EXTREMUM PROBLEMS FOR EIGENVALUES OF DISCRETE LAPLACE OPERATORS

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Abstract. The P1 discretization of the Laplace operator on a triangulated polyhedral surface is related to geometric properties of the surface. This paper studies extremum problems for eigenvalues of the P1 discretization of the Laplace operator. Among all triangles, an equilateral triangle has the maximal first positive eigenvalue. Among all cyclic quadrilateral, a square has the maximal first positive eigenvalue. Among all cyclic n-gons, a regular one has the minimal value of the sum of all positive eigenvalues and the minimal value of the product of all positive eigenvalues.

1. Introduction

1.1. **Dirichlet energy.** A polyhedral surface S is a surface obtained by gluing Euclidean triangles. It is associated with a triangulation T. We assume that T is simplicial. Suppose (Σ, T) is a polyhedral surface so that V, E, F are sets of all vertices, edges and triangles of T. We identify vertices of T with indices, edges of T with pairs of indices and triangles of T with triples of indices. This means $V = \{1, 2, ...|V|\}, E = \{ij \mid i, j \in V\}$ and $F = \{\triangle ijk \mid i, j, k \in V\}$. A vector $(f_1, f_2, ..., f_{|V|})^t$ indexed by the set of vertices V defines a piecewise-linear function over (S, T) by linear interpolation.

The Dirichlet energy of a function f on S is

$$E_S(f) = \frac{1}{2}\int_S |\nabla f|^2 dA.$$

When f is obtained by linear interpolation of $(f_1, f_2, ..., f_{|V|})^t$, the Dirichlet energy of f turns out to be

$$E_{(S,T)}(f) = \frac{1}{4}\sum_{ijk\in F}[\cot\alpha^i_{jk}(f_j - f_k)^2 + \cot\alpha^j_{ki}(f_k - f_i)^2 + \cot\alpha^k_{ij}(f_i - f_j)^2]$$

where the sum runs over all triangles of T and for a triangle $ijk \in F$, $\alpha^i_{jk}, \alpha^j_{ki}, \alpha^k_{ij}$ are angles opposite to the edges jk, ki, ij respectively.

Collecting the terms in the sum above according to edges, we obtain

$$E_{(S,T)}(f) = \frac{1}{2}\sum_{ij\in E} w_{ij}(f_i - f_j)^2 \tag{1}$$

where the sum runs over all edges of T and

$$w_{ij} = \begin{cases} \frac{1}{2}(\cot\alpha^k_{ij} + \cot\alpha^l_{ij}) & \text{if } ij \text{ is shared by two triangles } ijk \text{ and } ijl \\ \frac{1}{2}(\cot\alpha^k_{ij}) & \text{if } ij \text{ is an edge of only one triangle } ijk \end{cases}$$

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The Dirichlet energy of a piecewise linear function on a polyhedral surface was introduced and the formula (1) was derived by R. J. Duffin [6], G. Dziuk [7] and U. Pinkall & K. Polthier [15] in different context. For applications of the Dirichlet energy and formula (1) in characterization of Delaunay triangulations, see [16, 9, 3, 5]. For interesting applications of the Dirichlet energy and formula (1) in computer graphics, see the survey [4].

1.2. **Discrete Laplace operator.** By rewriting the Dirichlet energy using notation of matrices, we get

$$E_{(S,T)}(f) = \frac{1}{2}(f_1, ..., f_{|V|})L(f_1, ..., f_{|V|})^t$$

where each entry of the matrix L is

$$L_{ij} = \begin{cases} \sum_{ik\in E} w_{ik} & \text{if } i = j \\ -w_{ij} & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

The matrix L is the P1 discretization of the Laplace operator. By definition, the Dirichlet energy $E_S(f)$ is non-negative for any f. Consequently, $E_{(S,T)}(f)$ is non-negative for any f. Therefore L is positive semi-definite. The eigenvalues of L are denoted by

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{|V|-1}.$$

For the derivation of this discretization of the Laplace operator, see also [14] and [1]. The P1 discretization of the Laplace operator and its eigenvalues are related to the geometric properties of the polyhedral surface (S, T). For example, it is proved in [5] that among all triangulations, the Delaunay triangulation has the minimal eigenvalues. In [21], it is shown that a polyhedral surface is determined up to scaling by its P1 discretization of the Laplace operator.

There are many other possible discretization of the Laplace operator, for example, see [20, 2]. Discretization of the Laplace operator has important applications in spectral mesh processing, for example, see [17, 11, 19, 13].

1.3. **Pólya's theorem.** The spectral geometry is to relate geometric properties of a Riemannian manifold to the spectra of the Laplace operator on the manifold. One of the interesting results is the following one due to G. Pólya. For reference, for example, see [10], page 50.

Theorem (Pólya)**.** *The equilateral triangle has the least first eigenvalue among all triangles of given area. The square has the least first eigenvalue among all quadrilaterals of given area.*

It is conjectured that, for $n \geq 5$, the regular n-gon has the least first eigenvalue among all n-gons of given area.

