AN ABSTRACT OF THE THESIS OF

NABIL NASSIB GHANDOUR (Name) for the MASTER OF ARTS (Degree)

in MATHEMATICS (Major) presented on September 2, 1970 (Date)

Title: STABILITY OF FLOWS OF THERMO-VISCOELASTIC FLUIDS BETWEEN ROTATING COAXIAL CIRCULAR CYLINDERS.

Redacted for Privacy

Abstract approved: M.N.L. Narasimhan

We investigate in this thesis, the problem of stability of thermo-viscoelastic fluid flow between rotating coaxial cylinders. By using the thermo-viscoelastic constitutive equations given by Eringen, we reduce the equations of motion into a form suitable for stability analysis. The course of reduction which we follow yields some interesting intermediate results. The solution for the steady state couette flow problem is found. Interestingly enough, the velocity field for this problem is found to be identical with the classical viscous case and the case of a Reiner-Rivlin fluid, but the temperature and pressure fields are different. The non-dimensional forms of the equations of motion governing the couette flow of thermo-viscoelastic fluids in a heat reservoir are given. These equations are reduced further by considering a small

gap between the cylinders and by imposing some physically reasonable mechanical and geometrical restrictions on the flow. This results in a secular equation which forms a characteristic value problem. The solution of the characteristic value problem has been obtained and this yields a criterion for stability in terms of a critical Taylor number. In general, the critical values of Taylor numbers are found to be higher than corresponding ones in classical hydrodynamic stability problems, which implies that thermo-viscoelastic fluids are more stable, in a couette flow, than classical viscous fluids under a similar situation. Comparing this result with existing investigations in non-Newtonian fluids we find that, like Bingham fluids, thermo-viscoelastic fluids are more stable than viscous and Reiner-Rivlin fluids.

STABILITY OF FLOWS OF THERMO-VISCOELASTIC FLUIDS BETWEEN ROTATING COAXIAL CIRCULAR CYLINDERS

by

Nabil Nassib Ghandour

A THESIS
submitted to
Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Arts

June 1971

APPROVED:

Redacted for Privacy

Professor of Mathematics

in charge of major

Redacted for Privacy

Acting Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented September 2, 1970

Typed by Linda Knuth for Nabil Nassib Ghandour

ACKNOWLEDGEMENTS

The author wished to express his sincere appreciation to his major professor, Professor Narasimhan, for his suggestion of the problem, criticisms and suggestions during the course of the research, and especially his giving of unlimited time and effort during the writing of this thesis.

The author wishes to thank Linda Knuth for the excellent typing of this thesis.

The author wishes to thank Mary Berryman for her help in typing the program for the numerical computations.

The author is indebted to Professor A.T. Lonseth and the Atomic Energy Commission for their financial support, AT 30-262-2006. Also he wishes to thank Dr. L. Hunter and the computer center for providing the computer time. Finally, the author wishes to thank the Conseil National de la Recherche Scientifique (Lebanon) for their financial support.

TABLE OF CONTENTS

STABILITY OF FLOWS OF THERMO-VISCOELASTIC FLUIDS BETWEEN ROTATING COAXIAL CIRCULAR CYLINDERS.

I. INTRODUCTION

1.1 Hydrodynamic Stability

The equations of fluid dynamics, complex in nature as they are, admit steady state solutions to some viscometric flow problems. Each of these flows is characterized by a parameter involving the criterion for its stability and it can only be realized for a certain range of values of this parameter. Outside this range the original pattern of the flow cannot be maintained. This is owing to its inability to sustain itself in the presence of disturbances to which any physical system may be subject. The influence of disturbances superposed on a given steady laminar flow might, in course of time, change it into a turbulent flow or into another type of laminar flow. In both cases, the original flow is said to be unstable with respect to the superposed disturbances. This differentiation of stable from unstable patterns of possible flows is what originates the problems of hydrodynamical stability.

In considering the stability of a system, the question that arises would be whether a disturbance superposed on a steady laminar flow would gradually die down in time or whether the amplitude of disturbances continue to grow causing the system to depart from its initial state without ever reverting to it. In the former case the flow is said to be stable, while in the latter case it is said to be unstable.

Clearly, a system is stable if it is stable with respect to every mode of disturbance. The initial steady flow is characterized by a set of fluid parameters such as the channel geometry, velocity field, pressure gradient, thermal or magnetic fields, etc. When small disturbances are superposed on the flow, if instability sets in, the state which separates the original stable state from the final unstable stable state is known as the marginal state. Thus the marginal state is a state of neutral stability.

Instability could occur in two ways depending on the manner in which the disturbances are superposed. One, if the disturbances are aperiodic and the resulting flow is unstable, then this type of instability is known to set through steady motions. This phenomenon is known as the principle of exchange of stabilities. In the second type of instability, if the disturbances introduced are periodic, resulting in oscillatory motions in the fluid, then this phenomenon is known as overstability, Chandrashaker (1961).

1.2 Stability Problems of Classical Viscous Fluid Flows

Since the latter part of the nineteenth century, a variety of stability problems of ideal and viscous fluids have been attempted and solved. We give here a brief account of the most important of these problems for purposes of reference as well as our own investigations later in this thesis.

The Bénard Problem

Although the phenomenon of thermal convection of a viscous fluid was recognized as early as 1797 by Count Rumford, it was not until 1900, an experimental study of the stability of a horizontal fluid layer heated from below was made by Bénard. Rayleigh (1916, 1920) considered the same problem both experimentally and theoretically and observed the flow taking cellular patterns. Taylor (1917, 1921, 1923) discussed the above stability problem under the influence of rotation. He found theoretically that the effect of rotation on the thermal convectional flow was to stabilize it and confirmed this result by his own experiments. Chandrashaker (1954, 1956, 1957) investigated the stability of a viscous fluid layer heated from below in the presence of a magnetic field and concluded that the magnetic field inhibits the onset of instability in the above problem.

The Stability of Couette Flow

In the Bénard and related problems, the instability is caused by an outside adverse temperature gradient. For the Couette Flow such as the flow between rotating circular cylinders this is not the case. It is found that instability is caused by a prevailing adverse gradient of angular momentum. For this problem also, the effects of a magnetic and thermal fields have been studied. In addition, the effects of the presence of a pressure gradient on the stability of a Couette Flow were studied by Rayleigh (1920), Taylor (1923) and Lin (1955).

The Stability of General Flows in Curved Channels

The stability of viscous flows in curved channels presents a wide variety of problems of which the stability of couette flow is one, Dean (1928), Reid (1958) and DePrima.

The Stability of Superposed Fluids

In the case of superposed fluids, instability is caused by a different means than in the problems already mentioned. Two causes of instability exist in this case. One is the presence of a density gradient and the other is the existance of relative horizontal motion between two layers of fluids superposed over each other, Rayleigh (1900), Taylor (1950), Chandrashaker (1955) and Reid (1959).

The Stability of Jets and Cylindrical Flows

The onset of instability in jets and cylindrical flows is caused by surface tensions and the geometry of the channel,[Rayleigh (1954), Chandrashaker and Fermi (1953), Simon (1958) and Volkov (1959)].

1.3 Stability Problems of Non-Newtonian Fluid Flow

Non-Newtonian fluids such as high polymer solutions, paints, condensed milk, etc., generally have a more complex behavior than viscous fluids. These fluids require more complex constitutive equations which complicate the mathematical nature of the problem. Very few problems for non-Newtonian fluids have been investigated. We present in the following sections in brief some of the investigations of stability of non-Newtonian fluids which have appeared in recent years.

Stability of a Bingham Fluid in Couette Flow

Graebel (1962) by considering a constitutive equation given by Oldroyd, eq. (2.3.3), analyses the stability of a Bingham fluid (Bingham plastic) between two co-rotating circular cylinders. It was found that the non-Newtonian nature of the fluid acts as a stabilizing agent.

Stability of a Non-Newtonian Couette Flow in the Presence of a Circular Magnetic Field

Using a Reiner-Rivlin model, (eq. 2.3.2), Narasimhan (1963) investigated the stability of flow of a non-Newtonian liquid between two rotating cylinders in the presence of a circular magnetic field. He found that the non-Newtonian behavior of the fluid facilitates the onset of instability and hence the non-Newtonian Couette flow is less stable than the classical viscous Couette flow in the presence of a circular magnetic field.

1.4 Purpose and Need for the Present Investigation

Although stability problems of a wide variety of classical viscous flows have been investigated to date, there is, as mentioned before, very little work done on similar problems concerning varieties of non-Newtonian or viscoelastic fluids under different mechanical or electrical influences. Hence it seems reasonable and important to investigate stability problems of viscoelastic fluids in detail.

Since viscoelastic fluids exhibit many interesting thermal properties, it seems important to consider the thermal aspects of the stability as well. Hence, in the present investigation, we consider the stability problem of a thermo-viscoelastic fluid flow between two rotating coaxial circular cylinders which are maintained at different constant temperatures.

1.5 Plan of the Present Investigation

Chapter one is devoted to the introduction of hydrodynamical stability problems and to familiarize the reader with some of the existing investigations on the subject. Since the non-linear constitutive equations of thermo-viscoelasticity are relatively new, the second chapter will present a brief account of their derivation and of obtaining various approximations to suit different situations. Chapter three will bring us into the heart of the stability problem of a rotating circular annulus of thermo-viscoelastic fluids. The fourth chapter presents the solution of this stability problem in terms of a criterion for stability. Finally, the last chapter summarizes the method used and discusses the results and compares them with existing work in stability of Newtonian as well as non-Newtonian fluids. Further scope and suggestions for future research in the field are also given.

II. NON-LINEAR THERMO-VISCOELASTIC FLUIDS

2.1 Introduction

This chapter will be devoted to the foundations and development of the non-linear constitutive equations governing the behavior of thermo-viscoelastic fluids following Eringen (1963). First, we discuss the limitations of existing theories and methods used to handle problems of thermo-viscoelastic nature. Secondly, we give the foundations of the non-linear theory of thermo-viscoelasticity. We close this chapter by giving an account of the existing investigations of thermo-viscoelastic fluid flow problems.

2.2 Some Basic Concepts of Continuum Mechanics and Thermodynamics

Before we proceed to develop a set of constitutive equations for thermo-viscoelastic materials, it would be helpful to briefly review some concepts of continuum mechanics and thermodynamics and adapt them to the situation where we need them.

Consider a general coordinate system with $\underset{\sim}{X}$ or X_L to denote the undeformed state, $\underset{\sim}{x}$ or x_k to denote the deformed state. Any material, regardless of its mechanical and thermal properties, must satisfy certain conservation principles. Following is a list of the mathematical expressions of these principles.

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + (\rho v^{\ell})_{,\ell} = 0, \qquad (2.2.1)$$

where ρ is the mass density, t the time, and $v^{\ell} = \dot{x}^{\ell}$ is the velocity. A comma (,) preceeding a subscript indicates covariant differentiation with respect to the coordinates in the deformed state if the subscript is a miniscule (e.g. ℓ) or to that of the undeformed state if the subscript is a majuscule (e.g. L). Diagonally repeated indices imply summation over the range (1,2,3).

Conservation of the Linear Momentum

$$T^{\ell m}{}_{,\ell} + \rho(f^m - a^m) = 0, \qquad (2.2.2)$$

where $T^{\ell m}$, f^m and a^m are the contravariant components of the stress tensor $\underset{\sim}{T}$, the body force $\underset{\sim}{f}$ and the acceleration $\underset{\sim}{a} \equiv \dot{\underset{\sim}{v}}$.

Conservation of Angular Momentum

For a non-polar case (that is in the absence of couple stresses and body couples),

$$T^{m\ell} = T^{\ell m}, \qquad (2.2.3)$$

which implies that the stress tensor is symmetric.

Conservation of Energy

$$\rho\dot{\varepsilon} = T^{m\ell}d_{\ell m} - q^{\ell}{}_{,\ell} + \rho Q, \qquad (2.2.4)$$

where $\dot{\varepsilon}$ is the time rate of change of the specific internal energy ε, q^k are the contravariant components of the heat flux vector $\underset{\sim}{q}$; Q is the supply of energy; and $d_{\ell m}$ are the components of the deformation-rate tensor $\underset{\sim}{d}$ defined by

$$2d_{\ell m} = v^{\ell}{}_{,m} + v^{m}{}_{,\ell}. \qquad (2.2.5)$$

Bodies with the same geometry and mass but of different materials generally react differently to similar outside effects. This is owing to the fact that each material has its own special internal constitution. Mathematically, this is expressed by the so-called constitutive equations. To be properly formulated, a constitutive equation has to satisfy certain invariance principles. Of these, the following three are of particular importance in the course of this analysis.

(a) Principle of Determinism: For our purposes this principle can be stated as: The stress $\underset{\sim}{T}(\underset{\sim}{x}, t)$ and the heat flux $\underset{\sim}{q}(\underset{\sim}{x}, t)$ at the spatial point $\underset{\sim}{x}$ and time t are determined by the past history of the motion of an arbitrarily small neighborhood of the material point $\underset{\sim}{X}$ and the past thermodynamic history of this neighborhood. The stress $\underset{\sim}{T}$ and the heat flux $\underset{\sim}{q}$ are in general functionals of certain kinematic and thermodynamic variables which both characterize materials with memory. However, we should restrict our study only to the class of materials that are conscious of an initial state $\underset{\sim}{X}$ and a present state $\underset{\sim}{x}$ but oblivious to the intermediate configurations.

(b) Principle of Equipresence: Any independent variable appearing in either the stress equation or the heat flux equation--these equations make up a set of thermomechanical constitutive relations--must also appear in the other. In other words, this principle states that $\underset{\sim}{T}$ and $\underset{\sim}{q}$ should be functions of the same kinematic and thermodynamic variables.

(c) Principle of Material Objectivity: This principle in effect states that the response of the material to a given event must be independent of the observer, i.e., mathematically, constitutive equations must remain invariant under any rigid motion of spatial coordinates.

We consider now the concepts of thermodynamics that will be referred to later on. Consider a body B with volume V and mass M and an internal total energy E. In continuum mechanics we assume that the body possesses a 'specific internal energy' such that

$$E = \int_V \varepsilon dM = \int_V \rho\varepsilon dV, \qquad (2.2.6)$$

we also assume that the caloric equation of state has the form (Eringen 1961, 63)

$$\varepsilon = \hat{\varepsilon}(\eta,\ c_{\ell m}), \tag{2.2.7}$$

where $\hat{\varepsilon}$ is some prescribed function of the specific entropy η and the Cauchy deformation tensor $c_{\ell m}$ is defined as:

$$c_{\ell m} = X^L{}_{,\ell}\ X^L{}_{,m}. \tag{2.2.8}$$

By differentiating with respect to time, eq(2.2.7) we obtain

$$\dot{\varepsilon} = \left(\frac{\partial \varepsilon}{\partial \eta}\right)_{\underset{\sim}{c}} \dot{\eta} + \left(\frac{\partial \varepsilon}{\partial c_{\ell m}}\right)_{\eta} \dot{c}_{\ell m}. \tag{2.2.9}$$

In analogy with classical thermodynamics we set

$$\theta \equiv \left(\frac{\partial \hat{\varepsilon}}{\partial \eta}\right)_c ,$$

$$\gamma^{\ell m} \equiv \left(\frac{\partial \hat{\varepsilon}}{\partial c_{\ell m}}\right)_{\eta} , \tag{2.2.10}$$

and call θ, 'temperature' and $\gamma_{\ell m}$, the thermodynamic tension.

We know $\dot{c}_{\ell m} = -(c_{n\ell} v^n{}_{,m} + c_{nm} v^n{}_{,\ell})$, the proof of this is simple and is found in Eringen (1961). With this on hand we can rewrite eq. (2.2.9) as

$$\dot{\varepsilon} = \theta\dot{\eta} - 2\gamma^{\ell m} c_{n\ell} d^n{}_m - 2\gamma^{\ell m} c_{n\ell} w^n{}_m, \tag{2.2.12}$$

where $w^n{}_m$ is the spin tensor defined by

$$2w_{nm} = v_{n,m} - v_{m,n}. \tag{2.2.13}$$

Since the specific internal energy ε is an objective quantity we could show following Eringen (1961) that

$$\frac{\partial \varepsilon}{\partial c_{km}} c_{mn} = \frac{\partial \varepsilon}{\partial c_{nm}} c_{mk} \ . \tag{2.2.14}$$

The Cauchy deformation tensor $\underset{\sim}{c}$ is symmetric, which implies that γ^{km} is also symmetric. Using the above facts, in eq.(2.2.4) we obtain:

$$\rho\dot{\eta} = \frac{1}{\theta}(T_{km} + 2\rho\gamma_k{}^n c_{nm})d^{mk} - \left.\frac{q_k}{\theta}\right._{,k} - \frac{q^k\theta_{,k}}{\theta^2} + \frac{\rho Q}{\theta} \ , \tag{2.2.15}$$

Eq.(2.2.15) is known as the Entropy Production equation. The total entropy of the body B is given by

$$H = \int_V \rho\eta dV.$$

Eq.(2.2.15) can be rewritten as

$$\int_V \rho\eta dV + \int_V \left(\frac{q_k}{\theta}\right)_{,k} dV = \int_V \theta\Delta dV,$$

where $$\theta\Delta = (T_{km} + 2\rho\gamma_k{}^n c_{nm})d^{mk} - \frac{q^k\theta_{,k}}{\theta} + \rho Q. \tag{2.2.16}$$

Applying the Green-Gauss Theorem to eq.(2.2.16) we obtain

$$\dot{H} + \oint_S \frac{q^k}{\theta} ds_k = \int_V \theta\Delta dV \ . \tag{2.2.17}$$

Since $\theta > 0$, the Classius-Duhem inequality yields

$$\theta\Delta \geq 0. \tag{2.2.18}$$

This inequality provides some restrictive conditions on the constitutive equations.

From the caloric equations of state, eq.(2.2.7) and the definition of temperature eq.(2.2.10), we find that

$$\theta = \left(\frac{\partial \hat{\varepsilon}}{\partial \eta}\right)_{\underset{\sim}{c}} = f(\eta, \underset{\sim}{c}), \qquad (2.2.19)$$

which could be solved for η giving

$$\eta = \hat{\eta}(\theta, \underset{\sim}{c}). \qquad (2.2.19)$$

Substituting, eq.(2.2.19) in eq.(2.2.7) we obtain

$$\varepsilon = \hat{\varepsilon}[\eta(\theta, \underset{\sim}{c}), \underset{\sim}{c}] = \check{\varepsilon}(\underset{\sim}{\theta}, \underset{\sim}{c}), \qquad (2.2.20)$$

where $\check{\varepsilon}$ is some explicit function of θ and $\underset{\sim}{c}$, which is generally different from $\hat{\varepsilon}$.

From eq.(2.2.20) we obtain

$$\dot{\varepsilon} = \kappa\dot{\theta} + \lambda^{km}c_{km} , \qquad (2.2.21)$$

$$\kappa \equiv \left(\frac{\partial \check{\varepsilon}}{\partial \theta}\right)_{\underset{\sim}{c}}, \lambda^{km} \equiv \left(\frac{\partial \check{\varepsilon}}{\partial c_{km}}\right)_{\theta} ,$$

are respectively the 'specific heat' and a 'modified thermodynamic tension.' Substituting eq.(2.2.21) in eq.(2.2.4) we obtain

$$\rho\kappa\dot{\theta} = (T_{km} + 2\rho\lambda_k{}^n c_{nm})d^{mk} - q^k{}_{,k} + \rho Q. \qquad (2.2.22)$$

We now compare the thermodynamic coefficients κ and λ^{kn} with the corresponding coefficients in eq.(2.2.15). For this purpose we take the material derivative of eq.(2.2.21), obtaining:

$$\dot{\eta} = \left(\frac{\partial \hat{\eta}}{\partial \theta}\right)_{\underset{\sim}{c}}\dot{\theta} + \left(\frac{\partial \hat{\eta}}{\partial c_{km}}\right)\dot{c}_{km} . \qquad (2.2.23)$$

Substituting eq.(2.2.23) in eq.(2.2.9) we obtain the following relations:

$$\kappa = \theta\left(\frac{\partial\hat{\eta}}{\partial\theta}\right)_{\underset{\sim}{c}}, \quad \lambda_{km} = \gamma_{km} + \theta\left(\frac{\partial\hat{\eta}}{\partial c_{km}}\right)_{\theta} \qquad (2.2.24)$$

with θ and $\underset{\sim}{c}$ as independent variables. The heat conduction eq. (2.2.22) can now be written as:

$$\rho\kappa\dot{\theta} = [T_{km} + 2\rho\lambda_{k}{}^{n}c_{nm} + 2\rho\theta\left(\frac{\partial\hat{\eta}}{\partial c_{kn}}\right)_{\theta} c_{nm}]d^{mk} - q^{k}_{,k} + \rho Q. \qquad (2.2.25)$$

As a special case for Newtonian viscous fluids, the caloric equation of state takes the form

$$\varepsilon = \varepsilon(\eta, \frac{1}{\rho}) \qquad (2.2.26)$$

where $\frac{1}{\rho}$ is the specific volume. In this case the thermodynamic tension γ^{kn} is identified as the 'thermodynamic pressure' defined by:

$$\pi \equiv -\left(\frac{\partial\varepsilon}{\partial\nu}\right)_{\eta}, \qquad (2.2.27)$$

and the temperature

$$\theta \equiv \left(\frac{\partial\varepsilon}{\partial\eta}\right)_{\nu}. \qquad (2.2.27a)$$

The specific enthropy given by the heat conduction equation now becomes

$$\rho\kappa\dot{\theta} = \left\{T_{km} + \rho\left[\pi + \theta\left(\frac{\partial\pi}{\partial\theta}\right)_{\nu}\right]\delta_{km}\right\}d^{mk} - q^{k}_{,k} + \rho Q, \quad (2.2.28)$$

where δ_{km} is the usual Kronecker delta. The equation of continuity eq. (2.2.1) and the relation $\left(\frac{\partial\eta}{\partial\nu}\right)_{\theta} = -\left(\frac{\partial\pi}{\partial\theta}\right)_{\nu}$ have been used to obtain eq. (2.2.27).

