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In this note, we generalize recent work of Mizuhara, Sellers, and Swisher that gives a method for establishing restricted plane partition congruences based on a bounded number of calculations. Using periodicity for partition functions, our extended technique could be a useful tool to prove congruences for certain types of combinatorial functions based on a bounded number of calculations. As applications of our result, we establish new and existing restricted plane partition congruences and several examples of restricted partition congruences. Also, we define a restricted form of plane overpartitions called k-rowed plane overpartitions as plane overpartitions with at most k rows. We derive the generating function for this type of partition and obtain a congruence modulo 4. Next, we engage a combinatorial technique to establish plane and restricted plane overpartition congruences modulo small powers of 2. For each even integer k, we prove a set of k-rowed plane overpartition congruences modulo 4. For odd integer k, we prove an equivalence relation modulo 4 between k-rowed plane overpartitions and unrestricted overpartitions. As a consequence, using a result of Hirschhorn and Sellers, we obtain an infinite family of k-rowed plane overpartition congruences modulo 4 for each odd integer

$k \geq 1$. Also, we obtain a few unrestricted plane overpartition congruences modulo 4. We establish and prove several restricted plane overpartition congruences modulo 8. Some examples of equivalences modulo 4 and 8 between plane overpartitions and overpartitions are obtained. In addition, we find and prove an infinite family of 5-rowed plane overpartition congruences modulo 8.

Periodicity and Partition Congruences

by

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Major Professor, representing Mathematics

Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Ali Hammood Al-Saedi, Author

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PERIODICITY AND PARTITION CONGRUENCES

1 INTRODUCTION

This thesis consists of a chapter of background materials, beyond this introduction, and two chapters of our main work and results followed by a short conclusion. In Chapter 2, we give a summarized and general introduction of partition theory as follows: In Section 2.1, we give a very brief background of partitions, overpartitions, partition generating functions, some well-known congruences and their historical context. Indeed, these topics are considered as a regular part of any standard literature in Number Theory. In Sections 2.2 and 2.3, we go over the concepts of plane partitions and plane overpartitions in order to give the reader a technical introduction to the main area of this thesis. Also, we define a new object called k-rowed plane overpartition as a restricted form of plane overpartitions with at most k rows. Using a result of Vultic [41] that provides the generating function for plane overpartitions which can fit in a box with finite dimensions, we obtain the generating function for k-rowed plane overpartitions. In Section 2.4, we study the notion of periodicity of q-series and apply a result of Kwong [24] to calculate the minimum periodicity of certain type of combinatorial functions that are generated by finite multisets. Also, we include some results to be used later and mostly in Chapter 3.

In Chapter 3, we extend recent work of Mizuhara, Sellers and Swisher [33] which gives a method to prove congruences modulo ℓ of ℓ-rowed plane partitions where ℓ is a prime number. This method relies on the periodicity of the following finite product which exists as a factor in the generating function for such plane partitions,

$$F_\ell(q) := \prod_{n=1}^{\ell-1} \frac{1}{(1-q^n)^n}. \tag{1.1}$$

There is a finite positive integer $\pi_\ell(F_\ell)$ which represents the minimum periodicity of $F_\ell(q)$, and can be calculated by a formula of Kwong [24]. This is defined fully in Section 2.4.

One can obtain a congruence modulo ℓ by checking only a finite number of terms. In particular, let the function $pl_\ell(n)$ count the number of plane partitions with at most ℓ rows. Then Mizuhara, Sellers, and Swisher [33] show that if a_i, b_j are nonnegative integers less than ℓ such that for all $0 \leq n < \pi_\ell(F_\ell)/\ell$, the following holds

$$\sum_{i=1}^{s} pl_\ell(n\ell + a_i) \equiv \sum_{j=1}^{t} pl_\ell(n\ell + b_i) \pmod{\ell}, \tag{1.2}$$

then (1.2) holds for all $n \geq 0$.

Our first goal of Chapter 3 is to generalize this result to a wider class of partition functions and include prime power moduli, which we do in Theorem 3.3. Thus, for a prime ℓ and a positive integer N, our general technique is to consider a certain type of combinatorial functions of the form

$$G(q) := \sum_{n=0}^{\infty} \lambda(n) q^n \equiv \left(\sum_{n=0}^{\infty} \alpha(n) q^n \right) \cdot \left(\sum_{n=0}^{\infty} \beta(n) q^{\ell^N n} \right) \pmod{\ell^N},$$

where the q-series $A(q) := \sum_{n=0}^{\infty} \alpha(n) q^n$ is periodic modulo ℓ^N with minimum periodicity denoted by $\pi_{\ell^N}(A)$ that is divisible by ℓ^N. The following identity is a more general version of (1.2),

$$\sum_{i=1}^{s} \lambda(n\ell^N + a_i) \equiv \sum_{j=1}^{t} \lambda(n\ell^N + b_i) \pmod{\ell^N}. \tag{1.3}$$

We show that once the identity (1.3) holds up for $0 \leq n < \pi_{\ell^N}(A)/\ell^N$, then it holds for all $n \geq 0$.

Second, we apply our extended periodicity technique to prove and establish new and existing congruences for several combinatorial functions. For example, we show several new and existing k-rowed plane partition congruences for small primes. In fact, a few congruences for this type of partition have been discovered since 1964, for example see [11] and [17]. Our result is an automated formula to reach some of these congruences and establish new identities. As an example of new congruences, we prove that for all $n \geq 0$,

$$pl_8(8n) + pl_8(8n+1) \equiv pl_8(8n+3) \pmod 2,$$

$$pl_9(9n+1) \equiv pl_9(9n+8) \pmod 3.$$

This will be seen in Theorem 3.7.

For restricted plane overpartitions, we apply our periodic approach to obtain the following congruences for 4-rowed plane overpartitions which states that for all $n \geq 0$,

$$\overline{pl}_4(4n+1) + \overline{pl}_4(4n+2) + \overline{pl}_4(4n+3) \equiv 0 \pmod 4$$

which will be seen in Theorem 3.9.

Furthermore, we obtain a few equivalences modulo 3 and 5 for a restricted type of regular partitions as follows. For all integers $n \geq 0$,

$$p(3n+1,2) + p(3n+2,2) \equiv 0 \pmod 3,$$

$$p(10n+6,4) + p(10n+7,4) + p(10n+8,4) \equiv 0 \pmod 5,$$

$$p(10n+2,4)+p(10n+3,4)+p(10n+4,4)\equiv 0 \pmod 5,$$

where for positive integers n and m, $p(n,m)$ denotes the number of partitions of n into parts $\leq m$. This will be seen in Theorem 3.10.

In Chapter 4, we mainly focus on plane overpartition congruences modulo small powers of 2. Let the function $\overline{pl}(n)$ denote the number of unrestricted plane overpartitions of an integer n while the function $\overline{pl}_k(n)$ denotes the number of plane overpartitions of n with at most k rows. These are defined fully in Section 2.3.

In Section 4.1, we establish several examples of restricted and unrestriced plane overpartition congruences modulo 4. Let $\overline{p}_o(n)$ count the number of overpartitions of n into odd parts. A congruence relation modulo 4 between unrestricted plane partitions and overpartitions into odd parts is obtained as follows. For all $n \geq 0$,

$$\overline{pl}(n) \equiv \overline{p}_o(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n \text{ is a square or } n \text{ is twice a square,} \\ 0 \pmod 4 & \text{otherwise,} \end{cases} \tag{1.4}$$

where the second congruence in (1.4) is a result of Hirschhorn and Sellers [[21], Theorem 1.1]. As a consequence, we show that for all $n \geq 0$,

$$\overline{pl}(4n+3) \equiv 0 \pmod 4.$$

We establish a pattern of k-rowed plane overpartition congruences modulo 4 for each even $k \geq 2$. Specifically, let ℓ be the least common multiple of all odd integers between 1 and $k-1$. Then we prove that for any odd prime $p < k$, for all $n \geq 1$, and for $1 \leq r \leq ord_p(\ell)$, where $ord_p(\ell)$ is the unique highest power such that $p^{ord_p(\ell)} | \ell$,

$$\overline{pl}_k(\ell n + p^r) \equiv \begin{cases} 0 \pmod 4 & \text{if } r \text{ is odd,} \\ 2 \pmod 4 & \text{if } r \text{ is even.} \end{cases}$$

Moreover, for all $n \geq 1$,

$$\overline{pl}_k(\ell n) \equiv \begin{cases} 0 \pmod{4} & \text{if } k \equiv 0 \pmod{4}, \\ 2 \pmod{4} & \text{if } k \equiv 2 \pmod{4}. \end{cases}$$

Since $ord_p(\ell)$ is finite, then we get a finite family of congruences modulo 4 for each positive even integer k. This will be seen in Theorem 4.7.

For example, the following hold for all $n \geq 1$,

$$\overline{pl}_4(3n) \equiv 0 \pmod{4},$$

$$\overline{pl}_6(15n+3) \equiv 0 \pmod{4},$$

$$\overline{pl}_8(105n+5) \equiv 0 \pmod{4},$$

$$\overline{pl}_{10}(315n+7) \equiv 0 \pmod{4}.$$

In addition, for odd positive integers k, we obtain an equivalence relation modulo 4 between k-rowed plane overpartitions and unrestricted overpartitions. In other words, the following holds for each $n \geq 0$ and for each $k \geq 0$,

$$\overline{pl}_{2k+1}(2n+1) \equiv \overline{p}(2n+1) \pmod{4}.$$

As a consequence, along with a result of Hirschhorn and Sellers [20], we obtain the following infinite family of congruences modulo 4. That is, for all $k \geq 0$, $n \geq 0$, and $\alpha \geq 0$,

$$\overline{pl}_{2k+1}(9^\alpha(54n+45)) \equiv 0 \pmod{4}.$$

We prove a pattern of congruences modulo 4 between $\overline{pl}_k(n)$ and $\overline{p}(n)$ for each odd $k \geq 5$. specifically, let $k \geq 2$ and ℓ be the least common multiple of all positive even integers $\leq 2k$. Then for all integers $n \geq 1$,

$$\overline{pl}_{2k+1}(\ell n + 2^j) \equiv \overline{p}(\ell n + 2^j) \pmod 4,$$

where $j \geq 2, j \equiv 0 \pmod 2$ and $2^{j-1} \leq k$. Moreover, if $k \equiv 0 \pmod 2$, then for all integers $n \geq 0$

$$\overline{pl}_{2k+1}(\ell n) \equiv \overline{p}(\ell n) \pmod 4.$$

This will be seen in Theorem 4.12.

Also, we establish and prove several examples of k-rowed plane overpartition congruences modulo 8 for $k = 4, 8$. For example, we show that for all $n \geq 1$,

$$\overline{pl}_4(6n + 3) \equiv 0 \pmod 8,$$

$$\overline{pl}_8(210n + 3) \equiv 0 \pmod 8,$$

which will be seen in Theorem 4.16. We also prove a few congruences modulo 8 for unrestricted overpartitions. For example, we show that for all nonsquare odd integers $n \geq 0$,

$$\overline{p}(n) \equiv 0 \pmod 8, \tag{1.5}$$

which will be seen in Theorem 4.17.

As a consequence of (1.5), we obtain the following result which gives an infinite family of overpartition congruences modulo 8. For any integer $\alpha \geq 3$, and $\beta \geq 0$, we show that in Corollary 4.18 for each $n \geq 0$,

$$\overline{p}(2^\alpha 3^\beta n + 5) \equiv 0 \pmod 8. \tag{1.6}$$

For k-rowed plane overpartitions with odd k, we obtain the following equivalence modulo 8 with at most 5 rows. For all $n \geq 0$,

$$\overline{pl}_5(12n+1) \equiv \overline{p}(12n+1) \pmod{8}, \tag{1.7}$$

$$\overline{pl}_5(12n+5) \equiv \overline{p}(12n+5) \pmod{8}. \tag{1.8}$$

By combining the identities (1.6) and (1.7), we obtain an infinite family of 5-rowed plane overpartition congruences modulo 8. For any integers $\alpha \geq 3$ and $\beta \geq 1$, and $n \geq 0$,

$$\overline{pl}_5(2^\alpha 3^\beta n+5) \equiv 0 \pmod{8}.$$

Both (1.7) and (1.8) will be seen in Theorem 4.20.

Finally, in Chapter 5, we conclude with some final remarks.

2 BACKGROUND MATERIALS

2.1 Partitions and Overpartitions

A general problem in additive number theory is to write an integer n into a nonincreasing sequences of integers called parts that sum to n, where these parts come from a set or multiset of integers and to count all possible ways of obtaining such sequences.

A *partition* π of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1, \ldots, \lambda_k$ that sum to n. We write $n = \lambda_1 + \cdots + \lambda_k = |\pi|$, to denote the size of a partition, and call λ_i the parts of the partition π.

The total number of partitions of n is denoted by $p(n)$. We can define $p(n)$ on the set of all integers by setting $p(0) = 1$ and $p(n) = 0$ for all $n < 0$. For example, the partitions of $n = 5$ are given by

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

Thus, $p(5) = 7$.

The function $p(n)$ is also known as the unrestricted partition function to emphasize that no restrictions are imposed upon parts of the partitions counted.

We can represent a partition π of n graphically by a Ferrers-Young diagram. A *Ferrers-Young* diagram of a partition π of n is a left-justified rectangular array of n boxes, or cells, with a row of length λ_j for each part λ_j of π, ordered from top to bottom. For example, the Ferrers-Young diagram of $\pi = 6 + 4 + 3 + 1$ is as follows.

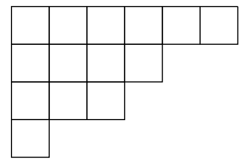

FIGURE 2.1: Ferrers-Young diagram of $|\pi| = 14$

Generating functions are important tools to study partition functions and their arithmetic properties. A *generating function* for a sequence of integers is a formal power series whose nth coefficient corresponds to the nth term of the sequence.

The generating function for $p(n)$ is due to Euler [15] and is given by the following infinite product representation

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{2.1}$$

Euler's idea was to expand each term of the product into formal power series for $|q| < 1$ and then grouping all coefficients associated with the same power of q. In other words,

$$\begin{aligned}
\frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdots =&(1+q+q^{1+1}+q^{1+1+1}+\cdots)(1+q^2+q^{2+2}+q^{2+2+2}+\cdots) \\
&(1+q^3+q^{3+3}+q^{3+3+3}+\cdots)\cdots \\
&= 1+q+q^2+q^{1+1}+q^3+q^{2+1}+q^{1+1+1}+\cdots \\
&= p(0)+p(1)q+p(2)q^2+\cdots \\
&= \sum_{n=0}^{\infty} p(n)q^n.
\end{aligned}$$

Note that we do not evaluate the power series at a particular value of q, so we do not need to deal with convergence.

Ramanujan's beautiful partition congruences [38], which state that for all $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod{5} \tag{2.2}$$

$$p(7n+5) \equiv 0 \pmod{7} \tag{2.3}$$

$$p(11n+6) \equiv 0 \pmod{11}, \tag{2.4}$$

have inspired a vast number of mathematicians to study and investigate special arithmetic properties of partitions, as well as interesting restricted partition functions and generalizations. For example see work of Andrews and Berndt [8], Atkin and Swinnerton-Dyer [9], Garvan [18], Ono [37] and many others [13], [29], [39] to mention a few.

Ramanujan conjectured [38]:
"It appears that there are no equally simple properties for any moduli involving primes other than these three (i.e. $m = 5, 7, 11$)."
In other words, the identities (2.2), (2.3), and (2.4) were the only congruences of form

$$p(\ell n + t) \equiv 0 \pmod{\ell}$$

for all integers $n \geq 0$, where ℓ is prime, and t some fixed integer. In 2003, Ramanujan's conjecture was proved by Ahlgren and Boylan [4].

Since Ramanujan, mathematicians have been searching for more examples of such congruences and many of the type

$$p(An + B) \equiv 0 \pmod{\ell}$$

have been found for integers A, B. The question then arises whether or not the partition function $p(n)$ is divisible by an arbitrary prime for some arithmetic progressions. The answer is addressed by Ono [37] who proves a surprising result which states for a prime

$m \geq 5$ and each positive integer k, a positive proportion of primes ℓ have the property that

$$p\left(\frac{m^k \ell^3 n + 1}{24}\right) \equiv 0 \pmod{m},$$

for all nonnegative integers n relatively prime to ℓ. This has been extended to all integers n coprime to 6 by Ahlgren [3].

One can also consider partitions where the parts are restricted to a specific set S of integers. For example, let S be a set containing positive integers, then $p(n; S)$ denotes the number of partitions of n into parts from S. Clearly $p(n) = p(n; \mathbb{N})$.

Let $p(n, k)$ count the partitions of n into parts each at most k. The generating function of such partitions [7] is given by

$$\sum_{n=0}^{\infty} p(n,k) q^n = \sum_{n=0}^{\infty} p(n; S_k) q^n = \prod_{n=1}^{k} \frac{1}{1-q^n}, \tag{2.5}$$

where $S_k = \{1, 2, \ldots, k\}$.

We can also consider partitions where parts are from a multiset S such that each repeated number is treated independently. The generating function of such partitions is given by

$$\sum_{n=0}^{\infty} p(n; S) q^n = \prod_{n \in S} \frac{1}{1-q^n}. \tag{2.6}$$

For example, consider the multiset

$$S = \{1_1, 1_2, 2_1, 2_2, 2_3, 3\},$$

where repeated numbers have different indices. Then $p(2; S) = 6$ since the partitions of 2 with parts from S are

$$2_1, 2_2, 2_3, 1_1 + 1_1, 1_1 + 1_2, 1_2 + 1_2.$$

Note that the order in the multiset gives an implied order to the repeated numbers. In particular,

$$\sum_{n=0}^{\infty} p(n;S)q^n = \frac{1}{(1-q)^2(1-q^2)^3(1-q^3)}. \tag{2.7}$$

In fact, it is not easy to determine generating functions for all restricted partitions. For example, consider the partitions of an integer n into parts the are neither repeated nor consecutive. Such a generating function is not easy to obtain directly. However, a Rogers-Ramanujan identity [7] which states that the number of partitions of n into parts congruent to 1 or 4 modulo 5 is equal to the number of partitions into parts that are neither repeated nor consecutive. Thus, the generating function for these partitions can be given by

$$\sum_{n=0}^{\infty} p(n;S)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

where $S = \{5n+1, 5n+4 \mid n \in \mathbb{N}\}$.

