Why or why not?*
Sofia: *Sometimes, but also I think it's important to make mistakes. That's just my opinion.*
Int: *That's great. Can you say more about that? Why is it important to make mistakes?*
Sofia: *Because you can identify where you messed up and you can try to learn from your mistakes and not mess up like that again.*

While Sofia acknowledged having experienced frustration and discouragement as the result of making a mistake while solving a counting problem, she also went on, in the same sentence, to stress the importance of making mistakes. In particular, she argued that making mistakes is helpful because "you can identify where you messed up and you can try to learn from your mistakes and not mess up like that again." Though negative affect towards mistakes would be typical of a student with a fixed mindset, this positive perception of mistakes is not. As discussed previously, students with a fixed mindset view mistakes as a threat to their competency (Turner et al., 1998). This is different from the frustration and discouragement Clarice reported as a result of failure, given that Sofia "sometimes" experiences negative affect, but recognized the beneficial nature of mistakes. Moreover, this was not the only instance in which Sofia made what seemed like a fixed mindset statement, but immediately stipulated it with a statement more typical of individuals with a growth mindset.

Int: *Do you lose confidence in your ability to count when you make a mistake when solving a counting problem?*
Sofia: *A little bit, yeah, if I'm not able to figure it out or find help. Then yeah, because I'm just like, "Oh, I can't do it."*
Int: *Okay, but it doesn't mean you're never gonna be able to be a good counter?*
Sofia: *Yeah, it just means I need to ask someone for help later.*

Here, we attempted to explore Sofia's beliefs and attitudes towards mistakes further. As in the last excerpt, we see Sofia make a statement that initially appears to suggest a fixed mindset, but is amended with a statement more typical of a growth mindset. In particular, in this excerpt, we see that Sofia reported a loss of confidence after making a mistake on a counting problem, which would be indicative of a fixed mindset, but because she specified that the loss occurs *if* she cannot "figure it out or find help," I would argue that it actually supports the claim that Sofia possessed a growth mindset. This is because the loss of confidence appears to result from a lack of progress or inability to find assistance, and not the mistake itself. This is an important distinction as an individual with a fixed mindset would experience negative affect as a result of the mistake (Turner et al., 1998), as we see in Clarice. Unlike Sofia, Clarice's loss of confidence was not because of a lack of progress, but because she perceived a mistake as an indication of low ability. In fact, both Sofia and Carl appeared to experience negative affect as the result of a lack of progress, rather than at the occurrence of an error. Furthermore, that she would seek help is also characteristic of a growth mindset (Turner et al., 1998).

To summarize, Sofia presented as a growth mindset, though it was much more subtle than Carl's case. However, we still see many of the same characteristics in Sofa as we did in Carl. In particular, we are able to witness Sofia exhibit self-regulatory behavior as she engaged in combinatorial problem solving, which included both implementing effective cognitive strategies as well as seeking and accepting help. In addition, like Carl, we can also discern positive affect towards mistakes as they are perceived as potential learning opportunities. As a result, it appears that despite finding some aspects of counting challenging, she possessed a positive view of counting, overall. In fact, Sofia actually perceived counting as quite applicable to her major, as the following exchange demonstrates:

Int: *…You mentioned feeling like you see counting's applicability to computer science. Can you say a little bit more about that?*

Sofia: *You can make a computer program that does this, and it might be right, and it might be wrong, but knowing how to do it by hand would make it so you could tell. For example, encryption or password cracking, this would be applicable to that.*

In the next student, a computer science major enrolled in MTH 231, we see our first instance of a mindset that is neither solely fixed nor solely growth.

5.4 Nathan (A mixed mindset)

Nathan provided an interesting example of a student who, unlike Carl and Clarice, did not lie on one extreme end of the mindset spectrum, but rather, fell somewhere in the middle. In fact, while it appeared as though much of Nathan's counting activity and his attitude towards mistakes reflected a growth mindset, Nathan also made several statements during the interview that suggested a fixed mindset. In exploring Nathan's case, we can gain some insights about mindsets. As noted in the Theoretical Perspectives section, some authors have accounted for some students having elements of both fixed and growth mindsets (Turner et al., 1998). In discussing Nathan (and the following student, Damien), I provide examples of this phenomenon. I will relate these students to the broader theory of mindset in the Discussion section.

To provide an accurate depiction of Nathan's mindset, I will discuss particularly interesting aspects of his interview and how those episodes demonstrate either a fixed mindset or growth mindset. Moreover, there were also several instances in which Nathan's actions suggested a fixed mindset and growth mindset, simultaneously. However, after further analysis, I will argue and demonstrate that the fixed aspects of Nathan's mindset were primarily motivated by his desire to appear competent via good grades.

