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The reliability of sequential machines is an important factor in their design and implementation. In this thesis, stochastic sequential machine models are employed to investigate some of the problems concerning the reliability of sequential machines. Two different methods are used to find the reliability required of individual elements. The first method is based on a property of stochastic matrices with a principal entry in every row, and the second one uses the notion of entropy for stochastic automata. These two approaches are then compared and the first one is found to be advantageous. A survey of the literature on stochastic automata and a method for increasing the reliability of sequential machines by simple redundancy at the element level are also included.

STOCHASTIC AUTOMATA AND THE PROBLEMS OF RELIABILITY IN SEQUENTIAL MACHINES

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STOCHASTIC AUTOMATA AND THE
PROBLEMS OF RELIABILITY
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I. INTRODUCTION

Stochastic sequential machine models have been used in the study of finite-state communication channels, sequential switching networks made of unreliable components, and learning systems. In this paper, we will employ the stochastic machine model to investigate some of the problems concerning the reliability of sequential machines. The present chapter is devoted to the introduction of basic notions and models in the theory of stochastic machines and to a survey of the literature in this rapidly growing field. We shall describe basic models of stochastic machines, and summarize the present state-of-the-art with respect to machine equivalence, minimality, and so on.

Stochastic Sequential Machines

Stochastic sequential machines are generalizations of finite deterministic machines in which the present state and input determine the next state and output in a probabilistic manner.

Definition 1. A stochastic sequential machine (SSM) is a quadruple $S=(X,Y,Q,p)$, in which X, Y, and Q are finite sets of inputs, outputs, and states, respectively, and p is a probability function. Elements of X, Y, and Q will be denoted by x, y, and q, respectively. The conditional probability density function $p(q_j,y/q_i,x)=p_{ij}(y/x)\geqslant 0$ is interpreted as the probability that the next state and output will be q_j and y, given that the present state is q_i and the input is x. Clearly, p must satisfy the following condition:

$$\sum_{q_j \epsilon Q} \sum_{y \epsilon Y} p_{ij}(y/x)=1 \tag{1}$$

As shown in Figure 1, stochastic sequential machines may be represented by state graphs similar to those for deterministic machines. Each transition arrow is labeled with a pair of input and output symbols and the probability of that transition. Arrows corresponding to transitions with zero probability are eliminated. Thus, from the graph point of view, the probability values (indicated in parentheses) are associated strictly with the transition arrows in the graph. An absence of a probability value is, in every case, interpreted as p=1.

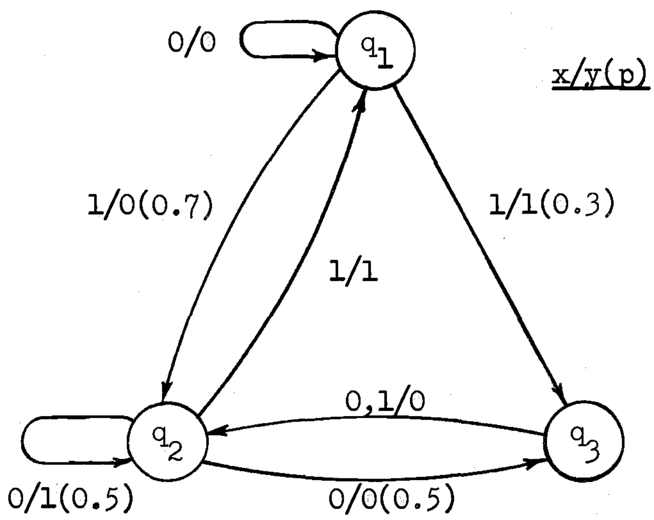

Figure 1. State graph of an SSM.

From the probabilities $p_{ij}(y/x)$, we can define other probability density functions. For instance

$$p(y/q_i,x)=p_i(y/x)=\sum_{q_j\epsilon Q} p_{ij}(y/x) \tag{2}$$

is the probability that the machine will respond to the input x by producing the output y if started in state q_i. The probability function $p_i(y/x)$ determines the input/output behaviour of state q_i. On the other hand, consider

$$p(q_j/q_i,x)=p_{ij}(x)=\sum_{y\epsilon Y} p_{ij}(y/x) \tag{3}$$

which is the probability that the next state will be q_j, given that the present state and input are q_i and x, respectively.

We can regard $p_{ij}(y/x)$ and $p_{ij}(x)$ as ij elements of nxn matrices M(y/x) and M(x), respectively, where n is the number of states in Q. Then, (3) may be written as:

$$M(x)=\sum_{y\epsilon Y} M(y/x) \tag{4}$$

Similarly, $p_i(y/x)$ can be viewed as the i-th element of a column vector c(y/x). If c_n is a column vector of length n with each component equal to one, then:

$$c(y/x)=M(y/x)c_n \tag{5}$$

To clarify the above ideas, we give the matrices corresponding to the SSM in Figure 1.

$$M(0/0)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix} \qquad M(0/1)=\begin{bmatrix} 0 & 0.7 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M(1/0)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0.5 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad M(1/1)=\begin{bmatrix}0 & 0 & 0.3\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$M(0)=\begin{bmatrix}1 & 0 & 0\\ 0 & 0.5 & 0.5\\ 0 & 1 & 0\end{bmatrix} \qquad M(1)=\begin{bmatrix}0 & 0.7 & 0.3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

Now we consider D, the set of all probability distributions on Q, as an extension to the state set Q. By a probability distribution on Q, we mean a row vector $d=(d_1,d_2,\ldots,d_n)$ with $d_i \geqslant 0$ and $\sum_i d_i=1$, in which d_i is the probability that the machine is in state q_i. This is a natural generalization since a state q_i can be considered as a state distribution d with $d_i=1$ and $d_j=0$ $(j\neq i)$. Note that although the automaton is stochastic, the transitions between state distributions are deterministic in the sense that the present state distribution d and the input x uniquely determine the next state distribution d'.

$$d'(d,x)=dM(x) \tag{6}$$

The input/output behaviour of a state distribution d is determined by the following function:

$$p_d(y/x)=\sum_i d_i p_i(y/x)=dc(y/x) \tag{7}$$

Now consider the input and output sequences $u=x_1x_2\ldots x_k$ and $v=y_1y_2\ldots y_k$. We define $p(q_j,v/q_i,u)=p_{ij}(v/u)$ as the probability that the final state and output sequence will be q_j and v, given that the input and initial state are u and q_i, respectively. Similarly, $p_{ij}(u)$, $p_i(v/u)$, $M(v/u)$, $M(u)$, and $c(v/u)$ can be defined. The following relations are easily derived:

$$M(v/u)=M(y_1/x_1)M(y_2/x_2)\ldots M(y_k/x_k) \tag{8}$$

$$M(u)=M(x_1)M(x_2)\ldots M(x_k) \tag{9}$$

$$c(v/u)=M(v/u)c_n \tag{10}$$

$$d'(d,u)=dM(u) \tag{11}$$

$$p_d(v/u)=\sum_i d_i p_i(v/u)=dc(v/u) \tag{12}$$

<u>Definition 2</u>. State distributions d and d' of stochastic sequential machines S and S' are k-equivalent if their input/output behaviours, given by (12), are identical for all sequences of length k. If d and d' are k-equivalent for every k, they are equivalent and we denote this by $(S,d)\sim(S',d')$. Equivalence of states can be defined analogously.

Carlyle (1963) proves that $(n+n'-1)$-equivalence of d and d' is sufficient for their equivalence, where n and n' denote the number of states of S and S', respectively. Definition 2 is valid for states and state distributions of the same machine if we let $S=S'$. Then, $(n-1)$-equivalence is needed for equivalence. This is the stochastic generalization of the result previously obtained for deterministic machines.

At this point, we introduce a matrix H which will be of subsequent importance. Columns of H are those elements of the set

$\{c(v/u)\colon v\epsilon Y^k, u\epsilon X^k, k=1,2,\ldots\}$ which are linearly independent and any other element of the set is a linear combination of them (X^k and Y^k are the sets of all input and output sequences of length k). It can be shown (Carlyle, 1963) that equivalent states correspond to identical rows in H, and that state distributions d and d' are equivalent iff dH=d'H.

Definition 3. A stochastic sequential machine S' covers S if for each state distribution d in S there exists d' in S' such that $(S,d)\sim(S',d')$. We denote this by $S' \geqslant S$. If $S \geqslant S' \geqslant S$, then S and S' are equivalent. Two SSM's are state equivalent if each state in one has at least one equivalent state in the other.

