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Title:

On Shortest Disjoint Paths and Hamiltonian Cycles in Some Interconnection Networks

Abstract approved: ______________________________

Bella Bose

Parallel processors are classified into two classes: shared-memory multiprocessors and distributed-memory multiprocessors. In the shared-memory system, processors communicate through a common memory unit. However, in the distributed multiprocessor system, each processor has its own memory unit and the communications among the processors are performed through an interconnection network. Thus, the interconnection topology plays an important role in the performance of these parallel systems.

Recently, some new classes of interconnection networks, referred as Gaussian and Eisenstein-Jacobi networks, have been introduced. In this dissertation, we study the problem of finding the shortest node disjoint paths in the Gaussian and the Eisenstein-Jacobi networks. Moreover, we also describe how to generate edge disjoint Hamiltonian cycles in Eisenstein-Jacobi and Generalized Hypercube networks. Node disjoint paths are paths between any given source and destination nodes such that the paths have no common nodes except the endpoints. Similarly, edge disjoint Hamiltonian cycles are cycles in a given graph where each node is visited once and returns to the starting node and every edge is in at most one cycle.

On Shortest Disjoint Paths and Hamiltonian Cycles in Some Interconnection Networks

by

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APPROVED:

Major Professor, representing Computer Science

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Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Zaid Hussain, Author

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Chapter 1

Introduction

Parallel processors are classified into two classes: shared-memory multiprocessors and distributed-memory multiprocessors. In the shared-memory system, processors communicate through a common memory unit. However, in the distributed multiprocessor system, each processor has its own memory unit and the communications among the processors are performed through an interconnection network. Thus, the interconnection topology plays an important role in the performance of these parallel systems. Many types of interconnection topologies have been proposed in the past, Hypercube [20], k-ary n-cube [34], torus [13], De Bruijn [33], etc.

Recently, two new classes of interconnection networks, Gaussian [30][28] and Eisenstein-Jacobi (EJ) [29][16] networks, have been introduced, and these networks have better diameter than the known toroidal networks [13]. The formal definitions of these networks are given in Chapter 2. The Gaussian networks are based on the quotient rings of Gaussian integers and are regular symmetric networks of degree four. Similarly, the EJ networks are based on the quotient rings of EJ integers and are regular symmetric networks of degree six. The hexagonal network is a special case of EJ network.

In addition, another class of interconnection topology, called Generalized Hypercube [9], has been proposed. It is a regular graph of degree $\sum_{i=0}^{n-1}(D_i - 1)$, where n is the number of dimensions and D_i is the size of dimension i. The formal definition of this network is given in Chapter 4.

In this dissertation, we address problems related to finding node disjoint paths and generating edge disjoint Hamiltonian cycles in these classes of interconnection networks. In the rest of this chapter, we summarize the research problems to be discussed in this dissertation.

1.1 All Shortest Node Disjoint Paths in Gaussian and EJ networks

Given two distinct nodes s and d of a graph $G = (V, E)$ where V is the set of nodes and E is the set of edges, two distinct paths from node s to node d are said to be node disjoint paths if and only if they have no common node except the endpoints. A node-to-node disjoint paths problem is to find all node disjoint paths from the given source node s to the destination node d. The problem of finding the shortest node disjoint paths considers that all paths are chosen to be the shortest paths among all other paths. Routing using node disjoint paths improves the throughput and the reliability of the network since it avoids congestion, increases the transmission rate, and provides fault-tolerance [31]. For example, disjoint paths can be used to divide a huge packet into smaller pieces where each piece can be sent through different node disjoint paths in order to avoid contention and increase the transmission rate. In Chapter 2, we investigate the problem of finding all shortest node disjoint paths in Gaussian and EJ networks.

1.2 Edge Disjoint Hamiltonian Cycles in EJ and GHC networks

Hamiltonian cycle (HC) in a graph is a cycle that visits each node exactly once and returns to the starting point. The problem of edge disjoint HC in a graph is finding the maximum number of HCs so that an edge can be in at most one HC, i.e., the edge that is used in one HC is not used in other HC. Edge disjoint Hamiltonian cycles are used in some efficient communication algorithms, such as the all-to-all communication [23][15] where each node broadcasts its message to all other nodes in the network. This problem is already presented in [16] for EJ networks when $\gcd(a, b) = 1$. In Chapter 3, we present our solution to this problem when $\gcd(a, b) > 1$. In addition, Chapter 4 describes how to construct edge disjoint Hamiltonian cycles in GHC.

1.3 Dissertation Structure

This dissertation is organized as follows.

Chapter 2 is divided into five sections. In Section 2.1 and 2.3, we review some topological and distance properties of both Gaussian and Eisenstein-Jacobi networks, respectively. Based on their distance properties, Section 2.2 and 2.4 introduce the solution for finding the shortest node disjoint paths, respectively, in Gaussian and Eisenstein-Jacobi networks. The chapter is conculded in Section 2.5.

In Chapter 3, Section 3.1 reviews the problem of finding edge disjoint Hamiltonian cycles in Eisenstein-Jacobi networks, which are generated by $\alpha = a + b\rho$, when $\gcd(a, b) = 1$. In Section 3.2, we introduce the rectangular representation for Eisenstein-Jacobi networks. In addition, using the rectangular representation, we demonstrate how to generate three edge disjoint Hamiltonian cycles for these networks when $\gcd(a, b) > 1$. Section 3.3 is the conclusion of the chapter.

Chapter 4 is divided into four sections. Section 4.1 reviews some topological properties of Generalized Hypercube networks. In Section 4.2, we illustrate how to construct edge disjoint Hamiltonian cycles in Generalized Hypercube when the number of dimensions is $n = 2^r$, $r \geq 0$, and all the dimensions size are p, p is a prime ≥ 3. In addition, these cycles are described in terms of Gray codes. Section 4.3 describes how to generate these cycles when the number of dimensions is $n \geq 1$, and all the dimension sizes ≥ 3. In both Sections, 4.2 and 4.3, we assume that all the dimensions have the same size. The chapter is concluded in Section 4.4.

Finally, Chapter 5 discusses some open research problems related to the topics discussed in this dissertation.

Chapter 2

Shortest Node Disjoint Paths in Gaussian and EJ Networks

As previously explained in the introduction, in this problem, the paths are chosen to be shortest and they have no common node except the endpoints. Using these node disjoint paths it is possible to improve the throughput of the network and also to provide fault-tolerance. Solutions to this problem are given for other types of network in [12][11][24][35][17][18][27][26].

Recently, two new classes of interconnection networks were introduced, Gaussian [30][28] [16] and Eisenstein-Jacobi [16]. In this chapter, first we review some topological properties of both Gaussian and Eisenstein-Jacobi networks. After that, we describe the solutions for finding all shortest node disjoint paths in these networks based on their distance distribution properties.

2.1 Gaussian Networks

2.1.1 Quotient Rings of Gaussian Integers

The *Gaussian integer* $\mathbb{Z}[i]$, as described in [28][19][21], is the subset of complex numbers with integer real and integer imaginary parts, that is

$$\mathbb{Z}[i] = \{x + yi \mid x, y \in \mathbb{Z}\} \tag{2.1}$$

where $i = \sqrt{-1}$ and $\mathbb{Z}[i]$ is a Euclidean domain and the norm of a Gaussian integer $\alpha = a+bi$ is known as $N(\alpha) = a^2 + b^2$. Then for $\alpha \neq 0$ and $\alpha, \beta \in \mathbb{Z}[i]$ there exist $q, r \in \mathbb{Z}[i]$ such that $\beta = q\alpha + r$ with $N(r) < N(\alpha)$. This means that there exists a Euclidean division algorithm for Gaussian integers analogous to that of the rational integers. Usually, the set of remainders of the division by any integer N is denoted as $\mathbb{Z}_N$, which is called the integers modulo N. Analogously, we can consider $\mathbb{Z}[i]_\alpha$, i.e., the Gaussian integers modulo α. Given a Gaussian integer $0 \neq \alpha = a + bi$, it is well known that $N(\alpha) = a^2 + b^2$ is equal to the number of residue classes modulo α and that various representations of these residue classes are given in [22].

2.1.2 Definition and Topological Properties of Gaussian Network

Gaussian networks are based on quotient rings of Gaussian integers and defined as follows.

Definition 2.1 ([28]). Given $0 \neq \alpha = a + bi \in \mathbb{Z}[i]$ with $0 \leq a \leq b$, the Gaussian network is defined as a graph $G_\alpha(V, E)$ where:

1. $V = \mathbb{Z}[i]_\alpha$ is the node set.
2. $E = \{(\beta, \gamma) \in V \times V \mid (\beta - \gamma) \equiv \pm 1, \pm i \ mod\ \alpha\}$ is the edge set as shown in Figure 2.1.

we refer G_α as the Gaussian network generated by α.

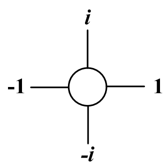

Figure 2.1: Edges for Each Node in G_α.

Gaussian networks are regular symmetric networks of degree four, i.e., each node has four neighbors. The total number of nodes in the network is as $N(\alpha) = a^2 + b^2$. Each node in the network is labeled by $x + yi$. The nodes A and B are said to be adjacent, i.e., neighbors, if and only if $(A - B)\ mod\ \alpha$ is equal to ± 1 or $\pm i$.

The distance between any two nodes β and γ in the network is defined as

$$D_\alpha(\beta, \gamma) = \min\{|x| + |y| \mid (\beta - \gamma) \equiv x + yi\ (mod\ \alpha)\} \tag{2.2}$$

Since G_α is vertex-symmetric the weight of vertex β, which is the distance of this node from node 0, is defined as

$$WT(\beta) = \min\{|x| + |y| \mid (\beta) \equiv x + yi \ (mod\ \alpha)\} \tag{2.3}$$

The following theorems describe the number of nodes at a distance s, for $s = 0, 1, \ldots, k$, where k is the diameter of the network. This number is denoted as $WT_G(s)$. We refer to [28] for the proofs.

Theorem 2.1 ([28])**.** *Let $0 \neq \alpha = a + bi \in \mathbb{Z}[i]$ such that $0 \leq a \leq b$, $N(\alpha) = a^2 + b^2$ an odd integer, and $t = \frac{a+b-1}{2}$. The distance distribution of the graph G_α is as follows:*

$$WT_G(s) = \begin{cases} 1 & if\ s = 0 \\ 4s & if\ 0 < s \leq t \\ 4(b-s) & if\ t < s < b \end{cases} \tag{2.4}$$

Theorem 2.2 ([28])**.** *Let $0 \neq \alpha = a + bi \in \mathbb{Z}[i]$ be such that $0 \leq a \leq b$, $N(\alpha) = a^2 + b^2$ an even integer, and $t = \frac{a+b}{2}$. When $a < b$, the distance distribution of the graph G_α is as follows:*

$$WT_G(s) = \begin{cases} 1 & if\ s = 0 \\ 4s & if\ 0 < s < t \\ 2(b-1) & if\ s = t \\ 4(b-s) & if\ t < s < b \\ 1 & if\ s = b \end{cases} \tag{2.5}$$

when $0 < a = b$, the distance distribution of the graph G_{b+bi} is as follows:

$$WT_G(s) = \begin{cases} 1 & if\ s = 0 \\ 4s & if\ 0 < s < b \\ 2b-1 & if\ s = b \end{cases} \tag{2.6}$$

Based on the above theorems we conclude that the diameter of the network is defined as follows.

Corollary 2.1 ([28]). *Let $0 \neq \alpha = a + bi \in \mathbb{Z}[i]$ be such that $0 \leq a \leq b$. Let $N(\alpha) = a^2 + b^2$ be the norm of α. The diameter k of the Gaussian graph G_{a+bi} is:*

$$k = \begin{cases} b & if\ N(\alpha)\ is\ even \\ b-1 & if\ N(\alpha)\ is\ odd \end{cases} \tag{2.7}$$

Figure 2.2 illustrates the tilling and the modulo operation in Gaussian network generated with $\alpha = 3 + 4i$, which is helpful in computing the shortest node disjoint paths in the next section.

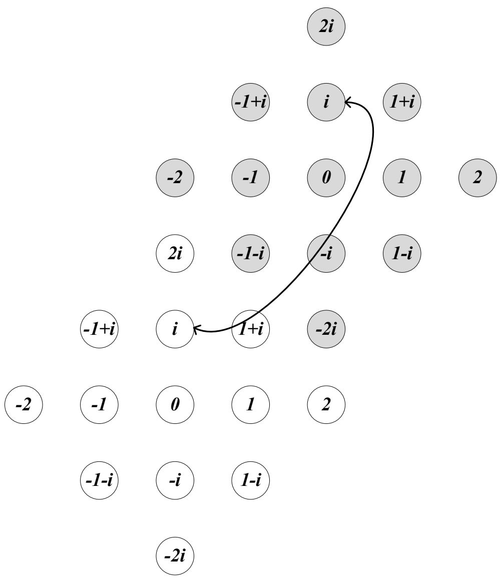

Figure 2.2: Tilling and Modulo Operation in Gaussian Generated with $\alpha = 3 + 4i$.

Figure 2.3 shows an example of a Gaussian network with $\alpha = 5 + 6i$. From the figure, it is obvious that we can divide the graph into four quadrants, Q_1, Q_2, Q_3 and Q_4. i.e., a node $\beta = x + yi$ in Q_1 when $x \geq 0$ and $y > 0$, in Q_2 when $x < 0$ and $y \geq 0$, in Q_3 when $x \leq 0$ and $y < 0$, and in Q_4 when $x > 0$ and $y \leq 0$, where the black node is node 0.

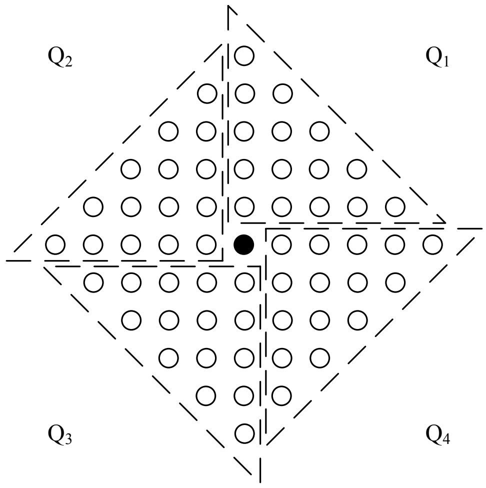

Figure 2.3: Quadrants in Gaussian Network with $\alpha = 5 + 6i$.

2.2 Shortest Node Disjoint Paths in Gaussian Networks

In this section, we consider how to construct shortest node disjoint paths in Gaussian networks. Suppose that the Gaussian network is generated by $\alpha = a + bi$ where $0 \le a \le b$. Let $\beta_1 = x_1 + y_1 i$ and $\beta_2 = x_2 + y_2 i$ be the given source and destination nodes, respectively. Our aim is to find the maximum number of node disjoint paths from β_1 to β_2. In order to do this, let $\beta_2 - \beta_1 = (x_2 - x_1) + (y_2 - y_1)i = x + yi$; then the distance from β_1 to β_2 is $|x| + |y|$ [28]. One of the shortest paths from β_1 to β_2 can be obtained by going x steps along the x dimension and then y steps along the y dimension. We first show the disjoint paths assuming the source node $\beta_1 = 0$ and the destination node $\beta_2 \neq 0$. When $\beta_1 \neq 0$, the node disjoint paths can be easily obtained from the paths 0 to $\beta_2 - \beta_1$. The following two theorems show that there are four node disjoint paths. The value of each node in every path must be computed by taking *mod* α.

Theorem 2.3. *Suppose the generator of a Gaussian network is* $\alpha = a + bi$ *with* $0 \le a \le b$. *Let* 0 *and* $\beta = x + yi$, *where* $|x| > 0$ *and* $|y| > 0$, *be the source and destination nodes, respectively. Then there are four node disjoint paths from* 0 *to* β, *with path lengths at most* $d + 4$, *where* $d = |x| + |y|$ *is the minimum distance from* 0 *to* β.

Proof. The four node disjoint paths are given as follows:

First quadrant, when $x > 0$ and $y > 0$:

$P_1 = 0, i, 2i, \ldots, yi, 1 + yi, 2 + yi, \ldots, x + yi.$

$P_2 = 0, 1, 2, \ldots, x, x + i, x + 2i, \ldots, x + yi.$

$P_3 = 0, -1, -1 + i, -1 + 2i, \ldots, -1 + (y + 1)i, (y + 1)i, 1 + (y + 1)i, \ldots, x + (y + 1)i, x + yi.$

$P_4 = 0, -i, 1 - i, 2 - i, \ldots, (x + 1) - i, x + 1, (x + 1) + i, \ldots, (x + 1) + yi, x + yi.$

Denoting $|P_i|$ as the length of the path P_i, then $|P_1| = |P_2| = d$ and $|P_3| = |P_4| = d + 4$, where d is the shortest path from 0 to β. The same proof applies to the other quadrants when we take x and y into consideration to the correspondent quadrants. □

Example 2.1. Let $\alpha = 5+6i$, the source node $S = 0$, and the destination node $D = 2+3i$. Figure 2.4 illustrates the four node disjoint paths from S to D as follows:

$P_1 = 0, i, 2i, 3i, 1 + 3i, 2 + 3i.$

$P_2 = 0, 1, 2, 2 + i, 2 + 2i, 2 + 3i.$

$P_3 = 0, -1, -1 + i, -1 + 2i, -1 + 3i, -1 + 4i, 4i, 1 + 4i, 2 + 4i \equiv -3 - 2i \ (mod\ \alpha), 2 + 3i.$

$P_4 = 0, -i, 1 - i, 2 - i, 3 - i, 3, 3 + i, 3 + 2i, 3 + 3i \equiv -2 - 3i \ (mod\ \alpha), 2 + 3i.$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 5$, and $|P_3| = |P_4| = 9$.

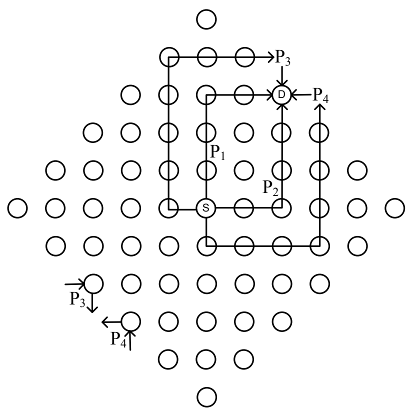

Figure 2.4: Node Disjoint Paths in Gaussian Network with $\alpha = 5 + 6i$ from $S = 0$ to $D = 2 + 3i$.

Theorem 2.4. *Suppose the generator of a Gaussian network is $\alpha = a + bi$ with $0 \leq a \leq b$. Let 0 and $\beta = x + yi$, where $x = 0$ and $|y| > 0$ for the first and third quadrants (or $|x| > 0$ and $y = 0$ for the second and fourth quadrants), be the source and destination* *nodes, respectively. Then there are four node disjoint paths from* 0 *to* β*, with path lengths at most* $d+8$*, where* $d = |x| + |y|$ *is the minimum distance from* 0 *to* β*.*

Proof. The four node disjoint paths are given as follows:

First quadrant, when $x = 0$ and $y > 0$:

$P_1 = 0, i, 2i, \ldots, yi.$

$P_2 = 0, -1, -1+i, -1+2i, \ldots, -1+yi, yi.$

$P_3 = 0, 1, 1+i, 1+2i, \ldots, 1+yi, yi.$

$P_4 = 0, -i, -1-i, -2-i, -2, -2+i, -2+2i, \ldots, -2+(y+1)i, -1+(y+1)i, (y+1)i, yi.$

Denoting $|P_i|$ as the length of the path P_i, then $|P_1| = d$, $|P_2| = |P_3| = d+2$, and $|P_4| = d+8$ where d is the shortest path from 0 to β. When $\beta = x + yi$ is in other quadrants, the proofs are similar to this one. □

Example 2.2. Let $\alpha = 5 + 6i$, the source node $S = 0$, and the destination node $D = 5i$. Figure 2.5 describes the four node disjoint paths from S to D as follows:

$P_1 = 0, i, 2i, 3i, 4i, 5i.$

$P_2 = 0, 1, 1+i, 1+2i, 1+3i, 1+4i, 1+5i \equiv -4-i \ (mod\ \alpha), 5i.$

$P_3 = 0, -1, -1+i, -1+2i, -1+3i, -1+4i, -1+5i \equiv 5 \ (mod\ \alpha), 5i.$

$P_4 = 0, -i, -1-i, -2-i, -2, -2+i, -2+2i, -2+3i, -2+4i \equiv 4-i \ (mod\ \alpha), -2+5i \equiv 4 \ (mod\ \alpha), -2+6i \equiv 4+i \ (mod\ \alpha), -1+6i \equiv -5i \ (mod\ \alpha), 6i \equiv -5 \ (mod\ \alpha), 5i.$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = 5$, $|P_2| = |P_3| = 7$, and $|P_4| = 13$.

In some cases, the maximum length of some paths can be reduced. These are described next. Two cases are considered, corresponding to when $N(\alpha)$ is *odd* and *even*. Additionally, two sub-cases, for each of *odd* and *even*, when $|x|, |y| > 0$ and $x = 0$, $|y| > 0$ for the first and third quadrants (or $|x| > 0$, $y = 0$ for the second and fourth quadrants) are described in the following theorems.

Theorem 2.5. *Suppose the generator of a Gaussian network is* $\alpha = a + bi$ *with* $0 \leq a \leq b$ *and* $N(\alpha) = a^2 + b^2$ *is odd. Let* 0 *and* $\beta = x + yi$*, where* $|x| > 0$ *and* $|y| > 0$*, be the source and destination nodes, respectively. Then there are four node disjoint paths from* 0 *to* β*,*

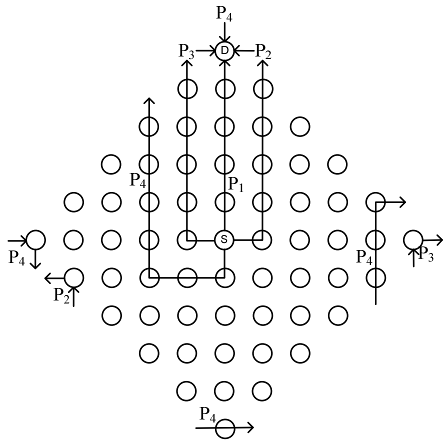

Figure 2.5: Node Disjoint Paths in Gaussian Network with $\alpha = 5 + 6i$ from $S = 0$ to $D = 5i$.

with path lengths strictly less than $d + 4$*, where* $d = |x| + |y|$ *is the minimum distance from* 0 *to* β*.*

Proof. Let $t = \frac{a+b-1}{2}$. The maximum length node disjoint paths can be reduced according to the following cases:

1. First quadrant, when $x > 0$ and $y > 0$:

   1.1) If $b > a + 1$ and $x < a$ and $t - x < 1.5$ and $t - d < 1.5$ and $4t - 2x + 2 < 2(4t - d - 2a + 2)$ and $4t - 2x + 2 < 2(2t - x + y + 3)$ and $4t - 2x + 2 < 2(2t - d + 1)$ then

   $P_3 = 0, -1, -1 + i, \ldots, -1 + (a + y)i, -2 + (a + y)i, \ldots, -(2t - x - a + 1) + (a + y)i.$

   $P_4 = 0, -i, -1 - i, \ldots, -(a - x) - i, -(a - x) - 2i, \ldots, -(a - x) - (2t - y - a + 1)i.$

   1.2) If $b > a + 1$ and $x \geq a$ and $t - x < 1.5$ and $t - y - a < 1.5$ and $4t - 2a + 2 < 2(4t - d - 2a + 2)$ and $4t - 2a + 2 < 2(2t - x + y + 3)$ and $4t - 2a + 2 < 2(2t + x - y - 2a + 3)$ then

   $P_3 = 0, -1, -1 + i, \ldots, -1 + (a + y)i, -2 + (a + y)i, \ldots, -(2t - x - a + 1) + (a + y)i.$

   $P_4 = 0, -i, 1 - i, \ldots, (x - a) - i, (x - a) - 2i, \ldots, (x - a) - (2t - y - a + 1)i.$

   1.3) If $b > a + 1$ and $x < a$ and $t - d < 1.5$ and $2t - d + 1 < 2t - x + y + 3$ and $2t - d + 1 < 4t - d - 2a + 2$ then

$P_3 = 0, -1, -2, \ldots, -(a-x), -(a-x)-i, \ldots, -(a-x)-(2t-y-a+1)i.$

$P_4 = 0, -i, -2i, \ldots, -(2t-y-a+1)i, -1-(2t-y-a+1)i, \ldots, -(a-x)-(2t-y-a+1)i.$

1.4) If $b > a+1$ and $x \geq a$ and $t-y-a < 0.5$ and $2t+x-y-2a+3 < 2t-x+y+3$ and $2t+x-y-2a+3 < 4t-d-2a+2$ then

$P_3 = 0, -1, -1-i, -1-2i, -2i, 1-2i, \ldots, (x-a)-2i, (x-a)-3i, \ldots, (x-a)-(2t-y-a+1)i$.

$P_4 = 0, -i, 1-i, 2-i, \ldots, (x-a+1)-i, (x-a+1)-2i, \ldots, (x-a+1)-(2t-y-a+1)i, (x-a)-(2t-y-a+1)i.$

1.5) If $b > a+1$ and $t-x < 0.5$ and $2t-x+y+3 < 4t-d-2a+2$ then

$P_3 = 0, -1, -1+i, -1+2i, \ldots, -1+(a+y+1)i, -2+(a+y+1)i, \ldots, -(2t-x-a+1)+(a+y+1)i, -(2t-x-a+1)+(a+y)i.$

$P_4 = 0, -i, -1-i, -2-i, -2, -2+i, \ldots, -2+(a+y)i, -3+(a+y)i, \ldots, -(2t-x-a+1)+(a+y)i.$

1.6) If $b = a+1$ and $t-d < 1.5$ then

$P_3 = 0, -1, -2, \ldots, -(t-x), -(t-x)-i, \ldots, -(t-x)-(t-y+1)i.$

$P_4 = 0, -i, -2i, \ldots, -(t-y+1)i, -1-(t-y+1)i, \ldots, -(t-x)-(t-y+1)i.$

1.7) If $b > a+1$ and $2t-d-a < 1$ then

$P_3 = 0, -1, -2, \ldots, -(2t-x+1), -(2t-x+1)-i, \ldots, -(2t-x+1)-(2t-y-2a+1)i.$

$P_4 = 0, -i, -2i, \ldots, -(2t-y-2a+1)i, -1-(2t-y-2a+1)i, \ldots, -(2t-x+1)-(2t-y-2a+1)i.$

The same proof applies to the other quadrants when we take x and y into consideration to the correspondent quadrants. □

Example 2.3. Let $\alpha = 3+8i$, the source node $S = 0$, and the destination node $D = 5+2i$. Then, $t = 5$ and $d = 7$. Since the condition of case 1.2 in Theorem 2.5 is satisfied, the paths are:

$P_1 = 0, i, 2i, 1+2i, 2+2i, 3+2i, 4+2i, 5+2i.$

$P_2 = 0, 1, 2, 3, 4, 5, 5+i, 5+2i$.

$P_3 = 0, -1, -1+i, -1+2i, -1+3i, -1+4i, -1+5i, -2+5i, -3+5i \equiv 5+2i \ (mod \ \alpha)$.

$P_4 = 0, -i, 1-i, 2-i, 2-2i, 2-3i, 2-4i, 2-5i, 2-6i \equiv 5+2i \ (mod \ \alpha)$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 7$ and $|P_3| = |P_4| = 8$. Figure 2.6 illustrates the shortest node disjoint paths from S to D.

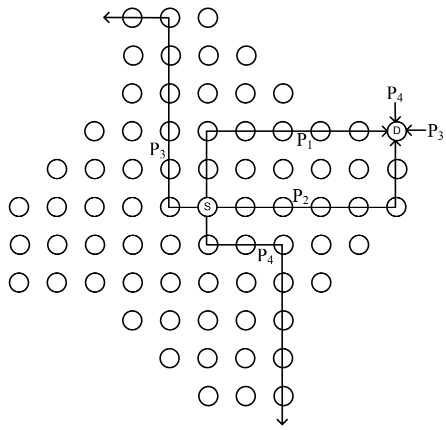

Figure 2.6: Shortest Node Disjoint Paths in Gaussian Network with $\alpha = 3+8i$ from $S = 0$ to $D = 5+2i$.

Example 2.4. Let $\alpha = 5+6i$, the source node $S = 0$, and the destination node $D = 2+3i$. Then, $t = 5$ and $d = 5$. Since the condition of case 1.6 in Theorem 2.5 is satisfied, the paths are:

$P_1 = 0, i, 2i, 3i, 1+3i, 2+3i$.