Much of the insights about Nathan's mindset emerged during his solving of counting problems, and much of his activity suggested a growth mindset. While trying to solve the Round Table problem, Nathan appeared to exhibit self-regulatory behavior:

Nathan: *Yeah, so there are 10 locations on this table. The first location we just look at it isolated. It can be populated by any number of those ten people. Let's just say each seat is matched by one person. So the first chair can have any number of those ten people, but now that we've chosen a person for that seat then one chair goes away. One person goes away, so now we have nine people for the next chair. Similarly two people have been now taken away. Two chairs have been taken away. So now we have eight more people, and we can just assume that this goes on until the last person fills the last chair. So the answer would be 10!*

Int: *So here's a question. Is that for around the circular table or around the line?*

Nathan: *That's around a line.*

Int: *Because you're thinking of ten options for the first positon. Okay, great.*

Nathan: *I feel like it should translate to a table as well. I'm just thinking about a real world example. Suppose this was down to four people instead of ten, or even three people. That would better to actually draw. So you can have 1 here, 2 here, 3 here. So 1, 2, 3,*

or you can have 1, 3, 2, right? Oh, it's a rotation.
Int: *So tell me what you saw.*
Nathan: *Okay, yeah, so suppose 1 is now placed over here, and 2 is over here. 3 is over here. We can start from over here, and that would be 1, 2, 3, and then if 2 was over here, and we went the other way we'd want 3, 2 again. Hmm, so there's only two possibilities or two ways. 2, 3, 1 is the same thing as 1, 2, 3, right?*

In particular, we see how Nathan used the information he ascertained from solving a different, but similar, problem (ten people in a line as opposed to around a table), as well as from breaking the problem down into a smaller case to make progress on the given problem. This was a strategy that both Nathan and Carl utilized in their efforts to be self-regulatory. According to Turner et al., (1998), employing self-regulatory behavior is typical of individuals with a growth mindset. Furthermore, in the following excerpt, we can see how Nathan's efforts enabled him to make progress on the problem:

Nathan: *Okay. So if we had 1, 2, 3, that's equivalent to 2, 3, 1, which is equivalent to 3, 1, 2.*
Int: *That's right. Nice.*
Nathan: *If we extrapolated this to four, then there would be four equivalent statements around the table.*
Int: *That's right. Why?*
Nathan: *You're just rotating once.*
[…]
Nathan: *So I'm assuming that if we had a table with n people around, is it n! divided by n?*
Int: *Okay, tell me how you get that.*
Nathan: *If there was three people, and they were just sitting down, there's three possible equivalents to how they can be arranged, and the total number of ways they can be arranged that we got from here is the n!, so then if you just divide that by the similar ways then that's you get uniqueness.*

Thus, Nathan not only solved the given problem, but he also actually went one step further to generalize the answer for *n* people sitting at a round table. This is significant as it demonstrates that Nathan did not avoid attempting more challenging problems. In fact, he chose to pursue this challenge on his own accord. This is typical of someone with a growth mindset as they tend to be learning oriented. In this way, challenges can be perceived as an opportunity to learn (Turner et al., 1998).

We can also see what happens when Nathan was confronted with an unexpected challenge during the problem solving process, rather than in pursuit of one. Specifically, Nathan had been trying to solve the third part of the Sequence problem. Although he had come up with a way to represent the combinations with an *e* in one particular place, he was experiencing difficulties obtaining a solution that accounted for all of the outcomes.

Int: *What are you thinking about?*
Nathan: *I think from over here we concluded that we can just divide by n, and it would just work, but that's not the case over here, because even though there are three elements, there are four separate cases, or four distinctions possible, right? Four for each possible letter combination.*
Int: *Mm-hmm.*
Nathan: *So is it multiplying by the factorial? No, that doesn't make any sense. That's a tough problem.*
Int: *I think you can treat these as three different cases. There's a case where the E is first, second, and third. Those are the only possibilities for what you actually want to count. This is gonna get counted in this situation here, which is good.*
Nathan: *So we did handle for that.*
Int: *Yeah, and in fact this is gonna count all of the situations where E is last.*
Nathan: *And this counts for all situations where E is first.*
Int: *Mm-hmm.*
Nathan: *And this counts for everything that's in the middle.*

So, similar to both Carl (while solving the Jupiter problem) and Sofia (while solving the Sequence problem), we see that Nathan was able to persist, and, when given guidance, accepted and effectively used this guidance to gain more insights. In fact, shortly after this exchange, Nathan was able to move on and solve the problem correctly. While it is not necessarily important that Nathan arrived at the correct solution, it is worth noting that Nathan persisted and accepted help. Again, this demonstrates that Nathan displayed traits typical of an individual with a growth mindset (Turner et al., 1998).

The above exchanges are representative of Nathan's counting activity, and so as one can see, much of his counting activity reflected aspects of a growth mindset. In fact, there were very few instances that indicated a fixed mindset. However, one instance that suggested a fixed mindset was when Nathan attempted to verify that his proposed solution to the Domino problem was correct. In particular, like Clarice, he appeared to be concerned with the accuracy of his answer, and not the comprehension of pertinent counting concepts a correct answer might suggest. Turner et al. (1998) would argue that this is indicative of a fixed mindset because it demonstrates a focus on performance, rather than understanding. Not only was this focus present throughout the interview, but as evidenced by the following excerpt, this attribute was often accompanied by a manifestation of his wavering beliefs about his ability.