Definition 4. An SSM is reduced if it is not state equivalent to another machine with fewer states. Any reduced SSM which is state equivalent to S is a reduced form for it. An SSM is minimal-state if no state q_i is equivalent to a state distribution d with $d_i=0$. Any minimal-state SSM which is equivalent to S is a minimal form for it.

A reduced form for S may not be unique. In fact, an SSM may have an infinite family of distinct reduced forms. By constructing a five-state machine with two non-isomorphic minimal forms, Even (1965) proves that minimal forms may not be unique either. Nieh (1970) gives a necessary and sufficient condition in terms of the matrix H for the minimal-state machine S to be unique. He proves that if S is not unique, then an infinite number of distinct minimal forms exist.

Bacon (1964a) proves that all minimal forms of equivalent machines are state equivalent, have the same number of states, and are not equivalent to any machine with fewer states. It can be shown

(Even, 1965) that a reduced deterministic machine is minimal-state in the stochastic sense.

Ott (1966) points out that a minimal-state machine may be covered by another machine with fewer states. This suggests that the state reduction of SSM's may be carried beyond the minimal forms; however no effective procedure has been found for such reductions. Some further results along this line are given by Paz (1968). It is interesting to note that a reduced deterministic machine may not be covered by another machine with fewer states (Ott, 1966). In view of the above statement, it is easy to see why Bacon chose the term "minimal-state" for SSM's which are not equivalent to any machine with fewer states. For other results on state reduction, see the papers by Paz (1967a) and Souza (1969).

The following algorithm for finding a minimal form for S is based on the work of Bacon (1964a) and Even (1965). First form the matrix H for S and identify as many rows as possible which are not convex combinations of the other rows. A row vector r is a covex combination of $r^1, r^2, \ldots, r^k$, if $r_j = \sum_i w_i r_j^i$, where r_j and r_j^i are the j-th components of r and r^i, respectively, and $w_i \geqslant 0$. Of the remaining rows, eliminate those which can be obtained by convex combinations of the others, since each such row corresponds to a state q_i which is equivalent to a state distribution d with $d_i = 0$. Finally, direct the transitions entering an eliminated state to an equivalent state or state distribution.

We now turn to a different model for stochastic machines which is due to Rabin (1963).

Stochastic Acceptors

Stochastic acceptors are generalizations of finite deterministic acceptors in which the present state and input determine the next state in a probabilistic manner.

Definition 5. A stochastic acceptor is a quintuple $A=(X,Q,p,d,F)$ in which X and Q are finite sets of inputs and internal states, p is a probability function, d is an initial state distribution, and $F \subseteq Q$ is a designated set of final states. The conditional probability density function $p(q_j/q_i,x)=p_{ij}(x) \geq 0$ is interpreted as the probability of a transition to state q_j, given that the present state is q_i and the input is x.

Let p(u) denote the probability that after the application of an input sequence u to A, the final state will be in F. To find p(u), we sum those components of the final state distribution, given by (11), which correspond to the states in F. Hence,

$$p(u)=d'(d,u)c_n(F)=dM(u)c_n(F) \tag{13}$$

where $c_n(F)$ is a column vector of length n, with the k-th element equal to one if $q_k \in F$ and zero otherwise.

Associated with each stochastic acceptor is a set of stochastic languages which we now proceed to define, since they reveal another difference between stochastic and deterministic sequential machines.

Definition 6. A tape (input sequence) u is accepted by A with cut-point C $(0 \leq C < 1)$ if

$$p(u) > C \tag{14}$$

The set U of all tapes which satisfy (14) is the set (language) accepted by A with cut-point C.

$$U(A,C)=\{u:\ p(u) > C\} \tag{15}$$

A language U is accepted by A if (15) holds for some C. Hence, A stochastic acceptor A orders the set of all tapes according to the value of p(u), and C, the cut-point, indicates some point in this ordering which separates the set of accepted tapes from those not accepted.

It can be shown that there exists a stochastic acceptor A and a cut-point C such that U(A,C) is not a regular language, i.e. it cannot be accepted by a deterministic acceptor. Rabin (1963) proves this by constructing a two-state machine with two input symbols which accepts a nondenumerable set of languages. The conclusion follows noting that the set of regular languages is denumerable. Hence the set of regular languages is a proper subset of the set of stochastic languages. However, if a cut-point C is isolated, i.e. if there exists $e > 0$ such that for any input sequence u

$$|p(u)-C| > e \tag{16}$$

then U(A,C) is a regular set accepted by a deterministic acceptor with no more than $(1+1/2e)^{n-1}$ states.

A necessary and sufficient condition for a language to be accepted by a stochastic acceptor and one which does not satisfy this criterion are given by Bukharaev (1966). Other aspects of stochastic languages have been treated by Nasu and Honda (1968, 1969), Salomaa (1968, 1969b), Turakainen (1968, 1969), Knast (1970), and Paz (1970).

Alternate Models and Other Topics

The two models for stochastic sequential machines which we have described so far are not exhaustive. Souza and Leake (1967, 1969) give another model which is of special importance in the realization of SSM's. This model consists of a state-assigned deterministic machine whose behaviour is made stochastic by supplying some of its inputs from random sources. Kelly and Schooley (1968) prove that any SSM can be realized in terms of binary memory units, combinational logic, and binary white sources (history-independent sequential sources of binary random digits with equal probability for the appearance of zeros and ones). Other results on the synthesis of stochastic machines have been obtained by Tsertsvadze (1963).

As we mentioned earlier, stochastic sequential machines are generalizations of deterministic machines. Because of this generalization, it is desirable to define various classes of stochastic machines by placing restrictions on the general model in order to facilitate further developments.

That all transitions between states for actual sequential machines have strictly positive (though sometimes very small) probabilities, leads Rabin (1963) to define the class of actual automata. A property which he investigates for machines of this class is their stability under slight changes in transition probabilities. More precisely, for each actual automaton A with an isolated cut-point C, there exists $e>0$ such that for every automaton A' with transition probabilities differing from those of A by less than e, we have $U(A,C)=U(A',C)$.

Carlyle (1965) studies the characteristics of state-calculable SSM's (SSM's for which the next state is uniquely determined from the knowledge of present state, input, and output). Definite stochastic automata (SSM's for which the final state distribution after the application of an input sequence of length k or more is independent of the initial state distribution) have been introduced by Paz (1965, 1966, 1967b) and further studied by Liu (1969), Chen and Sheng (1970). By putting some restrictions on the probability function p, Knast (1969a) introduces the class of linear probabilistic machines which are generalizations of linear deterministic machines. The class of m-adic stochastic automata (SSM's with two states and m input symbols) have been studied by Paz (1966), Salomaa (1967), Yasui and Yajima (1970).

Some attempts have been made to generalize the class of stochastic sequential machines. Varshavskii and Vorontsova (1963) study SSM models in which the transition probabilities vary in time. Knast (1969b) further generalizes this model by considering a continuous time scale instead of a discrete one. Turakainen (1969) gives a generalized model in which the components of $c_n(F)$, introduced in Equation (13), are arbitrary real numbers. However, he proves that such machines accept a language iff it is accepted by an SSM.

Page (1966) considers state-assigned SSM's with numerical input and output, an example of which is a slot-machine. He defines the expectation of output for an initial state distribution d and an input sequence u by $E(u,d)=dM(u)c_n(Y)$, where $c_n(Y)$ is an n-component column vector whose i-th element is the output corresponding to state q_i. Then, he defines the expectation-equivalence of two SSM's with initial state distributions d and d' as the equality of E(u,d) and E(u,d') for every u. Nieh (1968) defines the expected pay-off for each initial state distribution d and input sequence $u=x_1x_2...x_k$ by

$$P(u,d)=\sum_{v\epsilon Y^k} dM(v/u)c_nV(v/u)$$

where Y^k is the set of all output sequences of length k, and V(v/u) is defined as $\sum_i (y_i-x_i)$, in which y_i is the i-th element of v. The function P can be used to order SSM's with numerical input and output according to their value to the experimenter.

Bacon (1964b) generalizes the decomposition theory for deterministic sequential machines to the stochastic case. Arbib (1967), Carlyle and Paz (1969) study the realization of a given input/output behaviour by an SSM. Nieh and Carlyle (1968) suggest that the number of states should not be the only measure of complexity for SSM's, since a given machine may have many corresponding realizations with the same number of states and different logical complexities. Accordingly, they introduce a measure of complexity which allows us to compare any pair of SSM's. Some properties of sets of stochastic matrices have been investigated by Paz (1965), Yasui and Yajima (1969).