$P_2 = 0, 1, 2, 2+i, 2+2i, 2+3i$.

$P_3 = 0, -1, -2, -3, -3-i, -3-2i, -3-3i \equiv 2+3i \ (mod \ \alpha)$.

$P_4 = 0, -i, -2i, -3i, -1-3i, -2-3i, -3-3i \equiv 2+3i \ (mod \ \alpha)$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 5$ and $|P_3| = |P_4| = 6$. Figure 2.7 illustrates the shortest node disjoint paths from S to D.

Example 2.5. Let $\alpha = 3+8i$, the source node $S = 0$, and the destination node $D = 4+2i$. Then, $t = 5$ and $d = 6$. Since the condition of case 1.4 in Theorem 2.5 is satisfied, the paths are:

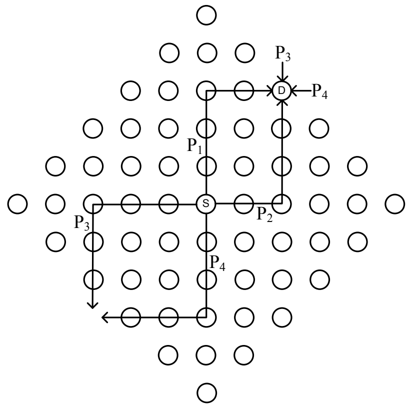

Figure 2.7: Shortest Node Disjoint Paths in Gaussian Network with $\alpha = 5 + 6i$ from $S = 0$ to $D = 2 + 3i$.

$P_1 = 0, i, 2i, 1 + 2i, 2 + 2i, 3 + 2i, 4 + 2i.$

$P_2 = 0, 1, 2, 3, 4, 4 + i, 4 + 2i.$

$P_3 = 0, -1, -1 - i, -1 - 2i, -2i, 1 - 2i, 1 - 3i, 1 - 4i, 1 - 5i, 1 - 6i \equiv 4 + 2i\ (mod\ \alpha).$

$P_4 = 0, -i, 1-i, 2-i, 2-2i, 2-3i, 2-4i, 2-5i, 2-6i \equiv 5+2i\ (mod\ \alpha), 1-6i \equiv 4+2i\ (mod\ \alpha).$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 6$ and $|P_3| = |P_4| = 9$. Figure 2.8 illustrates the shortest node disjoint paths from S to D.

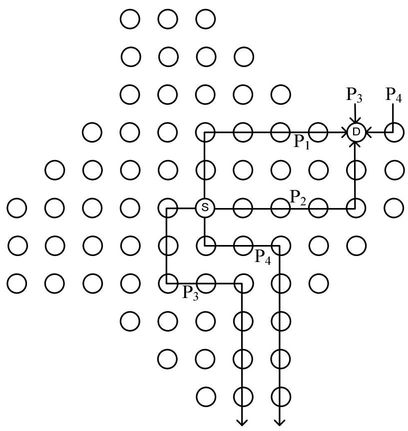

Figure 2.8: Shortest Node Disjoint Paths in Gaussian Network with $\alpha = 3 + 8i$ from $S = 0$ to $D = 4 + 2i$.

Example 2.6. Let $\alpha = 3 + 8i$, the source node $S = 0$, and the destination node $D = 5 + i$. Then, $t = 5$ and $d = 6$. Since the condition of case 1.5 in Theorem 2.5 is satisfied, the paths are:

$P_1 = 0, i, 1+i, 2+i, 3+i, 4+i, 5+i$.

$P_2 = 0, 1, 2, 3, 4, 5, 5+i$.

$P_3 = 0, -1, -1+i, -1+2i, -1+3i, -1+4i, -1+5i, -2+5i, -3+5i \equiv 5+2i\ (mod\ \alpha), -3+4i \equiv 5+i\ (mod\ \alpha)$.

$P_4 = 0, -i, -1-i, -2-i, -2, -2+i, -2+2i, -2+3i, -2+4i, -3+4i \equiv 5+i\ (mod\ \alpha)$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 6$ and $|P_3| = |P_4| = 9$. Figure 2.9 illustrates the shortest node disjoint paths from S to D.

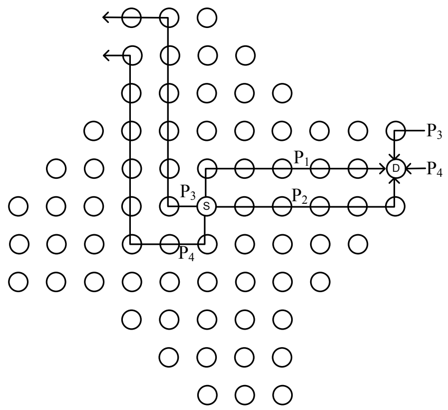

Figure 2.9: Shortest Node Disjoint Paths in Gaussian Network with $\alpha = 3+8i$ from $S = 0$ to $D = 5+i$.

Theorem 2.6. *Suppose the generator of a Gaussian network is $\alpha = a+bi$ with $0 \leq a \leq b$ and $N(\alpha) = a^2+b^2$ is odd. Let 0 and $\beta = x+yi$, where $x = 0$ and $|y| > 0$ for the first and third quadrants (or $|x| > 0$ and $y = 0$ for the second and fourth quadrants), be the source and destination nodes, respectively. Then there are four node disjoint paths from 0 to β, with path lengths strictly less than $d+8$, where $d = |x|+|y|$ is the minimum distance from 0 to β.*

Proof. Let $k = b-1$ be the network diameter and $t = \frac{a+b-1}{2}$. The maximum length of the paths can be reduced as follows.

1. <u>*First quadrant, when $x = 0$ and $y > 0$:*</u>

   1.1) If $y = k$ and $b = a+1$ then

   $P_2 = 0, -1, -2, \ldots, -a, -a-i$.

$P_3 = 0, 1, 2, \ldots, a+1.$

$P_4 = 0, -i, -1-i, -2-i, \ldots, -a-i.$

1.2) If $t - y < 3.5$ then

$P_4 = 0, -i, -1-i, -2-i, \ldots, -a-i, -a-2i, \ldots, -a-(2t-y-a+1)i.$

The same proof applies to the other quadrants when we take x and y into consideration to the correspondent quadrants. □

Example 2.7. Let $\alpha = 5+6i$, the source node $S = 0$, and the destination node $D = 5i$. Then, $t = 5$ and $d = 5$. Since the condition of case 1.1 in Theorem 2.6 is satisfied, the paths are:

$P_1 = 0, i, 2i, 3i, 4i, 5i.$

$P_2 = 0, -1, -2, -3, -4, -5, -5-i \equiv 5i \ (mod\ \alpha).$

$P_3 = 0, 1, 2, 3, 4, 5, 6 \equiv 5i \ (mod\ \alpha).$

$P_4 = 0, -i, -1-i, -2-i, -3-i, -4-i, -5-i \equiv 5i \ (mod\ \alpha).$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = 5$ and $|P_2| = |P_3| = |P_4| = 6$. Figure 2.10 illustrates the shortest node disjoint paths from S to D.

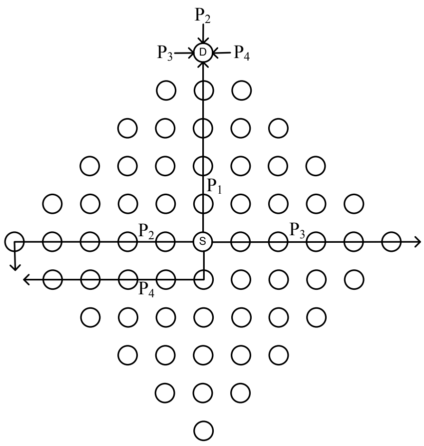

Figure 2.10: Shortest Node Disjoint Paths in Gaussian Network with $\alpha = 5+6i$ from $S = 0$ to $D = 5i$.

Example 2.8. Let $\alpha = 3+8i$, the source node $S = 0$, and the destination node $D = 5i$. Then, $t = 5$ and $d = 5$. Since the condition of case 1.2 in Theorem 2.6 is satisfied, the paths are:

$P_1 = 0, i, 2i, 3i, 4i, 5i.$

$P_2 = 0, -1, -1+i, -1+2i, -1+3i, -1+4i, -1+5i, 5i.$

$P_3 = 0, 1, 1+i, 1+2i, 1+3i, 1+4i, 1+5i \equiv -2-3i \ (mod\ \alpha), 5i.$

$P_4 = 0, -i, -1-i, -2-i, -3-i, -3-2i, -3-3i \equiv 5i \ (mod\ \alpha).$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = 5$, $|P_2| = |P_3| = 7$, and $|P_4| = 6$. Figure 2.11 illustrates the shortest node disjoint paths from S to D.

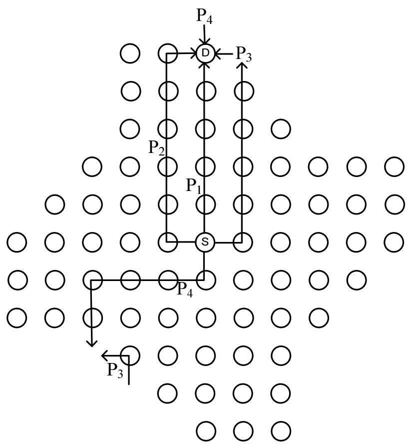

Figure 2.11: Shortest Node Disjoint Paths in Gaussian Network with $\alpha = 3+8i$ from $S = 0$ to $D = 5i$.

Theorem 2.7. *Suppose the generator of a Gaussian network is $\alpha = a+bi$ with $0 \le a \le b$ and $N(\alpha) = a^2+b^2$ is even. Let 0 and $\beta = x+yi$, where $|x| > 0$ and $|y| > 0$, be the source and destination nodes, respectively. Then there are four node disjoint paths from 0 to β, with path lengths strictly less than $d+4$, where $d = |x|+|y|$ is the minimum distance from 0 to β.*

Proof. Let $t = \frac{a+b}{2}$. The maximum length node disjoint paths can be reduced according to the following cases:

1. First quadrant, when $x > 0$ and $y > 0$:

   1.1) If $b \ge a+4$ and $x < a$ and $t-x < 2$ and $t-d < 2$ and $4t-2x < 2(2t-d)$ and $4t-2x < 2(2t-x+y+2)$ and $4t-2x < 2(4t-d-2a)$ then

$P_3 = 0, -1, -1+i, -1+2i, \ldots, -1+(a+y)i, -2+(a+y)i, \ldots, -(2t-x-a)+(a+y)i.$

$P_4 = 0, -i, -1-i, -2-i, \ldots, -(a-x)-i, -(a-x)-2i, \ldots, -(a-x)-(2t-y-a)i$

1.2) If $b \geq a+4$ and $x \geq a$ and $t-x < 2$ and $t-y-a < 2$ and $4t-2a < 2(2t+x-y-2a+2)$ and $4t-2a < 2(2t-x+y+2)$ and $4t-2a < 2(4t-d-2a)$ then

$P_3 = 0, -1, -1+i, -1+2i, \ldots, -1+(a+y)i, -2+(a+y)i, \ldots, -(2t-x-a)+(a+y)i.$

$P_4 = 0, -i, 1-i, 2-i, \ldots, (x-a)-i, (x-a)-2i, \ldots, (x-a)-(2t-y-a)i.$

1.3) If $b \geq a+4$ and $x < a$ and $t-d < 2$ and $2t-d < 2t-x+y+2$ and $2t-d < 4t-d-2a$ then

$P_3 = 0, -1, -2, \ldots, -(a-x), -(a-x)-i, \ldots, -(a-x)-(2t-y-a)i.$

$P_4 = 0, -i, -2i, \ldots, -(2t-y-a)i, -1-(2t-y-a)i, \ldots, -(a-x)-(2t-y-a)i.$

1.4) If $b \geq a+4$ and $x \geq a$ and $t-y-a < 1$ and $2t+x-y-2a+2 < 2t-x+y+2$ and $2t+x-y-2a+2 < 4t-d-2a$ then

$P_3 = 0, -1, -1-i, -1-2i, -2i, 1-2i, \ldots, (x-a)-2i, (x-a)-3i, \ldots, (x-a)-(2t-y-a)i.$

$P_4 = 0, -i, 1-i, 2-i, \ldots, (x-a+1)-i, (x-a+1)-2i, \ldots, (x-a+1)-(2t-y-a)i, (x-a)-(2t-y-a)i.$

1.5) If $b \geq a+4$ and $t-x < 1$ and $2t-x+y+2 < 4t-d-2a$ then

$P_3 = 0, -1, -1+i, -1+2i, \ldots, -1+(a+y+1)i, -2+(a+y+1)i, \ldots, -(2t-x-a)+(a+y+1)i, -(2t-x-a)+(a+y)i.$

$P_4 = 0, -i, -1-i, -2-i, -2, -2+i, \ldots, -2+(a+y)i, -3+(a+y)i, \ldots, -(2t-x-a)+(a+y)i.$

1.6) If $b \geq a+4$ and $3t-x-2y-2a < 4$ and $d = k$ then

$P_3 = 0, -1, -2, \ldots, -t, -t-i, \ldots, -t-(2t-y-2a)i.$

$P_4 = 0, -i, -2i, \ldots, -(2t-y-2a)i, -1-(2t-y-2a)i, \ldots, -t-(2t-y-2a)i.$

1.7) If $b \geq a+4$ and $2t-d-a < 2$ then

$P_3 = 0, -1, -2, \ldots, -(2t-x), -(2t-x)-i, \ldots, -(2t-x)-(2t-y-2a)i.$

$P_4 = 0, -i, -2i, \ldots, -(2t-y-2a)i, -1-(2t-y-2a)i, \ldots, -(2t-x)-(2t-y-2a)i.$

1.8) If $b = a + 2$ and $d = k$ and $t - d < 3$ then

$P_3 = 0, -1, -2, \ldots, -t, -t - i.$

$P_4 = 0, -i, -1 - i, -2 - i, \ldots, -t - i.$

1.9) If $b = a + 2$ and $x = t - 1$ and $y = 1$ and $t - d - y < 1$ then

$P_3 = 0, -1, -1 - i, -1 - 2i, -2i, \ldots, -(t - y + 1)i.$

$P_4 = 0, -i, 1 - i, 1 - 2i, \ldots, 1 - (t - y + 1)i, -(t - y + 1)i.$

1.10) If $b = a + 2$ and $t - d < 2$ then

$P_3 = 0, -1, -2, \ldots, -(t - x - 1), -(t - x - 1) - i, \ldots, -(t - x - 1) - (t - y + 1)i.$

$P_4 = 0, -i, -2i, \ldots, -(t - y + 1)i, -1 - (t - y + 1)i, \ldots, -(t - x - 1) - (t - y + 1)i.$

1.11) If $b = a$ and $t - d < 2$ then

$P_3 = 0, -1, -2, \ldots, -(t - x - 1), -(t - x - 1) - i, \ldots, -(t - x - 1) - (t - y + 1)i.$

$P_4 = 0, -i, -2i, \ldots, -(t - y + 1)i, -1 - (t - y + 1)i, \ldots, -(t - x - 1) - (t - y + 1)i.$

2. *Second quadrant, when $x < 0$ and $y > 0$:*

   The proof is similar to the first quadrant with the following cases: 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.9, 1.10, and 1.11.

3. *Third quadrant, when $x < 0$ and $y < 0$:*

   The proof is similar to the first quadrant with the following cases: 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.10, and 1.11.

4. *Fourth quadrant, when $x > 0$ and $y < 0$:*

   The proof is similar to the first quadrant with the following cases: 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.10, and 1.11.

The same proof of the first quadrant applies to the other quadrants when we take x and y into consideration to the correspondent quadrants. □

The paths in the above theorem are similar to the previous examples of odd case except that we take the even case of Gaussian networks into consideration for each quadrant.

Theorem 2.8. *Suppose the generator of a Gaussian network is $\alpha = a + bi$ with $0 \leq a \leq b$ and $N(\alpha) = a^2 + b^2$ is even. Let 0 and $\beta = x + yi$, where $x = 0$ and $|y| > 0$ for the first and third quadrants (or $|x| > 0$ and $y = 0$ for the second and fourth quadrants), be the source and destination nodes, respectively. Then there are four node disjoint paths from 0 to β, with path lengths strictly less than $d + 8$, where $d = |x| + |y|$ is the minimum distance from 0 to β.*

Proof. Let $k = b$ be the network diameter and $t = \frac{a+b}{2}$. The maximum length of the paths can be reduced as follows.

1. First quadrant, when $x = 0$ and $y > 0$:

   1.1) If $y = k$ and $b = a$ then

   $P_2 = 0, -1, -2, \ldots, -(t-1), -(t-1) - i.$

   $P_3 = 0, 1, 2, \ldots, t.$

   $P_4 = 0, -i, -1 - i, -2 - i, \ldots, -(t-1) - i.$

   1.2) If $y = k - 1$ and $b = a + 2$ then

   $P_2 = 0, -1, -2, \ldots, -(t-1), -(t-1) - i.$

   $P_3 = 0, 1, 1 + i, 2 + i, \ldots, (t+1) + i.$

   $P_4 = 0, -i, -1 - i, -2 - i, \ldots, -(t-1) - i.$

   1.3) If $t - y < 4$ and ($b = a$ or $b = a + 2$) then

   $P_4 = 0, -i, -1-i, -2-i, \ldots, -(t-1)-i, -(t-1)-2i, \ldots, -(t-1)-(t-y+1)i.$

   1.4) If $t - y < 4$ and $b \geq a + 4$ then

   $P_4 = 0, -i, -1 - i, -2 - i, \ldots, -a - i, -a - 2i, \ldots, -a - (2t - y - a)i.$

2. Second quadrant, when $x < 0$ and $y = 0$:

   2.1) If $|x| = k - 1$ and $b = a$ then

   $P_2 = 0, -i, -2i, \ldots, -ti, 1 - ti.$

   $P_3 = 0, i, 2i, \ldots, (t+1)i.$

   $P_4 = 0, 1, 1 - i, 1 - 2i, \ldots, 1 - ti.$

2.2) If $|x| = k - 1$ and $b = a + 2$ then

$P_2 = 0, -i, -2i, \ldots, -(t-1)i, 1-(t-1)i.$

$P_3 = 0, i, -1+i, -1+2i, \ldots, -1+(t+1)i.$

$P_4 = 0, 1, 1-i, 1-2i, \ldots, 1-(t-1)i.$

2.3) If $t - |x| < 4$ and $b = a$ then

$P_4 = 0, 1, 1-i, 1-2i, \ldots, 1-ti, 2-ti, \ldots, (t-|x|)-ti.$

2.4) If $t - |x| < 4$ and $b = a + 2$ then

$P_4 = 0, 1, 1-i, 1-2i, \ldots, 1-(t-1)i, 2-(t-1)i, \ldots, (t-|x|+1)-(t-1)i.$

2.5) If $t - |x| < 4$ and $b \geq a + 4$ then

$P_4 = 0, 1, 1-i, 1-2i, \ldots, 1-ai, 2-ai, \ldots, (2t-|x|-a)-ai.$

3. *Third quadrant, when $x = 0$ and $y < 0$:*

   The proof is similar to the first quadrant with the following cases: 1.3 and 1.4.

4. *Fourth quadrant, when $x > 0$ and $y = 0$:*

   The proof is similar to the first quadrant with the following cases: 2.1, 2.3, 2.4 and 1.4.

□

Example 2.9. Let $\alpha = 3 + 5i$, the source node $S = 0$, and the destination node $D = 4i$. Then, $t = 4$ and $d = 4$. Since the condition of case 1.2 in Theorem 2.8 is satisfied, the paths are:

$P_1 = 0, i, 2i, 3i, 4i.$

$P_2 = 0, -1, -2, -3, -3-i \equiv 4i \ (mod\ \alpha).$

$P_3 = 0, 1, 1+i, 2+i, 3+i, 4+i, 5+i \equiv 4i \ (mod\ \alpha).$

$P_4 = 0, -i, -1-i, -2-i, -3-i \equiv 4i \ (mod\ \alpha).$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = |P_4| = 4$ and $|P_3| = 6$. Figure 2.12 illustrates the shortest node disjoint paths from S to D.

The other cases have paths similar to the odd case.

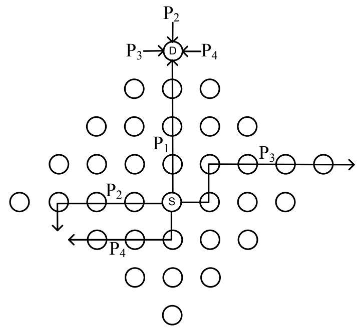

Figure 2.12: Shortest Node Disjoint Paths in Gaussian Network with $\alpha = 3 + 5i$ from $S = 0$ to $D = 4i$.

2.3 Eisenstein-Jacobi (EJ) Networks

2.3.1 Quotient Rings of EJ Integers

Similar to the Gaussian integer $\mathbb{Z}[i]$, *Eisenstein-Jacobi integer* $\mathbb{Z}[\rho]$, as described in [16], is defined as

$$\mathbb{Z}[\rho] = \{x + y\rho \mid x, y \in \mathbb{Z}\} \tag{2.8}$$

where $i = \sqrt{-1}$, $\rho = (1+i\sqrt{3})/2$, and $\rho^2 = \rho-1$. It is known that $\mathbb{Z}[\rho]$ is a Euclidean domain and the norm of an Eisenstein-Jacobi integer $\alpha = a+b\rho$ is given by $N(\alpha) = a^2+b^2+ab$. We can consider $\mathbb{Z}[\rho]_\alpha$, i.e., the EJ integers modulo α. Similar to Gaussian, it is well known that the number of residue classes in EJ integers modulo $\alpha \neq 0$ is equal to $N(\alpha) = a^2 + b^2 + ab$ [16].

2.3.2 Definition and Topological Properties of EJ Networks

Eisenstein-Jacobi networks are based on quotient rings of Eisenstein-Jacobi integers and defined as follows.

Definition 2.2 ([16])**.** Given $0 \neq \alpha = a + b\rho \in \mathbb{Z}[\rho]$ where $\rho = (1 + i\sqrt{3})/2$ and $0 \leq a \leq b$, the Eisenstein-Jacobi network is defined as a graph $EJ_\alpha(V, E)$ where:

1. $V = \mathbb{Z}[\rho]_\alpha$ is the node set.

2. $E = \{(\beta, \gamma) \in V \times V \mid (\beta - \gamma) \equiv \pm 1, \pm\rho, \pm\rho^2 \ mod \ \alpha\}$ is the edge set as shown in Figure 2.13.

we refer EJ_α as the Eisenstein-Jacobi network generated by α.

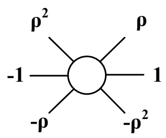

Figure 2.13: Edges for Each Node in EJ_α.

EJ networks are regular symmetric networks of degree six, that is each node has six neighbors. The total number of nodes in the network is as $N(\alpha) = a^2 + b^2 + ab$. Each node in the network is labeled by $x + y\rho$. Two nodes A and B are said to be adjacent if and only if $(A - B) \ mod \ \alpha$ is equal to ± 1, $\pm\rho$, or $\pm\rho^2$. The distance between any two nodes β and γ in the EJ network is defined as

$$D_\alpha(\beta, \gamma) = \min\{|x| + |y| + |z| \mid (\beta - \gamma) \equiv x + y\rho + z\rho^2 \ (mod \ \alpha)\} \tag{2.9}$$

Similar to the Gaussian network, the modulo operation in EJ network can be easily computed by tilling the EJ network.

The following theorem describes the number of nodes at distance s, for $s = 0, 1, ..., M$, where $M = (a + 2b)/3$. This number is denoted as $WT_{EJ}(s)$.

Theorem 2.9 ([16])**.** *Let* $0 \neq \alpha = a + b\rho \in \mathbb{Z}[\rho]$ *such that* $0 \leq a \leq b$, $T = (a + b)/2$, $M = (a + 2b)/3$. *The diameter of this network is at most* M *and the distance distribution* *of the graph EJ_α is as follows:*

$$WT_{EJ}(s) = \begin{cases} 1 & if\ s = 0 \\ 6s & if\ 1 \leq s < T \\ 18(M-s) & if\ T < s < M \\ 2 & if\ b \equiv a\ (mod\ 3)\ and\ s = M \\ 0 & if\ s > M \\ N(\alpha) - \sum\limits_{s=0, s\neq T}^{M} WT_{EJ}(s) & if\ s = T \end{cases} \tag{2.10}$$

For example, Figure 2.14 shows an Eisenstein-Jacobi network with $\alpha = 4 + 5\rho$. It is clear that we can divide the graph into six sectors, S_1, S_2, S_3, S_4, S_5, and S_6. i.e., a node $\beta = x\rho^{j-1} + y\rho^j$ is in sector j for $j = 1, 2, 3, 4, 5$, and 6.

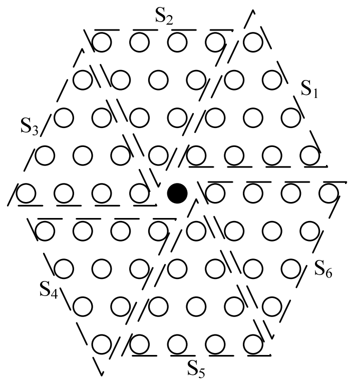

Figure 2.14: Sectors in EJ Network with $\alpha = 4 + 5\rho$.

2.4 Shortest Node Disjoint Paths in EJ Networks

The methodology described in Section 2.2 for the Gaussian networks can also be applied to EJ networks. Again, suppose that the EJ network is generated by $\alpha = a + b\rho$ where $0 \leq a \leq b$. Let $\beta_1 = x_1 + y_1\rho$ and $\beta_2 = x_2 + y_2\rho$ be the given source and destination nodes, respectively. Our aim is to find the maximum number of node disjoint paths from β_1 to β_2. In order to do this, let $\beta_2 - \beta_1 = (x_2 - x_1) + (y_2 - y_1)\rho = x\rho^{j-1} + y\rho^j$, where $x, y \geq 0$ and in this case the distance from β_1 to β_2 is $x + y$ [16]. One of the shortest paths from β_1 to β_2

can be obtained by going x steps along ρ^{j-1} direction and then y steps along ρ^j direction. Thus, in the following, without loss of generality, it is assumed that the source node is 0, and the destination node is $x + y\rho$, where $x \geq 0$ and $y > 0$, i.e., the destination node is in sector 1. The following two theorems show that there are six node disjoint paths. Similar to Gaussian, the value of each node in every path must be computed by taking *mod* α.

Theorem 2.10. *Suppose the generator of an EJ network is $\alpha = a + b\rho$ with $0 \leq a \leq b$. Let 0 and $\beta = x + y\rho$, where $x > 0$ and $y > 0$, be the source and destination nodes, respectively. Then there are six node disjoint paths from 0 to β, with path lengths at most $d + 6$, where $d = x + y$ is the minimum distance from 0 to β.*

Proof. The six node disjoint paths are given as follows:

$P_1 = 0, \rho, 2\rho, \ldots, y\rho, 1+y\rho, 2+y\rho, \ldots, x+y\rho$, i.e., going y steps along ρ^1 direction and then x steps along ρ^0 direction.

$P_2 = 0, 1, 2, \ldots, x, x + \rho, x + 2\rho, \ldots, x + y\rho$, i.e., going x steps along ρ^0 direction and then y steps along ρ^1 direction.

$P_3 = 0, \rho^2, \rho + \rho^2, 2\rho + \rho^2, \ldots, y\rho + \rho^2, (y + 1)\rho, 1 + (y + 1)\rho, 2 + (y + 1)\rho, \ldots, (x - 1) + (y + 1)\rho, x + y\rho$, i.e., going 1 step along ρ^2, y steps along ρ^1, x steps along ρ^0, and 1 step along ρ^5 directions.

$P_4 = 0, \rho^5, 1+\rho^5, 2+\rho^5, \ldots, x+\rho^5, x+1, (x + 1)+\rho, (x + 1)+2\rho, \ldots, (x + 1)+(y - 1)\rho, x+y\rho$, i.e., going 1 step along ρ^5, x steps along ρ^0, y steps along ρ^1, and 1 step along ρ^2 directions.

$P_5 = 0, \rho^3, \rho^2 + \rho^3, 2\rho^2, \rho + 2\rho^2, 2\rho + 2\rho^2, \ldots, y\rho + 2\rho^2, (y + 1)\rho + \rho^2, (y + 2)\rho, 1 + (y + 2)\rho, 2 + (y + 2)\rho, \ldots, (x - 1) + (y + 2)\rho, x + (y + 1)\rho, x + y\rho$, i.e., going 1 step along ρ^3, 1 step along ρ^2, $y + 1$ steps along ρ^1, $x + 1$ steps along ρ^0, 1 step along ρ^5, and 1 step along ρ^4 directions.

