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Allen Freedman defined a density space to be the ordered pair $(S, \mathfrak{F})$ where S is a certain kind of semigroup called an s-set and $\mathfrak{F}$ is a special type of family of finite subsets of S called a fundamental family on S. Several properties for density spaces are obtained, and the theory of a particular class of density spaces, called discrete density spaces, is developed in detail. The theory of discrete density spaces includes generalizations of some theorems of H. Mann, F. Dyson, and Freedman. Several inequalities for the density of the sum of two subsets of the s-set are obtained. Particular emphasis is placed on obtaining some sufficient conditions for the $\alpha + \beta$ property of additive number theory.

Density Space Theory: Discrete Spaces

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DENSITY SPACE THEORY: DISCRETE SPACES

CHAPTER I

INTRODUCTION

In 1930, L. Schnirelmann [13, 14] introduced the following density for a subset A of the non-negative integers. Let $A(n)$ denote the number of positive integers in the set A which do not exceed n. Then the Schnirelmann density of A is given by

$$\alpha = \text{glb}\ \{\frac{A(n)}{n} \mid n \geq 1\}. \tag{1}$$

Let A and B be two subsets of the non-negative integers. The sum of A and B, denoted by $A + B$, is the set $\{a + b \mid a \in A,\ b \in B\}$. Now let α, β, and γ denote the Schnirelmann densities of A, B, and $C = A + B$ respectively. Suppose $0 \in A \cap B$. Some of the results which have been obtained are:

(2) If $\alpha + \beta \geq 1$, then $\gamma = 1$ (Schnirelmann [14]).

(3) $\gamma \geq \alpha + \beta - \alpha\beta$ (E. Landau [9] and Schnirelmann [14]).

(4) If $\alpha + \beta < 1$, then $\gamma \geq \beta/(1-\alpha)$ (I. Schur [15]).

(5) $\gamma \geq \min\ \{1, \alpha+\beta\}$ (H. Mann [10], F, Dyson [2], and B. Kvarda Garrison [6]).

Inequality (3) is often called the Landau-Schnirelmann inequality,

(4) is called the Schur inequality, and (5) is the famous $\alpha + \beta$ property.

In 1965, A. Freedman [3] generalized the concept of Schnirelmann density to arbitrary sets. Let S be an arbitrary set with a special element called zero and denoted 0. For a subset X of S, and a finite subset D of S, let $X(D)$ denote the number of non-zero elements in the set $X \cap D$. Let $\mathcal{G}$ be any family of finite subsets of S such that $G \in \mathcal{G}$ implies $G \setminus \{0\} \neq \phi$. Then the density of a subset A of S with respect to $\mathcal{G}$ is

$$\alpha = \text{glb} \left\{ \frac{A(G)}{S(G)} \,\middle|\, G \in \mathcal{G} \right\}.$$

Freedman developed a general theory for density by introducing two sets of axioms, one set giving structure to S and the other set giving structure to $\mathcal{G}$. The set S is then called an s-set, the family $\mathcal{G}$ a fundamental family on S, and the ordered pair $(S, \mathcal{G})$ a density space. Freedman, by introducing additional restrictions where necessary, has extended many of the results which have been obtained for positive integers, including (2), (3), (4) and (5). In fact, (2) is valid for all density spaces.

In this thesis we study a special class of density spaces called discrete density spaces. We are primarily interested in knowing when the $\alpha + \beta$ property holds, but in seeking such a result we also extend results (3) and (4). We have results for several types of discrete density spaces. Freedman [3] has proved the $\alpha + \beta$ property for discrete density spaces of order 1 and 2. We show that the $\alpha + \beta$ property holds for all purely discrete density spaces, for all singularly discrete density spaces of order $n \leq 4$, for all nested singularly discrete density spaces of order $n \leq 5$, and for all nested singularly discrete density spaces having two or less essential points. We prove results (3) and (4) for all nested singularly discrete density spaces (actually more generally for all T-spaces). We give examples where the $\alpha + \beta$ property and results (3) and (4) all fail for discrete density spaces of order $n \geq 3$ and for singularly discrete density spaces of order $n \geq 5$. We conjecture that the $\alpha + \beta$ property holds for all nested singularly discrete density spaces. The above theory is developed in Chapters III through VIII. In Chapter V we obtain basic results which may be used to simplify the proofs of certain density space theorems. These results allow us to replace the density space by a simpler one. This chapter is of independent interest.

In the last part of Chapter VIII and in the final three chapters we look at several other interesting topics in discrete density space theory and the more general density space theory of Freedman. These topics include extensions of Mann's and Dyson's inequalities, Mann's Second Theorem, mixed density theory, the relationship between c-density and k-density, and relationships between the transformation properties of Freedman.

We use tne following numbering conventions in this thesis. All definitions, lemmas, theorems, corollaries, and conjectures are numbered consecutively in each section from x. y. 1, where x denotes the chapter and y denotes the section. Displayed material which is referred to later is numbered consecutively from (1) within each section.

We use the following notational conventions. Let A and B be any sets. We write $A \subset B$ if A is a proper subset of B. We write $A \subseteq B$ if A is a subset of B with equality allowed. We use the symbol $\setminus$ to denote set difference; that is, $A \setminus B = \{x | x \in A \text{ and } x \notin B\}$. If $\{a_1, a_2, \ldots, a_k, a\}$ is a set of non-negative integers, we denote by $\{a_1, a_2, \ldots, a_k, \overline{a}\}$ the set

$$\{a_i | 1 \leq i \leq k\} \cup \{m | m \text{ is an integer and } m \geq a\}.$$

For any real numbers a and b such that $a \leq b$, we write

$$[a, b] = \{x | a \leq x \leq b\},$$

$$(a, b] = \{x | a < x \leq b\},$$

$$[a, b) = \{x | a \leq x < b\},$$

$$(a, b) = \{x | a < x < b\}.$$

CHAPTER II

BACKGROUND

In this chapter we state those definitions and theorems from Freedman's work which are used in the remainder of this thesis. Proofs of the theorems are omitted and may be found in Freedman's thesis [3] or in his article [4]. There is a significant difference in convention between Freedman's thesis and his article. In his thesis he requires the 0 element to be missing from all s-sets, while in his article he requires the 0 element to be in all s-sets. We adopt the convention of his article.

2.1. s-sets

Throughout this section S is a non-empty subset of an abelian group G. The operation in G is denoted by $+$ and the identity element by 0. We also assume that $0 \in S$.

Definition 2.1.1. For x and y in G, we write $x \prec y$ (or $y \succ x$) whenever $y - x \in S \setminus \{0\}$.

Definition 2.1.2. For $x \in S \setminus \{0\}$, let $L(x)$ denote the set of all $y \in S$ for which $y \prec x$ or $y = x$. We call $L(x)$ the lower set of x with respect to S.

Definition 2.1.3. The set S is called an s-set whenever the following three axioms are satisfied:

Axiom s.1. $S \setminus \{0\}$ is closed under $+$.

Axiom s.2. S contains at least one element in addition to 0.

Axiom s.3. $L(x)$ is finite for each $x \in S \setminus \{0\}$.

Definition 2.1.4. Let I denote the set of all non-negative integers.

Theorem 2.1.5. The set I is an s-set where $+$ is the usual addition for integers.

Theorem 2.1.6. If S' is a closed subset of an s-set S, $0 \in S'$, and $S' \setminus \{0\} \neq \phi$, then S' is an s-set.

For example, the set of even non-negative integers is an s-set.

Definition 2.1.7. Let X be a subset of an s-set S. An element $x \in X$ is called a minimal point of X if $x = 0$ or $X \cap L(x) = \{x\}$. The set of all minimal points of X is denoted by Min (X).

Definition 2.1.8. Let S be an s-set and $x \in S$. Denote by $U(x)$ the set of all $y \in S$ such that $x \prec y$ or $x = y$. We call $U(x)$ the upper set of x with respect to S.

Definition 2.1.9. Let X be a subset of an s-set S. An element $x \in X$ is called a <u>maximal point</u> of X if $X \cap U(x) = \{x\}$. The set of all maximal points of X is denoted by $\mathrm{Max}(X)$.

<u>Theorem 2.1.10.</u> An s-set S is isomorphic to I if and only if $\mathrm{Min}(S \setminus \{0\})$ reduces to a singleton $\{x\}$.

<u>2.2. Fundamental Families</u>

<u>Definition 2.2.1.</u> For an arbitrary set S, let $\mathfrak{D} = \mathfrak{D}(S)$ denote the family of all sets $D \subseteq S$ with D finite, $0 \in D$, and $D \setminus \{0\} \neq \phi$.

<u>Definition 2.2.2.</u> The ordered pair $(S, \mathfrak{F})$ is called a <u>space</u> whenever S is an s-set and $\mathfrak{F}$ is a non-empty subfamily of $\mathfrak{D}(S)$.

<u>Definition 2.2.3.</u> Let $\mathfrak{F}$ be an arbitrary subfamily of $\mathfrak{D}$ and let F be a set in $\mathfrak{F}$. An element $x \in F$ is called a <u>corner point</u> of F (with respect to $\mathfrak{F}$) if $F \setminus \{x\} \in \mathfrak{F} \cup \{\{0\}\}$. The set of all corner points of F is denoted by F^*.

<u>Definition 2.2.4.</u> Let S be an s-set. A non-empty family $\mathfrak{F} \subseteq \mathfrak{D}(S)$ is called a <u>fundamental family</u> on S if the following four axioms are satisfied:

Axiom f.1. $S = \cup \{F \mid F \in \mathfrak{F}\}$.

Axiom f.2. If $F \in \mathfrak{F}$ and $G \in \mathfrak{F}$, then $F \cup G \in \mathfrak{F}$.

Axiom f.3. If $F \in \mathfrak{F}$ and $G \in \mathfrak{F}$, then $F \cap G \in \mathfrak{F} \cup \{\{0\}\}$.

Axiom f.4. If $F \in \mathfrak{F}$, then $\text{Max}(F) \subseteq F^*$.

Definition 2.2.5. The space $(S, \mathfrak{F})$ is called a density space whenever $\mathfrak{F}$ is a fundamental family on S.

The following theorem is useful in the actual construction of fundamental families on s-sets.

Theorem 2.2.6. Let S be an arbitrary s-set. Corresponding to each $x \in S \setminus \{0\}$, let $H(x)$ be a subset of S satisfying the following three conditions:

c.1. $\{0, x\} \subseteq H(x)$,

c.2. $H(x) \subseteq L(x)$,

c.3. If $y \in H(x) \setminus \{0\}$, then $H(y) \subseteq H(x)$.

Let $\mathfrak{F}_H = \{F \mid F \in \mathfrak{D}(S),\ x \in F \setminus \{0\} \text{ implies } H(x) \subseteq F\}$. Then $\mathfrak{F}_H$ is a fundamental family on S. Conversely, given any fundamental family $\mathfrak{F}$ on S, there exists a unique function $H(x)$ defined on $S \setminus \{0\}$ satisfying c.1, c.2, and c.3 such that $\mathfrak{F} = \mathfrak{F}_H$.

Definition 2.2.7. For any s-set S and any function $H(x) \subseteq S$ defined on $S \setminus \{0\}$, we denote by $\mathfrak{F}_H$ the family

$$\mathfrak{F}_H = \{F \mid F \in \mathfrak{D}(S),\ x \in F \setminus \{0\} \text{ implies } H(x) \subseteq F\}.$$

Definition 2.2.8. Let $x \in S \setminus \{0\}$. Denote by $[x]$ the intersection of all $F \in \mathfrak{F}$ such that $x \in F$. Then $[x]$ is called the

Cheo set of $\mathfrak{F}$ determined by x.

Theorem 2.2.9. Let S be an s-set and let H(x) satisfy c.1, c.2, and c.3 of Theorem 2.2.6 for each $x \in S \setminus \{0\}$. Then [x], the Cheo set of $\mathfrak{F}_H$ determined by x, is equal to H(x).

Definition 2.2.10. A point $x \in S \setminus \{0\}$ is an essential point of the density space $(S, \mathfrak{F})$ if $[x] = \{0, x\}$. If $[x] \neq \{0, x\}$ then x is called a non-essential point.

We define a special fundamental family as follows:

Definition 2.2.11. Let S be an s-set. We denote by $\mathcal{K} = \mathcal{K}(S)$ the fundamental family $\mathfrak{F}_H$ with $H(x) = L(x)$ for each $x \in S \setminus \{0\}$.

Theorem 2.2.12. For any fundamental family $\mathfrak{F}$ on an s-set S we have $\mathcal{K} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$.

Theorem 2.2.13. For any s-set S, the spaces $(S, \mathcal{K})$ and $(S, \mathfrak{D})$ are density spaces.

Theorem 2.2.14. Let $\mathfrak{F}$ be a fundamental family. If $F \in \mathfrak{F}$, $G \in \mathfrak{F}$, and $F^* \subseteq G$, then $F \subseteq G$.

Theorem 2.2.15. Let $(S, \mathfrak{F})$ be a density space and $X \subseteq S$ where X is finite and $X \setminus \{0\} \neq \phi$. Then the set

$F = \cup\{[x] | x \in X \setminus \{0\}\}$ is in $\mathfrak{F}$ and furthermore $F^* \subseteq X$.

Theorem 2.2.16. Let $\mathfrak{F}$ be a fundamental family. If $F \in \mathfrak{F}$, then $F = \cup\{[x] | x \in F \setminus \{0\}\}$.

2.3. Density

In this section let $(S, \mathfrak{F})$ be an arbitrary density space.

Definition 2.3.1. Let X be a subset of S. For any finite (possibly empty) subset D of S we let $X(D)$ be the number of non-zero elements in the set $X \cap D$. If $D \setminus \{0\}$ is non-empty, let $q(X, D)$ be the quotient $X(D)/S(D)$.

Definition 2.3.2. Let A be an arbitrary subset of S. The k-density of A with respect to $\mathfrak{F}$ is

$$d_k(A, \mathfrak{F}) = \text{glb}\{q(A, F) | F \in \mathfrak{F}\}.$$

Definition 2.3.3. Let A be an arbitrary subset of S. The c-density of A with respect to $\mathfrak{F}$ is

$$d_c(A, \mathfrak{F}) = \text{glb}\{q(A, [x]) | x \in S \setminus \{0\}\},$$

where $[x]$ is the Cheo set of $\mathfrak{F}$ determined by x.

The k-density generalizes the density defined by B. Kvarda [8] and the c-density generalizes the density used by L. Cheo [1] and

F. Kasch [7]. Both of these densities reduce to Schnirelmann density when the density space is $(I, \mathcal{K})$.

For any density space $(I, \mathfrak{F})$ we define two more counting functions in addition to the one given by Definition 2.3.1.

Definition 2.3.4. Let X be a subset of I and $x \in I$. Then $X(x)$ denotes the number of positive integers in X which do not exceed x. Let $a \in I$, $b \in I$, and $a \leq b$. Then $X(a, b)$ denotes the number of integers $x \in X$ such that $a \leq x \leq b$.

The function $X(a, b)$ is the only counting function in this thesis which can count 0. In particular, we have $X(0, b) = X(b) + 1$ if $0 \in X$.

Definition 2.3.5. Let A and B be subsets of S. The sum $A + B$ of A and B is the set $\{a + b \mid a \in A, b \in B\}$.

Let $C = A + B$ and set $d_k(A, \mathfrak{F}) = \alpha_k = \alpha$, $d_k(B, \mathfrak{F}) = \beta_k = \beta$, $d_k(C, \mathfrak{F}) = \gamma_k = \gamma$, $d_c(A, \mathfrak{F}) = \alpha_c$, $d_c(B, \mathfrak{F}) = \beta_c$, and $d_c(C, \mathfrak{F}) = \gamma_c$. If we refer to the density of a set we always mean k-density.

In the remainder of this chapter let A and B be subsets of S with $0 \in A \cap B$.

Theorem 2.3.6. We have $0 \leq \alpha \leq \alpha_c \leq 1$. Furthermore, $\alpha = \alpha_c = 1$ if and only if $A = S$.

Theorem 2.3.7. $\gamma \geq \max\{\alpha, \beta\}$.

Theorem 2.3.8. If $\alpha + \beta \geq 1$, then $\gamma = 1$.

Theorem 2.3.9. $\gamma_c \geq \max\{\alpha_c, \beta_c\}$.

Theorem 2.3.10. If $\mathfrak{F} = \mathcal{K}(S)$ and $\alpha_c + \beta_c \geq 1$, then $\gamma_c = 1$.

Definition 2.3.11. A fundamental family $\mathfrak{F}_H$ is separated if, whenever x and y are elements of $S \setminus \{0\}$ with $x \notin H(y)$ and $y \notin H(x)$, then $H(x) \cap H(y) = \{0\}$.

Theorem 2.3.12. If $\mathfrak{F}$ is a separated fundamental family, then k-density and c-density are identical; that is, $\alpha = \alpha_c$ for each $A \subseteq S$.

Theorem 2.3.13. If $S \setminus A$ is non-empty and the fundamental family $\mathfrak{F}$ is separated, then $\alpha_c = \text{glb}\{q(A,[x]) \mid x \in S \setminus A\}$.

Freedman [3, p. 43-4] defines two transformation properties, trans-1 and trans-2.

Definition 2.3.14. Let $F \in \mathfrak{F}$, $x \in F$, $D = F \cap U(x)$, and $T_1[D] = \{y - x \mid y \in D\}$. Then $\mathfrak{F}$ is trans-1 if $T_1[D] \in \mathfrak{F} \cup \{\{0\}\}$ for every $F \in \mathfrak{F}$ and $x \in F$.

Definition 2.3.15. Let $x \in S \setminus \{0\}$, $F \in \mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}$, $D = H(x) \setminus F$, and $T_2[D] = \{x - y \mid y \in D\}$. Then $\mathfrak{F}_H$ is trans-2 if $T_2[D] \in \mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}$ for every $x \in S \setminus \{0\}$ and $F \in \mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}$.

Theorem 2.3.16. The fundamental families $\mathfrak{K}(S)$ and $\mathfrak{D}(S)$ are both trans-1 and trans-2.

Theorem 2.3.17. Suppose $\mathfrak{F}$ is trans-1. Then for each $F \in \mathfrak{F}$ we have $C(F) \geq A(F) + \beta(S \setminus A)(F)$.

Theorem 2.3.18. If $\mathfrak{F}$ is trans-1, then $\gamma \geq \alpha + \beta - \alpha\beta$.

Theorem 2.3.19. If $\mathfrak{F}$ is trans-1, then $\gamma_c \geq \alpha_c + \beta - \alpha_c\beta$.

Theorem 2.3.20. Suppose $\mathfrak{F}$ is trans-2. If $S \setminus C \neq \phi$ and $F \in \mathfrak{F}$ where $F^* \subseteq S \setminus C$, then $C(F) \geq \alpha C(F) + B(F)$.

Theorem 2.3.21. If $\mathfrak{F}$ is trans-2 and $\alpha + \beta < 1$, then $\gamma \geq \beta/(1-\alpha)$.

If $S = I$ we have the following result.

Theorem 2.3.22. If $n \in I \setminus C$, then $A(n) + B(n) < n$.

CHAPTER III

DISCRETE, PURELY DISCRETE, AND SINGULARLY DISCRETE DENSITY SPACES

In this chapter we begin the study of discrete density spaces. We study discrete, purely discrete, and singularly discrete density spaces. In particular we are interested in determining when the $\alpha + \beta$ property holds for these density spaces.

3.1. Discrete Density Spaces in General

We define a discrete density space with the following definition due essentially to Freedman [3, p. 101]:

Definition 3.1.1. A density space $(S, \mathfrak{F})$ is called a discrete density space whenever $\mathfrak{F}$ is separated. The order of the space is given by

$$\max \{S([x]) \mid x \in S \setminus \{0\}\}$$

if this maximum exists. Otherwise, the space is said to be of infinite order.

The density space $(I, \mathfrak{X})$ is an example of a discrete density space of infinite order. The density space $(S, \mathfrak{D})$ is a discrete density space of order 1.

Since $\mathfrak{F}$ is separated in a discrete density space we known, by Theorem 2.3.12, that the k-density and c-density of any subset of S are identical. Therefore, whenever we are working with discrete density spaces we can say that the set A has density α, knowing that α is both the k-density and the c-density of A.

Freedman [3, p. 102-3] proves the following theorem whose proof we give here for completeness:

Theorem 3.1.2. Let $(S, \mathfrak{F})$ be any discrete density space of order 1 or 2, let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. Then $\gamma \geq \min \{1, \alpha + \beta\}$.

Proof: Suppose $(S, \mathfrak{F})$ is discrete of order 1. Then the only values possible for α, β, and γ are 0 and 1. If $\gamma = 1$, then $\gamma \geq \min \{1, \alpha + \beta\}$. If $\gamma = 0$, then by Theorem 2.3.7 we have $0 \geq \max \{\alpha, \beta\}$, and so $\alpha + \beta = 0$. Therefore, $\gamma \geq \min \{1, \alpha + \beta\}$.

Suppose $(S, \mathfrak{F})$ is discrete of order 2. Then the only values possible for α, β, and γ are $0, \frac{1}{2}$, and 1. If $\gamma = 0$ or 1, then we argue as above to obtain $\gamma \geq \min \{1, \alpha + \beta\}$. If $\gamma = \frac{1}{2}$ and $\alpha + \beta \leq \frac{1}{2}$, then $\gamma \geq \min \{1, \alpha + \beta\}$. If $\gamma = \frac{1}{2}$ and $\alpha + \beta > \frac{1}{2}$, then $\alpha + \beta \geq 1$ and so by Theorem 2.3.8 we have $\gamma = 1$, a contradiction. This completes the proof.

The following example shows that there is a discrete density space of any finite order greater than 2 for which the $\alpha + \beta$ property fails. In fact, it shows that Theorem 2.3.7, which is true for all density spaces, gives in a sense the strongest result. Let n be any positive integer greater than 2 and let $S = I$. Define $H(x)$ for all $x \in I \setminus \{0\}$ as follows:

$$H(x) = \begin{cases} \{0, 1, 3\} & \text{if } x = 3, \\ \{0, 1, 3, 4\} & \text{if } x = 4, \\ \{0, 2, 5\} & \text{if } x = 5, \\ \{0, 2, 5, 6\} & \text{if } x = 6, \\ \{0, 11, 12, \ldots, 10+n\} & \text{if } x = 10+n, \\ \{0, x\} & \text{otherwise.} \end{cases}$$

Now $H(x)$ satisfies conditions c.1, c.2, and c.3 of Theorem 2.2.6, so $\mathfrak{F}_H$ is a fundamental family on I. Also $\mathfrak{F}_H$ is separated. Furthermore, $I([10+n]) = n$ and $I([x]) \leq n$ for all $x \in I \setminus \{0\}$. Therefore, by Definition 3.1.1, the space $(I, \mathfrak{F}_H)$ is discrete of order n. Let $A = B = \{0, 1, 2, \overline{7}\}$. Then $C = \{0, 1, 2, 3, 4, \overline{7}\}$. Now $\alpha = \beta = \gamma = \frac{1}{3}$. Hence, Theorem 2.3.7 gives in a sense the strongest result.

We obtain a space $(I, \mathfrak{F}_H)$ which is discrete of infinite order and for which the $\alpha + \beta$ property fails by replacing the fifth line in the above definition for $H(x)$ by

$H(x) = \{0, 11, 12, \ldots, 10+n\}$ if $x = 10 + n$ for all $n \geq 3$.

In the remainder of this chapter and in Chapter IV we place various restrictions on our discrete density spaces so that we can prove results stronger than that given by Theorem 2.3.7.

3.2. Purely Discrete Density Spaces

Definition 3.2.1. A density space $(S, \mathfrak{F})$ is purely discrete if the following two conditions are satisfied:

(i) $(S, \mathfrak{F})$ is discrete,

(ii) If $x \in S \setminus \{0\}$ and $y \in [x] \setminus \{0, x\}$, then $[y] = \{0, y\}$.

Here we are able to obtain the $\alpha + \beta$ property.

Theorem 3.2.2. Let $(S, \mathfrak{F})$ be any purely discrete density space, let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. Then $\gamma \geq \min\{1, \alpha + \beta\}$.

Proof: Suppose that $(S, \mathfrak{F})$ is purely discrete of order n. From condition (ii) of Definition 3.2.1, we conclude that for any $x \in S \setminus \{0\}$, either

(a) $[x] = \{0, x\}$,

or

(b) $[x] = \{0, x_1, \ldots, x_{i-1}, x\}$ where $2 \leq i \leq n$ and

$$[x_j] = \{0, x_j\} \quad \text{for} \quad j = 1, 2, \ldots, i-1.$$

In case (a), we have $\alpha = 0$ if $x \notin A$. In case (b), we have $\alpha = 0$ if $x_j \notin A$ for some j where $1 \leq j \leq i-1$. Therefore, if $\alpha \neq 0$, we must have $q(A, [x]) = 1$ or $q(A, [x]) = \frac{i-1}{i}$. Therefore, the only values α, β, and γ can take on are 0, 1, or $\frac{i-1}{i}$ where $2 \leq i \leq n$. If either α or β is 0, then by Theorem 2.3.7, we have

$$\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}.$$

If either α or β is 1, then by Theorem 2.3.8, we have

$$\gamma = 1 = \min\{1, \alpha + \beta\}.$$

If $\alpha = \frac{i-1}{i}$ where $2 \leq i \leq n$ and $\beta = \frac{j-1}{j}$ where $2 \leq j \leq n$, then

$$\alpha + \beta = \frac{i-1}{i} + \frac{j-1}{j} \geq \frac{1}{2} + \frac{1}{2} = 1.$$

Therefore, by Theorem 2.3.8, we have $\gamma = 1 = \min\{1, \alpha + \beta\}$.