We conclude this section by refering to several books which supply the basic theory of stochastic sequential machines. Carlyle (1968) and Arbib (1969, p. 324-348) treat the general theory, the former emphasizing the input/output behaviour of machines. Salomaa (1969a, p. 71-113) treats the subject with a stress on stochastic languages. Booth (1967, p. 505-541) deals mainly with random processes in probabilistic machines.

<u>Relevance To Reliability Considerations</u>

The reliability of sequential machines is an important factor in their design and implementation. Many authors have investigated different problems concerning the reliability of sequential machines. The first major contribution to this area was made by von Neumann (1956).

A deterministic sequential machine (DSM) may be regarded as an SSM for which $p_{ij}(y/x)$ is either zero or one. Accordingly, the matrices $M(y/x)$ and $M(x)$ for a DSM have only zeros and ones as their elements. When realized in terms of actual gates and flip-flops, of course all sequential machines behave in a probabilistic manner. This is due to the unreliability of the elements of which the machine is constructed. As a result of this unreliability, the entries of $M(y/x)$ and $M(x)$ will no longer be only zeros and ones. However, if the probability of failure for each component is sufficiently small, these entries will be very close to their ideal values. The above argument suggests that SSM models may be useful in studying the reliability of sequential machines. Such studies have been made by Bruce and Fu (1963), Tsertsvadze (1964a), and Tou (1968).

The problems of reliability for sequential machines, discussed in subsequent chapters, can be divided into two major groups:

1. Finding the reliability of a given sequential machine.
2. Synthesizing a sequential machine with a given reliability.

Two different approaches for investigating these problems are presented in Chapters II and III. The results given in Chapter II are original while those given in Chapter III are based on the work by Tsertsvadze (1964a). In Chapter V, these two approaches are compared and that of

Chapter II is found to be advantageous.

For synthesizing a sequential machine with a given overall reliability, the procedures of Chapters II and III can be used to find the reliability required of individual elements. However, elements with this reliability may not be available or economical to use. The synthesis procedure of Chapter IV has been developed to handle such cases.

II. AN SSM MODEL FOR SEQUENTIAL MACHINES MADE OF UNRELIABLE COMPONENTS

In this chapter, we consider sequential machines which are synthesized in terms of gates and flip-flops. Our object is to develop a method for finding SSM's which can be used as models in studying the reliability of such machines. In order to treat both synchronous and asynchronous sequential machines with the same theory, we consider their operation in terms of cycles. By a cycle of operation, we mean the time interval between the application of two consecutive inputs (clock pulses in synchronous sequential machines are considered as part of the input).

Finding the SSM Model

The output y of an ideal element, gate or flip-flop, can be written in the form $y=f(x,z)$, where x is the input to the element, and z is the internal variable (z is a constant for elements without internal variables). For actual elements, the output is not always given by the above equation. In other words, with a probability p, which in general depends on x and z, a failure occurs and as a result $y\neq f(x,z)$. Such a failure is caused by a temporary or permanent breakdown of the element. In what follows, we assume that failures are caused only by temporary breakdowns of elements. We will also assume that p does not deped on x or z. The case where p is a function of x and z can be handled similarly if we choose $p=\max_{x,z} p(x,z)$. This value of p will result in a model which represents an even less reliable machine than the original one. Hence if the model satisfies a certain reliability criterion, then so does the original machine.

Now consider an arbitrary sequential machine realized by a set of elements $E_1, E_2, \ldots, E_k$. We assume that the probability of failure for each of these elements is sufficiently small such that the probability of two or more failures occuring at the same cycle is negligible. It is also convenient to assume that the probability of failure for each element is the same at different cycles of operation and does not depend on the operation of other elements. Let p_i denote the probability of failure for E_i. Then, the probability of having no failures in one cycle is equal to $\prod_i (1-p_i)$ and the probability of only E_j being faulty is $p_j \prod_{i \neq j} (1-p_i)$. Using linear approximations, we will have:

$$\prod_{i=1}^{k} (1-p_i) \simeq 1 - \sum_{i=1}^{k} p_i \tag{17}$$

$$p_j \prod_{i \neq j} (1-p_i) \simeq p_j \tag{18}$$

We study the behaviour of this sequential machine when its input is x. For each state q we can find the joint probability density function of the next state q' and the output y in the following way:

1. For all the cases in which no gate is faulty or only one is, find the next state and output as well as the probability of occurrence for that particular case.
2. The probability that the next state and output are q' and y is equal to the sum of the probabilities of all cases in step 1 which result in q' and y as the next state and output.

<u>Example 1</u>. Consider the three-state machine whose state table is given below. Figure 2 shows a realization for this machine which will be used as an example to illustrate and clarify the above procedure. The following encoding has been used for this realization.

States	Inputs 0	Inputs 1
q_1	q_1 0	q_2 0
q_2	q_1 0	q_3 0
q_3	q_1 0	q_3 1

State table

States	Internal variables z_1	z_2
q_1	0	0
q_2	0	1
q_3	1	0

Encoding table

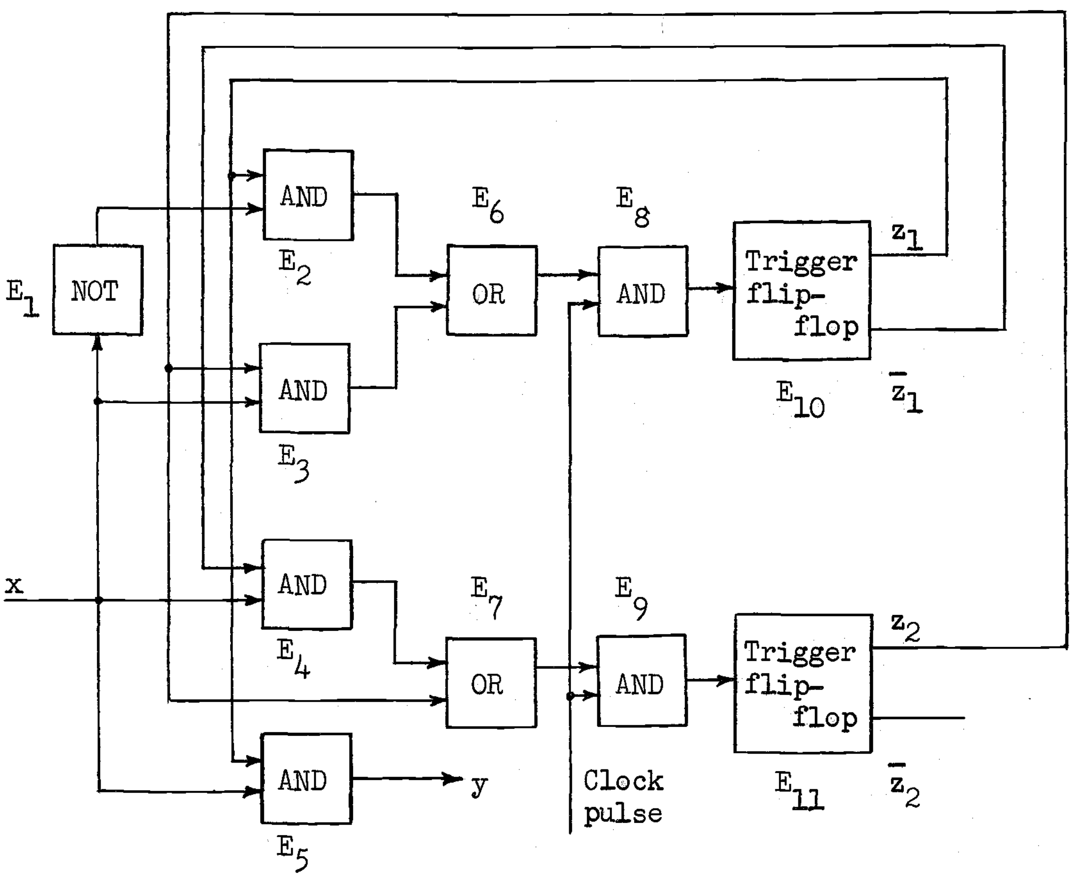

Figure 2. The sequential machine of Example 1.

We shall assume in this case that the probability of failure for each element is equal to p. This assumption will simplify the example and is sufficient to illustrate the process. When numerical values are used, a different probability of failure for each element can be easily handled. Furthermore, if we choose $p=\max(p_1,p_2,\ldots,p_{11})$, then the model will represent an even less reliable machine than the original one. Hence, if the model satisfies a certain reliability criterion, then so does the original machine.