$P_6 = 0, \rho^4, \rho^4 + \rho^5, 2\rho^5, 1 + 2\rho^5, 2 + 2\rho^5, \ldots, x + 2\rho^5, (x + 1) + \rho^5, x + 2, (x + 2) + \rho, (x + 2) + 2\rho, \ldots, (x + 2) + (y - 1)\rho, (x + 1) + y\rho, x + y\rho$, i.e., going 1 step along ρ^4, 1 step along ρ^5, $x + 1$ steps along ρ^0, $y + 1$ steps along ρ^1, 1 step along ρ^2, and 1 step along ρ^3 directions.

Denoting $|P_i|$ as the length of the path P_i, we have $|P_1| = |P_2| = d$, $|P_3| = |P_4| = d + 2$, and $|P_5| = |P_6| = d + 6$, where d is the shortest path from 0 to β. $\square$

Example 2.10. Let $\alpha = 4+5\rho$, the source node $S = 0$, and the destination node $D = 2+2\rho$.

The paths are as follows:

$P_1 = 0, \rho, 2\rho, 1+2\rho, 2+2\rho$.

$P_2 = 0, 1, 2, 2+\rho, 2+2\rho$.

$P_3 = 0, \rho^2, \rho+\rho^2, 2\rho+\rho^2, 3\rho, 1+3\rho, 2+2\rho$.

$P_4 = 0, \rho^5, 1+\rho^5, 2+\rho^5, 3, 3+\rho, 2+2\rho$.

$P_5 = 0, \rho^3, \rho^2+\rho^3, 2\rho^2, \rho+2\rho^2, 2\rho+2\rho^2, 3\rho+\rho^2, 4\rho, 1+4\rho \equiv 3\rho^3+\rho^4 \ (mod\ \alpha), 2+3\rho \equiv 2\rho^3+2\rho^4 \ (mod\ \alpha), 2+2\rho$.

$P_6 = 0, \rho^4, \rho^4+\rho^5, 2\rho^5, 1+2\rho^5, 2+2\rho^5, 3+\rho^5, 4, 4+\rho \equiv 4\rho^4 \ (mod\ \alpha), 3+2\rho \equiv \rho^3+3\rho^4 \ (mod\ \alpha), 2+2\rho$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 4$, $|P_3| = |P_4| = 6$, $|P_5| = |P_6| = 10$. Figure 2.15 illustrates all node disjoint paths from S to D.

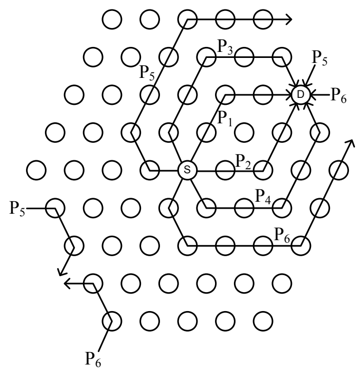

Figure 2.15: Node Disjoint Paths in EJ Network with $\alpha = 4+5\rho$ from $S = 0$ to $D = 2+2\rho$.

Theorem 2.11. *Suppose the generator of an EJ network is $\alpha = a+b\rho$ with $0 \le a \le b$. Let 0 and $\beta = x+y\rho$, where $x = 0$ and $y > 0$, be the source and destination nodes, respectively. Then there are six node disjoint paths from 0 to β, with path lengths at most $d+9$, where $d = x+y$ is the minimum distance from 0 to β.*

Proof. The six node disjoint paths are given as follows:

$P_1 = 0, \rho, 2\rho, \ldots, y\rho$, i.e., going y steps along ρ^1 direction.

$P_2 = 0, \rho^2, \rho+\rho^2, 2\rho+\rho^2, \ldots, (y-1)\rho+\rho^2, y\rho$, i.e., going 1 step along ρ^2, $y-1$ steps along ρ^1, and 1 step alon1g ρ^0 directions.

$P_3 = 0, 1, 1+\rho, 1+2\rho, \ldots, 1+(y-1)\rho, y\rho$, i.e., going 1 step along ρ^0, $y-1$ steps along ρ^1, and 1 step along ρ^2 directions.

$P_4 = 0, \rho^3, \rho^2+\rho^3, 2\rho^2, \rho+2\rho^2, 2\rho+2\rho^2, \ldots, (y-1)\rho+2\rho^2, y\rho+\rho^2, y\rho$, i.e., going 1 step along ρ^3, 1 step along ρ^2, y steps along ρ^1, 1 step along ρ^0, and 1 step along ρ^5 directions.

$P_5 = 0, \rho^5, 1+\rho^5, 2, 2+\rho, 2+2\rho, \ldots, 2+(y-1)\rho, 1+y\rho, y\rho$, i.e., going 1 step along ρ^5, 1 step along ρ^0, y steps along ρ^1, 1 step along ρ^2, and 1 step along ρ^3 directions.

$P_6 = 0, \rho^4, \rho^3+\rho^4, 2\rho^3, \rho^2+2\rho^3, 2\rho^2+\rho^3, 3\rho^2, \rho+3\rho^2, 2\rho+3\rho^2, \ldots, (y-1)\rho+3\rho^2, y\rho+2\rho^2, (y+1)\rho+\rho^2, (y+1)\rho, y\rho$, i.e., going 1 step along ρ^4, 1 steps along ρ^3, 2 steps along ρ^2, $y+1$ steps along ρ^1 2 steps along ρ^0, 1 step along ρ^5, and 1 step along ρ^4 directions.

Denoting $|P_i|$ as the length of the path P_i, we have $|P_1| = d$, $|P_2| = |P_3| = d+1$, $|P_4| = |P_5| = d+4$, and $|P_6| = d+9$, where d is the shortest path from 0 to β. $\square$

Example 2.11. Let $\alpha = 4+5\rho$, the source node $S = 0$, and the destination node $D = 4\rho$. Then,

$P_1 = 0, \rho, 2\rho, 3\rho, 4\rho$.

$P_2 = 0, \rho^2, \rho+\rho^2, 2\rho+\rho^2, 3\rho+\rho^2, 4\rho$.

$P_3 = 0, 1, 1+\rho, 1+2\rho, 1+3\rho, 4\rho$.

$P_4 = 0, \rho^3, \rho^2+\rho^3, 2\rho^2, \rho+2\rho^2, 2\rho+2\rho^2, 3\rho+2\rho^2 \equiv \rho^4+3\rho^5 \ (mod\ \alpha), 4\rho+\rho^2 \equiv 4\rho^5 (mod\ \alpha), 4\rho$.

$P_5 = 0, \rho^5, 1+\rho^5, 2, 2+\rho, 2+2\rho, 2+3\rho \equiv 2\rho^3+2\rho^4 \ (mod\ \alpha), 1+4\rho \equiv 3\rho^3+\rho^4 \ (mod\ \alpha), 4\rho$.

$P_6 = 0, \rho^4, \rho^3+\rho^4, 2\rho^3, \rho^2+2\rho^3, 2\rho^2+\rho^3, 3\rho^2, \rho+3\rho^2, 2\rho+3\rho^2 \equiv 2\rho^4+2\rho^5 \ (mod\ \alpha), 3\rho+3\rho^2 \equiv \rho^4+2\rho^5 \ (mod\ \alpha), 4\rho+2\rho^2 \equiv 3\rho^5 \ (mod\ \alpha), 5\rho+\rho^2 \equiv 1+3\rho^5 \ (mod\ \alpha), 5\rho \equiv 4\rho^3 \ (mod\ \alpha), 4\rho$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = 4$, $|P_2| = |P_3| = 5$, $|P_4| = |P_5| = 8$, and $|P_6| = 13$. These paths are shown in Figure 2.16.

Similar to the Gaussian network, in some cases the maximum length of the paths can be reduced. Two cases are considered and they when $b = a+2j+1$ and $b = a+2j$ for $j = 0, 1, 2, \ldots etc$, i.e., b is equal to a plus some odd integer (and plus an even integer). Furthermore, two sub-cases, for each of odd and even, are illustrated in the following theorems.

Theorem 2.12. *Suppose the generator of an EJ network is $\alpha = a+b\rho$ with $0 \leq a \leq b$ and*

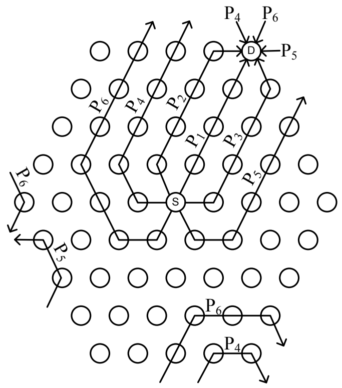

Figure 2.16: Node Disjoint Paths in EJ Network with $\alpha = 4 + 5\rho$ from $S = 0$ to $D = 4\rho$.

$b = a + 2j + 1$ for $j = 0, 1, 2, \ldots$ etc, i.e., b is equal to a plus some odd number. Let 0 and $\beta = x + y\rho$, where $x > 0$ and $y > 0$, be the source and destination nodes, respectively. Then there are six node disjoint paths from 0 to β, with path lengths strictly less than $d+6$, where $d = x + y$ is the minimum distance from 0 to β.

Proof. Let $h = \frac{a+b-1}{2}$. The length of the maximum node disjoint paths can be reduced according to the following cases:

1. <u>*When $d = M$:*</u>

   $P_3 = 0, \rho^3, 2\rho^3, \ldots, y\rho^3, \rho^2 + y\rho^3, 2\rho^2 + y\rho^3, \ldots, x\rho^2 + y\rho^3$, i.e., going y steps along ρ^3 direction and then x steps along ρ^2 direction.

   $P_4 = 0, \rho^2, 2\rho^2, \ldots, x\rho^2, x\rho^2 + \rho^3, x\rho^2 + 2\rho^3, \ldots, x\rho^2 + y\rho^3$, i.e., going x steps along ρ^2 direction and then y steps along ρ^3 direction.

   $P_5 = 0, \rho^5, 2\rho^5, \ldots, y\rho^5, \rho^4 + y\rho^5, 2\rho^4 + y\rho^5, \ldots, x\rho^4 + y\rho^5$, i.e., going y steps along ρ^5 direction and then x steps along ρ^4 direction.

   $P_6 = 0, \rho^4, 2\rho^4, \ldots, x\rho^4, x\rho^4 + \rho^5, x\rho^4 + 2\rho^5, \ldots, x\rho^4 + y\rho^5$, i.e., going x steps along ρ^4 direction and then y steps along ρ^5 direction.

2. <u>*When $x - a < 0$ and if $h - d < 2.5$ then*</u>

   $P_5 = 0, \rho^3, 2\rho^3, \ldots, (a-x)\rho^3, (a-x)\rho^3 + \rho^4, (a-x)\rho^3 + 2\rho^4, \ldots, (a-x)\rho^3 + (2h - y - a + 1)\rho^4$, i.e., going $a - x$ steps along ρ^3 direction and then $2h - y - a + 1$ steps along ρ^4 direction.

$P_6 = 0, \rho^4, 2\rho^4, \ldots, (2h-y-a+1)\rho^4, \rho^3+(2h-y-a+1)\rho^4, 2\rho^3+(2h-y-a+1)\rho^4, \ldots, (a-x)\rho^3+(2h-y-a+1)\rho^4$, i.e., going $2h-y-a+1$ steps along ρ^4 direction and then $a-x$ steps along ρ^3 direction.

3. *When $x - a \geq 0$:*

3.1) If $b = a+3$ and $x-a=0$ and $y=1$ then

Go to case 3.5.

3.2) If $2h-2d+y<1$ and $2h-2d+x-a<1$ then

$P_3 = 0, \rho^2, 2\rho^2, \ldots, (a+y)\rho^2, (a+y)\rho^2+\rho^3, (a+y)\rho^2+2\rho^3, \ldots, (a+y)\rho^2+(2h-d-a+1)\rho^3$, i.e., going $a+y$ steps along ρ^2 direction and then $2h-d-a+1$ steps along ρ^3 direction.

$P_4 = 0, \rho^5, 2\rho^5, \ldots, (x-a)\rho^5, \rho^4+(x-a)\rho^5, 2\rho^4+(x-a)\rho^5, \ldots, (2h-d+1)\rho^4+(x-a)\rho^5$, i.e., going $x-a$ steps along ρ^5 direction and then $2h-d+1$ steps along ρ^4 direction.

$P_5 = 0, \rho^3, 2\rho^3, \ldots, (2h-d-a+1)\rho^3, \rho^2+(2h-d-a+1)\rho^3, 2\rho^2+(2h-d-a+1)\rho^3, \ldots, (a+y)\rho^2+(2h-d-a+1)\rho^3$, i.e., going $2h-d-a+1$ steps along ρ^3 direction and then $a+y$ steps along ρ^2 direction.

$P_6 = 0, \rho^4, 2\rho^4, \ldots, (2h-d+1)\rho^4, (2h-d+1)\rho^4+\rho^5, (2h-d+1)\rho^4+2\rho^5, \ldots, (2h-d+1)\rho^4+(x-a)\rho^5$, i.e., going $2h-d+1$ steps along ρ^4 direction and then $x-a$ steps along ρ^5 direction.

3.3) If $2h-2d+y<5$ and $2h-2d+x-a<5$ and $4h-2d+x+y-a+2<2(2h-d+y+2)$ and $4h-2d+x+y-a+2<2(2h-d+x-a+2)$ and $x-a>0$ then

$P_5 = 0, \rho^3, \rho^2+\rho^3, 2\rho^2+\rho^3, \ldots, (a+y)\rho^2+\rho^3, (a+y)\rho^2+2\rho^3, \ldots, (a+y)\rho^2+(2h-d-a+1)\rho^3$, i.e., going 1 step along ρ^3, $a+y$ steps along ρ^2, and then $2h-d-a$ steps along ρ^3 directions.

$P_6 = 0, \rho^4, \rho^4+\rho^5, \rho^4+2\rho^5, \ldots, \rho^4+(x-a)\rho^5, 2\rho^4+(x-a)\rho^5, \ldots, (2h-d+1)\rho^4+(x-a)\rho^5$, i.e., going 1 step along ρ^4, $x-a$ steps along ρ^5, and then $2h-d$

steps along ρ^4 directions.

3.4) If $2h - 2d + y < 4$ and $2h - d + y + 2 < 2h - d + x - a + 2$ then

$P_5 = 0, \rho^3, \rho^2 + \rho^3, 2\rho^2 + \rho^3, \ldots, (a + y + 1)\rho^2 + \rho^3, (a + y + 1)\rho^2 + 2\rho^3, \ldots, (a + y + 1)\rho^2 + (2h - d - a)\rho^3, (a + y)\rho^2 + (2h - d - a + 1)\rho^3$, i.e., going 1 step along ρ^3, $a + y + 1$ steps along ρ^2, $2h - d - a - 1$ steps along ρ^3, and then 1 step along ρ^4 directions.

$P_6 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3, \rho^2 + 2\rho^3, 2\rho^2 + 2\rho^3, \ldots, (a + y)\rho^2 + 2\rho^3, (a + y)\rho^2 + 3\rho^3, \ldots, (a + y)\rho^2 + (2h - d - a + 1)\rho^3$, i.e., going 1 step along ρ^4, 1 steps along ρ^3, $a + y + 1$ steps along ρ^2, and then $2h - d - a - 1$ steps along ρ^3 directions.

3.5) If $2h - 2d + x - a < 4$ then

$P_5 = 0, \rho^3, \rho^3 + \rho^4, 2\rho^4, 2\rho^4 + \rho^5, 2\rho^4 + 2\rho^5, \ldots, 2\rho^4 + (x - a)\rho^5, 3\rho^4 + (x - a)\rho^5, \ldots, (2h - d + 1)\rho^4 + (x - a)\rho^5$, i.e., going 1 step along ρ^3, 1 step along ρ^4, $x - a + 1$ along ρ^5, and then $2h - d - 1$ steps along ρ^4 directions.

$P_6 = 0, \rho^4, \rho^4 + \rho^5, \rho^4 + 2\rho^5, \ldots, \rho^4 + (x - a + 1)\rho^5, 2\rho^4 + (x - a + 1)\rho^5, \ldots, (2h - d)\rho^4 + (x - a + 1)\rho^5, (2h - d + 1)\rho^4 + (x - a)\rho^5$, i.e., going 1 step along ρ^4, $x - a + 1$ steps along ρ^5, $2h - d - 1$ along ρ^4, and then 1 step along ρ^3 directions.

□

Example 2.12. Let $\alpha = 4+5\rho$, the source node $S = 0$, and the destination node $D = 2+2\rho$. Then, $h = 4$ and $d = 4$. Since the condition of case 2 in Theorem 2.12 is satisfied, the paths are:

$P_1 = 0, \rho, 2\rho, 1 + 2\rho, 2 + 2\rho$.

$P_2 = 0, 1, 2, 2 + \rho, 2 + 2\rho$.

$P_3 = 0, \rho^2, \rho + \rho^2, 2\rho + \rho^2, 3\rho, 1 + 3\rho, 2 + 2\rho$.

$P_4 = 0, \rho^5, 1 + \rho^5, 2 + \rho^5, 3, 3 + \rho, 2 + 2\rho$.

$P_5 = 0, \rho^3, 2\rho^3, 2\rho^3 + \rho^4, 2\rho^3 + 2\rho^4, 2\rho^3 + 3\rho^4 \equiv 2 + 2\rho \ (mod\ \alpha)$.

$P_6 = 0, \rho^4, 2\rho^4, 3\rho^4, \rho^3 + 3\rho^4, 2\rho^3 + 3\rho^4 \equiv 2 + 2\rho \ (mod\ \alpha)$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 4$, $|P_3| = |P_4| = 6$, $|P_5| =$

$|P_6| = 5$. Figure 2.17 illustrates all these shortest node disjoint paths from S to D.

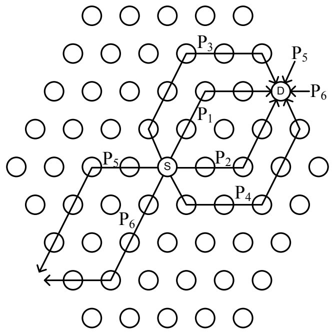

Figure 2.17: Shortest Node Disjoint Paths in EJ Network with $\alpha = 4 + 5\rho$ from $S = 0$ to $D = 2 + 2\rho$.

Example 2.13. Let $\alpha = 2 + 5\rho$, the source node $S = 0$, and the destination $D = 3 + \rho$. Then, $h = 3$ and $d = 4$. Since the condition of case 1 in Theorem 2.12 is satisfied, the paths are:

$P_1 = 0, \rho, 1 + \rho, 2 + \rho, 3 + \rho$.

$P_2 = 0, 1, 2, 3, 3 + \rho$.

$P_3 = 0, \rho^3, \rho^2 + \rho^3, 2\rho^2 + \rho^3, 3\rho^2 + \rho^3 \equiv 3 + \rho \ (mod\ \alpha)$.

$P_4 = 0, \rho^2, 2\rho^2, 3\rho^2, 3\rho^2 + \rho^3 \equiv 3 + \rho \ (mod\ \alpha)$.

$P_5 = 0, \rho^5, \rho^4 + \rho^5, 2\rho^4 + \rho^5, 3\rho^4 + \rho^5 \equiv 3 + \rho \ (mod\ \alpha)$.

$P_6 = 0, \rho^4, 2\rho^4, 3\rho^4, 3\rho^4 + \rho^5 \equiv 3 + \rho \ (mod\ \alpha)$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = |P_3| = |P_4| = |P_5| = |P_6| = 4$. Figure 2.18 illustrates all these shortest node disjoint paths from S to D.

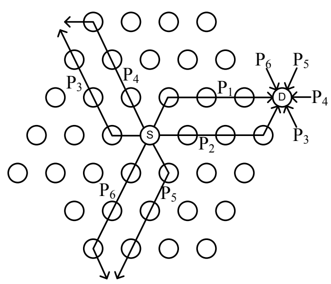

Figure 2.18: Shortest Node Disjoint Paths in EJ Network with $\alpha = 2 + 5\rho$ from $S = 0$ to $D = 3 + \rho$.

Example 2.14. Let $\alpha = 2+7\rho$, the source node $S = 0$, and the destination node $D = 4+\rho$. Then, $h = 4$ and $d = 5$. Since the condition of case 3.2 in Theorem 2.12 is satisfied, the paths are:

$P_1 = 0, \rho, 1+\rho, 2+\rho, 3+\rho, 4+\rho.$

$P_2 = 0, 1, 2, 3, 4, 4+\rho.$

$P_3 = 0, \rho^2, 2\rho^2, 3\rho^2, 3\rho^2+\rho^3, 3\rho^2+2\rho^3 \equiv 4+\rho \ (mod\ \alpha).$

$P_4 = 0, \rho^5, 2\rho^5, \rho^4+2\rho^5, 2\rho^4+2\rho^5, 3\rho^4+2\rho^5 \equiv 4\rho^2+\rho^3 \ (mod\ \alpha), 4\rho^4+2\rho^5 \equiv 4+\rho \ (mod\ \alpha).$

$P_5 = 0, \rho^3, 2\rho^3, \rho^2+2\rho^3, 2\rho^2+2\rho^3, 3\rho^2+2\rho^3 \equiv 4+\rho \ (mod\ \alpha).$

$P_6 = 0, \rho^4, 2\rho^4, 3\rho^4, 4\rho^4, 4\rho^4+\rho^5, 4\rho^4+2\rho^5 \equiv 4+\rho \ (mod\ \alpha).$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = |P_3| = |P_5| = 5$ and $|P_4| = |P_6| = 6$. Figure 2.19 illustrates all these shortest node disjoint paths from S to D.

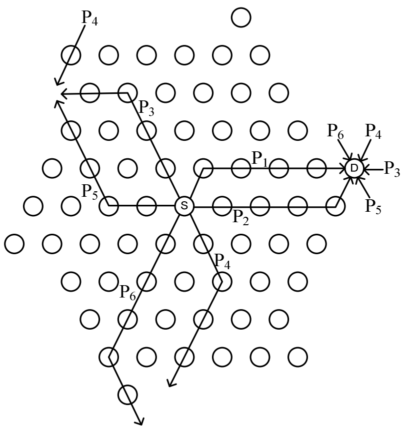

Figure 2.19: Shortest Node Disjoint Paths in EJ Network with $\alpha = 2+7\rho$ from $S = 0$ to $D = 4\rho$.

Example 2.15. Let $\alpha = 2+7\rho$, the source node $S = 0$, and the destination node $D = 2+2\rho$. Then, $h = 4$ and $d = 4$. Since the condition of case 3.5 in Theorem 2.12 is satisfied, the paths are:

$P_1 = 0, \rho, 2\rho, 1+2\rho, 2+2\rho.$

$P_2 = 0, 1, 2, 2+\rho, 2+2\rho.$

$P_3 = 0, \rho^2, \rho+\rho^2, 2\rho+\rho^2, 3\rho, 1+3\rho, 2+2\rho.$

$P_4 = 0, \rho^5, 1+\rho^5, 2+\rho^5, 3, 3+\rho, 2+2\rho.$

$P_5 = 0, \rho^3, \rho^3 + \rho^4, 2\rho^4, 3\rho^4, 4\rho^4, 5\rho^4 \equiv 2 + 2\rho \ (mod\ \alpha)$.

$P_6 = 0, \rho^4, \rho^4 + \rho^5, 2\rho^4 + \rho^5, 3\rho^4 + \rho^5, 4\rho^4 + \rho^5, 5\rho^4 \equiv 2 + 2\rho \ (mod\ \alpha)$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 4$, $|P_3| = |P_4| = 6$, $|P_5| = |P_6| = 6$. Figure 2.20 illustrates all node disjoint paths from S to D.

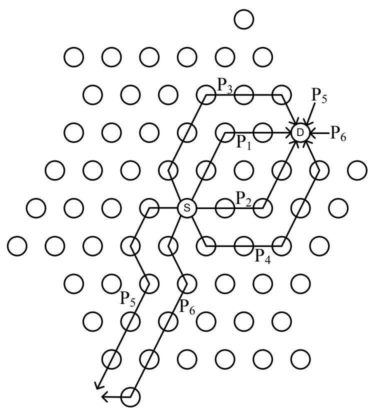

Figure 2.20: Shortest Node Disjoint Paths in EJ Network with $\alpha = 2 + 7\rho$ from $S = 0$ to $D = 2 + 2\rho$.

Theorem 2.13. *Suppose the generator of an EJ network is $\alpha = a + b\rho$ with $0 \le a \le b$ and $b = a + 2j + 1$ for $j = 0, 1, 2, \ldots$ etc, i.e., b is equal to a plus some odd number. Let 0 and $\beta = x + y\rho$, where $x = 0$ and $y > 0$, be the source and destination nodes, respectively. Then there are six node disjoint paths from 0 to β, with path lengths strictly less than $d + 9$, where $d = x + y$ is the minimum distance from 0 to β.*

Proof. Let $h = \frac{a+b-1}{2}$. The length of the maximum node disjoint paths can be reduced according to the following cases:

1. <u>When $b = a + 1$:</u>

   1.1) If $h - y < 1.5$ then
   $P_4 = 0, \rho^3, 2\rho^3, \ldots, a\rho^3, a\rho^3 + \rho^4, a\rho^3 + 2\rho^4, \ldots, a\rho^3 + (2h - y - a + 1)\rho^4$, i.e., going a steps along ρ^3 direction and then $2h - y - a + 1$ steps along ρ^4 direction.
   $P_5 = 0, \rho^5, \rho^4 + \rho^5, 2\rho^4 + \rho^5, \ldots, (a - y)\rho^4 + \rho^5, (a - y)\rho^4 + 2\rho^5, \ldots, (a - y)\rho^4 + (2h - a + 1)\rho^5$, i.e., going 1 step along ρ^5, $a - y$ steps along ρ^4, and then $2h - a$

steps along ρ^5 directions.

$P_6 = 0, \rho^4, 2\rho^4, \ldots, (2h-y-a+1)\rho^4, \rho^3+(2h-y-a+1)\rho^4, 2\rho^3+(2h-y-a+1)\rho^4, \ldots, a\rho^3+(2h-y-a+1)\rho^4$, i.e., going $2h-y-a+1$ steps along ρ^4 direction and then a steps along ρ^3 direction.

1.2) If $1.5 \le h-y < 4$ then

$P_6 = 0, \rho^4, \rho^3+\rho^4, 2\rho^3+\rho^4, \ldots, a\rho^3+\rho^4, a\rho^3+2\rho^4, \ldots, a\rho^3+(2h-y-a+1)\rho^4$, i.e., going 1 step along ρ^4, a steps along ρ^3, and then $2h-y-a$ steps along ρ^4 direction.

2. _When $b > a+1$:_

2.1) If $h-y < 1$ and $2h-y+1 < 2h-a$ then

$P_4 = 0, \rho^3, 2\rho^3, \ldots, (a+1)\rho^3, (a+1)\rho^3+\rho^4, (a+1)\rho^3+2\rho^4, \ldots, (a+1)\rho^3+(2h-y-a)\rho^4, a\rho^3+(2h-y-a+1)\rho^4$, i.e., going $a+1$ steps along ρ^3, $2h-y-a$ steps along ρ^4, and then 1 step along ρ^5 direction.

$P_5 = 0, \rho^5, \rho^4+\rho^5, 2\rho^4+\rho^5, \ldots, (2h-y-a)\rho^4+\rho^5, (2h-y-a+1)\rho^4, \rho^3+(2h-y-a+1)\rho^4, 2\rho^3+(2h-y-a+1)\rho^4, \ldots, a\rho^3+(2h-y-a+1)\rho^4$, i.e., going 1 step along ρ^5, $2h-y-a$ steps along ρ^4, and then $a+1$ steps along ρ^3 directions.

$P_6 = 0, \rho^4, \rho^3+\rho^4, 2\rho^3+\rho^4, \ldots, a\rho^3+\rho^4, a\rho^3+2\rho^4, \ldots, a\rho^3+(2h-y-a+1)\rho^4$, i.e., going 1 step along ρ^4, a steps along ρ^3, and then $2h-y-a$ steps along ρ^4 directions.

2.2) If $2h-y-a < 3$ then

$P_4 = 0, \rho^3, 2\rho^3, \ldots, a\rho^3, a\rho^3+\rho^4, a\rho^3+2\rho^4, \ldots, a\rho^3+(2h-y-a+1)\rho^4$, i.e., going a steps along ρ^3 direction and then $2h-y-a+1$ steps along ρ^4 direction.

$P_5 = 0, \rho^5, 1+\rho^5, 2+\rho^5, \ldots, (y-a)+\rho^5, (y-a)+2\rho^5, \ldots, (y-a)+(2h-y+1)\rho^5$, i.e., going 1 step along ρ^5, $y-a$ steps along ρ^0, and then $2h-y$ steps along ρ^5 directions.