For incompressible fluids where the equation of continuity becomes $d^{kk} = 0$ the heat conduction equation takes the final form:

$$\rho\kappa\dot{\theta} = T_{km}d^{mk} - q^{k}_{,k} + \rho Q. \qquad (2.2.29)$$

Now that we have developed all the necessary fundamental concepts we pass on to the development of a non-linear constitutive theory for thermo-viscoelastic materials.

2.3 Non-Linear Constitutive Theory of Thermo-viscoelasticity

In the last century, viscoelastic behavior of materials has been the subject of extensive studies by a number of workers. But it was not until the last two decades that these studies entered the non-linear realm. Although the desire for rigor motivated this extension, the failure of linear theories to explain such phenomena as the Poynting and Kelvin effects in elasticity was an important force which has brought accelerated developments in this field.

Before going into the basic concepts of thermo-viscoelasticity we give a brief account of existing constitutive equations of viscoelastic or general Maxwell-Voigt materials. Each of these theories has its own limitations. However, we discuss here only their limitations when used in problems involving thermal fields. For a detailed discussion of the limitations of these theories we refer the reader to Eringen (1962).

In the next paragraph, we list, for the sake of future reference, various phases of development of constitutive equations for non-linear materials.

Newtonian Fluids

$$\underset{\sim}{T} = (-p + \lambda\theta)\underset{\sim}{I} + 2\mu\underset{\sim}{D}, \qquad (2.3.1)$$

where $\underset{\sim}{I}$ = Identity tensor,

$\underset{\sim}{T}$ = Stress tensor,

μ, λ_1 = Coefficients of viscosity,

$\underset{\sim}{D}$ = Deformation-rate matrix,

θ = Dilatation

p = Hydrostatic pressure.

Eq.(2.3.1) successfully explains the behavior of some fluids, such as certain gases, water, alcohol and other similar fluids. But it fails to explain phenomena like the Merrington effect (the swelling of a fluid at the exit of a tube) Merrington(1943), and the Weissenberg effect (the climbing of a fluid on a rotating rod) Weissenberg (1947), both very common among industrial fluids such as high polymer solutions, pastes, paints, colloidal solutions, paper pulp, etc. These limitations give rise to the constitutive equations which follow, Coleman, Noll, and Markowitz (1965).

Reiner (1945) and Rivlin (1948) Fluids

$$\underset{\sim}{T} = -P\underset{\sim}{I} + \alpha_1 \underset{\sim}{D} + \alpha_2 \underset{\sim}{D}^2, \qquad (2.3.2)$$

where α_1 = Coefficient of viscosity,

α_2 = Coefficient of cross-viscosity,

$\underset{\sim}{T}$ = Stress tensor,

$\underset{\sim}{D}$ = Deformation-rate tensor,

P = hydrostatic pressure.

The above constitutive equation for the fluid is found to explain the normal stress effects mentioned above.

Oldroyd Fluids (1950)

$$(1 + \lambda_1 \frac{D}{Dt}) t'_{ij} - 2k_1 (d_{im} t'^{m}_{j} \; d_{jm} t'^{m}_{i})$$

$$= 2\mu (1 + \lambda_2 \frac{\partial}{\partial t}) d_{ij} - 8\mu k_2 \; d_{im} d^{m}{}_{j} \; ,$$

where λ_1 = relaxation-time constant

$$\frac{D}{Dt} A_{ij} = \frac{\partial}{\partial t} A_{ij} + A_{ij,m} v^m + A_{mj} v^m{}_{,i} + A_{im,j} \; , \qquad (2.3.4)$$

A_{ij} being a given tensor,

t'_{ij} = deviatoric part of the stress tensor,

k_1 and k_2 = arbitrary scalar constants,

λ_2 = retardation time constant.

Rivlin-Ericksen Fluids (1955)

$$T = a_0 \underset{\sim}{I} + \sum_{k=1}^{N} a_k(\pi_k + \pi_k^*), \tag{2.3.5}$$

where a_k $(k = 0, 1, \ldots, N)$ are unknown functions of the invariants of the kinematic tensors $\underset{\sim}{D}^{(1)}, \underset{\sim}{D}^{(2)}, \ldots, \underset{\sim}{D}^{(n)}$ which are defined as follows:

$$\underset{\sim}{D}^{(r)} = \| d_{ij}^{(r)} \| ,$$

$$d_{ij}^{(1)} = \frac{1}{2}(v_{i,j} + v_{j,i}) , \quad \text{and}$$

$$d_{ij}^{(r)} = \left(\frac{\partial}{\partial t} d_{ij}^{(n-1)} \right) + v^m d_{ij,m}^{(r-1)} + d_{im}^{(r-1)} v^m_{,j}$$

$$+ d_{im}^{(r-1)} v^m_{,i} \qquad (r \geq 2) \tag{2.3.5a}$$

and π_k are certain tensor products formed from the k kinematic matrices $\underset{\sim}{D}^{(1)}, \underset{\sim}{D}^{(2)}, \ldots, \underset{\sim}{D}^{(n)}$; π_k^* is the transpose of π_k.

Green-Rivlin Fluids (1957)

$$t_{ij} = \sum_{N=0}^{R} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \phi_{ijp_1q_1\ldots p_Nq_N}(t, \tau_1, \ldots, \tau_N)$$

$$\times\, g_{p_1q_1}(\tau_1)\ldots g_{p_Nq_N}(\tau_N)\, d\tau_1 \ldots d\tau_N, \tag{2.3.6}$$

where

the kernels $\phi_{ij \ldots p_N q_N}$ are continuous functions of the indicated arguments and also of $D^{(0)}_{pq}$, $D^{(1)}_{pq}$, ..., $D^{(\nu)}_{pq}$, defined in Rivlin-Ericksen's theory,

$$g_{pq}(\tau) \equiv x_{k,p}(\tau)x^{k}_{,q}(\tau)$$
$$t_{ij} \equiv \text{stress tensor.} \tag{2.3.7}$$

Noll Fluids (1958)

$$T^{1} = \underset{s=0}{\overset{\infty}{\mathcal{F}}}(\underset{\sim}{G}(s)), \tag{2.3.8}$$

where

$\mathcal{F}$ is the constitutive functional and $\underset{\sim}{G}(s)$ is the history of the relative deformation gradient.

Clearly these theories were not made to account for any thermal properties. So, when a problem involves a thermal gradient, the classical heat conduction equation has to be used. This leads to inaccuracies because there is no account of any interactions between the thermal field and other mechanical fields.

Some workers have given non-linear theories of thermoelasticity, Green and Adkins (1961), Green, England and Flavin, (1961). Earlier, Truesdell (1951), had given a theory of thermoviscous materials. But no work on a non-linear theory of thermo-viscoelasticity has appeared in the literature until 1963 when Eringen and Koh published their general non-linear theory of thermo-viscoelasticity, [Eringen, Koh (1963)].

If a viscoelastic material is characterized by an expression of the stress components T_{ij} as a polynomial in the gradients of the displacement, velocity, acceleration, ..., (n-1)th acceleration, then according to Rivlin and Ericksen (1955), each of these stress components can be expressed as a polynomial in the components of the kinematic tensors:

$$c_{ij}^{-1} \equiv x_{i,}{}^{K} x_{j,K} \quad , \tag{2.3.9}$$

where $\underset{\sim}{c}^{-1}$ is known as Finger's deformation tensor, and the Rivlin-Ericksen tensors:

$$a_{ij}^{(r)} \equiv \frac{\partial a_{ij}^{(r-1)}}{\partial t} + a_{ij,m}^{(r-1)} v^{m} + a_{mi}^{(r-1)} v^{m}_{,j}$$

$$+ a_{mj} v^{m}_{,i} \quad , \qquad n > r \geq 2 \ . \tag{2.3.10}$$

Furthermore, the stress matrix $\underset{\sim}{T} \equiv \|T_{ij}\|$ is a matrix polynomial in the matrix variables

$$\underset{\sim}{c}^{-1} \equiv \|c_{ij}^{-1}\| , \|\underset{\sim}{a}^{(1)}\| \equiv \|a_{ij}^{(1)}\| , \ \ldots , \ \underset{\sim}{a}^{(n)} \ \|a_{ij}^{(n)}\| , \tag{2.3.11}$$

with coefficients that are scalar polynomials in the simultaneous invariants of these matrix variables. Following Eringen (1963) we consider that a viscoelastic material in a thermal state is characterized by two sets of constitutive equations, one for stress and one for heat flux. The stress tensor $\underset{\sim}{T}$ and the heat flux bi-vector $\underset{\sim}{h}$ are polynomial functions of the kinematic tensors $\underset{\sim}{c}^{-1}$, $\underset{\sim}{d}$, the thermal gradiant bi-vector $\underset{\sim}{b}$, the density ρ, and the temperature θ where $\underset{\sim}{c}^{-1}$ and $\underset{\sim}{d}$ are respectively defined by eq.(2.3.9) and eq.(2.3.10) and the following:

$$\underset{\sim}{b} \equiv \| b_{ij} \| \equiv \| e^{ijk} \, \theta_{,k} \| \, , \tag{2.3.12}$$

$$\underset{\sim}{h} \equiv \| h_{ij} \| \equiv \| e^{ijk} \, q_k \| \, , \tag{2.3.13}$$

where e^{ijk} is the permutation symbol.

So in effect we have:

$$\underset{\sim}{T} = \hat{\underset{\sim}{f}}(\underset{\sim}{c}^{-1}, \underset{\sim}{d}, \underset{\sim}{b}; \rho, \theta) \, , \tag{2.3.14}$$

and

$$\underset{\sim}{h} = \check{\underset{\sim}{f}}(\underset{\sim}{c}^{-1}, \underset{\sim}{d}, \underset{\sim}{b}; \rho, \theta) . \tag{2.3.15}$$

Note that this satisfies the principles of determinism and equipresence discussed in section 2 of this chapter. As for the axiom of material objectivity, it can be easily shown that $\underset{\sim}{c}^{-1}$, $\underset{\sim}{d}$ and $\underset{\sim}{b}$ are all objective. This implies that $\underset{\sim}{T}$ and $\underset{\sim}{h}$ are invariant for every rigid motion of the spatial frame, i.e., T_{ij} and h_{ij} are both hemitropic functions of their arguments, whence $\underset{\sim}{T}$ and $\underset{\sim}{h}$ are objective.

The details for the reduction of the matrix polynomials in two symmetric matrices ($\underset{\sim}{c}^{-1}$ and $\underset{\sim}{d}$) and one antisymmetric matrix $\underset{\sim}{b}$ to represent a symmetric matrix $\underset{\sim}{T}$ and an antisymmetric matrix $\underset{\sim}{h}$ are lengthy and will not be given here but we refer the reader to Eringen (1963) for them.

The final results are:

$$
\begin{aligned}
\mathbf{T} = {} & \alpha_1 \mathbf{I} + \alpha_2 \mathbf{c}^{-1} + \alpha_3 \mathbf{d} + \alpha_4 \mathbf{c}^{-2} + \alpha_5 \mathbf{d}^2 + \alpha_6 \mathbf{b}^2 \\
& + \alpha_7 (\mathbf{c}^{-1}\mathbf{d} + \mathbf{d}\mathbf{c}^{-1}) + \alpha_8 (\mathbf{d}\mathbf{b} - \mathbf{b}\mathbf{d}) + \alpha_9 (\mathbf{c}^{-1}\mathbf{b} - \mathbf{b}\mathbf{c}^{-1}) \\
& + \alpha_{10} (\mathbf{c}^{-1}\mathbf{d}^2 + \mathbf{d}^2\mathbf{c}^{-1}) + \alpha_{11} (\mathbf{d}\mathbf{c}^{-2} + \mathbf{c}^{-2}\mathbf{d}) \\
& + \alpha_{12} (\mathbf{d}\mathbf{b}^2 + \mathbf{b}^2\mathbf{d}) + \alpha_{13} (\mathbf{c}^{-1}\mathbf{b}^2 + \mathbf{b}^2\mathbf{c}^{-1}) + \alpha_{14} (\mathbf{b}\mathbf{c}^{-2} - \mathbf{c}^{-2}\mathbf{b}) \\
& + \alpha_{15} (\mathbf{b}\mathbf{d}^2 - \mathbf{d}^2\mathbf{b}) + \alpha_{16} (\mathbf{c}^{-2}\mathbf{d}^2 + \mathbf{d}^2\mathbf{c}^{-2}) + \alpha_{17} (\mathbf{d}^2\mathbf{b}^2 + \mathbf{b}^2\mathbf{d}^2) \\
& + \alpha_{18} (\mathbf{b}^2\mathbf{c}^{-2} + \mathbf{c}^{-2}\mathbf{b}^2) + \alpha_{19} (\mathbf{c}^{-1}\mathbf{b}\mathbf{c}^{-2} - \mathbf{c}^{-2}\mathbf{b}\mathbf{c}^{-1}) \\
& + \alpha_{20} (\mathbf{d}\mathbf{b}\mathbf{d}^2 - \mathbf{d}^2\mathbf{b}\mathbf{d}) + \alpha_{21} (\mathbf{b}\mathbf{c}^{-1}\mathbf{b}^2 - \mathbf{b}^2\mathbf{c}^{-1}\mathbf{b}) \\
& + \alpha_{22} (\mathbf{b}\mathbf{d}\mathbf{b}^2 - \mathbf{b}^2\mathbf{d}\mathbf{b}) + \alpha_{23} (\mathbf{b}\mathbf{c}^{-2}\mathbf{b}^2 - \mathbf{b}^2\mathbf{c}^{-2}\mathbf{b}) \\
& + \alpha_{24} (\mathbf{b}\mathbf{d}^2\mathbf{b}^2 - \mathbf{b}^2\mathbf{d}^2\mathbf{b}) + \alpha_{25} (\mathbf{c}^{-1}\mathbf{d}\mathbf{b} - \mathbf{b}\mathbf{d}\mathbf{c}^{-1}) \\
& + \alpha_{26} (\mathbf{d}\mathbf{c}^{-1}\mathbf{b} - \mathbf{b}\mathbf{c}^{-1}\mathbf{d}) + \alpha_{27} (\mathbf{d}\mathbf{b}\mathbf{c}^{-1} - \mathbf{c}^{-1}\mathbf{b}\mathbf{d}) \\
& + \alpha_{28} (\mathbf{c}^{-2}\mathbf{d}\mathbf{b} - \mathbf{b}\mathbf{d}\mathbf{c}^{2}) + \alpha_{29} (\mathbf{c}^{-2}\mathbf{b}\mathbf{d} - \mathbf{d}\mathbf{b}\mathbf{c}^{-2}) \\
& + \alpha_{30} (\mathbf{d}^2\mathbf{b}\mathbf{c}^{-1} - \mathbf{c}^{-1}\mathbf{b}\mathbf{d}^2) + \alpha_{31} (\mathbf{d}^2\mathbf{c}^{-1}\mathbf{b} - \mathbf{b}\mathbf{c}^{-1}\mathbf{d}^2) \\
& + \alpha_{32} (\mathbf{b}^2\mathbf{c}^{-1}\mathbf{d} + \mathbf{d}\mathbf{c}^{-1}\mathbf{b}^2) + \alpha_{33} (\mathbf{b}^2\mathbf{d}\mathbf{c}^{-1} + \mathbf{c}^{-1}\mathbf{d}\mathbf{b}^2) \\
& + \alpha_{34} (\mathbf{c}^{-1}\mathbf{d}^2\mathbf{b} - \mathbf{b}\mathbf{d}^2\mathbf{c}^{-1}) + \alpha_{35} (\mathbf{d}\mathbf{c}^{-2}\mathbf{b} - \mathbf{b}\mathbf{c}^{-2}\mathbf{d}) \\
& + \alpha_{36} (\mathbf{c}^{-2}\mathbf{d}^2\mathbf{b} - \mathbf{b}\mathbf{d}^2\mathbf{c}^{-2}) + \alpha_{37} (\mathbf{d}^2\mathbf{c}^{-2}\mathbf{b} - \mathbf{b}\mathbf{c}^{-2}\mathbf{d}^2)
\end{aligned}
$$

$$\begin{aligned}
&+ \alpha_{38}(\underset{\sim}{d}^{2}\underset{\sim}{b}^{2}\underset{\sim}{c}^{-1} + \underset{\sim}{c}^{-1}\underset{\sim}{b}^{2}\underset{\sim}{d}^{2}) + \alpha_{39}(\underset{\sim}{b}^{2}\underset{\sim}{d}^{2}\underset{\sim}{c}^{-1} + \underset{\sim}{c}^{-1}\underset{\sim}{d}^{2}\underset{\sim}{b}^{2}) \\
&+ \alpha_{40}(\underset{\sim}{b}^{2}\underset{\sim}{c}^{-2}\underset{\sim}{d} + \underset{\sim}{d}\underset{\sim}{c}^{-2}\underset{\sim}{b}^{2}) + \alpha_{41}(\underset{\sim}{c}^{-2}\underset{\sim}{b}^{2}\underset{\sim}{d} + \underset{\sim}{d}\underset{\sim}{b}^{2}\underset{\sim}{c}^{-2}) \\
&+ \alpha_{42}(\underset{\sim}{c}^{-1}\underset{\sim}{d}\underset{\sim}{b}\underset{\sim}{c}^{-2} - \underset{\sim}{c}^{-2}\underset{\sim}{b}\underset{\sim}{d}\underset{\sim}{c}^{-1}) + \alpha_{43}(\underset{\sim}{c}^{-1}\underset{\sim}{b}\underset{\sim}{d}\underset{\sim}{c}^{-2} - \underset{\sim}{c}^{-2}\underset{\sim}{d}\underset{\sim}{b}\underset{\sim}{c}^{-1}) \\
&+ \alpha_{44}(\underset{\sim}{d}\underset{\sim}{b}\underset{\sim}{c}^{-1}\underset{\sim}{d}^{2} - \underset{\sim}{d}^{2}\underset{\sim}{c}^{-1}\underset{\sim}{b}\underset{\sim}{d}) + \alpha_{45}(\underset{\sim}{d}\underset{\sim}{c}^{-1}\underset{\sim}{b}\underset{\sim}{d}^{2} - \underset{\sim}{d}^{2}\underset{\sim}{b}\underset{\sim}{c}^{-1}\underset{\sim}{d}) \\
&+ \alpha_{46}(\underset{\sim}{b}\underset{\sim}{c}^{-1}\underset{\sim}{d}\underset{\sim}{b}^{2} - \underset{\sim}{b}^{2}\underset{\sim}{d}\underset{\sim}{c}^{-1}\underset{\sim}{b}) + \alpha_{47}(\underset{\sim}{b}\underset{\sim}{d}\underset{\sim}{c}^{-1}\underset{\sim}{b}^{2} - \underset{\sim}{b}^{2}\underset{\sim}{c}^{-1}\underset{\sim}{d}\underset{\sim}{b})
\end{aligned} \tag{2.3.16}$$