1.4. **Statements of results.** In this paper, similar results as Pólya's theorem are obtained for the P1 discretization of the Laplace operator L.

Theorem 1. *Among all triangles, an equilateral triangle has the maximal λ_1, the minimal λ_2 and the minimal $\lambda_1 + \lambda_2$.*

Proof. See section 2. □

In [18], there is a discussion of λ_2 for triangles and its significance in finite element modeling. In fact λ_2 is a scale-invariant indicator of the quality of a triangle's shape.

A cyclic polygon is a polygon whose vertices are on a common circle. By adding diagonals, a cyclic polygon is decomposed into a union of triangles. For each inner edge of any triangulation of a cyclic polygon, the weight w_{ij} is zero. Therefore the discrete Laplace operator is independent of the choice of a triangulation of a cyclic polygon. For example, there are two ways to decompose a cyclic quadrilateral into a union of two triangles. The two ways produce the same P1 discretization of the Laplace operator.

Theorem 2. *Among all cyclic quadrilaterals, a square has the maximal λ_1, the minimal $\lambda_1+\lambda_2+\lambda_3$, the minimal $\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1$ and the minimal $\lambda_1\lambda_2\lambda_3$.*

Proof. See section 3. □

Theorem 3. *For $n\geq 5$, among all cyclic n-gons, a regular n-gon has the minimal $\sum_{i=1}^{n-1}\lambda_i$ and the minimal $\prod_{i=1}^{n-1}\lambda_i$.*

Proof. See section 4. □

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2. Triangles

2.1. **Characteristic polynomial.** Let $\theta_1,\theta_2,\theta_3$ be the three angles of a triangle. Let $c_i:=\cot\theta_i$ for $i=1,2,3$. The condition $\theta_1+\theta_2+\theta_3=\pi$ implies that

$$c_1c_2+c_2c_3+c_3c_1=1. \tag{2}$$

The P1 discretization of the Laplace operator on the triangle is

$$L_3=\begin{pmatrix} c_1+c_3 & -c_1 & -c_3\\ -c_1 & c_1+c_2 & -c_2\\ -c_3 & -c_2 & c_2+c_3\end{pmatrix}.$$

The characteristic polynomial of L_3 is

$$\begin{aligned} P_3(x)=\det(L_3-xI_3) &= -x^3+2(c_1+c_2+c_3)x^2-3(c_1c_2+c_2c_3+c_3c_1)x\\ &= -x^3+2(c_1+c_2+c_3)x^2-3x\\ &= -x(x^2-2(c_1+c_2+c_3)x+3)\\ &= -x(x^2-(\lambda_1+\lambda_2)x+\lambda_1\lambda_2)\end{aligned}$$

by the equation (2).

The eigenvalues of L_3 are $0=\lambda_0\leq\lambda_1\leq\lambda_2$ satisfying $\lambda_1+\lambda_2=2(c_1+c_2+c_3)$.

2.2. **Sum of eigenvalues.** To verify that an equilateral triangle has the minimal $\lambda_1+\lambda_2$, we claim that $c_1+c_2+c_3\geq\sqrt{3}$ and the equality holds if and only if $\theta_1=\theta_2=\theta_3=\frac{\pi}{3}$.

Consider $f:=c_1+c_2+c_3=\cot\theta_1+\cot\theta_2+\cot\theta_3$ as a function defined on the domain

$$\Omega_3:=\{(\theta_1,\theta_2,\theta_3)\mid\theta_1+\theta_2+\theta_3=\pi,\theta_i>0,i=1,2,3\}.$$

To find the global minimum of f, we apply the method of Lagrange multiplier. Let

$$F := c_1 + c_2 + c_3 + y(\theta_1 + \theta_2 + \theta_3 - \pi).$$

Since

$$\frac{dc_i}{d\theta_i} = -\frac{1}{\sin^2\theta_i} = -(1+c_i^2),$$

we have

$$0 = \frac{\partial F}{\partial \theta_1} = -(1+c_1^2) + y,$$
$$0 = \frac{\partial F}{\partial \theta_2} = -(1+c_2^2) + y,$$
$$0 = \frac{\partial F}{\partial \theta_3} = -(1+c_3^2) + y,$$
$$0 = \frac{\partial F}{\partial y} = \theta_1 + \theta_2 + \theta_3 - \pi.$$

Therefore the function f has the unique critical point $(\theta_1, \theta_2, \theta_3) = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$.

Next, we investigate the behavior of the function f when the variables $(\theta_1, \theta_2, \theta_3)$ approach the boundary of the domain Ω_3. Let $(\theta_1(t), \theta_2(t), \theta_3(t)), t \in [0, \infty)$, be a path in the domain Ω_3. Without loss of generality, we assume

$$\lim_{t\to\infty}(\theta_1(t), \theta_2(t), \theta_3(t)) = (0, s_2, s_3)$$

where $s_2 \geq 0, s_3 \geq 0$ and $s_2 + s_3 = \pi$.