If $(S, \mathcal{F})$ is purely discrete of infinite order we replace $2 \leq i \leq n$ by $2 \leq i$ and we replace $2 \leq j \leq n$ by $2 \leq j$ everywhere they occur in the proof for the finite case.

This completes the proof.

Although we are able to obtain the $\alpha + \beta$ property here, we should note that to do so we have severely restricted the discrete density spaces involved. In the next section we introduce a different restriction which allows us to obtain the $\alpha + \beta$ property for some other discrete spaces.

3.3. Singularly Discrete Density Spaces

Definition 3.3.1. A density space $(S, \mathfrak{F})$ is singularly discrete if the following two conditions are satisfied:

(i) $(S, \mathfrak{F})$ is discrete,

(ii) For every integer $i \geq 2$ we have $S([x]) = i$ for at most one $x \in S \setminus \{0\}$.

Theorem 3.3.2. Let $(S, \mathfrak{F})$ be any singularly discrete density space of order 3, let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. Then $\gamma \geq \min\{1, \alpha + \beta\}$.

Proof: There are three ways in which $\mathfrak{F}_H$ can be defined so that $(S, \mathfrak{F}_H)$ is singularly discrete of order 3.

Space 1: Let $H(x)$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3$. The space $(S, \mathfrak{F}_H)$ is purely discrete of order 3 and, by Theorem 3.2.2, we have $\gamma \geq \min\{1, \alpha + \beta\}$.

Space 2: Let $H(x)$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x_4, x_5\} & \text{if } x = x_5, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3$ and $0 \prec x_4 \prec x_5$ and where $x_i \neq x_j$ if $i \neq j$. The space $(S, \mathfrak{F}_H)$ is purely discrete of order 3 and, by Theorem 3.2.2, we have $\gamma \geq \min\{1, \alpha + \beta\}$.

Space 3: Let $H(x)$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x_1, x_2\} & \text{if } x = x_2, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3$. The only values possible for α, β, and γ are $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.3.7, we have

$$\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}.$$

If $\alpha + \beta \geq 1$, then by Theorem 2.3.8, we have

$$\gamma = 1 = \min\{1, \alpha + \beta\}.$$

We may select our notation so that $\alpha \leq \beta$. Therefore, in the remainder of the proof we may assume that $0 < \alpha \leq \beta$ and $\alpha + \beta < 1$.

Since $0 < \alpha \leq \beta$, we known that all of the essential points are in $A \cap B$ and hence in C; that is,

$$S \setminus \{x_2, x_3\} \subseteq A \cap B \subseteq C.$$

We have two cases to consider.

Case 1 ($\alpha = \beta = \frac{1}{3}$): We have $\alpha + \beta = \frac{2}{3}$. We show that $\gamma \geq \frac{2}{3}$ by showing that $x_2 \in C$. Since $\alpha = \frac{1}{3}$, we have $x_2 \notin A$ and $x_3 \notin A$. Since $x_1 \prec x_2 \prec x_3$, we have $x_2 - x_1 \in S$. Also $x_2 - x_1 \prec x_2 \prec x_3$ because $x_2 - (x_2 - x_1) = x_1 \in S$. The relation $\prec$ is transitive and hence $x_2 - x_1 \neq x_2$ and $x_2 - x_1 \neq x_3$. Therefore, $x_2 - x_1 \in A$ because all elements except x_2 and x_3 are in A. Therefore, since $x_1 \in B$ we have $x_2 = (x_2 - x_1) + x_1 \in C$.

Case 2 ($\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$): Here $\alpha + \beta = \frac{5}{6}$. We show that $\gamma = 1 > \frac{5}{6}$ by showing that $x_2 \in C$ and $x_3 \in C$. Since $\alpha = \frac{1}{3}$, we have $x_2 \notin A$ and $x_3 \notin A$. Since $\beta = \frac{1}{2}$, we have $x_2 \notin B$ and $x_3 \in B$. Now $x_2 \in C$ exactly as in Case 1 and $x_3 \in B \subseteq C$.

This completes the proof.

We now show that the $\alpha + \beta$ property holds for all singularly discrete density spaces of order 4. We begin with the following lemma which is used in the proof for Space 5 of Theorem 3.3.4.

<u>Lemma 3.3.3.</u> Let $(S, \mathfrak{F}_H)$ be the density space determined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x_1, x_2\} & \text{if } x = x_2, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$. Let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$ and let α be the density of A.

(i) If $\alpha > 0$ and $x_1 \in B$, then $x_2 \in C$.

(ii) If $\alpha > 0$ and $x_1 \in B$ and $x_2 \in B$, then $x_3 \in C$.

Proof of (i): Since $x_1 \prec x_2 \prec x_3 \prec x_4$, we have $x_2 - x_1 \in S$ and $x_2 - x_1 \prec x_2 \prec x_3 \prec x_4$. Therefore, $x_2 - x_1 \neq x_2$, $x_2 - x_1 \neq x_3$, and $x_2 - x_1 \neq x_4$. Since $\alpha > 0$, only x_2, x_3, or x_4 could be missing from A. Therefore, $x_2 - x_1 \in A$. Now $x_1 \in B$, so $x_2 = (x_2 - x_1) + x_1 \in C$.

Proof of (ii): Since $x_1 \prec x_2 \prec x_3 \prec x_4$, we have $x_3 - x_2 \in S$, $x_3 - x_1 \in S$, $x_2 - x_1 \in S$, $x_3 - x_2 \prec x_3 \prec x_4$, and $x_3 - x_1 \prec x_3 \prec x_4$. Therefore, $x_3 - x_2 \neq x_3$, $x_3 - x_2 \neq x_4$, $x_3 - x_1 \neq x_3$, and $x_3 - x_1 \neq x_4$. Also $x_3 - x_2 \prec x_3 - x_1$ because $(x_3 - x_1) - (x_3 - x_2) = x_2 - x_1 \in S$, and so $x_3 - x_2 \neq x_3 - x_1$. Therefore, either $x_3 - x_2 \neq x_2$ or $x_3 - x_1 \neq x_2$. Since $\alpha > 0$, only x_2, x_3, or x_4 could be missing from A. Therefore, either $x_3 - x_2 \in A$ or $x_3 - x_1 \in A$. Since $x_1 \in B$ and $x_2 \in B$, then in either case $x_3 \in C$.

This completes the proof of Lemma 3.3.3.

Theorem 3.3.4. Let $(S, \mathfrak{F})$ be any singularly discrete density space of order 4, let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. Then $\gamma \geq \min \{1, \alpha + \beta\}$.

Proof: There are fourteen ways in which $\mathfrak{F}_H$ can be defined so that $(S, \mathfrak{F}_H)$ is singularly discrete of order 4. In the remainder of the proof we assume that $x_i \neq x_j$ whenever $i \neq j$.

Space 1: Let $\mathfrak{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$. The space $(S, \mathfrak{F}_H)$ is purely discrete of order 4 and, by Theorem 3.2.2, we have $\gamma \geq \min \{1, \alpha + \beta\}$.

Space 2: Let $\mathfrak{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_5, x_6\} & \text{if } x = x_6, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$ and $0 \prec x_5 \prec x_6$. The space

$(S, \mathfrak{F}_H)$ is purely discrete of order 4 and, by Theorem 3.2.2, we have $\gamma \geq \min\{1, \alpha + \beta\}$.

Space 3: Let $\mathfrak{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_5, x_6, x_7\} & \text{if } x = x_7, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$ and $0 \prec x_5 \prec x_6 \prec x_7$. The space $(S, \mathfrak{F}_H)$ is purely discrete of order 4 and, by Theorem 3.2.2, we have $\gamma \geq \min\{1, \alpha + \beta\}$.

Space 4: Let $\mathfrak{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_5, x_6, x_7\} & \text{if } x = x_7, \\ \{0, x_8, x_9\} & \text{if } x = x_9, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$ and $0 \prec x_5 \prec x_6 \prec x_7$ and $0 \prec x_8 \prec x_9$. The space $(S, \mathfrak{F}_H)$ is purely discrete of order 4 and, by Theorem 3.2.2, we have $\gamma \geq \min\{1, \alpha + \beta\}$.

Space 5: Let $\mathfrak{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x_1, x_2\} & \text{if } x = x_2, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$. The only values possible for α, β, and γ, listed in increasing order, are $0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.3.7, we have

$$\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}.$$

If $\alpha + \beta \geq 1$, then by Theorem 2.3.8, we have

$$\gamma = 1 = \min\{1, \alpha + \beta\}.$$

We may select our notation so that $\alpha \leq \beta$. Therefore, in the remainder of the proof for Space 5 we may assume that $0 < \alpha \leq \beta$ and $\alpha + \beta < 1$. Since $0 < \alpha \leq \beta$, we know that all of the essential points are in $A \cap B$ and hence in C; that is,

$$S \setminus \{x_2, x_3, x_4\} \subseteq A \cap B \subseteq C.$$

By part (i) of Lemma 3.3.3, we have $x_2 \in C$. Therefore, the only elements which could be missing from C are x_3 and x_4. This allows four possibilities for the set C. The following table lists the eight ways in which the positive densities of Space 5 can be obtained. Those entries where x_2 is in the set give the four possibilities for the set C.

Density	Subcase	Non-Essential Points x_2	x_3	x_4
$\frac{1}{4}$	1	0	0	0
$\frac{1}{3}$	1	0	0	+
$\frac{1}{2}$	1	0	+	0
	2	0	+	+
	3	+	0	0
$\frac{2}{3}$	1	+	0	+
$\frac{3}{4}$	1	+	+	0
1	1	+	+	+

KEY

0 Not in the set

\+ In the set

We have six cases to consider. We make frequent use of Lemma 3.3.3 and the above table of densities.

Case 1 ($\alpha = \beta = \frac{1}{4}$): We have $\alpha + \beta = \frac{1}{2}$. We know that $\gamma \geq \frac{1}{2}$ because $x_1 \in C$ and $x_2 \in C$.

Case 2 ($\alpha = \frac{1}{4}$, $\beta = \frac{1}{3}$): We have $\alpha + \beta = \frac{7}{12}$. We show that $\gamma \geq \frac{2}{3} > \frac{7}{12}$ by showing that $x_4 \in C$. Now $x_4 \in B \subseteq C$.

Case 3 ($\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$): We have $\alpha + \beta = \frac{3}{4}$. We show that $\gamma \geq \frac{3}{4}$ by showing that $x_3 \in C$.

Subcases 1 and 2 for $\beta = \frac{1}{2}$: Here $x_3 \in B \subseteq C$.

Subcase 3 for $\beta = \frac{1}{2}$: Here $x_3 \in C$ by part (ii) of Lemma 3.3.3.

Case 4 ($\alpha = \frac{1}{4}$, $\beta = \frac{2}{3}$): We have $\alpha + \beta = \frac{11}{12}$. We show that $\gamma = 1 > \frac{11}{12}$ by showing that $x_3 \in C$ and $x_4 \in C$. Now $x_3 \in C$ by part (ii) of Lemma 3.3.3 and $x_4 \in B \subseteq C$.

Case 5 ($\alpha = \beta = \frac{1}{3}$): We have $\alpha + \beta = \frac{2}{3}$. We show that $\gamma \geq \frac{2}{3}$ by showing that $x_4 \in C$. Now $x_4 \in B \subseteq C$.

Case 6 ($\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$): We have $\alpha + \beta = \frac{5}{6}$. We show that $\gamma = 1 > \frac{5}{6}$ by showing that $x_3 \in C$ and $x_4 \in C$.

Subcases 1 and 2 for $\beta = \frac{1}{2}$: Here $x_3 \in B \subseteq C$ and $x_4 \in A \subseteq C$.

Subcase 3 for $\beta = \frac{1}{2}$: Here $x_3 \in C$ by part (ii) of Lemma 3.3.3 and $x_4 \in A \subseteq C$.

This completes the proof for Space 5.

Space 6: Let $\mathcal{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_5, x_6, x_7\} & \text{if } x = x_7, \\ \{0, x_5, x_6\} & \text{if } x = x_6, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$ and $0 \prec x_5 \prec x_6 \prec x_7$. The only values possible for α, β, and γ are $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.3.7, we have

$$\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}.$$

If $\alpha + \beta \geq 1$, then by Theorem 2 3.8, we have

$$\gamma = 1 = \min \{1, \alpha + \beta\}.$$

We may select our notation so that $\alpha \leq \beta$. Therefore, in the remainder of the proof for Space 6 we may assume that $0 < \alpha \leq \beta$ and $\alpha + \beta < 1$. Since $0 < \alpha \leq \beta$, we know that all of the essential points are in $A \cap B$ and hence in C; that is,

$$S \setminus \{x_4, x_6, x_7\} \subseteq A \cap B \subseteq C.$$

We have two cases to consider.

Case 1 ($\alpha = \beta = \frac{1}{3}$): We have $\alpha + \beta = \frac{2}{3}$. We show that $\gamma \geq \frac{2}{3}$ by showing that $x_6 \in C$. Since $\alpha = \frac{1}{3}$, we have $x_6 \notin A$ and $x_7 \notin A$. Since $x_5 \prec x_6 \prec x_7$, we have $x_6 - x_5 \in S$ and $x_6 - x_5 \prec x_6 \prec x_7$. Therefore, $x_6 - x_5 \neq x_6$ and $x_6 - x_5 \neq x_7$. If $x_6 - x_5 \neq x_4$ then $x_6 - x_5 \in A$, and since $x_5 \in B$, we have $x_6 \in C$. If $x_6 - x_5 = x_4$ then $x_1 \prec x_4 \prec x_6$, and so $x_4 \neq x_6 - x_1 \prec x_6 \prec x_7$. Therefore, $x_6 - x_1 \in A$, and since $x_1 \in B$, we have $x_6 \in C$.

Case 2 ($\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$): We have $\alpha + \beta = \frac{5}{6}$. We show that $\gamma = 1 > \frac{5}{6}$ by showing that $x_6 \in C$, $x_7 \in C$, and $x_4 \in C$. Since $\beta = \frac{1}{2}$, we have $x_6 \notin B$ and $x_7 \in B$. Now $x_6 \in C$ exactly as in Case 1 and $x_7 \in B \subseteq C$. Since $x_1 \prec x_2 \prec x_3 \prec x_4$, we have

$$x_4 - x_3 \prec x_4 - x_2 \prec x_4 - x_1 \prec x_4,$$

and since x_4 and x_6 are the only elements which could be missing from B, we have at least two of the elements $x_4 - x_3$, $x_4 - x_2$, and $x_4 - x_1$ in B. Since x_1, x_2, and x_3 are all in A, we have $x_4 \in C$.

Spaces 7, 8, and 9: Let $\mathcal{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_i, x_j\} & \text{if } x = x_j, \\ \{0, x_5, x_6, x_7\} & \text{if } x = x_7, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$ and $0 \prec x_5 \prec x_6 \prec x_7$, and $i < j$ where $i, j \in \{1, 2, 3\}$. (Note: Spaces 7, 8, 9 refer respectively to $i = 1, j = 2$; $i = 1, j = 3$; $i = 2, j = 3$.) The only values possible for α, β, and γ are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.3.7, we have

$$\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}.$$

If both α and β are positive, then $\alpha + \beta \geq 1$ and, by Theorem 2.3.8, we have

$$\gamma = 1 = \min\{1, \alpha + \beta\}.$$

Space 10: Let $\mathcal{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x_5, x_6\} & \text{if } x = x_6, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$ and $0 \prec x_5 \prec x_6$. The only values possible for α, β, and γ are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.3.7, we have

$$\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}.$$

If both α and β are positive, then $\alpha + \beta \geq 1$ and, by Theorem 2.3.8, we have

$$\gamma = 1 = \min\{1, \alpha + \beta\}.$$

Spaces 11, 12, and 13: Let $\mathfrak{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_i, x_j\} & \text{if } x = x_j, \\ \{0, x\} & \text{otherwose,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$, and $i < j$ where $i, j \in \{1, 2, 3\}$. The only values possible for α, β, and γ are $0, \frac{1}{2}, \frac{3}{4}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.3.7, we have

$$\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}.$$

If both α and β are positive, then $\alpha + \beta \geq 1$ and, by Theorem 2.3.8, we have

$$\gamma = 1 = \min \{1, \alpha + \beta\}.$$

Space 14: Let $\mathfrak{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4$. The only values possible for α, β, and γ are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2 3.7, we have

$$\gamma \geq \max \{\alpha, \beta\} = \min \{1, \alpha + \beta\}.$$

If both α and β are positive, then $\alpha + \beta \geq 1$ and, by Theorem 2.3.8, we have

$$\gamma = 1 = \min \{1, \alpha + \beta\}.$$

This completes the proof of Theorem 3.3.4.

The following examples show that there are singularly discrete density spaces of all orders greater than 4 (and of infinite order) for which the $\alpha + \beta$ property fails. The Landau-Schnirelmann inequality and the Schur inequality also fail here.

Let $\mathfrak{F}_H$ be defined by

$$H(x) = \begin{cases} \{0, 2, 3, 4\} & \text{if } x = 4, \\ \{0, 2, 3, 4, 5\} & \text{if } x = 5, \\ \{0, 2, 3, 4, 5, 6\} & \text{if } x = 6, \\ \{0, 1, 7\} & \text{if } x = 7, \\ \{0, x\} & \text{otherwise.} \end{cases}$$

Then $(I, \mathfrak{F}_H)$ is a singularly discrete density space of order 5. Let $A = B = \{0, 1, 2, 3, \overline{8}\}$. Then $C = \{0, 1, 2, 3, 4, 5, 6, \overline{8}\}$. Now $\alpha = \beta = \frac{2}{5}$ while $\gamma = \frac{1}{2}$. Therefore, we have $\gamma = \frac{1}{2} < \frac{4}{5} = \min\{1, \alpha+\beta\}$, we have $\gamma = \frac{1}{2} < \frac{16}{25} = \alpha + \beta - \alpha\beta$, and we have $\gamma = \frac{1}{2} < \frac{2}{3} = \beta/(1-\alpha)$.

For any integer $n > 5$ we can modify the definition of $H(x)$ in the above example to make $(S, \mathfrak{F}_H)$ singularly discrete of order n. To do this we insert the following:

$$H(x) = \{0, 11, 12, \ldots, 10+n\} \quad \text{if} \quad x = 10 + n.$$

To make $(S, \mathfrak{F}_H)$ singularly discrete of infinite order we insert:

$$H(x) = \{0, 11, 12, \ldots, 10+n\} \text{ if } x = 10 + n \text{ for all } n \geq 6.$$

These insertions do not change the values of α, β, and γ for the sets A, B, and C in the above example. Therefore, the indicated properties still fail.

These failures motivate a further restriction on the discrete density spaces we wish to consider. We introduce such a restriction in the next chapter.

We have just seen that there are 14 types of singularly discrete density spaces of order 4. Similarly, it can be shown that there are 99 types of order 5. Surprisingly, the $\alpha + \beta$ property is valid for 98 of the 99 types of spaces. The only space where the $\alpha + \beta$ property can fail is of the type $(S, \mathfrak{F}_H)$ where

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4, x_5\} & \text{if } x = x_5, \\ \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x_6, x_7\} & \text{if } x = x_7, \\ \{0, x\} & \text{otherwise}, \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$ and $0 \prec x_6 \prec x_7$. Of course, the example we used earlier to show that the $\alpha + \beta$ property fails for some singularly discrete density spaces of order 5 is of this type.

CHAPTER IV

NESTED SINGULARLY DISCRETE DENSITY SPACES AND T-SPACES

In this chapter we introduce a special family of discrete density spaces called nested singularly discrete density spaces. We also introduce a class of density spaces called T-spaces which generalizes the family of nested singularly discrete density spaces. We show that both the Landau-Schnirelmann and Schur inequalities hold for all T-spaces.

4.1. Basic Definitions and Representations for Nested Singularly Discrete Density Spaces

Definition 4.1.1. A density space $(S, \mathfrak{F})$ is nested singularly discrete if the following three conditions are satisfied:

(i) $(S, \mathfrak{F})$ is singularly discrete.

(ii) If $2 \leq S([x]) < S([y])$, then $[x] \subset [y]$ where $x \in S \setminus \{0\}$ and $y \in S \setminus \{0\}$.

(iii) If $y \in S \setminus \{0\}$ and $S([y]) = i \geq 2$, then for each integer j $(1 \leq j \leq i)$, there is an element $x \in S \setminus \{0\}$ such that $S([x]) = j$.

Theorem 4.1.2. A space $(S, \mathfrak{F})$ is a nested singularly discrete density space of order $n \geq 1$ if an only if $\mathfrak{F} = \mathfrak{F}_H$ where

$H(x)$ is defined by

$$(1) \qquad H(x) = \begin{cases} \{0\} \cup \{x_i \mid i = 1, 2, \ldots, m\} & \text{if } x = x_m, \text{ for all } m = 1,2,\ldots,n, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec \cdots \prec x_n$.

Proof: Suppose $(S, \mathcal{F})$ is a nested singularly discrete density space of order $n \geq 1$. By Theorem 2.2.6, there is a function $H(x)$ such that $\mathcal{F} = \mathcal{F}_H$. Since the order is n, there is an element $x \in S \setminus \{0\}$ such that $S([x]) = n$. Now by condition (iii) of Definition 4.1.1, for each $i = 1, 2, \ldots, n$, there is an element $x_i \in S \setminus \{0\}$ such that $S([x_i]) = i$. Since $(S, \mathcal{F}_H)$ is singularly discrete by condition (i) of Definition 4.1.1, we know that for each $i = 2, 3, \ldots, n$ there is only one element $x_i \in S \setminus \{0\}$ such that $S([x]) = i$. By condition (ii) of Definition 4.1.1, we know that

$$[x_2] \subset [x_3] \subset \cdots \subset [x_n].$$

The set $[x_2]$ contains one nonzero element of S different from x_2. Call it x_1. Then $S([x_1]) = 1$ and

$$[x_1] \subset [x_2] \subset \cdots \subset [x_n].$$

Therefore, since $[x_i] = H(x_i) \subseteq L(x_i)$ for each $i = 1, 2, \ldots, n$ by condition c.2 of Theorem 2.2.6, we have

$$H(x_1) = \{0, x_1\},$$
$$H(x_2) = \{0, x_1, x_2\}$$
$$\vdots$$
$$H(x_n) = \{0, x_1, x_2, \ldots, x_n\},$$

where $0 \prec x_1 \prec x_2 \prec \ldots \prec x_n$. For all $x \in S \setminus \{0\}$ where $S([x]) = 1$, we have $H(x) = \{0, x\}$. Therefore, by condition (i) of Definition 4.1.1, we have formula (1).

Conversely, suppose $H(x)$ is given by formula (1). Now $H(x)$ satisfies conditions c.1, c.2, and c.3 of Theorem 2.2.6. Therefore, $(S, \mathfrak{F}_H)$ is a density space. We now confirm the three conditions of Definition 4.1.1.

Condition (i) holds: If $x \notin H(y)$ and $y \notin H(x)$ for $x \in S\setminus\{0\}$ and $y \in S \setminus\{0\}$, then $H(x) \cap H(y) = \{0\}$. Therefore, $\mathfrak{F}_H$ is separated and hence $(S, \mathfrak{F}_H)$ is discrete. Since $S([x]) = i$, $i \geq 2$, only if $x = x_i$, we have that $(S, \mathfrak{F}_H)$ is singularly discrete.

Condition (ii) holds: Suppose $x \in S \setminus \{0\}$, $y \in S \setminus\{0\}$, and

$$2 \leq S([x]) = i < j = S([y]).$$

Then $x = x_i$ and $y = x_j$ and so

$$[x] = [x_i] = H(x_i) = \{0, x_1, x_2, \ldots, x_i\}$$
$$\subset \{0, x_1, x_2, \ldots, x_j\} = H(x_j) = [x_j] = [y].$$

Condition (iii) holds: Suppose $y \in S \setminus \{0\}$ and $S([y]) = i \geq 2$. Then $S([x_j]) = j$ for each $j = 1, 2, \ldots, i$.

Therefore, $(S, \mathfrak{F}) = (S, \mathfrak{F}_H)$ is a nested singularly discrete density space of order n. This completes the proof.