Int: *... Which of the following two statements reflects your own experience?*
Nathan: *To be very, very honest the second.* [I can learn new things, but I can't really change my ability to count.]
Int: *How come?*
Nathan: *Usually when I do take a course involving counting problems, they're hard in that it usually doesn't account for too much of our letter grade, or anything like that, so I don't really pay too much attention to it. I'm just like, "I'm not gonna worry about learning this. I'm just gonna worry about the other stuff."*
Int: *So you just resign to not being able to get it?*
Nathan: *Yeah, because they're hard. I do think I could improve for sure, and I will need to obviously, but as of right now it's just an option.*

The fact that Nathan selected the second statement to be more representative of his experiences indicates a fixed mindset as he appeared to believe that his abilities were fixed. In fact, he went on to confirm that he was resigned to not understanding counting "because they are hard." Moreover, his statement, "it usually doesn't account for too much of our letter grade," indicates that Nathan was focused on performance and suggests a fixed mindset. I think it was this focus on performance that caused Nathan to act on his fixed mindsets tendencies. In particular, Nathan acknowledged that he "could improve for sure," but since it "doesn't account for too much of [his] letter grade," he was "just gonna worry about the other stuff." Thus, I argue that Nathan did believe in his capabilities to improve, which suggests a growth mindset, but his desire to perform well pushed Nathan towards behavior that reflected a fixed mindset. So, in some senses, he was like Carl and

Sofia, who did believe that he had the ability to change and improve if he really wanted to. However, he and Clarice have in common the desire perform rather than learn. These conflicting beliefs and mindsets were a common theme in his interview, as shown below.

Int: *Do you have any sense in what makes counting problems difficult for you?*
Nathan: *I think conceptually it's just really hard for me to grasp around. One thing I've always struggled with is whenever people say the difference between C or permutation combinations is the idea that there's an order. Give an example and then ask me whether this has order or not. I can't really tell you, because I'm not really sure what I'm looking for.*
Int: *Okay, that makes sense. How has it been in 231? Have you felt that still has persisted?*
Nathan: *Yeah, it's definitely persisted, but **I feel better at it. I'm just trying to slowly learn.***
Int: *Yeah, that's great. Do you think counting is difficult in general for people, too?*
Nathan: *Honestly, I'm not really sure, because at least my friends think it's not too bad, but I guess it's subjective. **Some people are just good at it. It's intuitive for them,** but I'm assuming that for a lot of it's pretty hard.*
Int: *I think it is challenging. That's what interesting to me about them. They're easy to understand and read, but they can be hard to solve.*
Nathan: *Yeah, the questions make sense to you, but **it's just trying to come up with the right answer.***

Here, we can see a significant amount of wavering between the two mindsets. On the one hand, his attention to performance, as indicated by his focus on "just trying to come up with the right answer," and his belief that some people are "just good at it" because counting is "intuitive for them," are suggestive of a fixed mindset. In fact, this latter statement indicates that he perceives counting as a sort of natural gift. However, later in the interview, Nathan said, "once you get [counting], it's always as if you've got it," which I infer to mean that Nathan believed counting is something that can be learned. Though these statements are indicative of a fixed mindset, it is important to note that like individuals with growth mindsets, or like Sofia and Carl, Nathan also appeared to believe that counting can be learned. This is further supported by his optimistic attitude towards past and potential improvement, as evidenced by his acknowledgement of his own improvement, dedication to "trying to slowly learn," and the following excerpt:

Int: *Do you lose confidence in your ability to count when you make a mistake?*

Nathan: *Yeah, for sure. Obviously you don't want to make a mistake, but I guess it shouldn't discourage me, but at the same time I wouldn't say I'm the worst math student, but something like this is almost new to me. So being that frustrated, like this question is a simple question, and so trying to think so deep about such a simple question and trying to figure out the answer is pretty frustrating.*
Int: *All that makes sense, and yet you still feel like generally you can learn from your mistakes, though. Even if it's frustrating it's still an opportunity to learn?*
Nathan: *Yeah, for sure, because I do recognize the importance of problems like these. These pop up in computer all the time, so you kind of get mad, but obviously you can learn it. It is a process like anything.*

His belief in his ability to learn is marked by his statement, "but obviously you can learn it."