$P_6 = 0, \rho^4, 2\rho^4, \ldots, (2h-y-a+1)\rho^4, \rho^3+(2h-y-a+1)\rho^4, 2\rho^3+(2h-y-a+1)\rho^4, \ldots, a\rho^3+(2h-y-a+1)\rho^4$, i.e., going $2h-y-a+1$ steps along ρ^4

direction and then a steps along ρ^3 direction.

2.3) If $3 \leq 2h - y - a$ and $1 \leq h - y < 4$ then

$P_6 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3 + \rho^4, \ldots, a\rho^3 + \rho^4, a\rho^3 + 2\rho^4, \ldots, a\rho^3 + (2h - y - a + 1)\rho^4$, i.e., going1 step along ρ^4, a steps along ρ^3, and then $2h - y - a$ steps along ρ^4 directions.

□

Example 2.16. Let $\alpha = 0 + 5\rho$, the source node $S = 0$, and the destination node $D = 2\rho$. Then, $h = 2$ and $d = 2$. Since the condition of case 2.1 in Theorem 2.13 is satisfied, the paths are:

$P_1 = 0, \rho, 2\rho$.

$P_2 = 0, \rho^2, \rho + \rho^2, 2\rho$.

$P_3 = 0, 1, 1 + \rho, 2\rho$.

$P_4 = 0, \rho^3, \rho^3 + \rho^4, \rho^3 + 2\rho^4 \equiv 2\rho + \rho^2 \ (mod\ \alpha), 3\rho^4 \equiv 2\rho \ (mod\ \alpha)$.

$P_5 = 0, \rho^5, \rho^4 + \rho^5, 2\rho^4 + \rho^5, 3\rho^4 \equiv 2\rho \ (mod\ \alpha)$.

$P_6 = 0, \rho^4, 2\rho^4, 3\rho^4 \equiv 2\rho \ (mod\ \alpha)$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = 2$, $|P_2| = |P_3| = 3$, $|P_4| = |P_5| = 4$, and $|P_6| = 3$. Figure 2.21 illustrates all these shortest node disjoint paths from S to D.

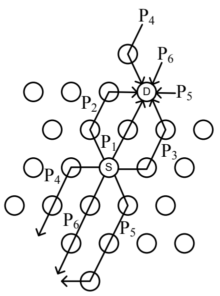

Figure 2.21: Shortest Node Disjoint Paths in EJ Network with $\alpha = 0 + 5\rho$ from $S = 0$ to $D = 2\rho$.

Example 2.17. Let $\alpha = 2 + 5\rho$, the source node $S = 0$, and the destination node $D = 3\rho$. Then, $h = 3$ and $d = 3$. Since the condition of case 2.2 in Theorem 2.13 is satisfied, the paths are:

$P_1 = 0, \rho, 2\rho, 3\rho.$

$P_2 = 0, \rho^2, \rho + \rho^2, 2\rho + \rho^2, 3\rho.$

$P_3 = 0, 1, 1 + \rho, 1 + 2\rho, 3\rho.$

$P_4 = 0, \rho^3, 2\rho^3, 2\rho^3 + \rho^4, 2\rho^3 + 2\rho^4 \equiv 3\rho \ (mod\ \alpha).$

$P_5 = 0, \rho^5, 1 + \rho^5, 1 + 2\rho^5, 1 + 3\rho^5 \equiv 3\rho^3 + \rho^4 \ (mod\ \alpha), 1 + 4\rho^5 \equiv 3\rho \ (mod\ \alpha).$

$P_6 = 0, \rho^4, 2\rho^4, \rho^3 + 2\rho^4, 2\rho^3 + 2\rho^4 \equiv 3\rho \ (mod\ \alpha).$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = 3$, $|P_2| = |P_3| = 4$, $|P_5| = 5$ and $|P_4| = |P_6| = 4$. Figure 2.22 illustrates all these shortest node disjoint paths from S to D.

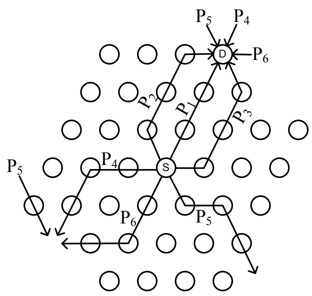

Figure 2.22: Shortest Node Disjoint Paths in EJ Network with $\alpha = 2 + 5\rho$ from $S = 0$ to $D = 3\rho$.

Example 2.18. Let $\alpha = 2 + 5\rho$, the source node $S = 0$, and the destination node $D = \rho$. Then, $h = 3$ and $d = 1$. Since the condition of case 2.3 in Theorem 2.13 is satisfied, the paths are:

$P_1 = 0, \rho.$

$P_2 = 0, \rho^2, \rho.$

$P_3 = 0, 1, \rho.$

$P_4 = 0, \rho^3, \rho^2 + \rho^3, 2\rho^2, \rho + \rho^2, \rho.$

$P_5 = 0, \rho^5, 1 + \rho^5, 2, 1 + \rho, \rho.$

$P_6 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3 + \rho^4, 2\rho^3 + 2\rho^4 \equiv 3\rho \ (mod\ \alpha), 2\rho^3 + 3\rho^4 \equiv 2\rho \ (mod\ \alpha), 2\rho^3 + 4\rho^4 \equiv \rho \ (mod\ \alpha).$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = 1$, $|P_2| = |P_3| = 2$, $|P_4| = |P_5| = 5$, and $|P_6| = 6$. Figure 2.23 illustrates all these shortest node disjoint paths from S to D.

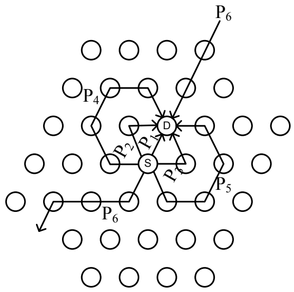

Figure 2.23: Shortest Node Disjoint Paths in EJ Network with $\alpha = 2 + 5\rho$ from $S = 0$ to $D = \rho$.

Theorem 2.14. *Suppose the generator of an EJ network is $\alpha = a + b\rho$ with $0 \le a \le b$ and $b = a + 2j$ for $j = 0, 1, 2, \ldots$ etc, i.e., b is equal to a plus some even number. Let 0 and $\beta = x + y\rho$, where $x > 0$ and $y > 0$, be the source and destination nodes, respectively. Then there are six node disjoint paths from 0 to β, with path lengths strictly less than $d+6$, where $d = x + y$ is the minimum distance from 0 to β.*

Proof. Let $h = \frac{a+b}{2} - 1$. The length of the maximum node disjoint paths can be reduced according to the following cases:

1. <u>*When $d = M$ and $a \neq b$:*</u>

   $P_3 = 0, \rho^3, 2\rho^3, \ldots, y\rho^3, \rho^2 + y\rho^3, 2\rho^2 + y\rho^3, \ldots, x\rho^2 + y\rho^3$, i.e., going y steps along ρ^3 direction and then x steps along ρ^2 direction.

   $P_4 = 0, \rho^2, 2\rho^2, \ldots, x\rho^2, x\rho^2 + \rho^3, x\rho^2 + 2\rho^3, \ldots, x\rho^2 + y\rho^3$, i.e., going x steps along ρ^2 direction and then y steps along ρ^3 direction.

   $P_5 = 0, \rho^5, 2\rho^5, \ldots, y\rho^5, \rho^4 + y\rho^5, 2\rho^4 + y\rho^5, \ldots, x\rho^4 + y\rho^5$, i.e., going y steps along ρ^5 direction and then x steps along ρ^4 direction.

   $P_6 = 0, \rho^4, 2\rho^4, \ldots, x\rho^4, x\rho^4 + \rho^5, x\rho^4 + 2\rho^5, \ldots, x\rho^4 + y\rho^5$, i.e., going x steps along ρ^4 direction and then y steps along ρ^5 direction.

2. <u>*When $x - a < 0$ and if $h - d < 2$ then*</u>

   $P_5 = 0, \rho^3, 2\rho^3, \ldots, (a-x)\rho^3, (a-x)\rho^3 + \rho^4, (a-x)\rho^3 + 2\rho^4, \ldots, (a-x)\rho^3 + (2h - y -$

$a+2)\rho^4$, i.e., going $a-x$ steps along ρ^3 direction and then $2h-y-a+2$ steps along ρ^4 direction.

$P_6 = 0, \rho^4, 2\rho^4, \ldots, (2h-y-a+2)\rho^4, \rho^3+(2h-y-a+2)\rho^4, 2\rho^3+(2h-y-a+2)\rho^4, \ldots, (a-x)\rho^3+(2h-y-a+2)\rho^4$, i.e., going $2h-y-a+2$ steps along ρ^4 direction and then $a-x$ steps along ρ^3 direction.

3. *When $x-a \geq 0$:*

3.1) If $b = a+2$ and $x-a=0$ and $y=1$ then

$P_3 = 0, \rho^2, 2\rho^2, \ldots, (h+1)\rho^2, (h+1)\rho^2+\rho^3$, i.e., going $h+1$ steps along ρ^2 direction and then 1 step along ρ^3 direction.

$P_4 = 0, \rho^5, \rho^4+\rho^5, 2\rho^4, 3\rho^4, \ldots, (h+1)\rho^4$, i.e., going 1 step along ρ^5, 1 step along ρ^4, 1 step along ρ^3, and then $h-1$ steps along ρ^4 directions.

$P_5 = 0, \rho^3, \rho^2+\rho^3, 2\rho^2+\rho^3, \ldots, (h+1)\rho^2+\rho^3$, i.e., going 1 step along ρ^3 direction and then $h+1$ steps along ρ^2 direction.

$P_6 = 0, \rho^4, \rho^3+\rho^4, \rho^3+2\rho^4, \ldots, \rho^3+h\rho^4, (h+1)\rho^4$, i.e., going 1 step along ρ^4,1 step along ρ^3, $h-1$ steps along ρ^4, and then 1 step along ρ^5 directions.

3.2) If $2h-2d+y<0$ and $2h-2d+x-a<0$ then

$P_3 = 0, \rho^2, 2\rho^2, \ldots, (a+y)\rho^2, (a+y)\rho^2+\rho^3, (a+y)\rho^2+2\rho^3, \ldots, (a+y)\rho^2+(2h-d-a+2)\rho^3$, i.e., going $a+y$ steps along ρ^2 direction and then $2h-d-a+2$ steps along ρ^3 direction.

$P_4 = 0, \rho^5, 2\rho^5, \ldots, (x-a)\rho^5, \rho^4+(x-a)\rho^5, 2\rho^4+(x-a)\rho^5, \ldots, (2h-d+2)\rho^4+(x-a)\rho^5$, i.e., going $x-a$ steps along ρ^5 direction and then $2h-d+2$ steps along ρ^4 direction.

$P_5 = 0, \rho^3, 2\rho^3, \ldots, (2h-d-a+2)\rho^3, \rho^2+(2h-d-a+2)\rho^3, 2\rho^2+(2h-d-a+2)\rho^3, \ldots, (a+y)\rho^2+(2h-d-a+2)\rho^3$, i.e., going $2h-d-a+2$ steps along ρ^3 direction and then $a+y$ steps along ρ^2 direction.

$P_6 = 0, \rho^4, 2\rho^4, \ldots, (2h-d+2)\rho^4, (2h-d+2)\rho^4+\rho^5, (2h-d+2)\rho^4+2\rho^5, \ldots, (2h-d+2)\rho^4+(x-a)\rho^5$, i.e., going $2h-d+2$ steps along ρ^4 direction and then $x-a$ steps along ρ^5 direction.

3.3) If $2h - 2d + y < 4$ and $2h - 2d + x - a < 4$ and $4h - 2d + x + y - a + 4 < 2(2h - d + y + 3)$ and $4h - 2d + x + y - a + 4 < 2(2h - d + x - a + 3)$ and $x - a > 0$ then

$P_5 = 0, \rho^3, \rho^2 + \rho^3, 2\rho^2 + \rho^3, \ldots, (a+y)\rho^2 + \rho^3, (a+y)\rho^2 + 2\rho^3, \ldots, (a+y)\rho^2 + (2h - d - a + 2)\rho^3$, i.e., going 1 step along ρ^3, $a + y$ steps along ρ^2, and then $2h - d - a + 1$ steps along ρ^3 directions.

$P_6 = 0, \rho^4, \rho^4 + \rho^5, \rho^4 + 2\rho^5, \ldots, \rho^4 + (x-a)\rho^5, 2\rho^4 + (x-a)\rho^5, \ldots, (2h-d+2)\rho^4 + (x-a)\rho^5$, i.e., going 1 step along ρ^4, $x - a$ steps along ρ^5, and then $2h - d + 1$ steps along ρ^4 directions.

3.4) If $2h - 2d + y < 3$ and $2h - d + y + 3 < 2h - d + x - a + 3$ then

$P_5 = 0, \rho^3, \rho^2 + \rho^3, 2\rho^2 + \rho^3, \ldots, (a+y+1)\rho^2 + \rho^3, (a+y+1)\rho^2 + 2\rho^3, \ldots, (a+y+1)\rho^2 + (2h - d - a + 1)\rho^3, (a+y)\rho^2 + (2h - d - a + 2)\rho^3$, i.e., going 1 step along ρ^3, $a + y + 1$ steps along ρ^2, $2h - d - a$ steps along ρ^3, and then 1 step along ρ^4 directions.

$P_6 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3, \rho^2 + 2\rho^3, 2\rho^2 + 2\rho^3, \ldots, (a+y)\rho^2 + 2\rho^3, (a+y)\rho^2 + 3\rho^3, \ldots, (a+y)\rho^2 + (2h - d - a + 2)\rho^3$, i.e., going 1 step along ρ^4, 1 steps along ρ^3, $a + y + 1$ steps along ρ^2, and then $2h - d - a$ steps along ρ^3 directions.

3.5) If $2h - 2d + x - a < 3$ then

$P_5 = 0, \rho^3, \rho^3 + \rho^4, 2\rho^4, 2\rho^4 + \rho^5, 2\rho^4 + 2\rho^5, \ldots, 2\rho^4 + (x-a)\rho^5, 3\rho^4 + (x-a)\rho^5, \ldots, (2h - d + 2)\rho^4 + (x-a)\rho^5$, i.e., going 1 step along ρ^3, 1 step along ρ^4, $x - a + 1$ along ρ^5, and then $2h - d$ steps along ρ^4 directions.

$P_6 = 0, \rho^4, \rho^4 + \rho^5, \rho^4 + 2\rho^5, \ldots, \rho^4 + (x - a + 1)\rho^5, 2\rho^4 + (x - a + 1)\rho^5, \ldots, (2h - d + 1)\rho^4 + (x - a + 1)\rho^5, (2h - d + 2)\rho^4 + (x - a)\rho^5$, i.e., going 1 step along ρ^4, $x - a + 1$ steps along ρ^5, $2h - d$ along ρ^4, and then 1 step along ρ^3 directions.

□

Example 2.19. Let $\alpha = 3 + 5\rho$, the source node $S = 0$, and the destination node $D = 3 + \rho$. Then, $h = 3$ and $d = 4$. Since the condition of case 3.1 in Theorem 2.14 is satisfied, the paths are:

$P_1 = 0, \rho, 1+\rho, 2+\rho, 3+\rho.$

$P_2 = 0, 1, 2, 3, 3+\rho.$

$P_3 = 0, \rho^2, 2\rho^2, 3\rho^2, 4\rho^2 \equiv 3\rho^4 + \rho^5\ (mod\ \alpha), 4\rho^2 + \rho^3 \equiv 3+\rho\ (mod\ \alpha).$

$P_4 = 0, \rho^5, \rho^4+\rho^5, 2\rho^4, 3\rho^4, 4\rho^4 \equiv 3+\rho\ (mod\ \alpha).$

$P_5 = 0, \rho^3, \rho^2+\rho^3, 2\rho^2+\rho^3, 3\rho^2+\rho^3, 4\rho^2+\rho^3 \equiv 3+\rho\ (mod\ \alpha).$

$P_6 = 0, \rho^4, \rho^3+\rho^4, \rho^3+2\rho^4, \rho^3+3\rho^4 \equiv 2+2\rho\ (mod\ \alpha), 4\rho^4 \equiv 3+\rho\ (mod\ \alpha).$

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_2| = 4$, $|P_3| = |P_5| = 5$ and $|P_4| = |P_6| = 5$. Figure 2.24 illustrates all these shortest node disjoint paths from S to D.

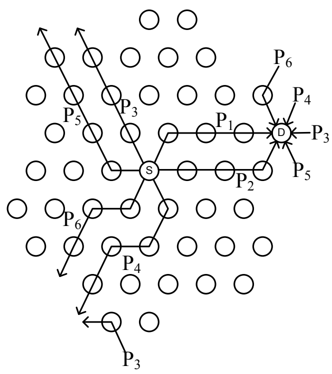

Figure 2.24: Shortest Node Disjoint Paths in EJ Network with $\alpha = 3 + 5\rho$ from $S = 0$ to $D = 3 + \rho$.

Theorem 2.15. *Suppose the generator of an EJ network is $\alpha = a + b\rho$ with $0 \le a \le b$ and $b = a + 2j$ for $j = 0, 1, 2, \ldots$ etc, i.e. b is equal to a plus some even number. Let 0 and $\beta = x + y\rho$, where $x = 0$ and $y > 0$, be the source and destination nodes, respectively. Then there are six node disjoint paths from 0 to β, with path lengths strictly less than $d+9$, where $d = x + y$ is the minimum distance from 0 to β.*

Proof. Let $h = \frac{a+b}{2} - 1$. The length of the maximum node disjoint paths can be reduced according to the following cases:

1. <u>*When $b = a$:*</u>

   1.1) If $y = M$ and ρ^j where $j = 1$ or 2 (i.e., sector 1 or 2) then

$P_3 = 0, \rho^3, 2\rho^3, \ldots, y\rho^3$, i.e., going y steps along ρ^3 direction.

$P_4 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3 + \rho^4, \ldots, (y-1)\rho^3 + \rho^4, y\rho^3$, i.e., going 1 step along ρ^4, $y-1$ steps along ρ^3, and then 1 step along ρ^2 directions.

$P_5 = 0, \rho^5, 2\rho^5, \ldots, y\rho^5$, i.e., going y steps along ρ^5 direction.

$P_6 = 0, 1, 1 + \rho^5, 1 + 2\rho^5, \ldots, 1 + (y-1)\rho^5, y\rho^5$, i.e., going 1 step along ρ^0, $y-1$ steps along ρ^5, and then 1 step along ρ^4 directions.

1.2) If $h - y < 1$ then

$P_4 = 0, \rho^3, 2\rho^3, \ldots, a\rho^3, a\rho^3 + \rho^4, a\rho^3 + 2\rho^4, \ldots, a\rho^3 + (h-y+1)\rho^4$, i.e., going a steps along ρ^3 direction and then $h-y+1$ steps along ρ^4 direction.

$P_5 = 0, \rho^5, \rho^4 + \rho^5, 2\rho^4 + \rho^5, \ldots, (a-y)\rho^4 + \rho^5, (a-y)\rho^4 + 2\rho^5, \ldots, (a-y)\rho^4 + (2h-a+2)\rho^5$, i.e., going 1 step along ρ^5, $a-y$ steps along ρ^4, and then $2h-a+1$ steps along ρ^5 directions.

$P_6 = 0, \rho^4, 2\rho^4, \ldots, (h-y+1)\rho^4, \rho^3 + (h-y+1)\rho^4, 2\rho^3 + (h-y+1)\rho^4, \ldots, a\rho^3 + (h-y+1)\rho^4$, i.e., going $h-y+1$ steps along ρ^4 direction and then a steps along ρ^3 direction.

1.3) If $1 \le h - y < 3.5$ then

$P_6 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3 + \rho^4, \ldots, a\rho^3 + \rho^4, a\rho^3 + 2\rho^4, \ldots, a\rho^3 + (h-y+1)\rho^4$, i.e., going 1 step along ρ^4, a steps along ρ^3, and then $h-y$ steps along ρ^4 directions.

2. *When $b > a$:*

2.1) If $h - y < 0.5$ and $2h - y + 3 < 2h - a + 2$ then

$P_4 = 0, \rho^3, 2\rho^3, \ldots, (a+1)\rho^3, (a+1)\rho^3 + \rho^4, (a+1)\rho^3 + 2\rho^4, \ldots, (a+1)\rho^3 + (2h-y-a+1)\rho^4, a\rho^3 + (2h-y-a+2)\rho^4$, i.e., going $a+1$ steps along ρ^3, $2h-y-a+1$ steps along ρ^4, and then 1 step along ρ^5 directions.

$P_5 = 0, \rho^5, \rho^4 + \rho^5, 2\rho^4 + \rho^5, \ldots, (2h-y-a+1)\rho^4 + \rho^5, (2h-y-a+2)\rho^4, \rho^3 + (2h-y-a+2)\rho^4, 2\rho^3 + (2h-y-a+2)\rho^4, \ldots, a\rho^3 + (2h-y-a+2)\rho^4$, i.e., going 1 step along ρ^5, $2h-y-a+1$ steps along ρ^4, and then $a+1$ steps along ρ^3 directions.

$P_6 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3 + \rho^4, \ldots, a\rho^3 + \rho^4, a\rho^3 + 2\rho^4, \ldots, a\rho^3 + (2h - y - a + 2)\rho^4$, i.e., going 1 step along ρ^4, a steps along ρ^3, and then $2h - y - a + 1$ steps along ρ^4 directions.

2.2) If $2h - y - a < 2$ then

$P_4 = 0, \rho^3, 2\rho^3, \ldots, a\rho^3, a\rho^3 + \rho^4, a\rho^3 + 2\rho^4, \ldots, a\rho^3 + (2h - y - a + 2)\rho^4$, i.e., going a steps along ρ^3 direction and then $2h - y - a + 2$ steps along ρ^4 direction.

$P_5 = 0, \rho^5, 1 + \rho^5, 2 + \rho^5, \ldots, (y - a) + \rho^5, (y - a) + 2\rho^5, \ldots, (y - a) + (2h - y + 2)\rho^5$, i.e., going 1 step along ρ^5, $y - a$ steps along ρ^0, and then $2h - y + 1$ steps along ρ^5 directions.

$P_6 = 0, \rho^4, 2\rho^4, \ldots, (2h - y - a + 2)\rho^4, \rho^3 + (2h - y - a + 2)\rho^4, 2\rho^3 + (2h - y - a + 2)\rho^4, \ldots, a\rho^3 + (2h - y - a + 2)\rho^4$, i.e., going $2h - y - a + 2$ steps along ρ^4 and then a steps along ρ^3 directions.

2.3) If $h - y < 3.5$ and $2h - y - a \geq 2$ then

$P_6 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3 + \rho^4, \ldots, a\rho^3 + \rho^4, a\rho^3 + 2\rho^4, \ldots, a\rho^3 + (2h - y - a + 2)\rho^4$, i.e., going 1 step along ρ^4, a steps along ρ^3, and then $2h - y - a + 1$ steps along ρ^4 directions.

$\square$

Example 2.20. Let $\alpha = 3 + 3\rho$, the source node $S = 0$, and the destination node $D = 3\rho$. Then, $h = 2$ and $d = 3$. Since the condition of case 1.1 in Theorem 2.15 is satisfied, the paths are:

$P_1 = 0, \rho, 2\rho, 3\rho$.

$P_2 = 0, \rho^2, \rho + \rho^2, 2\rho + \rho^2, 3\rho$.

$P_3 = 0, 1, 1 + \rho^5, 1 + 2\rho^5 \equiv \rho^2 + 2\rho^3 \ (mod\ \alpha), 3\rho^5 \equiv 3\rho \ (mod\ \alpha)$.

$P_4 = 0, \rho^3, 2\rho^3, 3\rho^3 \equiv 3\rho \ (mod\ \alpha)$.

$P_5 = 0, \rho^5, 2\rho^5, 3\rho^5 \equiv 3\rho \ (mod\ \alpha)$.

$P_6 = 0, \rho^4, \rho^3 + \rho^4, 2\rho^3 + \rho^4 \equiv 1 + 2\rho \ (mod\ \alpha), 3\rho^3 \equiv 3\rho \ (mod\ \alpha)$.

Let $|P_i|$ denote the length of the path i. Then, $|P_1| = |P_4| = |P_5| = 3$ and $|P_2| = |P_3| =$

$|P_6| = 4$. Figure 2.25 illustrates all these shortest node disjoint paths from S to D.

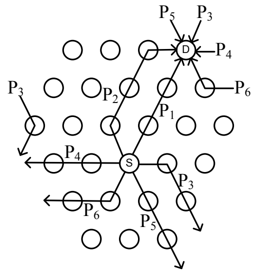

Figure 2.25: Shortest Node Disjoint Paths in EJ Network with $\alpha = 3 + 3\rho$ from $S = 0$ to $D = 3\rho$.

So far, we have explained how to find all the shortest node disjoint paths when the destination node is in sector 1. When the destination node is in one of the other sectors, say sector j, the proofs are same except we need to use ρ^j for that sector.

2.5 Conclusion

In this chapter, we reviewed two types of interconnection networks, the Gaussian and the Eisenstein-Jacobi networks, with their quotient rings and distance distribution properties. Based on the distance distribution properties of these networks, we proposed the solutions to finding all shortest node disjoint paths between a given source and destination nodes.

Chapter 3

Edge Disjoint Hamiltonian Cycles in EJ Networks

The problem of finding edge disjoint Hamiltonian cycles for some interconnection networks was considered in [2][3][4][8][7][6][32][25]. In this chapter, we first review how to construct three edge disjoint Hamiltonian cycles in EJ networks (defined in Section 2.3) when $\gcd(a, b) = 1$ as described in [16]. Then we show how to generate three edge disjoint Hamiltonian cycles even when the $\gcd(a, b) = d > 1$. In order to obtain this solution, first we introduce the rectangular representation of the EJ networks.

3.1 Previous Work

Given an EJ network generated with $\alpha = a + b\rho$ such that $0 \leq a \leq b$ and $\gcd(a, b) = 1$, the EJ network can be partitioned into three edge disjoint Hamiltonian cycles as follows [16].

For $j = 1, \rho$, or ρ^2, start at node 0 and traverse the path by adding $j \ mod \ \alpha$. Since the path has a limited number of nodes, at some point, a node, say node β, will repeat. i.e., there exists a node β and an integer $k > 0$ such that $\beta + kj \equiv \beta \ (mod \ \alpha)$. i.e., $k.j \equiv 0 \ (mod \ \alpha)$. Here, we only show that for $j = 1$, $k = |\alpha|^2 = N(\alpha)$ and hence the cycle is Hamiltonian. Similar proofs can be given when $j = \rho$ and $j = \rho^2$. Moreover, the cycles are edge disjoint since the edges used in the first, second, and the third cycles are respectively along the $\rho^0 = 1$, ρ, and ρ^2 dimensions.

Since $k \equiv 0 \ (mod \ \alpha)$, $\exists x + y\rho \in \mathbb{Z}[\rho]$ such that

$$\begin{aligned} k &= (x + y\rho)(a + b\rho) && (3.1) \\ &= ax + (bx + ay)\rho + by\rho^2 && (3.2) \end{aligned}$$

Since $\rho^2 = \rho - 1$, (3.2) gives

$$k = (ax - by) + (bx + ay + by)\rho \quad (3.3)$$

$$= (ax - by) + (ay + b(x + y))\rho \quad (3.4)$$

Since k is an integer, $ay + b(x + y) = 0$ with $\gcd(a, a + b) = \gcd(a, b) = 1$. Therefore, there exists an integer $c \geq 1$ such that $(x, y) = c(a + b, -b)$. Thus

$$k = ax - by = ca(a + b) + cb^2 \quad (3.5)$$

$$= c(a^2 + b^2 + ab) \quad (3.6)$$

$$\geq a^2 + b^2 + ab = |\alpha|^2 \quad (3.7)$$

This proves the cycle is Hamiltonian as described in [16]. □

Example 3.1. Figure 3.1 shows the first, second, and third edge disjoint Hamiltonian cycles in the EJ network generated with $\alpha = 3 + 4\rho$ by adding 1 *mod* α, ρ *mod* α, and ρ^2 *mod* α to node 0, respectively.