<u>Theorem 4.1.3.</u> A space $(S, \mathfrak{F})$ is a nested singularly discrete density space of infinite order if and only if $\mathfrak{F} = \mathfrak{F}_H$ where $H(x)$ is defined by

$$(2) \qquad H(x) = \begin{cases} \{0\} \cup \{x_i \mid i = 1, 2, \ldots, m\} & \text{if } x = x_m, \text{ for all integers } m \geq 1, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec \cdots$.

Proof: Suppose $(S, \mathfrak{F})$ is a nested singularly discrete density space of infinite order. By Theorem 2.2.6, there is a function $H(x)$ such that $\mathfrak{F} = \mathfrak{F}_H$. Since the order is infinite, for any integer N, as large as we wish to choose, there is an integer $n \geq N$ such that $S([x]) = n$ for some $x \in S \setminus \{0\}$. The proof from this point on is exactly like the proof of Theorem 4.1.2, except we use formula (2) in place of formula (1).

Theorems 4.1.2 and 4.1.3 are easier to use than Definition 4.1.1 when we are actually constructing examples of nested singularly discrete density spaces. This is particularly true of Theorem 4.1.2 which we use in Chapter VII when we prove that the $\alpha + \beta$ property holds for nested singularly discrete density spaces of order 5.

Definition 4.1.4. The point x_1 in formulas (1) and (2) is called the nested essential point of the nested singularly discrete density space $(S, \mathcal{F}_H)$.

It is clear that the nested essential point is an essential point. The nested essential point will be of importance in the next section and in Chapters VI through IX.

4.2. General T-spaces

In this section we introduce a family of density spaces over the s-set S which includes as a subfamily all nested singularly discrete density spaces over S.

Definition 4.2.1. Let S be any s-set and let T be any proper subset of $S \setminus \{0\}$. Denote by $(S, \mathcal{K}_T)$ the space $(S, \mathcal{F}_H)$ where $H(x)$ is defined by

$$H(x) = \begin{cases} \{0, x\} & \text{if } x \in T, \\ L(x) \setminus T & \text{if } x \in S \setminus \{0\} \text{ and } x \notin T. \end{cases}$$

Then $(S, \mathcal{K}_T)$ is called a T-space over S.

The choice of the notation $(S, \mathcal{K}_T)$ is motivated primarily by the fact that when $T = \phi$, we have $(S, \mathcal{K}_T) = (S, \mathcal{K})$.

Theorem 4.2.2. (i) Every T-space over S is a density space.

(ii) If $S \setminus T$ is linearly ordered by the partial ordering $\prec$ from S, then $(S, \mathcal{K}_T)$ is a nested singularly discrete density space.

(iii) For any nested singularly discrete density space $(S, \mathcal{H})$ there is a set $T \subset S \setminus \{0\}$ such that $(S, \mathcal{K}_T) = (S, \mathcal{H})$.

Proof of (i): To see that $(S, \mathcal{K}_T)$ is a density space, it suffices to verify the three conditions from Theorem 2.2.6.

Condition c.1 holds: Consider any $x \in S \setminus \{0\}$. If $x \in T$, then $\{0, x\} = H(x)$. If $x \notin T$, then $\{0, x\} \subseteq L(x) \setminus T = H(x)$.

Condition c.2 holds: Consider any $x \in S \setminus \{0\}$. If $x \in T$, then $H(x) = \{0, x\} \subseteq L(x)$. If $x \notin T$, then $H(x) = L(x) \setminus T \subseteq L(x)$.

Condition c.3 holds: Consider any $x \in S \setminus \{0\}$. If $x \in T$, then $H(x) = \{0, x\}$. Therefore, $y \in H(x) \setminus \{0\}$ implies $y = x$, and hence $H(y) = H(x)$. If $x \notin T$, then $H(x) = L(x) \setminus T$. Therefore, $y \in H(x) \setminus \{0\}$ implies $y \in L(x)$ and $y \notin T \cup \{0\}$. Hence $H(y) = L(y) \setminus T$. Since $y \in L(x)$ implies $L(y) \subseteq L(x)$, we have

$$H(y) = L(y) \setminus T \subseteq L(x) \setminus T = H(x).$$

Proof of (ii): Suppose $S \setminus (T \cup \{0\})$ has n elements. (We will handle the infinite case next.) Since the elements of $S \setminus T$ are linearly ordered, we can write

$$S \setminus (T \cup \{0\}) = \{x_1, x_2, \ldots, x_n\}$$

where $0 \prec x_1 \prec x_2 \prec \cdots \prec x_n$. If $x = x_m$ for an integer m $(1 \leq m \leq n)$, we have

$$H(x) = H(x_m) = L(x_m) \setminus T$$
$$= L(x_m) \cap (S \setminus T)$$
$$= \{0, x_1, x_2, \ldots, x_m\}$$
$$= \{0\} \cup \{x_i \mid i = 1, 2, \ldots, m\}.$$

Otherwise, if $x \neq 0$, then $x \in T$, and so we have $H(x) = \{0, x\}$. Therefore, by Theorem 4.1.2, the space $(I, \mathcal{K}_T)$ is nested singularly discrete of order n.

If $S \setminus (T \cup \{0\})$ has infinitely many elements we can write

$$S \setminus (T \cup \{0\}) = \{x_1, x_2, \ldots\}$$

where $0 \prec x_1 \prec x_2 \prec \ldots$. We consider $x = x_m$ for each integer $m \geq 1$ and proceed as above, except we apply Theorem 4.1.3 in place of Theorem 4.1.2 to conclude that $(S, \mathcal{K}_T)$ is nested singularly discrete of infinite order.

Proof of (iii): This part follows from the next theorem which demonstrates a specific set $T \subset S \setminus \{0\}$ such that $(S, \mathcal{H}) = (S, \mathcal{K}_T)$.

<u>Theorem 4.2.3</u>. Let $(S, \mathcal{H})$ be any nested singularly discrete density space. Then $(S, \mathcal{H}) = (S, \mathcal{K}_T)$ where

(i) if the order is n, then $T = S \setminus \{0, x_1, x_2, \ldots, x_n\}$,

(ii) if the order is infinite, then $T = S \setminus \{0, x_1, x_2, \ldots\}$.

We are using the fact that $(S, \mathcal{H}) = (S, \mathfrak{F}_{H'})$ where $H'(x)$ is defined as in formula (1) of Theorem 4.1.2 or formula (2) of Theorem 4.1.3.

Proof: Let $(S, \mathcal{H})$ be any nested singularly discrete density space of order $n \geq 1$. (We will handle the infinite case next.) By Theorem 4.1.2, we have $(S, \mathcal{H}) = (S, \mathfrak{F}_{H'})$ where $H'(x)$ is defined by

$$H'(x) = \begin{cases} \{0\} \cup \{x_i \mid i = 1,2,\ldots,m\} \text{ for } x = x_m, & \text{for all } m = 1,2,\ldots,n, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec \cdots \prec x_n$. Let $T = S \setminus \{0, x_1, x_2, \ldots, x_n\}$. Now $x_1 \in S \setminus \{0\}$ and $x_1 \notin T$, so $T \subset S \setminus \{0\}$. Also $x \in S \setminus \{0\}$ and $x \notin T$ if an only if $x = x_m$ for some $m = 1, 2, \ldots, n$. Therefore, if $x \in S \setminus \{0\}$ and $x \notin T$, we have

$$\begin{aligned} H(x) = H(x_m) &= L(x_m) \setminus T \\ &= L(x_m) \setminus (S \setminus \{0, x_1, x_2, \ldots, x_n\} \\ &= L(x_m) \cap \{0, x_1, x_2, \ldots, x_n\} \\ &= \{0, x_1, x_2, \ldots, x_m\} \\ &= H'(x_m) = H'(x). \end{aligned}$$

Otherwise, if $x \neq 0$, then $x \in T$, and so we have $H(x) = \{0, x\} = H'(x)$. Therefore, since $H(x) = H'(x)$ for all $x \in S \setminus \{0\}$, we have $(S, \mathcal{K}_T) = (S, \mathcal{H})$.

If $(S, \mathcal{H})$ is nested singularly discrete of infinite order we apply Theorem 4.1.3 in place of Theorem 4.1.2 and write $m = 1, 2, \ldots$ in place of $m = 1, 2, \ldots, n$, write $0 \prec x_1 \prec x_2 \prec \cdots$ in place of $0 \prec x_1 \prec x_2 \prec \cdots \prec x_n$, and write $\{0, x_1, x_2, \ldots\}$ in place of $\{0, x_1, x_2, \ldots, x_n\}$ in the proof for the finite case.

We now show that Theorem 4.2.3 can be reworded in the following way:

<u>Corollary 4.2.4.</u> Let $(S, \mathcal{H})$ be any nested singularly discrete density space. Then $(S, \mathcal{H}) = (S, \mathcal{K}_T)$ where T is the set of all essential points of $(S, \mathcal{H})$ except the nested essential point.

Proof: Using the notation of Theorem 4.2.3, we have that $x_2, x_3, \ldots$ are all non-essential points and all other points of $S \setminus \{0\}$ are essential points. Let E be the set of essential points of $(S, \mathcal{H})$. Then, by Theorem 4.2.3, we have

$$T = S \setminus \{0, x_1, x_2, \ldots, x_n\} \quad \text{or} \quad S \setminus \{0, x_1, x_2, \ldots\}$$
$$= E \setminus \{x_1\}.$$

This completes the proof.

Notice that we have been letting $(S, \mathcal{H})$ denote a nested singularly discrete density space. From here on, if a density space is denoted by $(S, \mathcal{H})$ it is understood to be nested singularly discrete.

We conclude one additional theorem which relates the set T to the set of essential points of a T-space $(S, \mathcal{K}_T)$.

<u>Theorem 4.2.5.</u> Let $(S, \mathcal{K}_T)$ be any T-space and let E be the set of essential points of $(S, \mathcal{K}_T)$. Let

$$V = \{x \mid x \in S \setminus (T \cup \{0\}) \text{ and } L(x) \setminus T = \{0, x\}\}.$$

Then (i) $E = T \cup V$ and (ii) $T \cap V = \phi$.

Proof of (i): Suppose $x \in E$, but $x \notin T$. Then $[x] = \{0,x\}$. Since 0 is not an essential point, we have $x \in S \setminus (T \cup \{0\})$. Therefore, by Definition 4.2.1, we have

$$L(x) \setminus T = H(x) = [x] = \{0, x\}.$$

Hence $x \in V$. Therefore, $E \subseteq T \cup V$.

Conversely, if $x \in T$, then $[x] = \{0, x\}$. If $x \in V$, then $x \in S \setminus \{0\}$ and $x \notin T$, and so $[x] = L(x) \setminus T = \{0, x\}$. Therefore, $T \cup V \subseteq E$.

Proof of (ii): If $x \in T$, then $x \notin S \setminus (T \cup \{0\})$ and so

$x \notin V$. Therefore, $T \cap V = \phi$.

We are now ready to extend some results of Freedman for density spaces $(S, \mathcal{K})$ to the more general T-spaces $(S, \mathcal{K}_T)$.

4.3. The Landau-Schnirelmann and Schur Inequalities for T-spaces

Freedman [3, 4] proved both the Landau-Schnirelmann and Schur inequalities for all density spaces $(S, \mathcal{K})$; that is, for all density spaces determined by $H(x) = L(x)$ for all $x \in S \setminus \{0\}$. The next two theorems encompass both of these results as special cases. They also provide new results for nested singularly discrete density spaces. Their proofs are modifications of the proofs of Freedman for $(S, \mathcal{K})$. First, however, we prove three lemmas.

Lemma 4.3.1. Let $(S, \mathcal{K}_T)$ be any T-space. Then $F \setminus T \in \mathcal{K}_T \cup \{\{0\}\}$ for every $F \in \mathcal{K}$.

Proof: Consider any $F \in \mathcal{K}$. Since $(S, \mathcal{K}) = (S, \mathcal{F}_H)$ where $H(x) = L(x)$ for all $x \in S \setminus \{0\}$, we have, by Theorem 2.2.16, that

$$F = \cup \{L(x) \mid x \in F \setminus \{0\}\}.$$

Therefore, since in $(S, \mathcal{K}_T)$ we have $[x] = L(x) \setminus T \in \mathcal{K}_T$ for each $x \in S \setminus (T \cup \{0\})$, it follows that

$$F \setminus T = \cup\{L(x) \mid x \in F \setminus \{0\}\} \setminus T$$
$$= \cup\{L(x) \setminus T \mid x \in F \setminus \{0\}\}$$
$$= \cup\{L(x) \setminus T \mid x \in F \setminus (T \cup \{0\})\} \cup \{0\}.$$

Hence $F \setminus T \in \mathcal{K}_T \cup \{\{0\}\}$.

<u>Lemma 4.3.2.</u> Let $(S, \mathcal{K}_T)$ be any T-space. Then for every set $F' \in \mathcal{K}_T$ there is a set $F \in \mathcal{K} \cup \{\{0\}\}$ and a set $T' \subseteq T$ such that

$$F' = (F \setminus T) \cup T'.$$

Proof: Consider any $F' \in \mathcal{K}_T$. By Definition 4.2.1, we have $(S, \mathcal{K}_T) = (S, \mathcal{F}_H)$ where $H(x)$ is defined by

$$H(x) = \begin{cases} \{0, x\} & \text{if } x \in T, \\ L(x) \setminus T & \text{if } x \in S \setminus \{0\} \text{ and } x \notin T. \end{cases}$$

By Theorem 2.2.16, we have

$$F' = \cup\{H(x) \mid x \in F' \setminus \{0\})$$
$$= \cup\{H(x) \mid x \in F' \cap T\} \cup (\cup\{H(x) \mid x \in F' \setminus (T \cup \{0\})\})$$
$$= \{0\} \cup (F' \cap T) \cup (\cup\{L(x) \setminus T \mid x \in F' \setminus (T \cup \{0\})\})$$
$$= \{0\} \cup (F' \cap T) \cup (\cup\{L(x) \mid x \in F' \setminus (T \cup \{0\})\} \setminus T).$$

If we let $F = \cup\{L(x) \mid x \in F' \setminus (T \cup \{0\})\} \cup \{0\}$ and $T' = F' \cap T$, then $F \in \mathcal{K} \cup \{\{0\}\}$ and $T' \subseteq T$. Substituting into the above equality, we obtain

$$F' = (F \setminus T) \cup T'.$$

Lemma 4.3.3. Let $(S, \mathcal{K}_T)$ be any T-space. Let A be any subset of S with $0 \in A$. Let α be the k-density of A in $(S, \mathcal{K})$ and let α' be the k-density of A in $(S, \mathcal{K}_T)$. Then $\alpha \geq \alpha'$.

Proof: If $T \nsubseteq A$, then $\alpha' = 0$ and so $\alpha \geq \alpha'$. Suppose $T \subseteq A$ and consider any $F \in \mathcal{K}$. If $F \setminus T = \{0\}$, then $F \subseteq A$ and so

$$\frac{A(F')}{S(F')} \leq \frac{A(F)}{S(F)} = 1$$

for any $F' \in \mathcal{K}_T$. If $F \setminus T \neq \{0\}$, then by Lemma 4.3.1 we have $F \setminus T \in \mathcal{K}_T$ and

$$\frac{A(F \setminus T)}{S(F \setminus T)} = \frac{A(F)-T(F)}{S(F)-T(F)} \leq \frac{A(F)}{S(F)} .$$

Therefore, for every $F \in \mathcal{K}$ there is an $F' \in \mathcal{K}_T$ such that

$$\frac{A(F')}{S(F')} \leq \frac{A(F)}{S(F)} ,$$

and so $\alpha \geq \alpha'$.

We are now ready to prove the Landau-Schnirelmann and Schur inequalities for general T-spaces.

Theorem 4.3.4. Let $(S, \mathcal{K}_T)$ be any T-space. Let A, B, and

$C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding k-densities in $(S, \mathcal{K}_T)$ be α', β', and γ'. Then $\gamma' \geq \alpha' + \beta' - \alpha'\beta'$.

Proof: Let the k-densities of A, B, and C in $(S, \mathcal{K})$ be α, β, and γ. If any essential point in $(S, \mathcal{K}_T)$ is missing from either A or B, then $\alpha' = 0$ or $\beta' = 0$ and the theorem is immediate. Therefore, suppose all essential points are in $A \cap B$ and hence in C. In particular, $T \subseteq A$ and $T \subseteq C$. Consider any $F' \in \mathcal{K}_T$. By Lemma 4.3.2, there is a set $F \in \mathcal{K} \cup \{\{0\}\}$ and a set $T' \subseteq T$ such that

$$F' = (F \setminus T) \cup T'.$$

If $F \setminus T = \{0\}$, then since $T' \subseteq T \subseteq C$, we have

$$\frac{C(F')}{S(F')} = \frac{C((F \setminus T) \cup T')}{S((F \setminus T) \cup T')} = \frac{C(T')}{S(T')} = 1$$
$$\geq \alpha' + \beta' - \alpha'\beta'.$$

Suppose $F \setminus T \neq \{0\}$. Then by Lemma 4.3.1, we have $F \setminus T \in \mathcal{K}_T$. Now since $F \setminus T$ and T' are disjoint and $T' \subseteq T \subseteq C$, we have

$$\frac{C(F')}{S(F')} = \frac{C((F \setminus T) \cup T')}{S((F \setminus T) \cup T')}$$

$$= \frac{C(F \setminus T)+C(T')}{S(F \setminus T)+S(T')}$$

$$= \frac{C(F \setminus T)+S(T')}{S(F \setminus T)+S(T')}$$

$$\geq \frac{C(F \setminus T)}{S(F \setminus T)}$$

$$= \frac{C(F)-T(F)}{S(F)-T(F)} .$$

By Theorems 2.3.16, 2.3.17, Lemma 4.3.3, and the above inequality, we have

$$\frac{C(F')}{S(F')} \geq \frac{C(F)-T(F)}{S(F)-T(F)}$$

$$\geq \frac{A(F)-T(F)+\beta(S \setminus A)(F)}{S(F)-T(F)}$$

$$= \frac{A(F)-T(F)}{S(F)-T(F)} + \beta \frac{S(F)-A(F)}{S(F)-T(F)}$$

$$\geq \frac{A(F)-T(F)}{S(F)-T(F)} + \beta' \frac{S(F)-A(F)}{S(F)-T(F)}$$

$$= \frac{A(F)-T(F)}{S(F)-T(F)} + \beta' \frac{S(F)-T(F)+T(F)-A(F)}{S(F)-T(F)}$$

$$= \frac{A(F)-T(F)}{S(F)-T(F)} (1-\beta') + \beta'.$$

Since $T \subseteq A$ and $F \setminus T \in \mathcal{K}_T$, we have, by the above inequality, that

$$\frac{C(F')}{S(F')} \geq \frac{A(F)-T(F)}{S(F)-T(F)} (1-\beta') + \beta'$$

$$= \frac{A(F \setminus T)}{S(F \setminus T)} (1-\beta') + \beta'$$

$$\geq \alpha'(1-\beta') + \beta'$$

$$= \alpha' + \beta' - \alpha'\beta'.$$

Now since

$$\frac{C(F')}{S(F')} \geq \alpha' + \beta' - \alpha'\beta'$$

for every $F' \in \mathcal{K}_T$, we have

$$\gamma' = \text{glb} \left\{\frac{C(F')}{S(F')} \,\middle|\, F' \in \mathcal{K}_T\right\} \geq \alpha' + \beta' - \alpha'\beta'.$$

<u>Theorem 4.3.5.</u> Let $(S, \mathcal{K}_T)$ be any T-space. Let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding k-densities in $(S, \mathcal{K}_T)$ be α', β' and γ'. If $\alpha' + \beta' < 1$, then $\gamma' \geq \beta'/(1-\alpha')$.

Proof: Let the k-densities of A, B, and C in $(S, \mathcal{K})$ be α, β, and γ. If any essential point in $(S, \mathcal{K}_T)$ is missing from either A or B, then $\alpha' = 0$ or $\beta' = 0$ and the theorem is immediate. Therefore, suppose all essential points are in $A \cap B$ and hence in C. In particular, $T \subseteq B$ and $T \subseteq C$. Consider any $F' \in \mathcal{K}_T$. By Lemma 4.3.2, there is a set $F \in \mathcal{K} \cup \{\{0\}\}$ and a set $T' \subseteq T$ such that

$$F' = (F \setminus T) \cup T'.$$

If $F \setminus T = \{0\}$, then since $T' \subseteq T \subseteq C$ and since $\beta'/(1-\alpha') < 1$, we have

$$(1) \qquad \frac{C(F')}{S(F')} = \frac{C((F \setminus T) \cup T')}{S((F \setminus T) \cup T')} = \frac{C(T')}{S(T')} = 1 > \beta'/(1-\alpha').$$

Suppose $F \setminus T \neq \{0\}$. Then by Lemma 4.3.1, we have $F \setminus T \in \mathcal{X}_T$. Now since $F \setminus T$ and T' are disjoint and $T' \subseteq T \subseteq C$, we have

$$(2) \qquad \frac{C(F')}{S(F')} = \frac{C((F \setminus T) \cup T')}{S((F \setminus T) \cup T')}$$

$$= \frac{C(F \setminus T)+C(T')}{S(F \setminus T)+S(T')}$$

$$= \frac{C(F \setminus T)+S(T')}{S(F \setminus T)+S(T')}$$

$$\geq \frac{C(F \setminus T)}{S(F \setminus T)}.$$

Also since $F \setminus T \neq \{0\}$ and $F \in \mathcal{X} \cup \{\{0\}\}$, we have $F \in \mathcal{X}$. If $C(F \setminus T) = S(F \setminus T)$, then by inequality (2) we have

$$(3) \qquad \frac{C(F')}{S(F')} \geq \frac{C(F \setminus T)}{S(F \setminus T)} = 1 > \beta'/(1-\alpha').$$

Suppose $C(F \setminus T) < S(F \setminus T)$. Since $T \subseteq C$ we have $C(F) < S(F)$. Therefore, $F \cap (S \setminus C) = \phi$. Let

$$G = \cup \{[x] \mid x \in F \cap (S \setminus C)\}$$

where we recall that $[x]$ is the Cheo set of x in $(S, \mathcal{K})$. Then by Theorem 2.2.15 we have $G \in \mathcal{K}$ and $G^* \subseteq F \cap (S \setminus C)$. Therefore, $G^* \subseteq S \setminus C$, and also $G^* \subseteq F$ which by Theorem 2.2.14 implies that $G \subseteq F$. Therefore,

(4) $$F = G \cup (F \setminus G).$$

By the way we constructed G we have $F \setminus G \subseteq C$. Therefore,

(5) $$F \setminus T = (G \setminus T) \cup (F \setminus (G \cup T)),$$

and

(6) $$C(F \setminus (G \cup T)) = S(F \setminus (G \cup T)).$$

Now by statements (2), (5), (4) and (6) are since $G \setminus T \neq \phi$, we have

(7) $$\frac{C(F')}{S(F')} \geq \frac{C(F \setminus T)}{S(F \setminus T)}$$

$$= \frac{C((G \setminus T) \cup (F \setminus (G \cup T)))}{S((G \setminus T) \cup (F \setminus (G \cup T)))}$$

$$= \frac{C(G \setminus T) + C(F \setminus (G \cup T))}{S(G \setminus T) + S(F \setminus (G \cup T))}$$

$$= \frac{C(G \setminus T) + S(F \setminus (G \cup T))}{S(G \setminus T) + S(F \setminus (G \cup T))}$$

$$\geq \frac{C(G \setminus T)}{S(G \setminus T)}.$$

Now since $G \in \mathcal{K}$ and $G^* \subseteq S \setminus C$ and since $T \subseteq B$ and $T \subseteq C$, we have by Theorems 2.3.16 and 2.3.20 and by Lemma 4.3.3 that

$$\frac{C(G \setminus T)}{S(G \setminus T)} = \frac{C(G)-T(G)}{S(G)-T(G)}$$

$$\geq \frac{\alpha C(G)+B(G)-T(G)}{S(G)-T(G)}$$

$$\geq \alpha' \frac{C(G)}{S(G)-T(G)} + \frac{B(G)-T(G)}{S(G)-T(G)}$$

$$\geq \alpha' \frac{C(G)-T(G)}{S(G)-T(G)} + \frac{B(G \setminus T)}{S(G \setminus T)}$$

$$\geq \alpha' \frac{C(G \setminus T)}{S(G \setminus T)} + \beta' .$$

Hence

$$\frac{C(G \setminus T)}{S(G \setminus T)}(1-\alpha') \geq \beta',$$

and so

(8) $$\frac{C(G \setminus T)}{S(G \setminus T)} \geq \beta'/(1-\alpha').$$

Combining inequalities (7) and (8), we obtain

(9) $$\frac{C(F')}{S(F')} \geq \beta'/(1-\alpha').$$

By inequalities (1), (3), and (9) we have

$$\frac{C(F')}{S(F')} \geq \beta'/(1-\alpha')$$

for all $F' \in \mathcal{K}_T$. Therefore,

$$\gamma' = \text{glb}\ \{\frac{C(F')}{S(F')}\ |\ F' \in \mathcal{K}_T\} \geq \beta'/(1-\alpha') .$$

This completes the proof.

