OPTIMAL HARVESTING TIME IN AQUACULTURE ASSUMING NONLINEAR SIZE-HETEROGENEOUS GROWTH[1]

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ABSTRACT

This study explores the optimal harvesting time in a size-heterogeneous population dynamics. The model includes the effect of population density in both the mortality rate and individual growth. An application to specific conditions of shrimp culture in Mexico is presented. The optimal harvesting rule is numerically found for different economic and productive scenarios. Parallel results are also obtained under the hypothesis of homogeneous population growth, which has been traditionally considered in the economic literature. In general, the discounted net revenue of the firm is underestimated if the size-heterogeneity phenomenon is not taken into account, while the calculated harvesting time shortens the predictions based on the homogeneous growth hypothesis. These results reveal that optimal management rules are significantly mistaken if the size-heterogeneity phenomenon is not taken into account.

Keywords: size heterogeneity, bioeconomic model, optimal harvesting time, shrimp

INTRODUCTION

Previous results on optimal management in aquaculture have been based on growth models which do not include size variability in their formulation. Instead of this, a homogeneous growth for all individuals is assumed [1-3]. The usefulness of these results in real practice has been questioned by several authors [4-5]. Critics are mainly focused on the high uncertainty existing in this economic activity from both productive and market aspects. Size variability is a common phenomenon among individuals of the same cohort [6-8]. Thus, the consideration of heterogeneity in the economic analysis would extend the previous results and the derived management recommendation would be more proximate to the real situation.

This paper aims to analyse the influence of size-heterogeneous population in the optimal management of fish culture. In particular, it centres on the optimal stopping or harvesting time of the culture (OHT). This problem has been extensively analysed in the literature, but mostly results are obtained with the (non realistic) assumption of homogeneous population. In this paper, a comparison of OHT obtained with both approaches (homogenous vs. heterogeneous population) is performed. The optimal harvesting time has already been calculated in [9] for the case of a simple linear size-heterogeneous model. This paper extends these results for a general non-linear model, that is, both growth and mortality are dependent on the total number of individuals.

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THE BIOLOGICAL MODELS

The paper presents two models for representing fish growth. The first model assumes that all organisms present identical weight and growth pattern along the culture span. This condition ignores the existence of size variability[2] in the same cage or pond[3]. Hence, the populations can be represented by one individual from the stocking to the harvesting time. This has been the most common way to model fish growth in aquaculture. Given the size of the representative at time t, $x(t)$, growth is defined throughout the following differential equation,

$$\dot{x} = g(x, \bar{N}), \quad x(t_0) = x_0, \tag{1}$$

where $\bar{N}(t)$ indicates the total number of individuals at time t, and x_0 is the stocking size. Thus, fish growth depends not only on the fish size, but also of the density in pond or cage. Some previous models have included this factor in fish growth [10]. In general a negative relationship among density and growth is expected. The dependence on size uses to be quadratic-shape, with two zeros in size $x=0$ and asymptotic size $x=\omega$, respectively.

In general, the mortality rate of individuals, $\mu(x, \bar{N})$, is also dependent on the fish size and density. So, the total number of individuals evolves following the expression,

$$\dot{\bar{N}} = -\mu(\bar{N})\bar{N}, \quad \bar{N}(t_0) = \bar{N}_0, \tag{2}$$

where $\bar{N}_0$ is the number of individuals at time $t=t_0$. The system (h)=(1)∪(2) defines the dynamic of fish size and the number of individuals jointly. It is called *homogeneous case* or *system (h)*.

