Time scale and intensity dependency in multiplicative cascades for temporal rainfall disaggregation

David E. Rupp,[1,2] Richard F. Keim,[3] Mina Ossiander,[4] Marcela Brugnach,[5] and John S. Selker[6]

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[1] Multiplicative random cascades (MRCs) can parsimoniously generate highly intermittent patterns similar to those in rainfall. The elemental MRC model parameter is the cascade weight, which determines how rainfall at one scale is partitioned at the next smallest scale in the cascade. While it is known that the probability density of these weights may vary with both time scale and rainfall intensity, nearly all previous studies have considered either time scale or intensity separately. We examined the simultaneous dependency of the weights on both factors and assessed the impacts of explicitly including these dependencies in the MRC model. On the basis of the observed relationships between cascade weights and time scale and intensity, four progressively more "dependent" models were constructed to disaggregate a long time series of daily rainfall to hourly intervals. We found that inclusion of the intensity dependency on the model parameters that generate dry intervals greatly improved performance. For the relatively small range of time scales over which the rainfall was disaggregated, varying model parameters with time scale resulted in minor improvement.

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1. Introduction

[2] Hydrological processes that control surface and subsurface water distribution can depend on rainfall forcing at subdaily time scales, therefore successful modeling of these processes often requires rainfall information at such scales. However, most long rainfall records are available at a daily time step. Additionally, retrieving lengthy output from physically based atmospheric models at hourly or subhourly time scales is, for many applications, computationally and cost prohibitive. Therefore, to generate fine-scale rainfall on the basis of historical records or long-term forecasts, often the best option is to stochastically disaggregate rainfall from coarser to finer scales.

[3] Among various methods of rainfall disaggregation (see, e.g., review by D. Koutsoyiannis, Rainfall disaggregation methods: Theory and applications, paper presented at Workshop on Statistical and Mathematical Methods for Hydrological Analysis, Università degli Studi di Roma "La Sapienza," Rome, 2003) is by means of a discrete multiplicative random cascade (MRC) [e.g., *Schertzer and Lovejoy*, 1987; *Mandelbrot*, 1989; *Gupta and Waymire*, 1993]. In this method, the rainfall R occurring over an interval in time (or space) is divided among a number of smaller intervals of equal size. The number of subintervals is known as the branching number and for our purposes will be 2, which is the most parsimonious case [*Gupta and Waymire*, 1993]. This method assumes that the amount of rain falling in one of two equal subintervals of a given interval is determined by multiplying the interval rainfall R by a dimensionless cascade weight W. This multiplication is repeated again and again through successively finer cascade levels. At the kth cascade level, the rainfall over a given time interval at position j in the time series can be expressed as

$$R_{j,k} = R_0 \prod_{i=1}^{k} W_{f(i,j),i} \tag{1}$$

where $j = 1, 2, \ldots, 2^k$. At the ith level, the function $f(i, j)$ indexes the position of the time interval and is given by rounding up $j/2^{k-i}$ to the nearest integer [e.g., *Gaume et al.*, 2007].

[4] The MRC model, first used in studies of turbulence [*Yaglom*, 1966; *Mandelbrot*, 1974], can produce fields and series that have statistically scale invariant properties. Over the last two decades a substantial body of literature has been created on the topic of generating simple fractal and multifractal rainfall fields and time series using multiplicative cascades. A small number of representative studies include, for example, *Schertzer and Lovejoy* [1987], *Gupta and Waymire* [1993], *Over and Gupta* [1996], *Menabde et al.* [1997], *Deidda et al.* [1999], *Deidda* [2000], and *Veneziano*

[1]DHI Water and Environment, Inc., Portland, Oregon, USA.
[2]National Institute of Water and Atmospheric Research, Christchurch, New Zealand.
[3]School of Renewable Natural Resources, Louisiana State University Agricultural Center, Baton Rouge, Louisiana, USA.
[4]Department of Mathematics and Statistics, Oregon State University, Corvallis, Oregon, USA.
[5]Institute for Environmental Systems Research, University of Osnabrück, Osnabrück, Germany.
[6]Department of Biological and Ecological Engineering, Oregon State University, Corvallis, Oregon, USA.

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et al. [2006b]. While the link to multifractality has been the motivation behind most of the research in MRC models [*Gaume et al.*, 2007], the MRC method is itself appealing in that in can parsimoniously generate complex intermittent and spiky patterns typical of rainfall time series, irrespective of whether the patterns are multifractal or not.

[5] Multiplicative random cascades can be constructed so that the weights of each branch of a cascade sum to 1 only on the average (canonical cascade), or so that they sum to exactly one in each split (microcanonical cascade) [*Schertzer and Lovejoy*, 1987]. In the microcanonical case, the weights are complementary so that where there are two branches, $W_1 = W$ and $W_2 = 1 - W$, where W is a random variable between 0 and 1, inclusive. Examples of microcanonical cascade models are given by *Olsson* [1998], *Menabde and Sivapalan* [2000], *Ahrens* [2003], and *Paulson and Baxter* [2007]. Two important attributes of the microcanonical model are that it conserves mass exactly at each branch and that the distribution of W can be extracted exactly from the data [*Cârsteanu and Foufoula-Georgiou*, 1996]. This latter attribute is particularly attractive because it permits a direct examination of the associations that the weights may have with other properties of rainfall [*Olsson*, 1998].

[6] The simplest random cascade is one in which W is assumed independent and identically distributed (IID) both in time and across all cascade levels (i.e., states). Over a finite range of time scales, e.g., less than 2 orders of magnitude, the assumption of time scale invariance of W has been shown to be reasonable in rainfall time series [*Harris et al.*, 1998; *Olsson*, 1998; *Cârsteanu et al.*, 1999; *Langousis and Veneziano*, 2007]. This is not universal, however: the cascade weights sometimes decrease in variance with decreasing time scale on and below the order of hours [*Olsson*, 1998; *Menabde and Sivapalan*, 2000; *Paulson and Baxter*, 2007]. Cascades in which the variance of the weights decrease with each level are called "bounded" [*Marshak et al.*, 1994; *Menabde et al.*, 1997; *Harris et al.*, 1998].

[7] For most time series at any given time scale, the weights are also neither independently nor identically distributed. For example, an analysis of high-resolution rainfall data by *Cârsteanu and Foufoula-Georgiou* [1996] revealed that the lag-one autocorrelation of W, or $\rho_W(1)$, in a microcanonical cascade was approximately -0.2 and not 0 as it would be for IID W. *Olsson* [1998] and *Güntner et al.* [2001] also found that the probability distribution of W associated with a given rainy interval depended on the state (wet or dry) of the intervals immediately preceding and following it. They separated time intervals into four classes: starting (preceded by dry and followed by wet), ending (preceded by wet and followed by dry), enclosed (bounded by wet), and isolated (bounded by dry). In their model they applied a distinct distribution function for W to each interval class.

[8] Last, a strong dependence of the weights on rainfall intensity has been observed in rainfall time series [*Olsson*, 1998; *Güntner et al.*, 2001; *Veneziano et al.*, 2006a], spatial rainfall fields [*Over and Gupta*, 1994] and in space-time [*Deidda*, 2000; *Deidda et al.*, 2004, 2006; *Veneziano et al.*, 2006a], though some have claimed intensity independence in temporal rainfall [e.g., *Venugopal et al.*, 1999, Figure 3]. Contributing to the dependency of the weights on rainfall intensity is that more and longer dry periods at fine scales are associated with lower rainfall intensities at the aggregated coarser scale and also that threshold amounts for measured rainfall amounts increase the sparseness of rainfall, particularly for low-intensity events [*Veneziano et al.*, 2006a].

[9] *Olsson* [1998] and *Güntner et al.* [2001] partially accounted for the intensity dependence by modeling W separately for rainfall intensities greater or less than the mean. In the case of space and space-time, cascade weight distribution parameters have been conditioned on the mesoscale rainfall intensity of a given event [*Over and Gupta*, 1994; *Deidda*, 2000; *Deidda et al.*, 2004, 2006]. *Veneziano et al.* [2006a], however, argued that the "mesoscale" as practically defined as the coverage of a radar frame (e.g., 256 km $\times$ 256 km) has no special physical or statistical significance and therefore chose to vary the cascade parameters with intensity at every level in the cascade.

[10] In addition to the evidence against IID weights, a difficulty in developing MRC models has been the accurate generation of dry intervals [e.g., *Schmitt et al.*, 1998]. In part, this issue arises because most probability density functions have zero probability of W being exactly zero. One simple technique to introduce dry intervals is to set a threshold below which any rainfall intensity is rounded to zero [e.g., *Pathirana et al.*, 2003], but this technique undesirably affects the statistics of the synthetic field. Another technique is to use a mixed distribution, in which a separate function is used to calculate the (nonzero) probability of W exactly equal to 0 (or also 1 for microcanonical cascades) [e.g., *Over and Gupta*, 1996; *Olsson*, 1998; *Langousis and Veneziano*, 2007; *Langousis et al.*, 2009; *Paulson and Baxter*, 2007; *Veneziano et al.*, 2007]. It has been argued by *Schmitt et al.* [1998] that neither approach is likely to generate the correct pattern of wet and dry periods.