This excerpt also demonstrates another characteristic that Nathan appeared to possess – we see that Nathan reported experiencing a loss of confidence in his ability to count. This characteristic, negative affect towards mistakes, is more typical of a student with a fixed mindset (Turner et al., 1998). This was something that could be seen in Clarice's counting activity as well. This, again, supports the argument that Nathan's emphasis on ability was pulling him towards a fixed mindset. His negative affective response was appropriate given that his goal was to perform. Moreover, this is not the only instance in which Nathan indicated experiencing negative affect towards mistakes in the context of counting:

Int: *Does making a mistake while solving a counting problem make you feel discouraged or frustrated?*
Nathan: *Yeah.*
Int: *Can you say more about that?*
Nathan: *Like this. Before you showed me that it was × 3, and you just told me it was incorrect but I was close, that always gets me. What am I missing? In a way I hate these problems, but they're also really challenging for me at least, and so that's why I come back to it.*
Int: *Sure, like a love-hate relationship.*
Nathan: *Yeah, pretty much.*

The fact that Nathan experienced discouragement and frustration as the result of making a mistake while solving counting problems is characteristic of a pupil with a fixed mindset. This is because Nathan's focus on ability, rather than understanding, caused him to see mistakes as detrimental to demonstrating competence (Turner et al., 1998). Again, though, we can see the conflicting mindsets at work as Nathan acknowledged that, "in a way," he hated counting problems, but also appeared to enjoy the challenge they present, which was why he came back to them. As the interviewer suggests, it is a "love-hate relationship." However, despite experiencing negative affect after making mistakes, the following excerpt indicates that Nathan was still able to see value in mistakes:

Int: *Yeah, that makes sense. I'm gonna ask you a little bit about mistakes, not because you make mistakes, but just because I'm curious about how people think about mistakes in counting. Do you think that make mistakes helps you learn more about counting? Do you make mistakes when you count, and does that help you learn?*

Nathan: *Yeah, I make a lot of mistakes. Now I know to look for this. If I were to see a similar problem again, this is obviously pretty helpful. I'll have that somewhere back there just saying, "Oh yeah, I just multiply by 3." You have to remember to always find the different possibilities or different ways to look for it. I think making mistakes definitely helps. That's how you learn. If you don't make any mistakes, then how do you really learn?*

Here, we can clearly see that Nathan perceived mistakes as a necessary part of learning. This is evidenced in his statement, "I think making mistakes definitely helps. That's how you learn." In fact, not only did he provide an instance in which he was able to learn from his mistakes during the interview, but he even went so far as to question how one would learn in their absence. As with Carl and Sofia, this demonstrates the growth aspect of Nathan's mindset.

In summary, Nathan represented a student who appeared to possess a mindset that has elements of both fixed and growth mindsets, which is not unprecedented (Turner et al., 1998; Boaler & Dweck, 2016). On the one hand, we can see that Nathan employed self-regulatory behavior as he attempted to solve counting problems, including cognitive strategies and accepting help. Furthermore, we also see that Nathan was not afraid take on a challenge, nor did he give up in the face of difficulties, and perceived mistakes as essential to learning. These attributes, as well as his belief that he could improve his abilities to count, are typical of a student with a growth mindset. Yet, despite the belief that he could improve, Nathan had resigned to not be able to understand counting because that particular unit did not count for a significant portion of the class grade. I interpret that this is because Nathan also appeared to place a strong emphasis on performance, rather than learning. Moreover, this intense focus motivated Nathan to act on characteristics more typical of a fixed mindset and, thus, there appears to be a reason for his fixed mindset. For instance, although he acknowledged mistakes as potential learning opportunities, he also reported frustration and a loss of confidence after making mistakes. While the positive perception is a result of the growth aspects of his mindset, this loss of confidence is likely due to his belief that errors are detrimental to his ability to perform well, which he valued. As a result of this wavering mindset, he seemed to experience a "love-hate relationship" with counting. He loved the challenge and even described counting problems as "really interesting problems" during the interview, but still, "in a way," hated counting. The next student, a MTH 231 student majoring in electrical engineering, also has a mixed mindset.

5.5 Damien (A mixed mindset) [2]

Damien provided another instance of an individual with neither a distinctly fixed mindset nor a uniquely growth mindset. However, unlike Nathan, there did not appear to be an obvious reason for the conflicting mindsets. Thus, as I attempt to portray an accurate representation of Damien's mindset, I will provide excerpts that demonstrate the fixed aspects of his mindset as well as those that serve to establish the growth aspects of his mindset.

On one hand, there were several exchanges, such as the one below, that indicated a growth mindset. In this particular excerpt, we are able to witness self-regulatory problem-solving behavior, similar to Carl, Sofia, and Nathan, as Damien worked on the Round Table problem.

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2 It is worth noting that parts of Damien's interview, such as his response to the question taken from Dweck (2006), were omitted due to misunderstandings as a result of a language barrier.

Damien: *So since one, two, three –would be just ten. So two people, just two way, three people it would be –three ways. I'm just gonna do with three people first.*
Int: *Okay, yep.*
Damien: *So three people are gonna be – so stay right here is gonna be one, two and if person two, this and this and person three went this and this. So it would be two, two to three way to – and I would say when I do it with ten people it can be two to the ten.*

In particular, we see that Damien attempted to break down the problem into a smaller, simpler case involving three people. He then attempted to extrapolate the insights he made while solving the smaller case to help him answer the given problem. As noted above, this is a typical characteristic of a growth mindset as they tend to exhibit self-regulatory behaviors (Turner et al., 1998). Moreover, we also see other aspects of a growth mindset in Damien's counting activity.