An *overpartition* of a positive integer n is a partition of n in which the first occurrence (equivalently, the final occurrence) of a part may be overlined. We denote the number of overpartitions of n by $\overline{p}(n)$ and define $\overline{p}(0) := 1$.

For example, the 14 overpartitions of 4 are

$$4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, 2+\overline{2}, 2+1+1,$$
$$\overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1.$$

An overpartition can be interpreted as a pair of partitions one into distinct parts corresponding with the overlined parts and the other unrestricted. Thus, it is easy to see that the generating function for overpartitions is given by

$$\overline{P}(q) := \sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots. \tag{2.8}$$

Overpartitions have been studied extensively by Corteel, Lovejoy, Osburn, Bringmann, Mahlburg, Hirschhorn, Sellers, and many other mathematicians. For example, see [10], [13], [19], [20], [21], [26], [27], [29] and [32] to mention a few.

The well-known Jacobi triple product identity [6] is given by

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1})=\sum_{n=-\infty}^{\infty}z^nq^{n^2}, \tag{2.9}$$

which converges when $z\neq 0$ and $|q|<1$. Letting $z=1$ in (2.9), one can observe one of Ramanujan's classical theta functions

$$\phi(q):=\sum_{n=-\infty}^{\infty}q^{n^2}=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2. \tag{2.10}$$

Replacing q by $-q$ in (2.10), we get

$$\phi(-q)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}=\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2=\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1+q^n)}=\frac{1}{\overline{P}(q)}. \tag{2.11}$$

Note that $\phi(q)$ can be written as

$$\phi(q)=1+2\sum_{n=1}^{\infty}q^{n^2}.$$

Thus, the generating function of overpartitions has the following 2-adic expansion,

$$\begin{aligned}\overline{P}(q)=&\frac{1}{\phi(-q)}=\frac{1}{1+2\sum_{n=1}^{\infty}(-1)^{n^2}q^{n^2}}\\ =&1+\sum_{k=1}^{\infty}2^k(-1)^k\left(\sum_{n=1}^{\infty}(-1)^{n^2}q^{n^2}\right)^k\\ =&1+\sum_{k=1}^{\infty}2^k\sum_{n_1^2+\cdots+n_k^2=n}(-1)^{n+k}q^n\\ =&1+\sum_{k=1}^{\infty}2^k\sum_{n=1}^{\infty}(-1)^{n+k}c_k(n)q^n, \end{aligned}\tag{2.12}$$

where $c_k(n)$ denotes the number of representations of n as a sum of k squares of positive integers.

Several overpartition congruences modulo small powers of 2 have been found using the 2-adic expansion formula (2.12). For example, Mahlburg [32] proves that

$$\{n \in \mathbb{N} \mid \overline{p}(n) \equiv 0 \pmod{64}\}$$

is a set of density $1^\star$. Later, Kim [22] generalized Mahlburg's result modulo 128.

Furthermore, Mahlburg conjectures [32] that for any integer $k \geq 1$,

$$\overline{p}(n) \equiv 0 \pmod{2^k},$$

for almost all integers n.

Overpartition congruences modulo small powers of 2 can be derived from the fact proved by Hirschhorn and Sellers [[21], Theorem 2.1] that states

$$\overline{P}(q) = \phi(q)\overline{P}(q^2)^2. \tag{2.13}$$

Iterating (2.13) yields that [[21], Theorem 2.2]

$$\overline{P}(q) = \phi(q)\ \phi(q^2)^2\ \phi(q^4)^4\ \phi(q^8)^8 \cdots.$$

Thus,

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)\left(1 + 2\sum_{n\geq 1} q^{2n^2}\right)^2\left(1 + 2\sum_{n\geq 1} q^{4n^2}\right)^4\left(1 + 2\sum_{n\geq 1} q^{8n^2}\right)^8 \cdots$$

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*The sequence A of positive integers $a1 < a2 < \cdots$ has a density $\delta(A)$ if

$$\delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}.$$

For more details about arithmetic density of integers, one may see [36].

In Chapter 4 [see Lemma 4.1], we will show that for all $k \geq 1$,

$$\phi(q)^{2^k} \equiv 1 \pmod{2^{k+1}}. \tag{2.14}$$

By (2.12) and (2.14), we obtain the following general equivalence modulo 2^k, for $k \geq 2$,

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n \equiv \prod_{j=0}^{k-2} \left(\phi(q^{2^j})\right)^{2^j} \equiv 1 + \sum_{j=1}^{k-1} 2^j \sum_{n=1}^{\infty} (-1)^{n+j} c_j(n) q^n \pmod{2^k}. \tag{2.15}$$

Thus, for the case $k = 2$, we obtain

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n \equiv \phi(q) \equiv 1 + 2\sum_{n=1}^{\infty} q^{n^2} \pmod{4}, \tag{2.16}$$

which yields for each nonsquare integer $n \geq 1$,

$$\overline{p}(n) \equiv 0 \pmod{4}. \tag{2.17}$$

Manipulating the generating function of overpartitions, Hirschhorn and Sellers [19] employed elementary dissection techniques of generating functions and derived a set of overpartition congruences modulo small powers of 2. For example, they prove that for all $n \geq 0$,

$$\overline{p}(9n+6) \equiv 0 \pmod{8},$$
$$\overline{p}(8n+7) \equiv 0 \pmod{64}.$$

For a modulus that is not a power of 2, Hirschhorn and Sellers [20] prove the first infinite family of congruences for $\overline{p}(n)$ modulo 12 by showing first that for all $n \geq 0$, and all $\alpha \geq 0$,

$$\overline{p}(9^\alpha(27n+18)) \equiv 0 \pmod{3}.$$

Together with the fact $9^\alpha(27n+18)$ is nonsquare for all $n \geq 0, \alpha \geq 0$, and hence by the help of (2.17), it follows that for all $\alpha, n \geq 0$,

$$\overline{p}(9^\alpha(27n+18)) \equiv 0 \pmod{12}. \tag{2.18}$$

Several examples of overpartition congruences have been found. For more examples of overpartition congruences, one may refer to work of Chen and Xia [12], Fortin, Jacob and Mathieu [16], Treneer [40] and Wang [42].

Now, let $\overline{p}_o(n)$ denote the number of overpartitions of n into odd parts. The generating function [21] is given by

$$\overline{P}_o(q) := \sum_{n=0}^{\infty} \overline{p}_o(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^{2n-1}}{1-q^{2n-1}}. \tag{2.19}$$

Similar to (2.13), The generating function $\overline{P}_o(q)$ can be written as [see [21], Theorem 2.3],

$$\overline{P}_o(q) = \phi(q)\overline{P}(q^2), \tag{2.20}$$

and the iteration of (2.20) yields [[21],Theorem 2.4],

$$\overline{P}_o(q) = \phi(q)\phi(q^2)\phi(q^4)^2\phi(q^8)^4 \cdots \tag{2.21}$$

For modulus 4, we then easily get

$$\sum_{n=0}^{\infty} \overline{p}_o(n) q^n \equiv \phi(q)\phi(q^2) \equiv 1 + 2\sum_{n\geq 1} q^{n^2} + 2\sum_{n \geq 1} q^{2n^2} \pmod 4.$$

As a consequence, Hirschhorn and Sellers obtain Theorem 1.1 of [21] as following.

Theorem 2.1 (Hirschhorn, Sellers, [21])**.** *For every integer* $n \geq 1$,

$$\overline{p}_o(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n \text{ is a square or twice a square}, \\ 0 \pmod 4 & \text{otherwise}. \end{cases}$$

Similar to (2.15), we have the following general equivalence modulo 2^k for all $k \geq 2$,

$$\sum_{n=0}^{\infty} \overline{p}_o(n)q^n \equiv \phi(q)\ \phi(q^2)\ \phi^2(q^4)\cdots\phi(q^{2^{k-1}})^{2^{k-2}} \pmod{2^k}.$$

Later in Chapter 4, we will revisit the equivalences (2.15), (2.17), and Theorem 2.1.

2.2 Plane and Restricted Plane Partitions

Each partition can be considered as a one dimensional array of parts, and MacMahon [30] extended this idea to a two-dimensional array. A *plane partition* λ of a positive integer n is a two-dimensional array of positive integers $n_{i,j}$ that sum to n, such that the array is the Ferrers-Young diagram of a partition, and the entries are nonincreasing from left to right and also from top to bottom. Letting i denote the row and j the column of $n_{i,j}$, this means that for all $i, j \geq 0$,

$$n_{i,j} \geq n_{i+1,j},$$
$$n_{i,j} \geq n_{i,j+1}.$$

Correspondingly, the entries $n_{i,j}$ are called the parts of λ, and the number of plane partitions of n is denoted by $pl(n)$.

For example, $pl(3) = 6$ since the plane partitions for $n = 3$ are as follows,

3

2	1

1	1	1

2
1

1	1
1	

1
1
1

FIGURE 2.2: The plane partitions of $n = 3$

One can visualize a plane partition as a pile of blocks by stretching each part $n_{i,j}$ in Ferrers-Young diagram into $n_{i,j}$ blocks that stack on top of each other. For example, the plane partition of $n = 31$ given in Figure 2.3

5	4	4	3	1
3	2	1		
2	2	1		
1	1			
1				

FIGURE 2.3: A plane partition of $n = 31$

can be visualized as in Figure 2.4,

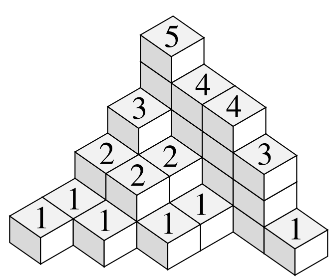

FIGURE 2.4: A plane partition of $n = 31$ in 3 dimensions

where the numbers on top correspond to the entries in the Ferrers-Young diagram.

Also, one can visualize plane partitions as a tuple of partitions where each entry in the tuple represents a Ferrers-Young diagram stacked on top each other. The corresponding decomposition for the plane partition of $n = 31$ in Figure 2.3 is as follows.

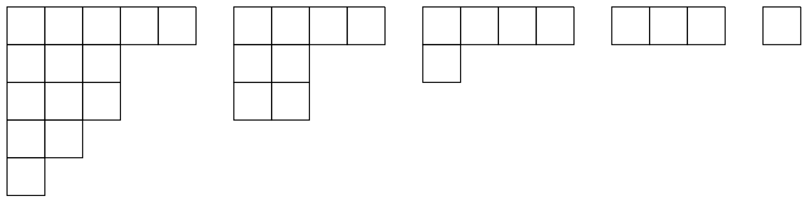

FIGURE 2.5: A tuple partition of $n = 31$

Thus, we get a tuple partition of $n = 31$ given by $(5+3+3+2+1, 4+2+2, 4+1, 3, 1)$.

MacMahon's challenge was to establish a nice generating function for $pl(n)$. However, it was not easy, it took him nearly twenty years (see [7],[30]) to prove that

$$PL(q) = \sum_{n=0}^{\infty} pl(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}. \tag{2.22}$$

He also considered [30] a restricted form of plane partition that has at most r rows and c columns. The generating function is given by

$$PL_{r,c}(q) = \sum_{n=0}^{\infty} pl_{r,c}(n)q^n = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1}{1-q^{i+j-1}}, \tag{2.23}$$

where $pl_{r,c}(n)$ denotes the number of plane partitions of n with at most r rows and c columns. By fixing r and letting $c \longrightarrow \infty$, one obtains the generating function for *r-rowed plane partitions*, which are plane partitions with at most r rows. The generating function is given by

$$PL_r(q) = \sum_{n=0}^{\infty} pl_r(n)q^n = \prod_{i=1}^{r}\prod_{j=1}^{\infty} \frac{1}{1-q^{i+j-1}} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{min\{r,n\}}}, \tag{2.24}$$

where $pl_r(n)$ denotes the number of r-rowed plane partitions of n.

The r-rowed plane partitions are also referred as r-line partitions, for examples one may see work of Cheema and Gordon [11], Gandhi [17].

Agarwal and Andrews [2] define a partition π of m allowing n copies of parts n to be a partition in which a part of size $n \geq 1$ comes in n different subscripts (or colors) denoted by $n_1, n_2, \ldots, n_n$. For example, the partitions with n copies of n of $m = 3$ are

$$3_3, 3_2, 3_1, 2_2 + 1_1, 2_1 + 1_1, 1_1 + 1_1 + 1_1.$$

Now, consider M to be the multiset of the positive integers

$$M = \{1, 2_1, 2_2, 3_1, 3_2, 3_3, 4_1, 4_2, 4_3, 4_4, \ldots\},$$

and let $P_M(m)$ be the number of partitions of m with n copies of n [2]. By standard techniques in partition theory (see [7]) the generating function of $P_M(m)$ is given by

$$\begin{aligned}\sum_{m=0}^{\infty} P_M(m)q^m = \sum_{m=0}^{\infty} p(m; M)q^m &= \frac{1}{(1-q)} \cdot \frac{1}{(1-q^2)^2} \cdot \frac{1}{(1-q^3)^3} \cdots \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}. \end{aligned} \tag{2.25}$$

Note that the left side of (2.25) is the generating function for MacMahon's plane partitions, thus indeed $pl(n) = P_M(n)$, for all $n \geq 0$. However, the combinatorial proof given by MacMahon that (2.25) is the generating function of $pl(n)$ is not easy [31].

Also, Agarwal [1] defines the k-color partitions as partitions of n copies of n with subscripts $\leq k$ and denotes $P_k(n)$ as the number of k-color partitions. For example, $P_2(3) = 5$ since the only partitions with subscripts ≤ 2 are

$$3_2, 3_1, 2_2 + 1_1, 2_1 + 1_1, 1_1 + 1_1 + 1_1.$$

He proves a bijection between k-rowed plane partitions enumerated by $pl_k(n)$ and k-color partitions numerated by $P_k(n)$ for all $n \geq 0$ [1]. This bijection makes it easier to look at the k-rowed plane partitions in terms of k-color partitions since it brings back the two dimensional partitions to one dimensional partitions.

Several congruences of Ramanujan type have been found for different partition functions. However, not many plane partition congruences have been found since MacMahon's discovery. In 1964, Cheema and Gordon [11] obtained the following congruences for 2 and 3-rowed plane partitions.

Theorem 2.2 (Cheema, Gordon, [11])**.** *For all* $n \geq 0$*,*

$$
\begin{aligned}
pl_2(5n+3) &\equiv 0 \pmod{5} \\
pl_2(5n+4) &\equiv 0 \pmod{5} \\
pl_3(3n+2) &\equiv 0 \pmod{3}.
\end{aligned}
$$

Three years later, Gandhi [17] found more congruences of this type and proved the following theorem.

Theorem 2.3 (Gandhi, [17])**.** *For all* $n \geq 0$*,*

$$pl_2(2n+1) \equiv pl_2(2n) \pmod{2}$$

$$pl_3(3n+1) \equiv pl_3(3n) \pmod{3}$$

$$pl_4(4n) \equiv pl_4(4n+1) \equiv pl_4(4n+2) \pmod{2}$$

$$pl_4(4n+3) \equiv 0 \pmod{2}$$

$$pl_5(5n+2) \equiv pl_5(5n+4) \pmod{5}$$

$$pl_5(5n+1) \equiv pl_3(5n+3) \pmod{5}.$$

2.3 Plane and Restricted Plane Overpartitions

Corteel, Savelief and Vuletić [14] define *plane overpartitions* as a generalization of the overpartitions as follows.

Definition 2.4 ([14]). *A plane overpartition is a plane partition where*

(1) in each row the last occurrence of an integer can be overlined or not and all the other occurrences of this integer in the row are not overlined and,

(2) in each column the first occurrence of an integer can be overlined or not and all the other occurrences of this integer in the column are overlined.

Similar to plane partitions, plane overpartitions can be represented in the form of Ferrers-Young diagrams. For example, a plane overpartition for $n = 31$ is given in Figure 2.6.

5	4	$\overline{4}$	3	$\overline{1}$
3	2	1		
2	$\overline{2}$	$\overline{1}$		
1	1			
$\overline{1}$				

FIGURE 2.6: A plane overpartition of $n = 31$

The total number of plane overpartitions of n is denoted by $\overline{pl}(n)$. For example, there are 16 plane overpartitions for $n = 3$ given in Figure 2.7.

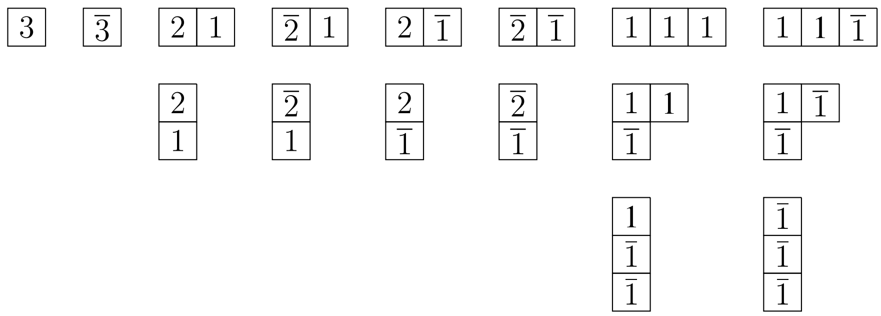

FIGURE 2.7: The plane overpartitions of $n = 3$.