and

$$\begin{aligned}
h = {} & \beta_{1}\underset{\sim}{b} + \beta_{2}(\underset{\sim}{c}^{-1}\underset{\sim}{d} - \underset{\sim}{d}\underset{\sim}{c}^{-1}) + \beta_{3}(\underset{\sim}{b}\underset{\sim}{d} + \underset{\sim}{d}\underset{\sim}{b}) + \beta_{4}(\underset{\sim}{b}\underset{\sim}{c}^{-1} + \underset{\sim}{c}^{-1}\underset{\sim}{b}) \\
&+ \beta_{5}(\underset{\sim}{c}^{-1}\underset{\sim}{d}^{2} - \underset{\sim}{d}^{2}\underset{\sim}{c}^{-1}) + \beta_{6}(\underset{\sim}{d}\underset{\sim}{b}^{2} - \underset{\sim}{b}^{2}\underset{\sim}{d}) + \beta_{7}(\underset{\sim}{c}^{-1}\underset{\sim}{b}^{2} - \underset{\sim}{b}^{2}\underset{\sim}{c}^{-1}) \\
&+ \beta_{8}(\underset{\sim}{b}\underset{\sim}{c}^{-2} + \underset{\sim}{c}^{-2}\underset{\sim}{b}) + \beta_{9}(\underset{\sim}{b}\underset{\sim}{d}^{2} + \underset{\sim}{d}^{2}\underset{\sim}{b}) + \beta_{10}(\underset{\sim}{d}\underset{\sim}{c}^{-2} - \underset{\sim}{c}^{2}\underset{\sim}{d}) \\
&+ \beta_{11}(\underset{\sim}{c}^{-2}\underset{\sim}{d}^{2} - \underset{\sim}{d}^{2}\underset{\sim}{c}^{-2}) + \beta_{12}(\underset{\sim}{d}^{2}\underset{\sim}{b}^{2} - \underset{\sim}{b}^{2}\underset{\sim}{d}^{2}) \\
&+ \beta_{13}(\underset{\sim}{c}^{-2}\underset{\sim}{b}^{2} - \underset{\sim}{b}^{2}\underset{\sim}{c}^{-2}) + \beta_{14}(\underset{\sim}{c}^{-1}\underset{\sim}{d}\underset{\sim}{c}^{-2} - \underset{\sim}{c}^{-2}\underset{\sim}{d}\underset{\sim}{c}^{-1}) \\
&+ \beta_{15}(\underset{\sim}{d}\underset{\sim}{c}^{-1}\underset{\sim}{d}^{2} - \underset{\sim}{d}^{2}\underset{\sim}{c}^{-1}\underset{\sim}{d}) + \beta_{16}(\underset{\sim}{c}^{-1}\underset{\sim}{d}^{2}\underset{\sim}{c}^{-2} - \underset{\sim}{c}^{-2}\underset{\sim}{d}^{2}\underset{\sim}{c}^{-1}) \\
&+ \beta_{17}(\underset{\sim}{d}\underset{\sim}{c}^{-2}\underset{\sim}{d}^{2} - \underset{\sim}{d}^{2}\underset{\sim}{c}^{-2}\underset{\sim}{d}) + \beta_{18}(\underset{\sim}{c}^{-1}\underset{\sim}{b}^{2}\underset{\sim}{c}^{-2} - \underset{\sim}{c}^{-2}\underset{\sim}{b}^{2}\underset{\sim}{c}^{-1}) \\
&+ \beta_{19}(\underset{\sim}{d}\underset{\sim}{b}^{2}\underset{\sim}{d}^{2} - \underset{\sim}{d}^{2}\underset{\sim}{b}^{2}\underset{\sim}{d}) + \beta_{20}(\underset{\sim}{c}^{-1}\underset{\sim}{d}\underset{\sim}{b} + \underset{\sim}{b}\underset{\sim}{d}\underset{\sim}{c}^{-1}) \\
&+ \beta_{21}(\underset{\sim}{d}\underset{\sim}{c}^{-1}\underset{\sim}{b} + \underset{\sim}{b}\underset{\sim}{c}^{-1}\underset{\sim}{d}) + \beta_{22}(\underset{\sim}{c}^{-2}\underset{\sim}{d}\underset{\sim}{b} + \underset{\sim}{b}\underset{\sim}{d}\underset{\sim}{c}^{-2}) \\
&+ \beta_{23}(\underset{\sim}{d}^{2}\underset{\sim}{c}^{-1}\underset{\sim}{b} + \underset{\sim}{b}\underset{\sim}{c}^{-1}\underset{\sim}{d}^{2}) + \beta_{24}(\underset{\sim}{b}^{2}\underset{\sim}{c}^{-1}\underset{\sim}{d} - \underset{\sim}{d}\underset{\sim}{c}^{-1}\underset{\sim}{b}^{2}) \\
&+ \beta_{25}(\underset{\sim}{b}^{2}\underset{\sim}{d}\underset{\sim}{c}^{-1} - \underset{\sim}{c}^{-1}\underset{\sim}{d}\underset{\sim}{b}^{2}) + \beta_{26}(\underset{\sim}{c}^{-1}\underset{\sim}{d}^{2}\underset{\sim}{b} + \underset{\sim}{b}\underset{\sim}{d}^{2}\underset{\sim}{c}^{-1})
\end{aligned}$$

$$+ \beta_{27}(\mathbf{d}\mathbf{c}^{-2}\mathbf{b} + \mathbf{b}\mathbf{c}^{-2}\mathbf{d}) + \beta_{28}(\mathbf{c}^{-1}\mathbf{b}^2\mathbf{d} - \mathbf{d}\mathbf{b}^2\mathbf{c}^{-1})$$

$$+ \beta_{29}(\mathbf{c}^{-2}\mathbf{d}^2\mathbf{b} + \mathbf{b}\mathbf{d}^2\mathbf{c}^{-2}) + \beta_{30}(\mathbf{d}^2\mathbf{c}^{-2}\mathbf{b} + \mathbf{b}\mathbf{c}^{-2}\mathbf{d}^2)$$

$$+ \beta_{31}(\mathbf{d}^2\mathbf{b}^2\mathbf{c}^{-1} - \mathbf{c}^{-1}\mathbf{b}^2\mathbf{d}^2) + \beta_{32}(\mathbf{b}^2\mathbf{d}^2\mathbf{c}^{-1} - \mathbf{c}^{-1}\mathbf{d}^2\mathbf{b}^2)$$

$$+ \beta_{33}(\mathbf{b}^2\mathbf{c}^{-2}\mathbf{d} - \mathbf{d}\mathbf{c}^{-2}\mathbf{b}^2) + \beta_{34}(\mathbf{c}^{-2}\mathbf{b}^2\mathbf{d} - \mathbf{d}\mathbf{b}^2\mathbf{c}^{-2})$$

$$+ \beta_{35}(\mathbf{c}^{-1}\mathbf{b}\mathbf{d}\mathbf{c}^{-2} + \mathbf{c}^{-2}\mathbf{d}\mathbf{b}\mathbf{c}^{-1}) + \beta_{36}(\mathbf{d}\mathbf{c}^{-1}\mathbf{b}\mathbf{d}^2 + \mathbf{d}^2\mathbf{b}\mathbf{c}^{-1}\mathbf{d})$$

$$+ \beta_{37}(\mathbf{b}\mathbf{d}\mathbf{c}^{-1}\mathbf{b}^2 + \mathbf{b}^2\mathbf{c}^{-1}\mathbf{d}\mathbf{b}). \qquad (2.3.17)$$

The constitutive coefficients α_i and β_i are polynomials of the following invariants:

$$\text{tr}\,\mathbf{c}^{-1},\ \text{tr}\,\mathbf{d},\ \text{tr}\,\mathbf{c}^{-1}\mathbf{d},$$

$$\text{tr}\,\mathbf{c}^{-2},\ \text{tr}\,\mathbf{d}^2,\ \text{tr}\,\mathbf{c}^{-3},\ \text{tr}\,\mathbf{d}^3,$$

$$\text{tr}\,\mathbf{c}^{-1}\mathbf{d}^2,\ \text{tr}\,\mathbf{c}^{-1}\mathbf{b}^2,\ \text{tr}\,\mathbf{d}\mathbf{c}^{-2},\ \text{tr}\,\mathbf{d}\mathbf{b}^2,$$

$$\text{tr}\,\mathbf{c}^{-2}\mathbf{d}^2,\ \text{tr}\,\mathbf{d}^2\mathbf{b}^2,\ \text{tr}\,\mathbf{b}^2\mathbf{c}^{-2},$$

$$\text{tr}\,\mathbf{b}\mathbf{c}^{-1}\mathbf{d}^2,\ \text{tr}\,\mathbf{c}^{-1}\mathbf{d}\mathbf{d}^2,\ \text{tr}\,\mathbf{d}\mathbf{b}\mathbf{c}^{-2},\ \text{tr}\,\mathbf{c}^{-1}\mathbf{b}\mathbf{d},$$

$$\text{tr}\,\mathbf{b}\mathbf{c}^{-2}\mathbf{d}^2,\ \text{tr}\,\mathbf{c}^{-1}\mathbf{d}^2\mathbf{b}^2,\ \text{tr}\,\mathbf{d}\mathbf{b}^2\mathbf{c}^{-2},$$

$$\text{tr}\,\mathbf{d}\mathbf{c}^{-1}\mathbf{b}\mathbf{c}^{-2},\ \text{tr}\,\mathbf{c}^{-1}\mathbf{d}\mathbf{b}\mathbf{d}^2,\ \text{tr}\,\mathbf{d}\mathbf{b}\mathbf{c}^{-1}\mathbf{b}^2,$$

$$\text{tr}\,\mathbf{c}^{-1}\mathbf{b}\mathbf{c}^{-2}\mathbf{b}^2,\ \text{tr}\,\mathbf{b}\mathbf{d}\mathbf{b}^2\mathbf{d}^2,\ \text{tr}\,\mathbf{b}\mathbf{d}\mathbf{c}^{-2}\mathbf{d}^2,$$

$$\text{tr}\,\mathbf{c}^{-1}\mathbf{b}\mathbf{d}^2\mathbf{b}^2,\ \text{tr}\,\mathbf{b}\mathbf{c}^{-1}\mathbf{d}^2\mathbf{c}^2,\ \text{tr}\,\mathbf{d}\mathbf{b}\mathbf{c}^{-2}\mathbf{b}^2\ . \qquad (2.3.18)$$

With secondary coefficients in general being functions of ρ and θ.

From these general results the constitutive equations for a large class of special materials can be obtained.

The constitutive equations for thermo-viscous and thermoelastic fluids are derivable without much difficulty. But these two fluids are limiting cases of a large class of fluids characterized by equations (2.3.16) and (2.3.17). Between these two cases, there lies a wide spectrum of materials which exhibit both elastic and dissipative (viscous) characteristics. One may classify these thermo-viscoelastic materials into different groups in various manners of classification depending on the extent of fluidity or elasticity of the material.

A method of defining special thermo-viscoelastic materials may be achieved by classifying the materials according to the combined degree of the independent variables $\underset{\sim}{c}^{-1}$, $\underset{\sim}{d}$ and $\underset{\sim}{b}$ appearing in each term of the constitutive equations. Let the degrees of $\underset{\sim}{c}^{-1}$, $\underset{\sim}{d}$ and $\underset{\sim}{b}$ appearing in a term be denoted respectively by M, N and P; then the combined degrees of that term is $|M + N + P|$.

To illustrate this, let's consider the case of the zero order theory.

<u>Zero Order Theory</u>: For this case $M = N = P = 0$. Therefore, all the constitutive coefficients, with the exception of α_1, are equal to zero. The constitutive equations are then:

$$\underset{\sim}{T} = \alpha_1 \underset{\sim}{I}$$

$$\underset{\sim}{h} = \underset{\sim}{0}$$

where α_1 is a function of ρ and θ. Clearly the materials characterized by these two equations are the ideal or non-viscous fluids, where α_1 is equal to the negative of the hydrostatic pressure.

Second Order Theory for the Rivlin-Ericksen Visco-elastic Materials: This is a class of materials which has been studied in recent years. We have mentioned about this class of materials in section 2 of this chapter. The stress tensor is assumed to be a matrix polynomial in the kinematic acceleration matrices $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$. However, for large n's the constitutive equations become too unwieldy. To avoid this we will make the assumption that $\underset{\sim}{T} = \underset{\sim}{f}(\underset{\sim}{a}_1, \underset{\sim}{a}_2)$. A strong motive for this assumption is that for viscometric flows, quite accidentally, the Rivlin-Ericksen tensors vanish for $n \geq 3$.

Generalization of the Rivlin-Ericksen viscoelastic theory into the thermo-viscoelastic theory following Eringen (1963) gives us the following modification:

$$\underset{\sim}{T} = \hat{\underset{\sim}{f}}(\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{b}; \rho, \theta). \qquad (2.3.19)$$

Since $\underset{\sim}{T}$ is a hemitropic function of the symmetric tensors $\underset{\sim}{a}_1, \underset{\sim}{a}_2$ and one anti-symmetric tensor $\underset{\sim}{b}$ we simply apply the results given by eqs.(2.3.16) and (2.3.17) by changing $\underset{\sim}{c}^{-1}$ into $\underset{\sim}{a}_1$ and $\underset{\sim}{d}$ into $\underset{\sim}{a}_2$. Then we apply the 'combined degrees' approximation technique to it with $\max |M + N + P| = 2$. We obtain the second order approximation of the constitutive equations of the general Rivlin-Ericksen fluid in viscometric flows exhibiting thermal effects. These equations are:

$$\mathbf{T} = \alpha_1 \mathbf{I} - \alpha_2 \mathbf{a}_1 + \alpha_3 \mathbf{a}_2 + \alpha_4 (\mathbf{a}_1)^2 + \alpha_5 (\mathbf{a}_2)^2 + \alpha_6 \mathbf{b}^2$$

$$+ \alpha_7 (\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1) + \alpha_8 (\mathbf{a}_2\mathbf{b} - \mathbf{b}\mathbf{a}_2) + \alpha_9 (\mathbf{a}_1\mathbf{b} - \mathbf{b}\mathbf{a}_1), \tag{2.3.20}$$

and

$$\mathbf{h} = \beta_1 \mathbf{b} + \beta_2 (\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}_2\mathbf{a}) + \beta_3 (\mathbf{b}\mathbf{a}_2 + \mathbf{a}_2\mathbf{b})$$

$$+ \beta_4 (\mathbf{b}\mathbf{a}_1 + \mathbf{a}_1\mathbf{b}), \tag{2.3.21}$$

where the coefficients α_i and β_i are polynomials in the invariants:

$$\mathrm{tr}\, \mathbf{a}_1,\ \mathrm{tr}\, \mathbf{a}_2,\ \mathrm{tr}\, \mathbf{a}_1\mathbf{a}_2,\ \mathrm{tr}\, (\mathbf{a}_1)^2$$

$$\mathrm{tr}\, (\mathbf{a}_2)^2,\ \mathrm{tr}\, \mathbf{b}^2.$$

These coefficients can be explicitly expressed as follows:

$$\alpha_1 = \alpha_{1000} + \alpha_{1100}\, \mathrm{tr}\, \mathbf{a}_1 + \alpha_{1010}\, \mathrm{tr}\, \mathbf{a}_2$$

$$+ \alpha_{1110}\, \mathrm{tr}\, \mathbf{a}_1\mathbf{a}_2 + \alpha^*_{1110}\, \mathrm{tr}\, \mathbf{a}_1\, \mathrm{tr}\, \mathbf{a}_2$$

$$+ \alpha_{1200}\, \mathrm{tr}\, (\mathbf{a}_1)^2 + \alpha^*_{1200}\, \mathrm{tr}\, (\mathbf{a}_1)^2 + \alpha_{1020}\, \mathrm{tr}\, (\mathbf{a}_2)^2$$

$$+ \alpha^*_{1020}\, \mathrm{tr}\, (\mathbf{a}_2)^2 + \alpha_{1002}\, \mathrm{tr}\, \mathbf{b}^2$$

$$\alpha_2 = \alpha_{2100} + \alpha_{2200}\, \mathrm{tr}\, \mathbf{a}_1 + \alpha_{2110}\, \mathrm{tr}\, \mathbf{a}_2$$

$$\alpha_3 = \alpha_{3010} + \alpha_{3020} \operatorname{tr} \underset{\sim}{a}_2 + \alpha_{3110} \operatorname{tr} \underset{\sim}{a}_1, \qquad (2.3.22)$$

$$\beta_1 = \beta_{1001} + \beta_{1101} \operatorname{tr} \underset{\sim}{a}_1 + \beta_{1011} \operatorname{tr} \underset{\sim}{a}_2,$$

with all the remaining coefficients α_i and β_i and all the secondary coefficients α_{irst} and β_{irst} being in general functions of ρ and θ.

The notations α_{irst}, β_{irst} are explained as follows: the first index $i = 1, 2, \ldots$ in these coefficients corresponds to the subscript of the primary coefficient α_i or β_i and the succeeding three indexes $r, s, t = 0, 1, 2, \ldots$ denote respectively the partial degrees of $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$, $\underset{\sim}{b}$, of the particular terms (matrix products) whose coefficient is α_{irst} or β_{irst}.

Using the above constitutive theory, Eringen (1963) solved the problem of a simple shearing flow of a thermoviscoelastic fluid.

In the next chapter, we apply this constitutive theory for the study of stability of thermoviscoelastic fluid flows between two rotating coaxial circular cylinders.

III. STABILITY OF THERMO-VISCOELASTIC FLUID FLOWS BETWEEN TWO ROTATING COAXIAL CYLINDERS

3.1 Introduction

In this chapter we formulate the stability problem physically and mathematically. In the course of solving the stability problem, we solve the steady state Couette flow for incompressible thermo-viscoelastic fluids. Also, we present the general equations in the non-dimensional form for the perturbed Couette flow. Before we pass on to the final solution of the stability problem in Chapter IV we present the plane layer approximation technique which is commonly used in the case of a narrow gap between two cylinders.

3.2 Formulation of the Problem

We consider the stability of incompressible laminar flow of a thermo-viscoelastic flow between two rotating coaxial cylinders with radii R_1 and R_2 $(R_2 > R_1)$. The inner and outer cylinders are maintained at constant temperature θ_1 and θ_2 respectively $(\theta_2 > \theta_1)$. We first investigate a steady state flow under the above conditions. Next, we investigate the influence of small periodic disturbances superposed on the steady flow.

We introduce a cylindrical coordinate system (r, ϕ, z) where z is chosen along the common axis of the cylinders, and r and ϕ are chosen as the radial and azimuthal coordinates respectively. The velocity components are u, v and w, along the radial, tangential and axial directions respectively.

The fluid dynamical equations of incompressible thermo-viscoelastic liquids are:

$$\rho \frac{Dv^i}{Dt} = T^{ij}_{,j} + \rho f^i \qquad (3.2.1)$$

and

$$\rho (v^i)_{,i} = 0, \qquad (3.2.2)$$

and the equation of heat conduction is:

$$\rho\kappa\dot{\theta} = T^{\ell m} d_{m\ell} - q^i_{,i} + \rho Q,$$

where ρ and κ are the mass density and the specific heat respectively. The external forces are represented by $\underset{\sim}{f}$ and the energy supply by Q and

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

We will adopt the following constitutive equations for the stress tensor $\underset{\sim}{T}$ and the heat flux bi-vector $\underset{\sim}{h}$ respectively:

$$\underset{\sim}{T} = \alpha_1 \underset{\sim}{I} + \alpha_2 \overset{1}{\underset{\sim}{a}} + \alpha_3 \overset{2}{\underset{\sim}{a}} + \alpha_4 \underset{\sim}{b} + \alpha_5 (\overset{1}{\underset{\sim}{a}}\underset{\sim}{b} - \underset{\sim}{b}\overset{1}{\underset{\sim}{a}}) + \alpha_6 (\overset{2}{\underset{\sim}{a}}\underset{\sim}{b} - \underset{\sim}{b}\overset{2}{\underset{\sim}{a}}), \qquad (3.2.3)$$

$$h = \beta_1 \underset{\sim}{b} + \beta_2 (\underset{\sim}{b}\overset{1}{\underset{\sim}{a}} + \overset{1}{\underset{\sim}{a}}\underset{\sim}{b}) + \beta_3 (\underset{\sim}{b}\overset{2}{\underset{\sim}{a}} + \overset{2}{\underset{\sim}{a}}\underset{\sim}{b}), \qquad (3.2.4)$$

where

$$\underset{\sim}{b} = \| e_{ijk}\theta,^{k} \|,$$

$$\underset{\sim}{h} = \| e_{ijk} q^{k} \|$$

are temperature gradient bivector and heat flux bivector respectively and

$$\overset{1}{a}_{ij} = 2d_{ij},$$

$$\overset{2}{a}_{ij} = a_{i,j} + a_{j,i} + 2v_{m,i}v^{m}_{,m},$$

where $\underset{\sim}{a}$ is the acceleration vector.