Damien: *–two and three and two and four and two and five and two and six. So it's gonna be a – this gonna be the same if you flip it. So I would say –it's gonna be seven times six, so it'll be seven factorial. Seven factorial dominoes if you can flip the side.*
Int: *Okay, if you can flip. And –okay. And one question. And so this is what? Seven times six times five times four times three times two times one?*
Damien: *Yeah.*
Int: *Okay, and why are you multiplying? Why does that make sense?*
Damien: *Because –oh, that's a good question. Actually, it would be seven plus – no, it would be seven plus six plus five, four, three, plus one.*
Int: *Okay, say more about that. And you can use another piece of paper if you want to.*
Damien: *Yeah, because since you can flip these numbers, so these numbers are gonna disappear again and again if you do like – keep doing that. So the number was just decreasing and since all the numbers, you add it up together, it's not multiply.*

Here, we see Damien accepted and effectively utilized the guidance offered by the interviewer as he worked on the Domino problem. Again, this a trait that he, Nathan, Sofia, and Carl had in common. This is significant as Damien not only displayed more self-regulatory behavior while engaged in combinatorial problem solving, but he also acknowledged an error in his thought process, something Clarice would not be likely to do due to her negative perception of mistakes. This acknowledgment is marked by his comment, "Because –oh, that's a good question. Actually, it would be seven plus – no, it would be seven plus six plus five, four, three, plus one." He recognized, as a result of the interviewer's intervention, that his original approach, using multiplication, was incorrect and that adding the numbers would yield the desired solution. Though he did not explicitly state it like Carl did, using a mistake, even if prompted by the interviewer's suggestion, to evaluate the validity of his mathematical approach is something Damien and Carl have in common. As with Carl, this is indicative of a growth mindset because someone with a fixed mindset would be more likely to avoid their mistakes as they are perceived as a threat to their attempts to demonstrate competence (Turner et al., 1998).

Furthermore, as we began exploring Damien's beliefs about mistakes during the second portion of the interview, I also saw elements of a growth mindset.

Int: *Okay, sure. Okay, I'm gonna ask some questions about mistakes, but not because you necessarily made mistakes, that's just the line of questioning. So don't – yeah. So one question is do mistakes help you learn more about counting? Like if you make a mistake does that help you learn more about counting?*
Damien: *Yeah, yeah. Totally.*
Int: *Okay, and can you say more about that?*
Damien: *Yeah, so my mistake and I would say the way I think might be wrong and if I know like is it wrong, so I would change it to like yeah, actually, that's how you should do it.*

His comment, "Yeah, so my mistake and I would say the way I think might be wrong and if I know like is it wrong, so I would change it to like yeah, actually, that's how you should do it," conveys that he did see value in mistakes as it would enable him to identify the error and change it. Recall that this is an attitude he, Carl, Nathan, and Sofia seem to share. This attitude is further supported by a later exchange still focused on attitude towards mistakes:

Int: *Yeah, so do you agree with this statement? To be considered good at counting, you have to make little or no mistakes.*
Damien: *No, I was thinking that you learn from your mistakes.*

However, not all of his attitudes and beliefs towards mistakes were indicative of a growth mindset.

Int: *Sure. Okay, so does making a mistake when solving a counting problem make you feel discouraged or frustrated? Or why or why not?*
Damien: *Yeah, like the goal is want to solve the problem right and do it – when you got it wrong, I don't know, you just feel like you didn't succeed to achieve something, yeah.*

Here, not only do we see that Damien appeared to experience negative affect towards mistakes, which is a typical attribute of individuals with fixed mindsets (Turner et al., 1998), but this excerpt also suggests that Damien associated success with solving the problem correctly. In other words, I interpret this to mean that Damien seemed to value correctness, or competency, over understanding, as suggested by his comment, “Yeah, like the goal is want to solve the problem right and do it – when you got it wrong, I don’t know, you just feel like you didn’t succeed to achieve something, yeah.” Both the negative affect towards mistakes and focus on performance are also attributes of a fixed mindset that Clarice and Nathan demonstrated. Moreover, although both Damien and Sofia admitted to experiencing negative affect in this scenario, Sofia indicated the importance of making mistakes and noted that the frustration and loss of confidence only occurred if she was unable to make progress. This is different than Damien, who simply acknowledged the frustration and, as previously discussed, associated success with correct answers rather than understanding.

There is other evidence of a fixed mindset that occurred during Damien’s interview. In particular, the following exchange suggests that Damien possessed one of the more prominent aspects of the fixed mindset, which is the belief that abilities are fixed entities (Boaler and Dweck, 2016).

Int: *…Okay, and then what does it mean to be considered good at counting?*
Damien: *Like the way you think of the problem. I would say if people thinking recursively it would be way easier.*
Int: *Okay, can you say a little bit more about that?*
Damien: *Like I would say they’re gonna do it from inside out, like the smallest case and go all the way to really big case.*
Int: *Okay, and you did some smaller cases. Is that a way you like to think and think about things?*
Damien: *I can’t tell. Sometimes I feel like the first thing in my mind to like do it.*
Int: *Sure, sure. Do you think people can develop the skill of thinking recursively? Or is it something that they kind of have?*
Damien: *I feel like it’s the thing that you were born with. So, yeah, it’s pretty hard to think*

recursively.