[11] An alternative is to employ a hybrid model in which the durations of consecutive dry and wet states and the mean wet period intensity are treated as random variables. The rain in each wet period is then disaggregated to the temporal resolution of interest as a MRC assuming no small-scale dry periods occur with a wet period [*Schmitt et al.*, 1998; *Menabde and Sivapalan*, 2000]. This technique is not well developed for disaggregation in which dry-wet sequences are predetermined at a specific time scale (i.e., daily), though this issue was addressed in a non-MRC framework by *Koutsoyiannis and Onof* [2001], who generated hourly scale dry intervals within existing daily scale wet intervals by applying a Bartlett-Lewis rectangular pulse model independently to periods of consecutive wet days. Through a sequence of repetition and adjustment, *Koutsoyiannis and Onof* [2001] assured that the synthetic rainfall series matched the observed rainfall when aggregated at the daily time interval.

[12] Given the dependencies between the cascade weight W on both time scale and rainfall intensity, we asked the following questions: What gains in performance could be made by explicitly incorporating these dependencies into an MRC model? Could we successfully recreate the probability densities of wet and dry periods without resorting to a hybrid model? Finally, were the gains worth the cost in model complexity? In this paper we address these questions

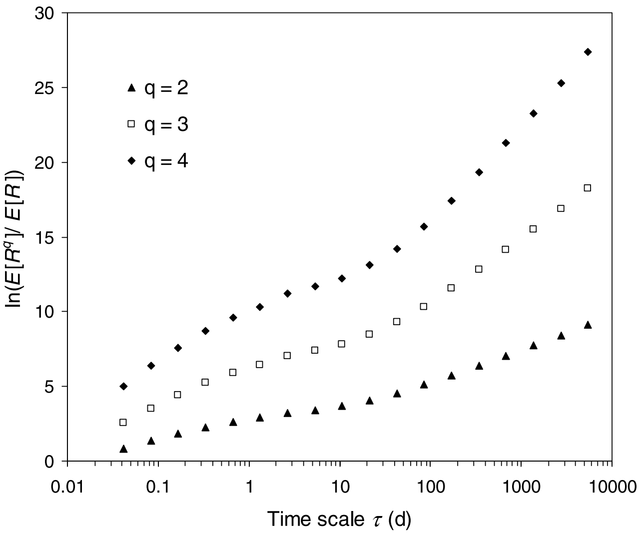

Figure 1. Log-log plots of the empirical q moments versus time scale (i.e., aggregation interval length).

by first examining how the cascade weights derived from measured hourly rainfall vary with time scale and intensity. We then use the observed dependencies of W to build four MRC models, each progressively more complex as they explicitly incorporated first time scale dependence then intensity dependence. The models were evaluated in their ability to reproduce the autocorrelation structure, frequency distribution of wet period duration, frequency distribution of rainfall intensity at the hourly time step and the scaling of the moments of intensity at time scales from 1 to 16 h. Last, we explored the dependency of the weights on interval class seen by *Olsson* [1998] and *Güntner et al.* [2001] and discuss how these dependencies may relate to intensity and the way in which rainfall data are sampled.

2. Analysis of Cascade Weights

2.1. Methods

[13] We analyzed hourly rainfall totals recorded at Christchurch Airport, New Zealand (43°29′S, 172°32′E; 37 m above mean sea level), from April 1960 through February 2006. Rainfall averaged 607 mm a^{-1} and had only a weak seasonal pattern. The data were recorded to a precision of 0.1 mm until 1994, and 0.2 mm thereafter.

[14] Power law scaling of the statistical q moments of observed rainfall R may indicate over which scales an MRC model is suitable [e.g., *Schertzer and Lovejoy*, 1987; *Gupta and Waymire*, 1993], though such an analysis should be limited to the analysis of lower moments ($q \leq 4$) because the higher empirical moments can be poor estimators of the true moments [*Ossiander and Waymire*, 2000, 2002]. Arguable breaks in power law scaling in the Christchurch data occur near 1 day and 1 month, where there is an apparent change in slope in the relationship between the moments, $E[R^q]$, and the temporal resolution of the measurement, or time scale, in log-log space (Figure 1). The presence of multiple ranges of scaling has been detected by others [*Fraedrich and Larnder*, 1993; *Hubert et al.*, 1993; *Olsson et al.*, 1993; *Lovejoy and Schertzer*, 1995; *Veneziano et al.*, 1996]. The apparent power law scaling from 1 day to 1 h suggest that an MRC model may be appropriate for disaggregating rainfall from 1 day to 1 h.

[15] The cascade weights W were calculated from non-overlapping adjacent pairs of rainfall measurements as

$$W_{j,\tau} = \frac{R_{j,\tau}}{R_{j,\tau} + R_{j+1,\tau}} \quad j = 1, 3, 5, \ldots, N_\tau - 1 \tag{2}$$

where $R_{j,\tau}$ is the rainfall depth recorded over the time interval of length τ at position j in the time series, and N_τ is the total number of records at time scale τ. The weights were calculated from the data at the originally recorded time step τ_{rec} = 1 h, and at aggregated intervals of length $2^m\tau_{rec}$ and $2^{m-1}(3\tau_{rec})$, where m is an integer greater than zero. When $R_j + R_{j+1}$ was zero, W was undefined and was excluded from the analysis.

[16] The relative frequencies, or probabilities, that the weights W_j equaled 0, 1, 0 or 1, or not 0 or 1, were calculated. We denote those weights that are equal to 0, 1, 0 or 1, and not 0 or 1, by W_0, W_1, W_{01} and W_x, respectively. The corresponding probabilities of each of these subsets of W are denoted as P_0, P_1, P_{01}, and P_x.

[17] Histograms of the cascade weights W_x (those not equal to zero or one) were generated at each time scale. We used least squares to fit a symmetrical beta probability density function (pdf) to each distribution of W_x:

$$p(W_x) = \frac{W_x^{r-1}(1 - W_x)^{r-1}}{\mathrm{B}(r)} \tag{3}$$

where $r > 0$ is a shape parameter and

$$\mathrm{B}(r) = \int_0^1 u^{r-1}(1 - u)^{r-1} du \tag{4}$$

is the single-parameter beta function evaluated at r. Useful properties of the one-parameter beta pdf are that it is bounded by 0 and 1, and that it can be U shaped ($r < 1$), uniform ($r = 1$), mounded ($r > 1$), or assume a Gaussian-like shape for large r. *Koutsoyiannis* [1988] and *Koutsoyiannis and Xanthopoulos* [1990] showed that W is beta distributed when R is gamma distributed.

[18] At each time scale τ, the weights were classified according to the rainfall intensity I at scale 2τ. The intensity classes were bounded below and above by $2^{n-1}I_{\min}$ and $2^n I_{\min}$, respectively, for n = 1, 2, 3, ... The minimum intensity $I_{\min}$ was chosen to be small enough so that no weights were left unclassified. The representative intensity for each class was assumed to be the exponent of the average of the log-transformed values of the interval bounds. As described above, a beta pdf was fitted to the empirical distribution of W for each time scale and intensity class.

2.2. Results

[19] The probability P_x varied strongly with time scale (Figure 2), with a minimum near 1 day. For time scales from 1 day to 1 h, P_x increased logarithmically with decreasing τ. The histograms of W_x also varied greatly with time scale (Figure 3) from highly centered at the large time scales, to slightly U shaped at τ of a few days, and more centered

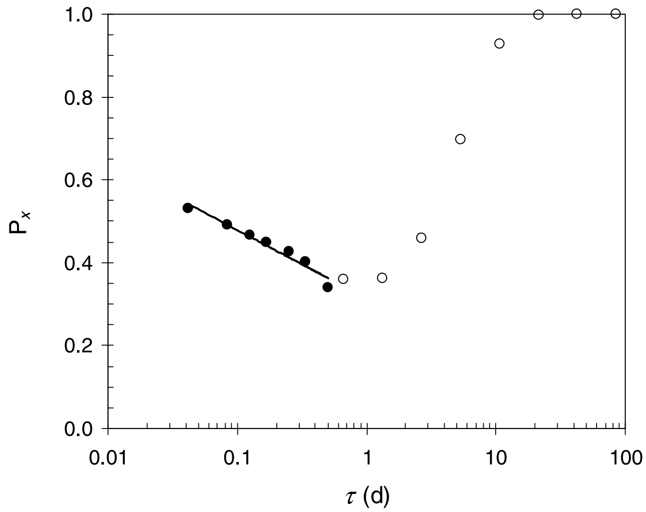

Figure 2. Probability that the cascade weight W is greater than 0 or less than 1 (P_x) against time scale τ. The solid circles indicate the time scales over which the models disaggregate rainfall, and the solid line is a fitted logarithmic function.

again at small time scales. For timescales ranging between 1 day and 1 h, r varied approximately as a power law with τ (Figure 4).