Corteel, Savelief and Vuletić [14] use various methods to obtain the following generating function for plane overpartitions,

$$\overline{PL}(q) := \sum_{n=0}^{\infty} \overline{pl}(n) q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)^n}{(1-q^n)^n}. \tag{2.26}$$

Using the notation of Lovejoy and Mallet [28], the generating function of plane overpartitions is also known as the generating function of n-color overpartitions. From the previous section, n-color partition is a partition in which each number n may appear in n colors, with parts ordered first according to size and then according to color *. An n-color overpartition is defined similarly to be an n-color partition in which the final occurrence of a part n_j may be overlined. For example, there are 16 n-color overpartitions of 3,

$$3_3, 3_2, 3_1, \overline{3}_3, \overline{3}_2, \overline{3}_1, 2_2 + 1_1, \overline{2}_2 + 1_1, 2_2 + \overline{1}_1, \overline{2}_2 + \overline{1}_1, 2_1 + 1_1, \overline{2}_1 + 1_1,$$
$$2_1 + \overline{1}_1, \overline{2}_1 + \overline{1}_1, 1_1 + 1_1 + 1_1, 1_1 + 1_1 + \overline{1}_1.$$

---

*We note that this is slightly different notation from what is usually meant by n-color partition, in which each part regardless of size may appear in one of n colors.

Similar to restricted plane partitions, we define a restricted form of plane overpartitions called *k-rowed plane overpartitions*.

Definition 2.5 (Al-Saedi,[5]). *For an integer $k \geq 1$, a k-rowed plane overpartition π of an integer n is a plane overpartition of n with at most k rows.*

The total number of k-rowed plane overpartitions of n is denoted by $\overline{pl}_k(n)$ and we define $\overline{pl}_k(0) := 1$. For example, $\overline{pl}_2(3) = 14$ as we see the 2-rowed plane overpartitions of $n = 3$ listed in Figure 2.8.

$$\begin{array}{|c|}\hline 3\\\hline\end{array}\quad \begin{array}{|c|}\hline \overline{3}\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline \overline{2}&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&\overline{1}\\\hline\end{array}\quad \begin{array}{|c|c|}\hline \overline{2}&\overline{1}\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&\overline{1}\\\hline\end{array}$$

$$\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline \overline{2}\\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline 2\\\hline \overline{1}\\\hline\end{array}\quad \begin{array}{|c|}\hline \overline{2}\\\hline \overline{1}\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline \overline{1}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&\overline{1}\\\hline \overline{1}\\\cline{1-1}\end{array}$$

FIGURE 2.8: The 2-rowed plane overpartitions of $n = 3$.

Now, let the function $\overline{pl}_{k,c}(n)$ count the total number of plane overpartitions with at most k rows and c columns. Vuletić [41] finds and proves the generating function for such plane overpartitions in the following theorem.

Theorem 2.6 (Vuletić, [41]). *The generating function for plane overpartitions which fit in an $k \times c$ box is*

$$\sum_{n=0}^{\infty} \overline{pl}_{k,c}(n) q^n = \prod_{i=1}^{k} \prod_{j=1}^{c} \frac{1+q^{i+j-1}}{1-q^{i+j-1}}.$$

We apply Theorem 2.6 to obtain the generating function of k-rowed plane overpartitions.

Lemma 2.7 (Al-Saedi,[5]). *For a fixed positive integer k, the generating function for k-rowed plane overpartitions is given by*

$$\overline{PL}_k(q) := \sum_{n=0}^{\infty} \overline{pl}_k(n) q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)^{min\{k,n\}}}{(1-q^n)^{min\{k,n\}}}. \tag{2.27}$$

Proof. By fixing k and letting $c \to \infty$ in Theorem 2.6, we get

$$\prod_{i=1}^{k}\prod_{j=1}^{\infty}\frac{1+q^{i+j-1}}{1-q^{i+j-1}} = \frac{(1+q)(1+q^2)^2\cdots(1-q^{k-1})^{k-1}}{(1-q)(1-q^2)^2\cdots(1-q^{k-1})^{k-1}}\cdot\prod_{n\geq k}\frac{(1+q^n)^k}{(1-q^n)^k}$$
$$= \prod_{n=1}^{\infty}\frac{(1+q^n)^{\min\{k,n\}}}{(1-q^n)^{\min\{k,n\}}}.$$

□

2.4 Periodicity and a Theorem of Kwong

In this Section, we shed light on the periodicity of a certain type of q-series, their minimum periodicity modulo integers and how to find such periodicity. Kwong and others have done extensive studies on the periodicity of certain rational functions, including partition generating functions, for example see [23], [24], [25], [34], and [35]. We will apply a result of Kwong [24] that provides us a systematic formula to calculate the minimum periodicity modulo prime powers of such periodic series.

Let

$$A(q) = \sum_{n=0}^{\infty}\alpha(n)q^n \in \mathbb{Z}[[q]]$$

be a formal power series with integer coefficients, and let d, ℓ and γ be positive integers. We say $A(q)$ is *periodic* with period d modulo ℓ if, for all $n \geq \gamma$,

$$\alpha(n+d) \equiv \alpha(n) \pmod{\ell}.$$

The smallest such period for $A(q)$, denoted $\pi_\ell(A)$, is called the *minimum period of* $A(q)$ modulo ℓ. $A(q)$ is called *purely periodic* if $\gamma = 0$. In this work, periodic always means purely periodic.

Note that if π is a period of $A(q)$, then $\pi \geq \pi_\ell(A)$. By Division Algorithm, there are two integers s and t such that $\pi = s\pi_\ell(A) + t$ and $0 \leq t < \pi_\ell(A)$. By the periodicity of $A(q)$, we observe that

$$\alpha(n+t) = \alpha(n + \pi - s\pi_\ell(A)) \equiv \alpha(n) \pmod{\ell},$$

and hence t is a period but $t < \pi_\ell(A)$, so it must be that $t = 0$. Thus, the minimum period of $A(q)$ divides all other periods of $A(q)$.

For example, consider the q-series $A(q) = \sum_{n=0}^{\infty} \alpha(n)q^n$, which generates the sequence $\alpha(n) := 4n+1$ for all $n \geq 0$. Note that $\alpha(n+2k) - \alpha(n) = 8k \equiv 0 \pmod{8}$ for all $n \geq 0$ and $k \geq 1$. Thus, $A(q)$ is periodic modulo 8 and for each k, there is a period of length $2k$. Thus, the minimum period modulo 8 is $\pi_8(A) = 2$.

Before we state a result of Kwong [24], we recall some necessary definitions from work of Mizuhara, Sellers, and Swisher [33]. For an integer n and prime ℓ, define $ord_\ell(n)$ to be the unique nonnegative integer such that

$$\ell^{ord_\ell(n)} \cdot m = n,$$

where m is an integer and $\ell \nmid m$. In addition, we call m the *ℓ-free part* of n.

For a finite multiset of positive integers S, we define $m_\ell(S)$ to be the ℓ-free part of $lcm\{n | n \in S\}$, and $b_\ell(S)$ to be the least nonnegative integer such that

$$\ell^{b_\ell(S)} \geq \sum_{n \in S} \ell^{ord_\ell(n)}.$$

We now state Kwong's theorem.

Theorem 2.8 (Kwong,[24]). *Fix a prime ℓ, and a finite multiset S of positive integers. Then for any positive integer N,*

$$A(q) = \sum_{n=0}^{\infty} p(n; S)q^n$$

is periodic modulo ℓ^N, with minimum period

$$\pi_{\ell^N}(A) = \ell^{N+b_\ell(S)-1} \cdot m_\ell(S).$$

For example, let $S = \{1_1, 1_2, 2_1, 2_2, 2_3, 4_1, 4_2, 5\}$. Then $p(n; S)$ is generated by the following q-series

$$A(q) := \sum_{n=0}^{\infty} p(n; S)q^n = \prod_{n\in S} \frac{1}{(1-q^n)} = \frac{1}{(1-q)^2(1-q^2)^3(1-q^4)^2(1-q^5)}.$$

Letting $\ell = 2$ in Theorem 2.8, we obtain

$$2^{b_2(S)} \geq \sum_{n\in S} 2^{ord_2(n)} = 2 \cdot 2^0 + 3 \cdot 2^1 + 2 \cdot 2^2 + 2^0 = 17.$$

Thus $b_2(S) = 5,\ lcm\{n : n \in S\} = 20$, and hence $m_2(S) = 5$. Using Theorem 2.8, for a positive integer N, the minimum period of $A(q)$ modulo 2^N is $\pi_{2^N}(A) = 2^{N+4} \cdot 5$.

We note that Theorem 2.8 can be applied to calculate the minimum periodicity modulo prime powers of any q-series of the form

$$R_k(e_1, e_2, \ldots, e_k; q) := \frac{1}{(1-q)^{e_1}(1-q^2)^{e_2}\cdots(1-q^k)^{e_k}}, \tag{2.28}$$

where k is a positive integer and e_i are nonnegative integers for $1 \leq i \leq k$. For the positive integers e_i, consider the multiset of positive integers i_j associated with e_i for $1 \leq j \leq e_i$, that is, define

$$S_{k,\overline{e}} := \{i_j | 1 \leq i \leq k, e_i \geq 1, 1 \leq j \leq e_i\},$$

where $\overline{e} := (e_1, e_2, \ldots, e_k)$. Then by standard partition theory arguments, we observe that

$$\sum_{n=0}^{\infty} p(n; S_{k,\overline{e}})q^n = R_k(e_1, e_2, \ldots, e_k; q).$$

Thus by using Theorem 2.8, we immediately obtain the following lemma.

Lemma 2.9 (Al-Saedi, [5]). *Fix a prime ℓ and a nonnegative integer N, then $R_k(e_1, e_2, \ldots, e_k; q)$ is a periodic q-series modulo ℓ^N with minimum period*

$$\pi_{\ell^N}(R_k) = \ell^{N+b_\ell(S_{k,\overline{e}})-1} \cdot m_\ell(S_{k,\overline{e}}).$$

For example, let $k = 4$, and $\overline{e} = (1, 0, 2, 3)$. Then

$$R_4(1, 0, 2, 3; q) = \frac{1}{(1-q)(1-q^3)^2(1-q^4)^3}$$

generates partitions with parts from the multiset

$$S_{4,\overline{e}} = \{1, 3_1, 3_2, 4_1, 4_2, 4_3\}.$$

In particular, for $\ell = 3$ and $N = 1$, we calculate $b_3(S_{4,\overline{e}})$ using that

$$3^{b_3(S_{4,\overline{e}})} \geq 3^0 + 2 \cdot 3^1 + 3 \cdot 3^0 = 10.$$

Hence $b_3(S_{4,\overline{e}}) = 3$. Also, we see that $m_3(S_{4,\overline{e}}) = 4$. Thus by Lemma 2.9, the minimum period modulo 3 of $R_4(1, 0, 2, 3; q)$ is given by

$$\pi_3(R_4) = 3^3 \cdot 4 = 108.$$

Letting ℓ be a prime, we consider the special case of Lemma 2.9 with $k = \ell - 1$ and $e_i = i$ for $1 \leq i \leq \ell - 1$. Then

$$F_\ell(q) = R_{\ell-1}(1, 2, \ldots, \ell - 1; q) = \prod_{n=1}^{\ell-1} \frac{1}{(1-q^n)^n},$$

where $F_\ell(q)$ was defined in (1.1).

We then have the following immediate corollary to Lemma 2.9 which is a particular case of Corollary 2.4 given by Mizuhara, Sellers, and Swisher in [33].

Corollary 2.10. *For a prime ℓ, and a positive integer N, $F_\ell(q)$ is periodic modulo ℓ^N with minimum period*

$$\pi_{\ell^N}(F_\ell) = \ell^{N+b_\ell(S_{\ell-1,\overline{e}})-1} \cdot m_\ell(S_{\ell-1,\overline{e}}),$$

where $\overline{e} = (1, \ldots, \ell - 1)$.

3 USING PERIODICITY TO OBTAIN PARTITION CONGRUENCES

The goal of this Chapter is to generalize a result of Mizuhara, Sellers, and Swisher [33] which uses periodicity to study plane partition congruences of the form

$$\sum_{i=1}^{s} pl_\ell(n\ell + a_i) \equiv \sum_{j=1}^{t} pl_\ell(n\ell + b_i) \pmod{\ell}, \quad \text{for all } \; n \geq 0. \tag{3.1}$$

Theorem 3.1 (Mizuhara, Sellers, Swisher [33]). *Fix positive integers s, t and nonnegative integers $0 \leq a_i, b_j \leq \ell - 1$ for each $1 \leq i \leq s, 1 \leq j \leq t$. For a prime ℓ, if*

$$\sum_{i=1}^{s} pl_\ell(n\ell + a_i) \equiv \sum_{j=1}^{t} pl_\ell(n\ell + b_i) \pmod{\ell}$$

holds for all $n < \pi_\ell(F_\ell)/\ell$, then it holds for all $n \geq 0$.

Theorem 3.1 states that for a prime ℓ, one can look only at a finite set of values of $pl_\ell(\ell n + a_i)$ and their finite sums for $0 \leq n < \pi_\ell(F_\ell)/\ell$ to determine if there is a congruence of the form (3.1) that holds for all n. For example, taking $\ell = 2$, if there is a congruence of the form (3.1) for $pl_2(n)$, then it must be one of the following possible choices

$$pl_2(2n) \equiv 0 \pmod{2}$$

$$pl_2(2n+1) \equiv 0 \pmod{2}$$

$$pl_2(2n) \equiv pl_2(2n+1) \pmod{2}.$$

If any of the congruences above holds for each $0 \leq n < \pi_2(F_2)/2$, then it holds for all $n \geq 0$. The reason this technique works is because $F_\ell(q)$ is periodic, as seen in Corollary 2.10.

Theorem 3.1 was used in [33] to prove several plane partition congruences, some previously known to Gandhi [17] and others previously unknown.

Theorem 3.2 (Mizuhara, Sellers, Swisher [33]). *The following hold for all $n \geq 0$,*

$$pl_2(2n+1) \equiv pl_2(2n) \pmod{2} \tag{3.2}$$

$$pl_3(3n+2) \equiv 0 \pmod{3} \tag{3.3}$$

$$pl_3(3n+1) \equiv pl_3(3n) \pmod{3} \tag{3.4}$$

$$pl_5(5n+2) \equiv pl_5(5n+4) \pmod{5} \tag{3.5}$$

$$pl_5(5n+1) \equiv pl_3(5n+3) \pmod{5} \tag{3.6}$$

$$pl_7(7n+2) + pl_7(7n+3) \equiv pl_7(7n+4) + pl_7(7n+5) \pmod{7}. \tag{3.7}$$

The identities (3.2), (3.4),(3.5) and (3.6) were originally shown by Gandhi [17], while (3.3) and (3.7) are proved in [33].

3.1 The Main Theorem

We now state and prove the main result of this Chapter. We generalize Theorem 3.1 to a wider class of q-series, and to include prime power moduli.

Theorem 3.3 (Al-Saedi, [5]). *Fix a prime ℓ, and let N, K, δ be any positive integers. Let $A(q), B(q) \in \mathbb{Z}[[q]]$ such that $A(q) := \sum_{n=0}^{\infty} \alpha(n)q^n$ is periodic modulo ℓ^N with* *minimum period* $\pi_{\ell^N}(A) = \delta K$ *and suppose that* $B(q) := \sum_{m=0}^{\infty} \beta(m)q^m$*, where* $\beta(0) \equiv 1 \pmod{\ell^N}$ *and* $\beta(m) \equiv 0 \pmod{\ell^N}$ *for* $m \not\equiv 0 \pmod{\delta}$. *Define*

$$G(q) := A(q) \cdot B(q) := \sum_{k \geq 0} \lambda(k)q^k.$$

Fix positive integers s, t *and nonnegative integers* $0 \leq a_i, b_j \leq \delta - 1$ *for each* $1 \leq i \leq s, 1 \leq j \leq t$. *If*

$$\sum_{i=1}^{s} \lambda(\delta n + a_i) \equiv \sum_{j=1}^{t} \lambda(\delta n + b_j) \pmod{\ell^N},$$

holds for all $0 \leq n < \pi_{\ell^N}(A)/\delta$*, then it holds for all* $n \geq 0$.

Proof. Let ℓ be a prime, and N, K, δ be any positive integers. Suppose that $A(q), B(q) \in \mathbb{Z}[[q]]$ such that $A(q) := \sum_{n=0}^{\infty} \alpha(n)q^n$ is periodic modulo ℓ^N with minimum period $\pi_{\ell^N}(A) = \delta K$ and $B(q) := \sum_{m=0}^{\infty} \beta(m)q^m$, where $\beta(0) \equiv 1 \pmod{\ell^N}$ and $\beta(m) \equiv 0 \pmod{\ell^N}$ for $m \not\equiv 0 \pmod{\delta}$. Let

$$G(q) := A(q) \cdot B(q) := \sum_{k \geq 0} \lambda(k)q^k.$$

Since $\beta(m) \equiv 0 \pmod{\ell^N}$ for $m \not\equiv 0 \pmod{\delta}$, then

$$B(q) \equiv \sum_{m \geq 0} \beta(m\delta)q^{m\delta} \pmod{\ell^N}.$$

Let $\beta'(m) := \beta(m\delta)$ for all $m \geq 0$. Thus

$$\begin{aligned}\sum_{k \geq 0} \lambda(k)q^k &\equiv \left(\sum_{n=0}^{\infty} \alpha(n)q^n\right) \cdot \left(\sum_{m \geq 0} \beta'(m)q^{m\delta}\right) \\ &= \sum_{k \geq 0} \left(\sum_{i=0}^{\lfloor \frac{k}{\delta} \rfloor} \alpha(k - i\delta)\beta'(i)\right) q^k \pmod{\ell^N}.\end{aligned}$$

Therefore, for $k \geq 0$,

$$\lambda(k) \equiv \sum_{i=0}^{\lfloor \frac{k}{\delta} \rfloor} \alpha(k - i\delta)\beta'(i) \pmod{\ell^N}. \tag{3.8}$$

Hence letting $k = n\delta + j$ in (3.8), for $n \geq 0$ and $0 \leq j \leq \delta - 1$, we obtain

$$\lambda(n\delta + j) \equiv \sum_{r=0}^{n} \alpha(r\delta + j)\beta'(n - r) \pmod{\ell^N}. \tag{3.9}$$

Notice that by (3.9), for any $n \geq 0$, the congruence

$$\sum_{i=1}^{s} \lambda(n\delta + a_i) \equiv \sum_{j=1}^{t} \lambda(n\delta + b_j) \pmod{\ell^N} \tag{3.10}$$

is equivalent to

$$\sum_{i=1}^{s} \sum_{r=0}^{n} \alpha(r\delta + a_i)\beta'(n - r) \equiv \sum_{j=1}^{t} \sum_{r=0}^{n} \alpha(r\delta + b_j)\beta'(n - r) \pmod{\ell^N},$$

or in particular to

$$\sum_{r=0}^{n} \beta'(n - r) \left(\sum_{i=1}^{s} \alpha(r\delta + a_i) \right) \equiv \sum_{r=0}^{n} \beta'(n - r) \left(\sum_{j=1}^{t} \alpha(r\delta + b_j) \right) \pmod{\ell^N}.$$

To prove (3.10) holds for all $n \geq 0$, it thus suffices to prove that the congruence

$$\sum_{i=1}^{s} \alpha(n\delta + a_i) \equiv \sum_{j=1}^{t} \alpha(n\delta + b_j) \pmod{\ell^N} \tag{3.11}$$

holds for all $n \geq 0$.