Then $\lim_{t\to\infty} \cot\theta_1(t) = \infty$ and $\cot\theta_2(t) + \cot\theta_3(t) \geq 0$ for $t \in [0, \infty)$. Hence

$$\lim_{t\to\infty}(\cot\theta_1(t) + \cot\theta_2(t) + \cot\theta_3(t)) = \infty.$$

If $\theta_i + \theta_j < \pi$, then $\cot\theta_i > \cot(\pi - \theta_j) = -\cot\theta_j$. Therefore $c_i + c_j > 0$. Hence $2f = (c_1+c_2) + (c_2+c_3) + (c_3+c_1) > 0$. Thus f has the global minimum. But the global minimum can not be achieved at a point on the boundary of Ω_3. It must be achieved at the unique critical point $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$.

This shows that $c_1 + c_2 + c_3 \geq \sqrt{3}$ and the equality holds if and only if $\theta_1 = \theta_2 = \theta_3 = \frac{\pi}{3}$.

2.3. **The first and second eigenvalues.** Since

$$\lambda_2 = c_1 + c_2 + c_3 + \sqrt{(c_1+c_2+c_3)^2 - 3},$$

we have $\lambda_2 \geq \sqrt{3}$ and the equality holds if and only if $\theta_1 = \theta_2 = \theta_3 = \frac{\pi}{3}$.

Since $\lambda_1\lambda_2 = 3$, we have $\lambda_1 \leq \sqrt{3}$ and the equality holds if and only if $\theta_1 = \theta_2 = \theta_3 = \frac{\pi}{3}$.

3. QUADRILATERALS

3.1. **characteristic polynomial.** In Figure 1, the vertices of a cyclic quadrilateral decompose its circumcircle into four arcs. We assume the radius of the circumcircle is 1 and the lengths of the four arcs are $2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4$. Let $c_i := \cot\theta_i$ for $i = 1, ..., 4$. The condition $\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi$ implies

$$c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4 = c_1 + c_2 + c_3 + c_4. \tag{3}$$

There are two ways to decompose a cyclic quadrilateral into a union of two triangles. The two ways produce the same P1 discretization of the Laplace operator:

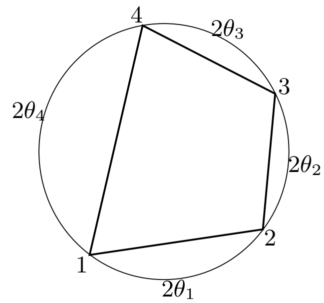

FIGURE 1

$$L_4 = \begin{pmatrix} c_1+c_4 & -c_1 & 0 & -c_4 \\ -c_1 & c_1+c_2 & -c_2 & 0 \\ 0 & -c_2 & c_2+c_3 & -c_3 \\ -c_4 & 0 & -c_3 & c_3+c_4 \end{pmatrix}.$$

The characteristic polynomial of L_4 is

$$\begin{aligned} P_4(x) &= x^4 - 2(c_1+c_2+c_3+c_4)x^3 \\ &\qquad + (3(c_1c_2+c_2c_3+c_3c_4+c_4c_1) + 4(c_1c_3+c_2c_4))x^2 \\ &\qquad\qquad - 4(c_1c_2c_3+c_1c_2c_4+c_1c_3c_4+c_2c_3c_4)x \\ &= x^4 - 2(c_1+c_2+c_3+c_4)x^3 \\ &\qquad + (3(c_1c_2+c_2c_3+c_3c_4+c_4c_1) + 4(c_1c_3+c_2c_4))x^2 \\ &\qquad\qquad - 4(c_1+c_2+c_3+c_4)x, \end{aligned}$$

by the equation (3).

The eigenvalues of L_4 are $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3$ satisfying $\lambda_1+\lambda_2+\lambda_3 = 2(c_1+c_2+c_3+c_4)$ and $\lambda_1\lambda_2\lambda_3 = 4(c_1+c_2+c_3+c_4)$.

3.2. **Sum and product of eigenvalues.** By the similar argument in the case of triangles, we can show that $c_1+c_2+c_3+c_4$ has the unique critical point at $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$. And we have $2(c_1+c_2+c_3+c_4) = (c_1+c_2)+(c_1+c_3)+(c_3+c_4)+(c_4+c_1) > 0$.

Next, we investigate the behavior of the function $c_1+c_2+c_3+c_4$ when the variable $(\theta_1, \theta_2, \theta_3, \theta_4)$ approaches the boundary of the domain

$$\Omega_4 := \{(\theta_1,\theta_2,\theta_3,\theta_4) \mid \theta_1+\theta_2+\theta_3+\theta_4 = \pi, \theta_i > 0, i = 1,2,3,4\}.$$

Let $(\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t))$, $t \in [0,\infty)$, be a path in the domain Ω_4.