[20] The dependence of P_x on rainfall intensity was pronounced across all time scales (Figure 5). Values of P_x ranged from 0 at the lowest intensity class to 1 at the largest class. P_x showed a distinct S-shaped relationship with the logarithm of intensity, suggesting that a lognormal cumulative distribution function is suitable for describing P_x as a

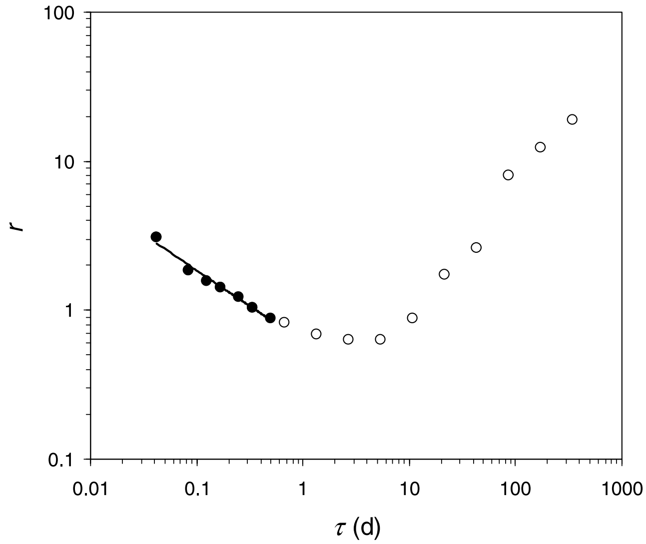

Figure 4. Beta distribution parameter r for the weights W_x against time scale τ. The solid circles indicate the time scales over which the model disaggregates rainfall, and the solid line is a fitted power law.

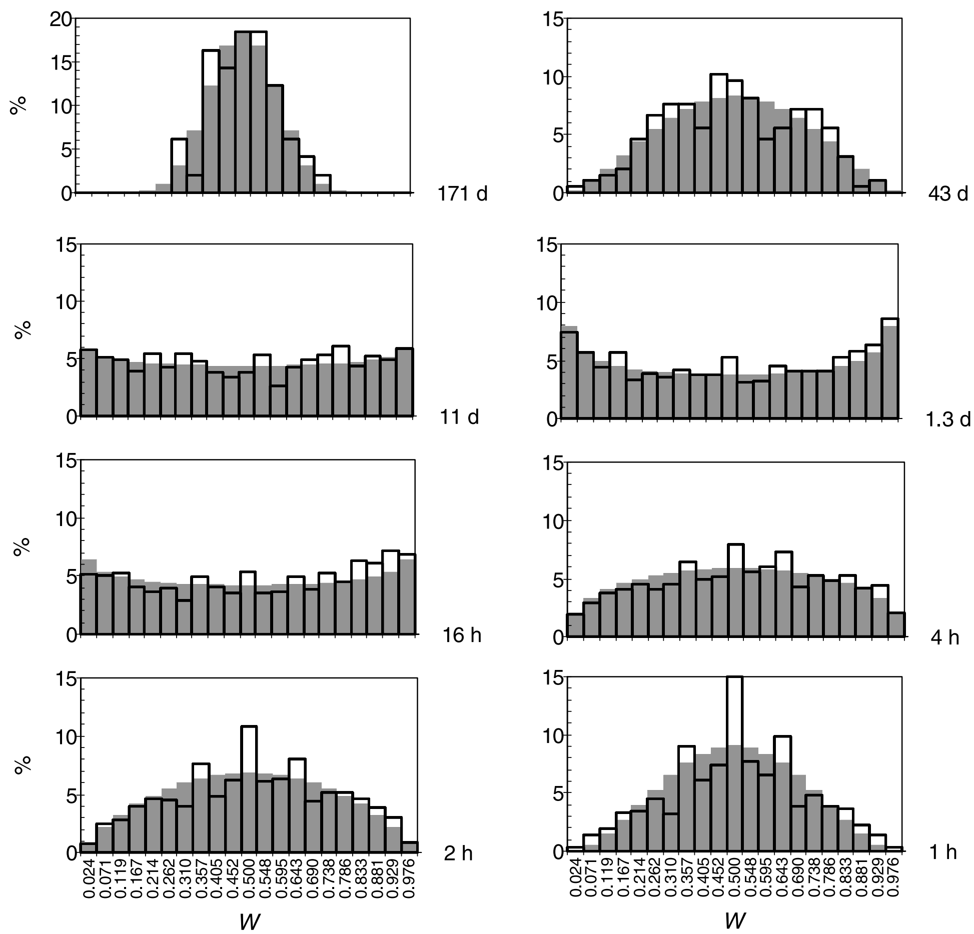

Figure 3. Histograms of the splitting weights W_x at various time scales. The shaded areas show fitted beta distributions.

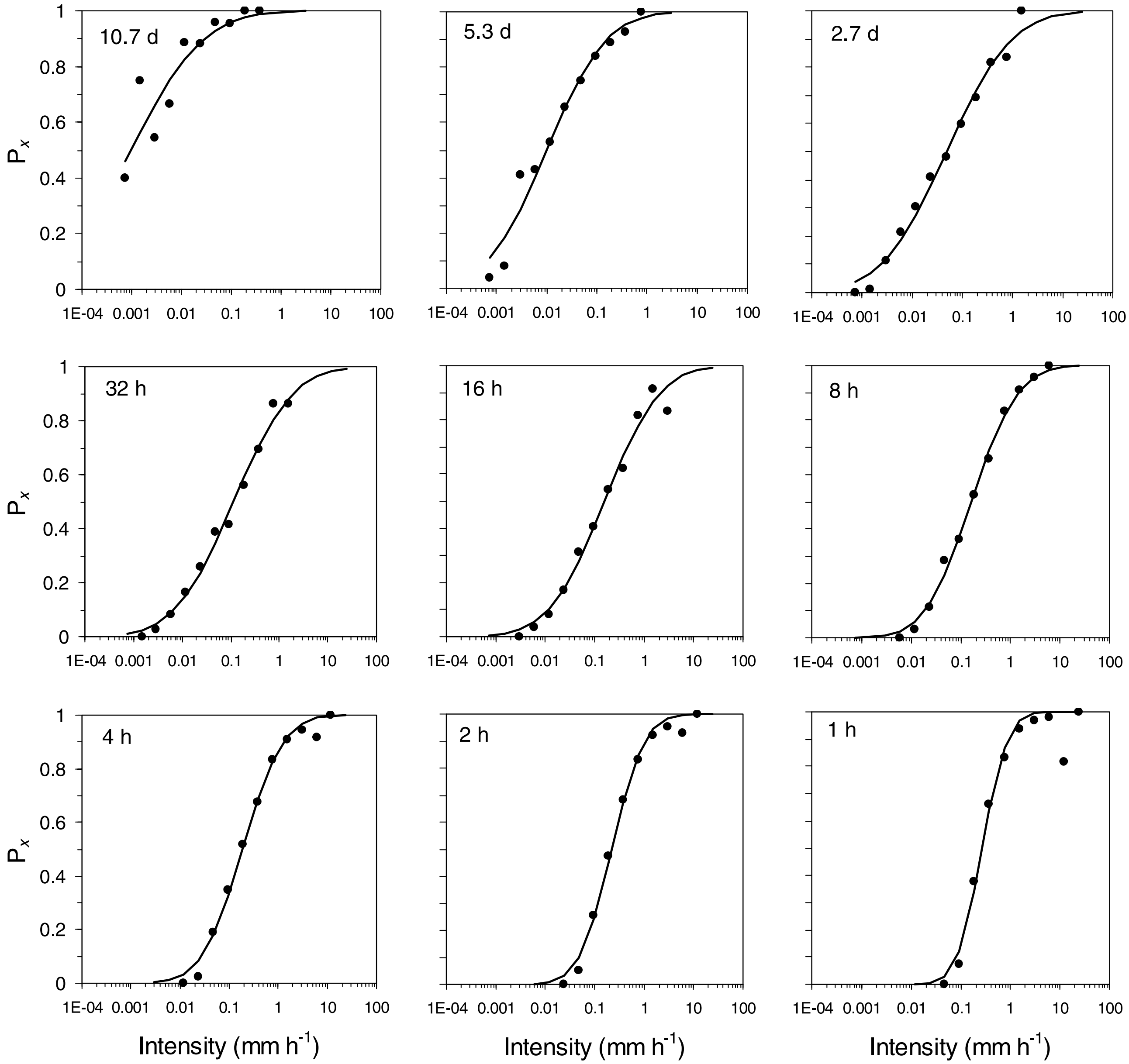

Figure 5. Probability that the cascade weight W is greater than 0 and less than 1 (P_x) against rainfall intensity class for various time scales. The lines are fitted lognormal cumulative distribution functions.

function of intensity. The mean μ and variance σ^2 of the fitted lognormal CDFs varied approximately as a power law with τ for $\tau \leq 0.5$ day (Figure 6). There was also strong dependency of the weights W_x on intensity across time scales (Figure 7). A pattern in this dependency emerged when the parameter r of the beta distribution was related to intensity. Ignoring the smallest two rainfall intensity classes, r smoothly decreased then increased with increasing intensity, with a minimum (highest variance) near $\sim$0.3 mm h^{-1}. This phenomenon was present at all time scales between 1 h and about 1 day. At time scales between 1 day and 1 week, r increased with decreasing τ (Figure 8).

[21] Some of the dependency on intensity was the result of instrument or recording precision. This precision artifact was manifested as relatively high frequencies of W = 1/2, 1/3, 2/3, and to a lesser extent of W = 1/4 and 3/4, at small time scales (Figure 3) and low intensities (Figure 7). This occurs because at low rainfall amounts there are only a small number of discrete possible values of W that can occur. For example, if an observed rainfall amount over two adjacent time intervals (R_j + R_{j+1}) is 0.2 mm and the precision is 0.1 mm, then the value of W for that pair of intervals can only be 0, 0.5, or 1. However, instrument precision does not account for the variability in W with intensity at moderate to high intensities.