By the hypothesis, (3.10) holds for all $0 \leq n < \pi_{\ell^N}(A)/\delta$. Thus for $0 \leq n < \pi_{\ell^N}(A)/\delta$, we see that

$$\sum_{r=0}^{n} \beta'(n-r) \left(\sum_{i=1}^{s} \alpha(r\delta + a_i) \right) \equiv \sum_{r=0}^{n} \beta'(n-r) \left(\sum_{j=1}^{t} \alpha(r\delta + b_j) \right) \pmod{\ell^N}. \tag{3.12}$$

Letting $n = 0$, (3.12) implies that $\beta'(0)(\sum_{i=1}^{s} \alpha(a_i)) \equiv \beta'(0)(\sum_{j=1}^{t} \alpha(b_j)) \pmod{\ell^N}$. Since $\beta'(0) \equiv 1 \pmod{\ell^N}$, thus $\sum_{i=1}^{s} \alpha(a_i) \equiv \sum_{j=1}^{t} \alpha(b_j) \pmod{\ell^N}$. For $n \geq 1$,

$$\sum_{i=1}^{s} \alpha(n\delta + a_i) + \sum_{r=0}^{n-1} \beta'(n-r) \left(\sum_{i=1}^{s} \alpha(r\delta + a_i) \right) \equiv \\ \sum_{j=1}^{t} \alpha(n\delta + b_j) + \sum_{r=0}^{n-1} \beta'(n-r) \left(\sum_{j=1}^{t} \alpha(r\delta + b_j) \right) \pmod{\ell^N}. \tag{3.13}$$

We see recursively from (3.13) that for all $0 \leq n < \pi_{\ell^N}(A)/\delta$,

$$\sum_{i=1}^{s} \alpha(n\delta + a_i) \equiv \sum_{j=1}^{t} \alpha(n\delta + b_j) \pmod{\ell^N}.$$

To finish the proof, it suffices to prove that (3.11) holds for all $n \geq \pi_{\ell^N}(A)/\delta$. By hypothesis, there is some $K \in \mathbb{N}$ such that $\pi_{\ell^N}(A) = K\delta$. Fix an arbitrary integer $n \geq \pi_{\ell^N}(A)/\delta = K$. By the Division Algorithm, we can write $n = xK + y$ where $0 \leq y < K$. Thus for each $1 \leq i \leq s$, and $1 \leq j \leq t$, we have

$$n\delta + a_i = x \cdot \pi_{\ell^N}(A) + (y\delta + a_i),$$

$$n\delta + b_j = x \cdot \pi_{\ell^N}(A) + (y\delta + b_j).$$

From this we see that

$$n\delta + a_i \equiv y\delta + a_i \pmod{\pi_{\ell^N}(A)},$$

$$n\delta + b_j \equiv y\delta + b_j \pmod{\pi_{\ell^N}(A)}.$$

Since $A(q)$ is periodic modulo ℓ^N with minimum period $\pi_{\ell^N}(A)$, then for each $1 \leq i \leq s$, and $1 \leq j \leq t$,

$$\alpha(n\delta + a_i) \equiv \alpha(y\delta + a_i) \pmod{\ell^N},$$

$$\alpha(n\delta + b_j) \equiv \alpha(y\delta + b_j) \pmod{\ell^N}.$$

Since $0 \leq y < K = \pi_{\ell^N}(A)/\delta$, we have by our hypotheses that

$$\sum_{i=1}^{s} \alpha(y\delta + a_i) \equiv \sum_{j=1}^{t} \alpha(y\delta + b_j) \pmod{\ell^N}.$$

Therefore,

$$\sum_{i=1}^{s} \alpha(n\delta + a_i) \equiv \sum_{i=1}^{s} \alpha(y\delta + a_i) \equiv \sum_{j=1}^{t} \alpha(y\delta + b_j) \equiv \sum_{j=1}^{t} \alpha(n\delta + b_j) \pmod{\ell^N},$$

as desired. $\square$

The generality of Theorem 3.3 gives potential for many more applications, which we discuss further in Section 3.2. Two such examples for plane partitions which we prove in Theorem 3.7 are as follows. For all $n \geq 0$,

$$pl_8(8n) + pl_8(8n+1) \equiv pl_8(8n+3) \pmod{2},$$

$$pl_9(9n+1) \equiv pl_9(9n+8) \pmod{3}.$$

Before we end this section, we state and prove inductively on N the following elementary lemmas to be used later.

Lemma 3.4. *For any prime ℓ and positive integers j and N,*

$$(1-q^j)^{\ell^N} \equiv (1-q^{j\ell^N}) \pmod{\ell}. \tag{3.14}$$

Proof. For $N = 1$. By the Binomial Theorem,

$$(1-q^j)^\ell = \sum_{n=0}^{\ell} \binom{\ell}{n} (-1)^n q^{jn}.$$

Since ℓ is prime, then for $1 \leq n \leq \ell - 1$, $\binom{\ell}{n} \equiv 0 \pmod{\ell}$ and hence

$$(1-q^j)^\ell \equiv (1-q^{j\ell}) \pmod{\ell}.$$

Suppose that (3.14) is true for all $1 \leq k \leq N-1$. Thus implies

$$(1-q^j)^{\ell^N} = \left((1-q^j)^{\ell^{N-1}}\right)^\ell \equiv (1-q^{j\ell^{N-1}})^\ell \equiv (1-q^{j\ell^N}) \pmod{\ell}.$$

□

Lemma 3.5. *For any prime ℓ and positive integers j and N,*

$$(1-q^j)^{\ell^N} \equiv (1-q^{j\ell})^{\ell^{N-1}} \pmod{\ell^N}. \tag{3.15}$$

Proof. Note that (3.15) is true for $N = 1$. Now suppose it is true for some $N > 1$. So we have for some polynomial $Y(q)$,

$$(1-q^j)^{\ell^N} = (1-q^{j\ell})^{\ell^{N-1}} + \ell^N Y(q).$$

Thus, we get

$$\begin{aligned}(1-q^j)^{\ell^{N+1}} &= \left((1-q^{j\ell})^{\ell^{N-1}} + \ell^N Y(q)\right)^\ell \\ &= (1-q^{\ell j})^{\ell^N} + \sum_{n=1}^{\ell} \binom{\ell}{n} \left(\ell^N Y(q)\right)^n \left((1-q^{j\ell})^{\ell^{N-1}}\right)^{\ell-n} \\ &\equiv (1-q^{\ell j})^{\ell^N} \pmod{\ell^{N+1}},\end{aligned}$$

as desired. □

3.2 Applications Of The Main Theorem

3.2.1 Plane Partition Congruences Involving Prime Powers

In work of Gandhi [17], Mizuhara, Sellers, and Swisher [33], elementary combinatorial methods were used to prove some plane partition congruences modulo primes and prime powers. With less effort and a different technique, we apply Theorem 3.3 to reprove some of these congruences and establish new equivalences.

We observe in the following lemma that the restricted plane partition generating functions $PL_{\ell^N}(q)$ are always of the shape needed in Theorem 3.3, where ℓ is a prime and N is a positive integer.

Lemma 3.6 (Al-Saedi, [5]). *For a prime ℓ and a positive integer N, then*

$$PL_{\ell^N}(q) \equiv F_{\ell^N}(q) \cdot \sum_{m\geq 0} \beta(m) q^{\ell^N m} \pmod{\ell},$$
$$PL_{\ell^N}(q) \equiv F_{\ell^N}(q) \cdot \sum_{m\geq 0} \beta'(m) q^{\ell m} \pmod{\ell^N},$$

where $\beta(m), \beta'(m) \in \mathbb{N}$.

Proof. We recall the generating function of ℓ^N-rowed plane partitions from (2.24)

$$PL_{\ell^N}(q) = F_{\ell^N}(q) \cdot \prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^n)^{\ell^N}}.$$

By Lemma (3.4) and Lemma(3.5), one can easily see that

$$\prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^n)^{\ell^N}} \equiv \prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^{n\ell^N})} \pmod{\ell},$$

where

$$\sum_{m\geq 0} \beta(m) q^{\ell^N m} := \prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^{n\ell^N})}.$$

Similarly,

$$\prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^n)^{\ell^N}} \equiv \prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^{n\ell})^{\ell^{N-1}}} \pmod{\ell^N},$$

where

$$\sum_{m\geq 0} \beta'(m) q^{\ell m} := \prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^{n\ell})^{\ell^{N-1}}}.$$

Therefore,

$$PL_\ell(q) \equiv F_{\ell^N}(q) \cdot \prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^{n\ell^N})} \pmod{\ell} \equiv F_{\ell^N}(q) \cdot \sum_{m\geq 0} \beta(m) q^{\ell^N m} \pmod{\ell},$$

$$PL_\ell(q) \equiv F_{\ell^N}(q) \cdot \prod_{n=\ell^N}^{\infty} \frac{1}{(1-q^{n\ell})^{\ell^{N-1}}} \pmod{\ell^N} \equiv F_{\ell^N}(q) \cdot \sum_{m\geq 0} \beta'(m) q^{\ell m} \pmod{\ell^N}.$$

□

Theorem 3.7 (Al-Saedi, [5]). *The following hold for all* $n \geq 0$,

$$pl_4(4n+3) \equiv 0 \pmod{2} \tag{3.16}$$

$$pl_4(4n) \equiv pl_4(4n+1) \equiv pl_4(4n+2) \pmod{2} \tag{3.17}$$

$$pl_8(8n) + pl_8(8n+1) \equiv pl_8(8n+3) \pmod{2} \tag{3.18}$$

$$pl_8(8n+5) \equiv pl_8(8n+6) \equiv pl_8(8n+7) \equiv 0 \pmod{2} \tag{3.19}$$

$$pl_9(9n+1) \equiv pl_9(9n+8) \pmod{3}. \tag{3.20}$$

We note that (3.16) and (3.17) are shown by Gandhi [17], and (3.19) is previously reported in [33], while (3.18) and (3.20) are new to the literature.

Proof. We note that

$$\sum_{n=0}^{\infty} pl_4(n)q^n = \frac{1}{(1-q)(1-q^2)^2(1-q^3)^3} \cdot \prod_{n=4}^{\infty} \frac{1}{(1-q^n)^4}$$
$$= \left(\frac{1}{(1-q)(1-q^3)^3}\right) \cdot \left(\frac{1}{(1-q^2)^2} \cdot \prod_{n=4}^{\infty} \frac{1}{(1-q^n)^4}\right).$$

By Lemma 2.9,

$$A(q) = \sum_{n=0}^{\infty} \alpha(n)q^n := \frac{1}{(1-q)(1-q^3)^3}$$
$$= 1 + q + q^2 + 4q^3 + 4q^4 + 4q^5 + 10q^6 + 10q^7 + 10q^8 + 20q^9 + 20q^{10} + 20q^{11} + 35q^{12} + \cdots$$

is periodic modulo 2 with minimum period $\pi_2(A) = 12$. Also, we use Lemma 3.4 to observe that

$$B(q) = \sum_{n=0}^{\infty} \beta(n)q^n := \frac{1}{(1-q^2)^2} \cdot \prod_{n=4}^{\infty} \frac{1}{(1-q^n)^4}$$
$$\equiv \frac{1}{(1-q^4)} \cdot \prod_{n=4}^{\infty} \frac{1}{(1-q^{4n})} \pmod{2}.$$

Thus $\beta(0) = 1$ and $\beta(n) \equiv 0 \pmod{2}$ for all $n \not\equiv 0 \pmod{4}$, and hence the series $B(q)$ and its coefficients satisfy the desired conditions of Theorem 3.3.

We see directly by expanding the generating function of $PL_4(q)$ that the congruences

$$pl_4(4n+3) \equiv 0 \pmod{2},$$

$$pl_4(4n) \equiv pl_4(4n+1) \equiv pl_4(4n+2) \pmod{2}$$

hold for $n = 0, 1$ and 2. For $\ell = 2, N = 1, \delta = 4$, we apply Theorem 3.3 and conclude that the equivalences (3.16) and (3.17) hold for all $n \geq 0$.

To prove the congruences (3.18) and (3.19), again we use Lemma 3.4 to observe that

$$\begin{aligned}\sum_{n=0}^{\infty} pl_8(n)q^n =& \frac{1}{(1-q)(1-q^2)^2(1-q^3)^3(1-q^4)^4(1-q^5)^5(1-q^6)^6(1-q^7)^7} \\ &\cdot \prod_{n=8}^{\infty} \frac{1}{(1-q^n)^8} \\ \equiv& \frac{1}{(1-q)(1-q^2)^2(1-q^3)^3(1-q^5)^5(1-q^6)^6(1-q^7)^7} \\ &\cdot \left(\frac{1}{(1-q^{16})} \cdot \prod_{n=8}^{\infty} \frac{1}{(1-q^{8n})} \right) \pmod{2}.\end{aligned}$$

By Lemma 2.9, the quotient

$$A(q) = \sum_{n=0}^{\infty} \alpha(n)q^n := \frac{1}{(1-q)(1-q^2)^2(1-q^3)^3(1-q^5)^5(1-q^6)^6(1-q^7)^7}$$

is periodic modulo 2 with minimum period $\pi_2(A) = 2^5 \cdot 105$. Maple programming shows that the congruences

$$pl_8(8n) + pl_8(8n+1) \equiv pl_8(8n+3) \pmod{2},$$

$$pl_8(8n+5) \equiv pl_8(8n+6) \equiv pl_8(8n+7) \equiv 0 \pmod{2}$$

hold for all $0 \leq n < \frac{\pi_2(A)}{8} = 420$. Thus for $\ell = 2, N = 1, \delta = 8$, Theorem 3.3 confirms that the congruences (3.18) and (3.19) hold for all $n \geq 0$.

To prove (3.20), we use the same method to see that

$$\begin{aligned}\sum_{n=0}^{\infty} pl_9(n)q^n =& \frac{1}{(1-q)(1-q^2)^2(1-q^3)^3(1-q^4)^4(1-q^5)^5(1-q^6)^6(1-q^7)^7(1-q^8)^8} \\ &\cdot \prod_{n=9}^{\infty} \frac{1}{(1-q^n)^9} \\ \equiv& \frac{1}{(1-q)(1-q^2)^2(1-q^4)^4(1-q^5)^5(1-q^6)^6(1-q^7)^7(1-q^8)^8}.\end{aligned}$$

$$\left(\frac{1}{(1-q^9)}\cdot\prod_{n=9}^{\infty}\frac{1}{(1-q^{9n})}\right) \pmod 3.$$

Again, by Lemma 2.9, the quotient

$$A(q)=\sum_{n\geq 0}\alpha(n)q^n := \frac{1}{(1-q)(1-q^2)^2(1-q^4)^4(1-q^5)^5(1-q^6)^6(1-q^7)^7(1-q^8)^8}$$

is periodic modulo 3 with minimum period $\pi_3(A)=3^4\cdot 280$. Again by maple programming, we confirm that for all $0\leq n<\frac{\pi_3(A)}{9}=2520$,

$$pl_9(9n+1)\equiv pl_9(9n+8) \pmod 3,$$

as desired. □

3.2.2 Plane Overpartition Congruences

Before we prove the main result of this section as an application of Theorem 3.3, we show below that for a prime ℓ and a positive integer N, the restricted plane overpartition generating function $\overline{PL}_{\ell^N}(q)$ is of the form $A(q)\cdot B(q)$ where $A(q)$ and $B(q)$ satisfy the conditions in Theorem 3.3.

Lemma 3.8 (Al-Saedi, [5])**.** *For a prime ℓ and a positive integer N, then*

$$\overline{PL}_{\ell^N}(q)\equiv R_k(m_1,\ldots,m_k;q)\cdot\sum_{m\geq 0}\beta(m)q^{\ell^N m} \pmod{\ell},$$

$$\overline{PL}_{\ell^N}(q)\equiv R_{k'}(m'_1,\ldots,m'_{k'};q)\cdot\sum_{m\geq 0}\beta'(m)q^{\ell m} \pmod{\ell^N},$$

for some positive integers k,k' and nonnegative integers $m_1,\ldots,m_k,m'_1,\ldots,m'_{k'},\beta(m),\beta'(m)$.