The Equations of Motion

The equations of motion can be written in physical components in cylindrical coordinates:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}$$

$$= \frac{1}{\rho}\left[\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\phi}}{\partial \phi} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\phi\phi}}{r}\right] + f_r, \qquad (3.2.5)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + \frac{uv}{r}$$

$$= \frac{1}{\rho}\left[\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{\phi z}}{\partial z} + \frac{2}{r}T_{\phi r}\right] + f_\phi, \qquad (3.2.6)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial z}$$

$$= \frac{1}{\rho}\left[\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r}\right] + f_z, \qquad (3.2.7)$$

and

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0, \qquad (3.2.8)$$

where

$$T_{rr} = \alpha_1 + \alpha_2\left[2\frac{\partial u}{\partial r}\right] + 2\alpha_3\left[\frac{\partial^2 u}{\partial t\partial r} + 2\left(\frac{\partial u}{\partial r}\right)^2 + u\frac{\partial^2 u}{\partial r^2}\right.$$

$$+ \frac{v}{r}\frac{\partial^2 u}{\partial r\partial\phi} - \frac{v}{r^2}\frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial r\partial z} - 2\frac{v}{r}\frac{\partial v}{\partial r}$$

$$\left. + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right] + \alpha_4\left[\left(\frac{\partial\theta}{\partial\phi}\right)^2 - \frac{1}{r^2}\left(\frac{\partial\theta}{\partial z}\right)^2\right]$$

$$+ \alpha_5\left[\frac{\partial\theta}{\partial\phi}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) - \frac{2}{r}\frac{\partial\theta}{\partial z}\left(\frac{1}{r}\frac{\partial u}{\partial\phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)\right]$$

$$+ 2\alpha_6\left[\frac{\partial\theta}{\partial\phi}\left(\frac{\partial^2 u}{\partial t\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial r} + u\frac{\partial^2 u}{\partial z\partial r} + \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial u}{\partial\phi}\right.\right.$$

$$+ \frac{v}{r}\frac{\partial^2 u}{\partial z\partial\phi} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial z^2} - \frac{1}{r}\frac{\partial v^2}{\partial z} + \frac{\partial^2 w}{\partial t\partial r} + \frac{\partial u}{\partial r}\frac{\partial w}{\partial r}$$

$$+ u\frac{\partial^2 w}{\partial r^2} + v\frac{\partial^2 w}{\partial r\partial\phi} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial w}{\partial\phi} - \frac{v}{r^2}\frac{\partial w}{\partial\phi} + \frac{\partial w}{\partial\phi} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial z}$$

$$\left. + 2\frac{\partial w}{\partial r}\frac{\partial w}{\partial z} - 2v\frac{\partial v}{\partial z}\right) - \left(\frac{1}{r}\frac{\partial^2 u}{\partial t\partial\phi} + \frac{1}{r}\frac{\partial u}{\partial\phi}\frac{\partial u}{\partial r} + \frac{1}{r}u\frac{\partial^2 u}{\partial\phi\partial r}\right.$$

$$+ \frac{1}{r^2}\frac{\partial v}{\partial\phi}\frac{\partial u}{\partial\phi} + \frac{v}{r^2}\frac{\partial^2 ur}{\partial\phi^2} + \frac{1}{r}\frac{\partial w}{\partial\phi}\frac{\partial u}{\partial z} + \frac{u}{r}\frac{\partial^2 u}{\partial\phi\partial z} - \frac{1}{r^2}\frac{\partial v^2}{\partial\phi}$$

$$+ 2 \frac{\partial^2 v}{\partial t \partial r} + \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + u \frac{\partial^2 v}{\partial \phi \partial r} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} - \frac{v}{r^2}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z}$$

$$+ w \frac{\partial^2 v}{\partial \phi \partial z} + v \frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial v}{\partial r} - 2 \frac{uv}{r^2} - \frac{1}{r}\frac{\partial v}{\partial r} - \frac{u}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\frac{\partial v}{\partial \phi}$$

$$- \frac{w}{r}\frac{\partial v}{\partial z} + \frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \phi} + \frac{2}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} + \frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \phi} + \frac{2}{r} u \frac{\partial v}{\partial r}$$

$$\left. - \frac{2}{r} v \frac{\partial u}{\partial r}\Big)\right] , \qquad (3.2.9)$$

$$T_{r\phi} = \alpha_2 \left[\frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{2v}{r^2}\right] + \alpha_3 \left[\frac{1}{r}\frac{\partial^2 u}{\partial t \partial \phi} + \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial r} + \frac{1}{r} u \frac{\partial^2 u}{\partial \phi \partial r}\right.$$

$$+ \frac{1}{r^2}\frac{\partial v}{\partial \phi}\frac{\partial u}{\partial \phi} + \frac{v}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r}\frac{\partial w}{\partial \phi}\frac{\partial u}{\partial z} + \frac{u}{r}\frac{\partial^2 u}{\partial \phi \partial z} - \frac{1}{r^2}\frac{\partial v^2}{\partial \phi}$$

$$+ \frac{\partial^2 v}{\partial t \partial r} + \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + u \frac{\partial^2 v}{\partial \phi \partial r} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} - \frac{v}{r^2}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z}$$

$$+ w \frac{\partial^2 v}{\partial \phi \partial z} + \frac{v}{r}\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial v}{\partial r} - 2 \frac{uv}{r^2} - \frac{1}{r}\frac{\partial v}{\partial r} - \frac{u}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\frac{\partial v}{\partial \phi}$$

$$- \frac{w}{r}\frac{\partial v}{\partial z} + \frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \phi} + \frac{2}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} + \frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \phi} + \frac{2}{r} u \frac{\partial v}{\partial r}$$

$$\left. - \frac{2}{r} v \frac{\partial u}{\partial r}\right] + \alpha_4 \left[\frac{1}{r}\frac{\partial \theta}{\partial \phi}\frac{\partial \theta}{\partial r}\right] + \alpha_5 \left[\frac{1}{r}\frac{\partial \theta}{\partial z}\left(2 \frac{\partial u}{\partial r} - \frac{2}{r}\frac{\partial v}{\partial \phi}\right.\right.$$

$$\left.\left. + \frac{\partial u}{r}\right) - \frac{1}{r}\frac{\partial \theta}{\partial \phi}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + 2 \frac{\partial \theta}{\partial \phi}\frac{\partial w}{\partial z}\right] + \alpha_6 \left[\frac{2}{r}\frac{\partial \theta}{\partial z}\left(\left(\frac{\partial^2 u}{\partial t \partial r}\right.\right.\right.$$

$$+ 2 \left(\frac{\partial u}{\partial r}\right)^2 + u \frac{\partial^2 u}{\partial r^2} + \frac{v}{r}\frac{\partial^2 u}{\partial r \partial \phi} - \frac{v}{r^2}\frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r}\frac{\partial u}{\partial z} + w \frac{\partial^2 u}{\partial r \partial z}$$

$$- 4v \frac{\partial v}{\partial r} + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\Bigg) - \Bigg(\frac{2}{r} \frac{\partial^2 v}{\partial t \partial \phi} + \frac{2}{r} \frac{\partial u}{\partial r} \frac{\partial v}{\partial r}$$

$$+ 2 \frac{u}{r} \frac{\partial^2 v}{\partial \phi \partial r} + \frac{2}{r^2} \left(\frac{\partial v}{\partial \phi}\right)^2 + \frac{2v}{r^2} \frac{\partial^2 v}{\partial \phi^2} + \frac{2}{r} \frac{\partial w}{\partial \phi} \frac{\partial v}{\partial z} + \frac{2}{r} u \frac{\partial^2 v}{\partial \phi \partial z}$$

$$+ \frac{2u}{r^2} \frac{\partial v}{\partial \phi} + \frac{2v}{r^2} \frac{\partial u}{\partial \phi} + \frac{2}{r} \frac{\partial u}{\partial t} + \frac{2u}{r} \frac{\partial u}{\partial r} + \frac{2v}{r^2} \frac{\partial u}{\partial \phi} + \frac{2w}{r} \frac{\partial u}{\partial z}$$

$$- 2 \frac{v^2}{r^2} + \frac{2}{r^2} \left(\frac{\partial u}{\partial \phi}\right)^2 + \frac{2}{r^2} \left(\frac{\partial u \phi}{\partial \phi}\right)^2 + \frac{2}{r^2} \left(\frac{\partial w}{\partial \phi}\right)^2 + \frac{2u^2}{r^2}$$

$$- 4 \frac{v}{r^2} \frac{\partial u}{\partial \phi} + \frac{4u}{r^2} \frac{\partial v}{\partial \phi}\Bigg)\Bigg) - \frac{1}{r} \frac{\partial \theta}{\partial r} \Bigg(\frac{\partial^2 u}{\partial t \partial r} + \left(\frac{\partial u}{\partial z}\right)^2 + \frac{\partial u}{\partial z} \frac{\partial u}{\partial r}$$

$$+ u \frac{\partial^2 u}{\partial z \partial r} + \frac{1}{r} \frac{\partial v}{\partial z} \frac{\partial u}{\partial \phi} + \frac{v}{r} \frac{\partial^2 u}{\partial z \partial \phi} + \frac{\partial w}{\partial z} \frac{\partial u}{\partial z} + w \frac{\partial^2 u}{\partial z^2} - \frac{1}{r} \frac{\partial v^2}{\partial z}$$

$$+ \frac{\partial^2 w}{\partial t \partial r} + \frac{\partial u}{\partial r} \frac{\partial w}{\partial r} + u \frac{\partial^2 w}{\partial r^2} + \frac{v}{r} \frac{\partial^2 w}{\partial r \partial \phi} + \frac{1}{r} \frac{\partial v}{\partial r} \frac{\partial w}{\partial \phi} - \frac{v}{r^2} \frac{\partial w}{\partial \phi}$$

$$+ \frac{\partial w}{\partial \phi} + \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} + w \frac{\partial^2 w}{\partial r \partial z} + 2 \frac{u}{r} \frac{\partial u}{\partial z} + 2 \frac{\partial v}{\partial r} \frac{\partial v}{\partial z} + 2 \frac{v}{r} \frac{\partial v}{\partial z}$$

$$+ 2 \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} - 2v \frac{\partial v}{\partial z}\Bigg) + \frac{\partial \theta}{\partial \phi} \Bigg(\frac{\partial^2 v}{\partial t \partial z} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial r} + u \frac{\partial^2 v}{\partial z \partial r}$$

$$+ \frac{v}{r} \frac{\partial^2 v}{\partial z \partial \phi} + \frac{1}{r} \frac{\partial v}{\partial z} \frac{\partial v}{\partial \phi} + 2 \frac{\partial w}{\partial z} \frac{\partial v}{\partial z} + w \frac{\partial^2 v}{\partial z^2} + \frac{u}{r} \frac{\partial v}{\partial z} + \frac{v}{r} \frac{\partial u}{\partial z}$$

$$+ \frac{1}{r} \frac{\partial^2 w}{\partial t \partial \phi} + \frac{1}{r} \frac{\partial u}{\partial \phi} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial v}{\partial \phi} \frac{\partial w}{\partial \phi} + \frac{v}{r} \frac{\partial^2 w}{\partial \phi^2} + w \frac{\partial^2 w}{\partial \phi \partial z}$$

$$+ \frac{\partial w}{\partial \phi} \frac{\partial w}{\partial z} + \frac{2}{r} \frac{\partial u}{\partial r} \frac{\partial u}{\partial z} + \frac{2}{r} \frac{\partial v}{\partial \phi} \frac{\partial v}{\partial z} + \frac{2}{r} \frac{\partial w}{\partial \phi} \frac{\partial w}{\partial z} + \frac{2u}{r} \frac{\partial v}{\partial z}$$

$$- \frac{2v}{r}\frac{\partial u}{\partial z}\Bigg)\Bigg], \qquad (3.2.10)$$

$$T_{\phi\phi} = \alpha_1 + 2\alpha_2\left[\frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{u}{r}\right] + 2\alpha_3\left[\frac{1}{r}\frac{\partial^2 v}{\partial t\partial \phi} + \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial r} + \frac{u}{r}\frac{\partial^2 v}{\partial \phi\partial r}\right.$$

$$+ \frac{1}{r^2}\left(\frac{\partial v}{\partial \phi}\right)^2 + \frac{2v}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{r}\frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z} + \frac{u}{r}\frac{\partial^2 v}{\partial \phi\partial z} + \frac{u}{r^2}\frac{\partial v}{\partial \phi}$$

$$+ \frac{v}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r}\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial r} + \frac{v}{r^2}\frac{\partial u}{\partial \phi} + \frac{w}{r}\frac{\partial u}{\partial \phi} - \frac{u^2}{r^2} + \frac{1}{r^2}\left(\frac{\partial u}{\partial \phi}\right)^2$$

$$\left. + \frac{1}{r^2}\;\frac{\partial v}{\partial \phi}^{\;2} + \frac{1}{r^2}\;\frac{\partial w}{\partial \phi}^{\;2} + \frac{1}{r^2}u^2 - 2\frac{v}{r^2}\frac{\partial u}{\partial \phi} + \frac{2u}{r^2}\frac{\partial v}{\partial \phi}\right]$$

$$+ \alpha_4\left[\frac{1}{r^2}\left(\frac{\partial \theta}{\partial r}\right)^2\right] + \alpha_5\left[2\frac{\partial \theta}{\partial \phi}\left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial \theta}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \phi}\right.\right.$$

$$\left.\left. + \frac{\partial v}{\partial r} - \frac{v}{r}\right) - \frac{1}{r}\frac{\partial \theta}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right)\right] + \alpha_6\left[\frac{1}{r}\frac{\partial \theta}{\partial z}\left(\frac{1}{r}\frac{\partial^2 u}{\partial t\partial \phi}\right.\right.$$

$$+ \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial^2 u}{\partial \phi\partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \phi}\frac{\partial u}{\partial \phi} + \frac{v}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r}\frac{\partial w}{\partial \phi}\frac{\partial u}{\partial z}$$

$$+ \frac{u}{r}\frac{\partial^2 u}{\partial \phi\partial z} - \frac{1}{r}\frac{\partial v^2}{\partial \phi} + \frac{\partial^2 v}{\partial t\partial \phi} + \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + u\frac{\partial^2 v}{\partial \phi\partial r} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi}$$

$$- \frac{v}{r^2}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial \phi\partial z} + \frac{v}{r}\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial v}{\partial r} - \frac{uv}{r^2} - \frac{1}{r}\frac{\partial v}{\partial r}$$

$$- \frac{u}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\frac{\partial v}{\partial \phi} - \frac{w}{r}\frac{\partial v}{\partial z} - \frac{uv}{r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \phi} + \frac{2}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi}$$

$$\left. + \frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \phi} + \frac{2}{r}u\frac{\partial v}{\partial r} - \frac{2}{r}v\frac{\partial u}{\partial r}\right) - \frac{2}{r}\frac{\partial \theta}{\partial r}\left(\frac{\partial^2 v}{\partial t\partial z} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial r}\right.$$

$$+ u \frac{\partial^2 v}{\partial z \partial r} + \frac{v}{r} \frac{\partial^2 v}{\partial z \partial \phi} + \frac{1}{r} \frac{\partial v}{\partial z} \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} \frac{\partial v}{\partial z} + w \frac{\partial^2 v}{\partial z^2} + \frac{u}{r} \frac{\partial v}{\partial z}$$

$$+ \frac{v}{r} \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial^2 w}{\partial t \partial \phi} + \frac{1}{r} \frac{\partial u}{\partial \phi} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial v}{\partial \phi} \frac{\partial w}{\partial \phi} + \frac{v}{r^2} \frac{\partial^2 w}{\partial \phi^2}$$

$$+ \frac{w}{r} \frac{\partial^2 w}{\partial \phi \partial z} + \frac{1}{r} \frac{\partial w}{\partial \phi} \frac{\partial w}{\partial z} + \frac{2}{r} \frac{\partial u}{\partial \phi} \frac{\partial u}{\partial z} + \frac{2}{r} \frac{\partial v}{\partial \phi} \frac{\partial v}{\partial z} + \frac{2}{r} \frac{\partial w}{\partial \phi} \frac{\partial w}{\partial z}$$

$$\left. + \frac{2u}{r} \frac{\partial v}{\partial z} - \frac{2v}{r} \frac{\partial u}{\partial r} \right], \qquad (3.2.11)$$

$$T_{rz} = \alpha_2 \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right] + \alpha_3 \left[\frac{\partial^2 u}{\partial t \partial z} + \frac{\partial u}{\partial z} \frac{\partial u}{\partial r} + w \frac{\partial^2 u}{\partial z^2} - \frac{1}{r} \frac{\partial^2 v}{\partial z} \right.$$

$$+ \frac{\partial^2 w}{\partial t \partial r} + \frac{\partial u}{\partial r} \frac{\partial w}{\partial z} + u \frac{\partial^2 w}{\partial r^2} + \frac{v}{r} \frac{\partial^2 w}{\partial r \partial \phi} + \frac{1}{r} \frac{\partial v}{\partial r} \frac{\partial w}{\partial \phi} - \frac{v}{r} \frac{\partial w}{\partial \phi}$$

$$+ \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} + w \frac{\partial^2 w}{\partial r \partial z} + 2 \frac{\partial u}{\partial r} \frac{\partial u}{\partial z} + 2 \frac{\partial v}{\partial r} \frac{\partial v}{\partial z} + 2 \frac{v}{r} \frac{\partial v}{\partial z}$$

$$\left. + 2 \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} - 2v \frac{\partial v}{\partial z} \right] + \alpha_5 \left[\frac{\partial \theta}{\partial \phi} \left(2 \frac{\partial w}{\partial \phi} - 2 \frac{\partial u}{\partial r} \right) \right.$$

$$\left. + \frac{1}{r} \frac{\partial \theta}{\partial r} \left(\frac{1}{r} \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) - \frac{1}{r} \frac{\partial \theta}{\partial z} \left(\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \phi} \right) \right]$$

$$+ \alpha_6 \left[2 \frac{\partial \theta}{\partial \phi} \left(\frac{\partial^2 w}{\partial t \partial z} + u \frac{\partial^2 w}{\partial z \partial r} + \frac{v}{r} \frac{\partial^2 v}{\partial z \partial \phi} + \frac{1}{r} \frac{\partial v}{\partial z} \frac{\partial w}{\partial \phi} \right. \right.$$

$$+ \left(\frac{\partial w}{\partial z} \right)^2 + w \frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 - \frac{\partial^2 u}{\partial t \partial r} - 2 \left(\frac{\partial u}{\partial r} \right)^2$$

$$- u \frac{\partial^2 u}{\partial r^2} - \frac{v}{r} \frac{\partial^2 u}{\partial r \partial \phi} + 2 \frac{v}{r^2} \frac{\partial u}{\partial \phi} - w \frac{\partial^2 u}{\partial r \partial \phi} - \frac{\partial w}{\partial r} \frac{\partial u}{\partial z}$$

$$- u \frac{\partial^2 u}{\partial r^2} + v \frac{\partial^2 u}{\partial r \partial \phi} + \frac{v}{r^2} \frac{\partial u}{\partial \phi} - \frac{\partial w}{\partial r} \frac{\partial u}{\partial z} - w \frac{\partial^2 u}{\partial r \partial z} + 2 \frac{v}{r} \frac{\partial v}{\partial r}$$

$$- \left(\frac{\partial v}{\partial r}\right)^2 - \left(\frac{\partial w}{\partial r}\right)^2 \Bigg) - \frac{1}{r} \frac{\partial \theta}{\partial z} \Bigg(\frac{\partial^2 v}{\partial t \partial z} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial r} + u \frac{\partial^2 v}{\partial z \partial r}$$

$$- \frac{v}{r} \frac{\partial^2 v}{\partial z \partial \phi} + \frac{1}{r} \frac{\partial v}{\partial z} \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} \frac{\partial v}{\partial z} + w \frac{\partial^2 v}{\partial z^2} + \frac{u}{r} \frac{\partial v}{\partial z} + \frac{u}{r} \frac{\partial u}{\partial z}$$

$$+ \frac{1}{r} \frac{\partial^2 w}{\partial t \partial \phi} + \frac{1}{r} \frac{\partial u}{\partial \phi} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial v}{\partial \phi} \frac{\partial w}{\partial \phi} + \frac{v}{r^2} \frac{\partial^2 w}{\partial \phi^2} + \frac{w}{r} \frac{\partial^2 w}{\partial \phi \partial z}$$