Specifically, Damien appeared to hold the belief that one's ability to think recursively is something they are born with. I infer this to mean that he thought the ability to think recursively is something someone either possesses or they do not, and it cannot be developed significantly. This was a belief that Clarice held onto tightly, especially as it pertained to herself. Because of this belief, and that fact that Damian perceived recursive thinking as essential to be good at counting, it is unclear whether Damien would argue that the ability to count is also a gift one is born with.

To summarize, Damien appears to fall among the 20% (Boaler & Dweck, 2016) whose mindset lies somewhere in the middle of the spectrum, as opposed to near one of the ends of the spectrum, which was the case for Carl, Clarice, and Sofia. In his counting activity, we were able to see that Damien employed metacognitive problem-solving strategies, as well as accepted and utilized guidance to make progress on tasks. Furthermore, as he reflected on prior experiences with counting, his comments suggested a positive perception mistakes as potentially helpful learning opportunities. These particular attributes are demonstrative of the growth aspects of Damien's mindset. On the other hand, Damien also reported negative affect towards mistakes and expressed that the ability to think recursively is a static entity. In particular, he argued that the ability to think in this manner is something one is born with. As it has already been discussed, these represent the fixed aspects of his mindset.

Chapter 6: Discussion

In this chapter, I address further the research questions posed in Chapter 1. In particular, as it relates to the research questions, I will summarize and discuss the main conclusions that can be drawn from the vignettes and immediate analysis surrounding the vignettes contained in Chapter 5. This includes an examination of the interesting results pertaining to mindsets in the context of undergraduate mathematics that this particular study afforded. In addition, it also includes an examination of the role mistakes and mindsets can play in combinatorics. Ultimately, I intend to clarify the primary contributions this work yielded in the field of mathematics education research as well as explore points of discussion.

As a reminder, the research questions are:

1. How does self-reported information about mindset and mistakes manifest itself during undergraduate STEM majors' combinatorial activity?
2. What evidence can be found to support that the concept of mindset applies to a spectrum, and not a dichotomy?
3. What role do mindsets and mistakes play in undergraduates' learning of enumerative combinatorics?

In Section 6.1, I discuss results that relate to my first research question, in Section 6.2 I present the results as they pertain to my second research question, and lastly, in Section 6.3, I discuss findings related to my third research question.

6.1 Attitudes and Beliefs Manifest Themselves in Undergraduate Students' Counting

Unlike many of the previous studies aimed at understanding mindsets, also known as achievement goals (e.g. Boaler & Dweck, 2016; Elliot & Dweck, 1988; Turner et al., 1998), this study was completed in the context of undergraduate mathematics. Specifically, I examined undergraduates' self-reported beliefs about and attitudes towards mistakes, and also asked to students to engage in combinatorial problem solving. This allowed me to make claims about the students' mindsets based both on their self-reported beliefs and what was observed of their problem solving activity. In particular, each sheds light on how the students' self-reported beliefs and attitudes manifested themselves in their efforts to count. For instance, Carl denied losing confidence as the result of making mistakes because of his perception of mistakes as learning opportunities. This was evident in his counting as he remained persistent in the Digits problem and used setbacks as a way to evaluate the validity of his approach. Another example was Clarice's comment regarding the confusing nature of errors in counting, "when people finally tell you you're wrong and then they actually tell you the answer, you're like ooh, it makes sense. But until then, you're like, 'oh, I don't know what to do.'" This certainly manifested itself in her chaotic counting attempts as she quickly changed answers, appearing to only be relating relevant numbers with operations typically used in counting (e.g. multiplication, addition, exponentiation, etc.). So, the design of this particular study not only permitted insight into the mindsets of undergraduates engaged in mathematics, but it also provided the opportunity to examine both self-reported affect and observed affect as they problem solved. As it pertains to this latter affordance, we can see very distinctive manifestations of their self-reported affect in their counting activity. This would not have been possible if the data had been derived from only self-reported information nor a task-based interview.

By studying undergraduates, I can shed further light on the constructs put forth by Turner et al. (1998), Elliot and Dweck (1988), Boaler and Dweck (2016), and others. In particular, it is noteworthy that we see mindsets persist even among undergraduate students, which indicates that older students are affected by their beliefs and experiences as well. It is interesting to wonder about what factors contribute to mindsets and how they might be changed. For instance, it is worth considering how Clarice's experiences might have differed from Carl's experiences over the span of their academic careers, and what that that might suggest about the importance of what we should do for students in early education. Perhaps someone could have helped shape Clarice's negative mindset to make her more comfortable with making mistakes or accepting assistance from her teachers and peers. By observing and identifying the same constructs among undergraduates as among young children, I feel that I am providing support for these constructs. Indeed, we can see how important it is to foster growth mindsets among students.