The second model relaxes the hypothesis of homogeneous growth. So, at any time different sizes are presented in the culture and therefore a representative fish can not be gathered. It is assumed that the total number of individuals at the initial time $t=t_0$, follows a (probabilistic) density function in the interval of possible sizes, $\upsilon_0(x), x \in [0, \omega]$. Following the size-structured model presented in [11], the number of individuals at time t with size x, $N(t,x)$, follows the non-linear PDE,

$$N_t(t, x)+ \big(g(x, \bar{N})N(t, x) \big)_x = -\mu(\bar{N})N(t, x), \quad 0 < x < \omega, \quad t > t_0$$

$$N(t_0, x) = \bar{N}_0 \upsilon_0(x), \tag{3}$$

$$N(t, 0) = 0.$$

The latter equation indicates that there is not reproduction or replacement of individuals along the culture. So, the mortality rate is identical to the *homogeneous case* and every individual follows the same growth pattern described in equation (1), with

$$\bar{N}(t) = \int_0^{\omega} N(t, x)dx.$$

The system (H)=(1)∪(3) is an extension of system (h) by assuming an initial distribution size $N(t_0,x)$. From this initial distribution, every individual follows an identical growing pattern. In fact, equation (2) is obtained by integrating equation (3) with respect size x. This second model is called the *heterogeneous case* or *system (H)*.

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[2] The term "size" and "weight" will be considered as synonymous in the paper.
[3] For notational convenience, the text would refer to fish culture, although the model can be translated directly to other cases.

Existence and uniqueness results of solutions for system (H) have been stated by [12] in a broader context. The numerical solution has recently appeared in [13]. The necessary analytic conditions over functions in the models above are identical to the most restrictive ones in these papers.

OPTIMAL HARVESTING TIME ASSUMING SIZE-HETEROGENEITY

Optimal management is dependent on the structure of revenues and costs in the farm. In theses models, a positive relationship between fish price and size is assumed, that is, higher sizes are more valued by the market. Function $p(x)$ represents price per gram of a fish with size x. Additionally, farms incur in operation cost per individual during the culture span. This cost includes feeding and energy costs, which are dependent on fish size and density in the cage. It is represented by $C(x,\bar{N})$.

For simplicity, a single culture cycle is considered. The evaluation of the economic conditions depends also on the chosen model. In the *homogeneous case*, the accumulated costs are given by

$$C_h(t) = \int_0^t e^{-r\tau} C(x(\tau),\bar{N}(\tau))\bar{N}(\tau)d\tau,$$

where parameter r represents the discount rate in the economy. Therefore, the farmer problem is to find the harvesting time t when the present value of the net revenue by harvesting all the biomass is maximal, that is,

$$\underset{t>0}{Max}\ \pi = p(x(t))x(t)\bar{N}(t)e^{-rt} - \int_0^t e^{-r\tau} C(x(\tau),\bar{N}(\tau))\bar{N}(\tau)d\tau.$$

The first order condition for the solution of the problem above is obtained by differentiating the expression on the right hand side and equating to zero. After some calculation and simplifications, the optimal harvesting time $t=t^h$ satisfies necessarily the equation,

$$
\begin{aligned}
p'(x(t^h))g(x(t^h),\bar{N}(t^h))x(t^h) &+ p(x(t^h))g(x(t^h),\bar{N}(t^h)) \\
&= r + \mu(\bar{N}(t^h))\ p(x(t^h))x(t^h) + C(x(t^h),\bar{N}(t^h)),
\end{aligned}
\tag{4}
$$

which is a simple extension of the results presented in [1]. The left-hand side of the expression represents the marginal revenue obtained by leaving one day more the individual fish growing in the cage, while the second-hand side represents the marginal cost.

For the *heterogeneous case*, the accumulated operational cost a time t is given by the formula,

$$C_H(t) = \int_0^t e^{-r\tau} \int_0^\omega C(x(\tau),\bar{N}(\tau))N(\tau,x)dx\,d\tau,$$

and the revenue at time t is given by,

$$R_H(t) = e^{-rt} \int_0^\omega p(x)xN(t,x)dx.$$

Identically, the farmer problem is to find the harvesting time t^H when the net revenue $\Pi = R_H(t) - C_H(t)$ is maximal. Applying the first order condition to Π and using equation (3), the following equation is obtained,

$$\int_0^\omega p(x)xN_t(t^H,x)dx - \int_0^\omega rp(x)x + C(x,\bar{N}(t^H))\ N(t^H,x)dx = 0 \tag{4}$$