If we let $T = \phi$, Theorems 4.3.4 and 4.3.5 yield the Landau-Schnirelmann and Schur inequalities for the density space $(S, \mathcal{K})$.

Let $(S, \mathcal{H})$ be any nested singularly discrete density space. By Theorem 4.2.2 part (iii), there is a set $T \subset S \setminus \{0\}$ such that $(S, \mathcal{H}) = (S, \mathcal{K}_T)$. Therefore, by Theorems 4.3.4 and 4.3.5 we have the Landau-Schnirelmann and Schur inequalities for all nested singularly discrete density spaces. In Chapter VI we obtain these inequalities by a different method of proof.

CHAPTER V

REDUCTION FROM $(S, \mathcal{H})$ TO $(I, \mathcal{H})$ FOR NESTED SINGULARLY DISCRETE DENSITY SPACES

In this chapter we present two theorems of independent interest about nested singularly discrete density spaces $(S, \mathcal{H})$. The first theorem shows that the verification of certain density inequalities in $(S, \mathcal{H})$ can be reduced to their verification in $(I, \mathcal{H})$. The only applications we make of this theorem are in Chapter VI where we obtain results which we proved in Chapter IV by different methods. The second theorem shows that for certain nested singularly discrete density spaces $(S, \mathcal{H})$ the verification of a density inequality can be reduced to its verification in $(I, \mathcal{H})$. This theorem is used in Chapter VIII.

5.1. Reduction from $(S, \mathcal{H})$ to $(I, \mathcal{H})$ for Admissible Inequalities

Definition 5.1.1. An inequality $*$ is admissible if for every nested singularly discrete density space $(S, \mathcal{H})$ and all subsets A, B, and $C = A + B$ of S, with $0 \in A \cap B$ and with densities α, β, and γ, the following three properties hold:

(i) The only variables in $*$ are α, β, and γ.

(ii) If $\alpha = 0$ or $\beta = 0$, then $*$ holds.

(iii) If $*$ holds for α, β, and γ, then $*$ holds for all

λ, μ, and ν such that $0 \le \lambda \le \alpha$, $0 \le \mu \le \beta$, and $\gamma \le \nu \le 1$.

Theorem 5.1.2. The following inequalities are admissible:

(1) $\gamma \ge \alpha + \beta - \alpha\beta$.

(2) If $\alpha + \beta < 1$, then $\gamma \ge \beta/(1-\alpha)$.

(3) $\gamma \ge \min\{1, \alpha + \beta\}$.

Proof: Property (i) is immediate for inequalities (1), (2), and (3). Property (ii) follows from Theorem 2.3.7. It remains to confirm property (iii).

Proof of property (iii) for inequality (1): Suppose $\gamma \ge \alpha + \beta - \alpha\beta$, $0 \le \lambda \le \alpha \le 1$, $0 \le \mu \le \beta \le 1$, and $\gamma \le \nu \le 1$. Then

$$\begin{aligned} 1 - \gamma &\le 1 - \beta - \alpha + \alpha\beta = (1-\alpha)(1-\beta) \\ &\le (1-\lambda)(1-\mu) = 1 - \lambda - \mu + \lambda\mu \end{aligned}$$

and so $\nu \ge \gamma \ge \lambda + \mu - \lambda\mu$.

Proof of property (iii) for inequality (2): Suppose $\alpha + \beta < 1$, $\gamma \ge \beta/(1-\alpha)$, $0 \le \lambda \le \alpha < 1$, $0 \le \mu \le \beta < 1$, and $\gamma \le \nu \le 1$. Then

$$\nu \ge \gamma \ge \beta/(1-\alpha) \ge \mu/(1-\alpha) \ge \mu/(1-\lambda) .$$

Proof of property (iii) for inequality (3): Suppose $\gamma \ge \min\{1, \alpha + \beta\}$, $0 \le \lambda \le \alpha \le 1$, $0 \le \mu \le \beta \le 1$, and $\gamma \le \nu \le 1$.

Then

$$\nu \geq \gamma \geq \min\{1, \alpha + \beta\} \geq \min\{1, \lambda + \mu\}.$$

Before stating and proving the main result of this chapter we introduce four lemmas.

<u>Lemma 5.1.3.</u> Let S be any s-set and consider any $x \in S$ and $y \in S$ such that $x \prec y$. Then $y - x \in \mathrm{Min}(S \setminus \{0\})$ if and only if there exists no $z \in S$ such that $x \prec z \prec y$.

Proof: Suppose $y - x \in \mathrm{Min}(S \setminus \{0\})$ and there is an element $z \in S$ such that $x \prec z \prec y$. Then $y - z \in S \setminus \{0\}$ and $y - z \prec y - x$ since

$$(y-x)-(y-z) = z - x \in S \setminus \{0\}.$$

This contradicts our assumption that $y - x \in \mathrm{Min}(S \setminus \{0\})$.

On the otherhand, suppose $y - x \notin \mathrm{Min}(S \setminus \{0\})$. Then there is an element $w \in S \setminus \{0\}$ such that $w \prec y - x$. Now since $w \prec y - x \preccurlyeq y$, we have $y - w \in S \setminus \{0\}$. Moreover,

$$(y-w) - x = (y-x) - w \in S \setminus \{0\},$$

so $x \prec y - w \prec y$. Therefore, there exists an element $z \in S$ such that $x \prec z \prec y$; namely, $z = y - w$.

<u>Lemma 5.1.4.</u> Let S be any s-set and consider any

$x \in S \setminus \{0\}$. Then there exists a finite sequence

$$x_0 \prec x_1 \prec \dots \prec x_t$$

of elements in S such that $x_0 = 0$, $x_t = x$, and $x_{i+1} - x_i \in \text{Min}\,(S \setminus \{0\})$ for $i = 0, 1, 2, \dots, t-1$.

Proof: Since $L(x)$ is finite and since $0 \in S$ and $x \in S$, there exists a sequence

$$0 = x_0 \prec x_1 \prec \dots \prec x_t = x$$

of elements of S for which t is maximal. For this sequence and for any integer i where $0 \leq i \leq t-1$ there exists no element $z \in S$ such that $x_i \prec z \prec x_{i+1}$. Therefore, by Lemma 5.1.3, we have $x_{i+1} - x_i \in \text{Min}\,(S \setminus \{0\})$ for $i = 0, 1, 2, \dots, t-1$.

Lemma 5.1.5. Let S be any s-set. Any sequence of elements of S of the form $0 = a_0 \prec a_1 \prec a_2 \prec \dots$ is a subsequence of some sequence of elements of S of the form $0 = b_0 \prec b_1 \prec b_2 \prec \dots$ where $b_{i+1} - b_i \in \text{Min}\,(S \setminus \{0\})$ for $i = 0, 1, 2, \dots$.

Proof: We apply Lemma 5.1.4 by setting $x = a_{i+1} - a_i$ and obtaining sequences

$$0 = x_{0i} \prec x_{1i} \prec \dots \prec x_{t_i i} = a_{i+1} - a_i$$

for each $i = 0, 1, 2, \ldots$. The sequence

$$\begin{aligned} 0 = x_{00} \prec x_{10} \prec \cdots \prec x_{t_0 0} \\ \prec a_1 + x_{11} \prec a_1 + x_{21} \prec \cdots \prec a_1 + x_{t_1 1} \prec \cdots \\ \prec a_i + x_{1i} \prec a_i + x_{2i} \prec \cdots \prec a_i + x_{t_i i} \prec \cdots \end{aligned}$$

has the desired properties because $a_i + x_{t_i i} = a_{i+1}$ for each $i = 0, 1, 2, \ldots$.

Lemma 5.1.6. Let $(S, \mathfrak{F}_H)$ be any nested singularly discrete density space and consider any element $a_1 \in \operatorname{Min}(S \setminus \{0\})$. Let $S' = \{ka_1 \mid k = 0, 1, 2, \ldots\}$. Let $H'(x) = H(x) \cap S'$ for each $x \in S' \setminus \{0\}$. Then $(S', \mathfrak{F}_{H'})$ is a nested singularly discrete density space of order not exceeding the order of $(S, \mathfrak{F}_H)$ and S' is isomorphic to I.

Proof: Since S' is a closed subset of S such that $0 \in S'$ and $S' \setminus \{0\} \neq \phi$, Theorem 2.1.6 assures us that S' is an s-set. To show that $(S', \mathfrak{F}_{H'})$ is a density space it suffices to show that $H'(x)$ satisfies conditions c.1, c.2, and c.3 of Theorem 2.2.6 for each $x \in S' \setminus \{0\}$. Let $L_S(x)$ and $L_{S'}(x)$ denote the lower set of x with respect to S and S' respectively.

Condition c.1 holds: Consider any $x \in S' \setminus \{0\}$. Since $S' \subseteq S$ we have $x \in S \setminus \{0\}$, and so by condition c.1 for the function $H(x)$

we have $x \in H(x)$. Therefore,

$$x \in H(x) \cap S' = H'(x).$$

Condition c. 2 holds: Consider any $x \in S' \setminus \{0\}$. By condition c. 2 for the function $H(x)$ and since $S' \subseteq S$, we have

$$H'(x) = H(x) \cap S' \subseteq L_S(x) \cap S' = L_{S'}(x).$$

Condition c. 3 holds: Consider any $x \in S' \setminus \{0\}$. If $y \in H'(x) \setminus \{0\}$, we have

$$y \in H'(x) \setminus \{0\} = (H(x) \cap S') \setminus \{0\} = (H(x) \setminus \{0\}) \cap S' \subseteq H(x) \setminus \{0\}.$$

Therefore, by condition c. 3 for the function $H(x)$ we have $H(y) \subseteq H(x)$. Hence

$$H'(y) = H(y) \cap S' \subseteq H(x) \cap S' = H'(x).$$

To show that $(S', \mathfrak{F}_{H'})$ is nested singularly discrete we proceed as follows: Since $H'(x) \subseteq H(x)$ for each $x \in S' \setminus \{0\}$, we know that each element of S' which is an essential point in $(S, \mathfrak{F}_H)$ is also an essential point in $(S', \mathfrak{F}_{H'})$. Let E denote the set of non-essentials in $(S, \mathfrak{F}_H)$ together with the nested essential point of $(S, \mathfrak{F}_H)$. By Theorem 4. 1. 2 (or Theorem 4. 1. 3), for any $x \in E$ we have

(4) $$H(x) = \{0\} \cup \{y \mid y \in E \text{ and } y \preccurlyeq x\} \subseteq E \cup \{0\}.$$

Let $E' = E \cap S'$. Then E' contains all of the non-essentials of $(S', \mathfrak{F}_{H'})$. Since S' is linearly ordered, we can write

$$E' = \{e_1, e_2, \ldots, e_n\}$$

where $e_1 \prec e_2 \prec \cdots \prec e_n$ if E' has n elements. We write

$$E' = \{e_1, e_2, \ldots\}$$

where $e_1 \prec e_2 \prec \cdots$ if E' has infinitely many elements. Now by statement (4) and since $E' = E \cap S'$ we have

$$H'(e_1) = H(e_1) \cap S' = \{0, e_1\},$$

$$H'(e_2) = H(e_2) \cap S' = \{0, e_1, e_2\},$$

$$\vdots$$

$$H'(e_i) = H(e_i) \cap S' = \{0, e_1, e_2, \ldots, e_i\},$$

$$\vdots$$

and if $x \in S' \setminus (E' \cup \{0\})$, we have $H'(x) = \{0, x\}$. Therefore, by Theorem 4.1.2 (or Theorem 4.1.3), we have that $(S', \mathfrak{F}_{H'})$ is nested singularly discrete. Since $H'(x) \subseteq H(x)$ for each $x \in S' \setminus \{0\}$, we know that the order of $(S', \mathfrak{F}_{H'})$ does not exceed the order of $(S, \mathfrak{F}_H)$. Clearly, S' is isomorphic to I.

We are now ready for the main result of this chapter.

Theorem 5.1.7. Let $*$ be any admissible inequality and let

$(S, \mathcal{H})$ be any nested singularly discrete density space. If $*$ holds for all nested singularly discrete density spaces $(I, \mathcal{F})$ of order not exceeding the order of $(S, \mathcal{H})$, then $*$ holds for $(S, \mathcal{H})$.

Proof: Suppose $(S, \mathcal{H})$ is nested singularly discrete of finite order n. (We handle this infinite case next.) By Theorem 4.1.2, we have $\mathcal{H} = \mathcal{H}_H$ where

$$H(x) = \begin{cases} \{0\} \cup \{x_i \mid i = 1,2,\ldots,m\} & \text{if } x = x_m \text{ for all } m = 1,2,\ldots,n, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

and where $0 \prec x_1 \prec x_2 \prec \ldots \prec x_n$. Let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ.

If either $\alpha = 0$ or $\beta = 0$ then since $*$ is admissible we know that $*$ holds for $(S, \mathcal{H})$. Hence we may assume that $\alpha > 0$ and $\beta > 0$. Therefore, all essential points are in $A \cap B$, and so

(5) $$S \setminus \{x_i \mid i = 2, \ldots, n\} \subseteq A \cap B \subseteq C.$$

Let $x_0 = 0$. By Lemma 5.1.5, we know that the sequence

(6) $$x_0, x_1, x_2, \ldots, x_n, 2x_n, 3x_n, \ldots$$

is a subsequence of some sequence $0 = a_0 \prec a_1 \prec a_2 \prec \cdots$ of elements of S where $a_{i+1} - a_i \in \operatorname{Min}(S \setminus \{0\})$ for $i = 0, 1, 2, \ldots$.

Note that the part of sequence (6) following x_n could just as well be any infinite sequence of elements $y_{n+1}, y_{n+2}, \ldots$ of S where $x_n \prec y_{n+1} \prec y_{n+2} \prec \ldots$. Now the sequence $a_0, a_1, \ldots$ can be of only two types.

Type 1: $a_k = ka_1$ for $k = 0, 1, 2, \ldots$.

Type 2: $a_k = ka_1$ for $k = 0, 1, \ldots, K$, where $K \geq 1$.

$a_{K+1} = a_K + b$ where $b \in \text{Min}(S \setminus \{0\})$ and $b \neq a_1$.

In either case let $S' = \{ka_1 | k = 0, 1, 2, \ldots\}$. Let $H'(x) = H(x) \cap S'$ for each $x \in S' \setminus \{0\}$. By Lemma 5.1.6 we know that $(S', \mathcal{H}_{H'})$ is a nested singularly discrete density space of order not exceeding the order of $(S, \mathcal{H}_H)$ and that S' is isomorphic to I. Therefore, by hypothesis, inequality * holds for $(S', \mathcal{H}_{H'})$. Also remember that for nested singularly discrete density spaces k-density and c-density are equal.

Suppose our sequence is of Type 1. Then

(7) $$\{x_i | i = 1, 2, \ldots, n\} \subseteq S'.$$

Let $A' = A \cap S'$, $B' = B \cap S'$, and $C' = A' + B'$. Then $C' \subseteq C \cap S'$. Let α', β', and γ' be the densities of A', B', and C' for the space $(S', \mathcal{H}_{H'})$. Since $\mathcal{H}_{H'}$ is separable we have by Theorem 2.3.13 and statements (5) and (7) that the densities α', β', and γ' are determined on $\{x_i | i = 1, 2, \ldots, n\}$. By Theorem 2.3.13 and statement (5) the densities α, β, and γ are also determined on $\{x_i | i = 1, 2, \ldots, n\}$. Now $H(x_i) \subseteq S'$ and so

$H'(x_i) = H(x_i)$ for $i = 1, 2, \ldots, n$. Therefore, $\alpha = \alpha'$, $\beta = \beta'$, and since $C' \subseteq C \cap S'$ we have $\gamma' \leq \gamma$. Now since $*$ holds for $(S', \mathcal{H}_{H'})$ it holds for α', β', and γ'. Since $*$ is admissible and since $\alpha = \alpha'$, $\beta = \beta'$, and $\gamma' \leq \gamma$ we have by condition (iii) of Definition 5.1.1 that $*$ holds for α, β, and γ. Therefore, $*$ holds for $(S, \mathcal{H})$.

Suppose our sequence is of Type 2. First we show that $a_k \in C$ for all $k > K$: Consider any a_{K+j} where $j \geq 1$. Then since $a_{K+1} \preccurlyeq a_{K+j}$ we have $a_{K+j} - b \in S \setminus \{0\}$ and $a_{K+j} - a_1 \in S \setminus \{0\}$. Now at least one of the elements $a_{K+j} - b$ and $a_{K+j} - a_1$ is an essential point for if not then since $(S, \mathcal{H})$ is nested singularly discrete we would have either $a_{K+j} - b \prec a_{K+j} - a_1$ (hence $a_1 \prec b$) or $a_{K+j} - a_1 \prec a_{K+j} - b$ (hence $b \prec a_1$). But since $a_1, b \in \mathrm{Min}\,(S \setminus \{0\})$ this is impossible. Now since a_1, b, and either $a_{K+j} - a_1$ or $a_{K+j} - b$ are essential points, they are in $A \cap B$ and hence $a_{K+j} \in C$. Therefore, $a_k \in C$ for all $k > K$. Now let $p = \max\{i \mid x_i \preccurlyeq a_K\}$. Then by statement (5) and since $a_k \in C$ for all $k > K$, we have

(8) $$S \setminus \{x_i \mid i = 2, 3, \ldots, p\} \subseteq C.$$

Furthermore,

(9) $$\{x_i \mid i = 1, 2, \ldots, p\} \subseteq S'.$$

Define sets A_0, B_0, and C_0 by

(10) $$A_0 = (A \cap S') \cup (S' \setminus \{x_i \mid i = 1, 2, \ldots, p\})$$

(11) $$B_0 = (B \cap S') \cup (S' \setminus \{x_i \mid i = 1, 2, \ldots, p\})$$

and $C_0 = A_0 + B_0$. Now we show that $C_0 \leq C \cap S'$. Since $A_0 \cup B_0 \subseteq S'$ and S' is an s-set then $C_0 = A_0 + B_0 \subseteq S'$. If $c_0 \in C_0$ then $c_0 = a_0 + b_0$ where $a_0 \in A_0$ and $b_0 \in B_0$. If $a_0 \in A$ and $b_0 \in B$ then $c_0 \in A + B = C$. If $a_0 \notin A$ then by statement (5) we have $a_0 \in \{x_i \mid 1 \leq i \leq n\}$. By statement (10) we have $a_0 \notin \{x_i \mid 1 \leq i \leq p\}$. Hence $a_0 \in \{x_i \mid p+1 \leq i \leq n\}$ and so $c_0 \succcurlyeq a_0 \succ x_p$. Therefore, by statement (8) we have $c_0 \in C$. Similarly if $b_0 \notin B$ then $c_0 \in C$. Let α_0, β_0, and γ_0 be the densities of A_0, B_0, and C_0 for the space $(S', \mathcal{H}_{H'})$. By Theorem 2.3.13, statements (9), (10), and (11), and the equation $C_0 = A_0 + B_0$, the densities α_0, β_0, and γ_0 are determined on $\{x_i \mid i = 1, 2, \ldots, p\}$. By Theorem 2.3.13 and statement (8) the density γ is determined on $\{x_i \mid i = 1, 2, \ldots, p\}$. Now by statement (9),

$$H(x_i) \subseteq \{0, x_1, x_2, \ldots, x_p\} \subseteq S'$$

and so $H'(x_i) = H(x_i)$ for each $i = 1, 2, \ldots, p$. Hence we have $\alpha \leq \alpha_0$, $\beta \leq \beta_0$, and since $C_0 \subseteq C$ we have $\gamma_0 \leq \gamma$. Now since * holds for $(S', \mathcal{H}_{H'})$ it holds for α_0, β_0, and γ_0. Since * is admissible and since $\alpha \leq \alpha_0$, $\beta \leq \beta_0$, and $\gamma_0 \leq \gamma$ we have by condition (iii) of Definition 5.1.1 that * holds for α, β, and γ. Therefore, * holds for $(S, \mathcal{H})$.

If $(S, \mathcal{H})$ is nested singularly discrete of infinite order, we use Theorem 4.1.3 in place of Theorem 4.1.2, we write $m = 1, 2, \ldots$ in place of $m = 1, 2, \ldots, n$, we write $0 \prec x_1 \prec x_2 \prec \cdots$ in place of $0 \prec x_1 \prec x_2 \prec \cdots \prec x_n$, and we write $i = 1, 2, \ldots$ in place of $i = 1, 2, \ldots, n$ in the above proof. We also replace sequence (6) by $x_0, x_1, x_2, \ldots$.

5.2. Another Reduction from $(S, \mathcal{H})$ to $(I, \mathcal{H})$

The final theorem of this chapter is used to generalize an important result in Chapter VIII.

Theorem 5.2.1. Let $(S, \mathcal{H})$ be any nested singularly discrete density space. If $(S, \mathcal{H})$ has a finite number of essential points, then S is isomorphic to I.

Proof: Suppose $(S, \mathcal{H})$ has a finite number of essential points and S is not isomorphic to I. Then by Theorem 2.1.10 the set $\text{Min}\,(S \setminus \{0\})$ has at least two members. Let $a, b \in \text{Min}\,(S \setminus \{0\})$ where $a \neq b$. Now since S, being an s-set, has an infinite number of elements, it must have an infinite number of non-essentials. Moreover, since $(S, \mathcal{H})$ is nested singularly discrete these non-essentials can be written as a countable partially ordered sequence $c_1 \prec c_2 \prec \cdots$ (Theorem 4.1.3). We have two cases to consider.

Case 1: Either $a \not\preccurlyeq c_i$ for all $i = 1, 2, \ldots$ or $b \not\preccurlyeq c_i$ for all $i = 1, 2, \ldots$.

Case 2: For some m and n we have $a \preccurlyeq c_m$ and $b \preccurlyeq c_n$.

Suppose we have Case 1. Assume without loss of generality that $a \not\preccurlyeq c_i$ for all $i = 1, 2, \ldots$. Suppose there is a positive integer k such that ka is a non-essential. Then for some i we must have $a \preccurlyeq ka = c_i$, a contradiction. Therefore, the infinite sequence $a, 2a, 3a, \ldots$ consists entirely of essential points of S.

Suppose we have Case 2; that is, $a \preccurlyeq c_m$ and $b \preccurlyeq c_n$ for some m and n. Let $N = \max\{m, n\}$. Then since the c_i's are partially ordered we have $a \prec c_i$ and $b \prec c_i$ for all $i > N$. Now for all $i > N$ let $d_i = c_i - a \in S$ and $e_i = c_i - b \in S$. For each $i > N$, either d_i or e_i is an essential point: Suppose both d_i and e_i are non-essentials. Then because all non-essentials are elements of the sequence $c_1 \prec c_2 \prec c_3 \prec \cdots$ we must have either $d_i \prec e_i$ or $e_i \prec d_i$. Without loss of generality suppose $d_i \prec e_i$. Then $c_i - a \prec c_i - b$ and hence $b \prec a$. But $a, b \in \mathrm{Min}\,(S \setminus \{0\})$ and we have a contradiction. Therefore, at least one of the sequences $\{d_i | i > N\}$ or $\{e_i | i > N\}$ contains an infinite number of essential points.

In both cases we have obtained an infinite set of essential points which contradicts our original hypothesis. Therefore, S must be isomorphic to I and the proof is complete.

CHAPTER VI

AN APPLICATION

In this chapter we prove the Landau-Schnirelmann and Schur inequalities for all nested singularly discrete density spaces $(S, \mathcal{H})$ by proving them for the case $S = I$ and using the results of Chapter V. In Chapter IV we proved these inequalities by another method.

6.1. Basic Theorems for $(I, \mathcal{H})$

Theorem 6.1.1. For any nested singularly discrete density space $(I, \mathcal{H})$ let A be a subset of I with $0 \in A$. Consider any set $T \subset I \setminus \{0\}$ such that $(I, \mathcal{H}) = (I, \mathcal{K}_T)$. If $n \in I \setminus (T \cup \{0\})$, then $A([n]) = (A \setminus T)(n)$.

Proof: We have $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ for some set $T \subset I \setminus \{0\}$ by Theorem 4.2.2 part (iii). Now since $n \in I \setminus (T \cup \{0\})$, we have by Definition 4.2.1 that $[n] = L(n) \setminus T$. Therefore,

$$
\begin{aligned}
A([n]) &= A(L(n) \setminus T) \\
&= (A \setminus T)(L(n)) \\
&= (A \setminus T)(n).
\end{aligned}
$$

Theorem 6.1.2. For any nested singularly discrete density space $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ let A, B, and $C = A + B$ be subsets of I

with $0 \in A \cap B$. If $n \in I \setminus C$ and $T \subseteq A \cap B$, then $A([n]) + B([n]) < I([n])$.