$$+ \frac{1}{r} \frac{\partial w}{\partial \phi} \frac{\partial w}{\partial z} + \frac{2}{r} \frac{\partial u}{\partial \phi} \frac{\partial u}{\partial z} + \frac{2}{r} \frac{\partial v}{\partial \phi} \frac{\partial v}{\partial z} + \frac{2}{r} \frac{\partial w}{\partial \phi} \frac{\partial w}{\partial z} + \frac{2u}{r} \frac{\partial v}{\partial z}$$

$$- 2 \frac{v}{r} \frac{\partial u}{\partial z} \Bigg) + \frac{1}{r} \frac{\partial \theta}{\partial r} \Bigg(\frac{1}{r} \frac{\partial^2 u}{\partial t \partial \phi} + \frac{1}{r} \frac{\partial u}{\partial \phi} \frac{\partial u}{\partial r} + \frac{u}{r} \frac{\partial^2 u}{\partial \phi \partial r}$$

$$+ \frac{1}{r^2} \frac{\partial v}{\partial \phi} \frac{\partial u}{\partial \phi} + \frac{v}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r} \frac{\partial w}{\partial \phi} \frac{\partial u}{\partial z} + \frac{u}{r} \frac{\partial^2 u}{\partial \phi \partial z} - \frac{2v}{r^2} \frac{\partial v}{\partial \phi}$$

$$+ \frac{\partial^2 v}{\partial t \partial r} + \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + u \frac{\partial^2 v}{\partial \phi \partial r} + \frac{1}{r} \frac{\partial v}{\partial r} \frac{\partial v}{\partial \phi} - \frac{v}{r^2} \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \phi} \frac{\partial v}{\partial z}$$

$$+ w \frac{\partial^2 v}{\partial \phi \partial z} + \frac{v}{r} \frac{\partial u}{\partial r} + \frac{u}{r} \frac{\partial v}{\partial r} - 2 \frac{vu}{r^2} - \frac{1}{r} \frac{\partial v}{\partial t} - \frac{u}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \frac{\partial v}{\partial \phi}$$

$$- \frac{w}{r} \frac{\partial v}{\partial z} + \frac{2}{r} \frac{\partial u}{\partial r} \frac{\partial u}{\partial \phi} + \frac{2}{r} \frac{\partial v}{\partial r} \frac{\partial v}{\partial \phi} + \frac{2}{r} \frac{\partial w}{\partial z} \frac{\partial w}{\partial \phi} + \frac{2w}{\partial r} \frac{\partial v}{r}$$

$$- \frac{2v}{r} \frac{\partial u}{\partial r} \Bigg) \Bigg] , \qquad (3.2.12)$$

$$T_{z\phi} = \alpha_2 \left[\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \phi} \right] + \alpha_3 \Bigg[\frac{\partial^2 v}{\partial t \partial z} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial r} + u \frac{\partial^2 v}{\partial z \partial r} + \frac{v}{r} \frac{\partial^2 v}{\partial z \partial \phi}$$

$$+ \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial z^2} + \frac{u}{r}\frac{\partial v}{\partial z} + \frac{v}{r}\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial^2 w}{\partial t \partial \phi}$$

$$+ \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \phi}\frac{\partial w}{\partial \phi} + \frac{v}{r}\frac{\partial^2 w}{\partial \phi} + w\frac{\partial^2 w}{\partial \phi \partial z} + \frac{\partial w}{\partial \phi}\frac{\partial w}{\partial z}$$

$$+ \frac{2}{r}\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial z} + \frac{2}{r}\frac{\partial v}{\partial \phi}\frac{\partial v}{\partial z} + \frac{2}{r}\frac{\partial w}{\partial \phi}\frac{\partial w}{\partial z} + \frac{2v}{r}\frac{\partial v}{\partial z} - \frac{2v}{r}\frac{\partial u}{\partial z}\Bigg]$$

$$+ \alpha_4\left[\frac{1}{r}\frac{\partial \theta}{\partial z}\frac{\partial \theta}{\partial \phi}\right] + \alpha_5\Bigg[\frac{2}{r}\frac{\partial \theta}{\partial r}\left(\frac{\partial v}{\partial \phi} - \frac{\partial w}{\partial z}\right) - \frac{\partial \theta}{\partial \phi}\Bigg(\frac{1}{r}\frac{\partial u}{\partial \phi}$$

$$+ \frac{\partial v}{\partial r} - \frac{2v}{r}\Bigg) + \frac{1}{r}\frac{\partial \theta}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\Bigg] + \alpha_6\Bigg[\frac{2}{r}\frac{\partial \theta}{\partial r}\Bigg(\frac{1}{r}\frac{\partial^2 v}{\partial t \partial \phi}$$

$$+ \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial r} + \frac{u}{r}\frac{\partial^2 v}{\partial \phi \partial r} + \frac{1}{r^2}\left(\frac{\partial v}{\partial \phi}\right)^2 + \frac{v}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{r}\frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z}$$

$$+ \frac{1}{r}w\frac{\partial^2 v}{\partial \phi \partial z} + \frac{u}{r}\frac{\partial v}{\partial \phi} + \frac{v}{r}\frac{\partial u}{\partial \phi} + \frac{1}{r}\frac{\partial u}{\partial t} + \frac{1}{r}u\frac{\partial u}{\partial r} + \frac{v}{r^2}\frac{\partial u}{\partial \phi}$$

$$+ \frac{w}{r}\frac{\partial u}{\partial z} - \frac{v^2}{r^2} + \frac{1}{r^2}\left(\frac{\partial u}{\partial \phi}\right)^2 + \frac{1}{r^2}\left(\frac{\partial v}{\partial \phi}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \phi}\right)^2 + \frac{u^2}{r^2}$$

$$- \frac{2v}{r^2}\frac{\partial u}{\partial \phi} + \frac{2u}{r^2}\frac{\partial v}{\partial \phi} - \frac{\partial^2 w}{\partial t \partial z} - \frac{\partial u}{\partial z}\frac{\partial w}{\partial r} - \frac{u\partial^2 w}{\partial r^2} - \frac{v}{r}\frac{\partial^2 w}{\partial z \partial \phi}$$

$$- \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial w}{\partial \phi} - \left(\frac{\partial w}{\partial z}\right)^2 - w\frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\Bigg)$$

$$- \frac{\partial \theta}{\partial \phi}\Bigg(\frac{1}{r}\frac{\partial^2 u}{\partial t \partial \phi} + \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial^2 u}{\partial \phi \partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \phi}\frac{\partial u}{\partial \phi} + \frac{v}{r^2}\frac{\partial^2 u}{\partial \phi^2}$$

$$+ \frac{1}{r}\frac{\partial w}{\partial \phi}\frac{\partial u}{\partial z} + \frac{w}{r}\frac{\partial^2 u}{\partial \phi \partial z} - 3\frac{v}{r^2}\frac{\partial v}{\partial \phi} + \frac{\partial^2 v}{\partial t \partial r} + \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{u\partial^2 v}{\partial \phi \partial r}$$

$$+ \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} - \frac{v}{r^2}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial \phi \partial z} + \frac{v}{r}\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial r}$$

$$- 2\frac{uv}{r^2} - \frac{1}{r}\frac{\partial v}{\partial t} - \frac{u}{r}\frac{\partial v}{\partial r} - \frac{w}{r}\frac{\partial v}{\partial z} + \frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \phi} + \frac{2}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi}$$

$$+ \frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \phi} + \frac{2}{r}u\frac{\partial v}{\partial r} - \frac{2}{r}v\frac{\partial u}{\partial r}\Bigg) + \frac{1}{r}\frac{\partial \theta}{\partial r}\Bigg(\frac{\partial^2 u}{\partial t \partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial r}$$

$$+ u\frac{\partial^2 u}{\partial z \partial r} + \frac{1}{r}\frac{\partial v}{\partial \phi}\frac{\partial u}{\partial \phi} + \frac{v}{r}\frac{\partial^2 u}{\partial z \partial \phi} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial z^2} - 2\frac{v}{r}\frac{\partial v}{\partial z}$$

$$+ \frac{\partial^2 w}{\partial t \partial r} + \frac{\partial u}{\partial r}\frac{\partial w}{\partial r} + u\frac{\partial^2 w}{\partial r^2} + \frac{v}{r}\frac{\partial^2 w}{\partial r \partial \phi} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial w}{\partial \phi} - \frac{v}{r}\frac{\partial w}{\partial \phi}$$

$$+ \frac{\partial w}{\partial r}\frac{\partial w}{\partial z} + w\frac{\partial^2 w}{\partial r \partial z} + 2\frac{\partial u}{\partial r}\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\frac{\partial v}{\partial z} + 2\frac{v}{r}\frac{\partial v}{\partial z}$$

$$+ 2\frac{\partial v}{\partial r}\frac{\partial v}{\partial z} - v\frac{\partial v}{\partial z}\Bigg)\Bigg], \qquad (3.2.13)$$

$$T_{zz} = \alpha_1 + \alpha_2\left[2\frac{\partial w}{\partial z}\right] + 2\alpha_3\Bigg[\frac{\partial^2 w}{\partial t \partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + u\frac{\partial^2 w}{\partial r} + \frac{v}{r}\frac{\partial^2 w}{\partial z \partial \phi}$$

$$+ \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial w}{\partial \phi} + \left(\frac{\partial w}{\partial z}\right)^2 + w\frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\Bigg]$$

$$- \alpha_4\left[\left(\frac{\partial \theta}{\partial \phi}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \theta}{\partial r}\right)^2\right] + 2\alpha_5\Bigg[\frac{\partial \theta}{\partial r}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right)$$

$$- \frac{\partial \theta}{\partial \phi}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\right)\Bigg] + \alpha_6\Bigg[\frac{2}{r}\frac{\partial \theta}{\partial r}\Bigg(\frac{\partial^2 v}{\partial t \partial z} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial r} + u\frac{\partial^2 v}{\partial z \partial r}$$

$$+ \frac{v}{r}\frac{\partial^2 v}{\partial z \partial \phi} + \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial z^2} + \frac{u}{r}\frac{\partial v}{\partial z} + \frac{v}{r}\frac{\partial u}{\partial z}$$

$$+ \frac{1}{r}\frac{\partial^2 w}{\partial t\partial\phi} + \frac{1}{r}\frac{\partial u}{\partial\phi}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial v}{\partial\phi}\frac{\partial w}{\partial\phi} + \frac{w}{r}\frac{\partial^2 w}{\partial\phi^2} + w\frac{\partial^2 w}{\partial\phi\partial z}$$

$$+ \frac{\partial w}{\partial\phi}\frac{\partial w}{\partial z} + \frac{2}{r}\frac{\partial u}{\partial\phi}\frac{\partial u}{\partial z} + \frac{2}{r}\frac{\partial v}{\partial\phi}\frac{\partial v}{\partial z} + \frac{2}{r}\frac{\partial w}{\partial\phi}\frac{\partial w}{\partial z} + 2\frac{v}{r}\frac{\partial v}{\partial z}$$

$$- 2\frac{\partial\theta}{\partial\phi}\left(\frac{\partial^2 u}{\partial t\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial r} + u\frac{\partial^2 u}{\partial z\partial r} + \frac{1}{r}\frac{\partial v}{\partial\phi}\frac{\partial u}{\partial\phi} + v\frac{\partial^2 u}{\partial z\partial\phi}\right.$$

$$+ \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial z^2} - \frac{2v}{r}\frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial t\partial r} + \frac{\partial u}{\partial r}\frac{\partial w}{\partial r} + u\frac{\partial^2 w}{\partial r^2}$$

$$+ \frac{v}{r}\frac{\partial^2 w}{\partial r\partial\phi} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial w}{\partial\phi} - \frac{v}{r}\frac{\partial w}{\partial\theta} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial z} + w\frac{\partial^2 w}{\partial r\partial z} + 2\left(\frac{\partial u}{\partial r}\right)^2$$

$$\left. + 2\frac{\partial v}{\partial r}\frac{\partial v}{\partial z} + 2\frac{v}{r}\frac{\partial v}{\partial z} + 2\frac{\partial w}{\partial r}\frac{\partial w}{\partial z} - 2v\frac{\partial v}{\partial z}\right)\Bigg] . \qquad (3.2.14)$$

The heat conduction equation is:

$$\rho\kappa\frac{\partial\theta}{\partial t} = T_{mk}d^{km} - q^k_{,k} + \rho Q, \qquad (3.2.15)$$

where

$$T^{mk}d_{km} = 2\,T_{rr}\frac{\partial u}{\partial r} + 2\,T_{\phi\phi}\left[\frac{1}{r}\frac{\partial v}{\partial\phi} + \frac{u}{r}\right] + 2\,T_{zz}\frac{\partial w}{\partial z}$$

$$+ 2\,T_{r\phi}\left[\frac{1}{r}\frac{\partial u}{\partial\phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right] + 2\,T_{rz}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]$$

$$+ 2\,T_{\phi z}\left[\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial\phi}\right],$$

$T_{rr}, \ldots, T_{zz}$ have been defined in equations 2.3.8 through 2.3.13,

$$q^k_{,k} \qquad \text{div } q = \frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{1}{r}\frac{\partial q_\phi}{\partial\phi} + \frac{\partial q_z}{\partial z}\,, \qquad (3.2.16)$$

$$q_r = rh_{\phi z},$$

$$q_\phi = -\frac{1}{r}h_{rz},$$

$$q_z = rh_{r\phi}, \qquad (3.2.17)$$

where

$$h_{r\phi} = \beta_1\left[\frac{1}{r}\frac{\partial\theta}{\partial z}\right] + \beta_2\left[\frac{2}{r}\frac{\partial\theta}{z}\left(\frac{\partial^2 u}{\partial t\partial r} + 2\left(\frac{\partial u}{\partial r}\right)^2 + u\frac{\partial^2 u}{\partial r^2} + \frac{v}{r}\frac{\partial^2 u}{\partial r\partial\phi}\right.\right.$$

$$- \frac{v}{r^2}\frac{\partial u}{\partial\phi} + \frac{\partial w}{\partial r}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial r\partial z} - 2\frac{v}{r}\frac{\partial v}{\partial r} + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2$$

$$- \frac{1}{r}\frac{\partial^2 v}{\partial t\partial\phi} - \frac{1}{r}\frac{\partial u}{\partial\phi}\frac{\partial v}{\partial r} - \frac{v}{r}\frac{\partial^2 v}{\partial\phi\partial r} - \frac{1}{r^2}\left(\frac{\partial v}{\partial\phi}\right)^2 + \frac{v}{r^2}\frac{\partial^2 v}{\partial\phi^2}$$

$$- \frac{1}{r}\frac{\partial u}{\partial\phi}\frac{\partial v}{\partial z} - \frac{u}{r}\frac{\partial^2 v}{\partial\phi\partial z} - \frac{u}{r^2}\frac{\partial v}{\partial\phi} - \frac{v}{r^2}\frac{\partial u}{\partial\phi} - \frac{1}{r}\frac{\partial u}{\partial t} - \frac{u}{r}\frac{\partial u}{\partial r}$$

$$- \frac{v}{r^2}\frac{\partial u}{\partial\phi} - \frac{v}{r}\frac{\partial u}{\partial z} + \frac{v^2}{r^2} - \frac{1}{r^2}\left(\frac{\partial u}{\partial\phi}\right)^2 - \frac{1}{r^2}\left(\frac{\partial v}{\partial\phi}\right)^2 - \frac{1}{r^2}\left(\frac{\partial w}{\partial\phi}\right)^2$$

$$\left. + \frac{2v}{r^2}\frac{\partial u}{\partial\phi} - \frac{2u}{r^2}\frac{\partial v}{\partial\phi}\right) - \frac{1}{r}\frac{\partial\theta}{\partial r}\left(\frac{\partial^2 u}{\partial t\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial r} + u\frac{\partial^2 u}{\partial z\partial r}\right.$$

$$+ \frac{1}{r}\frac{\partial v}{\partial\phi}\frac{\partial w}{\partial z} + \frac{v}{r}\frac{\partial^2 u}{\partial z\partial\phi} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial z^2} - \frac{2v}{\partial r}\frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial t\partial r}$$

$$+ \frac{\partial u}{\partial r}\frac{\partial w}{\partial r} + u\frac{\partial^2 w}{\partial r^2} + \frac{v}{r}\frac{\partial^2 w}{\partial r\partial\phi} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial w}{\partial\phi} - \frac{v}{r}\frac{\partial w}{\partial\phi} + 3\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}$$

$$\left. + w\frac{\partial^2 w}{\partial r\partial z} + 2\frac{\partial u}{\partial r}\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\frac{\partial v}{\partial z} + \frac{2v}{\partial r}\frac{\partial v}{\partial z} - 2v\frac{\partial v}{\partial z}\right)$$

$$+ \frac{\partial\theta}{\partial\phi}\left(\frac{\partial^2 v}{\partial t\partial z} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial r} + u\frac{\partial^2 v}{\partial z\partial r} + \frac{v}{r}\frac{\partial^2 v}{\partial z\partial\phi} + \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial v}{\partial\phi}\right.$$

$$+ \frac{\partial w}{\partial z}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial z^2} + \frac{u}{r}\frac{\partial v}{\partial z} + \frac{v}{r}\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial^2 w}{\partial t\partial\phi} + \frac{1}{r}\frac{\partial u}{\partial\phi}\frac{\partial w}{\partial r}$$

$$+ \frac{1}{r^2}\frac{\partial v}{\partial\phi}\frac{\partial w}{\partial\phi} + \frac{v}{r^2}\frac{\partial^2 w}{\partial\phi^2} + \frac{w}{r}\frac{\partial^2 w}{\partial\phi\partial z} + \frac{1}{r}\frac{\partial w}{\partial\phi}\frac{\partial w}{\partial z} + \frac{2}{r}\frac{\partial u}{\partial\phi}\frac{\partial u}{\partial z}$$

$$+ \frac{2}{r}\frac{\partial v}{\partial\phi}\frac{\partial v}{\partial z} + \frac{2}{r}\frac{\partial w}{\partial\phi}\frac{\partial w}{\partial z} + \frac{2u}{\partial r}\frac{\partial v}{\partial z} - 2\frac{v}{r}\frac{\partial u}{\partial z}\Bigg)\Bigg]$$

$$+ \beta_4\Bigg[\frac{2}{r}\frac{\partial\theta}{\partial z}\left(\frac{1}{r}\frac{\partial v}{\partial\phi} + \frac{u}{r} + \frac{\partial u}{\partial r}\right) - \frac{\partial\theta}{\partial\phi}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial\phi}\right)$$

$$- \frac{1}{r}\frac{\partial\theta}{\partial r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\Bigg], \qquad (3.2.18)$$

$$h_{rz} = -\beta_1\frac{\partial\theta}{\partial\phi} + \beta_2\Bigg[\frac{1}{r}\frac{\partial\theta}{\partial z}\Bigg(\frac{\partial^2 v}{\partial t\partial z} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial r} + u\frac{\partial^2 v}{\partial z\partial r} + \frac{v}{r}\frac{\partial^2 v}{\partial z\partial\phi}$$

$$+ \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial v}{\partial\phi} + \frac{\partial w}{\partial z}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial z^2} + \frac{u}{r}\frac{\partial v}{\partial z} + \frac{v}{r}\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial^2 w}{\partial t\partial\phi}$$

$$+ \frac{1}{r}\frac{\partial u}{\partial\phi}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial v}{\partial\phi}\frac{\partial w}{\partial\phi} + \frac{v}{r^2}\frac{\partial^2 w}{\partial\phi^2} + \frac{w}{r}\frac{\partial^2 w}{\partial\phi\partial z} + \frac{1}{r}\frac{\partial w}{\partial\phi}\frac{\partial w}{\partial z}$$

$$+ \frac{2}{r}\frac{\partial u}{\partial\phi}\frac{\partial u}{\partial z} + \frac{2}{r}\frac{\partial v}{\partial\phi}\frac{\partial v}{\partial z} + \frac{2}{r}\frac{\partial w}{\partial\phi}\frac{\partial w}{\partial z} + 2\frac{u}{r}\frac{\partial v}{\partial z} - 2\frac{v}{r}\frac{\partial u}{\partial z}\Bigg)$$

$$- \frac{1}{r}\frac{\partial\theta}{\partial r}\Bigg(\frac{1}{r}\frac{\partial^2 u}{\partial t\partial\phi} + \frac{1}{r}\frac{\partial u}{\partial\phi}\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial^2 u}{\partial\phi\partial r} + \frac{1}{r^2}\frac{\partial v}{\partial\phi}\frac{\partial u}{\partial\phi}$$

$$+ \frac{v}{r^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{1}{r}\frac{\partial w}{\partial\phi}\frac{\partial u}{\partial z} + \frac{w}{r}\frac{\partial^2 u}{\partial\phi\partial z} - \frac{2v}{r^2}\frac{\partial v}{\partial\phi} + \frac{\partial^2 v}{\partial t\partial r} + \frac{\partial u}{\partial r}\frac{\partial v}{\partial r}$$