Using a similar procedure followed in [9], we obtain

$$\int_0^\omega \left[p'(x)x + p(x) \ g(x, \bar{N}(t^H)) - \ r + \mu(\bar{N}(t^H)) \ p(x)x - C(x, \bar{N}(t^H)) \right] N(t^H, x) dx = 0. \quad (5)$$

This condition is the extension of equation (4) by assuming heterogeneous sizes in the culture. To find t^H, it is necessary to previously integrate equation (3). A more direct condition can be obtained by simplifying equation (5). Let us call $x(t;t_0,x_0)$ or *characteristic curve in (t_0,x_0)*, the solution of the growth equation (2)[4]. Making the change of variable $x=x(t;t_0,x_0)$, $dx=\partial x(t;t_0,x_0)/\partial x_0 \, dx_0$ in equation (5) and applying the general solution of (3) and Lemmas in [9], it gives,

$$\int_0^\omega [\ p'(x(t^H;t_0,x_0))x(t^H;t_0,x_0) + p(x(t^H;t_0,x_0)) \ g(x(t^H;t_0,x_0), \bar{N}(t^H))$$

$$- \ r + \mu(\bar{N}(t^H)) \ p(x(t^H;t_0,x_0))x(t^H;t_0,x_0) \quad (6)$$

$$- C(x(t^H;t_0,x_0), \bar{N}(t^H))] \upsilon_0(x_0) dx_0 = 0.$$

The optimal harvesting time t^H can therefore calculated from the characteristic curves and the total number of individuals, that is, from the solution of system (h), and the initial distribution of the individuals, $\upsilon_0(x)$, $x \in [0, \omega]$.

EMPIRICAL APPLICATION

The models estimation
The models were calibrated to represent the situation of shrimp farming in freshwater in Mexico. Initially, the growth function (1) was estimated from the data. Several traditional expressions were tested, as von Bertalannfy's and Gompertz's, lightly modified to include the density effect. The best statistical results were obtained with the function

$$g(x, \bar{N}) = a_0 e^{-a_1(Ln(\bar{N}/A))^2} x^{a_2} - a_3 x Ln(x), \quad (7)$$

where a_0, a_1, a_2 and a_3 are convenient parameters and A is the total culture area ($A=40.132$ m^2). Table 1 presents the statistical results of the model. All the parameters are significant and with the expected sign.

Table 1: Parameters, standard error (SE) and p-value for the estimation of growth model (7) with experimental data.

Parameter	Value	SE	p-value
a_0	0.055793	0.012266	6.66E-06
a_1	0.029512	0.007359	6.91E-05
α_2	0.598851	0.131886	6.91E-06
a_3	0.003076	0.001525	0.044256

The estimated mortality rate was exclusively dependent on the total number of individuals, that is, $\mu(\bar{N}) = 1/A \ b_0 + Ln(\bar{N}/A)^{b_1}$, with b_0 and b_1 parameters. Thus, size does not affect the individual mortality rate. Table 2 presents the parameter estimation of equation (2).

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[4] In fact, the characteristic curve also depends on $\bar{N}_0$. We omitted this argument for simplicity.

Table 2: Parameters, standard error (SE) and p-value for the estimation of mortality rate with experimental data.