Proof: Since $n \in I \quad C$ and $T \cup \{0\} \subseteq A \cap B \subseteq C$ we have $n \in I \setminus (T \cup \{0\})$. Therefore by Theorems 6.1.1 and 2.3.22 and since $T \subseteq A \cap B$ we have

$$\begin{aligned} A([n]) + B([n]) &= (A \setminus T)(n) + (B \setminus T)(n) \\ &= A(n) - T(n) + B(n) - T(n) \\ &\leq A(n) + B(n) - T(n) \\ &< n - T(n) = I([n]). \end{aligned}$$

Theorem 6.1.1 is particularly useful in the remainder of this chapter. Both theorems are useful in Chapter VIII.

6.2. The Landau-Schnirelmann Inequality for $(S, \mathcal{H})$

Theorem 6.2.1. Let $(S, \mathcal{H})$ be any nested singularly discrete density space, let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. Then $\gamma \geq \alpha + \beta - \alpha\beta$.

Proof: By Theorems 5.1.2 and 5.1.7 it suffices to prove the theorem for $S = I$. If $A = I$, then $\alpha = \gamma = 1$ and the theorem follows. Therefore, suppose $A \neq I$. Let m be any positive integer not in A. We construct integers a_i and b_i where

$$0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_{k-1} < b_{k-1} < a_k < m$$

as follows. Let $a_1 + 1$ be the least positive integer missing from A. Let $b_1 + 1$ be the least integer greater than $a_1 + 1$ which is in A. In general, let $a_i + 1$ be the least integer greater than $b_{i-1} + 1$ which is not in A and let $b_i + 1$ be the least integer greater than $a_i + 1$ which is in A. This process terminates when we reach $a_k < m$ and find that either b_k does not exist or $b_k \geq m$.

We can also assume that all essential points are in $A \cap B$ and hence in C. Otherwise, $\alpha = 0$ or $\beta = 0$, and the theorem follows from Theorem 2.3.7. By Corollary 4.2.4, we have $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ where T is the set of all essential points of $(I, \mathcal{H})$ except the nested essential point. Therefore, $T \subsetneq A$, $T \subsetneq C$, and $m \notin T$.

Let β' be the density of B in the space $(I, \mathcal{K})$. We have

$$C(m) \geq A(m) + B(b_1 - a_1) + \cdots + B(b_{k-1} - a_{k-1}) + B(m - a_k),$$

and so

$$(1) \qquad \frac{C(m) - T(m)}{m - T(m)} \geq \frac{A(m) - T(m) + B(b_1 - a_1) + \cdots + B(b_{k-1} - a_{k-1}) + B(m - a_k)}{m - T(m)} .$$

Since $m \notin T$ and $T \subsetneq A$, we have by Theorem 6.1.1 that

$$(2) \qquad \alpha \leq \frac{A([m])}{I([m])} = \frac{(A \setminus T)(m)}{(I \setminus T)(m)} = \frac{A(m) - T(m)}{m - T(m)} .$$

By Lemma 4.3.3, we have

(3) $$\beta \le \beta' \le \frac{B(m-a_k)}{m-a_k},$$

and for $i = 1, 2, \ldots, k-1$, we have

(4) $$\beta \le \beta' \le \frac{B(b_i-a_i)}{b_i-a_i}.$$

Combining inequalities (1), (3), and (4), we obtain

(5) $$\frac{C(m)-T(m)}{m-T(m)} \ge \frac{A(m)-T(m)+\beta((b_1-a_1)+\ldots+(b_{k-1}-a_{k-1})+(m-a_k))}{m-T(m)}.$$

Now $A(m) = a_k - (b_1-a_1) - \ldots - (b_{k-1}-a_{k-1})$, so by inequalities (5) and (2) and since $\beta \le 1$, we have

$$\begin{aligned}\frac{C(m)-T(m)}{m-T(m)} &\ge \frac{A(m)-T(m)+\beta(m-A(m))}{m-T(m)} \\ &= \frac{A(m)-T(m)+\beta(m-T(m)+T(m)-A(m))}{m-T(m)} \\ &= \beta + (1-\beta)\frac{A(m)-T(m)}{m-T(m)} \\ &\ge \beta + (1-\beta)\alpha = \alpha + \beta - \alpha\beta.\end{aligned}$$

Therefore,

(6) $$\frac{C(m)-T(m)}{m-T(m)} \ge \alpha + \beta - \alpha\beta$$

for all $m \in I \setminus A$. Now since $T \subsetneq A$, we have $m \in I \setminus A$ implies

$m \notin T$. Therefore, by Theorem 6.1.1 and inequality (6), we have

$$(7) \qquad \frac{C([m])}{I([m])} = \frac{(C \setminus T)(m)}{(I \setminus T)(m)} = \frac{C(m)-T(m)}{m-T(m)}$$

$$\geq \alpha + \beta - \alpha\beta$$

for all $m \in I \setminus A$ and hence for all $m \in I \setminus C$. If $I \setminus C = \phi$, then $\gamma = 1 \geq \alpha + \beta - \alpha\beta$. If $I \setminus C \neq \phi$, then by Theorems 2.3.12, 2.3.13 and by inequality (7) we have

$$\gamma = \text{glb}\,\{\frac{C([m])}{I([m])}\,|\,m \in I \setminus C\} \geq \alpha + \beta - \alpha\beta .$$

This completes the proof.

6.3. The Schur Inequality for $(S, \mathcal{H})$

Theorem 6.3.1. Let $(S, \mathcal{H})$ be any nested singularly discrete density space, let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. If $\alpha + \beta < 1$, then $\gamma \geq \beta/(1-\alpha)$.

Proof: By Theorems 5.1.2 and 5.1.7 it suffices to prove the theorem for $S = I$. We can assume that all essential points are in $A \cap B$ and hence in C. Otherwise, $\alpha = 0$ or $\beta = 0$, and the theorem follows from Theorem 2.3.7. By Corollary 4.2.4, we have $(I, \mathcal{H}) = (I, \mathcal{X}_T)$ where T is the set of all essential points of $(I, \mathcal{H})$ except the nested essential point. Therefore, $T \subseteq A$,

$T \subseteq B$, and $T \subseteq C$. If $\gamma = 1$, we have $\gamma > \alpha + \beta = \alpha\gamma + \beta$, and so $\gamma \geq \beta/(1-\alpha)$. Hence we can assume that $\gamma < 1$, and so at least one positive integer is missing from C. Let $x_1, x_2, \ldots$ be the positive integers missing from C, listed in increasing order. Let $x_0 = 0$. First, we show that

$$(1) \qquad x_i - x_{i-1} - 1 \geq B(x_i) - B(x_{i-1}) + A(x_i - x_{i-1} - 1)$$

for each $i = 1, 2, \ldots$.

Now $B(x_i) - B(x_{i-1})$ is the number of integers in B which lie in the interval $(x_{i-1}, x_i]$. Assume there are p such integers $b_1, b_2, \ldots, b_p$. Since $x_i \notin C$, we know that $x_i - b_j \notin A$ for each $j = 1, 2, \ldots, p$. We also know that

$$0 < x_i - b_j \leq x_i - x_{i-1} - 1$$

for each $j = 1, 2, \ldots, p$. Therefore, $A(x_i - x_{i-1} - 1) \leq x_i - x_{i-1} - 1 - p$. Hence

$$x_i - x_{i-1} - 1 = p + x_i - x_{i-1} - 1 - p \geq B(x_i) - B(x_{i-1}) + A(x_i - x_{i-1} - 1).$$

For any h such that $x_h \notin C$, we can sum inequality (1) from 1 to h obtaining

$$(2) \qquad x_h - h \geq B(x_h) + \sum_{i=1}^{h} A(x_i - x_{i-1} - 1).$$

Let α' be the density of A in the space $(I, \mathcal{K})$. By Lemma 4.3.3, we have

(3) $$\alpha \leq \alpha' \leq \frac{A(m)}{m}$$

for every $m \in I \setminus \{0\}$. Therefore, by inequalities (2) and (3) and since $C(x_h) = x_h - h$, we have

$$C(x_h) \geq B(x_h) + \sum_{i=1}^{h} \alpha(x_i - x_{i-1} - 1)$$

$$= B(x_h) + \alpha(x_h - h)$$

$$= B(x_h) + \alpha C(x_h).$$

Hence

(4) $$(1-\alpha)C(x_h) \geq B(x_h).$$

Since $T \subseteq B \subseteq C$ and $x_h \notin C$ we have $x_h \notin T$, and so by Theorem 6.1.1 and inequality (4), we have

(5) $$\frac{(1-\alpha)C(x_h) - T(x_h)}{x_h - T(x_h)} \geq \frac{B(x_h) - T(x_h)}{x_h - T(x_h)}$$

$$= \frac{(B \setminus T)(x_h)}{(I \setminus T)(x_h)}$$

$$= \frac{B([x_h])}{I([x_h])} \geq \beta .$$

However $0 < 1-\alpha \leq 1$, so

$$(1-\alpha)T(x_h) \leq T(x_h).$$

Hence inequality (5) becomes

$$(1-\alpha)\frac{C(x_h)-T(x_h)}{x_h - T(x_n)} \geq \beta,$$

and so

$$\frac{C(m)-T(m)}{m-T(m)} \geq \beta/(1-\alpha) \tag{6}$$

for all $m \in I \setminus C$. Now since $T \subseteq C$, we have $m \notin T$. Therefore, by Theorem 6.1.1 and inequality (6), we have

$$\frac{C([m])}{I([m])} = \frac{(C \setminus T)(m)}{(I \setminus T)(m)} = \frac{C(m)-T(m)}{m-T(m)}$$
$$\geq \beta/(1-\alpha)$$

for all $m \in I \setminus C$. Therefore, by Theorems 2.3.12, 2.3.13, and since we have assumed that $I \setminus C \neq \phi$, we have

$$\gamma = \mathrm{glb}\left\{\frac{C([m])}{I([m])} \,\middle|\, m \in I \setminus C\right\} \geq \beta/(1-\alpha).$$

This completes the proof.

CHAPTER VII

NESTED SINGULARLY DISCRETE DENSITY SPACES OF ORDER 5

In this chapter we prove the $\alpha + \beta$ property for all nested singularly discrete density spaces of order 5. We also make a conjecture about the validity of the $\alpha + \beta$ property for orders greater than 5.

7.1. The $\alpha + \beta$ Property for Nested Singularly Discrete Density Spaces of Order 5

In Chapter III we proved the $\alpha + \beta$ property for all singularly discrete density spaces of order 4 or less. We also gave an example of a singularly discrete density space of order 5 for which the $\alpha + \beta$ property fails. However, in this section we show that the $\alpha + \beta$ property holds for all nested singularly discrete density spaces of order 5. We begin with the following lemma:

Lemma 7.1.1. Let $(S, \mathfrak{F}_H)$ be the density space determined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4, x_5\} & \text{if } x = x_5, \\ \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x_1, x_2\} & \text{if } x = x_2, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$. Let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$ and let α be the density of A.

(i) If $\alpha > 0$ and $x_1 \in B$, then $x_2 \in C$.

(ii) If $\alpha > 0$ and $x_1 \in B$ and $x_2 \in B$, then $x_3 \in C$.

(iii) If $\alpha > 0$ and $x_1 \in B$, $x_2 \in B$, and $x_3 \in B$, then $x_4 \in C$.

(iv) If $\alpha > 0$ and $x_1 \in B$ and $x_4 \in A \cap B$, then $x_5 \in C$.

(v) If $\alpha > 0$ and $x_1 \in B$, $x_i \in A$ and $x_j \in B$ for some i and j where $i \in \{2, 3\}$ and $j \in \{2, 3\}$, then $x_4 \in C$.

Proof of (i): Since $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$, we have $x_2 - x_1 \in S$ and $x_2 - x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$. Therefore $x_2 - x_1 \neq x_2$, $x_2 - x_1 \neq x_3$, $x_2 - x_1 \neq x_4$, and $x_2 - x_1 \neq x_5$. Since $\alpha > 0$, only x_2, x_3, x_4, or x_5 could be missing from A. Therefore, $x_2 - x_1 \in A$. Now $x_1 \in B$, so $x_2 = (x_2 - x_1) + x_1 \in C$.

Proof of (ii): Since $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$, we have $x_3 - x_2 \in S$, $x_3 - x_1 \in S$, $x_2 - x_1 \in S$, $x_3 - x_2 \prec x_3 \prec x_4 \prec x_5$, and $x_3 - x_1 \prec x_3 \prec x_4 \prec x_5$. Therefore, $x_3 - x_2 \neq x_3$, $x_3 - x_2 \neq x_4$, $x_3 - x_2 \neq x_5$, $x_3 - x_1 \neq x_3$, $x_3 - x_1 \neq x_4$, and $x_3 - x_1 \neq x_5$. Also $x_3 - x_2 \prec x_3 - x_1$ because $(x_3 - x_1) - (x_3 - x_2) = x_2 - x_1 \in S$, and so $x_3 - x_2 \neq x_3 - x_1$. Therefore, either $x_3 - x_2 \neq x_2$ or $x_3 - x_1 \neq x_2$. Since $\alpha > 0$, only x_2, x_3, x_4, or x_5 could be missing from A. Therefore, either $x_3 - x_2 \in A$ or $x_3 - x_1 \in A$. Since $x_1 \in B$ and $x_2 \in B$, then in either case $x_3 \in C$.

Proof of (iii): Since $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$, we have $x_4 - x_3 \prec x_4 - x_2 \prec x_4 - x_1 \prec x_4 \prec x_5$. Now at least one of $x_4 - x_3$, $x_4 - x_2$, $x_4 - x_1$ must be different from x_2 and x_3. Suppose it is $x_4 - x_j$, $j = 1, 2,$ or 3. Then $x_4 - x_j \neq x_2$, $x_4 - x_j \neq x_3$, $x_4 - x_j \neq x_4$, and $x_4 - x_j \neq x_5$. Since $\alpha > 0$, only x_2, x_3, x_4, or x_5 could be missing from A. Therefore, $x_4 - x_j \in A$. Now $x_j \in B$, so $x_4 \in C$.

Proof of (iv): Suppose $x_5 \notin C$. Since $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$, we have $x_5 - x_4 \in S$, $x_5 - x_1 \in S$, and $x_5 - x_4 \prec x_5 - x_1 \prec x_5$. Since $x_1 \in B$, $x_4 \in B$, and $x_5 \notin C$, we have $x_5 - x_4 \notin A$ and $x_5 - x_1 \notin A$. Since $\alpha > 0$ and $x_4 \in A$, only x_2, x_3, or x_5 could be missing from A. Therefore, $x_5 - x_4 = x_2$ and $x_5 - x_1 = x_3$. Hence $x_2 + x_4 = x_5 = x_1 + x_3$. However, $x_1 \prec x_2$ and $x_3 \prec x_4$ so $x_2 - x_1 \in S$, $x_4 - x_3 \in S$, $(x_2 - x_1) + (x_4 - x_3) \in S$, $(x_2 + x_4) - (x_1 + x_3) \in S$ and therefore $x_1 + x_3 \prec x_2 + x_4$, a contradiction. Hence $x_5 \in C$.

Proof of (v): Suppose $x_4 \notin C$. Since $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$, we have $x_4 - x_i \in S$, $x_4 - x_1 \in S$, and $x_4 - x_i \prec x_4 - x_1 \prec x_4$. Since $x_1 \in B$, $x_j \in B$, and $x_4 \notin C$, we have $x_4 - x_j \notin A$ and $x_4 - x_1 \notin A$. Since $\alpha > 0$ then $x_1 \in A$. Since also $x_i \in A$, at most one element of S less than x_4 could be missing from A, a contradiction. Hence $x_4 \in C$. Therefore, $x_4 - x_j \in A$. Now $x_j \in B$, so $x_4 \in C$.

This completes the proof of the lemma.

Theorem 7.1.2. Let $(S, \mathcal{H})$ be any nested singularly discrete density space of order 5, let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. Then $\gamma \geq \min\{1, \alpha + \beta\}$.

Proof: Since $(S, \mathcal{H})$ is nested singularly discrete of order 5, by Theorem 4.1.2 we have $(S, \mathcal{H}) = (S, \mathcal{F}_H)$ where $H(x)$ is defined by

$$H(x) = \begin{cases} \{0, x_1, x_2, x_3, x_4, x_5\} & \text{if } x = x_5, \\ \{0, x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{0, x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{0, x_1, x_2\} & \text{if } x = x_2, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

where $0 \prec x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$. The only values possible for α, β, and γ, listed in increasing order, are $0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.3.7, we have

$$\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}.$$

If $\alpha + \beta \geq 1$, then by Theorem 2.3.8, we have

$$\gamma = 1 = \min\{1, \alpha + \beta\}.$$

We may select our notation so that $\alpha \leq \beta$. Therefore, in the remainder of the proof we may assume that $0 < \alpha \leq \beta$ and $\alpha + \beta < 1$.

There are twenty cases to consider, but first we make a few observations. By Theorem 6.2.1, we have $\gamma \geq \alpha + \beta - \alpha\beta$. By Theorem 6.3.1 and since $\alpha + \beta < 1$, we have $\gamma \geq \beta/(1-\alpha)$. Since $0 < \alpha \leq \beta$, we know that all of the essential points are in $A \cap B$ and hence in C; that is,

$$S \setminus \{x_2, x_3, x_4, x_5\} \subseteq A \cap B \subseteq C.$$

By part (i) of Lemma 7.1.1, we have $x_2 \in C$, Therefore, the only elements which could be missing from C are x_3, x_4, and x_5. This allows eight possibilities for the set C. The following table lists the sixteen ways in which the positive densities of the space $(S, \mathcal{H})$ can be obtained. Those entries where x_2 is in the set give the eight possibilities for the set C.

Density	Subcase	Non-Essential Points			
		x_2	x_3	x_4	x_5
$\frac{1}{5}$	1	0	0	0	0
$\frac{1}{4}$	1	0	0	0	+
$\frac{1}{3}$	1	0	0	+	0
	2	0	0	+	+
$\frac{2}{5}$	1	+	0	0	0
	2	0	+	0	0
$\frac{1}{2}$	1	+	0	0	+
	2	0	+	0	+
	3	0	+	+	0
	4	0	+	+	+

KEY

0 Not in the set

+ In the set

Density	Subcase	Non-Essential Points x_2	x_3	x_4	x_5
$\frac{3}{5}$	1	+	+	0	0
	2	+	0	+	0
$\frac{2}{3}$	1	+	0	+	+
$\frac{3}{4}$	1	+	+	0	+
$\frac{4}{5}$	1	+	+	+	0
1	1	+	+	+	+

KEY

0 Not in the set

\+ In the set

We now treat the twenty cases mentioned above. We make frequent use of Lemma 7.1.1 and the above table of densities.

Case 1 ($\alpha = \beta = \frac{1}{5}$): Since $\gamma \geq \alpha + \beta - \alpha\beta$, we have $\gamma \geq \frac{1}{5} + \frac{1}{5} - \frac{1}{25} = \frac{9}{25}$. Since $\frac{1}{3} < \frac{9}{25} < \frac{2}{5}$ we can conclude that $\gamma \geq \frac{2}{5}$. Therefore, $\gamma \geq \alpha + \beta$.

Case 2 ($\alpha = \frac{1}{5}$, $\beta = \frac{1}{4}$): We have $\alpha + \beta = \frac{9}{20}$. We show that $\gamma \geq \frac{1}{2} > \frac{9}{20}$ by showing that $x_5 \in C$. Now $x_5 \in B \subseteq C$.

Case 3 ($\alpha = \frac{1}{5}$, $\beta = \frac{1}{3}$): We have $\alpha + \beta = \frac{8}{15}$. We show that $\gamma \geq \frac{3}{5} > \frac{8}{15}$ by showing that $x_4 \in C$. Now in both subcases for $\beta = \frac{1}{3}$ we have $x_4 \in B \subseteq C$.

Case 4 ($\alpha = \frac{1}{5}$, $\beta = \frac{2}{5}$): Since $\gamma \geq \alpha + \beta - \alpha\beta$, we have $\gamma \geq \frac{1}{5} + \frac{2}{5} - \frac{2}{25} = \frac{13}{25}$. Now since $\frac{1}{2} < \frac{13}{25} < \frac{3}{5}$ we can conclude that $\gamma \geq \frac{3}{5}$.. Therefore, $\gamma \geq \alpha + \beta$.

Case 5 ($\alpha = \frac{1}{5}$, $\beta = \frac{1}{2}$): We have $\alpha + \beta = \frac{7}{10}$. We show that $\gamma \geq \frac{3}{4} > \frac{7}{10}$ by showing that $x_3 \in C$ and either $x_4 \in C$ or $x_5 \in C$.

Subcase 1 for $\beta = \frac{1}{2}$: Here $x_3 \in C$ by part (ii) of Lemma 7.1.1 and $x_5 \in B \subseteq C$.

Subcase 2 for $\beta = \frac{1}{2}$: Here $x_3 \in B \subseteq C$ and $x_5 \in B \subseteq C$.

Subcase 3 for $\beta = \frac{1}{2}$: Here $x_3 \in B \subseteq C$ and $x_4 \in B \subseteq C$.

Subcase 4 for $\beta = \frac{1}{2}$: Here $x_3 \in B \subseteq C$ and $x_4 \in B \subseteq C$.

Case 6 ($\alpha = \frac{1}{5}$, $\beta = \frac{3}{5}$): We have $\alpha + \beta = \frac{4}{5}$. We show that $\gamma \geq \frac{4}{5}$ by showing that $x_3 \in C$ and $x_4 \in C$.

Subcase 1 for $\beta = \frac{3}{5}$: Here $x_3 \in B \subseteq C$ and $x_4 \in C$ by part (iii) of Lemma 7.1.1.

Subcase 2 for $\beta = \frac{3}{5}$: Here $x_3 \in C$ by part (ii) of Lemma 7.1.1 and $x_4 \in B \subseteq C$.

Case 7 ($\alpha = \frac{1}{5}$, $\beta = \frac{2}{3}$): Since $\gamma \geq \beta/(1-\alpha)$, we have $\gamma \geq \frac{5}{6}$. Now since $\frac{4}{5} < \frac{5}{6} < 1$ we can conclude that $\gamma = 1$. Therefore, $\gamma \geq \alpha + \beta$.

Case 8 ($\alpha = \frac{1}{5}$, $\beta = \frac{3}{4}$): Since $\gamma \geq \beta/(1-\alpha)$, we have $\gamma \geq \frac{15}{16}$. Now since $\frac{4}{5} < \frac{15}{16} < 1$ we can conclude that $\gamma = 1$. Therefore, $\gamma \geq \alpha + \beta$.

Case 9 ($\alpha = \beta = \frac{1}{4}$): Since $\gamma \geq \alpha + \beta - \alpha\beta$, we have $\gamma \geq \frac{1}{4} + \frac{1}{4} - \frac{1}{16} = \frac{7}{16}$. Now since $\frac{2}{5} < \frac{7}{16} < \frac{1}{2}$ we can conclude that $\gamma \geq \frac{1}{2}$. Therefore, $\gamma \geq \alpha + \beta$.

Case 10 ($\alpha = \frac{1}{4}$, $\beta = \frac{1}{3}$): We have $\alpha + \beta = \frac{7}{12}$. We show that $\gamma \geq \frac{2}{3} > \frac{7}{12}$ by showing that $x_4 \in C$ and $x_5 \in C$. Now $x_5 \in A \subseteq C$ and in both subcases for $\beta = \frac{1}{3}$ we have $x_4 \in B \subseteq C$.

Case 11 ($\alpha = \frac{1}{4}$, $\beta = \frac{2}{5}$): We have $\alpha + \beta = \frac{13}{20}$. We show that $\gamma \geq \frac{3}{4} > \frac{13}{20}$ by showing that $x_3 \in C$ and $x_5 \in C$.

Subcase 1 for $\beta = \frac{2}{5}$: Here $x_3 \in C$ by part (ii) of Lemma 7.1.1 and $x_5 \in A \subseteq C$.

Subcase 2 for $\beta = \frac{2}{5}$: Here $x_3 \in B \subseteq C$ and $x_5 \in A \subseteq C$.

Case 12 ($\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$): We have $\alpha + \beta = \frac{3}{4}$. We show that $\gamma \geq \frac{3}{4}$ by showing that $x_3 \in C$ and either $x_4 \in C$ or $x_5 \in C$.

Subcase 1 for $\beta = \frac{1}{2}$: Here $x_3 \in C$ by part (ii) of Lemma 7.1.1 and $x_5 \in A \subseteq C$.

In the remaining three subcases for $\beta = \frac{1}{2}$ we have $x_3 \in B \subseteq C$ and $x_5 \in A \subseteq C$.

Case 13 ($\alpha = \frac{1}{4}$, $\beta = \frac{3}{5}$): We have $\alpha + \beta = \frac{17}{20}$. We show that $\gamma = 1 > \frac{17}{20}$ by showing that $x_3 \in C$, $x_4 \in C$, and $x_5 \in C$.

Subcase 1 for $\beta = \frac{3}{5}$: Here we have $x_3 \in B \subseteq C$, $x_4 \in C$ by part (iii) of Lemma 7.1.1, and $x_5 \in A \subseteq C$.