Consider the case when the machine is in state q_1 and receives the input $x=0$. If all the elements function properly, the next state will be q_1 and the output will be $y=0$. This event has the probability $(1-p)^{11} \approx 1-11p$. If E_2 fails to function properly, the next state and output will be q_3 and zero, respectively. In the same way, we can find all the entries of Table 1.

From Table 1, we can deduce the probability of having q_j and y as the next state and output, given that the present state and input are q_i and x, respectively. We simply look for the entries $q_j y$ in the row corresponding to q_i and x and sum their probabilities. This will result in Table 2.

From Table 2, different matrices corresponding to the machine of Figure 2 can be found. Remember that the ij entry of $M(y/x)$ is the probability of a transition from q_i to q_j with input x and output y. Similarly, the ij element of $M(x)$ is the probability of a transition from q_i to q_j with input x.

TABLE 1

Faulty element		None	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}
Probability		1-11p	p	p	p	p	p	p	p	p	p	p	p
Input	Present state												
x=0	q_1	$q_1$0	$q_1$0	$q_3$0	$q_3$0	$q_2$0	$q_1$1	$q_3$0	$q_2$0	$q_3$0	$q_2$0	$q_3$0	$q_2$0
	q_2	$q_1$0	$q_1$0	$q_3$0	$q_3$0	$q_1$0	$q_1$1	$q_3$0	$q_2$0	$q_3$0	$q_2$0	$q_3$0	$q_2$0
	q_3	$q_1$0	$q_3$0	$q_3$0	$q_1$0	$q_2$0	$q_1$1	$q_3$0	$q_2$0	$q_3$0	$q_2$0	$q_3$0	$q_2$0
	q_4	$q_1$0	$q_3$0	$q_3$0	$q_1$0	$q_1$0	$q_1$1	$q_3$0	$q_2$0	$q_3$0	$q_2$0	$q_3$0	$q_2$0
x=1	q_1	$q_2$0	$q_2$0	$q_4$0	$q_4$0	$q_1$0	$q_2$1	$q_4$0	$q_1$0	$q_4$0	$q_1$0	$q_4$0	$q_1$0
	q_2	$q_3$0	$q_3$0	$q_3$0	$q_1$0	$q_3$0	$q_3$1	$q_1$0	$q_4$0	$q_1$0	$q_4$0	$q_1$0	$q_4$0
	q_3	$q_3$1	$q_1$1	$q_1$1	$q_1$1	$q_4$1	$q_3$0	$q_1$1	$q_4$1	$q_1$1	$q_4$1	$q_1$1	$q_4$1
	q_4	$q_1$1	$q_1$1	$q_1$1	$q_3$1	$q_1$1	$q_1$0	$q_3$1	$q_2$1	$q_3$1	$q_2$1	$q_3$1	$q_2$1

TABLE 2

Output		y=0				y=1			
Next state		q_1	q_2	q_3	q_4	q_1	q_2	q_3	q_4
Input	Present state								
x=0	q_1	1-10p	4p	5p	0	p	0	0	0
	q_2	1-9p	3p	5p	0	p	0	0	0
	q_3	1-10p	4p	5p	0	p	0	0	0
	q_4	1-9p	3p	5p	0	p	0	0	0
x=1	q_1	4p	1-10p	0	5p	0	p	0	0
	q_2	4p	0	1-8p	3p	0	0	p	0
	q_3	0	0	p	0	6p	0	1-11p	4p
	q_4	p	0	0	0	1-8p	3p	4p	0

$$M(0/0)=\begin{bmatrix} 1-10p & 4p & 5p & 0 \\ 1-9p & 3p & 5p & 0 \\ 1-10p & 4p & 5p & 0 \\ 1-9p & 3p & 5p & 0 \end{bmatrix} \qquad M(0/1)=\begin{bmatrix} 4p & 1-10p & 0 & 5p \\ 4p & 0 & 1-8p & 3p \\ 0 & 0 & p & 0 \\ p & 0 & 0 & 0 \end{bmatrix}$$

$$M(1/0)=\begin{bmatrix} p & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ p & 0 & 0 & 0 \end{bmatrix} \qquad M(1/1)=\begin{bmatrix} 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 6p & 0 & 1-11p & 4p \\ 1-8p & 3p & 4p & 0 \end{bmatrix}$$

$$M(0)=\begin{bmatrix} 1-9p & 4p & 5p & 0 \\ 1-8p & 3p & 5p & 0 \\ 1-9p & 4p & 5p & 0 \\ 1-8p & 3p & 5p & 0 \end{bmatrix} \qquad M(1)=\begin{bmatrix} 4p & 1-9p & 0 & 5p \\ 4p & 0 & 1-7p & 3p \\ 6p & 0 & 1-10p & 4p \\ 1-7p & 3p & 4p & 0 \end{bmatrix}$$

Examples of Applications

We illustrate the applications of the matrices $M(y/x)$ and $M(x)$ for a sequential machine in the following examples.

Example 2. Consider the three-state sequential machine of Example 1. It is clear that $u=0$ is a synchronizing sequence for this machine; that is u will take the machine into state q_1 regardless of the initial state. Figure 2 shows a realization for this machine. Suppose that we have no knowledge about the initial state. Hence we can assume that the machine is started with the state distribution $d=(1/4\ 1/4\ 1/4\ 1/4)$. Using (11) and the results of Example 1, the state distribution after the application of u can be found.

$$d'(d,0)=dM(0)=(1-8.5p \quad 3.5p \quad 5p \quad 0)$$

Hence, the final state is q_1 with probability 1-8.5p.

For homing experiments (those in which the knowledge of u and v, the input and output sequences, is sufficient to determine the final state), a similar method can be used. The probability of ending up in the correct state can be found from the corresponding entry in the final state distribution which is given by:

$$d'(d,u,v)=dM(v/u) \tag{19}$$

<u>Example 3</u>. The sequential machine of Example 1 accepts an input sequence u (produces y=1 as the last output) if u ends in a block of three or more consecutive ones. The initial state is assumed to be q_1. Consider the input sequence u=10111. The probability that u is accepted by the unreliable version, can be found by using (12), (10), and (8).

$$\begin{aligned} p(u) &= \sum_v (1 \quad 0 \quad 0 \quad 0)M(v1/10111)c_n \\ &= (1 \quad 0 \quad 0 \quad 0)(\sum_v M(v/1011))M(1/1)c_n \\ &= (1 \quad 0 \quad 0 \quad 0)M(1011)M(1/1)c_n \simeq 1-14p \end{aligned}$$

Hence, u is accepted with probability 1-14p or more.

A more interesting problem is the following: given the matrices of a sequential machine S, find a number m such that if we apply an input sequence of length k or less to S, with a probability of at least 1-p a majority of the final outputs will be correct. A possible solution is given by the following algorithm.

1. Use the above procedure to find the probability of a correct final output for all sequences of length k or less.
2. Denote the smallest probability obtained in step 1 by $1-p_o$.
3. Use the following inequality to find m:

$$\sum_{j=(m+1)/2}^{m} \binom{m}{j} p_o^j (1-p_o)^{m-j} \leqslant p$$

Although the above procedure is an algorithm, it is very tedious to carry out even for relatively small values of k. I believe that a more efficient solution to this problem can be found.

The Reliability Required of Individual Elements

In this section, we introduce a method for finding the reliability required of individual elements of a sequential machine for a given overall reliability.

Definition 7. An entry in M(x) is a principal entry if it is greater than 1/2. We assume that the probabilities of failure are sufficiently small such that principal entries appear in locations where there are ones for the corresponding ideal sequential machine.

Theorem 1. Let p_h denote the smallest principal entry in an nxn stochastic matrix M_h which has a principal entry in every row (h=1,2,....,m). If $\sum_h (1-p_h) < 1/2$, then the smallest principal entry in $M=M_1M_2 \ldots M_m$ satisfies the following inequality:

$$p \geqslant p_1 p_2 \cdot\cdot\cdot\cdot p_m \tag{20}$$

Proof. We prove this statement for m=2. A simple induction on m will then establish the theorem. Let the principal entry of the i-th row of M_1 be in the k-th place. There is one, and only one, column in M_2 whose k-th element is a principal entry. Let this be the j-th column. Then the ij entry of $M=M_1M_2$ satisfies the inequality $p_{ij} \geqslant p_1p_2$. From the relation $(1-p_1)(1-p_2) \geqslant 0$, we can conclude that:

$$p_1p_2 \geqslant p_1+p_2-1=1-((1-p_1)+(1-p_2)) > 1/2$$

Hence, p_{ij} is a principal entry. This is the only principal entry in the i-th row since M is stochastic. Obviously, this argument is valid for all the rows in M_1, and hence the theorem is proved.