[22] Because of the apparent similarity of the curves of r versus I for time scales shorter than 1 day, we collapsed the curves into one single curve through normalization. The curves appeared to be offset from each along both the r axis and τ axis (Figure 8). Assuming that the offset in r is dominant, $r(I, \tau)$ scaled as

$$r^*(I) = r(I,\tau)/r(\tau) \quad (5)$$

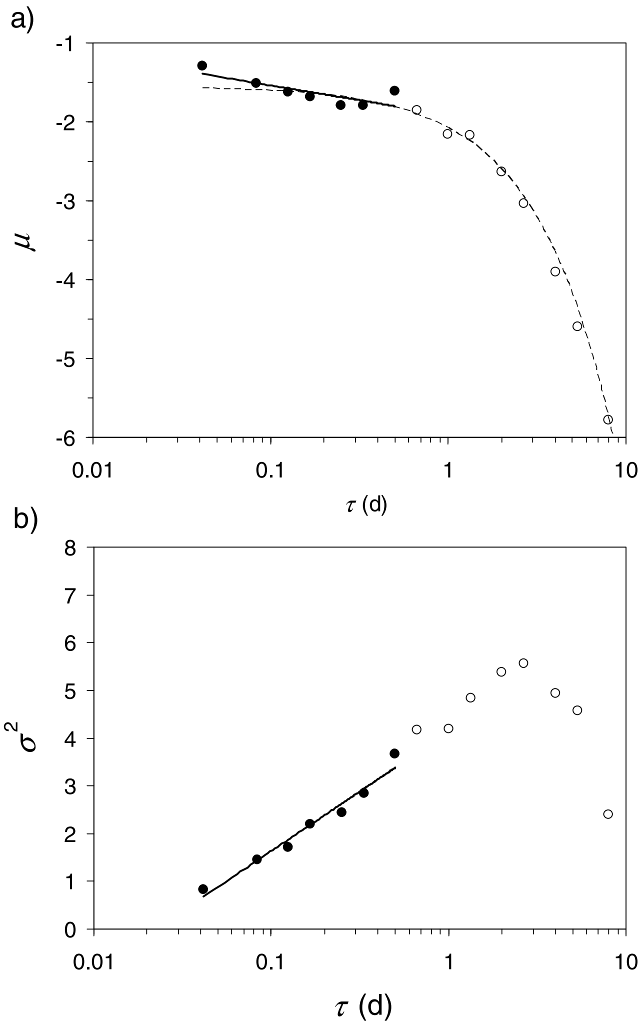

Figure 6. Model parameters (a) μ and (b) σ^2 for P_x against time scale τ. The solid circles indicate the time scales over which the model disaggregates rainfall. Solid lines are fitted logarithmic functions, and the dashed line is a fitted linear function.

where $r^*(I)$ is the scaled beta distribution parameter (Figure 9) and $r(\tau)$ is the beta distribution parameter dependent on time scale alone (Figure 4).

3. Simulating Rainfall With Dependent Cascade Weights

3.1. Model Description

[23] Four models, named I through IV, of progressively increasing complexity were used to disaggregate the 46-year-long daily Christchurch rainfall time series to hourly intervals (Table 1). The more complex models (models III and IV) are conceptually similar to the space-time model of *Veneziano et al.* [2006a], though the temporal models presented here are microcanonical and assume a different distribution function for the cascade weights based on the observed rainfall intensities

[24] This daily record was generated by aggregating the observed hourly rainfall to a resolution of 1 day. The disaggregation of the total rainfall depth of 1 day into 2^k intervals of interval length 2^{-k} days was accomplished with (1). In our case, $k = 0$ corresponded to the time scale of 1 day and the cascade was generated down to level $k = 5$, or 1/32 day. Afterward, the time series was resampled with a time step of 1/24 day so that the synthetic data had the same temporal resolution as the observed data [*Güntner et al.*, 2001]. The following metrics were computed for the synthetic and observed time series: the autocorrelation structure, frequency distribution of wet period duration, and frequency distribution of rainfall intensity at the hourly time step, and the scaling of the moments of intensity at time scales from 1 to 16 h.

[25] In model I, the pdf of W was assumed to be IID. In other words, the probability P_x and the beta distribution parameter r for W_x were constant across time scales and intensities. The probability of W_{01} was given as $P_{01} = 1 - P_x$, and P_0 and P_1 were assumed equal to $P_{01}/2$ (though not shown here, this symmetry in P_0 and P_1 was observed in the data and was also assumed for all the models hereafter). Although it was expected that this IID model would perform poorly given the findings in section 2, it served as a reference for which to assess the gains made by explicitly incorporating cascade weight dependencies.

[26] In model II, P_x varied with time scale as

$$P_x(\tau) = a_0 \ln(\tau) + b_0 \tag{6}$$

where a_0 and b_0 were constants. Because the time scale τ refers to the temporal resolution to which the disaggregation is occurring at any level within the cascade, (6) and subsequent equations, were fitted to data corresponding to time scales ranging from 1 to 12 h, inclusive (see Figure 2). To model time scale dependence of W_x (Figures 3 and 4), the dependence of the beta distribution parameter r on τ was given by the power law

$$r(\tau) = r_0 \tau^H \tag{7}$$

where r_0 and H were constants (Figure 4) [*Menabde and Sivapalan*, 2000; *Paulson and Baxter*, 2007].

[27] In model III, P_x was conditioned on both time scale and the rainfall intensity I at the immediately previous time scale as

$$P_x(I, \tau) = \frac{1}{2}\left\{1 + erf\left[\frac{\ln(I) - \mu}{\sqrt{2\sigma^2}}\right]\right\} \tag{8}$$

where the parameters μ and σ^2 are functions of τ and erf is the error function. Equation (8) is equivalent in form to the cumulative distribution function for the lognormal probability distribution. Its selection was based on good visual fit to the data over a wide range of time scales and intensities (Figure 5). Note that μ and σ^2 are simply fitting parameters and do not represent the mean and variance of $\ln(I)$. Following patterns in the Christchurch data (Figure 6), the parameters μ and σ^2 were varied logarithmically with time scale:

$$\mu = a_\mu \ln(\tau) + b_\mu \tag{9}$$

$$\sigma^2 = a_\sigma \ln(\tau) + b_\sigma \tag{10}$$

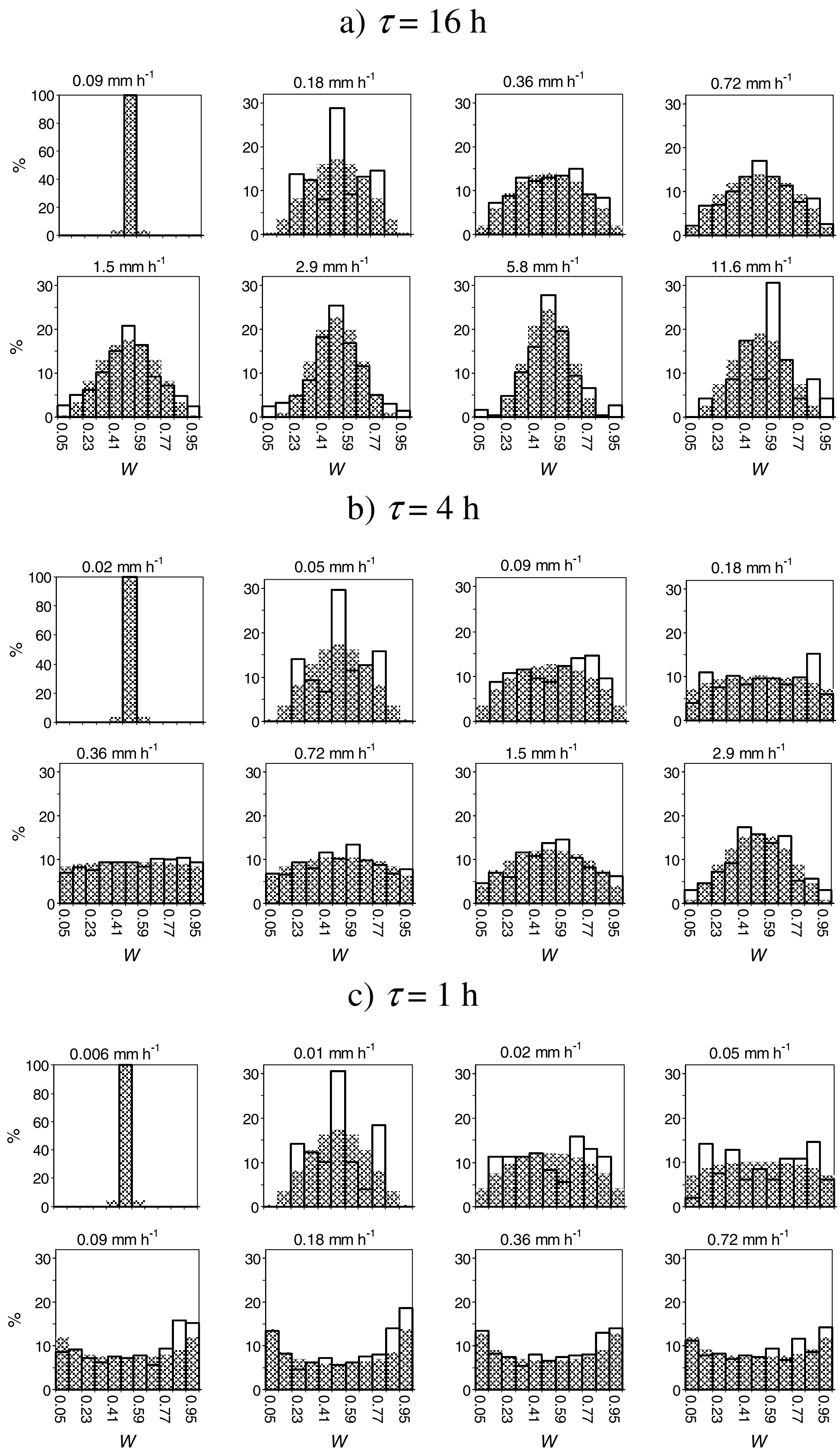

Figure 7. Histograms of W_x by rainfall intensity class for rainfall aggregated at (a) 16, (b) 4, and (c) 1 h intervals. The shaded areas show fitted beta distributions.