Subcase 2 for $\beta = \frac{3}{5}$: Here we have $x_3 \in C$ by part (ii) of Lemma 7.1.1, $x_4 \in B \subseteq C$, and $x_5 \in A \subseteq C$.

Case 14 ($\alpha = \frac{1}{4}$, $\beta = \frac{2}{3}$): Since $\gamma \geq \beta/(1-\alpha)$, we have $\gamma \geq \frac{8}{9}$. Now since $\frac{4}{5} < \frac{8}{9} < 1$ we can conclude that $\gamma = 1$. Therefore, $\gamma \geq \alpha + \beta$.

Case 15 ($\alpha = \beta = \frac{1}{3}$): We have $\alpha + \beta = \frac{2}{3}$. We show that $\gamma \geq \frac{2}{3}$ by showing that $x_4 \in C$ and $x_5 \in C$.

Subcase 1 for $\alpha = \frac{1}{3}$ and Subcase 1 for $\beta = \frac{1}{3}$: Here we have $x_4 \in B \subseteq C$ and $x_5 \in C$ by part (iv) of Lemma 7.1.1.

In the remaining three subcase combinations we have $x_4 \in B \subseteq C$ and $x_5 \in A \cup B \subseteq C$.

Case 16 ($\alpha = \frac{1}{3}$, $\beta = \frac{2}{5}$): We have $\alpha + \beta = \frac{11}{15}$. We show that $\gamma \geq \frac{4}{5} > \frac{11}{15}$ by showing that $x_3 \in C$ and $x_4 \in C$.

Subcase 1 for $\beta = \frac{2}{5}$: Here we have $x_3 \in C$ by part (ii) of Lemma 7.1.1 and $x_4 \in A \subseteq C$ for both subcases of $\alpha = \frac{1}{3}$.

Subcase 2 for $\beta = \frac{2}{5}$: Here we have $x_3 \in B \subseteq C$ and $x_4 \in A \subseteq C$ for both subcases of $\alpha = \frac{1}{3}$.

Case 17 ($\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$): We have $\alpha + \beta = \frac{5}{6}$. We show that $\gamma = 1 > \frac{5}{6}$ by showing that $x_3 \in C$, $x_4 \in C$, and $x_5 \in C$.

Subcase 1 for $\beta = \frac{1}{2}$: Here we have $x_3 \in C$ by part (ii) of Lemma 7.1.1, $x_4 \in A \subseteq C$ for both subcases of $\alpha = \frac{1}{3}$, and $x_5 \in B \subseteq C$.

Subcase 2 for $\beta = \frac{1}{2}$: Here we have $x_3 \in B \subseteq C$, $x_4 \in A \subseteq C$ for both subcases of $\alpha = \frac{1}{3}$, and $x_5 \in B \subseteq C$.

Subcase 3 for $\beta = \frac{1}{2}$: Here we have $x_3 \in B \subseteq C$, $x_4 \in B \subseteq C$, and $x_5 \in C$ by part (iv) of Lemma 7.1.1 for both subcases of $\alpha = \frac{1}{3}$.

Subcase 4 for $\beta = \frac{1}{2}$: Here we have $x_3 \in B \subseteq C$, $x_4 \in B \subseteq C$, and $x_5 \in B \subseteq C$.

Case 18 ($\alpha = \frac{1}{3}$, $\beta = \frac{3}{5}$): Since $\gamma \geq \beta/(1-\alpha)$, we have $\gamma \geq \frac{9}{10}$.

Now since $\frac{4}{5} < \frac{9}{10} < 1$ we can conclude that $\gamma = 1$. Therefore, $\gamma \geq \alpha + \beta$.

Case 19 ($\alpha = \beta = \frac{2}{5}$): We have $\alpha + \beta = \frac{4}{5}$. We show that $\gamma \geq \frac{4}{5}$ by showing that $x_3 \in C$ and $x_4 \in C$.

Subcase 1 for $\alpha = \frac{2}{5}$: Here we have $x_3 \in C$ by part (ii) of Lemma 7.1.1 and $x_4 \in C$ by part (v) of Lemma 7.1.1 for both subcases of $\beta = \frac{2}{5}$.

Subcase 2 for $\alpha = \frac{2}{5}$: Here we have $x_3 \in A \subseteq C$ and $x_4 \in C$ by part (v) of Lemma 7.1.1.

Case 20 ($\alpha = \frac{2}{5}$, $\beta = \frac{1}{2}$): Since $\gamma \geq \beta/(1-\alpha)$ we have $\gamma \geq \frac{5}{6}$. Now since $\frac{4}{5} < \frac{5}{6} < 1$ we can conclude that $\gamma = 1$. Therefore, $\gamma \geq \alpha + \beta$.

This completes the proof of Theorem 7.1.2.

7.2. A Conjecture

Conjecture 7.2.1. Let $(S, \mathcal{H})$ be any nested singularly discrete density space, let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. Then $\gamma \geq \min\{1, \alpha + \beta\}$.

Theorems 3.1.2, 3.3.2, 3.3.4, and 7.1.2 verify this conjecture for nested singularly discrete density spaces of order $n \leq 5$. However, the method of proof used in these theorems becomes unmanageable for larger values of n. For example, when $n = 5$ we needed to consider 20 cases, but if $n = 6$ the number of cases increases to 30 and if $n = 7$ to 72. Moreover, as n increases each case tends to have more subcases. We have attempted to extend the proofs of Mann [10], Dyson [2], and Garrison [6] for the $\alpha + \beta$ property in the space $(I, \mathcal{K})$ to nested singularly discrete density spaces. With the exception of Dyson's method, we have had no success. We are able to apply Dyson's method to obtain the $\alpha + \beta$ property for some nested singularly discrete density spaces of infinite order. These and related density results are discussed in the next chapter.

CHAPTER VIII

THEOREMS OF MANN AND DYSON

In this chapter we examine generalizations to nested singularly discrete density spaces of some density related theorems of Mann and Dyson for the space $(I, \mathcal{K})$.

8.1. Mann's First Theorem

For the density space $(I, \mathcal{K})$, H. B. Mann [10] proved the following theorem which yields the $\alpha + \beta$ property.

Theorem 8.1.1. Let A, B, and $C = A + B$ be subsets of I with $0 \in A \cap B$. For any $n \in I \setminus C$, there exists an integer m where $1 \leq m \leq n$ and $m \notin C$ such that

$$\frac{C(n)}{n} \geq \frac{A(m)+B(m)}{m}.$$

A natural extension of this theorem to nested singularly discrete density spaces is the following: For any nested singularly discrete density space $(I, \mathcal{H})$ let A, B, and $C = A + B$ be subsets of I with $0 \in A \cap B$. For any $n \in I \setminus C$, there exists an integer m where $1 \leq m \leq n$ and $m \notin C$ such that

$$\frac{C([n])}{I([n])} \geq \frac{A([m])+B([m])}{I([m])}.$$

This is clearly a generalization of Theorem 8.1.1, and hence is true when $(I, \mathcal{H})$ has just the essential point 1; that is, when $\mathcal{H} = \mathcal{K}$. However, the following example shows that this extension is not valid when $(I, \mathcal{H})$ has two essential points.

Let $T = \{2\}$. By Theorem 4.2.2 part (ii), the space $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ is nested singularly discrete. The essential points are $\{1, 2\}$. Let $A = B = \{0, 1, \overline{4}\}$. Then $C = \{0, 1, 2, \overline{4}\}$. Now $3 \notin C$ and so the above extension, if valid, would claim that

$$\frac{C([3])}{I([3])} \geq \frac{A([3])+B([3])}{I([3])} .$$

However, $A([3]) = B([3]) = C([3]) = 1$ and so the inequality fails. In this example $T \subseteq A$ and $T \subseteq B$, so $\alpha = \beta = 0$. However, $\gamma = \frac{1}{2}$ and so the $\alpha + \beta$ property holds. In fact, by Theorem 2.3.7, the $\alpha + \beta$ property holds whenever $T \subseteq A \cap B$. Hence we modify our generalization of Theorem 8.1.1 to the following which, if true, would yield the $\alpha + \beta$ property for nested singularly discrete density spaces.

Conjecture 8.1.2. For any nested singularly discrete density space $(I, \mathcal{H})$ let A, B, and $C = A + B$ be subsets of I with $T \cup \{0\} \subseteq A \cap B$. For any $n \in I \setminus C$, there exists an integer m where $1 \leq m \leq n$ and $m \notin C$ such that

$$\frac{C([n])}{I([n])} \geq \frac{A([m])+B([m])}{I([m])} .$$

The author's efforts to prove this conjecture have been unsuccessful. He has attempted to extend Mann's proof for $(I, \mathcal{K})$ using Mann's construction but has been unable to resolve certain difficulties.

8. 2. Dyson's Theorem

For the density space $(I, \mathcal{K})$, F. Dyson [2] proved the following theorem which, although weaker than Theorem 8. 1. 1, also yields the $\alpha + \beta$ property.

Theorem 8. 2. 1. Let A, B, and $C = A + B$ be subsets of I with $0 \in A \cap B$. For any $n \in I \setminus C$, there exists an integer m such that $1 \leq m \leq n$ and

$$\frac{C(n)}{n} \geq \frac{A(m)+B(m)}{m} .$$

A natural extension of this theorem to nested singularly discrete density spaces is the following: For any nested singularly discrete density space $(I, \mathcal{H})$ let A, B, and $C = A + B$ be subsets of I with $0 \in A \cap B$. For any $n \in I \setminus C$, there exists an integer m such that $1 \leq m \leq n$ and

$$\frac{C([n])}{I([n])} \geq \frac{A([m])+B([m])}{I([m])} .$$

This is clearly a generalization of Theorem 8.2.1, and hence is true when $(I, \mathcal{H})$ has just the essential point 1; that is, when $\mathcal{H} = \mathcal{K}$. It is also true when $(I, \mathcal{H})$ has two essential points. We state these results formally in the following theorem, the proof of which is delayed until Section 8.3.

Theorem 8.3.2. For any nested singularly discrete density space $(I, \mathcal{H})$ having two or less essential points let A, B, and $C = A + B$ be subsets of I with $0 \in A \cap B$. For any $n \in I \setminus C$, there exists an integer m such that $1 \leq m \leq n$ and

$$\frac{C([n])}{I([n])} \geq \frac{A([m])+B([m])}{I([m])} .$$

The following example shows that this extension is not valid when $(I, \mathcal{H})$ has three essential points.

Let $T = \{7, 9\}$. By Theorem 4.2.2 part (ii), the space $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ is nested singularly discrete. The essential points are $\{1, 7, 9\}$. Let $A = \{0, 1, 3, 6, 7, 9, \overline{12}\}$ and $B = \{0, 6, \overline{12}\}$. Then $C = \{0, 1, 3, 6, 7, 9, \overline{12}\}$. Now $11 \notin C$ and so the above extension, if valid, would claim that

$$\frac{C([11])}{I([11])} \geq \frac{A([m])+B([m])}{I([m])}$$

for some integer m where $1 \leq m \leq 11$. However,

$$\frac{C([11])}{I([11])} = \frac{3}{9} = \frac{1}{3}$$

while the right side of the inequality always exceeds $\frac{1}{3}$. Therefore, the inequality fails. In this example $T \not\subseteq B$, so $\beta = 0$. However, $\alpha = \gamma = \frac{1}{3}$ and so the $\alpha + \beta$ property holds. In fact, by Theorem 2.3.7, the $\alpha + \beta$ property holds whenever $T \not\subseteq A \cap B$. Hence we modify our generalization of Theorem 8.2.1 to the following which, if true, would yield the $\alpha + \beta$ property for nested singularly discrete density spaces.

Conjecture 8.2.2. For any nested singularly discrete density space $(I, \mathcal{H})$ let A, B, and $C = A + B$ be subsets of I with $T \cup \{0\} \subseteq A \cap B$. For any $n \in I \setminus C$, there exists an integer m such that $1 \leq m \leq n$ and

$$\frac{C([n])}{I([n])} \geq \frac{A([m])+B([m])}{I([m])} .$$

By Theorem 8.3.2 of the next section, this conjecture is true if $(I, \mathcal{H})$ has two or less essential points. However, the author's efforts to prove this conjecture in general have been unsuccessful. He has attempted to extend proofs using Dyson's transformation and the recent non-transformation proof of B. Kvarda Garrison [6]. In each case, he has been unable to resolve certain difficulties. With Garrison's method he has been unable to extend the proof even when

$(I, \mathcal{H})$ has two essential points.

8.3. The $\alpha + \beta$ Property for Nested Singularly Discrete Density Spaces Having Two or Less Essential Points

In this section we prove that the $\alpha + \beta$ property holds for all nested singularly discrete density spaces having two or less essential points.

Theorem 8.3.1. Let n be any positive integer and η be any real number such that $0 \leq \eta \leq 1$. Let A and B be subsets of $I \cap [0, n]$ with $0 \in A \cap B$, let $C = A + B$, and consider any nested singularly discrete density space $(I, \mathcal{H})$ having two or less essential points. Then

(1) $$A([m]) + B([m]) \geq \eta\, I([m]), \quad \text{for} \quad m = 1, 2, \ldots, n,$$

implies

(2) $$C([m]) \geq \eta\, I([m]), \quad \text{for} \quad m = 1, 2, \ldots, n.$$

Proof: If $B = \{0\}$, then statement (1) implies statement (2) because $A \subseteq C$.

Suppose the theorem fails for some η and n. Choose A and B in such a way that statement (1) is true, statement (2) is false, and $B(n)$ is minimal. Then

(3) $$B(n) \geq 1.$$

Let

$$a^* = \min \{a \mid a \in A, \{a\} + B \not\subseteq A\}. \tag{4}$$

Now a^* exists because by inequality (3) we have

$$\max \{a \mid a \in A\} + \max \{b \mid b \in B\} > \max \{a \mid a \in A\}.$$

Let

$$B'' = \{b'' \mid b'' \in B,\ a^* + b'' \notin A\}. \tag{5}$$

By equality (4), we have

$$B'' \neq \phi. \tag{6}$$

Since $a^* \in A$, we also have

$$0 \notin B''. \tag{7}$$

We define new sets A', B', and $C' = A' + B'$ where

$$A' = A \cup ((\{a^*\} + B'') \cap [0, n]) \tag{8}$$

and

$$B' = B \setminus B''. \tag{9}$$

Now $0 \in B$ and hence by (7), we have

$$0 \in B'. \tag{10}$$

Also, by equalities (5) and (9), we have

$$\{a^*\} + B' \subseteq A. \tag{11}$$

We now prove that

$$A'([m]) + B'([m]) \geq \eta I([m]), \quad \text{for} \quad m = 1, 2, \ldots, n,$$

which makes up the major portion of the proof of Theorem 8.3.1:

If $a^* = 0$, then by equalities (5), (8), and (9) and by statement (1), we have

$$A'([m]) + B'([m]) \geq A([m]) + B([m])$$
$$\geq \eta I([m]),$$

for $m = 1, 2, \ldots, n$.

Suppose $a^* > 0$. By Corollary 4.2.4 we have $(I, \mathcal{H}) = (I, \chi_T)$ where T is the set of all essential points of $(I, \mathcal{H})$ except the nested essential point. Consider any $t \in T \cap [1, n]$. Suppose $t \notin A$. If $t \notin B$, then

$$A([t]) + B([t]) = 0,$$

and since statement (1) is true, we must have $\eta = 0$. However, if $\eta = 0$, then statement (2) is also true, a contradiction. If $t \in B$, then $\{0\} + B \not\subseteq A$ and so $a^* = 0$, a contradiction. Therefore, we must have $t \in A$ and hence

(12) $$T \cap [1, n] \subseteq A.$$

Let m be any integer in $[1, n]$. If $m \in T$, then by statement

(12) and the fact that $A \subseteq A'$, which follows from equality (8), we have

$$A'([m]) + B'([m]) \geq 1 \geq \eta$$
$$= \eta I([m]).$$

Suppose $m \notin T$. Let

(13) $$B_0 = B \cap [m-a^*+1, m].$$

If $B_0(m) = 0$; that is, if B_0 contains no positive integers, then

(14) $$B''(m) = (\{a^*\}+B'')(m).$$

By equality (5) we have

(15) $$A \cap (\{a^*\}+B'') = \phi$$

and

(16) $$B'' \subseteq B.$$

Now by statements (12) and (16) and since $m \notin T$, we have

(17) $$(\{a^*\}+B'')([m]) = (\{a^*\}+B'')(m).$$

By statements (9) and (16) we have

(18) $$B([m]) = B''([m]) + B'([m]).$$

Therefore, by statements (8), (15), (17), and (14) we have

$$\begin{aligned} A'([m]) + B'([m]) &= A([m]) + (\{a^*\}+B'')([m]) + B'([m]) \\ &= A([m]) + (\{a^*\}+B'')(m) + B'([m]) \\ &= A([m]) + B''(m) + B'([m]). \end{aligned}$$

Since $B''(m) \geq B''([m])$ we have by the preceding equality and statements (18) and (1) that

$$\begin{aligned} A'([m]) + B'([m]) &\geq A([m]) + B''([m]) + B'([m]) \\ &= A([m]) + B([m]) \\ &\geq \eta I([m]). \end{aligned}$$

<u>Suppose $B_0(m) > 0$.</u> Let

(19) $$b_1 = \min \{b \mid b \in B_0, b \geq 1\}.$$

Consider any integer r satisfying $0 \leq r < a^*$. Then by equality (4), we have

$$(A \cap [0, r]) + \{b\} \subseteq A$$

for every $b \in B$. Therefore, $A(r) + 1 \leq A(b, b+r)$ where we have defined $A(0, r) = A(r) + 1$ (see Definition 2.3.4). Hence, we have

(20) $$A(r) + 1 \leq A(b, b+r)$$

for every integer r satisfying $0 \leq r < a^*$ and for every $b \in B$. Let $r_1 = m - b_1$. By equalities (13) and (19), we have $b_1 \in [m-a^* + 1, m]$. Therefore,

(21) $$0 \leq r_1 < a^*.$$

We now prove that the following inequality holds:

(22) $$A(r_1) + 1 \geq \eta(r_1+1) + (1-\eta)T(b_1, m)$$

Suppose r^* is the least integer satisfying $0 \leq r^* < a^*$ for which

(23) $$A(r^*) + 1 < \eta(r^*+1) + (1-\eta)T(b_1, m).$$

If r^* does not exist of if $r^* > r_1$ then inequality (22) holds. Therefore, suppose $0 \leq r^* \leq r_1$. If $r^* = 0$, then inequality (23) becomes

$$1 < \eta + (1-\eta)T(b_1, m),$$

and so

$$1 - \eta < (1-\eta)T(b_1, m) \leq 1 - \eta,$$

a contradiction. Notice that we have used for the first time the fact that $(I, \mathcal{H})$ has two or less essential points; that is, that T has one or less members. Therefore, suppose $r^* \geq 1$. Now by statement (12), inequality (23), and since $T(b_1, m) \leq 1$ and $0 \leq \eta \leq 1$, we have the following string of inequalities:

$$\begin{aligned}(A \setminus T)(r^*) &= A(r^*) - T(r^*) \\ &< \eta(r^*+1) + (1-\eta)T(b_1, m) - 1 - T(r^*) \\ &\leq \eta(r^*+1) + (1-\eta) - 1 - T(r^*) \\ &= \eta r^* - T(r^*) \\ &\leq \eta r^* - \eta T(r^*) \\ &= \eta(I \setminus T)(r^*).\end{aligned}$$

Hence, we have

$$(A \setminus T)(r^*) < \eta(I \setminus T)(r^*). \tag{24}$$

Consider any integer i where $1 \leq i \leq n$. If $i \notin T$, then by Theorem 6.1.1 and statement (1), we have

$$\begin{aligned}(A \setminus T)(i) + (B \setminus T)(i) &= A([i]) + B([i]) \\ &\geq \eta I([i]) \\ &= \eta(I \setminus T)(i).\end{aligned}$$

If $i \in T$, let j be the largest integer such that $j < i$ and $j \notin T$. If $j = 0$, then

$$(A \setminus T)(i) + (B \setminus T)(i) = 0 = \eta(I \setminus T)(i).$$

If $j > 0$, then

$$\begin{aligned}(A \setminus T)(i) + (B \setminus T)(i) &= (A \setminus T)(j) + (A \setminus T)(j) \\ &= A([j]) + B([j]) \\ &\geq \eta I([j]) \quad =\end{aligned}$$

$$= \eta(I \setminus T)(j)$$

$$= \eta(I \setminus T)(i).$$

Therefore, we have the following inequality which depends only on statement (1):

$$(25) \qquad (A \setminus T)(i) + (B \setminus T)(i) \geq \eta(I \setminus T)(i)$$

for each integer i where $1 \leq i \leq n$.

In particular, since $1 \leq r^* < n$, we have by inequality (25) that

$$(A \setminus T)(r^*) + (B \setminus T)(r^*) \geq \eta(I \setminus T)(r^*).$$

Therefore, by inequality (24), there exists a positive integer

$$b_0 \in B \cap [1, r^*].$$

By the definition of r^* we have

$$A(b_0 - 1) + 1 \geq \eta b_0 + (1-\eta)T(b_1, m),$$

which combined with inequality (23) yields

$$(26) \qquad A(b_0, r^*) < \eta(r^* - b_0 + 1).$$

Now since $0 \leq r^* < a^*$ and $1 \leq b_0 \leq r^*$, we have

$$0 \leq r^* - b_0 < a^*.$$

Therefore, by inequalities (20) and (26), we have

$$\begin{aligned} A(r^{*}-b_0) + 1 &\leq A(b_0, b_0+r^{*}-b_0) \\ &= A(b_0, r^{*}) \\ &< \eta(r^{*}-b_0+1) \\ &\leq \eta(r^{*}-b_0+1) + (1-\eta)T(b_1, m) \end{aligned}$$

which contradicts the minimal property of r^{*}. Hence, inequality (22) is valid.

Recall that we are in the process of proving that

$$A'([m]) + B'([m]) \geq \eta I([m]), \quad \text{for} \quad m = 1, 2, \ldots, n.$$

We are also under the assumptions that $a^{*} > 0$, $m \notin T$, and $B_0(m) > 0$. By statements (8), (15), and (17) we have that

$$\begin{aligned} A'([m]) + B'([m]) &\geq A'([m]) + (B' \setminus T)(b_1 - 1) \\ &= A([m]) + (\{a^{*}\}+B'')([m]) + (B' \setminus T)(b_1 - 1) \\ &= A([m]) + (\{a^{*}\}+B'')(m) + (B' \setminus T)(b_1 - 1). \end{aligned}$$

By statements (13), (19), (9), and the preceding inequality we have

$$\begin{aligned} A'([m]) + B'([m]) &\geq A([m]) + (\{a^{*}\}+B'')(m) + (B' \setminus T)(b_1 - 1) \\ &\geq A([m]) + B''(m-a^{*}) + (B' \setminus T)(b_1 - 1) \\ &= A([m]) + B''(b_1 - 1) + (B' \setminus T)(b_1 - 1) \\ &= A([m]) + B''(b_1 - 1) + ((B \setminus B'') \setminus T)(b_1 - 1) \quad = \end{aligned}$$

$$= A([m]) + B''(b_1-1) + ((B \setminus T) \setminus B'')(b_1-1)$$

$$\geq A([m]) + B''(b_1-1) + (B \setminus T)(b_1-1) - B''(b_1-1)$$

$$= A([m]) + (B \setminus T)(b_1-1).$$

This inequality together with Theorem 6.1.1, the fact that $m \notin T$, and statements (25) and (12) yield the following string of inequalities.

$$A'([m]) + B'([m]) \geq A([m]) + (B \setminus T)(b_1-1)$$

$$= (A \setminus T)(b_1-1) + (A \setminus T)(b_1, m) + (B \setminus T)(b_1-1)$$

$$\geq \eta(I \setminus T)(b_1-1) + (A \setminus T)(b_1, m)$$

$$= \eta(I \setminus T)(b_1-1) + A(b_1, m) - T(b_1, m).$$

By statements (21), (20), and (22) this inequality becomes

$$A'([m]) + B'([m]) \geq \eta(I \setminus T)(b_1-1) + A(r_1) + 1 - T(b_1, m)$$

$$\geq \eta(I \setminus T)(b_1-1) + \eta(r_1+1) + (1-\eta)T(b_1,m) - T(b_1,m)$$

$$= \eta(I \setminus T)(b_1-1) + \eta(r_1+1-T(b_1, m)).$$

Since $r_1 = m - b_1$ and $m \notin T$ this inequality becomes

$$A'([m]) + B'([m]) \geq \eta(I \setminus T)(b_1-1) + \eta(I(b_1, m) - T(b_1, m))$$

$$= \eta(I \setminus T)(b_1-1) + \eta(I \setminus T)(b_1, m)$$

$$= \eta(I \setminus T)(m)$$

$$= \eta I([m]).$$

We now continue with the proof of the main theorem. Consider any element $c' \in C'$. Then $c' = a' + b'$ for some $a' \in A'$ and $b' \in B'$. If $a' \in A$, then since $B' \subset B$ we have

$$c' = a' + b' \in A + B = C.$$

If $a' \notin A$, then $a' = a^* + b''$ for some $b'' \in B''$. Therefore, by statements (11) and (16), we have

$$\begin{aligned} c' = a' + b' &= (a^*+b'') + b' \\ &= (a^*+b') + b'' \in A + B = C. \end{aligned}$$

Hence, we have $C' \subseteq C$ and since statement (2) fails to hold for C it also fails for C'. Moreover, by statements (16), (6), and (7), we have $B'(n) < B(n)$. By statements (8) and (10) and since $0 \in A$, we have $0 \in A' \cap B'$. Therefore, we have constructed sets A' and B', with $0 \in A' \cap B'$, which are subsets of $I \cap [0, n]$ and for which statement (1) holds, statement (2) fails, and $B'(n) < B(n)$. This contradicts the minimality of B and completes the proof of Theorem 8.3.1.