Example 4. A sequential machine must function reliably over a preselected time interval. Consider the operation of the machine in Example 1 for a sequence u of length k or less. The smallest principal entries in M(0) and M(1) are 1-9p and 1-10p, respectively. If $10pk < 1/2$, the conditions for Theorem 1 are satisfied and therefore the worst principal entry in M(u) is greater than or equal to $(1-10p)^k$. Hence, with a probability of at least $(1-10p)^k$ the machine will end up in the correct state after the application of u. If k is relatively small, the linear approximation given by (17) can be used.

Suppose we want to determine an upper bound for p such that the machine of Example 1 will end up in the correct state with a probability of at least 0.9 after the application of an input sequence of length 20 or less. Using Theorem 1 and the linear approximation given by (17) we find the following condition which is sufficient for the required reliability:

$$1-10pk \geqslant 0.9 \;\Rightarrow\; p \leqslant 0.01/k=0.0005$$

The above method is also useful in estimating the reliability required of individual elements without finding the matrices $M(x)$. The only thing needed is a lower bound for the worst principal entry. Such a bound can be easily obtained from (17). Using this method, we find the following condition which is sufficient for the required reliability:

$$1-11pk \geqslant 0.9 \Rightarrow p \leqslant 0.009/k = 0.00045$$

III. ENTROPY ESTIMATES FOR RELIABILITY

In this chapter, we introduce the notion of entropy for SSM's and give its applications in estimating the reliability of sequential machines. The results given in this chapter have been obtained by Tsertsvadze (1964a).

Definition 8. We say that a sequential machine functions with reliability R for k cycles if its probability of being in the correct state after the application of an input sequence of length k or less is at least R. Obviously, for a machine which is intended to be deterministic, R must be very close to one.

Basic Definitions and Theorems

We first introduce the notion of entropy for stochastic row vectors and matrices.

Definition 9. The entropy of a stochastic row vector $r=(r_1,r_2,\ldots,r_k)$ is defined by

$$e(r)= -\sum_i r_i \log r_i \tag{21}$$

The entropy of a stochastic matrix M, e(M), is the maximum of the entropies of its rows.

Theorem 2. The entropy of a stochastic row vector $r=(r_1,r_2,\ldots,r_k)$, in which some of the elements are fixed, is maximum if the remaining elements are equal.

Proof. without loss of generality, we can assume that the first j elements of r are fixed ($j\leqslant k-2$). Since $\sum_i r_i=1$, we can write:

$$r_k=1-\sum_{i=1}^{k-1} r_i=1-\sum_{i=1}^{j} r_i-\sum_{h=j+1}^{k-1} r_h \tag{22}$$

Equation (21) may be written as:

$$e(r)= -\sum_{i=1}^{j} r_i\log r_i-\sum_{h=j+1}^{k-1} r_h\log r_h-r_k\log r_k \tag{23}$$

We can find the maximum of e(r) by differentiating (23) with respect to r_h ($j<h<k$) and equating the result to zero.

$$de(r)/dr_h= -\log r_h-1-(dr_k/dr_h)\log r_k-dr_k/dr_h=0 \tag{24}$$

Differentiating (22) with respect to r_h, we obtain $dr_k/dr_h= -1$. Then Equation (24) results in $r_h=r_k$ which proves the theorem since the second derivative of e(r) with respect to r_h is always negative. The maximum value of e(r) can be easily found from (23) and (22).

$$\max\ e(r)= -\sum_{i=1}^{j} r_i\log r_i-(1-\sum_{i=1}^{j} r_i)\log((1-\sum_{i=1}^{j} r_i)/(k-j)) \tag{25}$$

Lemma 1. If $r=(r_1,r_2,\ldots,r_k)$ is a stochastic row vector and M is a stochastic matrix with k rows, then the entropy of the product rM satisfies the following inequality:

$$e(rM)\leqslant e(r)+e(M) \tag{26}$$

Proof. The proof is rather straightforward and can be found in the paper by Tsertsvadze (1964a).

Theorem 3. If $M_1, M_2, \ldots,$ and M_m are arbitrary stochastic matrices and if $M_1 M_2 \ldots M_m$ is defined, then

$$e(M_1 M_2 \ldots M_m) \leq \sum_i e(M_i) \tag{27}$$

Proof. We prove this statement for m=2. A simple induction on m will then establish the theorem. Let M_1 be a stochastic matrix with rows $r^1, r^2, \ldots,$ and r^k. Then the j-th row of $M_1 M_2$ will be equal to $r^j M_2$. Hence, by definition

$$e(M_1 M_2) = \max_j e(r^j M_2)$$

Using (26), we can write:

$$e(M_1 M_2) \leq \max_j (e(r^j) + e(M_2)) = \max_j e(r^j) + e(M_2) = e(M_1) + e(M_2)$$

Definition 10. The entropy of a stochastic automaton at any instant is defined as the entropy of its state distribution $d=(d_1, d_2, \ldots, d_n)$.

If the state of an SSM is known to be q_i, i.e. if $d_i=1$ and $d_j=0$ $(j \neq i)$, then its entropy is equal to zero. This is always true for a DSM. On the other hand, if we have no knowledge about the state of an SSM, i.e. if $d_i = d_j$ $(1 \leq i, j \leq n)$, then by Theorem 2, its entropy is maximum. Hence, the entropy of an SSM can be used as a measure of the indeterminacy of its present state.

The Entropy of an SSM after k Cycles

If an SSM is started with state distribution d and supplied with the input sequence $u=x_1x_2\ldots x_k$, the final state distribution can be found from (11). Using (26), (9), and (27), we can write:

$$e(d'(d,u)) \leqslant e(d)+e(M(u)) \leqslant e(d)+\sum_i e(M(x_i)) \tag{28}$$

Obviously, $e(M(x_i)) \leqslant \max_{x\epsilon X} e(M(x))$. Hence,

$$e(d'(d,u)) \leqslant e(d)+k \max_{x\epsilon X} e(M(x)) \tag{29}$$

which is useful if only the length k of the input sequence u is known.

Let $1-p_w$ be the smallest principal entry in the set of matrices $\{M(x)\colon x\epsilon X\}$. Using (25), we can write

$$\max_{x\epsilon X} e(M(x)) \leqslant -(1-p_w)\log(1-p_w)-p_w\log(p_w/(n-1)) \tag{30}$$

where n is the number of states. Let the function g(t) be defined by

$$g(t)= -t \log t-(1-t)\log(1-t) \tag{31}$$

Then, Equation (30) may be written as

$$\max_{x\epsilon X} e(M(x)) \leqslant g(p_w)+p_w\log(n-1) \tag{32}$$

Combining (29) and (32), we obtain:

$$e(d'(d,u)) \leqslant e(d)+k(g(p_w)+p_w\log(n-1)) \tag{33}$$

Inequality (33) gives an upper bound for the entropy after the application of an input sequence u of length k.

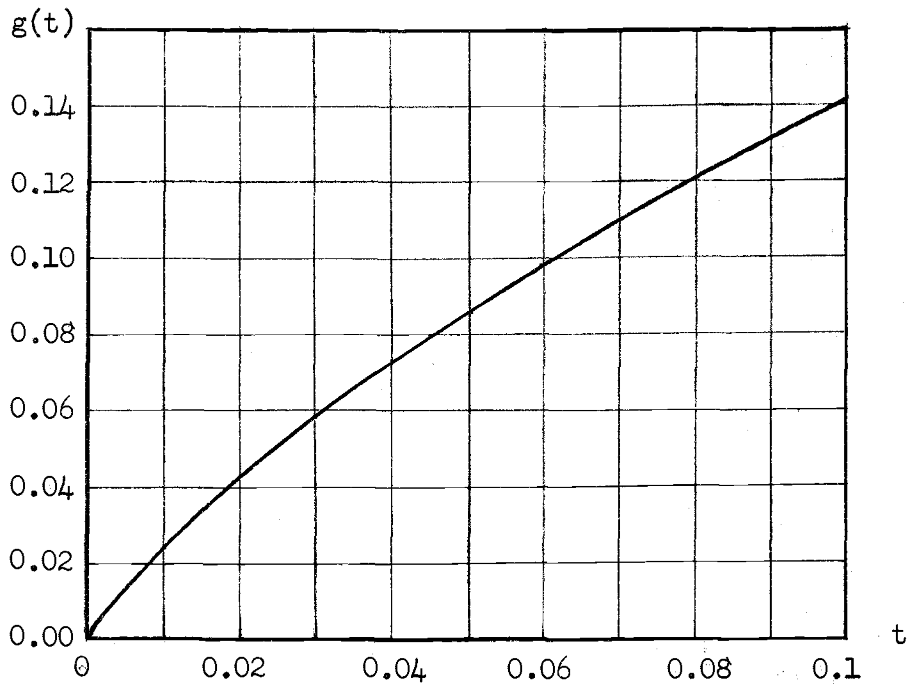

Figure 3. The curve representing g(t).