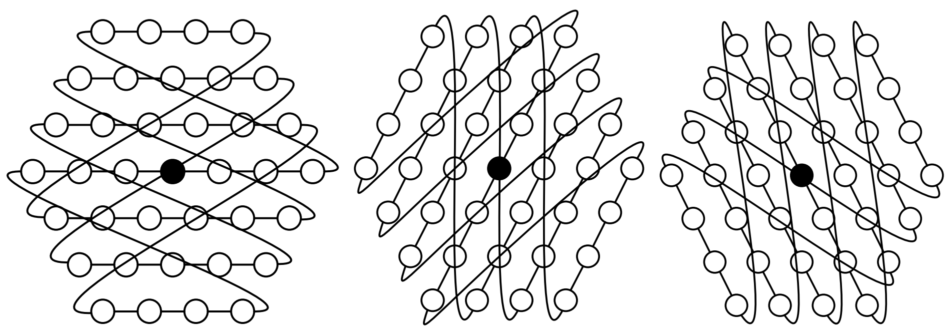

Figure 3.1: Three Edge Disjoint Hamiltonian Cycles in EJ Generated with $\alpha = 3 + 4\rho$.

3.2 Edge Disjoint Hamiltonian Cycles when $gcd(a, b) = d > 1$

3.2.1 Rectangle Representation

A new representation, called the rectangular representation, of the EJ network generated by $\alpha = a + b\rho$, where $\gcd(a, b) = d > 1$ is described in this section. This representation is useful in finding three edge disjoint Hamiltonian cycles.

The idea of this representation is to arrange $a^2 + b^2 + ab$ nodes in a rectangle of size $d \times \frac{a^2+b^2+ab}{d}$, where $d = \gcd(a, b) > 1$. The node zero is located at the lower left corner of the rectangle. Each internal node is connected to six other nodes, called neighbors. The nodes at the boundary have wrap-around links with other boundary nodes. The following lemmas and theorem are useful in describing the rectangular representation.

Lemma 3.1. *The numbers $i\rho^2 + k$ such that $0 \le i \le d-1$ and $0 \le k \le r-1$ give a complete residue class mod α, where $\alpha = a + b\rho$, $a = a_1 d$, $b = b_1 d$, and $gcd(a_1, b_1) = 1$.*

Proof. Suppose $(i_1\rho^2 + k_1) \equiv (i_2\rho^2 + k_2)\ mod\ \alpha$. Then

$$\begin{aligned}
\rho^2(i_1 - i_2) + (k_1 - k_2) &= (x + y\rho)(a + b\rho) && (3.8)\\
&= ax + (ay + bx)\rho + by\rho^2 && (3.9)\\
&= (ax + ay + bx) + (ay + bx + by)\rho^2 && (3.10)\\
&= d(x(a_1 + b_1) + a_1 y) + d(b_1 x + (a_1 + b_1)y)\rho^2 && (3.11)
\end{aligned}$$

Thus, $i_1 - i_2 = d[b_1 x + (a_1 + b_1)y]$. This implies $d | i_1 - i_2$. Since $|i_1 - i_2| < d$

$$\begin{aligned}
i_1 - i_2 &= 0 && (3.12)\\
i_1 &= i_2 && (3.13)
\end{aligned}$$

We get

$$b_1 x + (a_1 + b_1) y = 0 \quad (3.14)$$

$$\Rightarrow y = -c b_1 \quad (3.15)$$

$$\Rightarrow x = c(a_1 + b_1) \quad (3.16)$$

And

$$k_1 - k_2 = cd(a_1^2 + b_1^2 + 2a_1 b_1 - a_1 b_1) \quad (3.17)$$

$$= cd(a_1^2 + b_1^2 + a_1 b_1) \quad (3.18)$$

$$= cr \quad (3.19)$$

Since $|k_1 - k_2| < r$, we get $k_1 = k_2$. Thus, $i\rho^2 + k$, $0 \leq i \leq d-1$, $0 \leq k \leq r-1$, represent all the elements under $mod\ \alpha$. $\square$

The above lemma shows that the nodes in an EJ network can be arranged in a rectangular format of size d rows and $r = d(a_1^2 + b_1^2 + a_1 b_1)$ columns as shown in Figure 3.2.

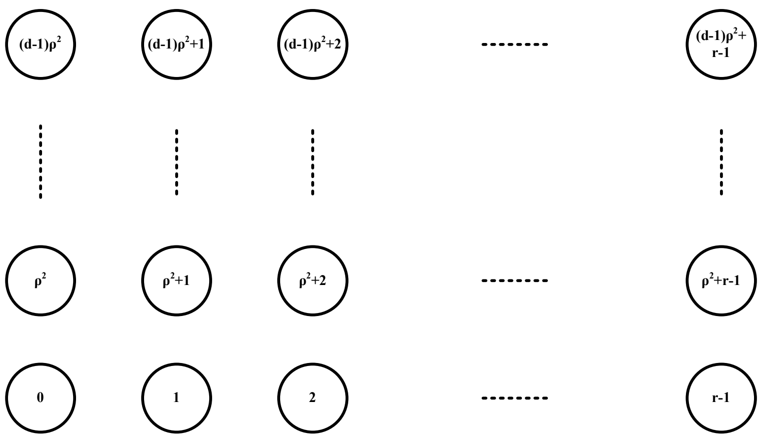

Figure 3.2: Rectangular Representation of an EJ Network.

A node at the i–th row and j–th column is given the address $i\rho^2 + j$, where $0 \leq i < d$ and $0 \leq j < r$. Thus, an internal node $i\rho^2 + j$ is connected to $i\rho^2 + j \pm 1$, $i\rho^2 + j \pm \rho^2$, and

$i\rho^2 + j \pm \rho = i\rho^2 + j \pm (\rho^2 + 1)$ as shown in Figure 3.3.

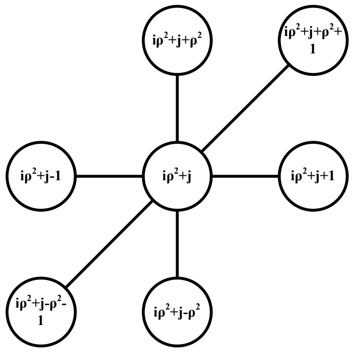

Figure 3.3: Edges of Node $i\rho^2 + j$.

Now, we need to find out how the boundary nodes are connected. For this, the following lemma is useful.

Lemma 3.2. *Let* $r = d(a_1^2 + b_1^2 + a_1 b_1)$ *with* $gcd(a, b) = d$, $a = a_1 d$, *and* $b = b_1 d$. *Then* $r \equiv 0 \ mod \ \alpha$ *where* $\alpha = a + b\rho$.

Proof. Suppose $r \equiv (p + q\rho^2) \ mod \ \alpha$, where $0 \leq p < r$ and $0 \leq q < d$. Then

$$\begin{aligned} r - (p + q\rho^2) &= (x + y\rho)(a + b\rho) && (3.20) \\ (r - p) - q\rho^2 &= d[a_1 x + (a_1 y + b_1 x)\rho + b_1 y \rho^2] && (3.21) \\ &= d[(x(a_1 + b_1) + a_1 y) + (b_1 x + (a_1 + b_1)y)\rho^2] && (3.22) \end{aligned}$$

This implies $d(b_1 x + (a_1 + b_1)y) = -q$ and so $d|q$. Since $0 \leq q < d$, we get $q = 0$. Therefor $b_1 x + (a_1 + b_1)y = 0$ implies $x = c(a_1 + b_1)$ and $y = -cb_1$ for some integer $c \geq 0$. Thus,

$$\begin{aligned} r - p &= d(x(a_1 + b_1) + a_1 y) && (3.23) \\ &= d(c(a_1 + b_1)(a_1 + b_1) + a_1(-cb_1)) && (3.24) \\ &= cd(a_1^2 + b_1^2 + a_1 b_1) = cr && (3.25) \end{aligned}$$

Since $0 \leq p < r$, $c \geq 1$. Furthermore, $r|(r - p)$ and so $p = 0$. This implies $r \equiv 0 \ mod \ \alpha$. $\square$

The above lemma implies that the boundary node $i\rho^2 + r - 1$, $0 \leq i < d$, is connected to the node $(i\rho^2 + r)\ mod\ \alpha \equiv i\rho^2\ mod\ \alpha$ as its $+1$ edge. In addition, the ρ-edge of the node $i\rho^2 + r - 1$ for $0 \le i \le d-2$ is connected to $(i\rho^2 + r - 1) + \rho = i\rho^2 + r - 1 + \rho^2 + 1 = (i+1)\rho^2 + r \equiv ((i+1)\rho^2)\ mod\ \alpha$. These edges are shown in Figure 3.4.

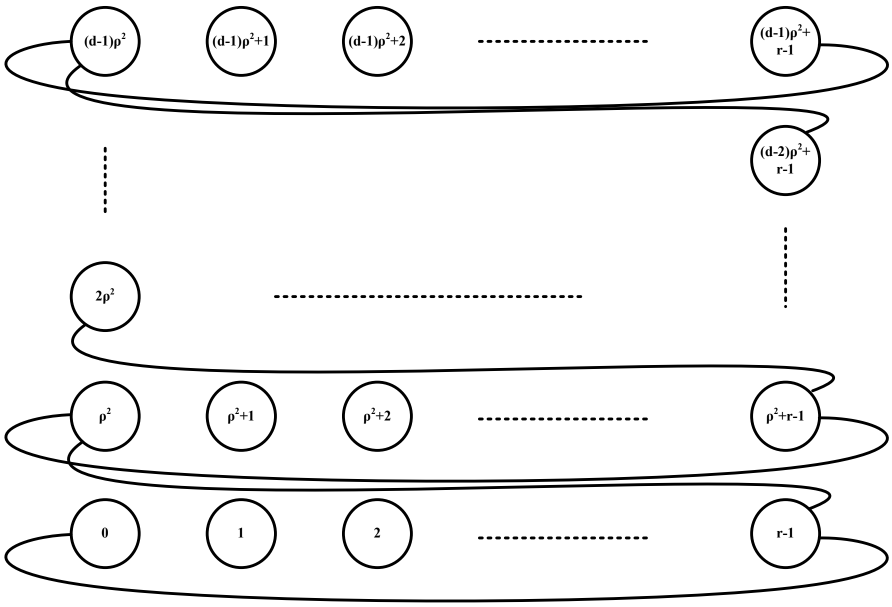

Figure 3.4: Wrap-around Links of Node $i\rho^2 + r - 1$ for $0 \le i < d$.

Now, we need to show how the $-\rho^2$ edges from node k, $0 \le k < r$, are connected. For this, we need to find $-\rho^2\ (mod\ \alpha)$. Let $-\rho^2 \equiv (p + q\rho^2)\ mod\ (a + b\rho)$ where $0 \le p < r$ and $0 \le q < d$. Then

$$-\rho^2 - p - q\rho^2 \equiv 0\ mod\ \alpha \tag{3.26}$$

$$-p - (1+q)\rho^2 = (x + y\rho)(a + b\rho) \tag{3.27}$$

$$= d[a_1x + (a_1y + b_1x)\rho + b_1y\rho^2] \tag{3.28}$$

$$= d[(a_1x + a_1y + b_1x) + (a_1y + b_1x + b_1y)\rho^2] \tag{3.29}$$

$$= d[(a_1(x+y) + b_1x) + (a_1y + b_1(x+y))\rho^2] \tag{3.30}$$

This implies $-(1+q) = d[a_1y + b_1(x+y)]$. Since $d|(-(1+q))$ and $0 \le q \le d-1$, we get

$q = d - 1$. Thus

$$
\begin{aligned}
-(1 + d - 1) &= d[a_1 y + b_1(x + y)] && (3.31) \\
1 &= -(a_1 y + b_1(x + y)) && (3.32)
\end{aligned}
$$

Since $gcd(a_1, b_1) = 1$, using the extended GCD algorithm we get

$$
\begin{aligned}
a_1 A + b_1 B &= 1 && (3.33) \\
\Rightarrow A &= -y \Rightarrow y = -A && (3.34) \\
\Rightarrow B &= -x - y \Rightarrow x = A - B && (3.35)
\end{aligned}
$$

Equating the $\rho^0 = 1$ term in (3.30), we get

$$
\begin{aligned}
-p &= d[a_1(x + y) + b_1 x] && (3.36) \\
p &= -d[-a_1 B + b_1(A - B)] && (3.37) \\
&= d[a_1 B + b_1(B - A)] && (3.38)
\end{aligned}
$$

Note that $(A + ib_1)$ and $(B - ia_1)$ also satisfy the equations (3.33) and (3.38). This is because

$$
\begin{aligned}
a_1(A + ib_1) + b_1(B - ia_1) &= a_1 A + b_1 B = 1 && (3.39) \\
p &= d[a_1(B - ia_1) + b_1(B - ia_1 - A - ib_1)] && (3.40) \\
&= d[(a_1 B + b_1(B - A)) - i(a_1^2 + b_1^2 + a_1 b_1)] && (3.41) \\
&= d(a_1 B + b_1(B - A)) - id(a_1^2 + b_1^2 + a_1 b_1) && (3.42) \\
&= d(a_1 B + b_1(B - A)) - ir && (3.43) \\
&\equiv d(a_1 B + b_1(B - A)) \ mod \ r \ where \ 0 \leq p < r. && (3.44)
\end{aligned}
$$

The above argument shows that the $-\rho^2$ edge from node $k = 0$ is connected to $p + q\rho^2$

where $q = d-1$ (i.e., row $d-1$) and $p = d[a_1B + b_1(B-A)]$, i.e., we need to choose A and B such that $0 \leq d[a_1B + b_1(B-A)] < r$. In general, the $-\rho^2$ edge from node $k+j$, $0 \leq j < r$, is connected to the node $((d-1)\rho^2 + d[a_1B + b_1(B-A)] + j)\ mod\ r$.

Similarly, the $-(\rho^2+1) = -\rho$ edge from node $k+j$ is connected to $((d-1)\rho^2 + d[a_1B + b_1(B-A)] + j - 1)\ mod\ r$.

In summary, based on (3.38), let $L \equiv p\ mod\ r$, then the position of node $-\rho^2$ in the rectangle representation is $(d-1)\rho^2 + L$. Now, let $C = x + y\rho^2 \in S$. Then the wrap-around links are defined as:

1. If $0 \leq x < r - L$ then C is connected to $(x+L) + (d-1)\rho^2$ as its $-\rho^2$ edge and to $(x+L-1) + (d-1)\rho^2$ as its $-\rho$ edge.

2. If $r - L \leq x \leq r-1$ then C is connected to $(x-r-L) + (d-1)\rho^2$ as its $-\rho^2$ edge and to $((x-r-L-1)\ mod\ r) + (d-1)\rho^2$ as its $-\rho$ edge.

3. If $0 \leq y \leq d-1$ then C is connected to $(r-1) + y\rho^2$ as its -1 edge and to $(r-1) + ((y-1)\ mod\ (d-1))\rho^2$ as its $-\rho$ edge.

For example, Figure 3.5 illustrates the rectangular representation for EJ network generated with $\alpha = 3 + 3\rho$.

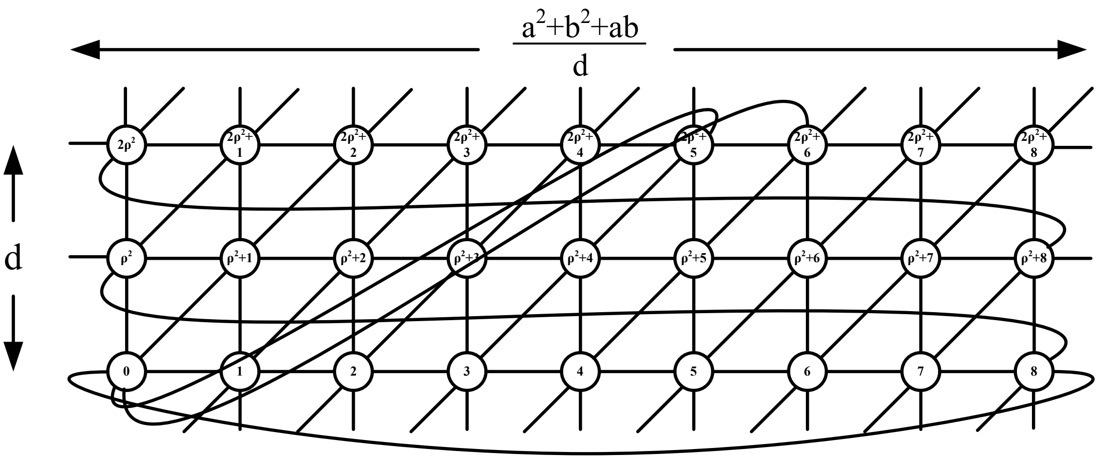

Figure 3.5: Rectangular Representation for EJ Network Generated with $\alpha = 3 + 3\rho$.

For simplicity, in a rectangular representation, we call the dimensions 1, $\rho^2 + 1 = \rho$, and ρ^2, in respective order, as horizontal, diagonal, and vertical dimensions. In addition, from the above argument it can be seen that there exist three sets of d disjoint cycles corresponding to these dimensions. For clarity, this is formaly proved in the following theorem.

Theorem 3.1. *Given an EJ network generated with $\alpha = a + b\rho$ where $gcd(a, b) = d > 1$, there exist three sets of d disjoint cycles where each cycle is of length $r = \frac{N(\alpha)}{d}$. The edges in the first, second, and third sets of d disjoint cycles are along, in respective order, the horizontal, the vertical, and the diagonal dimensions.*

Proof. We follow the same proof as in [1] to prove that there exists a set of d disjoint cycles in the horizontal dimension. Similar proofs can be followed for the vertical and diagonal dimensions. The proof of the horizontal dimension is as follows.

Since the path has a limited number of nodes, there exists a node at distance j, j is an integer > 0, from node β_1 such that $\beta_1 + j \equiv \beta_1$ $(mod\ \alpha)$. i.e., we visited all the nodes in the path and returned to node β_1. This implies that $j \equiv 0\ mod\ \alpha$. Thus, j is the smallest EJ integer multiple of α. Furthermore, we conclude that $j = \beta\alpha = (x + y\rho)(a + b\rho) = (xa - yb) + (xb + (a + b)y)\rho$. Since j is an integer, the coefficient of ρ is zero and so $xb + (a+b)y = 0$. Since $\gcd(a, b) = d$, we get $a = a_1 d$, $b = b_1 d$, and $\gcd(a_1, b_1) = 1$. So, $d(xb_1 + (a_1 + b_1)y) = 0$. Since $\gcd(a_1, b_1) = 1$, we get $-(a_1 + b_1)y = xb_1$. Thus, $y = -gb_1$, $x = g(a_1 + b_1)$. Therefore, $j = (xa - yb) = (ag(a_1 + b_1) + bgb_1) = gd(a_1^2 + b_1^2 + a_1 b_1) = g(\frac{a^2+b^2+ab}{d})$.

We need to prove that $g = 1$. If $\frac{a^2+b^2+ab}{d}$ $mod\ \alpha = 0$, then $g = 1$. Now consider

$$\begin{aligned}
\frac{a^2 + b^2 + ab}{d} &= d(a_1^2 + b_1^2 + a_1 b_1) && (3.45)\\
&= d(a_1 + b_1\rho)(a_1 + b_1 - b_1\rho) && (3.46)\\
&= (a_1 d + b_1 d\rho)(a_1 + b_1 - b_1\rho) && (3.47)\\
&= (a + b\rho)(a_1 + b_1 - b_1\rho) && (3.48)\\
&= \alpha(a_1 + b_1 - b_1\rho) && (3.49)
\end{aligned}$$

This implies $\alpha(a_1 + b_1 - b_1\rho)\ mod\ \alpha = 0$. Thus, $g = 1$. This proves that by adding 1 to the current node address to traverse the successive nodes we get a cycle of length $\frac{a^2+b^2+ab}{d}$.

Furthermore, suppose β_2 is not in the above cycle. Then we can prove that the nodes β_2+j, $0 \leq j < r$, form another cycle of length r. Now, consider a node β_3 which is not in the above two cycles. Then all the nodes β_3+j, $0 \leq j < r$, form another disjoint cycle of length r. This way we can get d disjoint cycles of length r. This can be proved as follows. Suppose $\gamma_1 \equiv (k_1\rho^2 + j_1)\ mod\ \alpha$, $0 \leq k_1 < d$, $0 \leq j_1 < r$ (i.e., γ_1 is in the k_1–th cycle). We need to prove that γ_1 is not in any other cycle. Suppose γ_1 is also equal to $\gamma_2 \equiv (k_2\rho^2 + j_2)\ mod\ \alpha$, $0 \leq k_2 < d$, $0 \leq j_2 < r$. Then $(k_1\rho^2 + j_1) \equiv (k_2\rho^2 + j_2)\ mod\ \alpha$. Then

$$
\begin{aligned}
\rho^2(k_1 - k_2) + (j_1 - j_2) &\equiv 0\ mod\ \alpha && (3.50)\\
&= (x + y\rho)(a + b\rho) && (3.51)\\
&= d[(a_1x - b_1y) + \rho(b_1x + y(a_1 + b_1))] && (3.52)\\
&= d[(a_1x - b_1y) + b_1x + y(a_1 + b_1) + \\
&\quad \rho^2(b_1x + y(a_1 + b_1))] && (3.53)
\end{aligned}
$$

Thus, $(k_1 - k_2) = d[b_1x + y(a_1 + b_1)]$. This implies $d|(k_1 - k_2)$. But $|k_1 - k_2| < d$ and hence $k_1 = k_2$ and $j_1 = j_2$. That is, we get d disjoint cycles which are of the form $\{i\rho^2 + j \mid 0 \leq i < d, 0 \leq j < r\}$. □

Moreover, the horizontal (and vertical and diagonal) d disjoint cycles are numbered from 0 to $d - 1$ starting from the bottom to the top (left to right) of the rectangle. Figure 3.6, for example, illustrates the three sets of d disjoint cycles in EJ network generated with $\alpha = 3 + 3\rho$. The graph has three node disjoint cycles in each dimension and each cycle contains 9 nodes.

Note that, as shown in Figure 3.6, the rectangular representation can be divided into $\frac{r}{d}$ blocks of size $d \times d$, which is helpful in exchanging edges, described in the next section.

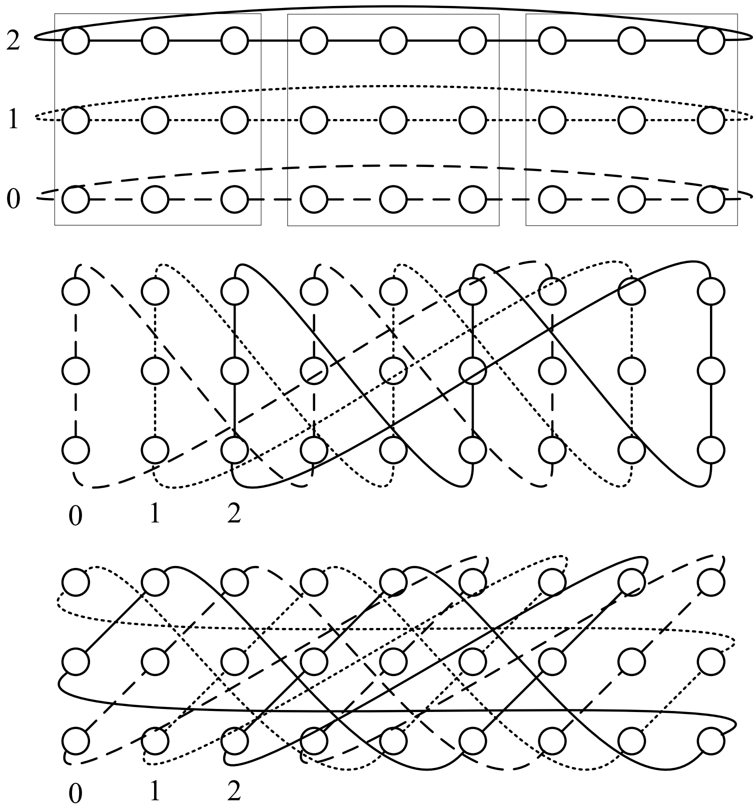

Figure 3.6: Three Sets of 3 Disjoint Cycles in EJ Generated with $\alpha = 3 + 3\rho$.

3.2.2 Edge Disjoint Hamiltonian Cycles

In the above, we have illustrated how to represent EJ networks in a rectangular representation. Next, using this representation, we describe how to generate three edge disjoint Hamiltonian cycles in EJ networks generated with $\alpha = a + b\rho$ where $\gcd(a, b) = d > 1$.

So far, we have explained that there exists three sets of d node disjoint cycles in the rectangular representation. In order to get three edge disjoint Hamiltonian cycles, some edges need to be exchanged between the first and the second sets of d node disjoint cycles to get the first two edge disjoint Hamiltonian cycles. In addition, we also need to do some edge exchanges between the first and third sets of d node disjoint cycles to get the third edge disjoint Hamiltonian cycle while the first cycle remains edge disjoint Hamiltonian cycle. However, selecting proper edges to be exchanged is not straightforward.

In order to get the first and second edge disjoint Hamiltonian cycles, we follow Table

3.1 to do proper edge exchanges between the first (horizontal) and second (vertical) sets of d node disjoint cycles in the first block of the rectangular representation.

Table 3.1: Proper Edge Exchanges between Horizontal and Vertical d Disjoint Cycles for $i = d-2, d-3, \ldots, 1$ as Horizontal and for $j = 1, 2, \ldots, d-2$ as Vertical Cycles.

Horizontal edges	Cycle #	Exchange with	Cycle #	Vertical edges
$(d-1)\rho^2$, $(d-1)\rho^2+1$	$d-1$		0	$(d-1)\rho^2$, $(d-2)\rho^2$
$(d-2)\rho^2$, $(d-2)\rho^2+1$, $(d-2)\rho^2+2$	$d-2$		1	$(d-1)\rho^2+1$, $(d-2)\rho^2+1$, $(d-3)\rho^2+1$
$\ldots$	$\ldots$		$\ldots$	$\ldots$
$i\rho^2+(d-2-i)$, $i\rho^2+(d-2-i)+1$, $i\rho^2+(d-2-i)+2$	i		j	$(d-j)\rho^2+j$, $(d-j-1)\rho^2+j$, $(d-j-2)\rho^2+j$
$\ldots$	$\ldots$		$\ldots$	$\ldots$
$\rho^2+(d-3)$, $\rho^2+(d-2)$, $\rho^2+(d-1)$	1		$d-2$	$2\rho^2+(d-2)$, $\rho^2+(d-2)$, $(d-2)$
$(d-2)$, $(d-1)$	0		$d-1$	$\rho^2+(d-1)$, $(d-1)$

By applying Table 3.1 to the horizontal and vertical d disjoint cycles, we get two edge disjoint Hamiltonian cycles. The first Hamiltonian cycle (in the horizontal dimension) can be obtained by letting $t = 0$ if d is odd (or $t = 1$ if d is even) and letting k be the cycle number in the horizontal dimension. The node visiting sequence is as follows:

1. In cycle $k = 0$, starting from node 0 visit $d-2$ nodes by stepping forward along dimension 1 and then 2 steps up (using the two exchanged vertical edges).

2. For $k = 2, 4, 6, \ldots, d-1-t$, visit all the nodes in cycles k by going forward along dimension 1 except for the last node (unless $k = d-1$ and d is odd) and then go 2 steps up, i.e., use two exchanged vertical edges, (or 1 step down if d is odd and $k = d-1$ or 1 step up if d is even and $k = d-2$).

3. For $k = d-2+t, d-4+t, d-6+t, \ldots, 1$, visit all the nodes in cycles k by going forward along dimension -1, i.e., backward of dimension 1, except for the last node (unless $k = d-1$ and d is even), and then go 2 steps down, i.e., use two exchanged vertical edges, (or 1 step down if $k = 1$).

4. For $k = 0$, visit all the remaining nodes in cycle k until reaching node 0.

The second Hamiltonian (vertical dimension) cycle can be obtained by following the above algorithm but considering the vertical dimension instead. The following lemma proves that these two cycles are edge disjoint Hamiltonian cycles.

Lemma 3.3. *Given an EJ network generated with $\alpha = a + b\rho$ with $gcd\,(a, b) = d$. By applying Table 3.1 to exchange proper edges between the horizontal and the vertical d disjoint cycles, we get the first two edge disjoint Hamiltonian cycles.*

Proof. Using the above argument, we described how to obtain two Hamiltonian cycles by applying proper edge exchanges as shown in Table 3.1. Now, we need to prove that these two Hamiltonian cycles are edge disjoint. The proof is as follows.

At the beginning, we showed that we have two sets of d disjoint cycles, horizontal and vertical. If we remove $2(d-1)$ edges from both the horizontal and the vertical d disjoint cycles, i.e., if a total of $4(d-1)$ edges are removed, and we exchange these edges in a proper way, then we have two edge disjoint Hamiltonian cycles. Thus, the $2(d-1)$ vertical edges, which are placed in the horizontal d disjoint cycles to form the first Hamiltonian cycle, are not used in the vertical d disjoint cycles. Similarly, the $2(d-1)$ horizontal edges are not used in the first Hamiltonian cycle. Thus, we get the first two edge disjoint Hamiltonian cycles in EJ network. □

Example 3.2. Given an EJ generator $\alpha = 9 + 9\rho$ and $\gcd\,(9, 9) = 9$ that is odd, the first and second Hamiltonian cycles can be obtained by using Table 3.1 as shown in Table 3.2.