Proof. First, let $N=1$. By Lemma 2.7, the generating function of ℓ-rowed plane overpartitions is given by

$$\overline{PL}_\ell(q)=\frac{(1+q)(1+q^2)^2\cdots(1+q^{\ell-1})^{\ell-1}}{(1-q)(1-q^2)^2\cdots(1-q^{\ell-1})^{\ell-1}}\cdot\prod_{n\geq\ell}\frac{(1+q^n)^\ell}{(1-q^n)^\ell}$$

$$= \frac{(1+q)(1+q^2)^2\cdots(1+q^{\frac{\ell-1}{2}})^{\frac{\ell-1}{2}}}{(1-q)(1-q^2)^2\cdots(1-q^{(\ell-1)})^{\ell-1}} \cdot (1+q^{\frac{\ell-1}{2}+1})^{\frac{\ell-1}{2}+1}\cdots(1+q^{\ell-1})^{\ell-1} \cdot \prod_{n\geq \ell} \frac{(1+q^n)^\ell}{(1-q^n)^\ell}.$$

By factorizing the denominator of the front quotient above, observing that for any $k \in \mathbb{N}$,

$$\frac{(1+q^k)^k}{(1-q^{2k})^{2k}} = \frac{1}{(1-q^k)^k(1-q^{2k})^k},$$

we note that there exist nonnegative integers $s_1, s_2, \ldots, s_{\ell-1}$ so that

$$\frac{(1+q)(1+q^2)^2\cdots(1+q^{\frac{\ell-1}{2}})^{\frac{\ell-1}{2}}}{(1-q)(1-q^2)^2\cdots(1-q^{(\ell-1)})^{\ell-1}} = \frac{1}{(1-q)^{s_1}(1-q^2)^{s_2}\cdots(1-q^{\ell-1})^{s_{\ell-1}}}.$$

Furthermore,

$$\begin{aligned}
&(1+q^{\frac{\ell-1}{2}+1})^{\frac{\ell-1}{2}+1}\cdots(1+q^{\ell-1})^{\ell-1} \cdot \prod_{n\geq \ell} \frac{(1+q^n)^\ell}{(1-q^n)^\ell} \\
&= (1+q^{\frac{\ell-1}{2}+1})^{\frac{\ell-1}{2}+1}\cdots(1+q^{\ell-1})^{\ell-1} \cdot \frac{(1+q^\ell)^\ell(1+q^{\ell+1})^\ell\cdots}{(1-q^\ell)^\ell(1-q^{\ell+1})^\ell\cdots(1-q^{2(\ell-1)})^\ell\cdots(1-q^{2\ell})^\ell\cdots} \\
&= \left(\frac{(1+q^{\frac{\ell+1}{2}})^{\frac{\ell+1}{2}}}{(1-q^{\ell+1})^\ell}\cdots\frac{(1+q^{\ell-1})^{\ell-1}}{(1-q^{2(\ell-1)})^\ell}\right) \cdot \left(\frac{(1+q^\ell)^\ell(1-q^{\ell+1})^\ell\cdots}{(1-q^\ell)^\ell(1-q^{\ell+2})^\ell\cdots(1-q^{2\ell})^\ell\cdots(1-q^{2(\ell+1)})^\ell\cdots}\right) \\
&= \frac{1}{(1-q^{t_1})^{r_1}\cdots(1-q^{t_j})^{r_j}} \cdot \prod_{i\geq 1}\frac{1}{(1-q^{n_i})^\ell},
\end{aligned}$$

for some nonnegative integers r_i and positive integers t_i and n_i. Therefore,

$$\overline{PL}_\ell(q) = \frac{1}{(1-q)^{s_1}(1-q^2)^{s_2}\cdots(1-q^{\ell-1})^{s_{\ell-1}}(1-q^{t_1})^{r_1}\cdots(1-q^{t_j})^{r_j}} \cdot \prod_{i\geq 1}\frac{1}{(1-q^{n_i})^\ell}.$$

Combining terms, we see that there exist nonnegative integers $m_1, m_2, \ldots, m_k$ such that

$$\begin{aligned}
\overline{PL}_\ell(q) &= \frac{1}{(1-q)^{m_1}(1-q^2)^{m_2}\cdots(1-q^k)^{m_k}} \cdot \prod_{i\geq 1}\frac{1}{(1-q^{n_i})^\ell} \\
&= R_k(m_1,\ldots,m_k;q) \cdot \prod_{i\geq 1}\frac{1}{(1-q^{n_i})^\ell}.
\end{aligned}$$

We can repeat the same process for $N > 1$ to obtain

$$\overline{PL}_{\ell^N}(q) = \frac{(1+q)(1+q^2)^2\cdots(1+q^{\ell^N-1})^{\ell^N-1}}{(1-q)(1-q^2)^2\cdots(1-q^{\ell^N-1})^{\ell^N-1}} \cdot \prod_{n\geq \ell^N}\frac{(1+q^n)^{\ell^N}}{(1-q^n)^{\ell^N}}$$

$$= \frac{(1+q)(1+q^2)^2\cdots(1+q^{\frac{\ell^N-1}{2}})^{\frac{\ell^N-1}{2}}}{(1-q)(1-q^2)^2\cdots(1-q^{(\ell^N-1)})^{\ell^N-1}} \cdot (1+q^{\frac{\ell^N-1}{2}+1})^{\frac{\ell^N-1}{2}+1}\cdots(1+q^{\ell^N-1})^{\ell^N-1}\cdot$$

$$\prod_{n\geq \ell^N} \frac{(1+q^n)^{\ell^N}}{(1-q^n)^{\ell^N}}.$$

Observing that the terms

$$(1+q^{\frac{\ell^N-1}{2}+1})^{\frac{\ell^N-1}{2}+1}\cdots(1+q^{\ell^N-1})^{\ell^N-1}\left(\prod_{n\geq \ell^N}(1+q^n)^{\ell^N}\right)$$

can be canceled by some terms in the denominator of the infinite product

$$\prod_{n\geq \ell^N} \frac{1}{(1-q^n)^{\ell^N}}.$$

Thus, there exist nonnegative integers $m_1, \ldots, m_k$ and positive integers n_i such that

$$\begin{aligned}\overline{PL}_{\ell^N}(q) &= \frac{1}{(1-q)^{m_1}(1-q^2)^{m_2}\cdots(1-q^k)^{m_k}} \cdot \prod_{i\geq 1}\frac{1}{(1-q^{n_i})^{\ell^N}} \\ &= R_k(m_1,\ldots,m_k;q)\cdot\prod_{i\geq 1}\frac{1}{(1-q^{n_i})^{\ell^N}}.\end{aligned}$$

Using Lemma 3.4 and Lemma 3.5, the rest follows. □

As an example of Lemma 3.8, for $\ell = 2, 3, 5$, we have the following generating functions,

$$\overline{PL}_\ell(q) = \begin{cases} \frac{1}{(1-q^2)}\cdot\left(\frac{1}{(1-q)^2}\cdot\prod_{n=2}^{\infty}\frac{(1+q^n)^2}{(1-q^{n+1})^2}\right) & \text{if } \ell = 2 \\ \frac{1}{(1-q)^2(1-q^4)}\cdot\left(\frac{1}{(1-q^2)^3(1-q^3)^3}\cdot\prod_{n=3}^{\infty}\frac{(1+q^n)^3}{(1-q^{n+2})^3}\right) & \text{if } \ell = 3 \\ \frac{1}{(1-q)^2(1-q^2)^3(1-q^3)(1-q^4)^2(1-q^8)^2}\cdot\left(\frac{1}{(1-q^3)^5(1-q^4)^5(1-q^5)^5(1-q^7)^5}\cdot\prod_{n=5}^{\infty}\frac{(1+q^n)^5}{(1-q^{n+3})^5}\right) & \text{if } \ell = 5. \end{cases}$$

Note that

$$\sum_{n=0}^{\infty}\overline{pl}_k(n)q^n = \prod_{n=1}^{\infty}\frac{(1+q^n)^{\min\{k,n\}}}{(1-q^n)^{\min\{k,n\}}} = \prod_{n=1}^{\infty}\left(\frac{1-q^n}{1-q^n}\right)^{\min\{k,n\}}$$

$$\equiv 1 \pmod 2.$$

Thus, for any $k, n \geq 1$,

$$\overline{pl}_k(n) \equiv 0 \pmod 2.$$

Theorem 3.9 (Al-Saedi, [5]). *The following holds for all* $n \geq 0$*,*

$$\overline{pl}_4(4n+1) + \overline{pl}_4(4n+2) + \overline{pl}_4(4n+3) \equiv 0 \pmod 4. \tag{3.21}$$

Proof. Observe that

$$\begin{aligned}
\overline{PL}_4(q) &= \left(\frac{(1+q)(1+q^2)^2(1+q^3)^3}{(1-q)(1-q^2)^2(1-q^3)^3}\right) \cdot \left(\prod_{n=4}^{\infty} \frac{(1+q^n)^4}{(1-q^n)^4}\right) \\
&= \left(\frac{1}{(1-q)(1-q^3)^3} \cdot \frac{(1+q)}{(1-q^2)^2} \cdot \frac{(1+q^2)^2}{(1-q^4)^4} \cdot \frac{(1+q^3)^3}{(1-q^6)^4}\right) \cdot \\
&\quad \left((1+q^4)^4(1+q^6)^4 \cdot \frac{(1+q^5)^4}{(1-q^5)^4} \cdot \prod_{n=7}^{\infty} \frac{(1+q^n)^4}{(1-q^n)^4}\right) \\
&= \left(\frac{1}{(1-q)^2(1-q^2)^3(1-q^3)^6(1-q^6)}\right) \cdot \\
&\quad \left((1+q^6)^4 \cdot \frac{(1+q^4)^4(1+q^5)^4}{(1-q^4)^2(1-q^5)^4} \cdot \prod_{n=7}^{\infty} \frac{(1+q^n)^4}{(1-q^n)^4}\right).
\end{aligned}$$

Note that for all $n \geq 1$,

$$\frac{(1+q^n)^4}{(1-q^n)^4} = \left(1 + \frac{2q^n}{1-q^n}\right)^4 \equiv 1 \pmod 4.$$

Therefore,

$$\frac{(1+q^5)^4}{(1-q^5)^4} \cdot \prod_{n=7}^{\infty} \frac{(1+q^n)^4}{(1-q^n)^4} \equiv 1 \pmod 4. \tag{3.22}$$

Also, we note that,

$$(1+q^6)^4 \equiv 1 + 2q^{12} + q^{24} \pmod 4, \tag{3.23}$$

$$\frac{(1+q^4)^4}{(1-q^4)^2} \equiv (1+q^4)^2 \pmod 4. \tag{3.24}$$

So, we let

$$A(q) = \sum_{n=0}^{\infty} \alpha(n)q^n := \frac{1}{(1-q)^2(1-q^2)^3(1-q^3)^6(1-q^6)},$$

$$B(q) = \sum_{n=0}^{\infty} \beta(n)q^n := (1+q^6)^4 \cdot \frac{(1+q^4)^4(1+q^5)^4}{(1-q^4)^2(1-q^5)^4} \cdot \prod_{n=7}^{\infty} \frac{(1+q^n)^4}{(1-q^n)^4}.$$

We see that by (3.22), (3.23) and (3.24) that

$$B(q) \equiv (1+2q^{12}+q^{24})(1+q^4)^2 \pmod 4.$$

Then by Lemma 2.9,

$$A(q) = R_6(2,3,6,0,0,1)$$

is periodic modulo 4 with minimum period $\pi_4(A) = 2^5 \cdot 3$. By a calculation in Maple, we observe that

$$\overline{pl}_4(4n+1) + \overline{pl}_4(4n+2) + \overline{pl}_4(4n+3) \equiv 0 \pmod 4,$$

for all $0 \le n \le \pi_4(A)/4$. Hence letting $\ell = N = 2, \delta = 4$ and applying Theorem 3.3, the congruence (3.21) holds for all $n \ge 0$. $\square$

3.2.3 Congruences of partitions with parts at most m

Let m and n be positive integers. Define $p(n,m)$ to be the number of partitions of n into parts with size at most m. By (2.5), the generating function of $p(n,m)$ is given by

$$Q(q,m) := \sum_{n=0}^{\infty} p(n,m)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)},$$

which follows using the notation in Section 2.1 that

$$Q(q,m) = \sum_{n=0}^{\infty} p(n;T_m)q^n,$$

where $T_m = \{1, 2, \cdots, m\}$. Using Lemma 2.9 for a prime ℓ, one can see that

$$\pi_\ell(Q(q, \ell - 1)) = \ell \cdot m_\ell(T_{\ell-1}), \tag{3.25}$$

$$\pi_\ell(Q(q, \ell)) = \ell^2 \cdot m_\ell(T_\ell). \tag{3.26}$$

Theorem 3.10 (Al-Saedi, [5]). *The following holds for all $n \geq 0$,*

$$p(3n + 1, 2) + p(3n + 2, 2) \equiv 0 \pmod 3 \tag{3.27}$$

$$p(10n + 6, 4) + p(10n + 7, 4) + p(10n + 8, 4) \equiv 0 \pmod 5 \tag{3.28}$$

$$p(10n + 2, 4) + p(10n + 3, 4) + p(10n + 4, 4) \equiv 0 \pmod 5. \tag{3.29}$$

Proof. Observe that by (3.25), $\pi_3(Q(q, 2)) = 6$ and note for $0 \leq n \leq \pi_3(Q(q, 2))/3 = 2$,

$$p(1, 2) + p(2, 2) = 3 \equiv 0 \pmod 3,$$
$$p(4, 2) + p(5, 2) = 6 \equiv 0 \pmod 3.$$

Therefore, by Theorem 3.3 for $A(q) = Q(q, 2), B(q) = 1, N = 1, \ell = \delta = 3$, the identity (3.27) holds for all $n \geq 0$.

For $\ell = 5$, $\pi_5((Q(q, 4)) = 60$. By a calculation in Maple, we verify the coefficients of $Q(q, 4)$ modulo 5 in the following tables.

TABLE 3.1: Restricted Partitions Modulo 5 for (3.28)

n	$p(10n+6,4)$	$p(10n+7,4)$	$p(10n+8,4)$
0	4	1	0
1	4	2	4
2	1	0	4
3	3	1	1
4	0	2	3
5	0	0	0

From Table 3.1 we note the congruence

$$p(10n+6,4)+p(10n+7,4)+p(10n+8,4) \equiv 0 \pmod 5$$

holds for all $n = 0, 1, 2, 3, 4, 5$.

Furthermore, the following table contains the values of $p(10n+i,4)$ modulo 5 for $i = 2, 3, 4$ and for $0 \leq n \leq 5$.

TABLE 3.2: Restricted Partitions Modulo 5 for (3.29)

n	$p(10n+2,4)$	$p(10n+3,4)$	$p(10n+4,4)$
0	2	3	0
1	4	4	2
2	1	0	4
3	1	3	1
4	0	4	1
5	0	0	0

We have from Table 3.2 for all $n = 0, 1, 2, 3, 4, 5$,

$$p(10n + 2, 4) + p(10n + 3, 4) + p(10n + 4, 4) \equiv 0 \pmod 5.$$

By applying Theorem 3.3 for $A(q) = Q(q, 4), B(q) = 1, \delta = 10$ and $N = 1$ we deduce that (3.28) and (3.29) hold for all $n \geq 0$. $\square$

4 PLANE OVERPARTITION CONGRUENCES MODULO POWERS OF 2

This chapter contains two sections. In Section 4.1, we will investigate congruences modulo 4 for restricted and unrestricted plane overpartitions. Also, we will revisit the overpartition function $\overline{p}(n)$ and provide several examples of congruence relations modulo 4 between overpartitions and plane overpartitions. In Section 4.2, we establish and prove several examples of restricted plane overpartition congruences modulo 8. Also, we prove a few congruence relations modulo 8 between overpartitions and the 5-rowed plane overpartions.

4.1 Plane Overpartition Congruences Modulo 4

First, we define throughout the formal power series

$$f(q) := \frac{1+q}{1-q} = 1 + 2q + 2q^2 + 2q^3 + \cdots . \tag{4.1}$$

Note that for every positive integer $n \geq 1$,

$$f(q^n) = \frac{1-q^n}{1-q^n} \equiv 1 \pmod{2}.$$

Thus, we obtain

$$\sum_{n=0}^{\infty} \overline{pl}(n) q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)^n}{(1-q^n)^n} = \prod_{n=1}^{\infty} f(q^n)^n \equiv 1 \pmod{2}.$$

Before we prove next result, we give the following lemmas.

Lemma 4.1. *For all $k \geq 1$,*

$$(1 + 2S(q))^{2^k} \equiv 1 \pmod{2^{k+1}}, \tag{4.2}$$

where $S(q) \in \mathbb{Z}[[q]]$ is a q-series with integer coefficients.

Proof. We induct on k. It is easy to see that (4.2) is true for $k = 1$. Now suppose that (4.2) is true for $1 \leq j \leq k-1$. Then by induction there is a q-series $T(q) \in \mathbb{Z}[[q]]$ such that $(1 + 2S(q))^{2^{k-1}} = 1 + 2^k T(q)$. Thus,

$$\begin{aligned} (1 + 2S(q))^{2^k} &= \left((1 + 2S(q))^{2^{k-1}}\right)^2 \\ &= \left(1 + 2^k T(q)\right)^2 \\ &\equiv 1 \pmod{2^{k+1}}, \end{aligned}$$

as desired. □

Lemma 4.2. *For all integers $n, k \geq 1$,*

$$f(q^n)^{2^k} \equiv 1 \pmod{2^{k+1}}.$$

Proof. Let $S(q) := \sum_{m \geq 1} q^m$. We observe that $S(q) = \frac{q}{1-q}$, and so

$$f(q^n) = \frac{1 + q^n}{1 - q^n} = 1 + \frac{2q^n}{1 - q^n} = 1 + 2S(q^n). \tag{4.3}$$

The conclusion then follows by Lemma 4.1. □

Recall that $\overline{p}_o(n)$ denotes the number of overpartitions of a positive integer n into odd parts and $\overline{p}_o(0) = 1$.