Parameter	Value	SE	*p*-value
b_0	-0.990044	0.003591	0.000000
b_1	-0.007771	0.002196	0.000436

The size-structured system (H) assumes that exclusively the initial distribution $\upsilon_0(x)$, the mortality rate and the total number of individuals determines the future distribution of sizes along time. The weight of some individuals does not influence on mortality or other sizes' growth. This is quite a strong assumption. To check its reliability, the relationship among the Coefficient of Variation ($CV(t)$) of fish sizes and the growth rate is tested. Given $\bar{x}(t)$ the mean size of individuals at time t, $CV(t)$ is defined by the quotient between the standard deviation of sizes at time t, $\sigma_{\bar{x}}(t)$, and the mean size, that is, $CV(t) = \sigma_{\bar{x}}(t)/\bar{x}(t)$. If the growth pattern of individuals in the tank follows the linear version of equation (3), the "relative size variation will change in proportion to the relative change in the per unit size growth rate" [15]. In mathematical terms, this assertion means,

$$\frac{CV(t)}{CV(t_0)} \cong \frac{g(\bar{x})/\bar{x}}{g(\bar{x}_0)/\bar{x}_0} . \tag{8}$$

The equation (8) was proof by [14] if the mortality rate is zero. The extension for the more general non-linear mortality rate in (3) is also true, but the relationship is not assured if the growth function is density-dependent. Nevertheless, equation (8) will be used to have and indication of the empirical data fitness to a size-structured model. Both sides of the equation can be estimated from the data and, in case of adopting statistically similar values in the culture period, the size-structured model (3) with linear g can be accepted. The non-linear version was tested by validation.

However, the size-structured model does not fit the data from the beginning of the culture. A period of cohort *accommodation period* from the stocking date is assumed, that is, a period of time where individuals start to interact, establish hierarchies and show different growing rates. From that time, an identical growing pattern for all the individuals is observed. Results are presented in Table 3 for the different treatments. As can be observed, the accommodation period varies with the treatments, but the corresponding initial mean size falls between 2.14 and 2.55g. The determination coefficient R^2 was higher than 80%. These results suggest that a cohort *accommodation period* is presented until the mean size reaches values lightly over 2g.

Table 3: Mean size $(\bar{x}_0)$ and accommodation period (t_o).

$\bar{N}_0/A$	R²	$\bar{x}_0$	t_o
90	0.869	2.21	35
130	0.833	2.14	42
180	0.866	2.50	70
230	0.913	2.55	77
280	0.897	2.43	84
330	0.904	2.38	84

The estimated initial time t_0 is positively dependent on the initial density. The higher the initial number of individuals in the tank, the longer the accommodation period is. The sizes distribution at this time represents the initial distribution $\upsilon_0(x), x \in [0,\omega]$, in system (H). A beta function was chosen to describe the density distribution for all the treatments, that is,

$$\upsilon_0(x) = \frac{1}{x_0^1 - x_0^0} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{x - x_0^0}{x_0^1 - x_0^0} \right)^{\alpha-1} \left(1 - \frac{x - x_0^0}{x_0^1 - x_0^0} \right)^{\beta-1}, \; x_0^0 < x < x_0^1,$$

and zero in the rest of values x. $\Gamma(\cdot)$ is the gamma function, α and β are the beta function parameters and x_0^0 and x_0^1 are the minimum and maximum size at the initial time t_0.