Figures 3.7 and 3.8 show the first and the second edge disjoint Hamiltonian cycles after performing the proper edge exchanges between the horizontal and vertical sets of 9 disjoint cycles.

Example 3.3. Given an EJ generator $\alpha = 8 + 8\rho$ and $\gcd(8, 8) = 8$ that is even, the first and second Hamiltonian cycles can be obtained by using Table 3.1 as shown in Table 3.3.

Table 3.2: Proper Edge Exchanges between Horizontal and Vertical 9 Disjoint Cycles in EJ Generated with $\alpha = 9 + 9\rho$.

Horizontal edges	Cycle #	Exch. with	Cycle #	Vertical edges
$8\rho^2$, $8\rho^2+1$	8		0	$8\rho^2$, $7\rho^2$
$7\rho^2$, $7\rho^2+1$, $7\rho^2+2$	7		1	$8\rho^2+1$, $7\rho^2+1$, $6\rho^2+1$
$6\rho^2+1$, $6\rho^2+2$, $6\rho^2+3$	6		2	$7\rho^2+2$, $6\rho^2+2$, $5\rho^2+2$
$5\rho^2+2$, $5\rho^2+3$, $5\rho^2+4$	5		3	$6\rho^2+3$, $5\rho^2+3$, $4\rho^2+3$
$4\rho^2+3$, $4\rho^2+4$, $4\rho^2+5$	4		4	$5\rho^2+4$, $4\rho^2+4$, $3\rho^2+4$
$3\rho^2+4$, $3\rho^2+5$, $3\rho^2+6$	3		5	$4\rho^2+5$, $3\rho^2+5$, $2\rho^2+5$
$2\rho^2+5$, $2\rho^2+6$, $2\rho^2+7$	2		6	$3\rho^2+6$, $2\rho^2+6$, ρ^2+6
ρ^2+6, ρ^2+7, ρ^2+8	1		7	$2\rho^2+7$, ρ^2+7, 7
7, 8	0		8	ρ^2+8, 8

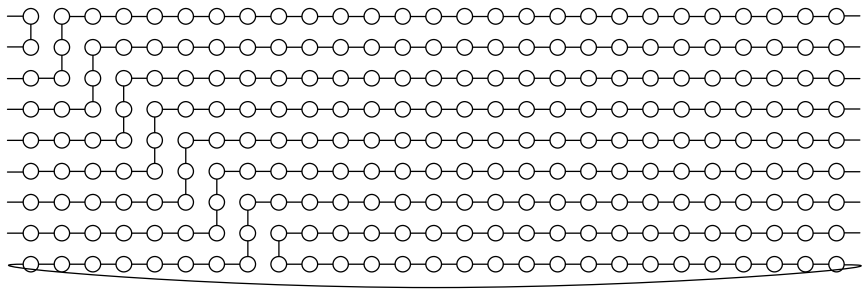

Figure 3.7: First Edge Disjoint Hamiltonian Cycle in EJ with $\alpha = 9 + 9\rho$ after First Edge Exchanges.

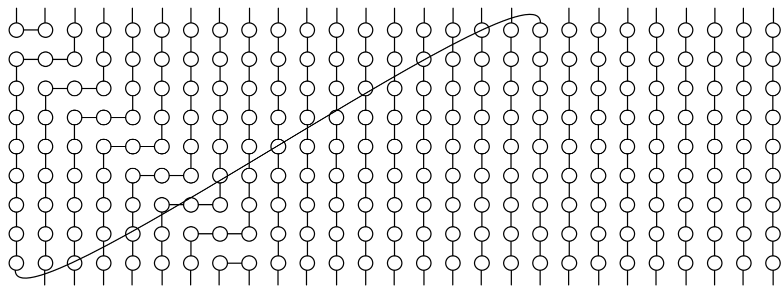

Figure 3.8: Second Edge Disjoint Hamiltonian Cycle in EJ with $\alpha = 9 + 9\rho$ after First Edge Exchanges.

Figures 3.9 and 3.10 show the first and the second edge disjoint Hamiltonian cycles after performing the proper edge exchanges between the horizontal and vertical sets of 8 disjoint cycles.

Now, to get the third edge disjoint Hamiltonian cycle, we have two cases, one in which d is odd and one in which d is even. When d is odd, we follow Table 3.4 to perform the proper

Table 3.3: Proper Edge Exchanges between Horizontal and Vertical 8 Disjoint Cycles in EJ Generated with $\alpha = 8 + 8\rho$.

Horizontal edges	Cycle #	Exch. with	Cycle #	Vertical edges
$7\rho^2$, $7\rho^2+1$	7		0	$7\rho^2$, $6\rho^2$
$6\rho^2$, $6\rho^2+1$, $6\rho^2+2$	6		1	$7\rho^2+1$, $6\rho^2+1$, $5\rho^2+1$
$5\rho^2+1$, $5\rho^2+2$, $5\rho^2+3$	5		2	$6\rho^2+2$, $5\rho^2+2$, $4\rho^2+2$
$4\rho^2+2$, $4\rho^2+3$, $4\rho^2+4$	4		3	$5\rho^2+3$, $4\rho^2+3$, $3\rho^2+3$
$3\rho^2+3$, $3\rho^2+4$, $3\rho^2+5$	3		4	$4\rho^2+4$, $3\rho^2+4$, $2\rho^2+4$
$2\rho^2+4$, $2\rho^2+5$, $2\rho^2+6$	2		5	$3\rho^2+5$, $2\rho^2+5$, ρ^2+5
ρ^2+5, ρ^2+6, ρ^2+7	1		6	$2\rho^2+6$, ρ^2+6, 6
6, 7	0		7	ρ^2+7, 7

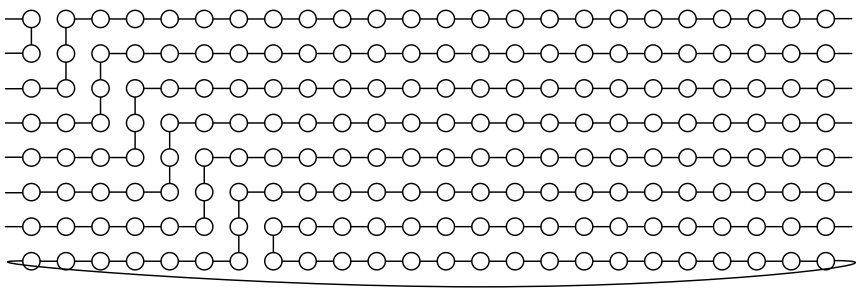

Figure 3.9: First Edge Disjoint Hamiltonian Cycle in EJ with $\alpha = 8 + 8\rho$ after First Edge Exchanges.

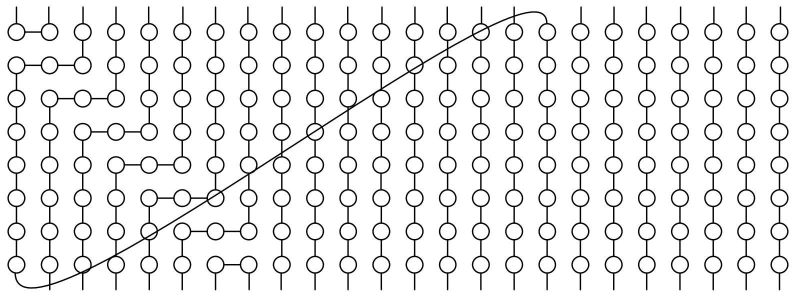

Figure 3.10: Second Edge Disjoint Hamiltonian Cycle in EJ with $\alpha = 8 + 8\rho$ after First Edge Exchanges.

edge exchanges between the first edge disjoint Hamiltonian cycle and the third (diagonal) set of d node disjoint cycles (in the last block of the rectangular representation) to get the third edge disjoint Hamiltonian cycle. Note that after the edges have been exchanged the first Hamiltonian cycle remains edge disjoint Hamiltonian cycle.

By applying Table 3.4 to the first Hamiltonian cycle and the diagonal d disjoint cycles where d is odd, we get the modified first Hamiltonian cycle and the third Hamiltonian cycle.

Table 3.4: Proper Edge Exchanges between Horizontal and Diagonal d Disjoint Cycles when d is Odd for $i = d-2, d-3, \ldots, 1$ as Horizontal Cycles and $A = r - d$.

Horizontal edges	Cycle # (Horizontal)	Diagonal edges
$(d-1)\rho^2 + A + 1$, $(d-1)\rho^2 + A + 2$	$d-1$	$(d-2)\rho^2 + A$, $(d-1)\rho^2 + A + 1$
$(d-2)\rho^2 + A$, $(d-2)\rho^2 + A + 1$, $(d-2)\rho^2 + A + 2$	$d-2$	$(d-3)\rho^2 + A$, $(d-2)\rho^2 + A + 1$, $(d-1)\rho^2 + A + 2$
$\ldots$	$\ldots$	$\ldots$
$i\rho^2 + A$, $i\rho^2 + A + 1$, $i\rho^2 + A + 2$	i	$(i-1)\rho^2 + A$, $i\rho^2 + A + 1$, $(i+1)\rho^2 + A + 2$
$\ldots$	$\ldots$	$\ldots$
$\rho^2 + A$, $\rho^2 + A + 1$, $\rho^2 + A + 2$	1	A, $\rho^2 + A + 1$, $2\rho^2 + A + 2$
A, $A+1$	0	$A+1$, $\rho^2 + A + 2$

The third Hamiltonian cycle (in the diagonal dimension) can be obtained as follows. Let k be the cycle number in the diagonal dimension. The node visiting sequence is as follows:

1. When $k = 0$, starting from node 0 visit d nodes by stepping along dimension $\rho^2 + 1$, i.e., diagonal, and then 1 step forward along dimension 1. Then, we are at cycle $k = 1$.

2. When $k = 1$, visit all nodes in cycle $k = 1$, and then go 2 steps along dimension -1, i.e., backward of dimension 1. Then we are at cycle $k = d - 1$.

3. For $k = d-1, d-3, d-5, \ldots, 2$, visit all the nodes in the cycle k except the last node (unless $k = 2$; then visit all the nodes), and then go 2 steps along dimension -1 (or if $k = 2$, go 1 step along dimension 1). Then, we are at cycle $k = 3$.

4. For $k = 3, 5, 7, \ldots, d-2$, visit all the nodes in cycle k except last node, and then go 2 steps along dimension 1. Then we are at cycle $k = 0$, go to the next step.

5. Visit all the nodes in cycle $k = 0$ that are not visited in (1) until reaching node 0.

Further, note that the edge exchanges are performed in the last block of the rectangular representation. Here, we will only explain the modification process on the first Hamiltonian cycle since it follows the same algorithm of the first Hamiltonian cycle previously described. The edge exchanges, i.e., the modification, on the first Hamiltonian cycle are as follows:

1. The cycle $d-2$ is connected to the cycle $d-1$ through the diagonal edge $(L+(d-2)\rho^2, (L+1)+(d-1)\rho^2)$.
2. For $k = d-3, d-4, \dots, 0$, the cycle k is connected to the cycle $k+2$ through the 2 diagonal edges $(L+k\rho^2, (L+1)+(k+1)\rho^2)$ and $((L+1)+(k+1)\rho^2, (L+2)+(k+2)\rho^2)$ passing through the node $(L+1)+(k+1)\rho^2$.
3. The cycle 0 is connected to the cycle 1 through the edge $((L+1), (L+2)+\rho^2)$.

From the above argument, we see that the first Hamiltonian cycle, after being modified, remains a Hamiltonian cycle. Thus, we get the modified first Hamiltonian cycle and the third Hamiltonian cycle in EJ network when d is odd. The following lemma proves that these two Hamiltonian cycles are edge disjoint.

Lemma 3.4. *Given an EJ network generated with $\alpha = a + b\rho$ with $gcd(a, b) = d$ is odd. By applying Table 3.4 to exchange proper edges between the horizontal edges of the first Hamiltonian cycle and the diagonal d disjoint cycles, we get the third edge disjoint Hamiltonian cycle and the first Hamiltonian cycle remains Hamiltonian cycle, after being modified.*

Proof. In the above argument, we described how to obtain the third Hamiltonian cycle when d is odd by applying proper edge exchanges with the horizontal edges of the first Hamiltonian cycle, as shown in Table 3.4. Now, we need to prove that these two Hamiltonian cycles are edge disjoint. The proof is as follows.

First, note that the first Hamiltonian cycle and the diagonal d disjoint cycles are edge disjoint since no diagonal edge is used in the first Hamiltonian cycle. If we remove $2(d-1)$ edges from both the first Hamiltonian cycle and the diagonal d disjoint cycles, i.e., if a total of $4(d-1)$ edges are removed, and exchange these edges in the proper way, then we have two edge disjoint Hamiltonian cycles. Thus, the $2(d-1)$ diagonal edges, which are placed in the first Hamiltonian cycle, are not used in the diagonal d disjoint cycles. Similarly, the

$2(d-1)$ horizontal edges, which are taken from the first Hamiltonian cycle, are not used in the first Hamiltonian cycle and are instead placed to the diagonal set of d disjoint cycles to form a Hamiltonian cycle. Thus, we get the third edge disjoint Hamiltonian cycle in EJ network when d is odd. $\square$

Example 3.4 (Continuing from Example 3.2)**.** Since $\gcd(9, 9) = 9$ is odd, the third Hamiltonian cycle can be obtained by using Table 3.4 as shown in Table 3.5.

Table 3.5: Proper Edge Exchanges between Horizontal and Diagonal 9 Disjoint Cycles.

Horizontal edges	Cycle # (Horizontal)	Diagonal edges
$8\rho^2+19$, $8\rho^2+20$	8	$7\rho^2+18$, $8\rho^2+19$
$7\rho^2+18$, $7\rho^2+19$, $7\rho^2+20$	7	$6\rho^2+18$, $7\rho^2+19$, $8\rho^2+20$
$6\rho^2+18$, $6\rho^2+19$, $6\rho^2+20$	6	$5\rho^2+18$, $6\rho^2+19$, $7\rho^2+20$
$5\rho^2+18$, $5\rho^2+19$, $5\rho^2+20$	5	$4\rho^2+18$, $5\rho^2+19$, $6\rho^2+20$
$4\rho^2+18$, $4\rho^2+19$, $4\rho^2+20$	4	$3\rho^2+18$, $4\rho^2+19$, $5\rho^2+20$
$3\rho^2+18$, $3\rho^2+19$, $3\rho^2+20$	3	$2\rho^2+18$, $3\rho^2+19$, $4\rho^2+20$
$2\rho^2+18$, $2\rho^2+19$, $2\rho^2+20$	2	ρ^2+18, $2\rho^2+19$, $3\rho^2+20$
ρ^2+18, ρ^2+19, ρ^2+20	1	18, ρ^2+19, $2\rho^2+20$
18, 19	0	19, ρ^2+20

Figures 3.11 and 3.12 show the modified first and third edge disjoint Hamiltonian cycles after performing the proper edge exchanges between the horizontal and diagonal 9 cycles.

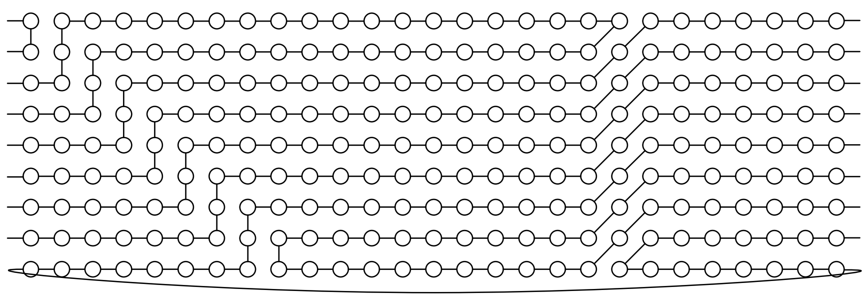

Figure 3.11: Modified First Edge Disjoint Hamiltonian Cycle in EJ with $\alpha = 9 + 9\rho$ after Second Edge Exchanges.

When d is even we follow Table 3.6 to perform a proper edge exchanges between the first edge disjoint Hamiltonian cycle and the third (diagonal) set of d node disjoint cycles (in the first and last d (horizontal) disjoint cycles of the rectangular representation) to get the third

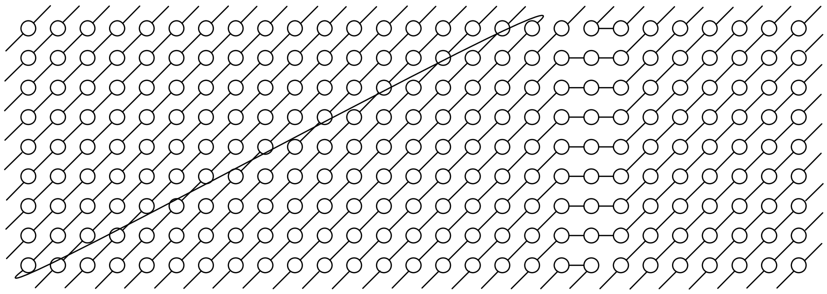

Figure 3.12: Third Edge Disjoint Hamiltonian Cycle in EJ with $\alpha = 9 + 9\rho$ after First Edge Exchanges.

edge disjoint Hamiltonian cycle. Again, note that after the edges have been exchanged the first Hamiltonian cycle remains edge disjoint Hamiltonian cycle.

Table 3.6: Proper Edge Exchanges between Horizontal and Diagonal d Disjoint Cycles when d is Even and $A = r - d$ Based on Horizontal d Disjoint Cycles for $j = 0, 1, \ldots, \frac{r}{2} - 1$, where r and L are as Described in Section 3.2.1.

Horizontal edges	Cycle # (Horizontal)	Diagonal edges
$(d-1)\rho^2 + (2j+1 \ mod \ r)$, $(d-1)\rho^2 + (2j+2 \ mod \ r)$	$d-1$	$(d-1)\rho^2 + 2j$, $r - L + 2j + 1 \ mod \ r$
$(2j+1 \ mod \ r)$, $(2j+2 \ mod \ r)$	0	$(d-1)\rho^2 + (L + 2j - 1 \ mod \ r)$, j

By applying Table 3.6 on the first Hamiltonian cycle and the diagonal d disjoint cycles, where d is even, we get the modified first Hamiltonian cycle and the third Hamiltonian cycle. The third Hamiltonian cycle (in the diagonal dimension) can be obtained as follows. Let *currentnode* $= 0$ be the current node during the visiting sequence. The node visiting sequence is as follows:

1. From the *currentnode*, we go $d-1$ steps along dimension $\rho^2 + 1$, i.e., diagonal, and then 1 step along dimension 1. Then we are at the next cycle.

2. From the *currentnode*, we go $d-1$ steps along dimension $-(\rho^2 + 1)$, i.e., backward on the diagonal dimension, and then 1 step along dimension 1.

3. Repeat (1) and (2) $\frac{r}{2}$ times until we reach node 0.

In the odd case, the edge exchanges are performed in the last block of the rectangular representation. In the even case, instead, the edge exchanges are performed in the first and last d (horizontal) disjoint cycles of the rectangular representation. Here, we describe the first edge disjoint Hamiltonian cycle after being modified as follows:

1. Let $currentnode = 0$ be the current node during the visiting sequence.

2. From the $currentnode$, we go 1 step along dimension 1, 1 step along dimension $-(\rho^2+1)$ (i.e., backward of dimension ρ^2+1), and then 1 step along dimension ρ^2+1.

3. We continue repeating (2) $\frac{d-2}{2}$ times until we reach node $d-2$, then we continue visiting all the nodes in the even cycles as previously described until we reach node $(d-1)\rho^2$; then we go 1 step along dimension (ρ^2+1).

4. From the $currentnode$, we go 1 step along dimension -1, 1 step along dimension $-(\rho^2+1)$, 1 step along dimension -1, and 1 step along dimension (ρ^2+1).

5. Repeat (4) $\frac{(r-L+1)-(d-1)}{2}$ times until we reach node $d-1$. Then we continue visiting all the nodes in the odd cycles, as previously described, until we reach node $(d-1)\,\rho^2+1$. Then, we go 1 step along dimension (ρ^2+1).

6. From the $currentnode$, we go 1 step along dimension 1, 1 step along dimension $-(\rho^2+1)$, 1 step along dimension 1, and 1 step along dimension (ρ^2+1).

7. Repeat (6) $\frac{L-2}{2}$ times until we reach node 0.

From the above argument, we see that the first Hamiltonian cycle, after being modified, remains a Hamiltonian cycle. Thus, we get the modified first Hamiltonian cycle and the third Hamiltonian cycle in EJ network when d is even. The following lemma proves that these two Hamiltonian cycles are edge disjoint.

Lemma 3.5. *Given an EJ network generated with $\alpha = a+b\rho$ with $gcd(a,b) = d$ is even. By applying Table 3.6 to exchange proper edges between the horizontal edges of the first Hamiltonian cycle and the diagonal d disjoint cycles, we get the third edge disjoint Hamiltonian cycle, and the first Hamiltonian cycle remains Hamiltonian cycle, after being modified.*

Proof. From the above argument, we described how to obtain the third Hamiltonian cycle when d is even by applying proper edge exchanges, as shown in Table 3.6, with the horizontal edges of the first Hamiltonian cycles. Now, we need to prove that these two Hamiltonian cycles are edge disjoint. The proof is as follows.

Similar to the odd case, the first Hamiltonian cycle and the diagonal d disjoint cycles are edge disjoint since no diagonal edges are used in the first Hamiltonian cycle. Now, if we remove r edges from both the first Hamiltonian cycle and the diagonal d disjoint cycles, i.e., if a total of $2r$ edges are removed, and exchange these edges in the proper way, then we have two edge disjoint Hamiltonian cycles. Thus, the r diagonal edges, which are placed in the first Hamiltonian cycle, are not used in the diagonal d disjoint cycles. Similarly, the r horizontal edges, which are taken from the first Hamiltonian cycle, are not used in the first Hamiltonian cycle and placed to the diagonal set of d disjoint cycles to form a Hamiltonian cycle. Thus, we get the third edge disjoint Hamiltonian cycle in EJ network when d is even. $\square$

Example 3.5 (Continuing from Example 3.3)**.** Since $\gcd(8, 8) = 8$ is even, the third Hamiltonian cycle can be obtained by using Table 3.6 as shown in Table 3.7.

Figures 3.13 and 3.14 show the modified first and third edge disjoint Hamiltonian cycles after performing the proper edge exchanges between the horizontal and diagonal 8 cycles.

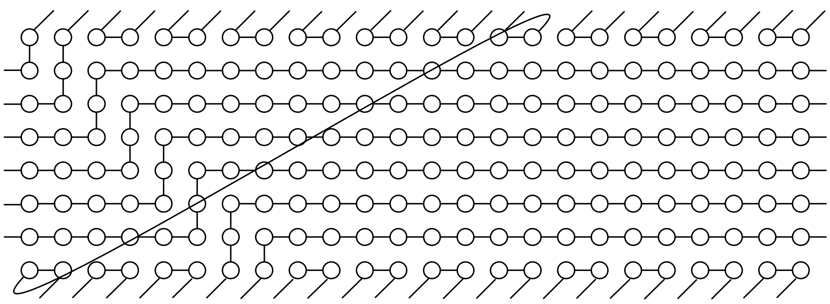

Figure 3.13: Modified First Edge Disjoint Hamiltonian Cycle in EJ with $\alpha = 8 + 8\rho$ after Second Edge Exchanges.

Theorem 3.2. *Given an EJ network generated with $\alpha = a + b\rho$ and $gcd(a, b) = d > 1$, then there exists three edge disjoint Hamiltonian cycles.*

Table 3.7: Proper Edge Exchanges between Horizontal and Diagonal 8 Disjoint Cycles.

Horizontal edges	Cycle # (Horizontal)	Diagonal edges
$7\rho^2+1$, $7\rho^2+2$, $7\rho^2+3$, $7\rho^2+4$, $7\rho^2+5$, $7\rho^2+6$, $7\rho^2+7$, $7\rho^2+8$, $7\rho^2+9$, $7\rho^2+10$, $7\rho^2+11$, $7\rho^2+12$, $7\rho^2+13$, $7\rho^2+14$, $7\rho^2+15$, $7\rho^2+16$, $7\rho^2+17$, $7\rho^2+18$, $7\rho^2+19$, $7\rho^2+20$, $7\rho^2+21$, $7\rho^2+22$, $7\rho^2+23$, $7\rho^2$	7	$7\rho^2$, 9, $7\rho^2+2$, 11, $7\rho^2+4$, 13, $7\rho^2+6$, 15, $7\rho^2+8$, 17, $7\rho^2+10$, 19, $7\rho^2+12$, 21, $7\rho^2+14$, 23, $7\rho^2+16$, 1, $7\rho^2+18$, 3, $7\rho^2+20$, 5, $7\rho^2+22$, 7,
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 0	0	$7\rho^2+15$, 0, $7\rho^2+17$, 2, $7\rho^2+19$, 4, $7\rho^2+21$, 6, $7\rho^2+23$, 8, $7\rho^2+1$, 10, $7\rho^2+3$, 12, $7\rho^2+5$, 14, $7\rho^2+7$, 16, $7\rho^2+9$, 18, $7\rho^2+11$, 20, $7\rho^2+13$, 22

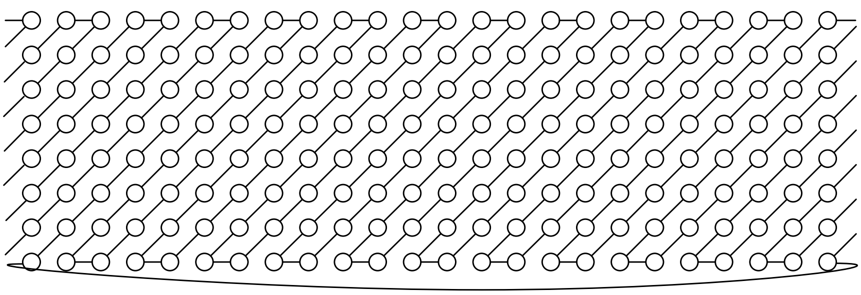

Figure 3.14: Third Edge Disjoint Hamiltonian Cycle in EJ with $\alpha = 8+8\rho$ after First Edge Exchanges.

Proof. In Lemma 3.3, we proved that the first two edge disjoint Hamiltonian cycles could be generated by performing proper edge exchanges, as shown in Table 3.1, between the horizontal and the vertical sets of d disjoint cycles. In order to find the third edge disjoint Hamiltonian cycle, we divided the problem into two cases, when d is odd and when it is even.

In Lemma 3.4, when d is odd, we proved that the third edge disjoint Hamiltonian cycle could be obtained by performing some edge exchanges, as shown in Table 3.4, between the horizontal edges of the first Hamiltonian cycle and the edges of the diagonal set of d disjoint cycles while the first Hamiltonian cycle remains Hamiltonian cycle. Now we need to prove that the first Hamiltonian cycle, after being modified, is also an edge disjoint Hamiltonian cycle with the second Hamiltonian cycle. In addition, we need to prove that the second Hamiltonian cycle is also edge disjoint with the third Hamiltonian cycle. The proof is as follows.

The first Hamiltonian cycle is based on the horizontal set of d disjoint cycles and contains $2(d-1)$ vertical edges that are not in the second and third Hamiltonian cycles. In addition, it also contains $2(d-1)$ diagonal edges that are not in the second and third Hamiltonian cycles. Thus, the first Hamiltonian cycle is edge disjoint with the second and third Hamiltonian cycles. Further, the second Hamiltonian cycle is based on the vertical set of d disjoint cycles and contains $2(d-1)$ horizontal edges that are not used in the first and third Hamiltonian cycles, i.e., no common edges. Thus, the second Hamiltonian cycle is edge disjoint with the third Hamiltonian cycle. Since each Hamiltonian cycle is edge disjoint with the others we get a three edge disjoint Hamiltonian cycles in EJ network when d is odd.