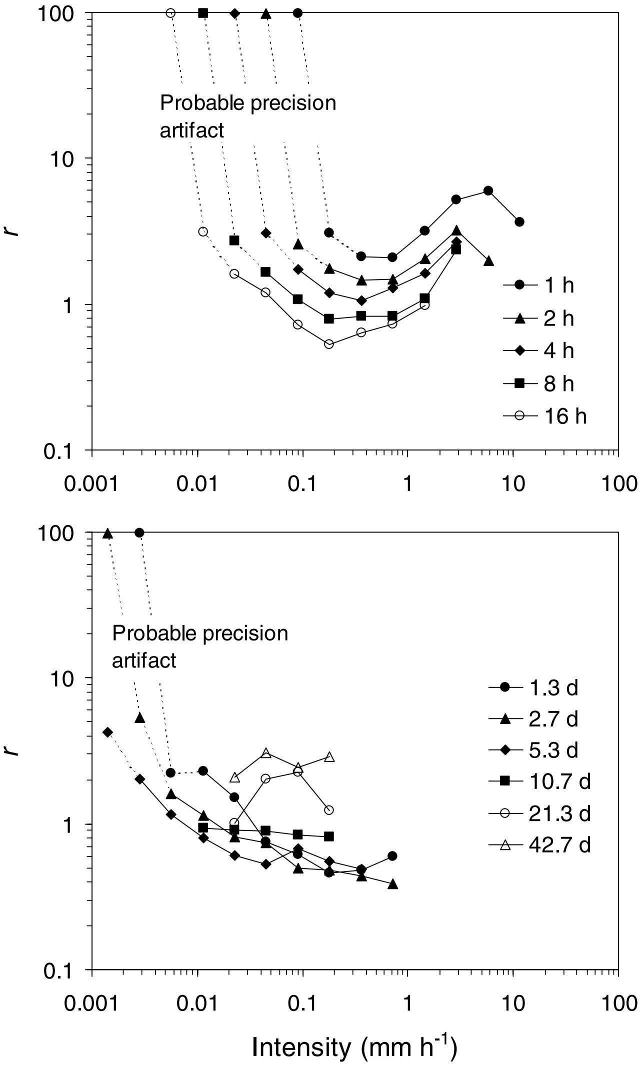

Figure 8. Beta distribution parameter r against rainfall intensity class for various time scales τ.

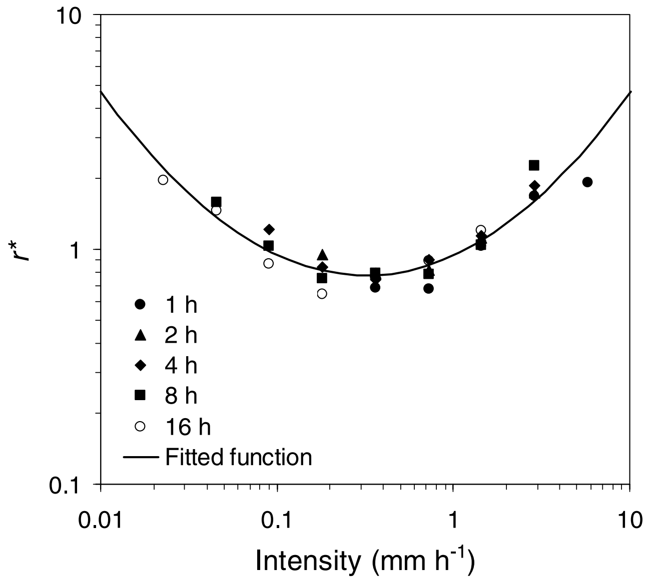

Figure 9. Scaled model parameter r^* for $p(W_x)$ against rainfall intensity class. The solid line is a fitted second-degree polynomial function.

with constants a_μ, b_μ, a_σ, and b_σ. As in model II, (7) modeled the time scale dependence of W_x from the relationship between r and τ, independent of rainfall intensity.

[28] Model IV used the same dependency of P_x on time scale and intensity as did model III ((8) through (10)), but added intensity dependence of W_x. The dependency of r on time scale and intensity was accounted for first by assuming that $r(I, \tau)$ scales by $r(\tau)$ as $r^*(I)$ independent of time scale, as in (5). The log transformation of $r^*(I)$ was modeled as a quadratic function of ln I:

$$\ln[r^*(I)] = c_0 + c_1 \ln(I) + c_2[\ln(I)]^2 \tag{11}$$

where c_0, c_1, and c_2 are constants (Figure 9). Finally, substituting (7) into (5) and rearranging gives $r(I, \tau)$ as the product of two functions, one of intensity and one of time scale:

$$r(I, \tau) = r^*(I) r_0 \tau^H \tag{12}$$

3.2. Parameter Estimation

[29] We fitted all parameters in (6), (7), and (9)–(11) using least square means and all of the observation data (Table 2). The value of $H = -0.478$ for (7) is very close to the value of -0.48 estimated by *Menabde and Sivapalan* [2000] in Melbourne, Australia, for scales ranging from a few hours to several minutes.

[30] When estimating the parameters in (11), we excluded the two smallest intensity classes, therefore did not attempt reproduce explicitly the artifact of instrument precision (Figures 7 and 8). The scaled beta parameter $r^*(I)$ for model IV is generally parabolic (Figure 9), however, the relationship between r^* and intensity is slightly dependent on time scale. For example, the minimum r^* for τ = 1 h occurs at about 0.7 mm h^{-1}, while the minimum r^* for τ = 16 days occurs at about 0.15 mm h^{-1}. When applying (11), we set the maximum allowable value for r^* to 2.5, which was the largest observed value.

4. Results and Discussion

[31] Model I, with IID cascade weights, did poorly at reproducing any of the metrics evaluated: scaling of the q moments (Figure 10), autocorrelation of hourly rainfall (Figure 11), frequency of the wet state duration (Figure 12), and frequency of hourly rainfall intensity (Figure 13). The underrepresentation of the longer periods of continuous rain simulated by this model (Figure 12) was similar to results *Schmitt et al.* [1998, Figure 4] obtained using the IID "β" model proposed by *Over and Gupta* [1994].

Table 1. Dependencies of the Cascade Weights W for the Various Disaggregation Models

Model	Time Scale–Dependent	Intensity-Dependent	
		W = 0 or 1	$0 < W < 1$
1	no	no	no
2	yes	no	no
3	yes	yes	no
4	yes	yes	yes

Table 2. Parameters for Models I, II, III, and IV[a]

Parameter	Model I	Model II	Model III	Model IV
		W = 0 or 1		
a_0	0^b	−0.072	NA	NA
b_0	0.444^c	0.313	NA	NA
a_μ	NA	NA	−0.166	−0.166
b_μ	NA	NA	−1.929	−1.929
a_σ	NA	NA	1.096	1.096
b_σ	NA	NA	4.145	4.145
		0 < W < 1		
r_0	1.583^d	0.615	0.615	0.615
H	0^b	−0.478	−0.478	−0.478
c_0	0^b	0^b	0^b	−0.061
c_1	0^b	0^b	0^b	0.344
c_2	0^b	0^b	0^b	0.151

[a]NA means not applicable.
[b]Assumed value.
[c]Equivalent to P_x.
[d]Equivalent to r.

[32] There was only minor improvement in model performance by introducing time scale dependence of the model parameters. While the q moments from model II did decrease more with decreasing time scale than did the Model I q moments, they still deviated considerably from the observed moments for time scales much shorter than 1 day (Figure 10). The autocorrelation and the distributions of wet state duration and hourly intensity were about the same for models I and II (Figures 11–13).

[33] Much more improvement was made when W_{01} was conditioned on intensity than when conditioned on time scale alone. For model III, the modeled q moments scaled correctly (Figure 10). The modeled autocorrelation structure was also consistent with the observed structure (Figure 11). Model III also did much better at reproducing the duration of wet state though it still did not generate the longest periods of continuous rain present in the observed data (Figure 12).

[34] Conditioning W_x on intensity actually degraded model performance as compared to conditioning on time scale alone (Figures 10–13). Why this was so is uncertain, though it is clear that the simple method of rescaling the beta distribution parameter used in model IV was ineffective. Normalizing intensity by time scale (i.e., plotting r versus depth), offset the curves too far along the horizontal axis (data not shown). An alternative would be to normalize intensity by τ to some power, however this, or some other, method should be balanced by the need to define and estimate more parameters.