Theorem 4.3. *For every integer $n \geq 1$,*

$$\overline{pl}(n) \equiv \overline{p}_o(n) \equiv \begin{cases} 2 \pmod 4 & \textit{if } n \textit{ is a square or twice a square}, \\ 0 \pmod 4 & \textit{otherwise}. \end{cases}$$

Proof. We observe by (2.26) and Lemma 4.2, the generating function for plane overpartitions

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pl}(n) q^n &= \prod_{n=1}^{\infty} \frac{(1+q^n)^n}{(1-q^n)^n} = \prod_{n=1}^{\infty} f(q^n)^n \\
&= \prod_{n=1}^{\infty} f(q^{2n})^{2n} f(q^{2n-1})^{2n-1} \\
&\equiv \prod_{n=1}^{\infty} f(q^{2n-1}) \pmod 4 \\
&= \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})}{(1-q^{2n-1})} \pmod 4 \\
&= \sum_{n=0}^{\infty} \overline{p}_o(n) q^n \pmod 4.
\end{aligned}$$

Thus for all $n \geq 1$,

$$\overline{pl}(n) \equiv \overline{p}_o(n) \pmod 4.$$

By Theorem 2.1, for all $n \geq 1$,

$$\overline{p}_o(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n \text{ is a square or twice a square}, \\ 0 \pmod 4 & \text{otherwise}, \end{cases}$$

and the result follows. □

Corollary 4.4. *The following holds for all $n \geq 0$,*

$$\overline{pl}(4n+3) \equiv 0 \pmod 4.$$

Proof. Note that for all $n \geq 0$, $4n+3$ is not a square since positive odd squares are 1 modulo 4. Also, $4n+3$ is odd so it can not be twice a square for all $n \geq 0$. The result then follows by Theorem 4.3. $\square$

The following theorem gives a congruence relation modulo 4 between $\overline{pl}(n)$ and $ord_p(n)$ for each odd prime $p|n$.

Theorem 4.5. *For any integer* $n \geq 1$,

$$\overline{pl}(n) \equiv 2 \cdot \prod_{odd\ prime\ p|n} (ord_p(n)+1) \pmod 4. \tag{4.4}$$

Proof. Following the same procedure in Theorem 4.3, we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{pl}(n) q^n &\equiv f(q) \cdot f(q^3) \cdot f(q^5) \cdots \pmod 4 \\ &\equiv \left(1 + 2\sum_{m\geq 1} q^m\right) \cdot \left(1 + 2\sum_{m\geq 1} q^{3m}\right) \cdot \left(1 + 2\sum_{n\geq 1} q^{5m}\right) \cdots \pmod 4 \\ &\equiv 1 + 2\sum_{m\geq 1} q^m + 2\sum_{m\geq 1} q^{3m} + 2\sum_{m\geq 1} q^{5m} + \cdots \pmod 4 \\ &\equiv 1 + 2\sum_{m\geq 1} (q^m + q^{3m} + q^{5m} + \cdots) \pmod 4. \end{aligned} \tag{4.5}$$

Now for any integer $n \geq 1$, by the fundamental theorem of arithmetic, n can be written as a product of prime powers. Thus,

$$n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \tag{4.6}$$

where p_i are primes and α_0, α_i are nonnegative integers for each $i = 1, \ldots, k$. Thus $ord_{p_i}(n) = \alpha_i$ for each $i = 1, \ldots, k$. Note that the term q^n will occur in the series

$$\sum_{m\geq 1} (q^m + q^{3m} + q^{5m} + \cdots)$$

when $m = n/d$ in q^{dm} where d is an odd divisor of n. In terms of the prime factorization of n in (4.6), the number of odd divisors of n is given by

$$\prod_{i=1}^{k}(\alpha_i + 1) = \prod_{i=1}^{k}(ord_{p_i}(n) + 1) = \prod_{odd\ prime\ p|n}(ord_p(n) + 1).$$

Thus the coefficient of q^n in (4.5) is then given by

$$2 \cdot \prod_{odd\ prime\ p|n}(ord_p(n) + 1).$$

□

As a consequence of Theorem 4.3 and Theorem 4.5, we obtain the following result.

Theorem 4.6. *If n is neither a square nor twice a square, then*

$$\prod_{odd\ prime\ p|n}(ord_p(n) + 1)$$

is an odd number.

Next Theorem gives a systematic pattern of congruences modulo 4 for even rowed plane overpartitions.

Theorem 4.7. *Let $k \geq 2$ be a positive even integer, $S_k := \{j \mid j \text{ odd }, 1 \leq j \leq k-1\}$ and ℓ be the least common multiple of the integers in S_k. Then for any odd prime $p < k$, $1 \leq r \leq ord_p(\ell)$, and $n \geq 1$,*

$$\overline{pl}_k(\ell n + p^r) \equiv \begin{cases} 0 \pmod 4 & \text{if } r \text{ is odd}, \\ 2 \pmod 4 & \text{if } r \text{ is even}. \end{cases}$$

Moreover, for all $n \geq 1$,

$$\overline{pl}_k(\ell n) \equiv \begin{cases} 0 \pmod 4 & \text{if } k \equiv 0 \pmod 4, \\ 2 \pmod 4 & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Proof. Let $k \geq 2$ be even. We first observe that the generating function of the k-rowed plane overpartitions can be rewritten modulo 4 using (2.27) and Lemma 4.2 to obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pl}_k(n) q^n &= \prod_{n=1}^{\infty} \frac{(1+q^n)^{\min\{k,n\}}}{(1-q^n)^{\min\{k,n\}}} = \prod_{n=1}^{\infty} f(q^n)^{\min\{k,n\}} \\
&= f(q) f(q^2)^2 \cdots f(q^{k-1})^{k-1} \cdot \prod_{n \geq k} f(q^n)^k \\
&\equiv f(q) f(q^3) \cdots f(q^{k-1}) \pmod 4 \\
&\equiv \left(1 + 2\sum_{n\geq 1} q^n\right) \cdot \left(1 + 2\sum_{n\geq 1} q^{3n}\right) \cdots \left(1 + 2\sum_{n\geq 1} q^{(k-1)n}\right) \pmod 4 \\
&\equiv 1 + 2\sum_{n\geq 1} q^n + 2\sum_{n\geq 1} q^{3n} + \cdots + 2\sum_{n\geq 1} q^{(k-1)n} \pmod 4.
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pl}_k(n) q^n &\equiv 1 + 2\sum_{i\in S_k} \sum_{n\geq 1} q^{in} \pmod 4 \\
&\equiv 1 + 2\sum_{i\in S_k} \sum_{in \not\equiv 0 \pmod{\ell}} q^{in} + 2\sum_{i\in S_k}\sum_{n\geq 1} q^{\ell n} \pmod 4 \\
&\equiv 1 + 2\sum_{i\in S_k} \sum_{in \not\equiv 0 \pmod{\ell}} q^{in} + 2|S_k|\sum_{n\geq 1} q^{\ell n} \pmod 4 \\
&\equiv 1 + 2\sum_{i\in S_k} \sum_{in \not\equiv 0 \pmod{\ell}} q^{in} + k\sum_{n\geq 1} q^{\ell n} \pmod 4,
\end{aligned}$$

where the last congruence is obtained using the fact that $|S_k| = k/2$. Thus, we obtain that

$$\overline{pl}_k(\ell n) \equiv \begin{cases} 0 \pmod 4 & \text{if } k \equiv 0 \pmod 4 \\ 2 \pmod 4 & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Now for a prime $p \in S_k$ and $s := ord_p(\ell)$, we let

$$\sum_{n\geq 1} \alpha(n) q^n := \sum_{n\geq 1} \left(q^n + q^{pn} + q^{p^2 n} + \cdots + q^{p^s n}\right).$$

For any $m \geq 1$ and $1 \leq r \leq s$, the term $q^{m\ell+p^r}$ will occur in the above series when $n = m\ell + p^r, \frac{m\ell+p^r}{p}, \ldots, \frac{m\ell+p^r}{p^r}$, arising from the terms $q^n, q^{pn}, \ldots, q^{p^r n}$, respectively. The term $q^{m\ell+p^r}$ can not be obtained from $\sum_{n\geq 1} q^{p^i n}$ for $i > r$, since p^i does not divide $m\ell + p^r$. Thus

$$\alpha(m\ell + p^r) = r + 1 \equiv \begin{cases} 0 \pmod 2 & \text{if } r \text{ is odd} \\ 1 \pmod 2 & \text{if } r \text{ is even.} \end{cases}$$

Observe that

$$\begin{aligned} \sum_{j\in S_k} \sum_{n\geq 1} q^{jn} &= \sum_{n\geq 1} \left(q^n + q^{pn} + \cdots + q^{p^s n}\right) + \sum_{j\in S_k - \{p^i:0\leq i\leq s\}} \sum_{n\geq 1} q^{jn} \\ &= \sum_{n\geq 1} \alpha(n) q^n + \sum_{j\in S_k - \{p^i:0\leq i\leq s\}} \sum_{n\geq 1} q^{jn}. \end{aligned}$$

Thus, we obtain

$$\sum_{n=0}^{\infty} \overline{pl}_k(n) q^n \equiv 1 + 2\sum_{n\geq 1} \alpha(n) q^n + 2 \sum_{j\in S_k - \{p^i:0\leq i\leq s\}} \sum_{n\geq 1} q^{jn} \pmod 4. \tag{4.7}$$

Also, we note that for all $n, m \geq 1$, if $j \in S_k - \{p^i : 0 \leq i \leq s\}$, then $jn \neq \ell m + p^r$ for all $1 \leq r \leq s$. If not, then there are two positive integers n_0, m_0 such that $jn_0 = \ell m_0 + p^r$, and thus $n_0 = (\ell m_0 + p^r)/j$. Since by the choice of ℓ, we have that j divides ℓ, then j must divide p^r which contradicts that $j \neq p^i$ for all $0 \leq i \leq s$. Thus terms of the form $q^{\ell n+p^r}$ will arise in $\sum_{j\in S_k} \sum_{n\geq 1} q^{jn}$ only from $\sum_{n\geq 1} \alpha(n) q^n$.

Now, If we extract the terms of the form $q^{\ell n+p^r}$ and replace n with $\ell n + p^r$ in (4.7), we find that,

$$\sum_{n\geq 1} \overline{pl}_k(\ell n + p^r) q^n \equiv 2 \cdot \sum_{n\geq 1} \alpha(\ell n + p^r) q^n \equiv 2(r+1) \sum_{n\geq 1} q^n \pmod 4.$$

Thus, modulo 4,

$$\overline{pl}_k(\ell n + p^r) \equiv 2\alpha(\ell n + p^r) = 2(r+1) \equiv \begin{cases} 0 \pmod 4 & \text{if } r \text{ is odd,} \\ 2 \pmod 4 & \text{if } r \text{ is even.} \end{cases}$$

$\square$

As an application of Theorem 4.7, we give a few examples in the following corollary.

Corollary 4.8. *The following hold for all $n \geq 1$,*

$$\overline{pl}_4(3n) \equiv 0 \pmod 4,$$

$$\overline{pl}_6(15n + b) \equiv 0 \pmod 4, \quad for \;\; b \in \{3, 5\},$$

$$\overline{pl}_6(15n) \equiv 2 \pmod 4,$$

$$\overline{pl}_8(105n + b) \equiv 0 \pmod 4, \quad for \;\; b \in \{0, 3, 5, 7\},$$

$$\overline{pl}_{10}(315n + b) \equiv 0 \pmod 4, \quad for \;\; b \in \{3, 5, 7\},$$

$$\overline{pl}_{10}(315n + b) \equiv 2 \pmod 4, \quad for \;\; b \in \{0, 9\},$$

$$\overline{pl}_{12}(3465n + b) \equiv 0 \pmod 4, \quad for \;\; b \in \{0, 3, 5, 7, 11\},$$

$$\overline{pl}_{12}(3465n + 9) \equiv 2 \pmod 4.$$

Proof. For the first congruence, letting $k = 4$, we have that $S_4 = \{1, 3\}$, and $\ell = 3$. Since $k \equiv 0 \pmod 4$, by Theorem 4.7, for all $n \geq 1$

$$\overline{pl}_4(3n) \equiv 0 \pmod 4.$$

Now to see the second and third congruences, let $k = 6$, then $S_6 = \{1, 3, 5\}$ and $\ell = 15$. The only primes in S_6 are 3 and 5 with $ord_p(\ell) = 1$ for $p = 3, 5$. Hence $r = 1$ is the only choice for $1 \leq r \leq ord_p(\ell)$. Thus by Theorem 4.7, for all $n \geq 1$,

$$\begin{aligned} \overline{pl}_6(15n+3) &\equiv 0 \pmod 4, \\ \overline{pl}_6(15n+5) &\equiv 0 \pmod 4. \end{aligned}$$

Moreover, $k = 6 \equiv 2 \pmod 4$ which yields that for all $n \geq 1$,

$$\overline{pl}_6(15n) \equiv 2 \pmod 4.$$

The rest of the identities can be proved similarly. $\square$

In addition, we prove the following theorem which gives an equivalence modulo 4 between the k-rowed plane overpartition function for odd integers k and the overpartition function.

Theorem 4.9. *Let k be a nonnegative integer. Then, for all $n \geq 0$,*

$$\overline{pl}_{2k+1}(2n+1) \equiv \overline{p}(2n+1) \pmod 4. \tag{4.8}$$

Proof. Clearly, for $k = 0$, $\overline{pl}_1(n) = \overline{p}(n)$, for all $n \geq 0$. Now, for $k \geq 1$, we first define

$$g(q) := \frac{1}{f(q)},$$

and note that by Lemma 4.2 and (4.3) that

$$g(q) \equiv f(q) = 1 + 2\sum_{n\geq 1} q^n \pmod 4. \tag{4.9}$$

Recall the generating function of $2k+1$-rowed plane overpartions (2.27),

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pl}_{2k+1}(n) q^n &= \prod_{n=1}^{\infty} \frac{(1+q^n)^{\min\{2k+1,n\}}}{(1-q^n)^{\min\{2k+1,n\}}} = \prod_{n=1}^{\infty} f(q^n)^{\min\{2k+1,n\}} \\
&= f(q) f(q^2)^2 \cdots f(q^{2k})^{2k} \cdot \prod_{n\geq 2k+1} f(q^n)^{2k+1} \\
&\equiv f(q) f(q^3) \cdots f(q^{2k-1}) \cdot \prod_{n\geq 2k+1} f(q^n) \pmod 4,
\end{aligned}$$

where the last congruence is by Lemma 4.2. Thus, we have by (4.9) that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pl}_{2k+1}(n) q^n &\equiv g(q^2) g(q^4) g(q^6) \cdots g(q^{2k}) \cdot \prod_{n=1}^{\infty} f(q^n) \pmod 4 \\
&\equiv f(q^2) f(q^4) f(q^6) \cdots f(q^{2k}) \cdot \prod_{n=1}^{\infty} f(q^n) \pmod 4 \\
&\equiv \left(1 + 2\sum_{n\geq 1} q^{2n}\right) \cdot \left(1 + 2\sum_{n\geq 1} q^{4n}\right) \cdot \left(1 + 2\sum_{n\geq 1} q^{6n}\right) \cdots \\
&\quad \left(1 + 2\sum_{n\geq 1} q^{2kn}\right) \cdot \left(\sum_{n=0}^{\infty} \overline{p}(n) q^n\right) \pmod 4.
\end{aligned}$$

Note that for all $n \geq 1, \overline{p}(n) \equiv 0 \pmod 2$, and hence $2\overline{p}(n) \equiv 0 \pmod 4$. Consequently,

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pl}_{2k+1}(n) q^n &\equiv \left(1 + 2\sum_{n\geq 1} \left(q^{2n} + q^{4n} + q^{6n} + \cdots + q^{2kn}\right)\right) \cdot \left(\sum_{n=0}^{\infty} \overline{p}(n) q^n\right) \pmod 4 \\
&\equiv 2\sum_{n\geq 1} \left(q^{2n} + q^{4n} + q^{6n} + \cdots + q^{2kn}\right) + \sum_{n=0}^{\infty} \overline{p}(n) q^n \pmod 4.
\end{aligned} \tag{4.10}$$

Thus, for all $k \geq 0, n \geq 0$,

$$\overline{pl}_{2k+1}(2n+1) \equiv \overline{p}(2n+1) \pmod 4,$$

as desired. $\square$

Corollary 4.10. *The following holds for every integer* $n \geq 0$,

$$\overline{pl}(2n+1) \equiv \overline{p}(2n+1) \pmod 4. \tag{4.11}$$

Proof. Note that for every integer $n \geq 1$, the plane overpartitions of n have at most n rows. Thus, we obtain for any $k \geq n$,

$$\overline{pl}(n) = \overline{pl}_k(n).$$

By Theorem 4.9, for $k \geq n$,

$$\overline{pl}(2n+1) = \overline{pl}_{2k+1}(2n+1) \equiv \overline{p}(2n+1) \pmod 4.$$

$\square$

Next result gives an infinite family of restricted plane overpartitions congruences modulo 4.

Corollary 4.11. *For all* $k, n \geq 0$, *and* $\alpha \geq 0$,

$$\overline{pl}(9^\alpha(54n+45)) \equiv \overline{pl}_{2k+1}(9^\alpha(54n+45)) \equiv 0 \pmod 4.$$

Proof. Recall that in [20], Hirschhorn and Sellers show that $9^\alpha(27n+18)$ is nonsquare for all $\alpha, n \geq 0$. Thus by (2.17), we have $\overline{p}(9^\alpha(27n+18)) \equiv 0 \pmod 4$ for all $n \geq 0$ and $\alpha \geq 0$. For any odd integer n, $9^\alpha(27n+18)$ is odd. Replacing the odd integer n by $2n+1$, the result follows by Theorem 4.9 and Corollary 4.10. $\square$

Next result gives a pattern of congruences modulo 4 between $\overline{pl}_k(n)$ and $\overline{p}(n)$ for odd k.