Similarly, when d is even, it follows the same proof but uses Lemma 3.5 (instead of Lemma 3.4) based on Table 3.6. That is, the first Hamiltonian cycle is based on the horizontal set of d disjoint cycles and contains $2(d-1)$ vertical edges that are not in the second and third Hamiltonian cycles. In addition, it also uses r diagonal edges that are not used in the second and third Hamiltonian cycles. Thus, the first Hamiltonian cycle is edge disjoint with the second and third Hamiltonian cycles. Further, the second Hamiltonian cycle is based on the vertical set of d disjoint cycles and contains $2(d-1)$ horizontal edges that are not used in the first and third Hamiltonian cycles, i.e., no common edges. Thus, the second Hamiltonian cycle is edge disjoint with the third Hamiltonian cycle. Since each Hamiltonian cycle is edge disjoint with the others we get a three edge disjoint Hamiltonian cycles in EJ network when d is even. $\square$

Figures 3.8, 3.11, and 3.12 show three edge disjoint Hamiltonian cycles in the EJ network generated with $\alpha = 9 + 9\rho$. In addition, Figures 3.10, 3.13, and 3.14 describe three edge disjoint Hamiltonian cycles in the EJ network generated with $\alpha = 8 + 8\rho$.

Finally, note that the first part of the solution of this problem, finding the first two edge disjoint Hamiltonian cycles in EJ, is a solution for finding the two edge disjoint Hamiltonian cycles in the Gaussian networks (using i instead of ρ^2) generated with $\alpha = a + bi$. Figures 3.15 and 3.16 show the first edge disjoint Hamiltonian cycle in the Gaussian network generated with $\alpha = 9 + 9i$ and $\alpha = 8 + 8i$, respectively. The solid line represents the first Hamiltonian cycle and the remaining edges form the second Hamiltonian cycle.

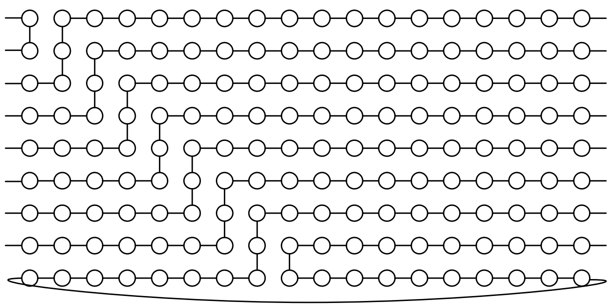

Figure 3.15: The First Edge Disjoint Hamiltonian Cycle in Gaussian Generated with $\alpha = 9 + 9i$.

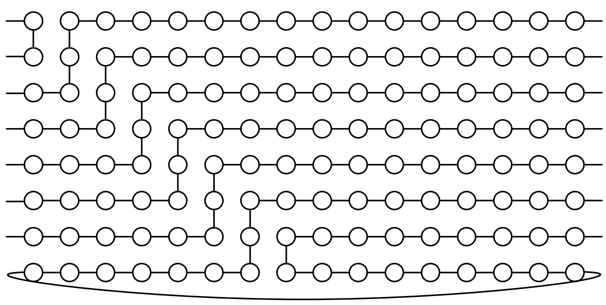

Figure 3.16: First Edge Disjoint Hamiltonian Cycle in Gaussian Generated with $\alpha = 8 + 8i$.

3.3 Conclusion

This chapter has discussed the construction of edge disjoint Hamiltonian cycles in EJ networks when $\gcd(a, b) = 1$. In addition, a new representation, called rectangular, was proposed to show that there exists three sets of d disjoint cycles in the 1, ρ, and ρ^2 dimensions. Moreover, we showed how to construct three edge disjoint Hamiltonian cycles when $\gcd(a, b) > 1$ by exchanging some proper edges between the three sets of d disjoint cycles.

Furthermore, we described that finding the first two edge disjoint Hamiltonian cycles in EJ networks is also a solution for finding edge disjoint Hamiltonian cycles in Gaussian networks for both d is odd and even.

Chapter 4

Edge Disjoint Hamiltonian Cycles in GHC Networks

The problem of finding edge disjoint Hamiltonian cycles in the Generalized Hypercube (GHC) networks is described here. First, we briefly review the structure and some topological properties of Generalized Hypercube (GHC) networks. Then, we describe how to construct edge disjoint Hamiltonian cycles in GHC when the dimension $n = 2^r$, $r \geq 0$, and each dimension is of size p, p a prime number ≥ 3. Finally, we extend this solution for any integer $n \geq 1$ and the size of each dimension p, is any integer, not necessarily a prime.

4.1 Definition and Topological Properties of GHC networks

In a Generalized Hypercube [9] $Q_{k_{n-1},k_{n-2},\ldots,k_0}$ each node is addressed with n-tuples, $(a_{n-1}, a_{n-2}, \ldots, a_0) \in \mathbb{Z}_{k_{n-1}} \times \mathbb{Z}_{k_{n-2}} \times \cdots \times \mathbb{Z}_{k_0}$, i.e., nodes are addressed using mixed radix number system, where $(a_{n-1}, a_{n-2}, \ldots, a_0) = a_{n-1}(k_{n-2} \times k_{n-3} \times \cdots \times k_0) + a_{n-2}(k_{n-3} \times k_{n-4} \times \cdots \times k_0) + \cdots + a_1 k_0 + a_0$. Each node $A = (a_{n-1}, a_{n-2}, \ldots, a_{i+1}, a_i, a_{i-1}, \ldots, a_1, a_0)$ is connected to the nodes $(a_{n-1}, a_{n-2}, \ldots, a_{i+1}, a'_i, a_{i-1}, \ldots, a_1, a_0)$ for all of $0 \leq i \leq n-1$, where a'_i takes all integer values between 0 to $k_i - 1$ except a'_i itself. This implies the two nodes $(a_{n-1}, a_{n-2}, \ldots, a_0)$ and $(b_{n-1}, b_{n-2}, \ldots, b_0)$ are adjacent if the Hamming distance between them is 1. The Hamming distance between two n-tuples is the number of positions they differ. For example, $D_H((3422),(3022)) = 1$ and $D_H((3422),(2202)) = 3$. Thus, suppose (324) is a node address in a Generalized Hypercube over $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5$. Then, the nodes adjacent to this node are [(224), (124), (024), (304), (314), (334), (320), (321), (322), (323)].

In a single dimensional Generalized Hypercube Q_n, the interconnection structure results in a complete graph. For example, Q_5 is as shown in Figure 4.1.

Figure 4.2 shows the interconnection topology of two-dimensional Generalized Hyper-

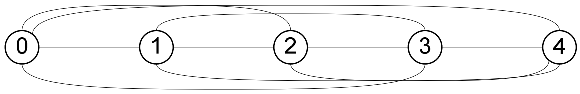

Figure 4.1: Generalized Hypercube Q_5.

cube over $\mathbb{Z}_5 \times \mathbb{Z}_5$ [complete graph along horizontal nodes and complete graph along vertical nodes].

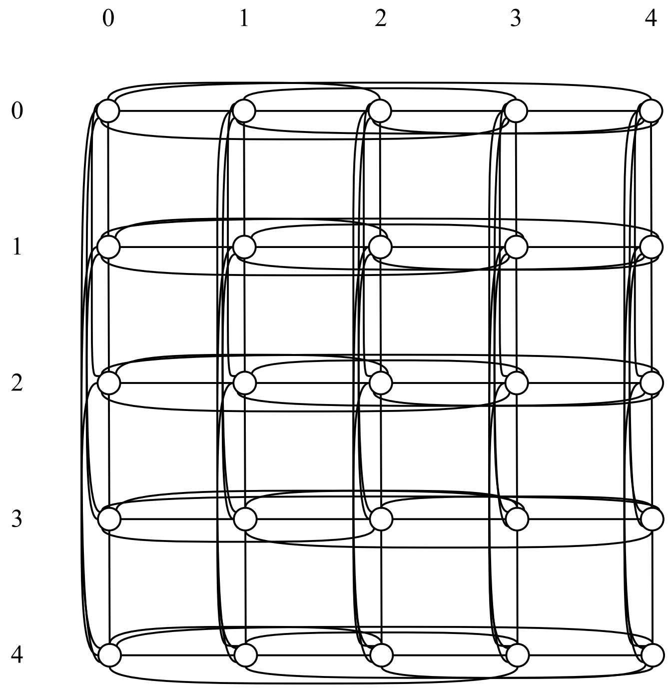

Figure 4.2: Two Dimensional Generalized Hypercube over $\mathbb{Z}_5 \times \mathbb{Z}_5$.

The Generalized Hypercube $Q_{k_{n-1},k_{n-2},\ldots,k_0}$ can also be represented in terms of the product of graphs, which is defined next.

The cross product between two graphs [7] $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is denoted by $G_1 \times G_2$ and is defined as the graph $G(V, E)$, where

1. $V = \{(u, v) \mid u \in V_1, v \in V_2\}$

2. $E = \{((u_1, v_1), (u_2, v_2)) | ((u_1, u_2) \in E_1 \text{ and } v_1 = v_2) \text{ or } ((v_1, v_2) \in E_2 \text{ and } u_1 = u_2)\}$

Thus, the Generalized Hypercube $Q_{k_{n-1},k_{n-2},\ldots,k_0} = Q_{k_{n-1}} \times Q_{k_{n-2}} \times \cdots \times Q_{k_0}$.

A cycle of length k is denoted by C_k, and each node in C_k is labeled with a radix

k number, $0, 1, \ldots, k-1$. There is an edge between vertices u and v if they differ by $\pm 1 (mod\ k)$.

Note that, a k-ary n-cube (C_k^n) and n-dimensional torus ($T_{k_1,k_2,\ldots,k_n}$) can be defined in terms of the product of cycles as follows.

$$C_k^n = \overbrace{C_k \times C_k \times \cdots \times C_k}^{n\ times} \tag{4.1}$$

$$T_{k_1,k_2,\ldots,k_n} = C_{k_1} \times C_{k_2} \times \cdots \times C_{k_n} \tag{4.2}$$

From the above we can define C_k^n recursively in terms of smaller k-ary n-cubes as follows.

$$C_k^n = \begin{cases} C_k & if n = 1 \\ C_k \times C_k^{n-1} & if n > 1 \end{cases} \tag{4.3}$$

4.2 Edge Disjoint Hamiltonian Cycles when $n = 2^r$

In this section, we describe how to find edge disjoint Hamiltonian cycles when all dimensions have the same size and the size is a prime number ≥ 3. i.e., $k_{n-1} = k_{n-2} = \cdots = k_0 = p$, p a prime ≥ 3.

4.2.1 Single Dimension

Case 1: $p = 3$. There is only one Hamiltonian cycle as shown in Figure 4.3.

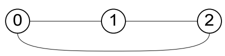

Figure 4.3: Hamiltonian Cycle in Q_3.

Case 2: $p \geq 5$. There are $\frac{p-1}{2}$ edge disjoint Hamiltonian cycles; $C_1, C_2, \ldots, C_{\frac{p-1}{2}}$. The i-th cycle is obtained by traversing along the edges $(ij, i(j+1))\ mod\ p$, for $j = 0, 1, \ldots, p-1$ and $i = 1, 2, \ldots, \frac{p-1}{2}$. Thus, $C_2 = 2 \times C_1, C_3 = 3 \times C_1, \ldots, C_{\frac{p-1}{2}} = \frac{p-1}{2} \times C_1$. For example, the edge disjoint cycles in Q_7 are shown in Figure 4.4.

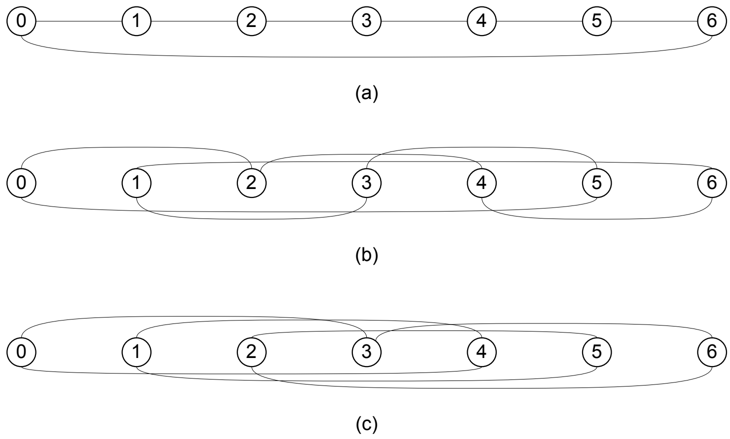

Figure 4.4: Edge Disjoint Hamiltonian Cycles in Q_7.

Note that, the first Hamiltonian cycle is given by $(0, 1, 2, 3, 4, 5, 6)$. The second and third Hamiltonian cycles are obtained by multiplying the node sequence in the first Hamiltonian cycle by 2 and 3 respectively, as shown in Table 4.1.

Table 4.1: Edge Disjoint Hamiltonian Cycles in Q_7.

1×HC	2×HC	3×HC
0	0	0
1	2	3
2	4	6
3	6	2
4	1	5
5	3	1
6	5	4

The above argument means that $p \geq 5$, Q_p can be partitioned into $\frac{p-1}{2}$ edge disjoint Hamiltonian cycles. i.e., $Q_p = H_1^p \oplus H_2^p \oplus \cdots \oplus H_{\frac{p-1}{2}}^p$, where H_i is the i-th Hamiltonian cycle in Q_p with p nodes.

4.2.2 Multi-dimensions

It is assumed the dimension $n = 2^r$, $r \geq 1$, and $k_{n-1} = k_{n-2} = \cdots = k_0 = p$ where p is a prime ≥ 3.

First, note that the cross product of two cycles, $H_1^p \times H_2^p$, gives a two dimensional toroidal network $T_{p\times p}$ and as described in [2][10][8] two edge disjoint Hamiltonian cycles can be obtained in $T_{p\times p}$. An example of the case of $T_{5\times 5}$ is shown in Figure 4.5.

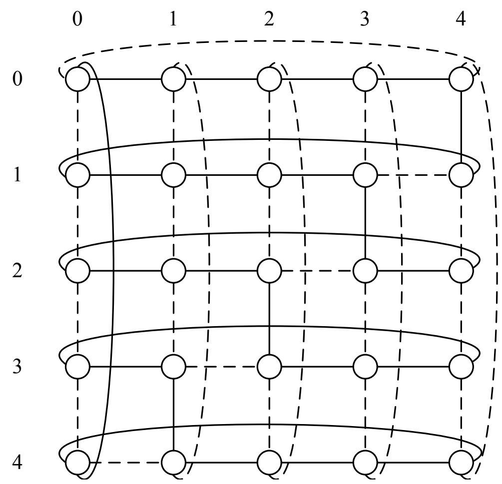

Figure 4.5: Two Edge Disjoint Hamiltonian Cycles in $T_{5\times 5}$.

In general, to get the first Hamiltonian cycle in $T_{p\times p}$, starting from node 0 we go forward to the right until we visit all the nodes on the horizontal dimension and then we step down. After that, we continue visiting the nodes on horizontal dimension to forward until all the nodes are visited and then we step down. We continue performing this process until we reach node 0. To get the second Hamiltonian cycle in $T_{p\times p}$, we perform the above process, but interchange the horizontal edge visits with vertical edge visits and vice versa.

Note that:

$$Q_{p\times p\times\cdots\times p} = \underbrace{Q_p \times Q_p \times \cdots \times Q_p}_{n\ times} \tag{4.4}$$

Now consider the case $Q_{p\times p} = Q_p \times Q_p$ (i.e., $n = 2$). Since Q_p can be partitioned into $\frac{p-1}{2}$ edge disjoint Hamiltonian cycles $Q_p \times Q_p$ can be written as:

$$(H_1^p \oplus H_2^p \oplus \cdots \oplus H_{\frac{p-1}{2}}^p) \times (H_1^{'p} \oplus H_2^{'p} \oplus \cdots \oplus H_{\frac{p-1}{2}}^{'p}) \tag{4.5}$$

Now, if we take the cross product between the i-th, $i = 1, 2, \ldots, \frac{p-1}{2}$, Hamiltonian cycles from the first and second $Q_p's$ then we get torus network as follows.

$$(H_1^p \times H_1'^p) \oplus (H_2^p \times H_2'^p) \oplus \cdots \oplus (H_{\frac{p-1}{2}}^p \times H_{\frac{p-1}{2}}'^p) \tag{4.6}$$

$$= T_{p\times p}^1 \oplus T_{p\times p}^2 \oplus \cdots \oplus T_{p\times p}^{\frac{p-1}{2}} \tag{4.7}$$

This means we partitioned $Q_p \times Q_p$ into $\frac{p-1}{2}$ edge disjoint torus networks, each of size $p \times p$. Then from each of the torus networks we find two edge disjoint Hamiltonian cycles of size p^2 as follows.

$$(H_1^{p^2} \oplus H_2^{p^2}) \oplus (H_3^{p^2} \oplus H_4^{p^2}) \oplus \cdots \oplus (H_{p-2}^{p^2} \oplus H_{p-1}^{p^2}) \tag{4.8}$$

For example, $Q_{5\times5}$ can be partitioned into two edge disjoint torus networks, $T_{5\times5}^1 \oplus T_{5\times5}^2$ as shown in Figure 4.6, and from these tori we can obtain a total of 4 edge disjoint Hamiltonian cycles, $H_1^{25} \oplus H_2^{25} \oplus H_3^{25} \oplus H_4^{25}$.

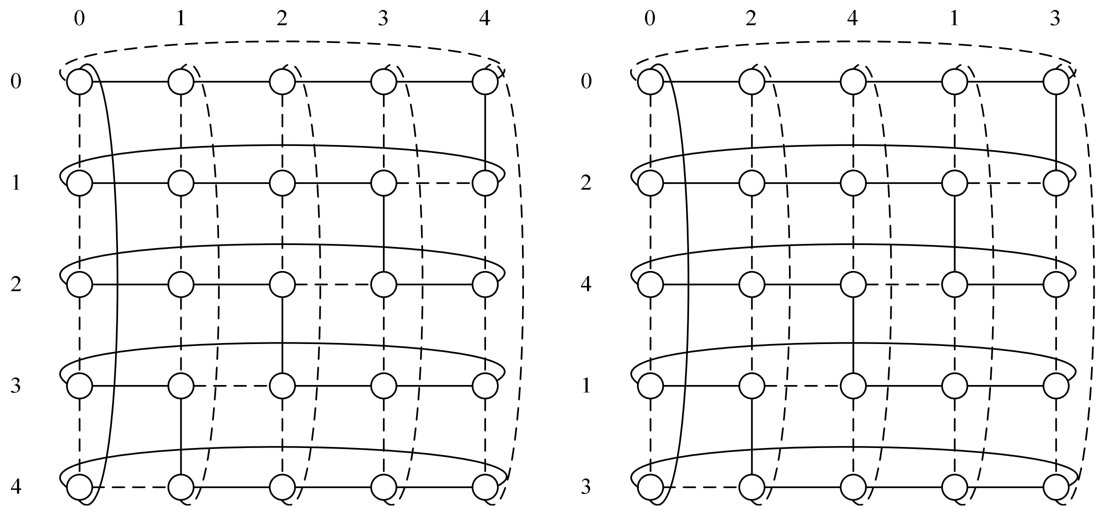

Figure 4.6: Two Edge Disjoint Torus Networks $T_{5\times5}^1 \oplus T_{5\times5}^2$.

Now consider the case for $n = 4$.

$$\begin{aligned} Q_{p\times p\times p\times p} &= Q_p \times Q_p \times Q_p \times Q_p && (4.9)\\ &= (T^1_{p\times p} \oplus T^2_{p\times p} \oplus \cdots \oplus T^{\frac{p-1}{2}}_{p\times p}) \times ({T'}^1_{p\times p} \oplus {T'}^2_{p\times p} \oplus \cdots \oplus {T'}^{\frac{p-1}{2}}_{p\times p}) && (4.10)\\ &= (T^1_{p\times p} \times {T'}^1_{p\times p}) \oplus (T^2_{p\times p} \times {T'}^2_{p\times p}) \oplus \cdots \oplus (T^{\frac{p-1}{2}}_{p\times p} \times {T'}^{\frac{p-1}{2}}_{p\times p}) && (4.11)\\ &= T^1_{p\times p\times p\times p} \oplus T^2_{p\times p\times p\times p} \oplus \cdots \oplus T^{\frac{p-1}{2}}_{p\times p\times p\times p} && (4.12) \end{aligned}$$

That is, each of the first and the second $Q_p \times Q_p$ is partitioned into $\frac{p-1}{2}$ edge disjoint torus networks of size $p \times p$. By taking the cross product of the corresponding two torus networks of size $p \times p$ from the first and second partitions, we obtain $\frac{p-1}{2}$ torus networks of type $T_{p\times p\times p\times p}$, and all these $\frac{p-1}{2}$ torus networks are edge disjoint. From each of $T_{p\times p\times p\times p}$, using the method given in [2], we can obtain two edge disjoint tori of type $T_{p^2\times p^2}$ and as explained before, from each of $T_{p^2\times p^2}$, we can obtain two edge disjoint Hamiltonian cycles of length p^4. This means from each of $T_{p\times p\times p\times p}$, we obtain four edge disjoint Hamiltonian cycles of length p^4; hence $Q_{p\times p\times p\times p}$ can be partitioned into $\frac{4(p-1)}{2} = 2(p-1)$ edge disjoint Hamiltonian cycles of length p^4.

By extending the above method, in general an n-dimensional GHC, where $n = 2^r, r \geq 1$, can be partitioned into $\frac{p-1}{2}$ edge disjoint torus networks as follows.

$$\overbrace{Q_p \times Q_p \times \cdots \times Q_p}^{2^r} = T^1_{p\times p\times\cdots\times p} \oplus T^2_{p\times p\times\cdots\times p} \oplus \cdots \oplus T^{\frac{p-1}{2}}_{p\times p\times\cdots\times p} \tag{4.13}$$

Then from each of $T_{\underbrace{p\times p\times\cdots\times p}_{2^r}}$, we can obtain $2^r = n$ edge disjoint Hamiltonian cycles of size p^n and hence a total of $\frac{n(p-1)}{2} = 2^{r-1}(p-1)$ edge disjoint Hamiltonian cycles of length p^n.

4.2.3 Gray Codes

The above method of generating edge disjoint Hamiltonian cycles can be described in terms of Hamming distance Gray codes. In a Hamming distance Gray code C, the set of p^n vectors over $\mathbb{Z}_p^n$ are arranged in a sequence such that two adjacent vectors are at a Hamming distance one. Furthermore, the first and the last vectors in this sequence are also at a distance 1. Thus, there is a one-to-one correspondence between the Gray codes and the Hamiltonian cycles in GHC. For example, the Gray codes in $\mathbb{Z}_5^2$, (00, 01, 02, 03, 04, 14, 10, 11, 12, 13, 23, 24, 20, 21, 22, 32, 33, 34, 30, 31, 41, 42, 43, 44, 40) and (00, 02, 04, 01, 03, 23, 20, 22, 24, 21, 41, 43, 40, 42, 44, 14, 11, 13, 10, 12, 32, 34, 31, 33, 30), correspond to the two different Hamiltonian cycles in a Generalized Hypercube $Q_{5\times 5}$.

Let $G_{p,1} = (a_0, a_1, \dots, a_{p-1})$ with $a_i \in \mathbb{Z}_p$ and all $a_i's$ are distinct. Note that $G_{p,1}$ is a Gray code with one digit. Define $G_{p,1}^t$ to be the cyclic t-shift of $G_{p,1}$ as described in [10]. For example:

$$G_{p,1}^0 = G_{p,1} = (a_0, a_1, \dots, a_{p-1}) \tag{4.14}$$

$$G_{p,1}^1 = (a_{p-1}, a_0, a_1, \dots, a_{p-2}) \tag{4.15}$$

$$G_{p,1}^2 = (a_{p-2}, a_{p-1}, a_0, a_1, \dots) \tag{4.16}$$

$$\vdots = \vdots$$

$$G_{p,1}^i = (a_{p-i}, a_{p-i+1}, \dots, a_{p-1}, a_0, a_1, \dots) \tag{4.17}$$

Furthermore, recursively define:

$$G_{p,2n} = G_{p,n} \otimes G_{p,n} \tag{4.18}$$

$$= \{A_i G_{p,n}^i \mid A_i \text{ is the i-th word in the } G_{p,n}, \text{for } i = 0, 1, 2, 3, \dots, p^r - 1\} \tag{4.19}$$

Example 4.1. Let $p = 5$, then:

$G_{5,1} = G_{5,1}^0 = (0, 2, 4, 1, 3)$

$G_{5,1}^1 = (3, 0, 2, 4, 1)$

$G_{5,1}^2 = (1,3,0,2,4)$

$G_{5,1}^3 = (4,1,3,0,2)$

$G_{5,1}^4 = (2,4,1,3,0)$

$G_{5,2} = G_{5,1} \otimes G_{5,1}$ as shown in Table 4.2.

$G_{5,4} = G_{5,2} \otimes G_{5,2}$ as shown in Table 4.3.

Table 4.2: Gray Code in $Q_{5\times 5}$.

$0G_{5,1}^0$	$2G_{5,1}^1$	$4G_{5,1}^2$	$1G_{5,1}^3$	$3G_{5,1}^4$
00	23	41	14	32
02	20	43	11	34
04	22	40	13	31
01	24	42	10	33
03	21	44	12	30

Claim 4.1 ([10]). *$G_{p,n}$ forms a Gray code over $\mathbb{Z}_p^n$ for n a power of 2, i.e., $n = 2^r$.*

Proof. The proof is by induction on the number of digits.

Base: For $n = 1$, $G_{p,1} = (a_0, a_1, \ldots, a_{p-1})$ forms a Gray code.

Induction: Assume that $G_{p,n}$ forms a Gray code. In the sequence $G_{p,2n} = G_{p,n} \otimes G_{p,n}$, consider two consecutive words $A_j = (A_{j,1}, A_{j,0})$ and $A_{j+1} = (A_{j+1,1}, A_{j+1,0})$, where $A_{i,1}$ and $A_{i,0}$ are respectively the first and the last n digits of the word A_i.

Case 1: If $j \neq sp^r - 1$, $s = 1, 2, \ldots, p^r - 1$, then $A_{j,1} = A_{j+1,1}$ (i.e., the first part of the words are the same), and $A_{j,0}$ and $A_{j+1,0}$ are two consecutive words in $G_{p,n}$. Thus $D_H(A_j, A_{j+1}) = 1$.

Case 2: Suppose $j = sp^r - 1$, $s = 1, 2, \ldots, p^r - 1$. In this case, $A_{j,0} = A_{j+1,0}$ (i.e., the second part of the code words are the same), and $A_{j,1}$ and $A_{j+1,1}$ are two consecutive words in $G_{p,n}$. Thus $D_H(A_j, A_{j+1}) = 1$.

Case 3: Finally, the distance between the first word, $(00\ldots0)$, and the last word, $(p-100\ldots0)$, is 1.

Therefore $G_{p,2n}$ forms a Gray code in $\mathbb{Z}_p^{2n}$. □

The Gray code sequence $G_{p,2n}$ described in the above theorem forms one Hamiltonian

Table 4.3: $G_{5,4} = G_{5,2} \otimes G_{5,2}$.

$00G^0_{5,2}$	$02G^1_{5,2}$	...	$33G^{23}_{5,2}$	$30G^{24}_{5,2}$
00 00	02 30	...	33 04	30 02
00 02	02 00	...	33 01	30 04
00 04	02 02	...	33 03	30 01
00 01	02 04	...	33 23	30 03
00 03	02 01	...	33 20	30 23
00 23	02 03	...	33 22	30 20
00 20	02 23	...	33 24	30 22
00 22	02 20	...	33 21	30 24
00 24	02 22	...	33 41	30 21
00 21	02 24	...	33 43	30 41
00 41	02 21	...	33 40	30 43
00 43	02 41	...	33 42	30 40
00 40	02 43	...	33 44	30 42
00 42	02 40	...	33 14	30 44
00 44	02 42	...	33 11	30 14
00 14	02 44	...	33 13	30 11
00 11	02 14	...	33 10	30 13
00 13	02 11	...	33 12	30 10
00 10	02 13	...	33 32	30 12
00 12	02 10	...	33 34	30 32
00 32	02 12	...	33 31	30 34
00 34	02 32	...	33 33	30 31
00 31	02 34	...	33 30	30 33
00 33	02 31	...	33 00	30 30
00 30	02 33	...	33 02	30 00

cycle in $\overbrace{Q_p \times Q_p \times \cdots \times Q_p}^{2n}$. All other edge disjoint Hamiltonian cycles can be obtained from $G_{p,2n}$ using some simple permutations and multiplications as described below.

The two-dimensional codes form the basis for generating disjoint Hamiltonian cycles in multi-dimensional GHC; how to generate edge disjoint Hamiltonian cycles in the $Q_p \times Q_p$ is described first.