The Relation Between Entropy and Reliability

Definition 8 gives us a measure for the reliability of sequential machines. Suppose that a given sequential machine is supplied with an arbitrary input sequence u of length k or less, starting in state q. We assume that the probability of failure for the elements of which the machine is built is sufficiently small such that the final state distribution has a principal entry, $1-p_u$. Then, by definition, the reliability of the machine for k cycles is

$$R = \min_{u \in U_k} (1-p_u) \tag{34}$$

where U_k is the set of all input sequences of length k or less.

From (33), we have an upper bound for the entropy after the application of u which is a monotone increasing function of k. Hence:

$$\max_{u \in U_k} e(d'(d,u)) \leq k(g(p_w)+p_w \log(n-1)) \tag{35}$$

The entropy of d is zero since we assume that the machine starts in a known state. Now, we can find a lower bound for R. Consider a state distribution d" whose entropy is equal to $k(g(p_w)+p_w \log(n-1))$. If d" has a principal entry $1-p_m$ and if all other entries are equal, Theorem 2 assures us that:

$$1-p_m \leq \min_{u \in U_k} (1-p_u) \tag{36}$$

On the other hand, we have by definition

$$e(d'')=(1-p_m)\log(1-p_m)+p_m \log(p_m/(n-1))$$

Hence, p_m can be found from the following equation:

$$g(p_m)+p_m \log(n-1)=k(g(p_w)+p_w \log(n-1)) \tag{37}$$

If we disregard the term $p_m \log(n-1)$, we obtain the result given by Tsertsvadze (1964a).

$$g(p_m) \leq k(g(p_w)+p_w \log(n-1)) \tag{38}$$

Since g(t) is a monotone increasing function for $t < 1/2$, Inequality (38) may be written as

$$p_m \leq g^{-1}[k(g(p_W)+p_W \log(n-1))]$$

where $g^{-1}(s)$ denotes the smaller root of $g(t)=s$. Using (36) and (34), we find:

$$R \geqslant 1-g^{-1}[k(g(p_w)+p_w \log(n-1))] \tag{39}$$

Inequality (39) is useful for finding a lower bound for the reliability R when p_w is known.

We now prove that

$$g(p_w)+p_w \log(n-1) \leqslant g(R)/k \tag{40}$$

is a sufficient condition for the reliability to be at least R. Combining (40) and (38), we obtain $g(p_m) \leqslant g(R)=g(1-R)$. Since $g(t)$ is monotone increasing for $t<1/2$, we can write $p_m < 1-R$ or $1-p_m > R$. Using (36), we obtain $1-p_u > R$ which proves that the reliability is at least R.

Inequality (40) can be used to obtain an upper bound for p_w for a given reliability R. Figure 4 shows a graphical method for solving Inequality (40) using the curve of Figure 3. The inequality is satisfied for $t \leqslant t_o$.

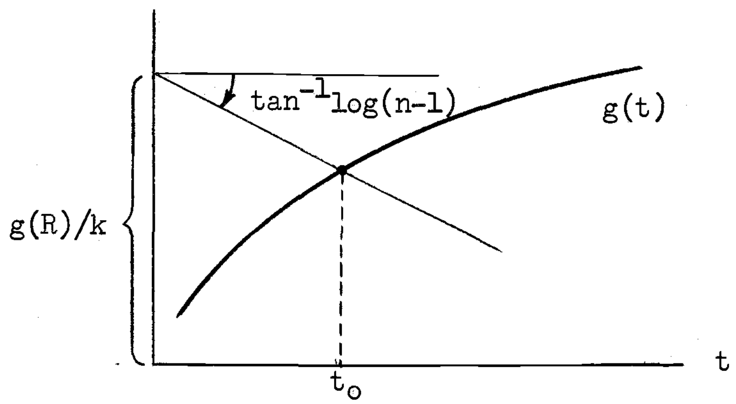

Figure 4. Graphical method for solving Inequality (40).

Example 5. Consider the four-state machine, given in Example 1, for which $p_w=10p$. Suppose we want the reliability of this machine to be at least 0.9 for input sequences of length $k=20$ or less. Substituting the known values in (40), we obtain

$$g(p_w)+0.477p_w \leqslant 0.0071$$

Solving this inequality by the graphical method of Figure 4 yields $p_w \leqslant 0.002$. Hence, if $p \leqslant 0.0002$, the reliability will be 0.9 or more for 20 cycles.

IV. RELIABLE SYNTHESIS OF SEQUENTIAL MACHINES

In the previous chapters, we developed two methods for estimating the reliability 1-p of individual elements of a sequential machine for a given overall reliability R. If 1-p is less than the reliability of available elements, and if these elements can be used economically, we can synthesize the given machine with these elements. However, if elements with higher reliability than 1-p are not available or are not economical to use, other methods for synthesis must be employed. One such method, which we will describe in this chapter, is simple redundancy at the element level. In this method, we first synthesize the machine in the usual way, assuming perfect reliability for the elements. Then, we substitute for each element a set of elements which perform the same function with the required reliability.

Redundancy Required for Single-Output Elements

Consider the configuration of Figure 5, in which M is a majority element with an odd number m of binary inputs and one binary output, and $E_1, E_2, \ldots, E_m$ are identical elements. The output y is, by definition, the same as the majority of $y_1, y_2, \ldots, y_m$. Let p_e and p_o be the probabilities of error for E_i and M, respectively, subject to the conditions given in the first part of Chapter II. Tsertsvadze (1964b) considers the more general case in which the probability of failure for a majority element depends on the number of its inputs which are equal to one.

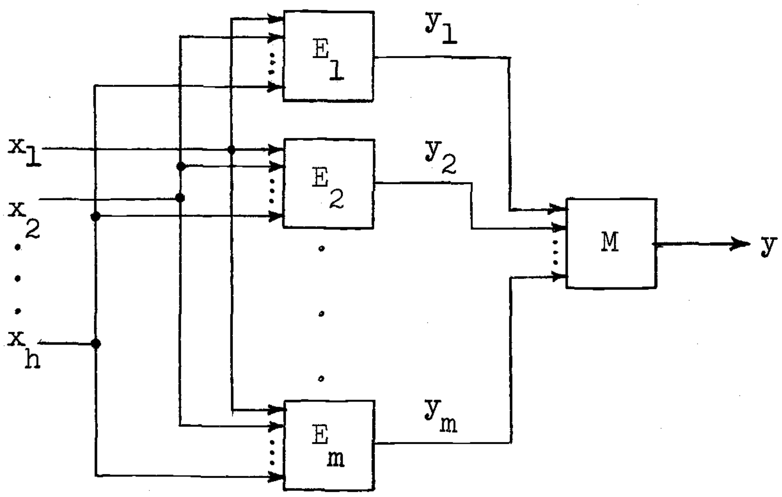

Figure 5. Redundancy of single-output elements.

Let $\bar{p}$ be the probability that a majority of $y_1, y_2, \ldots, y_m$ be erroneous. Then:

$$\bar{p} = \sum_{k=(m+1)/2}^{m} \binom{m}{k} p_e^k (1-p_e)^{m-k} \tag{41}$$

Figure 6 shows $\bar{p}$ as a function of p_e for different values of m. The overall probability of failure p for the circuit of Figure 5 can be easily found.

$$p = p_o(1-\bar{p}) + (1-p_o)\bar{p} \tag{42}$$

Since for any fixed $p_e < 1/2$, $\bar{p}$ is a decreasing function of m, Equation (42) suggests that for sufficiently large m, we can make p arbitrarily close to p_o. However, practical limitations on the number of inputs to a majority element makes this impossible. Verbeek (1962) notes that for a fixed m, we can usually reduce p by using several levels of majority elements.