Before analysing the economic implications of the size-heterogeneity, a simulation of systems (h) and (H) is presented. Figure 1 shows the size distributions after 200 days of culture for three different initial densities. The vertical line represents the size obtained with system (h). As can be observed, the higher densities reach lower sizes for the same culture period and also show higher variability.
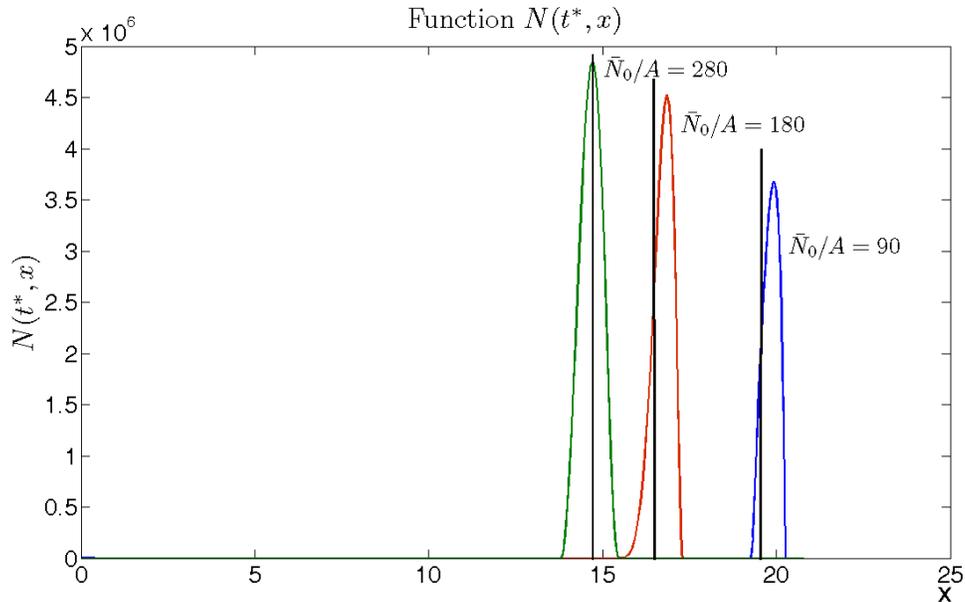
Figure 1. Size distribution after 400 days of culture for three different initial densities. The vertical line indicates the size reached in the homogeneous case.

Economic results

A re-circulation system for the intensive culture of shrimp in fresh water was assumed. In this systems, two main sources of costs are considered: a) Global harvesting cost, c_G, which is the sum of the harvesting cost (c_h), commercialization (c_c) and miscellanea (c_{mi}) per individual; b) operational cost $C(x, \bar{N})$, which are divided into fixed, maintenance, feeding and energy cost per individual. The global harvesting cost is constant and included in the price function, while the operational cost follows the expression

$$C(x, \bar{N}) = c_F \bar{N} + c_m x + c_f f(x, \bar{N}) + c_e \; E_p(x, \bar{N}) + E_a(x, \bar{N}) ,$$

where c_F represents the individual's fixed cost, c_m is the cost per gram of maintaining an x-size organism in the closed system of production (cost related to feeding management, replacements and registers of water quality, biometrics and equipment control), c_f is the feeding cost per gram and c_e is the unitary cost of energy. Function $f(x, \bar{N})$ represents the amount of food supplied to an organism of size x, which also

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depends on culture density. Functions $E_p(x,\bar{N})$ and $E_a(x,\bar{N})$ are the necessary pumping and aeration energy per individual of size x, respectively. These three functions were determined from specialized literature (see the extended version for details). The cost parameter estimations are shown in Table 4, together with some of the economic parameters.

Table 4. Economic parameters

Parameter	Description	Magnitude	Value	Source
c_f	Feeding cost	us\$ g^{-1}	0.00073	Local Market
c_e	Energy cost	us\$ kw-hr^{-1}	0.05	Local Market
c_m	Maintenance cost	us\$ g^{-1}	0.0000010	Calibration
c_F	Fixed cost	us\$ shrimp^{-1}	0.0000545	Calibration
c_h	Haversting cost	us\$ g^{-1}	0.0002	Local Market
c_c	Commercialization cost	us\$ g^{-1}	0.00015	Local Market
c_{mi}	Miscellanea cost	us\$ g^{-1}	0.00010	Local Market
R	Annual discount rate		0.08	Assumption
d_1	Maximum price	us\$ g-1	0.0009	Local Market
d_2	Price parameter		0.3	Calibration
d_3	Price parameter		3.438	Calibration

To find the optimal harvesting time of a single culture, the structure of prices is also needed. A growing relationship between price per g. and size was estimated by means of a logistic function. So,

$$p(x) = \frac{d_1}{1 + d_2 e^{-d_3 x}},$$

where d_1 represents the maximum price, d_2 and d_3 are function parameters. Prices information for different sizes of full shrimp from August to September 2008 was gathered to estimate the price function.