[35] On the basis of the chosen performance metrics, model III, which incorporated intensity dependency of the cascade weights W_{01}, was the superior model. For the remainder of the model evaluation, we considered it solely.

[36] As did *Olsson* [1998] and *Güntner et al.* [2001], we found that the empirical distributions of the cascade weights varied by interval class (i.e., starting, enclosed, ending, and isolated). It is important to note, however, that the distributions of intensities also varied by interval class. For example, the means (and variances) of intensity in the classes were 0.61 mm h^{-1} (1.00), 1.21 mm h^{-1} (1.66), 0.48 mm h^{-1} (1.66), and 0.35 mm h^{-1} (0.41), respectively. This suggests that, by conditioning the weights on intensity alone, we may account for much of the dependency on interval class. For example, it has been shown by *Olsson* [1998] that intervals of higher intensity and enclosed intervals are both less likely to form a 0–1 partition. Given that enclosed intervals are, on average, the most intense, model III therefore also effectively generates enclosed intervals with the smallest P_{01} (Figure 14), but the exact pattern of the observed data is not reproduced, illustrating that intensity is an incomplete substitute for interval class.

[37] A conspicuous feature of the cascade weights (W) is that they are asymmetric for the starting and ending interval classes [*Olsson*, 1998; *Güntner et al.*, 2001; *Veneziano and Iacobellis*, 2002]. In other words, $P_0 > P_1$ for the starting class, $P_1 > P_0$ for the ending class, and the histograms of W_x are skewed. Furthermore, the pdfs of the starting and ending classes are nearly mirror images of one another. This means, for example, that P_0 for the starting class can be considered equal to P_1 for the ending class. For the remaining analysis, we took advantage of this mirror symmetry to combine the starting and ending classes into a single starting/ending

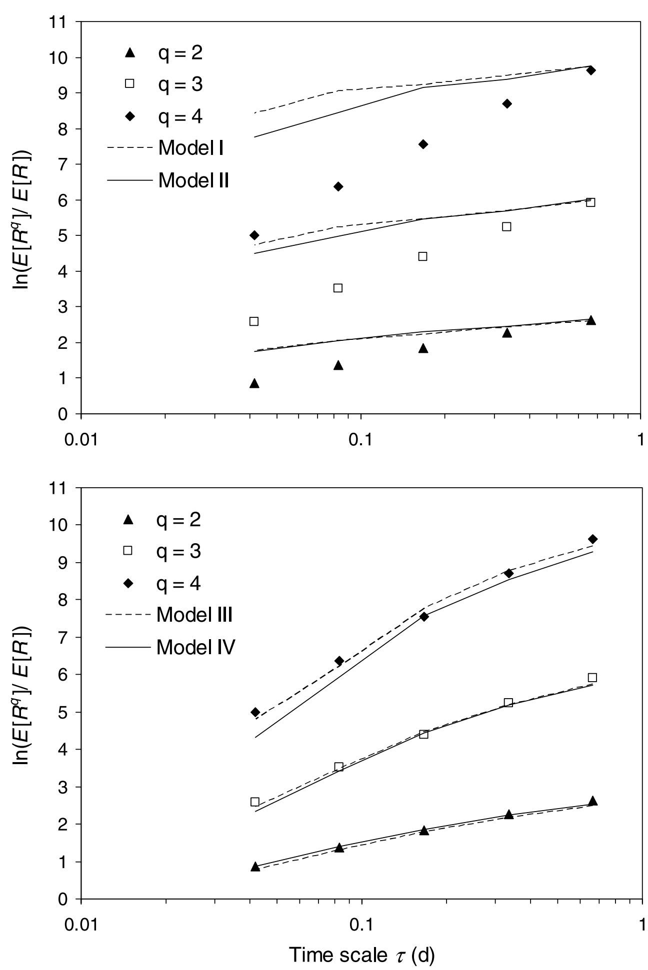

Figure 10. Log-log plots of the q moments versus time scale. The empirical moments are given by the symbols, and the modeled moments are given by the dashed (models I and III) and solid lines (models II and IV). Model results are of a single realization from the 46-year daily record.

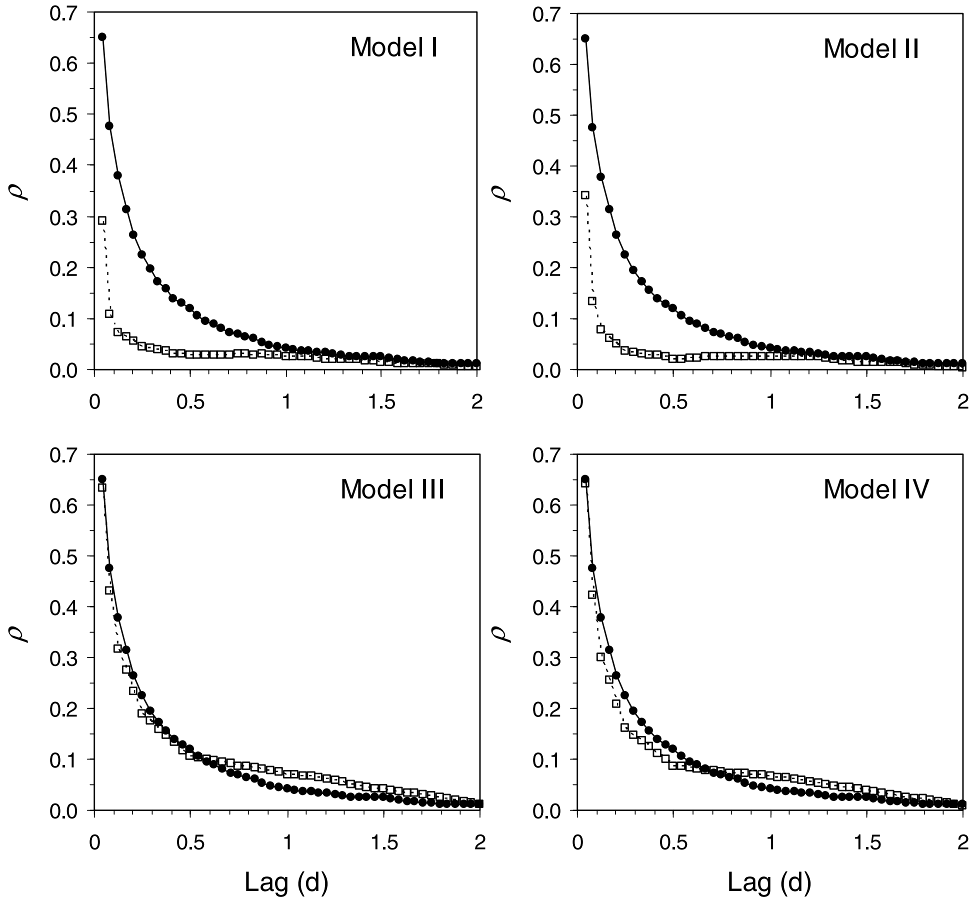

Figure 11. Autocorrelation coefficient ρ versus time lag for observed (solid circles) and simulated (open squares) hourly rainfall. Model results are of a 552-year simulation, based on repeating the 46-year observed daily record 12 times in series.

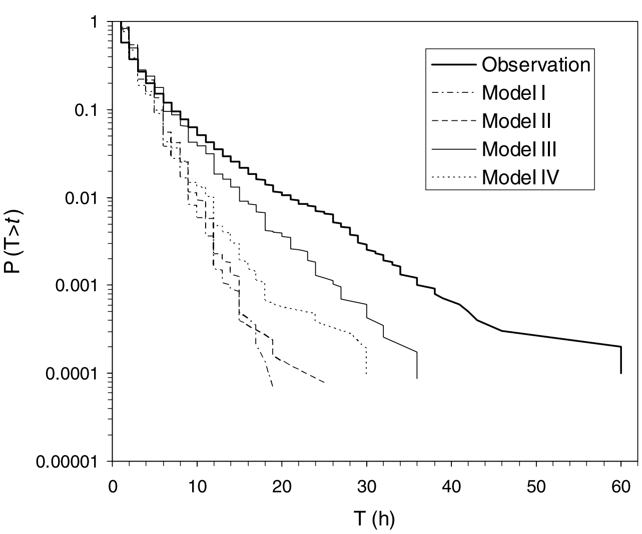

Figure 12. Complement of the cumulative probability distribution of duration T of the wet state for the observed and modeled hourly time series. Model results are of a single 46-year realization.

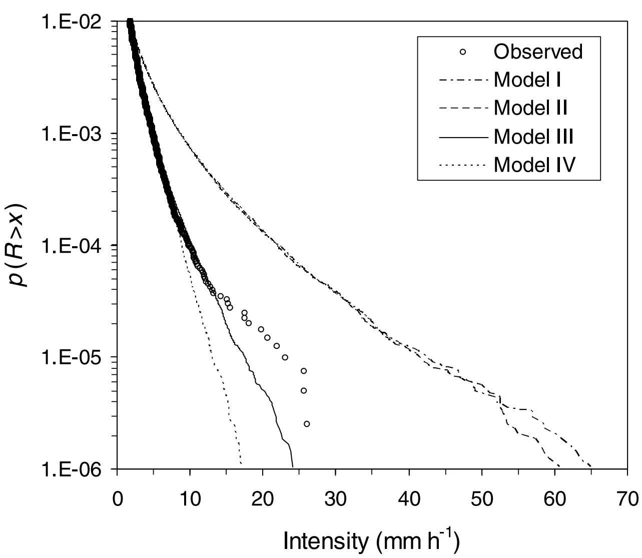

Figure 13. Complement of the cumulative probability distribution of observed and modeled hourly rainfall intensity. Model results are of a 552-year simulation, based on repeating the 46-year observed daily record 12 times in series.