Theorem 4.12. *Let $k \geq 2$ and ℓ be the least common multiple of all positive even integers $\leq 2k$. Then for all integers $n \geq 1$,*

$$\overline{pl}_{2k+1}(\ell n + 2^j) \equiv \overline{p}(\ell n + 2^j) \pmod 4, \tag{4.12}$$

where $j \geq 2, j \equiv 0 \pmod 2$ and $2^{j-1} \leq k$. Moreover, if $k \equiv 0 \pmod 2$, then for all integers $n \geq 0$

$$\overline{pl}_{2k+1}(\ell n) \equiv \overline{p}(\ell n) \pmod 4. \tag{4.13}$$

Proof. Recall from the proof of Theorem 4.9 and (4.10) that

$$\sum_{n=0}^{\infty} \overline{pl}_{2k+1}(n) q^n \equiv 2 \sum_{n \geq 1} \left(q^{2n} + q^{4n} + q^{6n} + \cdots + q^{2kn}\right) + \sum_{n=0}^{\infty} \overline{p}(n) q^n \pmod 4.$$

We note for odd $r > 1$, then $2r \nmid \ell m + 2^j$, as well $2^i \nmid \ell m + 2^j$ for $i > j$. Thus, we get for all $m \geq 1$, the term $q^{\ell m + 2^j}$ will occur in the series

$$2 \sum_{n \geq 1} \left(q^{2n} + q^{4n} + q^{6n} + \cdots + q^{4kn}\right),$$

only when $n = \ell m + 2^j/2, \ell m + 2^j/4, \ell m + 2^j/8, \ldots, \ell m + 2^j/2^j$, arising from the terms $q^{2n}, q^{4n}, q^{8n}, \ldots, q^{2^j n}$, respectively. Thus, the coefficient of $q^{\ell m + 2^j}$ in the above series is $2 \sum_{i=1}^{j} 1 = 2j \equiv 0 \pmod 4$ since $j \equiv 0 \pmod 2$. Therefor, for all $n \geq 1$,

$$\overline{pl}_{2k+1}(\ell n + 2^j) \equiv \overline{p}(\ell n + 2^j) \pmod 4,$$

as desired for (4.12). To prove (4.13), since $k \equiv 0 \pmod 2$, we replace k by $2k$ in (4.10) to obtain

$$\sum_{n=0}^{\infty} \overline{pl}_{4k+1}(n) q^n \equiv 2 \sum_{n \geq 1} \left(q^{2n} + q^{4n} + \cdots + q^{4kn}\right) + \sum_{n=0}^{\infty} \overline{p}(n) q^n \pmod 4. \tag{4.14}$$

Note that for all $m \geq 1$, the term $q^{\ell m}$ will occur in the series

$$2\sum_{n\geq 1}\left(q^{2n}+q^{4n}+\cdots+q^{4kn}\right),$$

when $n = \ell m/2, \ell m/4, \ldots, \ell m/4k$, arising from the terms $q^{2n}, q^{4n}, \ldots, q^{4kn}$, respectively. Thus, the coefficient of $q^{\ell m}$ in the above series is $2\sum_{i=1}^{2k} 1 = 4k \equiv 0 \pmod 4$. Therefor, for all $n \geq 0$,

$$\overline{pl}_{4k+1}(\ell n) \equiv \overline{p}(\ell n) \pmod 4,$$

where ℓ here is the least common multiple of all even positive integers $\leq 4k$. $\square$

As an application of Theorem 4.12, we give a few examples in the following corollary.

Corollary 4.13. *The following hold for every integer* $n \geq 1$,

$$\overline{pl}_5(4n) \equiv \overline{p}(4n) \pmod 4,$$

$$\overline{pl}_7(12n+4) \equiv \overline{p}(12n+4) \pmod 4,$$

$$\overline{pl}_9(24n+b) \equiv \overline{p}(24n+b) \pmod 4, \quad for \;\; b \in \{0,4\},$$

$$\overline{pl}_{11}(120n+4) \equiv \overline{p}(120n+4) \pmod 4,$$

$$\overline{pl}_{13}(120n+b) \equiv \overline{p}(120n+b) \pmod 4, \quad for \;\; b \in \{0,4\},$$

$$\overline{pl}_{15}(840n+4) \equiv \overline{p}(840n+4) \pmod 4,$$

$$\overline{pl}_{17}(1680n+b) \equiv \overline{p}(1680n+b) \pmod 4, \quad for \;\; b \in \{0,4,16\}.$$

Proof. Note that for $k = 2, 3, 4, 5, 6, 7, 8$, the least common multiple of the positive even integers $\leq 2k$ is $\ell = 4, 12, 24, 120, 120, 840, 1680$, respectively. By Theorem 4.12, the result follows. $\square$

4.2 Plane Overpartition Congruences Modulo 8

Before we give the main results of this section, we state and prove the following lemmas.

Lemma 4.14. *Let* $a, b, c \in \mathbb{N}$ *such that* $gcd(a, b) = 1$. *Then there are* $c - 1$ *pairs of positive integers* (n, m) *such that* $an + bm = abc$.

Proof. Suppose that $an + bm = abc$. Then $an = abc - bm$ and so $b|an$, and since $gcd(a, b) = 1$, we must have $b|n$. So $n = bN$ for some $N \in \mathbb{N}$. Similarly, $a|m$ and so $m = aM$ for some $M \in \mathbb{N}$. We see then that $abN + abM = abc$ and thus $N + M = c$. Hence, if $(n, m) \in \mathbb{N}^2$ satisfies $an + bm = abc$, then it is equivalent to say there is a pair $(N, M) \in \mathbb{N}^2$ such that $N + M = c$. Note that there are $c - 1$ pairs $(N, M) \in \mathbb{N}^2$ such that $N + M = c$ since the possible ways are $1 + (c - 1), 2 + (c - 2), \ldots, (c - 1) + 1$. □

The next lemma has a flavor of periodicity and restricted partitions.

Lemma 4.15. *Let* $a, b, c \geq 2$ *be integers such that* a, b *and* c *are pairwise relatively prime. Let* M_c *be the number of pairs of positive integers* $(n, m) \in \mathbb{N}^2$ *with* $an + bm = c$ *where* $M_c := 0$ *if no such pairs exists. Then,*

$$M_c = p(c; \{a, b\}),$$

where $p(c; \{a, b\})$ *is the number of partitions of* c *into parts from the set* $\{a, b\}$. *Moreover, for every integer* $N \geq 1$ *and a prime* ℓ,

$$M_{c+\pi_{\ell^N}} \equiv M_c \pmod{\ell^N},$$

where π_{ℓ^N} is the minimum period modulo ℓ^N of the q-series

$$\sum_{n=0}^{\infty} p(n;\{a,b\})q^n = \frac{1}{(1-q^a)(1-q^b)} \tag{4.15}$$

which generates the partitions with parts from $\{a,b\}$.

Proof. Note that if there are two positive integers n and m such that $an+bm=c$, then c can be partitioned into parts form $\{a,b\}$ as follows

$$\underbrace{a+\cdots+a}_{n\text{-times}} + \underbrace{b+\cdots+b}_{m\text{-times}} = c.$$

Thus, any pair of positive integers n and m that satisfy $an+bm=c$ corresponds to a partition of c into parts from $\{a,b\}$. Likewise, since $gcd(a,b)=gcd(a,c)=gcd(b,c)=1$, then any such partition of c must involve both a and b, and hence any corresponding integers n and m must be positive. By considering all such pairs (n,m), we then obtain

$$M_c = p(c;\{a,b\}).$$

By Theorem 2.8, the q-series (4.15) is periodic modulo ℓ^N for any integer $N \geq 1$ and a prime ℓ, with minimum period $\pi_{\ell^N} = \ell^{N+b_\ell(\{a,b\})-1} \cdot m_\ell(\{a,b\})$. Thus,

$$M_{c+\pi_{\ell^N}} = p(c+\pi_{\ell^N};\{a,b\}) \equiv p(c;\{a,b\}) = M_c \pmod{\ell^N}.$$

□

The first theorem in this section gives a few examples of 4 and 8-rowed plane overpartition congruences modulo 8. One may find more of this type using similar methods of proof.

Theorem 4.16. *For all integer $n \geq 1$,*

$$\overline{pl}_4(12n) \equiv 0 \pmod{8}, \tag{4.16}$$

$$\overline{pl}_4(6n+3) \equiv 0 \pmod 8, \tag{4.17}$$

$$\overline{pl}_8(210n) \equiv 0 \pmod 8, \tag{4.18}$$

$$\overline{pl}_8(210n+3) \equiv 0 \pmod 8, \tag{4.19}$$

$$\overline{pl}_8(210n+9) \equiv 0 \pmod 8, \tag{4.20}$$

$$\overline{pl}_8(210n+105) \equiv 0 \pmod 8. \tag{4.21}$$

Proof. Observe that by (2.27) and Lemma 4.2, we have that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pl}_4(n)q^n &= f(q)f(q^2)^2f(q^3)^3\prod_{n\geq 4}f(q^n)^4\\ &\equiv f(q)f(q^2)^2f(q^3)^3 \pmod 8\\ &\equiv \left(1+2\sum_{n\geq 1}q^n\right)\cdot\left(1+2\sum_{n\geq 1}q^{2n}\right)^2\cdot\left(1+2\sum_{n\geq 1}q^{3n}\right)^3 \pmod 8\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pl}_4(n)q^n \equiv 1+2\sum_{n\geq 1}q^n+4\sum_{n\geq 1}q^{2n}+6\sum_{n\geq 1}q^{3n}+\\ 4\sum_{m,n\geq 1}q^{2(n+m)}+4\sum_{m,n\geq 1}q^{3(n+m)}+4\sum_{m,n\geq 1}q^{n+3m} \pmod 8\end{aligned} \tag{4.22}$$

For any $k\geq 1$, the term q^{12k} will occur in the series

$$\sum_{n\geq 1}q^n, \sum_{n\geq 1}q^{2n}, \sum_{n\geq 1}q^{3n}$$

when $n = 12k, 6k, 4k$, arising from the terms q^n, q^{2n}, q^{3n}, respectively. Also, the term q^{12k} will occur in the series

$$\sum_{m,n\geq 1}q^{2(n+m)}, \sum_{m,n\geq 1}q^{3(n+m)}, \sum_{m,n\geq 1}q^{n+3m} \tag{4.23}$$

when $n+m=6k,4k$ and $n+3m=12k$, arising from the terms $q^{2(n+m)}, q^{3(n+m)}, q^{n+3m}$, respectively. We use Lemma 4.14 to count the appearances of q^{12k} in the three series of (4.23) and catalog the results in the following table.

TABLE 4.1: Coefficients of q^{12k} in the series of (4.23)

$an+bm=abc$	a	b	c	j	coefficient of q^{abcj} in $\sum_{n,m\geq 1} q^{j(an+bm)}$
$n+m=6k$	1	1	$6k$	2	$6k-1$
$n+m=4k$	1	1	$4k$	3	$4k-1$
$n+3m=12k$	1	3	$4k$	1	$4k-1$

Thus by Table 4.1, the coefficient of q^{12k} in the series on the right hand side of (4.22) is

$$2+4+6+4(6k-1)+4(4k-1)+4(4k-1)\equiv 0 \pmod 8,$$

which proves (4.16).

To prove (4.17), we observe that for any $k\geq 1$, the term q^{6k+3} will occur in the series

$$\sum_{n\geq 1} q^n, \sum_{n\geq 1} q^{3n}$$

from (4.22) when $n=6k+3, 2k+1$, arising from the terms q^n, q^{3n}, respectively. Also, the term q^{6k+3} will occur in the series

$$\sum_{m,n\geq 1} q^{3(n+m)}, \sum_{m,n\geq 1} q^{n+3m} \tag{4.24}$$

from (4.22) when $n+m=2k+1, n+3m=6k+3$ arising from the terms $q^{3(n+m)}, q^{n+3m}$ respectively. However, the term q^{6k+3} does not occur in the series

$$\sum_{m,n\geq 1} q^{2n}, \sum_{m,n\geq 1} q^{2(n+m)}, \tag{4.25}$$

because $6k+3$ is not divisible by 2 for every integer $k \geq 1$. Again, we use Lemma 4.14 to conclude the number of occurrences of q^{6k+3} in the series of (4.24) in the following table

TABLE 4.2: Coefficients of q^{6k+3} in the series of (4.24)

$an+bm=abc$	a	b	c	j	coefficient of q^{abcj} in $\sum_{n,m\geq 1} q^{j(an+bm)}$
$n+m=2k+1$	1	1	$2k+1$	3	$2k$
$n+3m=6k+3$	1	3	$2k+1$	1	$2k$

Thus by Table 4.2, the coefficient of q^{6k+3} in the series on the right hand side of (4.22) is

$$2+6+4\cdot 2k+4\cdot 2k \equiv 0 \pmod 8,$$

which proves (4.17).

We now prove (4.19) while (4.18) can be proved similarly with less effort. We observe that by (2.27) and Lemma 4.2 that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{pl}_8(n)q^n &= f(q)f(q^2)^2f(q^3)^3f(q^4)^4f(q^5)^5f(q^6)^6f(q^7)^7\prod_{n\geq 8}f(q^n)^8 \\ &\equiv f(q)\ f(q^2)^2\ f(q^3)^3\ f(q^5)\ f(q^6)^2\ f(q^7)^3 \pmod 8 \\ &\equiv \left(1+2\sum_{n\geq 1}q^n\right)\cdot\left(1+2\sum_{n\geq 1}q^{2n}\right)^2\cdot\left(1+2\sum_{n\geq 1}q^{3n}\right)^3\cdot \\ &\quad \left(1+2\sum_{n\geq 1}q^{5n}\right)\cdot\left(1+2\sum_{n\geq 1}q^{6n}\right)^2\cdot\left(1+2\sum_{n\geq 1}q^{7n}\right)^3 \pmod 8.\end{aligned}$$

Thus we have

$$\sum_{n=0}^{\infty}\overline{pl}_8(n)q^n \equiv 1+2\sum_{n\geq 1}q^n+4\sum_{n\geq 1}q^{2n}+6\sum_{n\geq 1}q^{3n}+2\sum_{n\geq 1}q^{5n}+4\sum_{n\geq 1}q^{6n}+6\sum_{n\geq 1}q^{7n}$$

$$+ 4\sum_{m,n\geq 1} q^{2(n+m)} + 4\sum_{m,n\geq 1} q^{3(n+m)} + 4\sum_{m,n\geq 1} q^{6(n+m)} + 4\sum_{m,n\geq 1} q^{7(n+m)}$$
$$+ 4\sum_{m,n\geq 1} q^{n+3m} + 4\sum_{m,n\geq 1} q^{n+5m} + 4\sum_{m,n\geq 1} q^{n+7m}$$
$$+ 4\sum_{m,n\geq 1} q^{3n+5m} + 4\sum_{m,n\geq 1} q^{3n+7m} + 4\sum_{m,n\geq 1} q^{5n+7m} \pmod 8.$$

For any $k \geq 1$, the term q^{210k+3} will occur in the series

$$\sum_{n\geq 1} q^n, \sum_{n\geq 1} q^{3n}$$

when $n = 210k+3, 70k+1$ arising from the terms q^n, q^{3n} respectively. Also, the term q^{210k+3} will occur in the series

$$\sum_{m,n\geq 1} q^{3(n+m)}, \sum_{m,n\geq 1} q^{n+3m}, \sum_{m,n\geq 1} q^{n+5m}$$
$$\sum_{m,n\geq 1} q^{n+7m}, \sum_{m,n\geq 1} q^{3n+5m}, \sum_{m,n\geq 1} q^{3n+7m}, \sum_{m,n\geq 1} q^{5n+7m},$$

when $3(n+m), n+3m, n+5m, n+7m, 3n+5m, 3n+7m, 5n+7m = 210k+3$ arising from the terms

$$q^{3(n+m)}, q^{n+3m}, q^{n+5m}, q^{n+7m}, q^{3n+5m}, q^{3n+7m}, q^{5n+7m}$$

respectively. Since $210k+3$ is not divisible by $2, 5, 6, 7$, so the term q^{210k+3} will not occur in any of the following q-series,

$$\sum_{n\geq 1} q^{2n}, \sum_{n\geq 1} q^{5n}, \sum_{n\geq 1} q^{6n}, \sum_{n\geq 1} q^{7n}, \sum_{m,n\geq 1} q^{2(n+m)}, \sum_{m,n\geq 1} q^{6(n+m)}, \sum_{m,n\geq 1} q^{7(n+m)}.$$

Again, by applying Lemma 4.14, the appearances of q^{210k+3} in the series

$$\sum_{m,n\geq 1} q^{3(n+m)}, \sum_{m,n\geq 1} q^{n+3m}, \tag{4.26}$$

are given in the following table.

TABLE 4.3: Coefficients of q^{210k+3} in the series of (4.26)

$an+bm=abc$	a	b	c	j	coefficient of q^{abcj} in $\sum_{n,m\geq 1} q^{j(an+bm)}$
$n+m=70k+1$	1	1	$70k+1$	3	$70k$
$n+3m=210k+3$	1	3	$70k+1$	1	$70k$

Now for $n+5m, n+7m=210k+3$, we have the following enumerations

$$5\cdot 1+(210k-5+3), 5\cdot 2+(210k-10+3), \dots, 5\cdot 42k+3,$$

$$7\cdot 1+(210k-7+3), 7\cdot 2+(210k-14+3), \dots, 7\cdot 30k+3.$$

Thus, we have $42k, 30k$ pairs of m and n for $n+5m, n+7m=210k+3$, respectively.