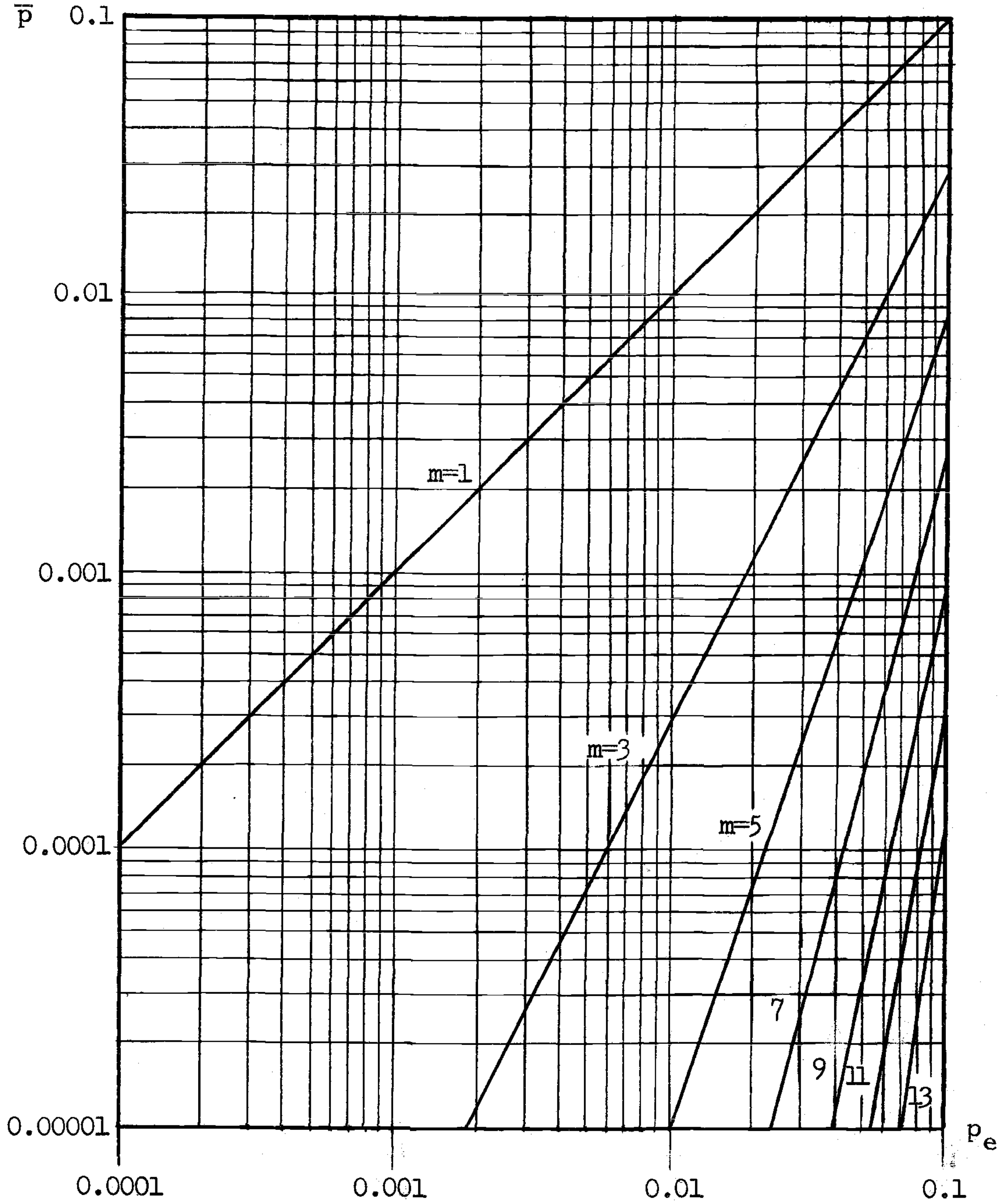

Figure 6. Probability of failure for a majority of m elements.

From Equation (42), we conclude that $p=p_o+(1-2p_o)\bar{p}>p_o$, since p_o is very small. Hence, for a given reliability 1-p, we can use the configuration of Figure 5 if majority elements with $p_o<p$ are available. This is also obvious from the fact that no circuit can be more reliable than the element which produces the final output. In what follows, we assume that this is always the case. Then, using (42), we obtain:

$$\bar{p}=(p-p_o)/(1-2p_o) \tag{43}$$

After finding $\bar{p}$ from (43), we use the following procedure to find the redundancy required. Let the reliability of each element be p_e. We find the point with coordinates p_e and $\bar{p}$ on Figure 6. If this point lies on one of the curves, the corresponding m indicates the required redundancy. If this point lies between two curves, the redundancy is obtained from the one with larger m.

The configuration of Figure 5, and hence Equations (42) and (43), can be used for gates and those memory elements with a single output line whose next states depend only on the input (delay flip-flops, for example).

Redundancy Required for Two-Output Elements

For memory elements with two output lines, the configuration of Figure 7 may be used. Then, the circuit functions properly if both of its outputs are correct. If the next state of each flip-flop depends only on its inputs, as is the case for R-S flip-flops, we have the following equations which correspond to (42) and (43):

$$p=(2p_o-p_o^2)(1-\bar{p})+(1-p_o^2)\bar{p} \tag{44}$$

$$\bar{p}=(p-2p_o+p_o^2)/(1-2p_o) \tag{45}$$

Hence, this configuration can be used if majority elements with $p_o < 1-(1-p)^{1/2}$ are available.

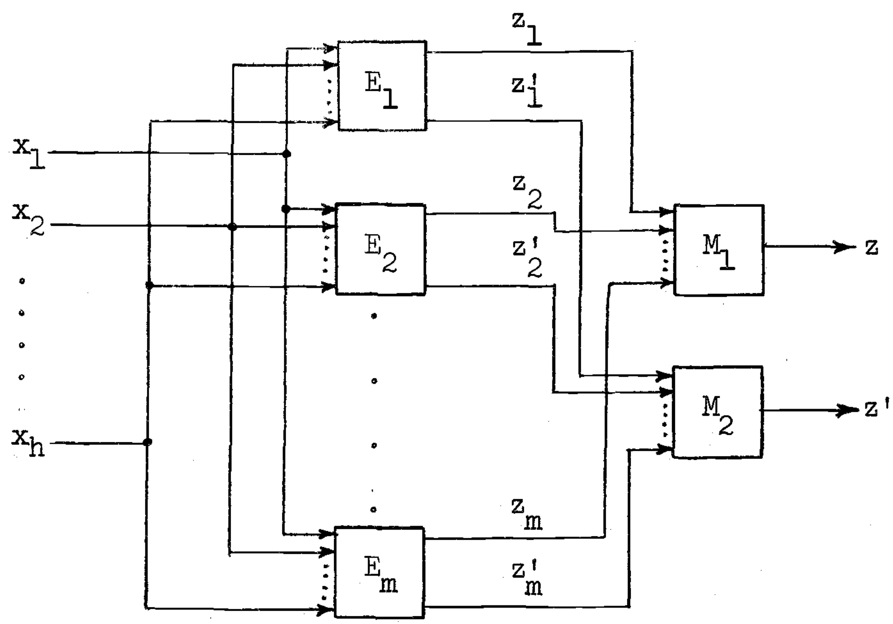

Figure 7. Redundancy of two-output elements.

For flip-flops whose next state depends on the present state, (trigger, J-K, and R-S-T flip-flops), Equation (41) is not valid. As an illustration, consider a set of m trigger flip-flops in a circuit similar to that of Figure 7. Let $z_1=z_2=\ldots=z_{m-1}=1$ while $z_m=0$ because of an error in the previous cycle. Suppose that the input $x=1$ is applied to these flip-flops. Then the probability that a majority of them produce an output of zero is not given by (41), since the m-th flip-flop produces a zero with probability $1-p_e$ instead of p_e.

The following method can be used to handle this case. Since the operation of such flip-flops in one cycle is not independent of their operation in preceeding cycles, the length of the input sequence will appear in our equations. Hence, we consider the operation of such flip-flops for input sequences of length k or less. If we assume that the probability of two or more failures for each element in k cycles is neglegible, the following inequality gives an upper bound for the probability that a flip-flop is in a different state than a majority of the flip-flops:

$$p'_e \leqslant kp_e(1-p_e)^{k-1}+(1-p_e)^k \sum_{j=(m+1)/2}^{m} \binom{m}{j}(kp_e)^j(1-kp_e)^{m-j}$$

$$\leqslant kp_e+\sum_{j=(m+1)/2}^{m} \binom{m}{j}(kp_e)^j \tag{46}$$

The probability that it produces an erroneous output is, therefore, less than

$$\bar{p} = p'_e(1-p_e)+(1-p'_e)p_e \tag{47}$$

By using $\bar{p}_e$ instead of p_e in our calculations, the methods described earlier remain valid in this case.