<u>Theorem 8.3.2.</u> For any nested singularly discrete density space $(I, \mathcal{H})$ having two or less essential points let A, B, and $C = A + B$ be any subsets of I with $0 \in A \cap B$. For any $n \in I \setminus C$, there exists an integer m such that $1 \leq m \leq n$ and

$$\frac{C([n])}{I([n])} > \frac{A([m])+B([m])}{I([m])} .$$

Proof: By Corollary 4.2.4, we have $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ where T is the set of all essential points of $(I, \mathcal{H})$ except the nested essential point. Also since $(I, \mathcal{H})$ has two or less essential points, T has one or less members. Let

$$\eta = \min_{1 \leq p \leq n} \frac{A([p])+B([p])}{I([p])} .$$

Clearly, $\eta \geq 0$. We now show that $\eta < 1$.

Consider any $t \in T \cap [1,n]$. If $t \notin A \cup B$, then

$$\frac{A([t])+B([t])}{I([t])} = 0 = \eta .$$

Suppose $t \in A \cup B$, and hence $t \in C$. Since $n \notin C$, and since T has one or less members, we have $n \notin T$. Therefore, by Theorems 6.1.1 and 2.3.22, and since $T \subseteq A \cup B$, we have

$$\begin{aligned} \frac{A([n])+B([n])}{I([n])} &= \frac{(A \setminus T)(n)+(B \setminus T)(n)}{(I \setminus T)(n)} \\ &\leq \frac{A(n)+B(n)-T(n)}{n-T(n)} \\ &< \frac{n-T(n)}{n-T(n)} = 1 . \end{aligned}$$

Therefore, $\eta < 1$.

Now

$$A([p]) + B([p]) = \frac{A([p])+B([p])}{I([p])} I([p])$$
$$\geq \eta I([p]) ,$$

for each $p = 1, 2, \ldots, n$. Let $A' = A \cap [0, n]$, $B' = B \cap [0, n]$, and $C' = A' + B'$. Then the preceding inequality yields

$$A'([p]) + B'([p]) \geq \eta I([p]),$$

for each $p = 1, 2, \ldots, n$, which is statement (1) of Theorem 8.3.1. Therefore, by Theorem 8.3.1, and since $C' \cap [0, n] = C \cap [0, n]$, we have

$$C([n]) = C'([n])$$
$$\geq \eta I([n]),$$

and so

$$\frac{C([n])}{I([n])} \geq \eta = \min_{1 \leq p \leq n} \frac{A([p])+B([p])}{I([p])} .$$

Hence there exists an integer m such that $1 \leq m \leq n$ and

$$\frac{C([n])}{I([n])} \geq \frac{A([m])+B([m])}{I([m])} .$$

Theorem 8.3.3. For any nested singularly discrete density space $(I, \mathcal{A})$ having two or less essential points let A, B, and $C = A + B$ be any subsets of I with $0 \in A \cap B$ and let the corresponding densities be α, β, and γ. Then $\gamma \geq \min \{1, \alpha + \beta\}$.

Proof: If $C = I$, then $\gamma = 1$ and the theorem follows. Therefore, suppose $I \setminus C \neq \phi$. If $n \in I \setminus C$, then by Theorem 8.3.2 there exists an integer m such that $1 \leq m \leq n$ and

$$\frac{C([n])}{I([n])} \geq \frac{A([m])+B([m])}{I([m])}$$

$$\geq \alpha + \beta .$$

Now $\frac{C([n])}{I([n])} \geq \alpha + \beta$ for all $n \notin C$. The fundamental family $\mathcal{H}$ is separated because $(I, \mathcal{H})$ is discrete. Hence we have by Theorems 2.3.12 and 2.3.13 that

$$\gamma = \underset{n \notin C}{\text{glb}} \frac{C([n])}{I([n])} \geq \alpha + \beta .$$

Therefore, $\gamma \geq \min \{1, \alpha + \beta\}$.

We may now use Theorem 5.2.1 to extend this result to the following:

<u>Theorem 8.3.4</u>. Let $(S, \mathcal{H})$ be any nested singularly discrete density space having two or less essential points, let A, B, and $C = A + B$ be any subsets of S with $0 \in A \cap B$, and let the corresponding densities be α, β, and γ. Then $\gamma \geq \min \{1, \alpha + \beta\}$.

Proof: Since $(S, \mathcal{H})$ has two or less essential points, we have, by Theorem 5.2.1, that S is isomorphic to I. Therefore, Theorem 8.3.4 follows immediately from Theorem 8.3.3.

8.4. An Extension of Mann's First Theorem

Consider the following statement: For any nested singularly discrete density space $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ let A, B, and $C = A + B$ be subsets of I with $T \cup \{0\} \subseteq A \cap B$. Let h be a real number. For any $n \in I \setminus C$, there exists an integer m where $1 \leq m \leq n$ and $m \notin C$ such that

(1) $$\frac{C([n])+h}{I([n])+h} \geq \frac{A([m])+B([m])+h}{I([m])+h} .$$

For what values of h is the above statement valid? When $h = 0$, the above statement reduces to Conjecture 8.1.2. R. Stalley [16] has shown that for the space $(I, \mathcal{K})$ the above statement is valid if and only if $-1 < h \leq 1$. Let K be any integer such that $K \geq 2$. In this section we show, by example, that there is a space $(I, \mathcal{H})$ with K essential points for which the above statement (1) is invalid when $h > \frac{1}{K}$.

Let $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ where

$$T = \{15+9k \mid k = 1, 2, \ldots, K-1\}.$$

By Corollary 4.2.4, there are K essential points; namely, $T \cup \{1\}$. Let A and B be defined as follows:

$$A = \{0,1,3,4,5,6\} \cup \bigcup_{k=0}^{K-1} \{9+9k, 10+9k, 12+9k, 13+9k, 14+9k, 15+9k\} \cup \overline{\{9+9K\}},$$

$$B = \{0, 1, 6\} \cup \bigcup_{k=0}^{K-1} \{9+9k, 10+9k, 15+9k\} \cup \overline{\{9+9K\}}.$$

Then

$$C = \{0,1,2,3,4,5,6,7\} \cup \bigcup_{k=0}^{K-1} \{9+9k, 10+9k, 11+9k, 12+9k, 13+9k, 14+9k, 15+9k, 16+9k\} \cup \overline{\{9+9K\}}.$$

The integers missing from C are $\{8+9k \mid k = 0, 1, \ldots, K\}$.

Now

$$\frac{A([8+9k])+([8+9k])+h}{I([8+9k])+h} = \begin{cases} \dfrac{7+h}{8+h} & \text{is } k = 0 \\[2ex] \dfrac{16+7(k-1)+h}{16+8(k-1)+h} & \text{if } k = 1, 2, \ldots, K \end{cases}$$

and

$$\frac{C([8+9K])+h}{I([8+9K])+h} = \frac{15+7(K-1)+h}{17+8(K-1)+h}.$$

To show that statement (1) fails for $h > \frac{1}{K}$ it suffices to show that

(i) $\dfrac{7+h}{8+h} > \dfrac{15+7(K-1)+h}{17+8(K-1)+h}$ for $h > \dfrac{1}{K}$,

and

(ii) $\dfrac{16+7(k-1)+h}{17+8(k-1)+h} > \dfrac{15+7(K-1)+h}{17+8(K-1)+h}$ for $k = 1, 2, \ldots, K$ and $h > \dfrac{1}{K}$.

Proof of (i): Suppose $h > \dfrac{1}{K}$ and

$$\frac{7+h}{8+h} \le \frac{15+7(K-1)+h}{17+8(K-1)+h} .$$

then $(7+h)(17+8(K-1)+h) \le (8+h)(15+7(K-1)+h)$, and so

$$119 + 56(K-1) + 7h + 17h + 8(K-1)h + h^2$$
$$\le 120 + 56(K-1) + 8h + 15h + 7(K-1)h + h^2.$$

Simplifying we obtain $Kh \le 1$, which contradicts our assumption that $h > \frac{1}{K}$.

Proof of (ii): Suppose $h > \frac{1}{K}$ and

$$\frac{16+7(k-1)+h}{17+8(k-1)+h} \le \frac{15+7(K-1)+h}{17+8(K-1)+h}$$

for some $k = 1, 2, \ldots, K$. Then

$$(16+7(k-1)+h)(17+8(K-1)+h) \le (17+8(k-1)+h)(15+7(K-1)+h)$$

and so

$$272 + 128(K-1) + 16h + 119(k-1) + 56(k-1)(K-1) + 7(k-1)h$$
$$+ 17h + 8(K-1)h + h^2$$
$$\le 255 + 119(K-1) + 17h + 120(k-1) + 56(k-1)(K-1) + 8(k-1)h$$
$$+ 15h + 7(K-1)h + h^2.$$

Simplifying we obtain $9 + 9K - k + h(1+K-k) \le 0$. However, $K \ge k \ge 1$ and $h > 0$, and hence

$$9 + 9K - k + h(1+K-k) > 0,$$

a contradiction. Therefore,

$$\frac{16+7(k-1)+h}{17+8(k-1)+h} > \frac{15+7(K-1)+h}{17+8(K-1)+h},$$

for $k = 1, 2, \ldots, K$.

8.5. An Extension of Dyson's Theorem

Consider the following statement: For any nested singularly discrete density space $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ let A, B, and $C = A + B$ be subsets of I with $T \cup \{0\} \subseteq A \cap B$. Let h be a real number. For any $n \in I \setminus C$, there exists an integer m such that $1 \leq m \leq n$ and

$$(1) \qquad \frac{C([n])+h}{I([n])+h} \geq \frac{A([m])+B([m])+h}{I([m])+h}.$$

For what values of h is the above statement valid? When $h = 0$, the above statement reduces to Conjecture 8.2.2 which by Theorem 8.3.2 is valid when $(I, \mathcal{H})$ has two or less essential points. R. Stalley [16] has shown that for the space $(I, \mathcal{K})$ the above statement is valid if and only if $-1 < h \leq 1$. Let K be any integer such that $K \geq 2$. We now show that the space $(I, \mathcal{H})$ with K essential points which we introduced in the previous section is a space for which statement (1) is invalid when $h > \frac{1}{K}$.

By straightforward computation and by Theorem 6.1.2 we have for each $i = 1, 2, \ldots, 6$ and $k = 0, 1, 2, \ldots, K$ that

$$(2) \qquad \frac{A([8+9k+i])+B([8+9k+i])+h}{I([8+9k+i])+h} \geq \frac{A([8+9k])+B([8+9k])+h+i}{I([8+9k])+h+i}$$

$$\geq \frac{A([8+9k])+B([8+9k])+h}{I([8+9k])+h},$$

and for $i = 8$ and $k = 0, 1, 2, \ldots, K$ we have

$$(3) \qquad \frac{A([8+9k+i])+B([8+9k+i])+h}{I([8+9k+i])+h} \geq \frac{A([8+9k])+B([8+9k])+h+7}{I([8+9k])+h+7}$$

$$\geq \frac{A([8+9k])+B([8+9k])+h}{I([8+9k])+h}.$$

For $i = 1, 2, \ldots, 7$ we have

$$(4) \qquad \frac{A([i])+B([i])+h}{I([i])+h} \geq \frac{i+h}{i+h} = 1.$$

In the previous section we proved that for $k = 0, 1, 2, \ldots, K$ we have

$$(5) \qquad \frac{A([8+9k])+B([8+9k])+h}{I([8+9k])+h} > \frac{C([8+9K])+h}{I([8+9K])+h}.$$

Combining results (2), (3), (4), and (5), and using $T \subseteq A \cap B$, we have

$$\frac{A([m])+B([m])+h}{I([m])+h} > \frac{C([8+9K])+h}{I([8+9K])+h}$$

for all $m = 1, 2, \ldots, 8+9K$.

8.6. Mann's Second Theorem

Definition 8.6.1. Consider any nested singularly discrete density space $(I, \mathcal{H})$ and any proper subset A of I. The Erdös density of A with respect to $\mathcal{H}$ is

$$d_1(A, \mathcal{H}) = \text{glb}\left\{ \frac{A([n])}{I([n])+1} \,\middle|\, n > 0,\ A([n]) < I([n]) \right\}.$$

Paul Erdös introduced the density $d_1(A, \mathcal{K})$ for the density space $(I, \mathcal{K})$. H. Mann [11] proved the following theorem for the space $(I, \mathcal{K})$ where A and B are subsets of I.

Theorem 8.6.2 (Mann's Second Theorem). If $0 \in A \cap B$ and $C = A + B$, then $C(n) \geq d_1(A, \mathcal{K})(n+1) + B(n)$ for every $n \in I \setminus C$.

We now look at the following generalization of Mann's Second Theorem to the nested singularly discrete density space $(I, \mathcal{H})$ where A and B are subsets of I and where $\alpha_1 = d_1(A, \mathcal{H})$: If $0 \in A \cap B$ and $C = A + B$, then for every $n \in I \setminus C$ we have

$$\text{(1)} \qquad C([n]) \geq \alpha_1 (I([n])+1) + B([n]).$$

We show that this generalization does not hold for all nested singularly discrete density spaces $(I, \mathcal{H})$, but that it does hold under special conditions which include the result for $(I, \mathcal{K})$.

Theorem 8.6.3. For any nested singularly discrete density space $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ with nested essential point e, let A, B, and $C = A + B$ be subsets of I with $0 \in A \cap B$ and let $\alpha_1 = d_1(A, \mathcal{H})$. Then inequality (1) holds for every $n \in I \setminus C$ if either of the following hold:

(i) $T \cup \{e\} \not\subseteq A$,

(ii) $T \subseteq B$.

Proof for (i): By Corollary 4.2.4, the set $T \cup \{e\}$ is the set of all essential points of $(I, \mathcal{H})$. If $T \cup \{e\} \not\subseteq A$, then $\alpha_1 = 0$, and since $C([n]) \geq B([n])$ for all n, inequality (1) follows.

Proof for (ii): If $T \cup \{e\} \not\subseteq A$, the inequality (1) follows by part (i). Therefore, suppose $T \cup \{e\} \subseteq A$. Hence for every $n > 0$ such that $A([n]) < I([n])$, we have by Theorem 6.1.1 that $A([n]) = A(n) - T(n)$ and so

$$\frac{A([n])}{I([n])+1} = \frac{A(n)-T(n)}{n+1-T(n)} \leq \frac{A(n)}{n+1}.$$

Therefore, $\alpha_1 \leq d_1(A, \mathcal{K})$. Consider any $n \in I \setminus C$. Since $T \subseteq B \subseteq C$, we have $n \in I \setminus B$, $B([n]) = B(n) - T(n)$, and $C([n]) = C(n) - T(n)$. Therefore, by Mann's Second Theorem for $(I, \mathcal{K})$, we have

$$\begin{aligned} C([n]) &= C(n) - T(n) = d_1(A,\mathcal{K})(n+1) + B(n) - T(n) \\ &= d_1(A,\mathcal{K})(n+1) + B([n]) \\ &\geq \alpha_1(n+1) + B([n]) \\ &\geq \alpha_1(n-T(n)+1) + B([n]) \\ &= \alpha_1(I([n])+1) + B([n]). \end{aligned}$$

This completes the proof.

The following example shows that inequality (1) does not always hold when $(I,\mathcal{H}) = (I, \mathcal{K}_T)$ is a nested singularly discrete density space, $T \cup \{e\} \subseteq A$, and $T \not\subseteq B$. Let $T = \{7\}$. Let $A = \{0, 1, 2, 5, 6, 7, 10, 11, 12, \overline{15}\}$ and $B = \{0, 1, 5, 6, 10, 11, \overline{15}\}$. Then $C = \{0, 1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, \overline{15}\}$. Straightforward calculations show that $\alpha_1 = \frac{2}{5}$, $C([14]) = 10$, $B([14]) = 5$, and $I([14]) = 13$, where $\alpha_1 = d_1(A, \mathcal{H})$. Therefore,

$$C([14]) = 10 < \frac{2}{5}\cdot 14 + 5 = \alpha_1(I([14])+1) + B([14]).$$

Since $14 \notin C$, inequality (1) fails.

We do have the following corollary to Theorem 8.6.3:

<u>Corollary 8.6.4.</u> For any nested singularly discrete density space $(I,\mathcal{H})$ let A, B, and $C = A + B$ be subsets of I with $0 \in A \cap B$. If $n \in I \setminus C$, at least one of the following inequalities holds:

(2) $$C([n]) \geq \alpha_1 (I([n])+1) + B([n]),$$

(3) $$C([n]) \geq A([n]) + \beta_1 (I([n])+1),$$

where $\alpha_1 = d_1(A, \mathcal{H})$ and $\beta_1 = d_1(B, \mathcal{H})$.

Proof: By Corollary 4.2.4, we have $(I, \mathcal{H}) = (I, \mathcal{X}_T)$. If $T \subseteq B$, then by Theorem 8.6.3 we have inequality (2). If $T \not\subseteq B$, then $T \cup \{e\} \not\subseteq B$, where e is the nested essential point of $(I, \mathcal{H})$. Therefore, by Theorem 8.6.3 we have inequality (3).

We also obtain the following important result.

Corollary 8.6.5. For any nested singularly discrete density space $(I, \mathcal{H})$ let A, B, and $C = A + B$ be proper subsets of I with $0 \in A \cap B$. Then we have the following two inequalities:

(i) $\gamma_1 \geq \alpha_1 + \beta_1$,

(ii) $\gamma \geq \min \{\alpha_1 + \beta, \alpha + \beta_1\}$,

where α, β, and γ are the k-densities (hence also c-densities) and α_1, β_1, γ_1 are the Erdös densities of A, B, and C in $(I, \mathcal{H})$.

Proof of (i): For any $n \in I \setminus C$ we have $n \notin A$ and $n \notin B$. Now by Corollary 8.6.4 we have either inequality (2) or (3). If inequality (2) holds then

$$\frac{C([n])}{I([n])+1} \geq \alpha_1 + \frac{B([n])}{I([n])+1}$$

$$\geq \alpha_1 + \beta_1 .$$

If inequality (3) holds then

$$\frac{C([n])}{I([n])+1} \geq \frac{A([n])}{I([n])+1} + \beta_1$$

$$\geq \alpha_1 + \beta_1 .$$

Hence in either case

$$\frac{C([n])}{I([n])+1} \geq \alpha_1 + \beta_1 . \tag{4}$$

Consider any $m \in I \setminus \{0\}$ such that $C([m]) < I([m])$. Then there is an $n \in [m] \setminus \{0\}$ such that $n \notin C$ and $i \in C$ for all $i \in [m]$ such that $n < i \leq m$. Hence by inequality (4) we have

$$\frac{C([m])}{I([m])+1} = \frac{C([n])+C([m]\setminus[n])}{I([n])+I([m]\setminus[n])+1}$$

$$= \frac{C([n])+I([m]\setminus[n])}{I([n])+1+I([m]\setminus[n])}$$

$$\geq \frac{C([n])}{I([n])+1} \geq \alpha_1 + \beta_1 .$$

Therefore, we have $\gamma_1 \geq \alpha_1 + \beta_1$.

Proof of (ii): For any $n \in I \setminus C$ we have $n \notin A$ and $n \notin B$. Now by Corollary 8.6.4 we have either inequality (2) or (3). If inequality (2) holds then

$$\frac{C([n])}{I([n])} \geq \alpha_1 \frac{I([n])+1}{I([n])} + \frac{B([n])}{I([n])}$$

$$\geq \alpha_1 + \beta .$$

In inequality (3) holds then

$$\frac{C([n])}{I([n])} \geq \frac{A([n])}{I([n])} + \beta_1 \frac{I([n])+1}{I([n])}$$

$$\geq \alpha + \beta_1 .$$

Hence by Theorem 2. 3. 13 we have

$$\gamma = \text{glb}\, \{\frac{C([n])}{I([n])} \,|\, n \in I \setminus C\} \geq \min\, \{\alpha_1 + \beta, \alpha + \beta_1\} .$$

This completes the proof.

CHAPTER IX

MIXED DENSITY THEOREMS

In this chapter we briefly consider some density inequalities involving more than one density space.

9.1. General Discussion

Let S be an s-set and consider a collection of density spaces $\{(S, \mathcal{F}_i) \mid i = 1, 2, \ldots\}$. A mixed density theorem is a theorem which relates densities from more than one such density space. A typical mixed density question is the following: If A, B, and $C = A + B$ are subsets of S with $0 \in A \cap B$, then is it true that

(1) $$d_k(C, \mathcal{F}_3) \geq \min\{1, d_k(A, \mathcal{F}_1) + d_k(B, \mathcal{F}_2)\}?$$

Of course, if $S = I$ and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{K}$, then inequality (1) is just the $\alpha + \beta$ property proved for the space $(I, \mathcal{K})$. This question and many others could be studied in detail. However, we do not attempt to do so here. Rather, we restrict our attention to a very special result for nested singularly discrete density spaces.

9.2. Mixed Density Inequalities for Nested Singularly Discrete Density Spaces

Recall that for discrete density spaces k-density and c-density are equal.

Theorem 9.2.1. Let $t \in I$ with $t \geq 2$. Let $(I, \mathcal{H}) = (I, \mathcal{K}_T)$ where $T = \{t\}$. Let A, B, and $C = A + B$ be subsets of I with $0 \in A \cap B$ and let the densities of A, B, and C be α, β, and γ for $(I, \mathcal{K})$ and α', β', and γ' for $(I, \mathcal{H})$ respectively. Then

(i) $\gamma \geq \min\{1, \alpha + \beta\}$,

(ii) $\gamma \geq \min\{1, \alpha + \beta'\}$,

(iii) $\gamma \geq \min\{1, \alpha' + \beta\}$,

(iv) $\gamma \geq \min\{1, \alpha' + \beta'\}$,

(v) $\gamma' \geq \min\{1, \alpha' + \beta'\}$,

(vi) there are examples for which $\gamma' < \min\{1, \alpha + \beta\}$,

(vii) there are examples for which $\gamma' < \min\{1, \alpha' + \beta\}$,

(viii) there are examples for which $\gamma' < \min\{1, \alpha + \beta'\}$.

Proof: Note that by part (ii) of Theorem 4.2.2 the space $(I, \mathcal{H})$ is nested singularly discrete. We first show that $\alpha \geq \alpha'$. If $t \notin A$, then $\alpha' = 0$ and hence $\alpha \geq \alpha'$. Suppose $t \in A$. Since $T = \{t\}$, we have $T(m) \leq 1$ for every $m \in I \setminus \{0, t\}$. Therefore, by Theorem 6.1.1 and since $t \geq 2$ and $T \subseteq A$, we have

$$\frac{A(m)}{m} \geq \frac{A(m)-T(m)}{m-T(m)} = \frac{(A \setminus T)(m)}{(I \setminus T)(m)} = \frac{A([m]')}{I([m]')}$$

for every $m \in I \setminus \{0,t\}$ where $[m]'$ denotes the Cheo set of m in the space $(I, \mathcal{H})$. Since $t \in A$, we have

$$\begin{aligned} \alpha &= \text{glb}\left\{\frac{A(m)}{m} \,\middle|\, m \in I \setminus \{0,t\}\right\} \\ &\geq \text{glb}\left\{\frac{A([m]')}{I([m]')} \,\middle|\, m \in I \setminus \{0,t\}\right\} \\ &= \alpha'. \end{aligned}$$

Therefore, $\alpha \geq \alpha'$. Likewise, $\beta \geq \beta'$.

Proof of (i): This is the $\alpha + \beta$ property for the space $(I, \mathcal{K})$ which was proved by Mann [10], Dyson [2], and Garrison [6].

Proof of (ii): By part (i) and since $\beta \geq \beta'$, we have

$$\gamma \geq \min\{1, \alpha + \beta\} \geq \min\{1, \alpha + \beta'\}.$$

Proof of (iii): By part (i) and since $\alpha \geq \alpha'$, we have

$$\gamma \geq \min\{1, \alpha + \beta\} \geq \min\{1, \alpha' + \beta\}.$$

Proof of (iv): By part (iii) and since $\beta \geq \beta'$, we have

$$\gamma \geq \min\{1, \alpha' + \beta\} \geq \min\{1, \alpha' + \beta'\}.$$

Proof of (v): By Corollary 4.2.4, the space $(S, \mathcal{H})$ has two essential points. Therefore, by Theorem 8.3.4, we have $\gamma' \geq \min\{1, \alpha' + \beta'\}$.