$$+ u\frac{\partial^2 v}{\partial\phi\partial r} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial u}{\partial\phi} - \frac{v}{r^2}\frac{\partial v}{\partial\phi} + \frac{\partial w}{\partial\phi}\frac{\partial v}{\partial z} + u\frac{\partial^2 v}{\partial\phi\partial r}$$

$$+ \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} - \frac{v}{r^2}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial \phi \partial z} + \frac{v}{r}\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial v}{\partial r}$$

$$- 2\frac{uv}{r^2} - \frac{1}{r}\frac{\partial v}{\partial t} - \frac{u}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\frac{\partial v}{\partial \phi} - \frac{w}{r}\frac{\partial v}{\partial z} - \frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \phi}$$

$$+ \frac{2}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} + \frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \phi} + \frac{2u}{\partial r}\frac{\partial v}{\partial r} - \frac{2v}{\partial r}\frac{\partial u}{\partial r}\Bigg) - 2\frac{\partial \theta}{\partial \phi}\Bigg(\frac{\partial^2 u}{\partial t \partial r}$$

$$+ 2\left(\frac{\partial u}{\partial r}\right)^2 + u\frac{\partial^2 u}{\partial r^2} + \frac{v}{r}\frac{\partial^2 u}{\partial r \partial \phi} - \frac{v}{r^2}\frac{\partial u}{\partial \phi} + 2\frac{\partial w}{\partial r}\frac{\partial u}{\partial z}$$

$$+ w\frac{\partial^2 u}{\partial r \partial z} - 2\frac{v}{r}\frac{\partial v}{\partial r} + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2 + \frac{\partial^2 w}{\partial t \partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial r}$$

$$+ u\frac{\partial^2 w}{\partial r \partial z} + \frac{v}{r}\frac{\partial^2 w}{\partial z \partial \phi} + \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial w}{\partial \phi} + \left(\frac{\partial w}{\partial z}\right)^2 + w\frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial u}{\partial z}\right)^2$$

$$+ \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\Bigg)\Bigg] + \beta_3\Bigg[\frac{1}{r}\frac{\partial \theta}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right) - 2\frac{\partial \theta}{\partial \phi}\Bigg(\frac{\partial w}{\partial z}$$

$$+ \frac{\partial u}{\partial r}\Bigg) + \frac{1}{r}\frac{\partial \theta}{\partial r}\ \Bigg|\ \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} - \frac{2v}{r}\Bigg)\Bigg] \quad , \qquad (3.2.19)$$

$$h_{\phi z} = \beta_1\left[\frac{1}{r}\frac{\partial \theta}{\partial r}\right] + \beta_2\Bigg[\frac{2}{r}\frac{\partial \theta}{\partial r}\Bigg(\frac{1}{r}\frac{\partial^2 v}{\partial t \partial \phi} + \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial r} + \frac{u}{r}\frac{\partial^2 v}{\partial \phi \partial r}$$

$$+ \frac{1}{r^2}\left(\frac{\partial v}{\partial \phi}\right)^2 + \frac{v}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{1}{r}\frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z} + \frac{u}{r}\frac{\partial^2 v}{\partial \phi \partial z} + \frac{u}{r^2}\frac{\partial v}{\partial \phi}$$

$$+ \frac{v}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r}\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial r} + \frac{v}{r^2}\frac{\partial u}{\partial \phi} + \frac{w}{r}\frac{\partial u}{\partial z} - \frac{v^2}{r^2} + \frac{1}{r^2}\left(\frac{\partial u}{\partial \phi}\right)^2$$

$$+ \frac{1}{r^2}\left(\frac{\partial v}{\partial \phi}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \phi}\right)^2 + \frac{u^2}{r^2} - \frac{2v}{r^2}\frac{\partial u}{\partial \phi} + \frac{2u}{r^2}\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial t \partial z}$$

$$+ \frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + 2\,u\,\frac{\partial^2 u}{\partial r^2} + \frac{v}{r}\frac{\partial^2 w}{\partial z \partial \phi} + \frac{1}{r}\frac{\partial v}{\partial z}\frac{\partial w}{\partial \phi} + \left(\frac{\partial w}{\partial z}\right)^2 + w\,\frac{\partial^2 w}{\partial z^2}$$

$$+ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\Bigg) - \frac{1}{r}\frac{\partial \theta}{\partial z}\Bigg(\frac{\partial^2 u}{\partial t \partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial r}$$

$$+ u\,\frac{\partial^2 u}{\partial z \partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \phi}\frac{\partial u}{\partial \phi} + \frac{v}{r}\frac{\partial^2 u}{\partial z \partial \phi} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} + u_z\,\frac{\partial^2 u}{\partial z^2}$$

$$- \frac{2v}{\partial r}\frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial t \partial r} + \frac{\partial u}{\partial r}\frac{\partial w}{\partial r} + u\,\frac{\partial^2 w}{\partial r^2} + \frac{v}{r}\frac{\partial^2 w}{\partial r \partial \phi} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial w}{\partial \phi}$$

$$- \frac{v}{r^2}\frac{\partial w}{\partial \phi} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial z} + w\,\frac{\partial^2 w}{\partial r \partial z} + 2\,\frac{\partial u}{\partial r}\frac{\partial u}{\partial z} + 2\,\frac{\partial v}{\partial r}\frac{\partial v}{\partial z}$$

$$+ 2\,\frac{v}{r}\frac{\partial v}{\partial z} + 2\,\frac{\partial w}{\partial r}\frac{\partial w}{\partial z} - 2v\,\frac{\partial v}{\partial z}\Bigg) - \frac{\partial \theta}{\partial \phi}\Bigg(\frac{1}{r}\frac{\partial^2 u}{\partial t \partial \phi} + \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial r}$$

$$+ \frac{u}{r}\frac{\partial^2 u}{\partial \phi \partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \phi}\frac{\partial u}{\partial \phi} + \frac{v}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial z} + \frac{w}{r}\frac{\partial^2 u}{\partial \phi \partial z}$$

$$- \frac{2v}{r^2}\frac{\partial v}{\partial \phi} + \frac{\partial^2 v}{\partial t \partial r} + \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + u\,\frac{\partial^2 v}{\partial \phi \partial r} + \frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} - \frac{v}{r^2}\frac{\partial v}{\partial \phi}$$

$$+ \frac{\partial w}{\partial \phi}\frac{\partial v}{\partial z} + \frac{w\partial^2 v}{\partial \phi \partial z} + \frac{v}{r}\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial v}{\partial r} - 2\,\frac{uv}{r^2} - \frac{1}{r}\frac{\partial v}{\partial t} - \frac{u}{r}\frac{\partial v}{\partial r}$$

$$- \frac{v}{r^2}\frac{\partial v}{\partial \phi} - \frac{w}{r}\frac{\partial v}{\partial z} + \frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial \phi} + \frac{2}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \phi} + \frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \phi}$$

$$+ \frac{2}{r}\,u\,\frac{\partial v}{\partial r} - \frac{2v}{\partial r}\frac{\partial u}{\partial r}\Bigg)\Bigg] + \beta_4\Bigg[\frac{2}{r}\frac{\partial \theta}{\partial r}\left(\frac{\partial w}{\partial z} + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{2u}{r}\right)$$

$$- \frac{1}{r}\frac{\partial \theta}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) - \frac{\partial \theta}{\partial \phi}\left(\frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)\Bigg] \quad . \qquad (3.2.20)$$

Since the cylinders are concentric, we have a symmetry about their common axis. This implies that the physical properties of the problem do not depend on the azimuthal coordinate, i.e., $\frac{\partial}{\partial\phi} \equiv 0$ throughout the whole system. This simplifies the foregoing equations considerably.

Boundary conditions:

We assume that the inner and outer cylinders revolve with constant angular velocities Ω_1 and Ω_2 respectively. The no-slip boundary conditions imply that at the boundaries

$$u = 0 = w \tag{3.2.21}$$

and

$$v(R_1) = -\Omega_1 R_1 \quad v(R_2) = \Omega_2 R_2 \ . \tag{3.2.22}$$

The heat resevoir situation implies that

$$\theta(R_1) = \theta_1, \ \theta(R_2) = \theta_2. \tag{3.2.23}$$

3.3 Steady State Flow of Thermoviscoelastic Fluids Through Rotating Coaxial Cylinders

We consider the problem which was formulated in section 3.2, and assume we have a fully developed flow in the annulus formed by two infinite rotating coaxial cylinders. This assumption relieves the mathematical system formed by this problem of its dependence on z, so we have:

$$\frac{\partial}{\partial t} = 0 \qquad \text{(steady flow)},$$

$$\frac{\partial}{\partial \phi} = 0 \qquad \text{(axial symmetry)},$$

and $$\frac{\partial}{\partial z} = 0 \qquad \text{(infinite channel)}.$$

Also, we assume that there does not exist any outside forces imposed on the system and there is no outside supply of energy into the system. This means that

$$f = 0,$$

$$\text{and} \quad Q = 0. \tag{3.3.1}$$

By comparison with Newtonian fluids, α, is the negative of the hydrostatic pressure, i.e.,

$$\alpha_1 = - P. \tag{3.3.2}$$

and α_2 is the coefficient of viscosity.

Let the temperature at any point inside the annulus be represented by ψ such that:

$$\theta = \theta_1 + \psi(r, \phi, z, t), \tag{3.3.3}$$

which for the steady state case reduces to:

$$\theta = \theta_1 + \psi(r). \tag{3.3.3a}$$

For the steady state flow, the physical components of the stress matrix become:

$$T_{rr} = -P + \alpha_3 \left[2 \left(\frac{\partial v}{\partial r}\right)^2 - \frac{\partial}{\partial r}\left(\frac{v^2}{r}\right) \right], \tag{3.3.4}$$

$$T_{r\phi} = \alpha_2 \left[r \frac{\partial}{\partial r} \left(\frac{v}{r}\right) \right], \tag{3.3.5}$$

$$T_{rz} = -\alpha_5 \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\frac{\partial\psi}{\partial r} \quad , \tag{3.3.6}$$

$$T_{\phi\phi} = -P + \alpha_4\left(\frac{\partial\psi}{\partial r}\right)^2 \quad , \tag{3.3.7}$$

$$T_{\phi z} = 0 \quad , \tag{3.3.8}$$

$$T_{zz} = -P - \frac{1}{r^2}\left(\frac{\partial\psi}{\partial r}\right)^2 \quad . \tag{3.3.9}$$

The equations of motion become:

$$-\rho\frac{v^2}{r} = -\frac{\partial p}{\partial r} + \alpha_3\left[\frac{\partial}{\partial r}\left[2\left(\frac{\partial v}{\partial r}\right)^2 - \frac{\partial}{\partial r}\left(\frac{v^2}{r}\right)\right] + \frac{2}{r}\left(\frac{\partial v}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{v^2}{r}\right)\right] - \frac{\alpha_4}{r}\left(\frac{\partial\psi}{\partial r}\right)^2 \quad , \tag{3.3.10}$$

$$0 = \alpha_2\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right) + 2\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right] , \tag{3.3.11}$$

$$0 = -\alpha_5\left[\frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right) + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right] . \tag{3.3.12}$$

The unknowns are p, v, and ψ, so we do not need the heat conduction equation to solve the system. But it provides us with a means of checking for the correctness of solutions obtained. The boundary conditions are:

$$r = R_1, \quad u = 0 = w, \quad v = R_1\Omega_1, \quad \theta = \theta_1 \; ; \tag{3.3.13}$$

$$r = R_2, \quad u = 0 = w, \quad v = R_2\Omega_2, \quad \theta = \theta_2 \; . \tag{3.3.14}$$

The steady state solutions of the equations (3.3.10) to (3.3.14) are:

$$v = \frac{A}{r} + Br, \tag{3.3.15}$$

$$\psi = \frac{1}{2}\frac{C}{B}\,\ell n(r^2 + \frac{A}{B}) + D, \tag{3.3.16}$$

$$A = \Omega_1 \alpha R_1^2 R_2^2 \;/\; (R_1^2 - R_2^2), \tag{3.3.17}$$

$$B = \Omega_1 \frac{R_2^2 - (\alpha+1)R_1^2}{(R_1^2 - R_2^2)}, \tag{3.3.18}$$

$$\alpha = \frac{\Omega_2}{\Omega_1} - 1, \tag{3.3.19}$$

$$C = \frac{2B(\theta_2 - \theta_1)}{\ell n\left(\frac{A + BR_2^2}{A + BR_1^2}\right)}$$

$$D = \frac{-(\theta_2 - \theta_1)\,\ell n(A + BR_2^2)}{\ell n\left(\frac{A + BR_2^2}{A + BR_1^2}\right)}, \tag{3.3.20}$$

Now to obtain an expression for the pressure gradient, we substitute eqs. (3.3.15) and (3.3.16) into eq. (3.3.10) and the result is:

$$\frac{dP}{dr} = \frac{\rho}{r}\left(\frac{A}{r} + Br\right)^2 + \alpha_3\left[4\frac{AB}{r^3} - A^2\left(\frac{12}{r^5} + \frac{2}{r^4}\right) + 2B^2\right] - \frac{\alpha_4}{r}\frac{C^2}{(A + Br^2)^2}. \tag{3.3.21}$$

A notably interesting result comes out of the solution of the steady state problem, namely, the velocity field is the same as that of a similar problem for Newtonian fluids, while the thermal and the pressure gradient field are different.

3.4 Non-dimensional Form of the Perturbed Problem

We assume a steady state flow and superpose small disturbances on the system such that

$$\underset{\sim}{v}_p = \underset{\sim}{v}' + \underset{\sim}{v}_s, \qquad \underset{\sim}{v}' = (u', v', w'), \tag{3.4.1}$$

$$p_p = p' + p_s, \tag{3.4.2}$$

$$\theta_p = \theta' + \theta_s, \tag{3.4.3}$$

where $\underset{\sim}{v}'$, p', θ' are small perturbations in the velocity field, the pressure field and the thermal field respectively; the subscript s indicates the steady state fields while the subscript p indicates the final perturbed fields. All perturbations are assumed to be functions of the coordinates r, ϕ and z and of the time t. We recall that

$$\underset{\sim}{v}_s = \underset{\sim}{v}_s(r), \qquad \underset{\sim}{v}_s = (0, v, 0), \tag{3.4.4}$$

$$p_s = p_s(r), \tag{3.4.5}$$

$$\theta_s = \theta_s(r). \tag{3.4.6}$$

A very helpful and physically realistic assumption to make is to take the disturbances as infinitesimal and axisymmetric. Thus $\frac{\partial}{\partial\theta} \equiv 0$ and quadratic terms and higher order terms in disturbances should be negligible. The assumption that the disturbances are infinitesimal allows us to write the following expressions for the perturbations.

$$u' = e^{iwt}\, f(r) \cos \lambda z, \qquad (3.4.7)$$

$$v' = e^{iwt}\, g(r) \cos \lambda z, \qquad (3.4.8)$$

$$w' = e^{iwt}\, h_1(r) \sin \lambda z, \qquad (3.4.9)$$

$$\theta' = e^{iwt}\, q_1(r) \cos \lambda z, \qquad (3.4.10)$$

$$p' = e^{iwt}\, \pi(r) \cos \lambda z, \qquad (3.4.11)$$

where f, g, h_1, q, and π are the amplitudes of the corresponding disturbances; w is a constant which can be complex, λ is the wave number of the disturbance. It is simple to see that for damped disturbances the real part of w has to be zero. With these assumptions on hand, and making use of equations (3.3.10), (3.3.11), (3.3.12), (3.4.7), (3.4.8), (3.4.10) and (3.4.11) in the equations of motion (3.2.7), (3.2.7a) and the heat conduction equation (2.3.13a) we obtain in non-dimensional form for steady disturbances $(w = 0)$:

$$- f_1 + (DD^* - a^2)\, g + G_1\left[\left(D^2 + \frac{2}{\zeta} D + \frac{1}{-}\left(\frac{3}{\zeta^4} + \frac{2a^2}{\zeta^2}\right)\right) f_1\right]$$

$$+ \frac{G_2}{1 - k\zeta^2}\, DD^*\, f_1 = 0, \qquad (3.4.12)$$

$$Ta^2\left(\frac{1}{\zeta^2} - \gamma\right) g + \left(DD^* - a^2\right)^2 f_1 + G_3 a^2\left[\left[\left(\gamma + \frac{5}{\zeta^2}\right) D^2\right.\right.$$

$$\left.\left. + 2\left(\frac{a^2}{\zeta^2} + \frac{3}{\zeta^4}\right)\right]\right] g \;+ G_4\left[\left[\frac{\gamma}{(1 - \gamma\zeta^2)}\left(\frac{1}{1 - \gamma\zeta^2} - 1\right) D\right.\right.$$

$$+ \frac{a^2}{\zeta(1 - \gamma\zeta^2)}\Big] q_1 \Big] = 0, \qquad (3.4.13)$$

and

$$- \frac{8}{\zeta^2} (D - \frac{1}{\zeta}) g - \phi_1(\zeta) f_1 + \phi_2(\zeta) D^* f_1 - \phi_3(\zeta) g$$

$$+ G_5 (D^* D - a^2) q_1 + \phi_4(\zeta) D f_1 + \phi_5(\zeta) f_1 - \phi_6(\zeta) D^*$$

$$+ a^2 \phi_7(\zeta) f_1 + a^2 \phi_8(\zeta) g - a^2 \phi_9(\zeta) q_1 = 0, \qquad (3.4.14)$$

where

$$\zeta = \frac{r}{R_2}, \quad D = \frac{D}{D\zeta}, \quad D^* = \frac{D}{D\zeta} + \frac{1}{\zeta},$$

$$a = \lambda R_2,$$

$$f_1 = \frac{2BR_2^2\rho}{\alpha_2} f, \qquad (3.4.15)$$

$$T = \frac{-4ABR_2^2\rho}{\alpha_2}, \qquad \text{(Taylor number)}$$

$$\gamma = - \frac{B}{A} R_2^2,$$

$$G_1 = \alpha_3 \mid \rho R_2^2,$$

$$G_2 = \alpha_5 c \mid \rho ABR_2^2,$$

$$G_3 = G_1 T, \qquad \text{(Modified Taylor number)}$$

$$G_4 = \frac{4\alpha_4 cB^2 R_2^2 \rho}{(\alpha_2)^2 A(\theta_2 - \theta_1)},$$

$$G_5 = \frac{\beta_1 B R_2^2}{A\alpha_2(\theta_2-\theta_1)} ,$$

$$\phi_1(\zeta) = \frac{1}{\rho AB}\left[2ABR_2 \ln\zeta - \frac{A^2}{2R_2^2\zeta^2} + \frac{1}{2} B^2R_2^2\zeta^2 + \alpha_3\left[\frac{gA^2}{2R_2^4}\frac{1}{\zeta^4} \right.\right.$$

$$\left. - 2B^2R_2\ln\zeta - 2\frac{AB}{R_2^2}\frac{1}{\zeta^2}\right] - \alpha_4\left[\frac{c}{2A(A+B\zeta^2R_2^2)}\right.$$

$$\left.\left. - \frac{c^2}{2A^2}\ln\left(\frac{BR_2^2\zeta^2}{A+BR_2^2\zeta^2}\right)\right]\right] ,$$

$$\phi_2(\zeta) = \frac{\alpha_3}{2\rho}\frac{c^2}{BA^3(1-\gamma\zeta^2)^2} ,$$

$$\phi_3(\zeta) = \frac{4\alpha_3}{\alpha_2\zeta}\frac{c^2R_2^2}{A^3(1-\gamma\zeta^2)^2} ,$$

$$\phi_4(\zeta) = \frac{\beta_2}{\rho}\frac{c}{2A^2B(1-\gamma\zeta^2)} ,$$

$$\phi_5(\zeta) = \frac{\beta_2}{\rho}\left[\frac{1}{A^2B\zeta(1-\gamma\zeta^2)} + \frac{2c\gamma\zeta}{A^2B(1-\gamma\zeta^2)^2}\right] ,$$

$$\phi_6(\zeta) = -\frac{\beta_2}{\rho}\left[\frac{2c}{A^2B(1-\gamma\zeta^2)} + \frac{2c\gamma\zeta^2}{A^2B(1-\gamma\zeta^2)^2}\right] ,$$

$$\phi_7(\zeta) = \frac{\beta_2}{\rho}\frac{c\zeta}{2A^2B(1-\gamma\zeta^2)} ,$$

$$\phi_8(\zeta) = \frac{8\beta_3 c}{\alpha_2 A\zeta(1-\gamma\zeta^2)} ,$$

$$\phi_9(\zeta) = -\frac{B_3BR_2}{\alpha_2(\theta_2 - \theta_1)}\left(\frac{B^2R_2}{A} + \frac{4AB}{R_2^3\zeta^4}\right) .$$

Equations (3.4.12), (3.4.13) and (3.4.14) define in non-dimensional form of the general perturbed problem where the perturbations are steady.