In fact, Clarice provides an interesting example of the importance of mindsets and, moreover, raises questions how such a mindset might have developed. The fact that she so thoroughly exemplifies numerous characteristics of a fixed mindset was rather surprising. In particular, she not only lacked confidence in her abilities to improve, but she had come to expect failure. She noted that this expectation developed after years of performing poorly in previous math courses. It is natural for me to ask whether this cycle could have been stopped if an intervention had occurred in an early math course, or would she have even been open to such an intervention. Perhaps one of the most notable traits of a fixed mindset that Clarice possessed was her interpretation of assistance as an attack on her ability. Though the researcher had never made a remark against Clarice and, in fact, only encouraged Clarice, a small suggestion was construed that she "sucked" on a problem. This is a striking interpretation that is concerning to me as a mathematics educator, and it emphasizes how pervasive negative mindsets can be for undergraduate students, who have had many years to solidify these attitudes and beliefs. I speculate that just as it took several years to cultivate this fixed mindset, it might similarly require many positive experiences for Clarice to begin shifting her mindset toward a growth mindset.

6.2 Evidence Indicative of a Mindset Spectrum

This study also provides insight into the concept of mindsets, in general, especially as it relates to the mindset spectrum. First, it is important to note the extreme cases of a fixed mindset and growth mindset in Clarice and Carl, respectively. In these two cases alone, we are able to witness the prominent aspects of each mindset. As it pertains to Clarice, we see the desire to perform, seemingly effortlessly, a negative perception of mistakes, negative affect towards mistakes, and a total lack of belief in her abilities, in general, as well as in her capability to improve (Boaler & Dweck, 2016; Turner et al., 1998). In Carl, we see nearly the opposite. We are able to witness one who not only perceives mistakes positively, but appears to take advantage of the information they can provide, and one who exhibits persistence, self-regulatory behavior, and unwavering confidence in his abilities (Turner et al., 1998). While Clarice perceives mathematics as a rote, hopeless subject, Carl appears to view mathematics as a process that can be learned. Thus, even in their differing affect, we can further see the significant impact mindset can have, and furthermore, the importance of understanding it. To be more explicit, it appears as though Carl's perception of mistakes and belief in his abilities to improve yield more confidence and a positive perception of mathematics. This is further supported in my examination of the other cases as well.

If mindsets were simply a dichotomy, we would expect all of the participants to align exclusively with one mindset or the other. It could be an explicit alignment, as in the case of Carl and Clarice, or in a more subtle nature, as in the case of Sofia. However, that we have students such as Nathan and Damien indicates that, indeed, the concept of mindsets should be treated as a spectrum, as argued in Turner et al. (1998) and Dweck (2006). As it pertains to the latter, Dweck indicated that "many people have elements of both [mindsets]" and furthermore, "whatever mindset people have in a particular area will guide them in that area" (p. 47). This appeared to be particularly true for Nathan, whose desire to demonstrate competence seemed to motivate the fixed aspects of his mindset. Moreover, I would argue that even Damien and Nathan lie on different parts of the spectrum, despite the fact that they both possess what I called a "mixed mindset." This is because although both possess a mixed mindset, the fixed aspects of Nathan's mindset appear to be motivated by his desire to perform. Otherwise, Nathan's self-reported affect and counting activity were indicative of a growth mindset. Damien's mindset, on the other hand, wavered, without an apparent cause, between a growth mindset and a fixed mindset. Thus, this study not only serves to demonstrate that the concept of mindset should be treated as a spectrum, but it also serves to establish the complex nature of this spectrum and some of the nuances in students' language and activity that can inform where on the spectrum they might lie.

6.3 Insights into Mindset and Combinatorial Activity

Earlier, the importance of understanding mindsets was noted because of its powerful influence on affective factors such as confidence, perception of mistakes, and view of mathematics. Because of the nature of combinatorics, I argue that understanding mindsets is particularly useful in the teaching and learning of combinatorics. Specifically, because counting has been described as "one of the most difficult mathematical topics to teach and to learn" (Eizenberg & Zaslavsky, 2004, p. 16), and in addition, as "strewn with pitfalls" (Hadar & Hadass, 1981, p. 435), it is especially important to recognize how these results extend to field of combinatorics. For instance, let us examine the case of Carl. In Carl, we see a student who seeks to find errors in his work, correct these mistakes, and use the information to evaluate his current trajectory. Thus, he was able to identify and self-correct errors in his work, employ effective problem solving strategies, develop confidence in his current abilities as well as capabilities to improve, and develop a positive disposition towards mathematics. So, in Carl, we can see the advantages of leveraging mistakes and encouraging a growth mindset in students. This is unlike Clarice who avoids acknowledging her mistakes, rejects assistance due to her perception of it as an attack on her ability, and lacks confidence in her capabilities to improve. This is significant as she will likely not learn from her mistakes on her own accord nor will she learn to leverage mistakes with the aid of others. She truly does not have any hope of improvement unless she is to begin changing her mindset, as mentioned earlier.