Without loss of generality, we assume

$$\lim_{t\to\infty} (\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t)) = (0, s_2, s_3, s_4)$$

where $s_i \ge 0$ for $i = 2,3,4$ and $s_2+s_3+s_4 = \pi$. Since two of s_2, s_3 and s_4 must be less than $\frac{\pi}{2}$, we assume that $s_2 < \frac{\pi}{2}$ and $s_3 < \frac{\pi}{2}$. Then $\lim_{t\to\infty} \cot\theta_1(t) = \infty$, $\cot\theta_2(t) > 0$ when t is sufficiently large and $\cot\theta_3(t) + \cot\theta_4(t) > 0$ for any $t \in [0,\infty)$. Hence

$$\lim_{t\to\infty} (\cot\theta_1(t) + \cot\theta_2(t) + \cot\theta_3(t) + \cot\theta_4(t)) = \infty.$$

Therefore $c_1+c_2+c_3+c_4$ achieves its absolute minimum at the unique critical point $(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4})$. Hence $c_1+c_2+c_3+c_4 \geq 4$ and the equality holds if and only if $\theta_1=\theta_2=\theta_3=\theta_4=\frac{\pi}{4}$.

Therefore $\lambda_1+\lambda_2+\lambda_3 \geq 8$, $\lambda_1\lambda_2\lambda_3 \geq 16$ and the equality holds if and only if $\theta_1=\theta_2=\theta_3=\theta_4=\frac{\pi}{4}$.

3.3. **Polynomial of degree 2.** In the subsection, we verify that a square has the minimal $\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1$. By the formula of the characteristic polynomial $P_4(x)$, it is enough to show

$$3(c_1c_2+c_2c_3+c_3c_4+c_4c_1)+4(c_1c_3+c_2c_4) \geq 20$$

and the equality holds if and only if $\theta_1=\theta_2=\theta_3=\theta_4=\frac{\pi}{4}$.

In fact, consider $g := 3(c_1c_2+c_2c_3+c_3c_4+c_4c_1)+4(c_1c_3+c_2c_4)$ as a function defined on the domain Ω_4.

To find the absolute minimum of g, we apply the method of Lagrange multiplier. Let

$$G := 3(c_1c_2+c_2c_3+c_3c_4+c_4c_1)+4(c_1c_3+c_2c_4)+y(\theta_1+\theta_2+\theta_3+\theta_4-\pi).$$

Then

$$\begin{aligned}
0 &= \frac{\partial G}{\partial \theta_1} = -(3c_2+3c_4+4c_3)(1+c_1^2)+y,\\
0 &= \frac{\partial G}{\partial \theta_2} = -(3c_1+3c_3+4c_4)(1+c_2^2)+y,\\
0 &= \frac{\partial G}{\partial \theta_3} = -(3c_2+3c_4+4c_1)(1+c_3^2)+y,\\
0 &= \frac{\partial G}{\partial \theta_4} = -(3c_3+3c_1+4c_2)(1+c_4^2)+y,\\
0 &= \frac{\partial G}{\partial y} = \theta_1+\theta_2+\theta_3+\theta_4-\pi.
\end{aligned}$$

The first and the third equation above imply that

$$(3c_2+3c_4+4c_3)(1+c_1^2) = (3c_2+3c_4+4c_1)(1+c_3^2)$$

which is equivalent to

$$(c_1-c_3)(3c_1c_2+3c_1c_4+3c_2c_3+3c_3c_4+4c_1c_3-4)=0.$$

We claim that the second factor is positive, i.e.,

$$3c_1c_2+3c_1c_4+3c_2c_3+3c_3c_4+4c_1c_3 > 4.$$

In fact, since $\theta_1+\theta_2+\theta_3<\pi$, then $\cot(\theta_1+\theta_2) > \cot(\pi-\theta_3)$. Then

$$\frac{c_1c_2-1}{c_1+c_2} > -c_3$$

which is equivalent to

$$c_1c_2+c_2c_3+c_3c_1 > 1 \tag{4}$$

since $c_1+c_2>0$.

By the similar reason,

$$c_1c_4+c_4c_3+c_3c_1 > 1.$$

At least one of c_2 and c_4 is positive. If $c_2 > 0$, then

$$\begin{aligned}
&3c_1c_2 + 3c_1c_4 + 3c_2c_3 + 3c_3c_4 + 4c_1c_3 \\
&\qquad\qquad = 3(c_1c_4 + c_4c_3 + c_3c_1) + (c_1c_2 + c_2c_3 + c_3c_1) + 2(c_1 + c_3)c_2 \\
&\qquad\qquad > 3 + 1 + 0.
\end{aligned}$$

If $c_4 > 0$, then

$$\begin{aligned}
&3c_1c_2 + 3c_1c_4 + 3c_2c_3 + 3c_3c_4 + 4c_1c_3 \\
&\qquad\qquad = (c_1c_4 + c_4c_3 + c_3c_1) + 3(c_1c_2 + c_2c_3 + c_3c_1) + 2(c_1 + c_3)c_4 \\
&\qquad\qquad > 1 + 3 + 0.
\end{aligned}$$

Thus the only possibility is $c_1 = c_3$ which implies $\theta_1 = \theta_3$. By the similar argument, $0 = \frac{\partial G}{\partial \theta_2}$ and $0 = \frac{\partial G}{\partial \theta_4}$ imply $\theta_2 = \theta_4$. Since $\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi$, we have $\theta_1 + \theta_2 = \frac{\pi}{2}$ which implies $c_1c_2 = 1$.