Theorem 4.1. *The Gray codes $iG_{p,2}$ and $iP_1(G_{p,2})$, for $i = 1, 2, \ldots, \frac{p-1}{2}$, where $P_1(G_{p,2})$ is the permutation of the two digits in $G_{p,2}$, and $iG_{p,2}$ represents the multiplication of each digit of the sequence by i under modulo p operation, generate $p-1$ edge disjoint Hamiltonian cycles in $Q_p \times Q_p$.*

Proof. Consider a two-dimensional torus network with a row labeled as $< 0, 1, 2, \ldots, p-1 >= G_{p,1}$ and column also labeled $< 0, 1, 2, \ldots, p-1 >= G_{p,1}$. As shown in [10], $G_{p,2}$ and $P_1(G_{p,2})$ form two edge disjoint Hamiltonian cycles. In addition, the toroidal networks with row labeled $iG_{p,1}$ and column labeled $iG_{p,1}$ for $i = 1, 2, \ldots, \frac{p-1}{2}$, are edge disjoint. Thus, from each of these we get two edge disjoint Hamiltonian cycles and they are exactly $iG_{p,2}$ and $iP_1(G_{p,2})$, for $i = 1, 2, \ldots, \frac{p-1}{2}$. □

Example 4.2. The four edge disjoint Hamiltonian cycles in $Q_5 \times Q_5$ are shown in Figure 4.6 and they are $G_{5,2}$, $P_1(G_{5,2})$, $2G_{5,2}$, and $2P_1(G_{5,2})$.

Consider $Q_{p\times p\times p\times p}$. It has been shown above that $Q_{p\times p\times p\times p}$ can be written as:

$$
\begin{aligned}
Q_{p\times p\times p\times p} &= Q_p \times Q_p \times Q_p \times Q_p && (4.20)\\
&= (H_1^{p^2} \oplus H_2^{p^2} \oplus \cdots \oplus H_{p-1}^{p^2}) \times (H'^{p^2}_1 \oplus H'^{p^2}_2 \oplus \cdots \oplus H'^{p^2}_{p-1}) && (4.21)\\
&= (H_1^{p^2} \times H'^{p^2}_1) \oplus (H_2^{p^2} \times H'^{p^2}_2) \oplus \cdots \oplus (H_{p-1}^{p^2} \times H'^{p^2}_{p-1}) && (4.22)\\
&= T^1_{p^2\times p^2} \oplus T^2_{p^2\times p^2} \oplus \cdots \oplus T^{p-1}_{p^2\times p^2} && (4.23)
\end{aligned}
$$

That is, $Q_{p\times p\times p\times p}$ is partitioned into $p-1$ torus networks of size $T_{p^2\times p^2}$. First consider $T^1_{p^2\times p^2} \oplus T^2_{p^2\times p^2}$. For $T^1_{p^2\times p^2}$, use row label $G_{p,2}$ and also column labels $G_{p,2}$; for $T^2_{p^2\times p^2}$, use the row and column labels $P_1(G_{p,2})$. Then for the tori numbered $2i+1$, for $i = 1, 2, \ldots, \frac{p-3}{2}$, use the row and column labels $iG_{p,2}$ and for tori numbered $2i$, for $i = 2, 4, \ldots, \frac{p-1}{2}$, use the row and column labels $iP_1(G_{p,2})$. All these $p-1$ tori are edge disjoint and from each of these we can obtain two edge disjoint Hamiltonian cycles. The two edge disjoint Hamiltonian cycles obtained from the torus with row and column labeled $iG_{p,2}$ are nothing but $iG_{p,4}$ and $iP_2(G_{p,4})$. In addition, the row and column labeled as $iP_1(G_{p,2})$ are nothing but $iP_1(G_{p,4})$ and $iP_3(G_{p,4})$. Further, $P_2(G_{p,4})$ is the sequence obtained by permuting the first two digits and last two digits of $G_{p,4}$, and $P_3(G_{p,4})$ is the combination of $P_1(G_{p,4})$ and $P_2(G_{p,4})$. Thus, we get $\frac{4(p-1)}{2}$ edge disjoint Hamiltonian cycles, each of which is obtained from $G_{p,4}$ as follows:

1. The sequence of $G_{p,4}$.
2. The sequence obtained by exchanging 0^{th} with 1^{st} digits, and 2^{nd} with 3^{rd} digits of $G_{p,4}$.
3. The sequence obtained by exchanging the 0^{th} and 1^{st} digits with 2^{nd} and 3^{rd} digits of $G_{p,4}$.
4. Combination of (2) and (3).
5. Multiplying all these digits of the above sequences with i, for $i = 2, 3, \ldots, \frac{p-1}{2}$.

These permutations can be explained in a simple way, as shown in [10]. Let the columns of the Gray code of $G_{p,n=2^r}$ be $(e_{n-1}, e_{n-2}, \ldots, e_0)$; then the 2^r distinct Gray codes which give 2^r disjoint Hamiltonian cycles in GHC can be generated by the permutation $P_i(G_{p,n=2^r})$ for $i = 0, 1, 2, \ldots 2^r - 1$ as follows. To get $P_i(G_{p,2^r})$, let i in binary be $i = (i_{r-1}, i_{r-2}, \ldots, i_0)$; if $i_j = 1$, $j = 0, 1, 2, \ldots, r-1$, then permute the least 0-th 2^j elements of $G_{p,2^r}$ with the next (1-th) 2^j elements, the second 2^j elements with the third 2^j, etc. For example, let $P_0(G_{p,2^3}) = P_{000}(G_{p,2^3}) = (e_7, e_6, e_5, e_4, e_3, e_2, e_1, e_0)$. Then

$$P_1(G_{p,2^3}) = P_{001}(G_{p,2^3}) = (\underline{e_6}, \underline{e_7}, \underline{e_4}, \underline{e_5}, \underline{e_2}, \underline{e_3}, \underline{e_0}, \underline{e_1})$$

$$P_2(G_{p,2^3}) = P_{010}(G_{p,2^3}) = (\underline{e_5, e_4}, \underline{e_7, e_6}, \underline{e_1, e_0}, \underline{e_3, e_2})$$

$$P_3(G_{p,2^3}) = P_{011}(G_{p,2^3}) = (\underline{\underline{e_4}, \underline{e_5}}, \underline{\underline{e_6}, \underline{e_7}}, \underline{\underline{e_0}, \underline{e_1}}, \underline{\underline{e_2}, \underline{e_3}})$$

$$P_4(G_{p,2^3}) = P_{100}(G_{p,2^3}) = (\underline{e_3, e_2, e_1, e_0}, \underline{e_7, e_6, e_5, e_4})$$

$$P_5(G_{p,2^3}) = P_{101}(G_{p,2^3}) = (\underline{\underline{e_2}, \underline{e_3}, \underline{e_0}, \underline{e_1}}, \underline{\underline{e_6}, \underline{e_7}, \underline{e_4}, \underline{e_5}})$$

$$P_6(G_{p,2^3}) = P_{110}(G_{p,2^3}) = (\underline{\underline{e_1, e_0}, \underline{e_3, e_2}}, \underline{\underline{e_5, e_4}, \underline{e_7, e_6}})$$

$$P_7(G_{p,2^3}) = P_{111}(G_{p,2^3}) = (\underline{\underline{\underline{e_0}, \underline{e_1}}, \underline{\underline{e_2}, \underline{e_3}}}, \underline{\underline{\underline{e_4}, \underline{e_5}}, \underline{\underline{e_6}, \underline{e_7}}})$$

In addition, by multiplying each of these Gray codes by j, $j = 1, 2, \ldots, \frac{p-1}{2}$, we get all edge disjoint Hamiltonian cycles.

4.3 Edge Disjoint Hamiltonian Cycles when $n \geq 1$

So far, we have described how to construct edge disjoint Hamiltonian cycles when the number of dimensions is $n = 2^r$, $r \geq 0$, and each dimension is of size p, p a prime number ≥ 3. In this section, we extend the solution to generate these cycles when $n \geq 1$ and p is an integer ≥ 3.

4.3.1 Single Dimension

The problem of finding all edge disjoint Hamiltonian cycles in GHC when $n = 1$ (i.e., a complete graph) and $p \geq 3$ is solved in [14] when p is odd and in [5] when p is even. Here, we review the solution of this problem and describe the proofs clearly in other ways.

For any integer $p \geq 3$, Q_p can be partitioned into $\frac{p-1}{2}$ and $\frac{p-2}{2}$ edge disjoint Hamiltonian cycles when p is odd and when it is even, respectively. As shown in Figure 4.7, the edge disjoint Hamiltonian cycles when p is odd can be generated as follows. First, place the node 0 in the center, node 1 on the left of node 0, and node $p-1$ on the right of node 0. Second, place the odd numbered nodes, in ascending order, below the node 0 and the even numbered nodes, in ascending order, above the node 0. Finally, to get the first edge disjoint Hamiltonian cycle, place the edges between the nodes sequencially according to the nodes number. i.e., (0,1),(1,2), ..., $(p-2, p-1)$. To get the other remaining edge disjoint Hamiltonian cycles just rotate the node numbers to the clockwise direction. i.e., a total of $\frac{p-1}{2} - 1$ rotations. Figure 4.8 shows an example of edge disjoint Hamiltonian cycles in Q_7.

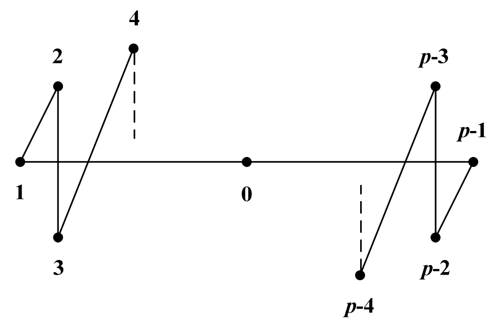

Figure 4.7: Hamiltonian Cycle in Q_p, p is Odd.

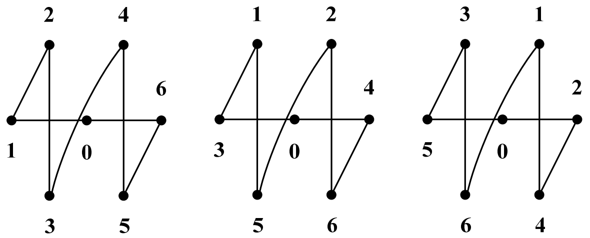

Figure 4.8: Edge Disjoint Hamiltonian Cycles in Q_7.

Further, the edge disjoint Hamiltonian cycles when p is even can be generated as follows. First, draw a graph similar to the case of p is odd except that replace node $p-1$ with $p-2$, and place the node $p-1$ on the right side of the node $p-2$. Second, connect node $j = \frac{p}{2} - 1$ to node $p-1$ and node $p-1$ to node $j+1$ as shown in Figures 4.9 and 4.10 when $\frac{p}{2}$ is Even and Odd, respectively.

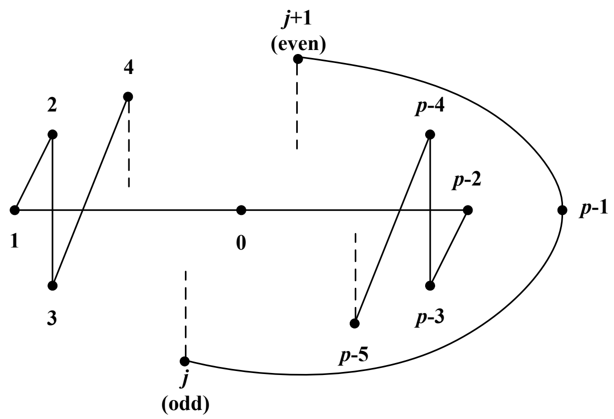

Figure 4.9: Hamiltonian Cycle in Q_p, p is Even and $\frac{p}{2}$ is Even.

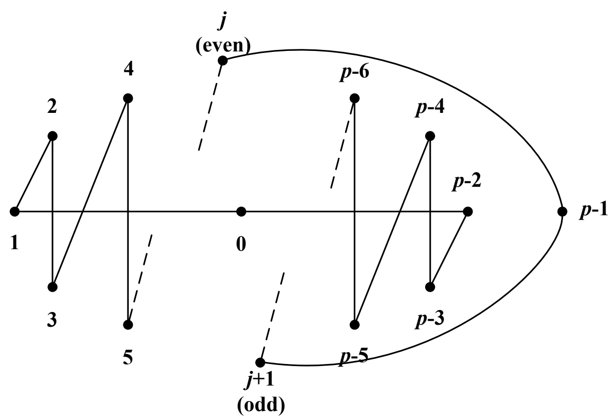

Figure 4.10: Hamiltonian Cycle in Q_p, p is Even and $\frac{p}{2}$ is Odd.

Moreover, similar to the case of p is odd, to get the other remaining edge disjoint Hamiltonian cycles just rotate the node numbers to the clockwise direction except node $p-1$. i.e., a total of $\frac{p-2}{2}-1$ rotations. Figure 4.11 and 4.12 are examples of Q_8 and Q_{10}, respectively.

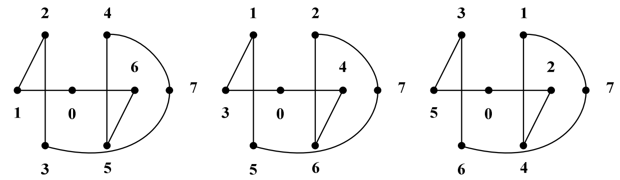

Figure 4.11: Edge Disjoint Hamiltonian Cycles in Q_8.

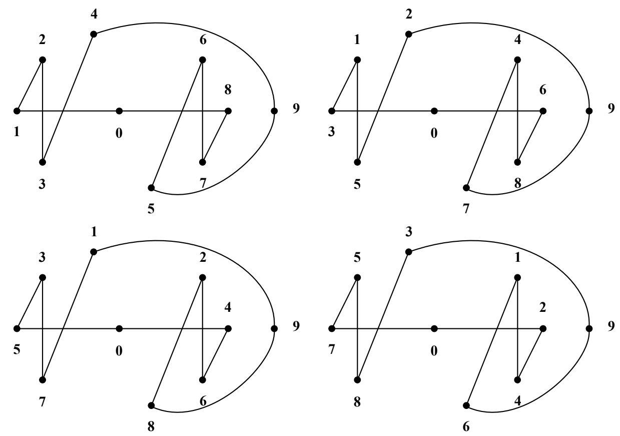

Figure 4.12: Edge Disjoint Hamiltonian Cycle in Q_{10}.

Table 4.4 and 4.5 describe these cycles when p is odd in a clear way such that each row describes a Hamiltonian cycle and each column represents a node address where every two adjacent columns correspond to an edge. Note that, a node has different neighbors (adjacent columns) in every row, i.e., there is no common edge between any two rows. In addition, when p is even just replace each p with $p-1$ and place a column of $p-1$ between columns j and $j+1$. Thus, we get $\frac{p-1}{2}$ (or $\frac{p-2}{2}$) edge disjoint Hamiltonian cycles in Q_p when p is odd (or p is even).

Table 4.4: Edge Disjoint Hamiltonian Cycles in Q_p, p is Odd and $\lceil \frac{p}{2} \rceil$ is Even. $r = i - \frac{j}{2}$, $s = i - \frac{p-j}{2}$, and $t = i - \frac{j+1}{2}$

Edge Disjoint Hamiltonian Cycles in Q_p														
Cycle i	Node j													
0	0	1	$2 = j - 2i$	3	$4 = j - 2i$	$\dots$	j (odd)	$j+1$ (even)	$\dots$	$p-5$	$p-4$	$p-3$	$p-2$	$p-1$
1	0	3	1	5	$2 = j - 2i$	$\dots$	$j+2i$	$(j+1)-2i$	$\dots$	$p-7$	$p-2$	$p-5$	$p-1$	$p-3$
2	0	5	3	7	1	$\dots$	$j+2i$	$(j+1)-2i$	$\dots$	$p-9$	$p-1$	$p-7$	$p-3$	$p-5$
3	0	7	5	9	3	$\dots$	$j+2i$	$(j+1)-2i$	$\dots$	$p-11$	$p-3$	$p-9$	$p-5$	$p-7$
4	0	9	7	11	5	$\dots$	$j+2i$	$(j+1)-2i$	$\dots$	$p-13$	$p-5$	$p-11$	$p-7$	$p-9$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$j+2i$	$1+2r$	$j+2i$	$1+2r$	$\dots$	$p-2$	2	$\dots$	$j-2i$	$p-1-2s$	$j-2i$	$p-1-2s$	$j-2i$
$\dots$	$\dots$	$j+2i$	$1+2r$	$j+2i$	$1+2r$	$\dots$	$p-1-2s$	$1 = 1+2t$	$\dots$	$j-2i$	$p-1-2s$	$j-2i$	$p-1-2s$	$j-2i$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$p-1-2s$	$1+2t$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\frac{p-1}{2}-4$	0	$p-8$	$p-10$	$p-6$	$p-12$	$\dots$	$\dots$	$\dots$	$\dots$	4	12	6	10	8
$\frac{p-1}{2}-3$	0	$p-6$	$p-8$	$p-4$	$p-10$	$\dots$	$\dots$	$\dots$	$\dots$	2	10	4	8	6
$\frac{p-1}{2}-2$	0	$p-4$	$p-6$	$p-2$	$p-8$	$\dots$	$\dots$	$\dots$	$\dots$	1	8	2	6	4
$\frac{p-1}{2}-1$	0	$p-2$	$p-4$	$p-1$	$p-6$	$\dots$	$\dots$	$\dots$	$\dots$	3	6	1	4	2

Table 4.5: Edge Disjoint Hamiltonian Cycles in Q_p, p is Odd and $\lceil \frac{p}{2} \rceil$ is Odd. $r = i - \frac{j}{2}$, $s = i - \frac{p-j}{2}$, and $t = i - \frac{p-j+1}{2}$

Edge Disjoint Hamiltonian Cycles in Q_p														
Cycle i	Node j													
0	0	1	$2 = j - 2i$	3	$4 = j - 2i$	$\dots$	j (even)	$j + 1$ (odd)	$\dots$	$p - 5$	$p - 4$	$p - 3$	$p - 2$	$p - 1$
1	0	3	1	5	$2 = j - 2i$	$\dots$	$j - 2i$	$(j + 1) + 2i$	$\dots$	$p - 7$	$p - 2$	$p - 5$	$p - 1$	$p - 3$
2	0	5	3	7	1	$\dots$	$j - 2i$	$(j + 1) + 2i$	$\dots$	$p - 9$	$p - 1$	$p - 7$	$p - 3$	$p - 5$
3	0	7	5	9	3	$\dots$	$j - 2i$	$(j + 1) + 2i$	$\dots$	$p - 11$	$p - 3$	$p - 9$	$p - 5$	$p - 7$
4	0	9	7	11	5	$\dots$	$j - 2i$	$(j + 1) + 2i$	$\dots$	$p - 13$	$p - 5$	$p - 11$	$p - 7$	$p - 9$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$j + 2i$	$1 + 2r$	$j + 2i$	$1 + 2r$	$\dots$	2	$q - 2$	$\dots$	$j - 2i$	$p - 1 - 2s$	$j - 2i$	$p - 1 - 2s$	$j - 2i$
$\dots$	$\dots$	$j + 2i$	$1 + 2r$	$j + 2i$	$1 + 2r$	$\dots$	$1 = 1 + 2r$	$p - 1 - 2t$	$\dots$	$j - 2i$	$p - 1 - 2s$	$j - 2i$	$p - 1 - 2s$	$j - 2i$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$1 + 2r$	$p - 1 - 2t$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\frac{p-1}{2} - 4$	0	$p - 8$	$p - 10$	$p - 6$	$p - 12$	$\dots$	$\dots$	$\dots$	$\dots$	4	12	6	10	8
$\frac{p-1}{2} - 3$	0	$p - 6$	$p - 8$	$p - 4$	$p - 10$	$\dots$	$\dots$	$\dots$	$\dots$	2	10	4	8	6
$\frac{p-1}{2} - 2$	0	$p - 4$	$p - 6$	$p - 2$	$p - 8$	$\dots$	$\dots$	$\dots$	$\dots$	1	8	2	6	4
$\frac{p-1}{2} - 1$	0	$p - 2$	$p - 4$	$p - 1$	$p - 6$	$\dots$	$\dots$	$\dots$	$\dots$	3	6	1	4	2

Further, Tables 4.4 and 4.5 are described in Algorithm 4.1 as follows. The algorithm takes two parameters, the current node and the length of the cycle, and produces the corresponding node to the next Hamiltonian cycle. Running this algorithm on a current Hamiltonian cycle will generate the next Hamiltonian cycle.

Algorithm 4.1: $f(j, p)$

---

input : Current node j and a cycle of length p
output: A corresponding node in the next cycle

if $j = 0$ **then** return 0
if $j = 2$ **then** return 1
if p *is odd and* $j = p - 2$ **then** return $p - 1$
if p *is even* **then**
    if $j = p - 3$ **then** return $p - 2$
    if $j = p - 1$ **then** return $p - 1$
end
if j *is odd* **then** return $j + 2$
if j *is even* **then** return $j - 2$

For example, Table 4.6 shows three edge disjoint Hamiltonian cycles in Q_8. The second and third Hamiltonian cycles are obtained after running Algorithm 4.1 on the first and second Hamiltonian cycles, respectively.

Table 4.6: Edge Disjoint Hamiltonian Cycles in Q_8.

j	0	1	2	3	7	4	5	6
$f(j, 8)$	0	3	1	5	7	2	6	4
$f(f(j, 8), 8)$	0	5	3	6	7	1	4	2

4.3.2 Multi-dimensions

So far, we have described how to solve edge disjoint Hamiltonian cycles when $n = 2^r$, $r \geq 0$. In this section, with the power of the cross product, we describe how to solve this problem when $n \geq 1$. Note that, taking the cross product between multiple cycles produces a multi-dimensional torus network. In addition, we refer to [2] to find edge disjoint Hamiltonian cycles in a multi-dimensional torus network.

Given a n-dimensional GHC $\underbrace{Q_{p\times p\times\cdots\times p}}_{n}$ where the size of p is odd, the edge disjoint Hamiltonian cycles can be found as follows.

$$\underbrace{Q_{p\times p\times\cdots\times p}}_{n} = \overbrace{Q_p \times Q_p \times \cdots \times Q_p}^{n} \tag{4.24}$$

$$\begin{aligned} &= (H_1^p \oplus H_2^p \oplus \cdots \oplus H_{\frac{p-1}{2}}^p) \times (H_1^{'p} \oplus H_2^{'p} \oplus \cdots \oplus H_{\frac{p-1}{2}}^{'p}) \\ &\quad \times \cdots \times (H_1^{''p} \oplus H_2^{''p} \oplus \cdots \oplus H_{\frac{p-1}{2}}^{''p}) \end{aligned} \tag{4.25}$$

$$\begin{aligned} &= (H_1^p \times H_1^{'p} \times \cdots \times H_1^{''p}) \oplus (H_2^p \times H_2^{'p} \times \cdots \times H_2^{''p}) \\ &\quad \oplus \cdots \oplus (H_{\frac{p-1}{2}}^p \times H_{\frac{p-1}{2}}^{'p} \times \cdots \times H_{\frac{p-1}{2}}^{''p}) \end{aligned} \tag{4.26}$$

$$\begin{aligned} &= T^1_{\underbrace{p\times p\times\cdots\times p}_{n}} \oplus T^2_{\underbrace{p\times p\times\cdots\times p}_{n}} \\ &\quad \oplus \cdots \oplus T^{\frac{p-1}{2}}_{\underbrace{p\times p\times\cdots\times p}_{n}} \end{aligned} \tag{4.27}$$

$$\begin{aligned} &= (H_1^{p^n} \oplus H_2^{p^n} \oplus \cdots \oplus H_n^{p^n}) \oplus \\ &\quad (H_1^{'p^n} \oplus H_2^{'p^n} \oplus \cdots \oplus H_n^{'p^n}) \oplus \cdots \oplus \\ &\quad (H_1^{''p^n} \oplus H_2^{''p^n} \oplus \cdots \oplus H_n^{''p^n}) \end{aligned} \tag{4.28}$$

[by solving each torus using the method in [2] we get the edge disjoint Hamiltonian cycles for n-dimensional GHC].

This gives a total of $\frac{n(p-1)}{2}$ edge disjoint Hamiltonian cycles in an n-dimensional Generalized Hypercube. When p is even, we just replace the value $p-1$ with $p-2$ in all the previous steps amd hence the total number of Hamiltonian cycles is $\frac{n(p-2)}{2}$.

For example, when $n = 3$ and $p = 8$ we can obtain 9 edge disjoint Hamiltonian cycles as follows:

$$
\begin{aligned}
Q_{8\times8\times8} &= Q_8 \times Q_8 \times Q_8 && (4.29)\\
&= (H_1^8 \oplus H_2^8 \oplus H_3^8) \times (H_1^{'8} \oplus H_2^{'8} \oplus H_3^{'8}) \times (H_1^{''8} \oplus H_2^{''8} \oplus H_3^{''8}) && (4.30)\\
&= (H_1^8 \times H_1^{'8} \times H_1^{''8}) \oplus (H_2^8 \times H_2^{'8} \times H_2^{''8}) \oplus (H_3^8 \times H_3^{'8} \times H_3^{''8}) && (4.31)\\
&= T^1_{8\times8\times8} \oplus T^2_{8\times8\times8} \oplus T^3_{8\times8\times8} && (4.32)\\
&= (H_1^{8^3} \oplus H_2^{8^3} \oplus H_3^{8^3}) \oplus (H_1^{'8^3} \oplus H_2^{'8^3} \oplus H_3^{'8^3}) \oplus (H_1^{''8^3} \oplus H_2^{''8^3} \oplus H_3^{''8^3}) && (4.33)\\
&= \frac{3(8-2)}{2} = 9 \text{ edge disjoint Hamiltonian cycles in } Q_{8\times8\times8}.
\end{aligned}
$$

[by solving each torus using the method in [2] we get 9 edge disjoint Hamiltonian cycles].

4.4 Conclusion

In this chapter, we have described how to construct edge disjoint Hamiltonian cycles for a higher dimensional Generalized Hypercube from a lower dimensional Generalized Hypercube and this was performed as follows. First, we decomposed the n-dimensional GHC into an n single dimensions GHC. Second, from each single dimensional GHC, we generated a $\frac{p-1}{2}$ edge disjoint Hamiltonian cycles if p is odd (or $\frac{p-2}{2}$ if p is even). Third, we took the cross product between these cycles to form edge disjoint torus networks. Finally, from each torus network, we found edge disjoint Hamiltonian cycles and then we combined them all to give $\frac{n(p-1)}{2}$ edge disjoint Hamiltonian cycles in GHC if p is odd (or $\frac{p-2}{2}$ if p is even). In addition, Gray codes for these cycles when $n = 2^r$, $r \geq 0$, are given. Finding the Gray code for these cycles when $n \geq 1$ is future work.

Chapter 5

Future Work

The investigations illustrated in this thesis lead to some open problems that should be pursued. This chapter is divided into three sections: 5.1, 5.2, and 5.3. In Section 5.1, we describe the problem of finding node-to-set and set-to-set disjoint paths. Section 5.2, we introduce the problem of finding edge disjoint Hamiltonian cycles in a Generalized Hypercube when the dimensions sizes are not equal. Finally, in Section 5.3, we describe the problem of representing edge disjoint Hamiltonian cycles in Generalized Hypercube networks in terms of Gray codes when dimension $n \geq 1$.

5.1 Node-to-Set and Set-to-Set Disjoint Paths

In Chapter 2 of this dissertation, some solutions for finding all shortest node-to-node disjoint paths in Gaussian and Eisenstein-Jacobi networks are presented.

However, in some situations, a node may want to send distinct messages to a subset of nodes. This is called node-to-set problem. A generalization of this problem is the set-to-set problem, where a subset of nodes wants to send their distinct messages to another subset of nodes. Finding node-to-set and set-to-set disjoint paths in Gaussian, Eisenstein-Jacobi, and Generalized Hypercube networks are challenging research topics.

5.2 Extended Edge Disjoint Hamiltonian Cycles in GHC

The problem of finding edge disjoint Hamiltonian cycles in a Generalized Hypercube when all dimensions are of equal size, either odd or even is solved in Chapter 4. However, finding edge disjoint Hamiltonian cycles in a Generalized Hypercube when some (or all) dimensions' sizes are different, i.e., some (or all) $k_i \neq k_j$ and $i \neq j$ for $i, j \geq 3$, is still under investigation. Extending our solution in Chapter 4 to generate all types of edge disjoint Hamiltonian cycles is an open research problem.

5.3 Gray Codes for Generalized Hypercube when $n \geq 1$

In Chapter 4, we described how to generate edge disjoint Hamiltonian cycles in Generalized Hypercube networks. These cycles are also represented in terms of Gray codes when the dimension $n = 2^r$. However, finding Gray codes for these cycles when the dimension $n \geq 1$ remains a challenging research topic.

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