Non-dimensional Form of the Boundary Conditions

The walls of the cylinders are rigid, which implies:

$$f = 0 = g,\ h_1 = 0 \quad \text{at} \quad \zeta = \frac{R_1}{R_2},\ 1.$$

The equation of continuity is:

$$D^*f + h_1 = 0,$$

which implies that $D^*f = 0$ at $\zeta = \frac{R_1}{R_2}, 1.$

Using eq. (3.4.14) we get

$$f_1 = 0 = D^*f_1,\ g = 0 \quad \text{at} \quad \zeta = \frac{R_1}{R_2},\ 1.$$

The heat reservoir situation gives:

$$q_1 = \theta_1 \quad \text{at} \quad \zeta = \frac{R_1}{R_2}$$

$$q_1 = \theta_2 \quad \text{at} \quad \zeta = 1$$

3.5 The Plane Layer Approximation

Solving eqs. (3.4.12), (3.4.13) and (3.4.14) proves to be a very difficult matter. Hence we consider the case of a narrow gap between the cylinders. In this case we find that we can apply the plane layer approximation technique. This changes the geometry of the problem from cylindrical to planar, which could be fairly well approximated when:

$$(R_2 - R_1) << \frac{1}{2}(R_2 + R_1) = \frac{R_0}{2} . \quad (3.5.1)$$

This assumption introduces the following simplifications in the derivatives:

$$D^* = D = \frac{d}{dr}, \quad DD^* = D^*D = D^2 = \frac{d^2}{dr^2} . \quad (3.5.2)$$

Also from eq. (3.5.1) it follows that we may neglect terms of order $\left[2\frac{(R_2 - R_1)}{R_0}\right]^2$ and smaller in all of our equations. As an illustration of this technique we consider

$$B + \frac{A}{r^2} = \Omega_1\left\{1 - \frac{2\alpha R_2^2}{R_2 + R_1}\frac{r - R_1}{(R_1 - R_2)R_1}\right\} .$$

Set $x = \dfrac{2(R_2 - R_1)}{R_1 + R_2}$ where $x << 1$.

Then $\dfrac{R_1}{R_2} = \dfrac{2 - x}{2 + x}$.

Consider $\dfrac{2R_2^2}{R_1(R_1 + R_2)} = \dfrac{4 + 4x + x^2}{4(1 - x/2)} = \dfrac{1}{4}(4 + 4x + x^2)$

$$(1 + \frac{x}{2} + \frac{x^2}{4} + \ldots) \cong 1 + \frac{3x}{2}$$

$$1 + \frac{3x}{2} = 1 + \frac{3(R_2 - R_1)}{R_1 + R_2}$$

$$= \left(2 - \frac{R_1}{R_2}\right)\left(\frac{2}{1 + R_1/R_2}\right)$$

$$= \left(2 - \frac{R_1}{R_2}\right)\left(1 + \frac{x}{2}\right) .$$

Thus $B + \frac{A}{r^2} \cong \Omega_1 \left\{ 1 + \alpha\left(2 - \frac{R_1}{R_2}\right)\left(1 + \frac{x}{2}\right)\left(\frac{r - R_1}{R_2 - R_1}\right)\right\}$

$$\cong \Omega_1 \left\{ 1 + \alpha\left(2 - \frac{R_1}{R_2}\right)\left(\frac{r - R_1}{R_2 - R_1}\right)\right\}.$$

The equations of the perturbed problem in dimensional form can be written as (for steady disturbances):

$$- 2 \quad B + \frac{A}{r^2} \quad g + \frac{\alpha_2}{\rho}\left[\frac{1}{\lambda^2}\ (DD^* - \lambda^2)^2 f\right] + \frac{\alpha_3}{\rho}\left[\left(2B - \frac{10A}{r^2}\right) D^2 + \frac{2A}{r^3} D - \frac{8B}{r^2} - \left(\frac{4A^2}{r^2} + \frac{12A}{r^4}\right)\right] g\Bigg]$$

$$+ \frac{\alpha_4}{\rho}\left[\left[\left(\frac{1}{r^2}\ \frac{\partial^2\theta_s}{\partial r^2} - \frac{3}{r^3}\ \frac{\partial\theta_s}{\partial r}\right) D + \frac{2\lambda^2}{r^2}\ \frac{\partial\theta_s}{\partial r}\right] q_1\right] = 0, \qquad (3.5.3)$$

$$- 2Bf + \frac{\alpha_2}{\rho}\left[(DD^* - \lambda^2) g\right] + \frac{\alpha_3}{\rho}\left[\left(2BD^2 + \frac{4B}{r} D - \left(\frac{3A}{r^4} + 2\ \frac{A\lambda^2}{r^2}\right)\right) f\right] + \frac{\alpha_5}{\rho}\left[\frac{2}{r}\ \frac{\partial\theta_s}{\partial r}\ DD^* f\right] = 0, \qquad (3.5.4)$$

and $- 8\ \frac{\alpha_2}{\rho}\left[\frac{A}{r^2}\left(D - \frac{1}{r}\right) g\right] - \frac{2}{\rho}\left\{\frac{2AB\ell nr}{r} - \frac{A^2}{2r^3} + \frac{1}{2} B^2 r\right.$

$$+ \alpha_3 \left[\frac{g}{2} \frac{A^2}{r^5} - 2 \frac{B^2 \ln r}{r} - 2 \frac{AB}{r^3} \right] - \frac{\alpha_4}{r} \left[\frac{c^2}{2Ar(A + Br^2)} \right.$$

$$\left. \left. - \frac{c^2}{rA^2} \ln \left(\frac{Br^2}{A + Br^2} \right) \right] \right\} f + \frac{\alpha_3}{\rho} \left[\left(\left(4 \frac{A^2}{r^4} - 2 \frac{AB}{r^2} \right) D \right. \right.$$

$$\left. \left. - \frac{8A^2}{r^5} \right) g \right] + \frac{\alpha_4}{\rho} \left[\left(\frac{2}{r^2} \left(\frac{\partial \theta_s}{\partial r} \right)^2 \right) D^* f - \frac{4}{r^3} \left(\frac{\partial \theta_s}{\partial r} \right)^2 g \right]$$

$$+ \frac{\beta_1}{\rho} \left[(D^* D - \lambda^2) q_1 \right] + \frac{\beta_2}{\rho} \left[\frac{2}{r} \frac{\partial \theta_s}{\partial r} Df + \frac{2}{r} \frac{\partial \theta_s}{\partial r^2} f \right.$$

$$\left. - \left(2 \frac{\partial^2 \theta_s}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_s}{\partial r} \right) D^* f + \lambda^2 \frac{\partial \theta_s}{\partial r} f \right] + \frac{\beta_3}{\rho} \left[8 \frac{A\lambda^2}{r^2} \frac{\partial \theta_s}{\partial r} g \right.$$

$$\left. - \left(B^2 + \frac{4A^2}{r^4} \right) \lambda^2 q_1 \right] = 0. \qquad (3.5.5)$$

If we apply the plane layer approximation technique to eqs. (3.5.3), (3.5.4) and (3.5.5), and non-dimensionalize we obtain:

$$(D^2 - a^2)^2 f + \left[\frac{1}{a^2} (N_1 + N_2 \xi) D^2 + (N_3 + N_4 \xi) D \right.$$

$$\left. + \left(\frac{N_5}{a^2} + M_1 \right) + \left(\frac{N_6}{a^2} + M_2 \xi \right) \right] g + \left[(N_7 + N_8 \xi) D \right.$$

$$\left. + a^2 (N_9 + N_{10} \xi) \right] q_1 = 0, \qquad (3.5.6)$$

$$a^2 Tf + (D^2 - a^2) g + \left[a^2 (N_{11} + N_{16} + N_{17} \xi) D^2 \right.$$

$$\left. + (N_{12} + N_{13} \xi) D + \left(a^2 N_{14} + N'_{14} + (a^2 N_{15} + N'_{15}) \xi \right) \right] f = 0,$$

$$(3.5.7)$$

$$\text{and}\ \left[(M_3 + M_4\xi)D + (M_5 + a^2N_{36}) + M_6 + a^2N_{37})\xi \right] g$$

$$+ \left[\left(a^2(N_{27} + N_{32}) + a^2(N_{28} + N_{33})\xi\right) D + \left(M_7 + a^2N'_{34} \right.\right.$$

$$\left.\left. + (M_8 + a^2N'_{35})\xi\right)\right] f + \left[N_{31}D^2 + a^2M_9 + N_{39}\xi \right] q_1 = 0, \tag{3.5.8}$$

where ξ, N_i, and M_j are non-dimensional quantities defined as:

$$\xi = (r - R_1)/(R_2 - R_1), \quad 0 \leq \xi \leq 1, \quad D = d/d\xi.$$

$$d = R_2 - R_1, \quad R_0 = R_1 + R_2,$$

$$\gamma_1 = \frac{2(\theta_2 - \theta_1)}{\ln(2\alpha + 2)}, \quad \alpha = \frac{\Omega_2}{\Omega_1} - 1,$$

$$\gamma_2 = \alpha\left(2 - \frac{R_1}{R_2}\right),$$

$$\gamma_3 = \frac{R_1}{R_1 + R_2},$$

$$\gamma_4 - \frac{R_1}{R_2 - R_1},$$

$$\gamma_5 = \frac{1}{\gamma_4},$$

$$\gamma_6 = \frac{R_2}{R_1 + R_2},$$

$$\gamma_7 = \frac{R_2}{R_2 - R_1},$$

$$\gamma_8 = \left(1 - \left(\frac{1}{\mu_1}\right)^2\right), \quad \mu_1 = \frac{R_2}{R_1},$$

$$N_1 = \frac{\alpha_3 \nu}{4\rho\Omega_1 d^4}\left[2 + \frac{3}{4}\gamma_7\right], \quad \nu = \alpha_2/\alpha = \text{kinematic viscosity},$$

$$N_2 = -\frac{\alpha_3 \nu}{4\rho\Omega_1 d^4}\gamma_2,$$

$$N_3 = -\frac{\alpha_3 \nu}{4\rho\Omega_1 d^4},$$

$$N_4 = \frac{3\alpha_3 \nu}{4\Omega_1 d^3 R_1}\gamma_2,$$

$$N_5 = -\frac{\alpha_3 \nu}{4\rho\Omega_1 d^4}\left[2\gamma_5(1 + \gamma_5) + 3\gamma_2\gamma_5\frac{d}{R_0}\right],$$

$$N_5' = \frac{\alpha_3 \nu}{\rho\Omega_1 d^4},$$

$$N_6 = \frac{\alpha_3 \nu}{\rho\Omega_1 d^4}\frac{(\gamma_6 + 3\gamma_2)}{\gamma_3},$$

$$N_6' = \frac{\alpha_3 \nu}{\rho\Omega_1 d^4}\gamma_6\gamma_7,$$

$$N_7 = -\frac{\alpha_4\gamma_1(\theta_2 - \theta_1)}{4\rho\Omega_1^2 d^3 R_1^3}\left[\frac{\alpha d}{4R_0} + \frac{(\alpha + 16)}{32}\right],$$

$$N_8 = -\frac{\alpha_4\gamma_1(\theta_2 - \theta_1)}{4\rho\Omega_1^2 d^3 R_1^3}\left[\frac{16d^2 + dR_0 + \alpha^2 R_1 R_0 + 32d^2 - \alpha^2 dR_0}{32 R_1 dR_0}\right],$$

$$N_9 = -\frac{\alpha_4\gamma_1(\theta_2 - \theta_1)}{4\rho\Omega_1^2 d^3 R_1^3}\frac{\alpha\gamma_3}{2},$$

$$N_{10} = \frac{\alpha_4\gamma_1(\theta_2 - \theta_1)}{8\rho\Omega_1^2 d^3 R_1^3} \left[\frac{d}{R_0} - \alpha^3\right],$$

$$N_{11} = \frac{4\alpha_3 d^2\Omega_1^2}{\rho\nu^2} (1 + \alpha\gamma_6\gamma_7),$$

$$N_{12} = \frac{8\alpha_3 d^2\Omega_1^2}{\rho\nu^2} (\gamma_5 - \gamma_2),$$

$$N_{13} = -\frac{8\alpha_3 d^2\Omega_1^2}{\rho\nu^2} (\gamma_5^2 + \gamma_2\gamma_5),$$

$$N_{14} = \frac{4\alpha_3 d^2\Omega_1^2}{\rho\nu^2} \alpha\gamma_6\gamma_7 ,$$

$$N'_{14} = -\frac{6\alpha_3 d^2\Omega_1^2}{\rho\nu^2} \gamma_2\gamma_5 \frac{d}{R_0} ,$$

$$N_{15} = 2N_{14} ,$$

$$N'_{15} = N'_{14}\gamma_3 ,$$

$$N_{16} = -\frac{\alpha_5\gamma_1 d}{\rho\nu^2 R_0} \alpha ,$$

$$N_{17} = -\frac{\alpha_5\gamma_1 d}{\rho\nu^2 R_0} \alpha^2,$$

$$N_{18} = \gamma_6\gamma_7 ,$$

$$N_{19} = 2\alpha\mu_1\gamma_6 ,$$

$$N_{20} = \alpha + 3\gamma_2\gamma_5,$$

$$N_{21} = \frac{\alpha^4}{4\nu^2} \frac{\Omega_1^2 \left(3\alpha d R_1(\mu_1-3) - 18\alpha^2(\mu_1 R_1^2) - \alpha^2 d R_1 - 3d^2\left(1+\frac{\alpha}{\alpha_8}\right)^2\right)}{3d^2}$$

$$+ \frac{\alpha_3\Omega_1^2}{\rho} \frac{\left(g\alpha^2 - 4(\gamma_8)^2 \ln R_1\left(1+\frac{\alpha}{(\gamma_8)^2}\right)^2 - 4\alpha\gamma_8\left(\frac{\alpha}{\gamma_8} - 1\right)\right)}{dR_1(\gamma_8)^2}$$

$$+ \frac{\alpha_4\gamma_1^2}{\rho}\left[\frac{R_1^2(\alpha-\gamma_8) + 4R_0 d \ln\left(\frac{\alpha\gamma_3}{8}\right)}{4R_1^5 R_0 d^2}\right]\Bigg\},$$

$$N_{22} = \frac{d^3}{4\nu^2}\Bigg\{2\Omega_1^2\Bigg[\alpha(\gamma_6 - 3\gamma_3)d - 3\alpha^2\mu_1 R_1(1 - \ln R_1)$$

$$- \frac{\alpha^2 d}{\alpha}\left(1 + \frac{\alpha}{\gamma^8}\right)^2\Bigg]$$

$$+ \frac{\alpha_3\Omega_1^2}{\rho}\left[\frac{9\alpha^2 d + rR_1(\gamma_8)^2\left(1+\frac{\alpha}{\gamma_8}\right)^2 - 24R_1\gamma_8\left(\frac{\alpha^7}{\gamma_8} - 1\right)}{R_1^2\ (\gamma_8)^2\ \gamma_4}\right],$$

$$+ \frac{\alpha_4\gamma_1^2}{8\rho\alpha R_1^6 R_0^2}\Bigg[8\alpha R_0^2 d\Bigg(2 - \ln\left(\frac{\alpha\gamma_3}{8}\right) + R_1^2 R_0(\gamma_8 - \alpha)^2$$

$$- 4R_1^2\alpha^2 d(\gamma_8 - \alpha)^2 - 2\alpha R_1^2 R_0(\gamma_8 - \alpha)\Bigg)\Bigg]\Bigg\},$$

$$N_{23} = \frac{\alpha_3\Omega_1}{8\rho\nu}\left[(1+\alpha^2)(3\gamma_3 - \gamma_6) + 3\alpha^2\gamma_7\right],$$

$$N_{24} = \frac{\alpha_3\Omega_1}{8\rho\nu}\left[3\alpha^2(7\gamma_7 - 4\gamma_4) - \alpha(\gamma_6 - \gamma_3) - \frac{3}{2}\alpha^2\mu_1\right],$$

$$N_{25} = -\frac{\alpha_3\Omega_1}{8\rho\nu}\alpha^2\gamma_3\frac{d}{R_0},$$

$$N_{26} = \frac{5\alpha_3\Omega_1}{8\rho\nu}\frac{(\gamma_6 - \gamma_3)d}{R_0},$$

$$N_{27} = \frac{\alpha_4\gamma_1^2}{32\rho\nu^2R_0^2}\alpha^2,$$

$$N_{28} = \frac{N_{27}}{2}\alpha,$$

$$N_{29} = -\frac{\alpha_4\gamma_1^2}{32\rho\nu\Omega_1}\frac{\alpha 2}{R_1R_2^2d},$$

$$N_{30} = -\frac{\alpha_4\gamma_1^2}{32\rho\nu\Omega_1}\left(\frac{\alpha^2}{R_1^2R_0^2} - \frac{\alpha^3}{R_0^3d}\right),$$

$$N_{31} = \beta_1\frac{(\theta_2 - \theta_1)}{8\rho\nu\Omega_1^2d^2}\gamma_5,$$

$$N_{32} = \frac{\beta_2\gamma_1}{4\rho\nu^2}\left[\frac{\alpha d}{2R_0} + \alpha^2\right],$$

$$N_{33} = -\frac{\beta_2\gamma_1}{128\rho\nu^2}(2\alpha^2\gamma_5 + \alpha^3),$$

$$N_{34} = - \frac{\beta_2\gamma_1}{32\rho\nu^2}\left[\frac{\alpha^2\gamma_5}{\alpha} + 4\gamma_5 \frac{d}{R_0}\right] ,$$

$$N'_{34} = - \frac{\beta_2\gamma_1}{32\rho\nu^2} \alpha ,$$

$$N_{35} = \frac{\beta_2\gamma_1}{4\rho\nu^2}\left[\frac{\alpha^3}{32}\frac{d}{R_0} + \frac{3\alpha^2(\gamma_5)^2}{32} - \frac{\alpha(\gamma_5)^2 d}{2R_0}\right] ,$$

$$N'_{35} = \frac{\beta_2\gamma_1}{4\rho\nu^2}\left[\frac{\alpha d}{4R_0} + \frac{\alpha^2}{32}\right] ,$$

$$N_{36} = - \frac{\beta_3\gamma_1}{4\rho\nu d^2}\frac{\alpha^2}{\gamma_8}\gamma_3 ,$$

$$N_{37} = \frac{\beta_3\gamma_1}{32\rho\nu d^2}\left[8\frac{\alpha^2 d}{\gamma_8 R_0} - \frac{\alpha^3}{\gamma_8}\right] ,$$

$$N_{38} = - \frac{\beta_3(\theta_2 - \theta_1)}{8\rho\nu d^2}\gamma_5\left[\frac{(\gamma_8 - \alpha)^2 - 4\alpha^2}{(\gamma_8)^2}\right] ,$$

$$N_{39} = - \frac{\beta_3(\theta_2 - \theta_1)}{2\rho\nu d^2}\gamma_5^2\left[\frac{100\alpha^2}{(\gamma_8)^2}\right] .$$

$$M_1 = N'_5 - 1 ,$$

$$M_2 = N'_6 + \gamma_2 ,$$

$$M_3 = N_{18} + N_{23} ,$$

$$M_4 = N_{19} + N_{24} ,$$

$$M_5 = N_{24} + N_{29},$$

$$M_6 = N_{20} + N_{26} + N_{30},$$

$$M_7 = N_{21} + N_{34},$$

$$M_8 = N_{22} + N_{35},$$

$$M_9 = N_{38} - N_{31}.$$

Equations (3.5.6), (3.5.7) and (3.5.8) will be used in the next chapter to establish the characteristic value problem the solution of which will yield the stability criterion for thermo-viscoelastic fluids between two corotating cylinders. The boundary conditions for this problem are:

$$f = 0 = Df \quad \text{at} \quad \xi = 0,\ 1 \tag{3.5.9}$$

$$g = 0 \quad \text{at} \quad \xi = 0,\ 1 \tag{3.5.10}$$

$$q_1 = \theta_1, \theta_2 \quad \text{at} \quad \xi = 0,\ 1 \text{ respectively.} \tag{3.5.11}$$

IV. CHARACTERISTIC VALUE PROBLEM OF THE STABILITY ANALYSIS

4.1 Introduction

Having deduced the non-dimensional equations which govern the flow of thermo-viscoelastic fluids between two concentric cylinders, we shall use them to reduce the stability problem into a characteristic value problem. This is done by expanding one of the unknowns involved by an infinite series of a set of orthogonal, complete functions, Chandrashaker, (1961). An infinite order secular equation results from this analysis involving T, the Taylor's number. We solve for T from this equation for each non-dimemsional wave number a by a suitable approximation technique. The least positive T we find is the critical Taylor's number which yields the criterion for stability.