Now $0 = \frac{\partial G}{\partial \theta_1}$ and $0 = \frac{\partial G}{\partial \theta_2}$ imply

$$(3c_2 + 3c_4 + 4c_3)(1 + c_1^2) = (3c_1 + 3c_3 + 4c_4)(1 + c_2^2).$$

Since $c_1 = c_3$ and $c_2 = c_4$, we have

$$(6c_2 + 4c_1)(1 + c_1^2) = (6c_1 + 4c_2)(1 + c_2^2).$$

Since $c_1c_2 = 1$, we have

$$(c_1 - c_2)(c_1^2 + c_2^2 + 2) = 0.$$

The only possibility is $c_1 = c_2$.

Therefore the function $g = 3(c_1c_2 + c_2c_3 + c_3c_4 + c_4c_1) + 4(c_1c_3 + c_2c_4)$ has the unique critical point $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$.

Next, we claim that $g > 0$. Since at least three of c_1, c_2, c_3, c_4 are positive, without loss of generality, we may assume that $c_1 > 0, c_2 > 0, c_3 > 0$. Let's write

$$g = 2(c_1c_2 + c_2c_4 + c_4c_1) + 2(c_2c_3 + c_3c_4 + c_4c_2) + (c_2 + c_4)c_1 + (c_1 + c_4)c_3 + 4c_1c_3.$$

Then each term of sum above is positive.

At last, we investigate the behavior of g when the variables approach the boundary of the domain Ω_4. Let $(\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t))$, $t \in [0, \infty)$, be a path in the domain Ω_4. Without loss of generality, we have

$$\lim_{t\to\infty} (\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t)) = (0, s_2, s_3, s_4)$$

where $s_i \geq 0$ for $i = 2, 3, 4$ and $s_2 + s_3 + s_4 = \pi$. And we can assume that $s_2 < \frac{\pi}{2}$ and $s_3 < \frac{\pi}{2}$.

Let's write

$$\begin{aligned}
g = {}& 2(\cot\theta_1(t)\cot\theta_2(t) + \cot\theta_2(t)\cot\theta_4(t) + \cot\theta_4(t)\cot\theta_1(t)) \\
&+ 2(\cot\theta_2(t)\cot\theta_3(t) + \cot\theta_3(t)\cot\theta_4(t) + \cot\theta_4(t)\cot\theta_2(t)) \\
&+ (\cot\theta_2(t) + \cot\theta_4(t))\cot\theta_1(t) + (\cot\theta_1(t) + \cot\theta_4(t))\cot\theta_3(t) \\
&+ 4\cot\theta_1(t)\cot\theta_3(t).
\end{aligned}$$

By the inequality (4),

$$\cot\theta_1(t)\cot\theta_2(t) + \cot\theta_2(t)\cot\theta_4(t) + \cot\theta_4(t)\cot\theta_1(t) > 1$$

and

$$\cot\theta_2(t)\cot\theta_3(t) + \cot\theta_3(t)\cot\theta_4(t) + \cot\theta_4(t)\cot\theta_2(t) > 1$$

for any $t \in [0, \infty)$. Since $\cot\theta_2(t) + \cot\theta_4(t) > 0$ any $t \in [0, \infty)$, $\lim_{t\to\infty}(\cot\theta_2(t) + \cot\theta_4(t))\cot\theta_1(t) = \infty$. And

$$(\cot\theta_1(t) + \cot\theta_4(t))\cot\theta_3(t) > 0, 4\cot\theta_1(t)\cot\theta_3(t) > 0$$

when t is sufficiently large. Hence g approaches ∞.

Therefore g has a lower bound and can not achieve its global minimum at a boundary point. It much achieve its global minimum at the unique critical point $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$.

3.4. **The first eigenvalue.** In this subsection, we verify that a square has the minimal λ_1. First, we verify that $\lambda_1 \leq 2$ as follows. Let

$$\begin{aligned} Q(x) := \frac{P_4(x)}{x} = x^3 - 2(c_1 + c_2 + c_3 + c_4)x^2 \\ + (3(c_1c_2 + c_2c_3 + c_3c_4 + c_4c_1) + 4(c_1c_3 + c_2c_4))x \\ - 4(c_1 + c_2 + c_3 + c_4). \end{aligned}$$

We have $Q(0) = -4(c_1 + c_2 + c_3 + c_4) \leq -16$.

If $Q(2) > 0$, then the first root of $Q(x)$ is less that 2, i.e., $\lambda_1 < 2$.

If $Q(2) \leq 0$, we claim that $Q'(0) > 0$ and $Q'(2) \leq 0$. Once the two statements are established, we have $\lambda_1 \leq \lambda_2 \leq 2$.

In fact $Q'(0) = 3(c_1c_2 + c_2c_3 + c_3c_4 + c_4c_1) + 4(c_1c_3 + c_2c_4) \geq 20$.