The optimal harvesting time for a single shrimp culture cycle in intensive re-circulation systems with fresh water was calculated. The two models above (*homogeneous* and *heterogeneous case*) were used independently, but assuming identical technical and economic conditions. The algorithm to solve equations (4) and (6) were implemented in MATLAB©. The time and size step was fixed in 1 day and 0.01 grams, respectively. The numerical solution of the non-linear size-structured model (3) was obtained by implementing the algorithm proposed in [4].

The results for the six densities tested in the experiment are presented in Table 5. The optimal harvesting time decreases for higher densities. The growth decrease derived from higher densities makes more profitable to shorten the harvest time. However, the harvesting time modifies if the size-heterogeneity is taken into account. The direction of the change depends on the stocking density. The harvesting time for the heterogeneous case is lower for low stocking densities, but lightly above the case of homogeneous growth for large stocking densities. In the latter culture strategy, the variability of sizes leads to leave the culture some days still growing in order to take advantage of revenue obtained with the higher sizes of the culture.

Table 5: Optimal harvesting time for six different initial densities of shrimp culture in Mexico.

$\bar{N}/A$	Homogeneous case		Heterogeneous case		
	t^h	x^h	t^H	$x^H_{min} - x^H_{max}$	%Utility
90	339	17.74	322	16.79-18.22	3.92
130	319	15.42	300	14.43-15.87	5.89
180	309	13.70	297	11.48-14.51	5.44
230	303	12.50	292	11.57-13.19	6.27
280	300	11.64	303	10.40-12.98	-1.73
330	298	10.97	300	9.70-11.96	-1.83

The last column of Table 5 presents the percentage gap between the discounted utility estimated by assuming the *homogeneous case* and the *heterogeneous case*, respectively. To calculate the net revenue for the heterogeneous case, system (h) was considered for the *accommodation period*, that is, from $t=0$ to $t=t_0$, being t_0 the initial time indicated in Table 3, and from that time, system (H) was used. The results indicate that the net revenue of the firm is generally underestimated if the homogeneous growth hypothesis is adopted. These results show the relevance of the size-heterogeneity in the optimal management of aquaculture farms.

CONCLUSIONS

This paper presents an estimation of the optimal harvesting time in aquaculture management by including the size-heterogeneity phenomenon in the calculations. A nonlinear size-structured model was built for this objective, where both the growth and the mortality rate depend on size and the total number of individuals. A necessary condition for the optimal harvesting time was obtained, which is an extension of other previous results based on a linear size-structured model.

A traditional (assuming size-homogeneity) and a size-structured model were adjusted to experimental data of shrimp culture in re-circulation systems in Mexico for six levels of culture density. The estimations show that the size-structured model fits the data from a certain time after the beginning of the culture, when the mean size surpasses a determined threshold, similar to any of the density levels. From this mean size, the nonlinear size-structured model is a good approximation of data.

The results with the empirical example show that the optimal harvesting time present decreasing values with respect to density. In general, the discounted net revenue of the firm is underestimated if the size-heterogeneity phenomenon is not taken into account, while the calculated harvesting time shortens the predictions based on the homogeneous growth hypothesis. This should not necessary be the case for other species, culture or economic conditions. Nevertheless, the empirical example illustrates that optimal management in intensive aquaculture farms could be significantly mistaken if a non-realistic homogeneous growth hypothesis is assumed.

In general, the market assigns different value to the same species according to its size, which determines several groups or classes. The size-structured model proposed here allows estimating the amount of the different classes in one culture cycle. By using this type of bioeconomic models, managers could design a segmentation strategy to allocate the product in the market. This is not possible by using the traditional models assuming size homogeneity.

The research presented in the paper could be extended in several aspects related with animal husbandry management. Particularly, selective harvesting (e.g. larger sizes of a same cohort) during the culture span is a common practice in the industry which has not been extensively analysed in the literature. Some recent results in the theory of optimal control for size-structured models could be used to solve the problem in the same framework presented in this paper. However, the numerical solution is still to appear.

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