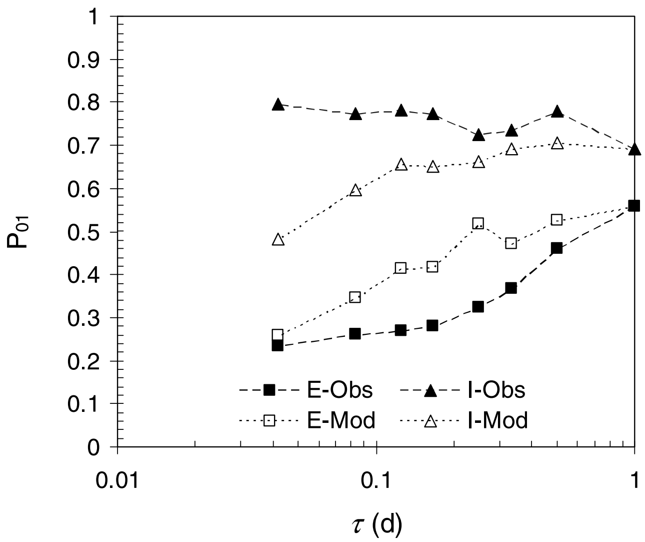

Figure 14. Relative frequency of $W = 0$ or 1 for observed (Obs) and model III (Mod) rainfall depth against time scale for the enclosed (E) and isolated (I) interval classes.

class, where W for the ending class was first transformed as $1 - W$.

[38] The Christchurch data also shows asymmetry in the probability density of W_x for the starting/ending class (Figure 15); W_x is on average less than 0.5 over the time scales 1 h to 1 day. *Güntner et al.* [2001] made a similar observation while grouping all time scales together (see their Figures 3 and 4). Here we show that the degree of asymmetry appears to decrease with decreasing time scale (Figure 15).

[39] Model III does not generate this asymmetry (Figure 15) and nor do any of the other MRC models tested. In fact, without some modification, a discrete MRC model cannot create such asymmetry at the same scales and interval discretization at which the model is applied. The asymmetry in the data could be explicitly reproduced by applying a distinct asymmetric distribution of W for starting and ending intervals, as was done by *Olsson* [1998] and *Güntner et al.* [2001], or by other similar binary disaggregation methods in which the rainfall is conditional on rainfall preceding and following it in time [*Koutsoyiannis*, 2002].

[40] We suspect however, that the asymmetry in the starting and ending distributions is largely an artifact of

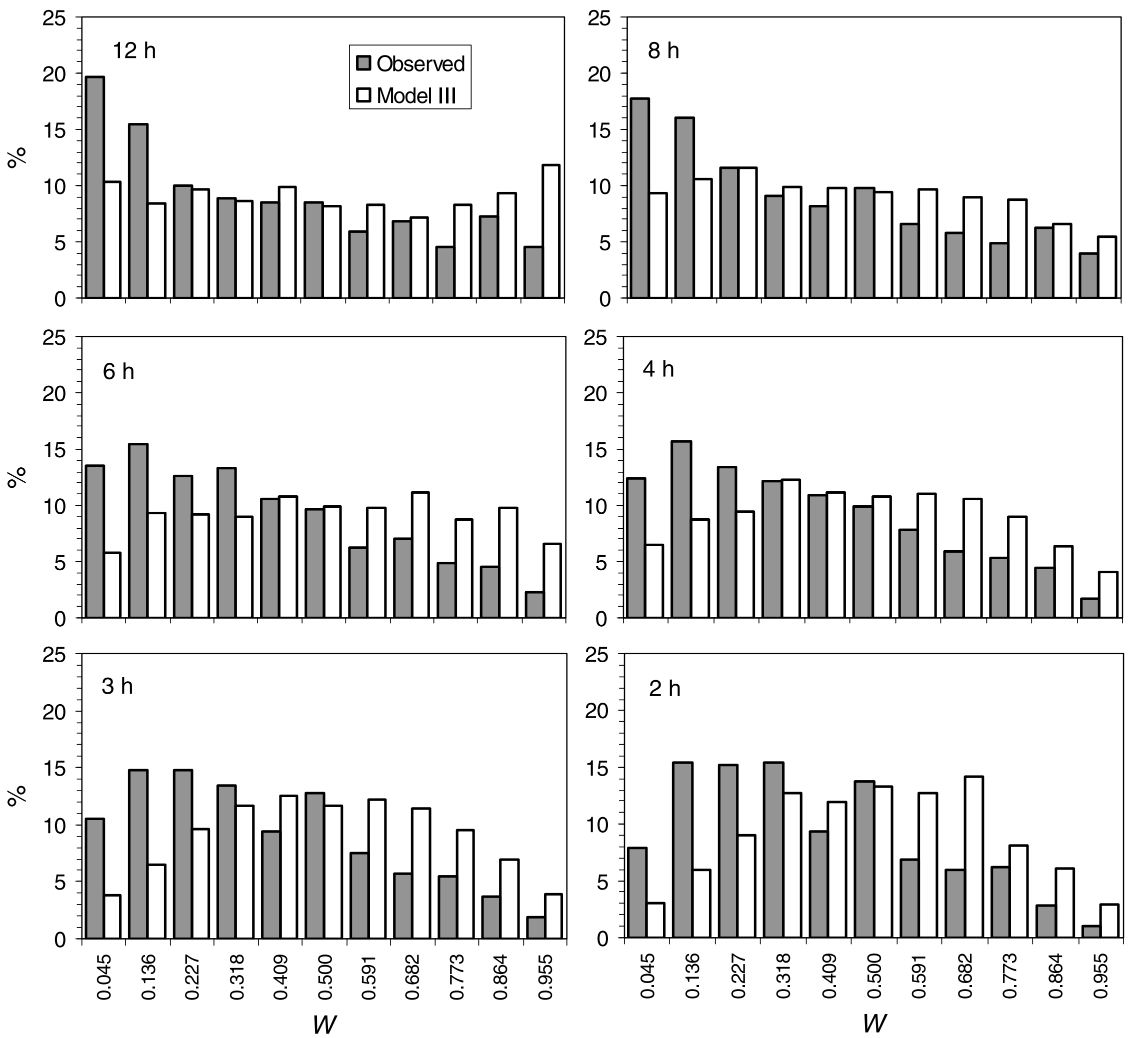

Figure 15. Histograms of the cascade weights W_x for the starting/ending interval class at various time scales for the observed rainfall and the synthetic rainfall from model III.