To verify $Q'(2) \leq 0$, we need to use the assumption $Q(2) \leq 0$. In fact $Q(2) \leq 0$ implies

$$3(c_1c_2 + c_2c_3 + c_3c_4 + c_4c_1) + 4(c_1c_3 + c_2c_4) \leq 6(c_1 + c_2 + c_3 + c_4) - 4.$$

Now

$$\begin{aligned} & Q'(2) \\ & = 12 - 8(c_1 + c_2 + c_3 + c_4) + 3(c_1c_2 + c_2c_3 + c_3c_4 + c_4c_1) + 4(c_1c_3 + c_2c_4) \\ & \leq 12 - 8(c_1 + c_2 + c_3 + c_4) + 6(c_1 + c_2 + c_3 + c_4) - 4 \\ & = 8 - 2(c_1 + c_2 + c_3 + c_4) \\ & \leq 0, \end{aligned}$$

since $c_1 + c_2 + c_3 + c_4 \geq 4$.

Second, we verify that $\lambda_1 = 2$ if and only if $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{\pi}{4}$. Since $\lambda_1 = 2$ is the first root of $Q(x)$, we have $Q'(2) \geq 0$. On the other hand, it is shown that $Q(2) \leq 0$ implies $Q'(2) \leq 0$. Hence the only possibility is $Q'(2) = 0$. This requires that $c_1 + c_2 + c_3 + c_4 = 4$. Therefore we must have $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{\pi}{4}$.

4. General cyclic polygons

4.1. **Discrete Laplace operator.** Assume $n \geq 5$. In Figure 2, the vertices of a cyclic n-gon decompose its circumcircle into n arcs. We assume the radius of the circumcircle is 1 and the lengths of the n arcs are $2\theta_1, 2\theta_2, ..., 2\theta_n$.

The P1 discretization of the Laplace operator of a cyclic n-gon is independent of the choice of a triangulation. It is

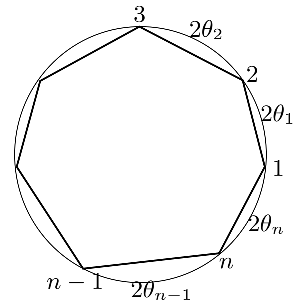

FIGURE 2

$$L_n = \begin{pmatrix} c_1+c_n & -c_1 & 0 & 0 & ... & -c_n \\ -c_1 & c_1+c_2 & -c_2 & 0 & ... & 0 \\ 0 & -c_2 & c_2+c_3 & -c_3 & ... & 0 \\ 0 & 0 & -c_3 & c_3+c_4 & ... & 0 \\ 0 & 0 & 0 & -c_4 & ... & 0 \\ & & & & \ddots & \\ -c_n & 0 & 0 & 0 & ... & c_{n-1}+c_n \end{pmatrix}.$$

The eigenvalues are $0 = \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{n-1}$.

4.2. **Sum of eigenvalues.** We have $\sum_{i=1}^{n-1} \lambda_i = 2\sum_{i=1}^{n} c_i$. By the similar argument in the case of triangles and cyclic quadrilaterals, we can show that $\sum_{i=1}^{n} c_i$ has the unique critical point $(\theta_1, \theta_2, ..., \theta_n) = (\frac{\pi}{n}, \frac{\pi}{n}, ..., \frac{\pi}{n})$.

Since there is at most one non-positive number in $c_1, ..., c_n$, without loss of generality, we may assume $c_1 > 0, ..., c_{n-1} > 0$. Since $c_{n-1} + c_n > 0$, we have $\sum_{i=1}^{n} c_i > 0$.

We investigate the behavior of $\sum_{i=1}^{n} c_i$ when the variables approach the boundary of the domain

$$\Omega_n = \{(\theta_1, \theta_2, ..., \theta_n) \mid \theta_1 + \theta_2 + ... + \theta_n = \pi, \theta_i > 0, i = 1, 2, ..., n\}.$$

Let $(\theta_1(t), \theta_2(t), ..., \theta_n(t))$, $t \in [0, \infty)$, be a path in the domain Ω_n.

Without loss of generality, we have

$$\lim_{t\to\infty} (\theta_1(t), \theta_2(t), ..., \theta_n(t)) = (0, s_2, ..., s_n)$$

where $s_i \geq 0$ for $i = 2, ..., n$ and $s_2 + ... + s_n = \pi$. And we can assume that $s_2 < \frac{\pi}{2}, ..., s_{n-1} < \frac{\pi}{2}$. Since $\cot\theta_i(t) > 0$ for $i = 2, ..., n-1$ and $\cot\theta_{n-1}(t) + \cot\theta_n(t) > 0$ when t is sufficiently large, $\lim_{t\to\infty} \cot\theta_1(t) = \infty$ implies that $\lim_{t\to\infty} \sum_{i=1}^{n} \cot\theta_i(t) = \infty$.

Thus $\sum_{i=1}^{n} c_i$ achieved the absolute minimum at $(\frac{\pi}{n}, ..., \frac{\pi}{n})$.

4.3. **Product of eigenvalues.** In this subsection we verify that a regular n-gon has the minimal $\prod_{i=1}^{n-1} \lambda_i$. We need the following result.