4.2 Reduction of the Stability Problem to a Characteristic Value Problem

Our starting point in the analysis of reducing the stability problem to a characteristic value problem is eqs. (3.5.6), (3.5.7) and (3.5.8). Let's represent f by a doubly infinite series of the form:

$$f = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin m\pi\xi \sin n\pi\xi \ . \qquad (4.2.1)$$

where c_{mn} are unknown coefficients and which satisfies the boundary conditions given by eq. (3.5.9). Substituting eq. (4.2.1) in eq. (3.5.7) and solving for g we get:

$$g = \begin{cases} \hat{g} & m \neq n \\ \check{g} & m = n, \end{cases} \qquad \text{where}$$

$$\hat{g} = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \Bigg\{ 2\hat{\alpha}_{mn} \cosh a\xi + 2\hat{\beta}_{mn} \sinh a\xi$$

$$+ \Bigg[a^2(N_{11} + N_{16})(m + n)^2\pi^2 - (a^2N_{14} + N'_{14} + a^2T)$$

$$+ a^2N_{17}(m + n)^2\pi^2\xi - (a^2N_{15} + N'_{15})\xi$$

$$+ \frac{2(m + n)^2\pi^2N_{13}}{a^2 + (m + n)^2\pi^2} \Bigg] \frac{\cos(m + n)\pi\xi}{a^2 + (m+n)^2\pi^2}$$

$$+ \Bigg[(a^2N_{14} + N'_{14} + a^2T) - a^2(N_{11} + N_{16})(m - n)^2\pi^2$$

$$- a^2N_{17}\pi^2(m - n)^2\xi - \frac{2(m-n)^2\pi^2N_{13}}{a^2 + (m-n)^2\pi^2} + (a^2N_{15} + N'_{15})\xi \Bigg]$$

$$\times \cos(m - n)\pi\xi/a^2 + \Bigg[(m - n)^2\pi^2a^2(N_{11} + N_{16})$$

$$+ (m + n)\pi(N_{12} + N_{13}) + \frac{(a^2N_{17}(m - n)^2(m + n)\pi^3)}{a^2 + (m + n)^2\pi^2}$$

$$- a^2N_{15} - N'_{15}\Big]\frac{\sin(m + n)\pi\xi}{a^2 + (m+n)^2\pi^2} + \Big[(m + n)\pi(N_{12} - N_{13})$$

$$+ \frac{a^2N_{17}(m + n)^2(m - n)\pi^3}{a^2 + (m - n)^2\pi^2} - a^2N_{15} - N'_{15}\Big]\frac{\sin(m - n)\pi\xi}{a^2+(m-n)^2\pi^2} , \quad (4.2.2)$$

$$\check{g} = \sum_{m=1}^{\infty} c_{mn}\Big\{\check{\alpha}_{mn}\cosh a\,\xi + \check{\beta}_{mn}\sinh a\xi + (a^2N_{15} + N'_{15}$$

$$- 2a^2m^2\pi^2N_{17})\frac{\xi}{a^2} + \Big[\frac{(a^2N_{15} + N'_{15})}{2m\pi} - m\pi N_{13}\Big]\xi\,\frac{\sin 2m\pi}{(a^2+4m^2\pi^2)}$$

$$+ \Big[(a^2N_{15} + N'_{15})\Big(\frac{2}{a^2 + 4m^2\pi^2} + \frac{1}{4m^2\pi^2}\Big) - \frac{4m^2\pi^2N_{13}}{(a^2+4m^2\pi^2)}\Big]\frac{\cos 2m\pi\xi}{(a^2+4m^2\pi^2)}$$

$$+ \Big[(a^2N_{14} + N'_{14} + a^2T) - m^2\pi^2(a^2N_{11} + N_{16})\Big]\,2\frac{(\sin m\pi\xi)^2}{a^2+4m^2\pi^2}$$

$$- m\pi N_{12}\frac{\sin 2m\pi\xi}{a^2 + 4m^2\pi^2}\Big\} , \quad (4.2.3)$$

where $\hat{g}$ indicates g when $m \neq n$ and $\check{g}$ indicates g when $m = n$, and, for $m \neq n$:

$$\hat{\alpha}_{mn} = -\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\Big\{\Big[a^2(N_{11} + N_{16})(m + n)^2\pi^2$$

$$- (a^2N_{14} + N'_{14} + a^2T) + \frac{2(m+n)^2\pi^2N_{13}}{a^2+(m+n)^2\pi^2}\Big]\frac{(-1)^{m+n}}{a^2+(m+n)^2\pi^2}$$

$$+ \left[(a^2 N_{14} + N'_{14} + a^2 T) - a^2 (N_{11} + N_{16}) (m - n)^2 \pi^2 - \frac{2(m-n)^2 \pi^2 N_{13}}{a^2 + (m-n)^2 \pi^2} \right] \frac{(-1)^{m-n}}{a^2 + (m-n)^2 \pi^2} \Bigg\}, \quad (4.2.4)$$

$$\hat{\beta}_{mn} = - \frac{\alpha_{mn}}{\sinh a} \Bigg\{ \cosh a + \left[a^2 N_{17} (m + n)^2 \pi^2 - (a^2 N_{15} + N'_{15}) \right] \times \frac{(-1)^{m+n}}{a^2 + (m+n)^2 \pi^2} + \left[(a^2 N_{15} + N'_{15}) - a^2 N_{17} (m + n)^2 \pi^2 \right] \times \frac{(-1)^{m-n}}{a^2 + (m+n)^2 \pi^2} \Bigg\}, \quad (4.2.5)$$

for $m = n$:

$$\check{\alpha}_{mn} = \frac{4m^2 \pi^2 N_{13}}{(a^2 + 4m^2 \pi^2)} - \frac{(a^2 N_{15} + N'_{15})}{(a^2 + 4m^2 \pi^2)} \left[\frac{1}{4m^2 \pi^2} + \frac{2}{(a^2 + 4m^2 \pi^2)} \right], \quad (4.2.6)$$

$$\check{\beta}_{mn} = \frac{1}{\sinh a} \Bigg\{ \frac{(a^2 N_{15} + N_{15})(\cosh a - 1)}{(a^2 + 4m^2 \pi^2)} \left[\frac{1}{4m^2 \pi^2} + \frac{2}{(a^2 + 4m^2 \pi^2)} \right] - \frac{(a^2 N_{15} + N'_{15}) - 2m^2 \pi^2 a^2 N_{17}}{a^2} \Bigg\}. \quad (4.2.7)$$

Eq. (3.5.8) can be written as:

$$\left[D^2 + \frac{M_9}{N_{31}} + \frac{N_{39}}{N_{31}} \xi \right] q_1 = - \frac{1}{N_{31}} \Bigg\{ \Big[(M_3 + M_4 \xi) D + (M_5 + a^2 N_{36}) + (M_6 + a^2 N_{37}) \xi \Big] g - a^2 \Big[\Big((N_{27} + N_{32}) + N_{28} + N_{33}) \xi \Big) D$$

$$+ (M_7 + N'_{34}) + (M_8 + N_{35})\xi \Big] f \Big\}. \qquad (4.2.8)$$

In order to illustrate the present stability analysis, we consider certain types of thermo-viscoelastic fluids governed by the following relations:

We set $N_{38} = N_{31}$ i.e. $M_9 = 0$. This reduces the above differential equations into the following form:

$$\left[D^2 + \frac{N_{39}}{N_{31}}\right] q_1 = Q(\xi), \qquad (4.2.9)$$

where $Q(\xi)$ is the right hand side of eq. (4.2.8). Eq. (4.2.9) is a non-homogeneous Airy's equation. Airy's functions $Ai(\sqrt[3]{N_{39}/N_{31}}\xi)$ and $Bi(\sqrt[3]{N_{39}/N_{31}}\xi)$ are solutions of the homogeneous equation and are defined by

$$Ai = c_1 \sum_{k=0}^{\infty} 3^k \left(\frac{1}{3}\right)_k \frac{Ai(\sqrt[3]{N_{39}/N_{31}}\xi)^{3k}}{3k!} - c_2 \sum_{k=0}^{\infty} 3^k \left(\frac{2}{3}\right)_k \times \frac{Ai(\sqrt[3]{N_{39}/N_{31}}\xi)^{3k+1}}{(3k+1)!}$$

$$Bi = \sqrt{3}\left[c_1 \sum_{k=0}^{\infty} 3^k \left(\frac{1}{3}\right)_k \frac{Bi(\sqrt[3]{N_{39}/N_{31}}\ )^{3k}}{3k!} + c_2 \sum_{k=0}^{\infty} 3^k \left(\frac{2}{3}\right)_k \times \frac{Bi(\sqrt[3]{N_{39}/N_{31}}\xi)^{3k+1}}{(3k+1)!}\right],$$

where $\left(\alpha + \frac{1}{3}\right)_k 3^k = (3\alpha + 1)(3\alpha + 4)\ldots(3\alpha + 3k - 2)$ for arbitrary α and $k = 1, 2, 3\ldots$, where c_1 and c_2 are

Airy's constants, Abramowitz (1967). Hence the solution of eq. (4.2.9) is:

$$q_1(\xi) = p_{mn}Ai(\sqrt[3]{N_{39}/N_{31}}\xi) + q_{mn}Bi(\sqrt[3]{N_{39}/N_{31}}\xi) + q_p(\xi), \tag{4.2.12}$$

where p_{mn} and q_{mn} are determined by the boundary conditions and q_p is the particular integral of eq. (4.2.9) and can be readily obtained by the method of variation of parameters involving the Wronskian of $Ai(\sqrt[3]{N_{39}/N_{31}}\xi)$ and $Bi(\sqrt[3]{N_{39}/N_{31}}\xi)$, [Elementary differential equations, Boyce and DePrima]. Eqs. (4.2.12), (4.2.13), and (4.2.14) actually stand for two separate sets of equations, one for the case $m = n$, the other for the case $m \neq n$. Finally, to obtain the characteristic value problem, we substitute eqs. (4.3.1), (4.3.2), (4.3.3) and (4.3.12) in eq. (3.5.6) and then integrate between 0 and 1 obtaining:

$$\sum_{m=1}^{\infty} \sum_{m=1}^{\infty} c_{mn}[F_{mn}(a, T, N_i, \alpha)] = 0, \quad i = 1 \text{ to } 39, \tag{4.2.15}$$

where F_{mn} is defined by:

$$\begin{aligned} F_{mn}(a, T, N_i, \alpha) = {} & a^4 \int_0^1 f + (N_1 + N_2\xi)Dg \Big|_0^1 \\ & + (N_5 - N_4 - 1)\int_0^1 g + (N_6 - \gamma_2)\left(\int g \Big|^1 - \iint g \Big|_0^1 \right) \\ & + N_7 q_1 \Big|_0^1 + N_8\left(q_1 \Big|^1\right) + (a^2 N_g - N_g) \int_0^1 q_1 \\ & + a^2 N_{12}\left(\int q_1 \Big|^1 - \iint q_1 \Big|_0^1 \right) , \end{aligned} \tag{4.2.16}$$

where q_1 and g and f are known.

Eqs.(4.2.15) form a set of linear homogeneous equations which can be solved for the coefficients c_{mn} provided there exist nontrivial solutions of

$$\|F_{mn}(a,\ T,\ N_i,\ \alpha)\| = 0 \qquad (4.2.17)$$

$$m,\ n = 1,\ 2,\ \ldots,\quad i = 1 \text{ to } 39.$$

The solving of eq. (4.2.17) for a minimum positive real value of T constitutes the characteristic value problem. For given values of N_i, $i = 1, 39$, R_2/R_1 and $(\theta_2 - \theta_1)$ a value of a is chosen and eq. (4.2.17) is solved for the lowest positive value of T. This procedure is repeated for different values of a until the minimum lowest value of T is found. The solving of the infinite order characteristic eq. (4.2.17) is accomplished by the approximate method of setting the finite determinant made up of the first k rows and columns equal to zero and solving for T. The usefulness of this method is determined by how rapidly the lowest positive value of T approaches its limit as $k \to \infty$. For the classical viscous fluid case, Chandrashaker (1954) has found that a very rapid convergence is expected. For non-Newtonian fluids, Narasimhan (1963) and Graebel (1962) have independently shown that this holds true also. In our present investigation also the above procedure has been found to be rapidly convergent.

4.3 Solution of the Characteristic Value Problem and Critical Taylor Numbers

As an illustration of the above stability investigation, we choose the following set of data:

$$R_1 = 1.0 \text{ cm} ,$$

$$R_2 = 1.03 \text{ cm} ,$$

$$\frac{\alpha_3 \nu}{4\rho\Omega_1 d^4} = 1.0 ,$$

$$\frac{16 d^6 \Omega_1^3}{\nu^3} = 1.0 \times 10^{-3} ,$$

$$\frac{\alpha_4 \gamma_1 (\theta_2 - \theta_1)}{4\rho\Omega_1^2 R_1^6} = 2.0 ,$$

$$\frac{\alpha_5 \gamma_1 d}{\rho\nu^2 R_0} = 2.0 ,$$

$$\frac{\beta_2 \gamma_1}{4\rho\nu^2} = 5.0 ,$$

$$\frac{\beta_3 \gamma_1}{4\rho\nu d^2} = 5.0 ,$$

$$\text{and} \quad 0.5 \le N_{31} \le 1.$$

The relation for N_{31} was chosen to facilitate comparison with the stability analysis of other workers in the case of classical viscous fluids.

Table I.

α	a	T
.5	4.2	1.785×10^3
1.	4.5	1.869×10^4
1.25	5.3	2.305×10^4

V. SUMMARY, DISCUSSION AND SCOPE OF FURTHER WORK

5.1 Summary and Discussion

In the previous chapters we have presented Eringen's theory of thermo-viscoelasticity and applied it to the problem of stability of couette flow between two heat reservoirs. In the course of this application we have solved the steady state problem and reduced the stability problem into a characteristic value problem and obtained the solution for it. Comparing the results given in Table I with the existing stability investigations, we find that thermo-viscoelastic fluids in a couette flow in the absence of a magnetic field are more stable than classical viscous fluids in the same situation. In the non-Newtonian realm, the results obtained physically indicate that thermo-viscoelastic fluids, like Bingham plastics, are more stable than viscous fluids under similar conditions, unlike Reiner-Rivlin fluids which have been found to be less stable than viscous fluids. This behavior of the flow is essentially due to the viscoelastic nature of the fluid under thermal as well as rotation effects.

5.2 Scope of Further Work

The present investigation of stability of thermo-viscoelastic fluid flow between two rotating coaxial cylinders maintained at constant temperatures has been restricted to narrow gap width between the cylinders in order to simplify the complex nature of the problem. But, it should be definitely possible to extend this problem to the wide gap case. Further, it should be interesting to consider the influence of a superposed electromagnetic field or density gradient field on the stability of the flow, in the case of conducting fluids. Also, further investigations into the interactions of thermo-viscoelastic character of the fluid with a variety of different combinations of rotational, thermal, magnetic and density fields should prove to be of great interest in technological as well as theoretical studies of stability of non-Newtonian fluids in general.

BIBLIOGRAPHY

Abramowitz, M. and Stegun, A. 1965. Handbook of Mathematical functions. New York, Dover. 446.

Bénard, H. 1900. Les tourbillons cellulaires dans une mappe liquide. Revue génêrales des Sciences pures et appliqûes. 11:1261-71 and 1309-28.

Chandrashaker, S. 1952. On the inhibition of convextion by a magnetic field. Phil. Mag. Sci. 7, 43:501-32.

Chandrashaker, S. and Fermi, E. 1953. Problems of gravitational stability in the presence of a magnetic field. Astrophys. J. London, 113:667-78.

Chandrashaker, S. 1954. On characteristic value problems in high order differential equations which arise in studies on hydrodynamic and hydromagnetic stability. American Math. Monthly, 61:32-45.

____________ 1955. The character of the equilibrium of an incompressible heavy viscous fluid of variable density. Proc. Camb. Phil. Soc. 51:162-78.

____________ 1956. On the stability of the simplest solutions of the equations of hydromagnetics. Proc. Nat. Acad. Sci. 42:273-6.

____________ 1957. Thermal convection. Daedalus. 86: 323-39.

____________ 1958. The thermodynamics of thermal instability in liquids. Max Plank Festschrift, Veb Deutscher Verlay det Wissenschafter, Berlin. 103-14.

____________ 1961. Hydrodynamic and hydromagnetic stability. Permagon Press, Oxford.

Coleman, B.D., H. Markowitz and W. Noll. Viscometric flows of non-Newtonian fluids.

Dean, W.R. 1928. Fluid motions in a curved channel. Proc. Roy. Soc. (London), Ser. A, 121:402-20.

DePrima, R.C. 1959. The stability of viscous flow between rotating concentric cylinders with a pressure gradient acting round the cylinder. J. Fluid Mech. 6: 462-8.

England, A.H. and A.E. Green, 1961. Steady-state thermoelasticity for initially stressed bodies. Phi. Trans. Ser. A. 253:517-542.

Eringen, A.C. 1962. Nonlinear theory of continuous media. New York, McGraw-Hill.

Eringen, A.C. and S.L. Koh 1963. On the foundations of non-linear thermo-viscoelasticity. Int. J. Engn. Sci. 1:199-229.

Eringen, A.C. 1967. Mechanics of continua. New York, Wiley.

Flavin, J.N. and A.E. Green 1961. Plane thermo-elastic waves in an initially stressed medium. J. Mech. Phys. Solids. 9:179-190.

Graebel, W.P. 1962. The hydrodynamic stability of a Bingham fluid in couette flow. Proceedings of the international symposium on second order effects in elasticity, plasticity and fluid dynamics, Haifa.

Green, A.E. and J.E. Adkins 1961. Large elastic deformations and non-linear continuum mechanics. Oxford, Clarendon Press.

Lin, C.C. 1955. The theory of hydrodynamic stability. Cambridge, England.

Merrington, A.C. 1943. Flow of viscoelastic materials in capillaries. Nature. 159:310-311.

Narasimhan, M.N.L. 1963. Stability of flow of a non-Newtonian liquid between two rotating cylinders in the presence of a circular magnetic field. Proceedings of the Fourth International Congress on Rheology, Brown University, Providence, R.I.

Oldroyd, J.G. 1951. The motion of elastico-viscous liquid contained between coaxial cylinders. I. Quarterly Journal of Mechanics and applied Mathematics. 4: 271-282.

Rayleigh, Lord 1900. Investigations of the character of the equilibrium of an incompressible heavy liquid of variable density. Cambridge, Scientific Papers: 200-7.

Rayleigh, Lord 1916. On convective currents in a horizontal layer of fluid when the higher temperature is on the under side. Phil. Mag. 32:529-46.

______________ 1954. Theory of sound. New York, Dover.

Reid, W.H. 1958. On the stability of viscous flow in a curved channel. Proc. Roy. Soc. (London), Ser. A, 244:186-98.

Reiner, M. 1945. A mathematical theory of dilatancy. American Journal of Math. 67:350-62.

Rivlin, R.S. 1948. The hydrodynamics of non-Newtonian fluids. I. Proc. Roy. Soc. (London), Ser. A, 193: 260-281.

Rivlin, R.S. and J.L. Ericksen 1955. Stress-deformation relations for isotropic materials. Journal of Rational Mechanics and Analysis. 4:323-425.

Simon, R. 1958. The hydromagnetic oscillations of an incompressible cylinder. Astrophys. J. 128:375-83.

Taylor, G.I. 1917. Motions of solids in fluids when the flow is not irrotational. Proc. Roy. Soc. (London), Ser. A. 93:99-113.

______________ 1921. Experiments with rotating fluids. ibid. 100:114-21.

______________ 1922. The motion of a sphere in rotating liquid. ibid. 102:180-0.

______________ 1923. Experiments of the motion of solid bodies in rotating fluids. ibid. 104:213-18.

Truesdell, C. 1951. A new definition of a fluid. II. The Maxwellian fluid. J. Math. Pures Appl. 30:111-158.

Volkov, T.F. 1959. On stability of a plasma cylinder in an external magnetic field. Plasma Physics and the Problem of Controlled Thermonuclear Reactions, ii, 216-22.

Weissenberg, K. 1947. A contiuum theory of rheological phenomena. Nature. 159:310-311.