For $(m+1)/2 \leqslant j \leqslant m$, we have $\max\binom{m}{j}=\binom{m}{(m+1)/2}$. Hence:

$$p' \leqslant kp_e+\binom{m}{(m+1)/2} \sum_{j=(m+1)/2}^{m} (kp_e)^j$$

$$\leqslant kp_e+\binom{m}{(m+1)/2}(kp_e)^{(m+1)/2}/(1-kp_e) \tag{48}$$

The second term in the right hand side of (48) is a function of m, f(m). We can write:

$$f(m+2)/f(m)=4kp_e(m+2)/(m+3)$$

Hence, for $4kp_e \leq 1$, $f(m)$ is a monotone decreasing function of m. Then:

$$p'_e \leq kp_e+kp_e/(1-kp_e)$$

Substituting this upper bound for p'_e in Equation (47), we obtain

$$\bar{p}_e \leq p_e+kp_e(1-2p_e)(2-kp_e)/(1-kp_e)$$

Finally, we find the following upper bound for $\bar{p}_e$:

$$\bar{p}_e \leq p_e+2kp_e/(1-kp_e) \tag{49}$$

Example 6. In Example 4, we found that for the machine of Figure 2 to function with a reliability of at least 0.9 for 20 cycles, the probability of failure of individual elements, p, should not exceed 0.0005. Suppose we want to synthesize this machine with elements whose reliability is 0.998, using majority elements with p_o=0.0002. From (43) we obtain:

$$\bar{p}=(0.0005-0.0002)/(1-0.0004) \simeq 0.0003$$

Using Figure 6, we find that the required redundancy for gates is m_g=3.

To find the redundancy required for the flip-flops, we first use (45) to obtain

$$\bar{p} \simeq (0.0005-0.0004)/(1-0.0004) \simeq 0.0001$$

Then, from (49), we find $\bar{p}_e \leq 0.085$. From Figure 6, the required redundancy for flip-flops is m_f=13.

V. SUMMARY AND CONCLUSIONS

In this thesis, we have developed techniques for estimating the reliability of sequential machines and for their reliable synthesis. The results obtained are in perfectly general form and can be applied to any sequential machine.

To summarize, in Chapter II, we introduced an SSM model for sequential machines made of unreliable components and showed its usefulness in estimating the reliability of such machines. Theorem 1 provided a simple method for estimating the reliability required of individual elements for a given overall reliability. Using the notation introduced in Chapter III the results given in Example 4 may be written as:

$$R \geqslant (1-p_w)^k \tag{50}$$

Therfore, the inequality

$$p_w \leqslant 1-R^{1/k} \tag{51}$$

can be used to find an upper bound for p_w if the reliability is to be at least R for k cycles. All the results obtained in Chapter II are original with this thesis.

In Chapter III, we introduced the notion of entropy for SSM's and its applications in estimating the reliability of sequential machines. The results given in Chapter III are due to Tsertsvadze (1964a).

In Chapter IV, we developed a method for reliable synthesis of sequential machines. The first part of this chapter is based on the works by von Neumann (1956), Verbeek (1962), and Tsertsvadze (1964b). The results obtained in the second part of this chapter, i.e. those concerning the redundancy required for memory elements, are original.

A comparison between the results of Examples 4 and 5, which represent the solution to the same problem using the methods of Chapters II and III, respectively, suggests that the method of Chapter II may be advantageous. The following argument shows that this is indeed the case.

Let p'_w and p''_w denote the bounds obtained for p_w from (51) and (40), respectively. We want to prove that $p'_w \geqslant p''_w$. It is obvious that if $p'_w \geqslant p''_w$, then p'_w yields a larger value for p (the probability of failure for each element) than p''_w and hence allows the use of less reliable elements.

To prove the above statement, we first note that

$$p'_w=1-R^{1/k} \tag{52}$$

$$h(p''_w)=g(R)/k \tag{53}$$

where $h(t)=g(t)+t\log(n-1)$. Using (52), we can write:

$$h(p'_w)\geqslant g(p'_w)=g(1-R^{1/k}) \tag{54}$$

On the other hand

$$1-R^{1/k}=1-(1-(1-R))^{1/k}$$

$$=1-\big[1-(1-R)/k+(1/k-1)(1-R)^2/(2!k)-(1/k-1)(1/k-2)$$

$$(1-R)^3/(3!k)+\ldots..\big]$$

$$=(1-R)/k+(1-1/k)(1-R)^2/(2!k)+(1-1/k)(2-1/k)$$

$$(1-R)^3/(3!k)+\ldots\ldots \geqslant (1-R)/k \qquad (55)$$

Since g(t) is a monotone increasing function of t, we conclude from (54) and (55) that:

$$h(p'_w) \geqslant g(1-R^{1/k}) \geqslant g((1-R)/k) \qquad (56)$$

From Figure 3, it is obvious that $g((1-R)/k) \geqslant g(1-R)/k$. Hence, (56) may be written as

$$h(p'_w) \geqslant g(1-R)/k = g(R)/k \qquad (57)$$

Comparing (57) and (53), we obtain $h(p'_w) \geqslant h(p''_w)$. Since h(t) is a monotone increasing function of t, the above inequality implies that $p'_w \geqslant p''_w$ which concludes the proof.

The following algorithms are direct consequences of the results obtained in preceeding chapters and the above comparison.

To find the reliability R of a given sequential machine for k cycles:

1. Use (17) to obtain a lower bound for the smallest principal entry in M(x).
2. Substitute this lower bound for $1-p_w$ in (50).

To synthesize a sequential machine with a given reliability R for k cycles:

1. Synthesize the machine in the usual way, assuming perfect reliability for the elements.
2. Use (52) and (17) to obtain two lower bounds for the smallest

principal entry in M(x).

3. By equating the results of Step 2, find the reliability required of individual elements.

4. If elements of higher reliability than the result of Step 3 are available, the synthesis of Step 1 provides the required reliability. Otherwise, go to Step 5.

5. Use the methods of Chapter IV to find the redundancy for each element which assures the given overall reliability, and replicate the original element in a majority scheme to achieve the required element reliability.

An example illustrating the use of these algorithms is given in the appendix.

The above procedures deal only with the number of elements in a sequential machine and do not use its concrete structure. As can be seen from Example 4, SSM models which use the concrete automaton structure may not give much better bounds. Furthermore, the procedure for finding such models is very tedious if the machine has a large number of elements. On the other hand, we know that sequential machines may be designed to have some error-correcting capabilities. Obviously, in such cases, the use of concrete structure of the machine will yield much better bounds for the reliability required of individual elements. Hence, further research in this area may be directed to special classes of sequential machines in order to obtain sharper bounds for reliability values.

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APPENDIX

APPENDIX

We illustrate the applications of the algorithms given in Chapter V in the following example.

Example. Suppose that for the synthesis of a sequential machine, 60 NAND gates and 15 R-S flip-flops are needed. Find a lower bound for the reliability of this machine for 10 cycles if it is synthesized with gates and flip-flops whose reliabilities are 0.9998 and 0.9995, respectively. Suppose that we want to use gates and flip-flops with reliabilities 0.99 and 0.97, respectively, and majority elements with $p_o =$ 0.000002. Find the redundancy required for each element if the overall reliability is to be at least 0.99 for 10 cycles.

Solution. From (17), we have

$$1-p_w \geqslant (0.9998)^{60}(0.9995)^{15} \simeq 0.9802$$

Then a lower bound for the reliability of this machine for 10 cycles can be obtained from (50).

$$R \geqslant (0.9802)^{10} \simeq 0.819$$

For the second part of the problem, we first use (52) to find:

$$1-p_w' = (0.99)^{1/10} \simeq 0.999$$

Denoting the reliability required of each gate and flip-flop by $1-p_g$ and $1-p_f$, respectively, we obtain from (17):

$$1-p_w \geqslant 1-60p_g-15p_f$$

Hence, the following condition is sufficient for the required reliability:

$$60p_g+15p_f=0.001$$

We choose $p_g=0.000012$ and $p_f=0.000019$, since our flip-flops are less reliable than the gates. From (43) and (45), we obtain $\bar{p} \simeq 0.00001$ for gates and $\bar{p} \simeq 0.000015$ for flip-flops. Hence, from Figure 6, the redundancies required for gates and flip-flops are $m_g=5$ and $m_f=7$, respectively.