The weighted matrix-tree Theorem. *Let M be an $n \times n$ matrix. If the sum of the entries of each row or each column of M vanishes, all principal $(n-1) \times (n-1)$ submatrices of M have the same determinant, and this value is equal to $\frac{1}{n}$ times the product of all nonzero eigenvalues of M.*

For the reference of the weighted matrix-tree Theorem, for example, see [12], page 450, Problem 34A or [8], Theorem 1.2.

In our case, according the weighted matrix-tree Theorem, to calculate $\prod_{i=1}^{n-1}\lambda_i$ of the matrix L_n, it is enough to calculate the determinant of a particular principal $(n-1)\times(n-1)$ submatrix.

Lemma 4. *Let N_n be the submatrix obtained by deleting the first row and first column of the matrix L_n. Then*

$$\det N_n = \sum_{i=1}^{n} c_1...\widehat{c_i}...c_n,$$

where $\widehat{c_i}$ means that c_i is missing.

Proof. We prove the statement by mathematical induction. It holds for $n=4$ as we see in the formula of the characteristic polynomial $P_4(x)$. We assume it holds for $n\leq m-1$. By the property of tridiagonal matrices, we have

$$\det N_m = (c_{m-1}+c_m)\det N_{m-1} - c_{m-1}^2 \det N_{m-2}.$$

Then by the assumption of mathematical induction,

$$\begin{aligned}\det N_m &= (c_{m-1}+c_m)\sum_{i=1}^{m-1} c_1...\widehat{c_i}...c_{m-1} - c_{m-1}^2\sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}\\ &= c_{m-1}\sum_{i=1}^{m-1} c_1...\widehat{c_i}...c_{m-1} - c_{m-1}\sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}c_{m-1}\\ &\qquad\qquad + c_m\sum_{i=1}^{m-1} c_1...\widehat{c_i}...c_{m-1}\\ &= c_1...c_{m-1} + c_m\sum_{i=1}^{m-1} c_1...\widehat{c_i}...c_{m-1}\\ &= \sum_{i=1}^{m} c_1...\widehat{c_i}...c_m.\end{aligned}$$

□

In the following, we prove that $\sum_{i=1}^{n} c_1...\widehat{c_i}...c_n$ achieves its global minimum when $\theta_1=...=\theta_n=\frac{\pi}{n}$. It is enough to show that

a. $\sum_{i=1}^{n} c_1...\widehat{c_i}...c_n$ has the unique critical point $(\frac{\pi}{n},...,\frac{\pi}{n})$;
b. $\sum_{i=1}^{n} c_1...\widehat{c_i}...c_n > 0$;
c. $\sum_{i=1}^{n} c_1...\widehat{c_i}...c_n$ approaches ∞ as the variables approach the boundary of the domain Ω_n.

When $n=4$, since $c_1c_2c_3+c_1c_2c_4+c_1c_3c_4+c_2c_3c_4 = c_1+c_2+c_3+c_4$, the three statements above are already shown to be true in section 3. We assume that the three statements above hold when $n\leq m-1$.

Let's check the three statements when $n=m$. Consider the function

$$H := \sum_{i=1}^{m} c_1...\widehat{c_i}...c_m - y(\theta_1+...\theta_m-\pi).$$

Then $0 = \frac{\partial H}{\partial \theta_1}$ and $0 = \frac{\partial H}{\partial \theta_2}$ imply that

$$(c_3...c_m + c_2 \sum_{i=3}^{m} c_3...\widehat{c_i}...c_m)(1+c_1^2) = (c_3...c_m + c_1 \sum_{i=3}^{m} c_3...\widehat{c_i}...c_m)(1+c_2^2).$$

Since $c_1 + c_2 > 0$, it is equivalent to

$$(c_1 - c_2)(c_1 + c_2)(c_3...c_m + \frac{c_1c_2 - 1}{c_1 + c_2} \sum_{i=3}^{m} c_3...\widehat{c_i}...c_m) = 0.$$

The third factor is

$$\cot\theta_3 ... \cot\theta_m + \cot(\theta_1 + \theta_2) \sum_{i=3}^{m} \cot\theta_3 ... \widehat{\cot\theta_i} ... \cot\theta_m$$

which is written as $\sum_{i=1}^{m} \widetilde{c_1}...\widehat{\widetilde{c_i}}...\widetilde{c}_{m-1}$, where $\widetilde{c_1} = \cot(\theta_1 + \theta_2), \widetilde{c_i} = \cot\theta_{i+1}$ for $i = 2, ..., m-1$. This expression corresponds to a cyclic $(m-1)$-gon. By assumption of the induction, $\sum_{i=1}^{m} \widetilde{c_1}...\widehat{\widetilde{c_i}}...\widetilde{c}_{m-1} > 0$.

Hence the only possibility is $c_1 = c_2$. By similar argument, we show that $c_i = c_j$ for any i, j. Hence the function $\sum_{i=1}^{m} c_1...\widehat{c_i}...c_m$ has the unique critical point such that $\theta_i = \frac{\pi}{m}$ for any $i = 1, ..., m$.