Furthermore, as it pertains to the teaching and learning of enumerative combinatorics, in particular, this study also affords insight as to why students find counting challenging. Though literature has recognized students' difficulties, this study highlights specific aspects they appear to struggle with. In fact, these struggles were evident in both the task-based portion of the interview as well as in their reflections of prior experiences. For example, many of the students we interviewed actually voiced a desire for more structure. While attempting to solve various counting problems, several students, especially Clarice, were prone to trying counting formulas at the problems without understanding. Perhaps this lack of structure in enumerative combinatorics (Annin & Lai, 2010; Martin, 2001) makes students more likely to resort to such an approach. Specifically, the lack of structure in counting problems, resulting confusion, and readily available formulas provide students with fixed mindsets opportunities to produce answers effortlessly. However, as one can see, this approach does not often yield correct answers. As a result, I hypothesize that the negative beliefs held by individuals with a fixed mindset are likely to be reinforced. In addition, as a result of this study, we also see that students struggle with, among other things, understanding concepts of permutations and combinations, understanding and interpreting problem statements, and ensuring all of the outcomes have been counted.

These findings suggest implications for the teaching and learning of combinatorics, namely that researchers and instructors should be aware of how strongly mindsets, attitudes, and beliefs can affect students' interactions with counting problems.

In summary, there are many significant results afforded by this particular study. These results are not only pertinent to all of mathematics education, but also to the teaching and learning of combinatorics. As it pertains to the research questions posed in Chapter 1, and, in particular, my second research question, we see that in this study alone, there is ample evidence to support that the concept of mindset is not a dichotomy, but rather, a spectrum. Related to my first research question, we can also see how students' self-reported attitude and beliefs manifest themselves in their counting activity to provide a clear and accurate portrayal of their mindset. In turn, this study demonstrated the significance of understanding and molding students' mindset in mathematics, and particularly, in combinatorics. For students like Clarice, and in such a notoriously difficult and pitfall strewn subject, it is importance to emphasize assistance as a means to gain understanding, which, rather than performance, should be the true objective, as well as one's capability to improve with leveraging mistakes as one of many primary avenues for doing so, which addresses my final research question.

Chapter 7: Conclusion

This study sought to fill in some of the gaps found in previous empirical research surrounding mistakes, student affect, combinatorics, and the intersection of these domains. In particular, I desired to leverage the concepts of mindsets (Boaler & Dweck, 2016; Dweck, 2006) and achievement goals (Elliot & Dweck, 1988; Turner et al., 1998) as a framework to analyze student affect towards mistakes in the context of counting. As a result of these efforts, we are able to discern particular struggles students face as they attempt to learn counting, as well as the importance of understanding mindsets, especially in the context of counting. As it was discussed in the previous chapter, there is much to benefit from teaching students to leverage mistakes, and in addition, help pushing individuals with a fixed mindset along the spectrum, towards a growth mindset. In addition, knowing specific aspects of counting that students find challenging can be helpful for educators as they can be particularly mindful in the classroom. Though this study demonstrates the importance of these results, it does not make any claims as to what would be appropriate ways in which to achieve them.

In addition, this study also afforded some research implications. In particular, this study provides significant evidence that suggests the immense impact mindset can have on a student's combinatorial problem solving process. Thus, while it is important to examine student thinking about counting, it also appears imperative to consider affect and mindsets when designing and conducting research. These influential factors should not be disregarded as it could lead to the preclusion of a comprehensive understanding or other vital insights. This study achieved its purpose in beginning to fill some of the aforementioned gaps in previous research, but there is more research to be done in this area of mathematics education research.

For example, future research could build off of this study by identifying appropriate methods to not only teach students the significance of leveraging mistakes in the context of counting, but also how to leverage errors in combinatorics, specifically. Again, given that counting is “strewn with pitfalls” (Hadar & Hadass, 1981), this would be an ideal setting to engage in such work. Furthermore, altering a students’ perception of mistakes is one step in shifting their mindset as well. In fact, this potential research could benefit undergraduate mathematics education research, as was the case in this study, but also in other mathematical contexts, or with younger participants.

Aside from the aforementioned research implications, this study also concedes several teaching implications. In particular, due to the demonstrated importance of leveraging mistakes in enumerative combinatorics, it might be worthwhile for educators to inform students of the potential benefits mistakes can yield. For instance, perhaps explicitly discussing the value of mistakes, facilitating dialogue about mindsets, or providing students with opportunities to evaluate errors in counting problems and learn from their mistakes. As it pertains to the latter, it might be beneficial to manufacture situations in which students are able to learn from mistakes. This might be letting students do test corrections, stipulating that they must indicate why their original answer was incorrect and why their new answer is correct, creating an assignment in which they identify and correct common counting errors, or providing them a challenging problem that necessitates persistence, cognitive strategies, and metacognitive strategies. However, if students are provided with a challenge problem, there must also be aid available so as to avoid reinforcing beliefs typical of a student with a fixed mindset. It is not only important to let them learn from their mistakes and struggles, but also discuss with them the significance of their efforts.

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