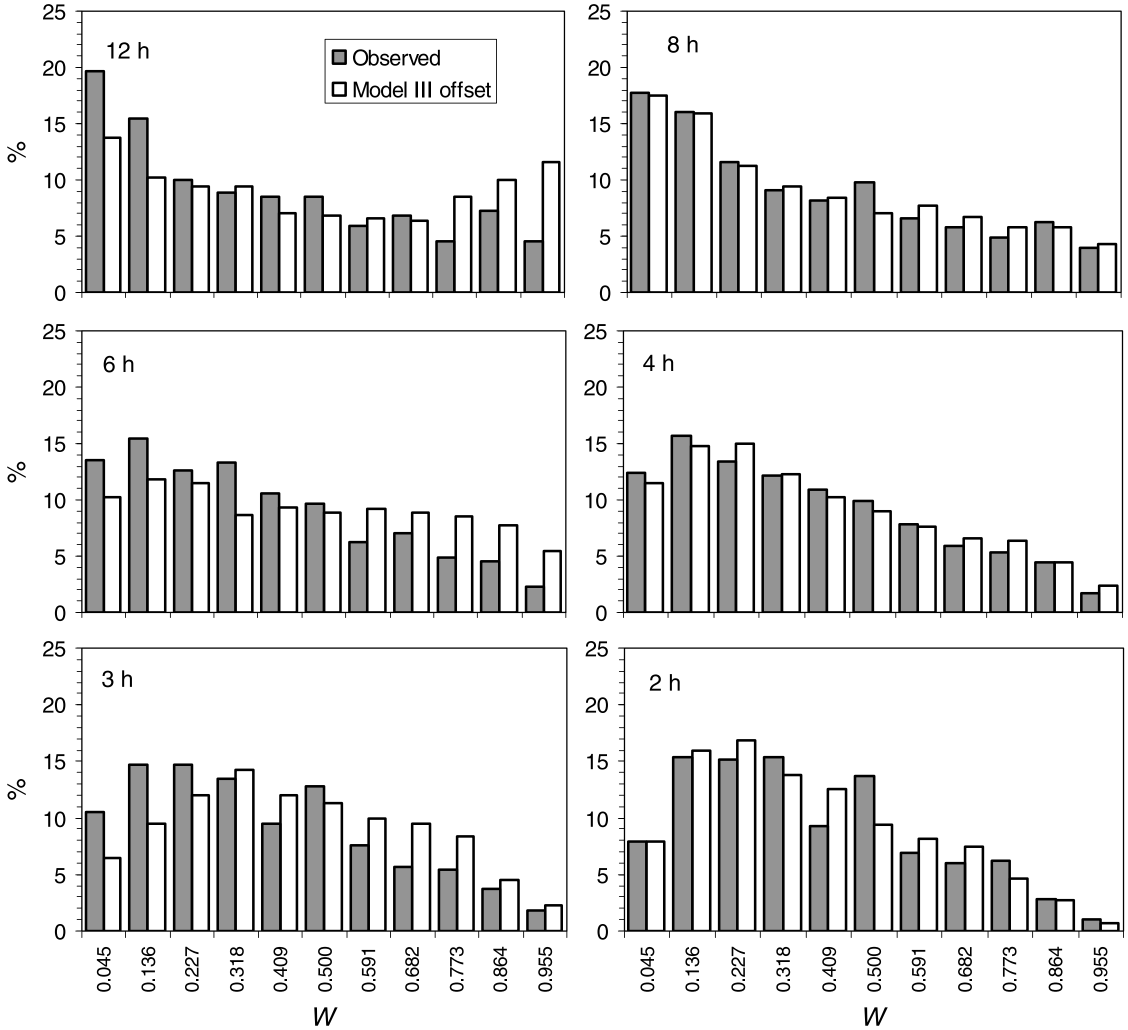

Figure 16. Histograms of the cascade weights W_x for the starting/ending interval class at various time scales for the observed rainfall and the synthetic rainfall from model III. Prior to determining W_x, the synthetic rainfall time series was resampled following an offset in time of $1/\pi$ h.

sampling a semicontinuous and irregularly timed process (rainfall) at discrete, regularly spaced, intervals. It is worth noting that the model of *Veneziano and Iacobellis* [2002], for example, generates asymmetry in the starting and ending classes (see their Figures 6 and 10) without any consideration of distinct interval classes. *Veneziano and Iacobellis* [2002] generated rainfall with a pulse-based model that did not have inherent discrete and regular time steps. In other words, their synthetic rain events could be of any length so did not have to begin or end at some time interval of discrete and regular length (e.g., precisely on the hour). However, when they sampled their results at regular, discrete intervals, the simulated data showed asymmetry in the starting and ending classes (see their Figures 6 and 10).

[41] To test for this sampling artifact, we offset, by $1/\pi$ h (an arbitrary amount), the time axis of the hourly rainfall time series simulated by model III. This created a time series of events that no longer began and ended at the beginning and end of the time intervals imposed by the MRC model. We then resampled the rainfall at 1-h intervals. A consequence of this offset was to smooth the rainfall pattern at the hourly time scale. For example, the rainfall amount in an isolated rainfall interval in the original time series would be spread across two adjacent time intervals in the new "offset" time series.

[42] Overall, it appeared that there are artifacts of discrete interval sampling in the structure of the data. The effect of offsetting the time axis was to introduce asymmetry to the probability density of W_x at all time scales for the starting/ending class (Figure 16). The distributions of W_x for the observed and modeled rainfall are similar the time scales of 1/12, 1/6, and 1/3 day though less alike for the time scales of 1/8, 1/4, and 1/2 day. These latter time scales correspond to the time scales of the cascade levels used in the MRC model.

[43] Offsetting the time axis also affected P_0 and P_1 for the starting/ending class (Figure 17). Prior to the offset, P_0 and P_1 were nearly identical for those time scales corresponding to the levels of the generated cascade (τ = 0.5, 0.25, and 0.125 day), yet P_0 was much greater than P_1 for t = 1, 2, 4, and 8 h. Following the offset, P_0 was much greater than P_1 at all time scales.

[44] While we do not present definitive evidence that the variability in the cascade weights among interval classes is

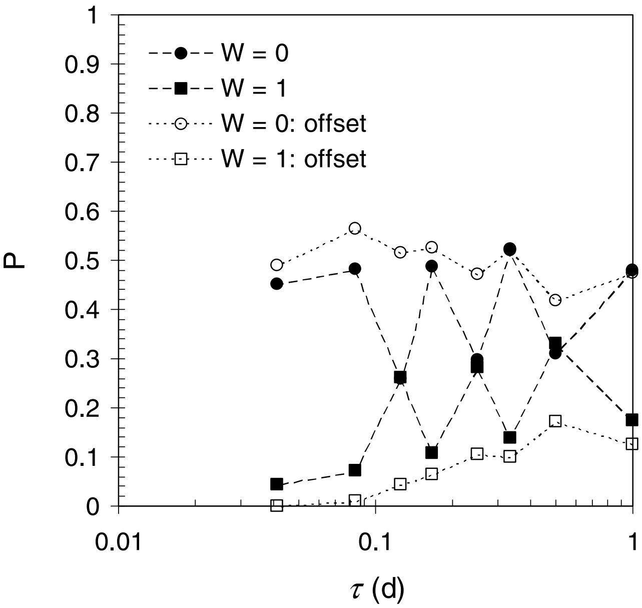

Figure 17. Relative frequencies P_0 and P_1 after one realization of model III for the starting/ending interval class. Open symbols show P_0 and P_1 after resampling the rainfall time series with offset of $1/\pi$ h.

completely an artifact of the sampling method, our preliminary analysis raises interesting issues that warrant further investigation. For one, if the dependence is largely an artifact, is the approach of *Olsson* [1998] and *Güntner et al.* [2001] to reproduce it explicitly warranted, particularly because it substantially increases the number of model parameters required? Also, if we sample our rainfall events such that the sampling intervals begin and end exactly when the rain actually begins and ends, will weights in the middle of an event still differ from those near its onset or termination?

[45] Last, the connection between cascade weights and intensity may be reflective of physically distinct processes in the atmosphere. If so, the mixing of these processes occurs when analyzing a time series as a whole leads to an undesirable averaging of model parameters [*Lovejoy and Schertzer*, 1991, p. 138; *Harris et al.*, 1997]. For example, a rainfall time series may consist of rainfall driven by convective cells, cyclones, and frontal passages, each of which may be associated with a different parameterization for the cascade process. Any one of these may also exhibit different behavior throughout its lifetime, as is the case for prefrontal and postfrontal rain [*Harris et al.*, 1997]. For spatial rainfall, correlation has been found between cascade parameters and properties of a storm system such as the mesoscale rainfall rate [*Over and Gupta*, 1994] and the convective available potential energy [*Perica and Foufoula-Georgiou*, 1996]. For temporal rainfall modeling, if the types or phases of a storm are characterized by their intensity, then the methods described here may partially account for these distinct processes without explicitly defining them.

5. Conclusions

[46] Accounting for time scale and rainfall intensity dependency in the cascade weights improved the ability of the multiplicative random cascade model to reproduce many characteristics of the rainfall time series, including the scaling of the moments, the frequency distributions of wet period duration and hourly rainfall depth, and the autocorrelation structure.

[47] Conditioning the model parameters on time scale alone resulted in minor improvements. The effect of time scale–dependent parameters would likely be greater if the rainfall was disaggregated over a large range of scales.

[48] By far the greatest gains occurred by conditioning on intensity the probability that a cascade weight equaled 0 or 1, or, in other words, where a dry interval first appears in a branch of the cascade. However, rainfall from this model was still too intermittent, meaning long periods of continuous rainfall were underrepresented. It may be that making the MRC more complex in order to accurately model dry intervals is ultimately a dead end and alternatives should be considered. Some alternatives have already been presented for creating complete times series [e.g., *Schmitt et al.*, 1998; *Menabde and Sivapalan*, 2000], but not for disaggregating existing time series wherein the dry-wet pattern is already given at certain time scale.

[49] Explicitly modeling the intensity dependence of cascade weights between 0 and 1 (exclusive) did not improve performance. This may be because our simple approach to parameterize the probability density of W_x as a function of both time scale and intensity was inappropriate. The apparent relationship between the cascade weights and intensity is complex and merits further investigation.

[50] Making the multiplicative random cascade a continuous function of time scale and intensity is a substantial departure from the simple model with independent and identically distributed weights. However, the total number of parameters for our most successful model was only six, which is equal to, or fewer than, other rainfall models that do not consider the dual dependency on time scale and intensity [*Olsson*, 1998; *Menabde and Sivapalan*, 2000; *Güntner et al.*, 2001; *Veneziano and Iacobellis*, 2002]. Thus, the model that incorporated intensity dependence of model parameters into the generation of dry intervals proved to be a useful, applied MRC model that is relatively easy to parameterize.

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M. Brugnach, Institute for Environmental Systems Research, University of Osnabrück, Barbarastrasse 12, D-49076 Osnabrück, Germany. (mbrugnac@usf.uni-osnabrueck.de)

R. F. Keim, School of Renewable Natural Resources, Louisiana State University Agricultural Center, School of Renewable Natural Resources Building, Baton Rouge, LA 70803, USA. (rkeim@lsu.edu)

M. Ossiander, Department of Mathematics and Statistics, Oregon State University, Kidder Hall, Corvallis, OR 97331, USA. (ossiand@math.orst.edu)

D. E. Rupp, DHI Water and Environment, Inc., 319 Southwest Washington Street, Suite 614, Portland, OR 97204, USA. (der@dhi.us)

J. S. Selker, Department of Biological and Ecological Engineering, Oregon State University, 116 Gilmore Hall, Corvallis, OR 97331, USA. (selkerj@engr.orst.edu)