Next, we claim that $\sum_{i=1}^{m} c_1...\widehat{c_i}...c_m > 0$. Without loss of generality, we assume that $c_1 > 0, c_2 > 0, ..., c_{m-1} > 0$. Now

$$\begin{aligned}
&\sum_{i=1}^{m} c_1...\widehat{c_i}...c_m \\
&= c_1c_2...c_{m-2}(c_{m-1} + c_m) + \sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}(c_{m-1}c_m) \\
&= c_1c_2...c_{m-2}(c_{m-1} + c_m) + \sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}(c_{m-1}c_m - 1) + \sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2} \\
&= (c_{m-1} + c_m)(c_1c_2...c_{m-2} + \sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}\frac{c_{m-1}c_m - 1}{c_{m-1} + c_m}) + \sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}.
\end{aligned}$$

Let $\widetilde{c}_{m-1} = \frac{c_{m-1}c_m - 1}{c_{m-1} + c_m} = \cot(\theta_{m-1} + \theta_m)$. Then

$$\begin{aligned}
&c_1c_2...c_{m-2} + \sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}\frac{c_{m-1}c_m - 1}{c_{m-1} + c_m} \\
&= c_1c_2...c_{m-2} + \sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}\widetilde{c}_{m-1}.
\end{aligned}$$

Consider an cyclic $(m-1)$-gon with angles $\theta_1, ..., \theta_{m-2}, \theta_{m-1} + \theta_m$. By the assumption of induction,

$$c_1c_2...c_{m-2} + \sum_{i=1}^{m-2} c_1...\widehat{c_i}...c_{m-2}\widetilde{c}_{m-1} > 0.$$

Therefore $\sum_{i=1}^{m} c_1...\widehat{c_i}...c_m > 0$.

At last, we investigate the behavior of the function $\sum_{i=1}^m c_1...\widehat{c_i}...c_m$ when the variables approach the boundary of the domain

$$\Omega_m := \{(\theta_1, ..., \theta_m) \mid \theta_1 + ... + \theta_m = \pi, \theta_i > 0, i = 1, ..., m\}.$$

Let $(\theta_1(t), ..., \theta_m(t))$, $t \in [0, \infty)$, be a path in the domain Ω_m. Without loss of generality, we assume

$$\lim_{t\to\infty}(\theta_1(t), ..., \theta_m(t)) = (0, s_2, ..., s_m),$$

where $s_2 \geq 0, ..., s_m \geq 0$ and $s_2 + ... + s_m = \pi$. And we can assume furthermore that $s_2 < \frac{\pi}{2}, ..., s_{m-1} < \frac{\pi}{2}$. Thus $\cot\theta_1(t) > 0, ..., \cot\theta_{m-1}(t) > 0$ when t is sufficiently large. Now

$$\begin{aligned}
&\sum_{i=1}^m \cot\theta_1(t)...\widehat{\cot\theta_i(t)}...\cot\theta_m(t)\\
&= [\cot\theta_{m-1}(t) + \cot\theta_m(t)]\\
&\qquad [\cot\theta_1(t)\cot\theta_2(t)...\cot\theta_{m-2}(t)\\
&\qquad\qquad + \sum_{i=1}^{m-2}\cot\theta_1(t)...\widehat{\cot\theta_i(t)}...\cot\theta_{m-2}(t)\frac{\cot\theta_{m-1}(t)\cot\theta_m(t) - 1}{\cot\theta_{m-1}(t) + \cot\theta_m(t)}]\\
&\qquad\qquad\qquad + \sum_{i=1}^{m-2}\cot\theta_1(t)...\widehat{\cot\theta_i(t)}...\cot\theta_{m-2}(t)\\
&= [\cot\theta_{m-1}(t) + \cot\theta_m(t)]\\
&\qquad [\cot\theta_1(t)\cot\theta_2(t)...\cot\theta_{m-2}(t)\\
&\qquad\qquad + \sum_{i=1}^{m-2}\cot\theta_1(t)...\widehat{\cot\theta_i(t)}...\cot\theta_{m-2}(t)\cot(\theta_{m-1}(t) + \theta_m(t))]\\
&\qquad\qquad\qquad + \sum_{i=1}^{m-2}\cot\theta_1(t)...\widehat{\cot\theta_i(t)}...\cot\theta_{m-2}(t).
\end{aligned}$$

By the assumption of induction,

$$\begin{aligned}
&\cot\theta_1(t)\cot\theta_2(t)...\cot\theta_{m-2}(t)\\
&\qquad + \sum_{i=1}^{m-2}\cot\theta_1(t)...\widehat{\cot\theta_i(t)}...\cot\theta_{m-2}(t)\cot(\theta_{m-1}(t) + \theta_m(t)) > 0
\end{aligned}$$

for any $t \in [0, \infty)$. Since $\cot\theta_{m-1}(t) + \cot\theta_m(t) > 0$ for any $t \in [0, \infty)$ and $\sum_{i=1}^{m-2}\cot\theta_1(t)...\widehat{\cot\theta_i(t)}...\cot\theta_{m-2}(t)$ approaches ∞, we see that

$$\sum_{i=1}^m \cot\theta_1(t)...\widehat{\cot\theta_i(t)}...\cot\theta_m(t)$$

approaches ∞.

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