For $3n+5m=210k+3$, then $5m=210k+3-3n$ and so $3|m$. Thus, counting for $3n+5m=210k+3$ is equivalent to counting for $3n+15m=210k+3$ which is equivalent to $n+5m=70k+1$ and the later has the following enumerations

$$5\cdot 1+(70k-5+3), 5\cdot 2+(70k-10+3), \dots, 5\cdot 14k+3.$$

Hence, we obtain $14k$ possible pairs n and m such that $3n+5m=210k+3$. Similarly, we have $10k$ pairs of positive integers m and n such that $3n+7m=210k+3$. Thus, the following table catalogs the coefficients of the term q^{210k+3} in the following series

$$\sum_{m,n\geq 1} q^{n+5m}, \sum_{m,n\geq 1} q^{n+7m}, \sum_{m,n\geq 1} q^{3n+5m}, \sum_{m,n\geq 1} q^{3n+7m}. \tag{4.27}$$

TABLE 4.4: Coefficients of q^{210k+3} in the series of (4.27)

$an + bm = 210k + 3$	a	b	coefficient of q^{210k+3} in $\sum_{n,m\geq 1} q^{an+bm}$
$n + 5m = 210k + 3$	1	5	$42k$
$n + 7m = 210k + 3$	1	7	$30k$
$3n + 5m = 210k + 3$	3	5	$14k$
$3n + 7m = 210k + 3$	3	7	$10k$

Now, we only need to check the coefficient of q^{210k+3} in the series $\sum_{m,n\geq 1} q^{5n+7m}$. Note that the integers $a = 5, b = 7$ and $c = 210k + 3$ satisfy the desired conditions of Lemma 4.15. Thus M_{210k+3} is the number of the possible pairs of positive integers (n, m) such that $5n + 7m = 210k + 3$ and

$$M_{210k+3} \equiv M_{210k+3+\pi_8} \pmod 8,$$

where π_8 is the minimum period modulo 8 of the following q-series

$$A(q) := \sum_{n=0}^{\infty} p(n; S)q^n = \frac{1}{(1-q^5)(1-q^7)}.$$

Letting $S = \{5, 7\}$, $\ell = 2$, and $N = 3$ in Theorem 2.8, then $\pi_8 = \pi_8(A) = 280$. In other words, for all $n \geq 0$,

$$M_{210k+3+\pi_8} = p(210k + 3 + \pi_8(A); S) \equiv p(210k + 3; S) = M_{210+3} \pmod 8.$$

If we let $k = 4j$ where $j \in \mathbb{N}$, then we observe by the periodicity of $A(q)$ that

$$M_{210k+3} = p(210k + 3; S) = p(3 + 3j \cdot \pi_8(A); S) \equiv p(3; S) \pmod 8 = 0 \pmod 8.$$

By a similar argument for $k = 4j-1, 4j-2, 4j-3$, we obtain the following

$$M_{210k+3} = p(210k+3; S) \equiv \begin{cases} p(3;S) \pmod 8 = 0 \pmod 8 & \text{if } k = 4j = 4, 8, 12, \ldots \\ p(73;S) \pmod 8 = 2 \pmod 8 & \text{if } k = 4j-1 = 3, 7, 11, \ldots \\ p(143;S) \pmod 8 = 4 \pmod 8 & \text{if } k = 4j-2 = 2, 6, 10, \ldots \\ p(213;S) \pmod 8 = 6 \pmod 8 & \text{if } k = 4j-3 = 1, 5, 9, \ldots \end{cases}$$

By summing all coefficients of q^{210k+3} and using Tables 4.3 and 4.4, we get

$$2 + 6 + 4 \cdot (70k + 70k + 42k + 30k + 14k + 10k + M_{210k+3}) \equiv 0 \pmod 8,$$

which proves (4.19). Similarly, the identities (4.20) and (4.21) can be proved using the same technique. However, for the sake of completeness, we show in Tables (4.5) and (4.6) the corresponding coefficients of the terms $q^{210k+9}, q^{210k+105}$ modulo 8 of the generating function of 8-rowed plane overpartitions, $\sum_{n=0}^{\infty} \overline{pl}_8(n)q^n$.

TABLE 4.5: Coefficients of $q^{210k+9}, q^{210k+105}$ modulo 8 in $c\sum_{n\geq 1} q^{jn}$

$c\sum_{n\geq 1} q^{jn}$	coefficient of q^{210k+9},	$q^{210k+105}$ in $c\sum_{n\geq 1} q^{jn}$
$2\sum_{n\geq 1} q^{n}$	2	2
$4\sum_{n\geq 1} q^{2n}$	0	0
$6\sum_{n\geq 1} q^{3n}$	6	6
$2\sum_{n\geq 1} q^{5n}$	0	2
$4\sum_{n\geq 1} q^{6n}$	0	0
$6\sum_{n\geq 1} q^{7n}$	0	6

TABLE 4.6: Coefficients of $q^{210k+9}, q^{210k+105}$ modulo 8 in $c\sum_{n,m\geq 1} q^{j(an+bm)}$

$c\sum_{n,m\geq 1} q^{j(an+bm)}$	coefficient of	q^{210k+9},	$q^{210k+105}$	in $c\sum_{n,m\geq 1} q^{j(an+bm)}$
$4\sum_{n,m\geq 1} q^{2(n+m)}$		0	0	
$4\sum_{n,m\geq 1} q^{3(n+m)}$		$4(70k+2)$	$4(70k+34)$	
$4\sum_{n,m\geq 1} q^{6(n+m)}$		0	0	
$4\sum_{n,m\geq 1} q^{7(n+m)}$		0	$4(30k+14)$	
$4\sum_{n,m\geq 1} q^{n+3m}$		$4(70k+2)$	$4(70k+34)$	
$4\sum_{n,m\geq 1} q^{n+5m}$		$4(42k+1)$	$4(42k+20)$	
$4\sum_{n,m\geq 1} q^{n+7m}$		$4(30k+1)$	$4(10k+14)$	
$4\sum_{n,m\geq 1} q^{3n+5m}$		$4\cdot 14k$	$4(14k+6)$	
$4\sum_{n,m\geq 1} q^{3n+7m}$		$4\cdot 10k$	$4(10k+4)$	
$4\sum_{n,m\geq 1} q^{5n+7m}$		$4\cdot M_{210k+9}$	$4(6k+2)$	

Now, we only need to show that the number M_{210k+9} is even. Similar to the argument above and by applying Lemma 4.15 for $a=5, b=7$ and $c=210k+9$, we obtain the following,

$$M_{210k+9} = p(210k+9;S) \equiv \begin{cases} p(9;S) \pmod 8 = 0 \pmod 8 & \text{if } k=4j=4,8,12,\dots \\ p(79;S) \pmod 8 = 2 \pmod 8 & \text{if } k=4j-1=3,7,11,\dots \\ p(149;S) \pmod 8 = 4 \pmod 8 & \text{if } k=4j-2=2,6,10,\dots \\ p(219;S) \pmod 8 = 6 \pmod 8 & \text{if } k=4j-3=1,5,9,\dots \end{cases}$$

Thus by M_{210k+9} is being even, the coefficient of q^{210k+9} modulo 8 in $\sum_{n=0}^{\infty} \overline{pl}_8(n)q^n$ is given by summing all coefficients of q^{210k+9} in Tables (4.5) and (4.6), so we obtain

$$2+6+4\cdot(70k+2+70k+2+42k+1+30k+1+14k+10k+M_{210k+9}) \equiv 0 \pmod 8,$$

as desired for the identity (4.20).

For the identity (4.21), by Tables (4.5) and (4.6), the corresponding coefficient modulo 8 of $q^{210k+105}$ in the series $\sum_{n=0}^{\infty} \overline{pl}_8(n)q^n$ is congruent to

$$16+4(252k+128) \equiv 0 \pmod 8.$$

□

Before we prove the next theorem, we recall from (2.15) that

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n \equiv \prod_{j=0}^{k-2} \left(\phi(q^{2^j})\right)^{2^j} \pmod{2^k},$$

where

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1+2\sum_{n=1}^{\infty} q^{n^2}$$

as in (2.10).

For the case $k=3$,

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n \equiv \phi(q)\cdot\phi(q^2)^2 \equiv 1+2\sum_{n\geq 1} q^{n^2}+4\sum_{n\geq 1} q^{2n^2}+4\sum_{n,m\geq 1} q^{2(n^2+m^2)} \pmod 8. \tag{4.28}$$

Thus yields the following useful theorem.

Theorem 4.17. *The following holds for all nonsquare odd integers $n \geq 0$,*

$$\overline{p}(n) \equiv 0 \pmod 8.$$

Proof. If n is a nonsquare odd integer, then n can not be written as $m^2, 2m^2$, or $2(m^2 + k^2)$ for all $m, k \geq 1$. Thus by (4.28), the result follows. $\square$

As a consequence, we obtain the following result which gives an infinite family of overpartition congruences modulo 8.

Corollary 4.18. *For any integer $\alpha \geq 3$, and $\beta \geq 0$, the following holds for each $n \geq 0$,*

$$\overline{p}(2^\alpha 3^\beta n + 5) \equiv 0 \pmod 8.$$

Proof. Clearly, for $\alpha \geq 3$, and $\beta \geq 0$, we have that $2^\alpha 3^\beta n + 5$ is an odd integer for each $n \geq 0$. Suppose that there is a positive integer m such that $2^\alpha 3^\beta n + 5 = (2m+1)^2$. Thus, we obtain $2^{\alpha-2} 3^\beta n + 1 = m(m+1)$. We know that $m(m+1)$ is even which contradicts the fact $2^{\alpha-2} 3^\beta n + 1$ is odd since $\alpha - 2 \geq 1$. Thus no such m exists, and $2^\alpha 3^\beta n + 5$ is not an odd square. $\square$

Next, we obtain a result of Hirschhorn and Sellers [19].

Corollary 4.19. *The following holds for all $n \geq 0$,*

$$\overline{p}(4n + 3) \equiv 0 \pmod 8. \tag{4.29}$$

Proof. Similar to the proof of Corollary 4.18, $4n + 3$ is a nonsquare odd integer for all $n \geq 0$. $\square$

For k-rowed plane overpartitions with odd k, we obtain the following equivalence modulo 8 for plane overpartitions with at most 5 rows.

Theorem 4.20. *The following holds for all* $n \geq 0$,

$$\overline{pl}_5(12n+1) \equiv \overline{p}(12n+1) \pmod 8, \tag{4.30}$$

$$\overline{pl}_5(12n+5) \equiv \overline{p}(12n+5) \pmod 8. \tag{4.31}$$

Proof. By Lemma 4.2 and the fact (2.16), $4\overline{p}(n) \equiv 0 \pmod 8$ for every integer $n \geq 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pl}_5(n)q^n &= f(q)f(q^2)^2 f(q^3)^3 f(q^4)^4 \prod_{n\geq 5} f(q^n)^5 \\
&\equiv f(q^2)\ f(q^3)^2\ f(q^4)^3\ \prod_{n=1}^{\infty} f(q^n) \pmod 8 \\
&\equiv \left(1+2\sum_{n\geq 1} q^{2n}\right)\left(1+2\sum_{n\geq 1} q^{3n}\right)^2\left(1+2\sum_{n\geq 1} q^{4n}\right)^3\left(1+\sum_{n\geq 1}\overline{p}(n)q^n\right) \pmod 8 \\
&\equiv 1+2\sum_{n\geq 1} q^{2n}+4\sum_{n\geq 1} q^{3n}+6\sum_{n\geq 1} q^{4n} \\
&\quad +4\sum_{n,m\geq 1} q^{3(n+m)}+4\sum_{n,m\geq 1} q^{4(n+m)}+4\sum_{n,m\geq 1} q^{2n+4m} \\
&\quad +\sum_{n\geq 1}\overline{p}(n)q^n+2\sum_{n,m\geq 1}\overline{p}(n)q^{n+2m}+6\sum_{n,m\geq 1}\overline{p}(n)q^{n+4m} \pmod 8.
\end{aligned}$$

We observe that

$$\begin{aligned}
2\sum_{n,m\geq 1}\overline{p}(n)q^{n+2m}+6\sum_{n,m\geq 1}\overline{p}(n)q^{n+4m} &= 2\sum_{n,m\geq 1}\overline{p}(n)q^{n+4m-2}+8\sum_{n,m\geq 1}\overline{p}(n)q^{n+4m} \\
&\equiv 2\sum_{n,m\geq 1}\overline{p}(n)q^{n+4m-2} \pmod 8.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pl}_5(n)q^n &\equiv 1+\sum_{n\geq 1}\left(2q^{2n}+4q^{3n}+6q^{4n}\right)+4\sum_{n,m\geq 1}\left(q^{3(n+m)}+q^{4(n+m)}+q^{2(n+2m)}\right) \\
&\quad +\sum_{n\geq 1}\overline{p}(n)q^n+2\sum_{n,m\geq 1}\overline{p}(n)q^{n+4m-2} \pmod 8. \qquad (4.32)
\end{aligned}$$

Note that $12k+1$ is not divisible by 2,3 and 4. So for any $k \geq 1$, the term q^{12k+1} will occur in the series

$$\sum_{n\geq 1} \overline{p}(n)q^n, \sum_{n,m\geq 1} \overline{p}(n)q^{n+4m-2},$$

when $n = 12k+1, 12k+1-(4m-2)$ arising from the terms $q^n, q^{12k+1-(4m-2)}$ respectively for $m = 1, \ldots, 3k$. Hence, the coefficient of q^{12k+1} in the series on the right hand side of (4.32) is then given by

$$\overline{p}(12k+1) + 2\sum_{m=1}^{3k} \overline{p}(12k-4m+3).$$

Note that by Corollary 4.19, for all $k \geq 1$ and $m = 1, \ldots, 3k$, we have

$$\overline{p}(12k-4m+3) = \overline{p}(4(3k-m)+3) \equiv 0 \pmod 8.$$

Thus,

$$\sum_{m=1}^{3k} \overline{p}(12k-4m+3) \equiv 0 \pmod 8.$$

Therefore, for all $k \geq 1$,

$$\overline{pl}_5(12k+1) \equiv \overline{p}(12k+1) + \sum_{m=1}^{3k} \overline{p}(12k-4m+3) \equiv \overline{p}(12k+1) \pmod 8.$$

For the case $k = 0$,

$$\overline{pl}_5(1) \equiv \overline{p}(1) \pmod 8.$$

Thus, for every integer $k \geq 0$,

$$\overline{pl}_5(12k+1) \equiv \overline{p}(12k+1) \pmod 8,$$

as desired for (4.30). The congruence (4.31) can be proved similarly. $\square$

We lastly end this chapter by combining Theorem 4.20 and Corollary 4.18 to obtain the following infinite family of 5-rowed plane overpartition congruences modulo 8.

Corollary 4.21. *For any integers $\alpha \geq 3$ and $\beta \geq 1$, the following holds for all $n \geq 0$,*

$$\overline{pl}_5(2^\alpha 3^\beta n + 5) \equiv 0 \pmod 8. \tag{4.33}$$

Proof. Note that by Theorem 4.20, for all $n \geq 0$,

$$\overline{pl}_5(2^\alpha 3^\beta n + 5) = \overline{pl}_5(12(2^{\alpha-2} 3^{\beta-1} n) + 5) \equiv \overline{p}_5(12(2^{\alpha-2} 3^{\beta-1} n) + 5) \pmod 8.$$

The rest follows by Corollary 4.18. $\square$

5 CONCLUSIONS

We have generalized the method of Mizuhara, Sellers, and Swisher in [33] to give a way to determine various congruences based on a bounded number of calculations. We note that as applications of Theorem 3.3, we obtain new plane partition and plane overpartition congruences. However, the results are limited to computing capabilities since, at least in our cases, increasing the primes leads to more involved coefficient calculations. We hope that further investigations may prove plane partition and plane overpartition congruences modulo higher primes and prime powers. In addition, it would be interesting to find examples of congruences for other types of combinatorial functions which can be proved by Theorem 3.3.

In Chapter 4, we established several examples of plane and restricted plane overpartition congruences modulo 4 and 8. Often, our technique is based on applying Lemma 4.2 up to a small power of 2, then collecting the coefficients of certain terms of the desired power. Lemma 4.2 can be a very powerful tool to find and prove additional congruences modulo powers of 2 for any partition function that involves products containing functions of the form $f(q^n)^m$ where f is defined by $f(q) = \frac{1+q}{1-q}$. For example, the overpartition function has this property.

Based on computational evidence, we conjecture that for each integer $r \geq 1$, and each $k \geq 1$, there exist infinitely many integers n such that

$$\overline{pl}_k(n) \equiv 0 \pmod{2^r}. \tag{5.1}$$

If this holds, then for infinitely many integers n,

$$\overline{pl}(n) \equiv 0 \pmod{2^r}. \tag{5.2}$$

Lemma 4.2 might be a powerful tool to tackle such congruences as (5.1). We note that Theorems 4.9 and 4.20 suggest there might be other arithmetic relations between plane overpartitions and overpartitions that are worth investigating. Furthermore, computational evidence suggests that there is a relation modulo powers of 2 between overpartitions and restricted plane overpartitions. Thus, we conjecture that for each $r \geq 1$ and each $k \geq 1$, there exist infinitely many integers n, such that

$$\overline{pl}_k(n) \equiv \overline{p}(n) \pmod{2^r}.$$

Another approach to establish congruences for plane overpartitions modulo powers of 2 is to look for an iteration formula for plane overpartitions similar to that of overpartitions given by Theorem 2.2 of [21]. That is, consider

$$\overline{P}(q) = \phi(q)\ \phi(q^2)^2\ \phi(q^4)^4\ \phi(q^8)^8 \cdots,$$

and let

$$G_n(q) := \prod_{i=n+1}^{\infty} \frac{1+q^i}{1-q^i} = \prod_{i=n+1}^{\infty} f(q^i).$$

Thus the generating function for plane overpartitions can be rewritten as

$$\begin{aligned} \overline{PL}(q) =& \overline{P}(q) \cdot G_1(q) \cdot G_2(q) \cdot G_3(q) \cdots \\ =& \prod_{n=1}^{\infty} \phi(q^{2^{n-1}})^{2^{n-1}} G_n(q). \end{aligned}$$

Investigating properties of $G_n(q)$ might yield congruences modulo higher powers of 2 for plane overpartitions.

One also may look for congruences modulo odd primes or powers of odd primes for plane and restricted plane overpartitions. Lemma 3.4 and Lemma 3.5 can be a key for establishing restricted plane overpartition congruences since generating functions for this type of partition involve functions of the form

$$f(q^n)^\ell \equiv f(q^{n\ell}) \pmod{\ell},$$
$$f(q^n)^{\ell^N} \equiv f(q^{n\ell})^{\ell^{N-1}} \pmod{\ell^N},$$

for any positive integer N and a prime ℓ.

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