Proof of (vi): Let $A = \{0, \overline{t+1}\}$ and $B = \{0, 1, \overline{t+1}\}$. Then $C = \{0, 1, \overline{t+1}\}$, $\alpha = 0$, $\beta = \gamma = \frac{1}{t}$, and $\alpha' = \beta' = \gamma' = 0$. Therefore,

$$0 = \gamma' < \min\{1, \alpha + \beta\} = \frac{1}{t}.$$

Proof of (vii): Use the same example as in part (vi). We have

$$0 = \gamma' < \min\{1, \alpha' + \beta\} = \frac{1}{t}.$$

Proof of (viii): Use the same example as in part (vi), except interchange sets A and B. We have

$$0 = \gamma' < \min\{1, \alpha + \beta'\} = \frac{1}{t}.$$

This completes the proof.

CHAPTER X

COMPARISON OF c-DENSITY AND k-DENSITY

In this chapter we compare c-density with k-density. In particular, we answer two questions posed by Freedman [4].

10.1. Fundamental Results

Let $(S, \mathcal{F})$ be any density space and let A be a subset of S with $0 \in A$. Recall (by Theorem 2.3.12) that for discrete density spaces the c-density and k-density of A are always equal. This is not true for density spaces in general. For general density spaces we have the following two results of Freedman [3, 4] (see Theorem 2.3.6). We include proofs for completeness.

Theorem 10.1.1. We have $0 \leq \alpha \leq \alpha_c \leq 1$.

Proof: We know that

$$\{q(A,F) \mid F \in \mathcal{F}\} \supseteq \{q(A,[x]) \mid x \in S \setminus \{0\}\}.$$

Therefore,

$$\begin{aligned} \alpha &= \text{glb}\,\{q(A,F) \mid F \in \mathcal{F}\} \\ &\leq \text{glb}\,\{q(A,[x]) \mid x \in S \setminus \{0\}\} \\ &= \alpha_c. \end{aligned}$$

Also for each $F \in \mathcal{F}$, we have $0 \leq A(F) \leq S(F)$ and so both

$0 \leq \alpha \leq 1$ and $0 \leq \alpha_c \leq 1$.

Theorem 10.1.2. The following three conditions are equivalent: (i) $\alpha = 1$, (ii) $\alpha_c = 1$, and (iii) $A = S$.

Proof that condition (i) implies condition (ii): Suppose $\alpha = 1$. By Theorem 10.1.1, we have $\alpha \leq \alpha_c \leq 1$. Therefore, $\alpha_c = 1$.

Proof that condition (ii) implies condition (iii): Suppose there is an $x \in S \setminus A$. Then we have $A([x]) < S([x])$ and so $\alpha_c \leq q(A,[x]) < 1$. Therefore, if $\alpha_c = 1$, we have $A = S$.

Proof that condition (iii) implies condition (i): Suppose $A = S$. Then $q(A, F) = 1$ for each $F \in \mathcal{F}$ and so $\alpha = 1$.

This completes the proof.

For the remainder of this chapter we suppose $\alpha_c < 1$. Freedman [4] poses the following question: Does there exist a density space $(S, \mathcal{F})$ and a subset A of S such that $\alpha_c > 0$ and $\alpha = 0$? The answer is yes, as we show in the next section.

10.2. An Example for which $\alpha_c > 0$ and $\alpha = 0$

Let d be any positive integer and let $(I, \mathcal{F}_H)$ be the density space determined by

$$H(x) = \begin{cases} \{0, 1, 2, \ldots, d\} \cup \{x\} & \text{if } x \geq d+1, \\ \{0, x\} & \text{otherwise.} \end{cases}$$

Let $A = \{0, 1, 2, \ldots, d\}$. Since A has finite cardinality, we have $\alpha = 0$. However, $\alpha_c = \frac{d}{d+1} > 0$.

This example does more than show that we can have $\alpha_c > 0$ and $\alpha = 0$. Since d is any positive integer we can, for any ε such that $0 < \varepsilon < 1$, choose d sufficiently large to have $\alpha_c > 1-\varepsilon$ and $\alpha = 0$. Of course, by Theorem 10.1.2 we can never have $\alpha = 0$ and $\alpha_c = 1$.

This example also shows that $0 \in A$ and $\alpha_c > 0$ does not imply that A is a basis for I. This question was also posed by Freedman [4]. The set A is not a basis because it has finite cardinality.

The space can also be used to show that Theorem 2.3.10 fails for general density spaces. Let $A = B = \{0, 1, 2, \ldots, d\}$. Then $C = \{0, 1, 2, \ldots, 2d\}$. Therefore, $\alpha_c = \beta_c = \gamma_c = \frac{d}{d+1}$ and so $\alpha_c + \beta_c = \frac{2d}{d+1} \geq 1$, whereas $\gamma_c = \frac{d}{d+1} < 1$.

10.3. A General Example

Theorem 10.3.1. Let ρ_1 and ρ_2 be any two real numbers such that $0 \leq \rho_1 \leq \rho_2 < 1$. Then there exists a density space $(I, \mathfrak{F}_H)$ and a set $A \subseteq I$ such that $\alpha = \rho_1$ and $\alpha_c = \rho_2$.

Proof: Let ρ_1 and ρ_2 satisfy the hypotheses of the theorem. Let $\{u_i | i = 1, 2, \ldots\}$ and $\{v_i | i = 1, 2, \ldots\}$ be strictly decreasing sequences of positive rational numbers less than 1 such that $\rho_1 = \text{glb}\,\{u_i | i = 1, 2, \ldots\}$, $\rho_2 = \text{glb}\,\{v_i | i = 1, 2, \ldots\}$, and $u_i \leq v_i$ for each $i = 1, 2, \ldots$.

Now since u_i and v_i are positive rationals, there exist positive integers a_i, b_i and d_i such that $u_i = \frac{a_i}{b_i}$ and $v_i = \frac{a_i}{d_i}$. Since $0 < u_i \leq v_i < 1$, we have $1 \leq a_i < d_i \leq b_i$.

Let $(I, \mathfrak{F}_H)$ be the space determined by $H(x)$ where

$$(1) \qquad H(x) = \begin{cases} \{0, 1, 2, \ldots, x\} & \text{if } 1 \leq x \leq d_1 - 1, \\ \{0, 1, 2, \ldots, d_1 - 1, x\} & \text{if } d_1 \leq x \leq b_1, \\ \{0, b_1+1, b_1+2, \ldots, x\} & \text{if } b_1+1 \leq x \leq b_1+d_2-1, \\ \{0, b_1+1, b_1+2, \ldots, b_1+d_2-1, x\} & \text{if } b_1+d_2 \leq x \leq b_1+b_2, \\ \vdots & \\ \{0, \sum_{i=1}^{j} b_i+1, \sum_{i=1}^{j} b_i+2, \ldots, x\} & \text{if } \sum_{i=1}^{j} b_i+1 \leq x \leq \sum_{i=1}^{j} b_i+d_{j+1}-1, \\ \{0, \sum_{i=1}^{j} b_i+1, \sum_{i=1}^{j} b_i+2, \ldots, \sum_{i=1}^{j} b_i+d_{j+1}-1, x\} & \text{if } \sum_{i=1}^{j} b_i+d_{j+1} \leq x \leq \sum_{i=1}^{j+1} b_i, \\ \vdots & \end{cases}$$

By Theorem 2.2.6, the space $(I, \mathfrak{F}_H)$ is a density space. Now let

$$A = \{0, 1, 2, \ldots, a_1\} \cup \{b_1+1, b_1+2, \ldots, b_1+a_2\}$$

$$\cup \ldots \cup \{ \sum_{i=1}^{j} b_i+1, \sum_{i=1}^{j} b_i+2, \ldots, \sum_{i=1}^{j} b_i+a_{j+1}\} \cup \ldots .$$

We now show that for the space $(I, \mathfrak{F}_H)$ and the set A we have $\alpha_c = \rho_2$ and $\alpha = \rho_1$. Recall that $H(x)$ and $[x]$ are identical and that they both denote the Cheo set of x for the space $(I, \mathfrak{F}_H)$.

For each integer $j \geq 0$, we have

$$\frac{A([\sum_{i=1}^{j} b_i + d_{j+1}])}{I([\sum_{i=1}^{j} b_i + d_{j+1}])} = \frac{a_{j+1}}{d_{j+1}} = v_{j+1}.$$

Since $\rho_2 = \text{glb}\,\{v_i \,|\, i = 1, 2, \ldots\} = \text{glb}\,\{v_{i+1} \,|\, i = 0, 1, 2, \ldots\}$, we have $\alpha_c \leq \rho_c$. Also for any positive integer m there is an integer $j \geq 0$ such that $\Sigma_{i=1}^{j} b_i + 1 \leq m \leq \Sigma_{i=1}^{j+1} b_i$. Hence we have

$$\frac{A([m])}{I([m])} \geq \frac{A([\sum_{i=1}^{j} b_i + d_{j+1}])}{I([\sum_{i=1}^{j} b_i + d_{j+1}])} = \frac{a_{j+1}}{d_{j+1}} = v_{j+1}.$$

Therefore,

$$\alpha_c = \text{glb}\,\{\frac{A([m])}{I([m])} \,|\, m = 1, 2, \ldots\} \geq \text{glb}\,\{v_{j+1} \,|\, j = 0, 1, 2, \ldots\} = \rho_2.$$

Hence we have $\alpha_c = \rho_2$.

It is more difficult to show that $\alpha = \rho_1$. For each integer $j \geq 0$, define the set F_j by

$$F_j = \{0, \sum_{i=1}^{j} b_i+1, \sum_{i=1}^{j} b_i+2, \ldots, \sum_{i=1}^{j+1} b_i\}.$$

By formula (1) we have

$$F_j = \bigcup_{m=m_1(j)}^{m_2(j)} [m],$$

where $m_1(j) = \Sigma_{i=1}^{j} b_i+1$ and $m_2(j) = \Sigma_{i=1}^{j+1} b_i$. Hence by Theorem 2.2.15 we have $F_j \in \mathcal{F}_H$. Now

$$\text{(2)} \qquad \frac{A(F_j)}{I(F_j)} = \frac{a_{j+1}}{b_{j+1}} = u_{j+1}.$$

Since $\rho_1 = \text{glb}\{u_i | i = 1, 2, \ldots\} = \text{glb}\{u_{i+1} | i = 0, 1, 2, \ldots\}$, we have $\alpha \leq \rho_1$. Now consider any $F \in \mathcal{F}_H$. For each integer $j \geq 0$, let $G_j = F \cap F_j$. Now since $F \in \mathcal{F}_H$ and $F_j \in \mathcal{F}_H$, we have $G_j \in \mathcal{F}_H \cup \{\{0\}\}$. Now $i \neq j$ implies $F_i \cap F_j = \{0\}$ and hence $G_i \cap G_j = \{0\}$. Also F is finite. Hence there is a finite integer

$$J = \max\{j | G_j \setminus \{0\} \neq \phi\}.$$

Now if $G_j \setminus \{0\} \neq \phi$, then $G_j \in \mathcal{F}_H$. If $G_j \setminus A \neq \phi$, then

$$\{\sum_{i=1}^{j} b_i+1, \sum_{i=1}^{j} b_i+2, \ldots, \sum_{i=1}^{j} b_i+a_{j+1}\} \subseteq G_j \subseteq F_j,$$

and so

$$\frac{A(G_j)}{I(G_j)} \geq \frac{A(F_j)}{I(F_j)} . \tag{3}$$

If $G_j \setminus A = \phi$, then $A(G_j) = I(G_j)$ and inequality (3) still holds. Therefore, by statements (3) and (2) and since $G_i \cap G_j = \{0\}$ for $i \neq j$, we have

$$\frac{A(F)}{I(F)} = \frac{A(\bigcup_{j=1}^{J} G_j)}{I(\bigcup_{j=1}^{J} G_j)} = \frac{\sum_{j=1}^{J} A(G_j)}{\sum_{j=1}^{J} I(G_j)}$$

$$\geq \frac{A(G_i)}{I(G_i)} \geq \frac{A(F_i)}{I(F_i)} = u_{i+1}$$

for some i $(1 \leq i \leq J)$. Therefore,

$$\alpha = \text{glb}\ \{\frac{A(F)}{I(F)} \mid F \in \mathfrak{F}_H\} \geq \text{glb}\ \{u_{i+1} \mid i = 0, 1, 2, \ldots\} = \rho_1 .$$

Hence we have $\alpha = \rho_1$.

This completes the proof.

CHAPTER XI

FREEDMAN'S TRANSFORMATION PROPERTIES

In this chapter we study some relationships between three transformation properties of Freedman and the Landau-Schnirelmann and Schur inequalities. Two of these transformation properties, trans-1 and trans-2, were introduced by Freedman in his Ph. D. thesis [3] and later in his article [4]. The third transformation property, which we call trans-3, was also introduced by Freedman but does not appear in the literature [5]. The relationships are summarized in Section 5 of this chapter.

11.1. The Landau-Schnirelmann Inequality and Trans-1

Let $(S, \mathfrak{F})$ be any density space where $\mathfrak{F}$ is trans-1 (see Definition 2.3.14). By Theorem 2.3.18, the Landau-Schnirelmann inequality holds; that is, $\gamma \geq \alpha + \beta - \alpha\beta$. In this section we show that $\gamma \geq \alpha + \beta - \alpha\beta$ does not imply that $\mathfrak{F}$ is trans-1. In order to show this result we prove the following theorem which is of independent interest.

Theorem 11.1.1. Consider any discrete density space $(I, \mathfrak{F})$ of finite order n. Then $\mathfrak{F}$ is trans-1 if and only if $\mathfrak{F} = \mathfrak{F}_H$ where $H(x) = \{0, x\}$ for all $x \in I \setminus \{0\}$; that is, if and only if $n = 1$.

Proof: Suppose $\mathfrak{F} = \mathfrak{F}_H$ where $H(x) = \{0, x\}$ for all $x \in I \setminus \{0\}$. By Theorem 2.2.6, we have $\mathfrak{F}_H = \mathfrak{D}$, and so $\mathfrak{F}_H$ is trans-1 by Theorem 2.3.16.

Now suppose $\mathfrak{F} = \mathfrak{F}_H$ where $\mathfrak{F}_H$ is trans-1 but $\mathfrak{F}_H$ is not defined by $H(x) = \{0, x\}$ for all $x \in I \setminus \{0\}$. Then there exists an integer $k \geq 2$ and an integer i where $1 \leq i \leq k-1$ such that $H(x) = \{0, x\}$ for all $x = 1, 2, \ldots, k-1$ and $\{0, i, k\} \subseteq H(k)$.

Suppose $H(k+1) = \{0, k+1\}$. Using the notation of Definition 2.3.14 and choosing $x = 1$ and $F = \{0, 1, k+1\}$, we have

$$F = H(1) \cup H(k+1) \in \mathfrak{F}_H,$$

$D = \{1, k+1\}$, and $T_1[D] = \{0, k\}$. Since $H(k) \supseteq \{0, i, k\}$, we have $H(k) \not\subseteq \{0, k\}$. Therefore, by Theorem 2.2.6, we have $T_1[D] \notin \mathfrak{F}_H \cup \{\{0\}\}$. Hence $\mathfrak{F}_H$ is not trans-1, a contradiction. Therefore, $k+1$ is not an essential point.

As our induction step we assume that $k+j$ is not an essential point for some integer $j \geq 1$. Suppose $H(k+j+1) = \{0, k+j+1\}$. Choosing $x = 1$ and $F = \{0, 1, k+j+1\}$, we have

$$F = H(1) \cup H(k+j+1) \in \mathfrak{F}_H,$$

$D = \{1, k+j+1\}$, and $T_1[D] = \{0, k+j\}$. Since $k+j$ is not an essential point, we have $H(k+j) \not\subseteq \{0, k+j\}$. Therefore, by Theorem 2.2.6, we have $T_1[D] \notin \mathfrak{F}_H \cup \{\{0\}\}$. Hence $\mathfrak{F}_H$ is not trans-1, a contradition. Therefore, $k+j+1$ is not an essential point. We have shown by induction that only the first $k-1$ positive integers are essential points.

By conditions c. 1 and c. 3 of Theorem 2. 2. 6, each $H(x)$ must contain at least one essential point; namely, the least positive integer in $H(x)$. Therefore, at least one of the first $k-1$ positive integers, say y, must occur in $H(x)$ for infinitely many $x \in I \setminus \{0\}$; that is, $y \in \bigcap_{i=1}^{\infty} H(x_i)$ for some strictly increasing infinite sequence of positive integers $x_1, x_2, \ldots$. Since $(I, \mathfrak{F}_H)$ is discrete, $\mathfrak{F}_H$ must be separated. Hence, since

$$y \in H(x_i) \cap H(x_{i+1})$$

and $x_i < x_{i+1}$, we must have $H(x_i) \subset H(x_{i+1})$. Therefore

$$H(x_1) \subset H(x_2) \subset \cdots ,$$

which contradicts the fact that $(I, \mathfrak{F}_H)$ is discrete of finite order n. Therefore, $\mathfrak{F}_H$ is determined by $H(x) = \{0, x\}$ for all $x \in I \setminus \{0\}$.

Also $(I, \mathfrak{F}_H)$ is discrete of order $n = 1$ if and only if $H(x) = \{0, x\}$ for all $x \in I \setminus \{0\}$. This completes the proof.

Consider any nested singularly discrete density space $(I, \mathfrak{F})$ of order $n \geq 2$. By Theorem 6. 2. 1 we have $\gamma \geq \alpha + \beta - \alpha\beta$. By Theorem 11. 1. 1 we have that $\mathfrak{F}$ is not trans-1. Therefore,

$\gamma \geq \alpha + \beta - \alpha\beta$ does not imply that $\mathfrak{F}$ is trans-1.

11.2. The Schur Inequality and Trans-2

Let $(S, \mathfrak{F})$ be any density space where $\mathfrak{F}$ is trans-2 (see Definition 2.3.15). By Theorem 2.3.21, the Schur inequality holds; that is, if $\alpha + \beta < 1$, then $\gamma \geq \beta/(1-\alpha)$. In this section we show that the Schur inequality does not imply that $\mathfrak{F}$ is trans-2.

Let $(I, \mathfrak{F}_H)$ be the density space determined by

$$H(x) = \begin{cases} \{0, 2, 4\} & \text{if } x = 4, \\ \{0, 1, 5\} & \text{if } x = 5, \\ \{0, x\} & \text{otherwise.} \end{cases}$$

If, using the notation of Definition 2.3.15, we let $x = 5$ and $F = \{0\}$, then $D = \{1, 5\}$ and $T_2[D] = \{0, 4\}$. Now $H(4) \not\subseteq \{0, 4\}$. Therefore, by Theorem 2.2.6, we have

$$T_2[D] \notin \mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}.$$

Hence $\mathfrak{F}_H$ is not trans-2. However, since $(I, \mathfrak{F}_H)$ is discrete of order 2, we have by Theorem 3.1.2 that

$$\gamma \geq \min\{1, \alpha + \beta\} = \alpha + \beta \geq \alpha\gamma + \beta$$

whenever $\alpha + \beta < 1$. Therefore, $\gamma - \alpha\gamma \geq \beta$ and so $\gamma \geq \beta/(1-\alpha)$. Hence the Schur inequality does not imply that $\mathfrak{F}_H$ is trans-2.

11.3. The Correspondence Between Trans-1 and Trans-2

In this section we show that the two transformation properties, trans-1 and trans-2, are incomparable. We begin by showing that trans-1 does not imply trans-2.

Let $(I, \mathfrak{F}_H)$ be the density space determined by

$$H(x) = \begin{cases} \{0, 1\} & \text{if } x = 1, \\ \{0, 2\} & \text{if } x = 2, \\ \{0, 1, 2, \ldots, x\} & \text{otherwise.} \end{cases}$$

If, using the notation of Definition 2.3.15, we let $x = 4$ and $F = \{0, 2\}$, then $D = \{1, 3, 4\}$ and $T_2[D] = \{0, 1, 3\}$. Now $H(3) \not\subseteq \{0, 1, 3\}$. Therefore, by Theorem 2.2.6, we have $T_2[D] \notin \mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}$. Hence $\mathfrak{F}_H$ is not trans-2. To show that $\mathfrak{F}_H$ is trans-1, consider any $F \in \mathfrak{F}_H$ and $x \in F$. Then either

(i) $F = \{0, 1, 2, \ldots, n\}$ for some $n \geq 1$

or

(ii) $F = \{0, 2\}$.

In case (i) we have

$$D = F \cap U(x) = \{x, x+1, \ldots, n\}$$

where $0 \leq x \leq n$. Therefore,

$$T_1[D] = \{0, 1, \ldots, n-x\} \in \mathfrak{F}_H \cup \{\{0\}\}.$$

In case (ii) if $x = 0$, we have $D = \{0, 2\}$, and so $T_1[D] = \{0, 2\} \in \mathfrak{F}_H$. If $x = 2$, we have $D = \{2\}$, and so $T_1[D] = \{0\} \in \mathfrak{F}_H \cup \{\{0\}\}$. Therefore, $\mathfrak{F}_H$ is trans-1.

Now we show that trans-2 does not imply trans-1. Let $(I, \mathfrak{F}_H)$ be the density space determined by

$$H(x) = \begin{cases} \{0, 1, 2\} & \text{if } x = 2 \\ \{0, x\} & \text{otherwise.} \end{cases}$$

Now $(I, \mathfrak{F}_H)$ is discrete of order 2 and hence, by Theorem 11.1.1, it is not trans-1. To show that $\mathfrak{F}_H$ is trans-2, consider any $x \in I \setminus \{0\}$ and any $F \in \mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}$. Now $D = H(x) \setminus F$. If $x = 2$, the only choices possible for D are ϕ, $\{2\}$, $\{1, 2\}$, and $\{0, 1, 2\}$, and so the only possibilities for $T_2[D]$ are ϕ, $\{0\}$, $\{0, 1\}$, and $\{0, 1, 2\}$ which are all in $\mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}$. If $x \neq 2$, the only choices possible for D are ϕ, $\{x\}$, and $\{0, x\}$, and so the only possibilities for $T_2[D]$ are ϕ, $\{0\}$, and $\{0, x\}$ which are all in $\mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}$. Therefore, $\mathfrak{F}_H$ is trans-2.

11.4. Transformation Property Trans-3

In Section 11.1 we saw that $\gamma \geq \alpha + \beta - \alpha\beta$ does not imply that $\mathfrak{F}$ is trans-1. A logical question to consider is whether there is a weaker property than trans-1 which still implies that $\gamma \geq \alpha + \beta - \alpha\beta$. Freedman [5] has recently found such a property which we call trans-3.

<u>Definition 11.4.1 (Freedman)</u>. The fundamental family $\mathcal{F}$ is called <u>trans-3</u> if for each $F \in \mathcal{F}$ and $x \in F$ there exists a set $G \in \mathcal{F} \cup \{\{0\}\}$ satisfying the following two properties:

(i) $x + G = \{x+g \mid g \in G\} \subseteq F$,

(ii) $F \setminus (x+G) \in \mathcal{F} \cup \{\{0\}\} \cup \{\phi\}$.

<u>Theorem 11.4.2 (Freedman)</u>. Property trans-3 is weaker than property trans-1.

Proof that trans-1 implies trans-3: Let $F \in \mathcal{F}$ and $x \in F$. For G choose

$$G = T_1[D] = \{y-x \mid y \in D\}$$

where $D = F \cap U(x)$. Since $\mathcal{F}$ is trans-1 we have $G \in \mathcal{F} \cup \{\{0\}\}$ and $x + G = D \subseteq F$. If

$$z \in F \setminus (x+G) = F \setminus D,$$

$z \neq 0$, and $y \in [z]$, then $y \in D$ implies $y \in U(x)$. Hence $x \preccurlyeq y \preccurlyeq z$ and so $z \in D$, a contradiction. Therefore, $y \notin D$ and so

$$[z] \subseteq F \setminus D = F \setminus (x+G)$$

and by Theorem 2.2.6, we have

$$F \setminus (x+G) \in \mathcal{F} \cup \{\{0\}\} \cup \{\phi\}.$$

Proof that trans-3 does not imply trans-1: Let $(I, \mathfrak{F}_H)$ be the density space determined by

$$H(x) = \begin{cases} \{0, x\} & \text{if } x \text{ is odd,} \\ \{0, x-1, x\} & \text{if } x \text{ is even,} \end{cases}$$

for all $x \in I \setminus \{0\}$. Then $(I, \mathfrak{F}_H)$ is discrete of order 2. Therefore, by Theorem 11.1.1, $\mathfrak{F}_H$ is not trans-1. Now let $F \in \mathfrak{F}_H$ and $x \in F$. If $x = 0$, choose $G = F$. Then $x + G = F$ and

$$F \setminus (x+G) = F \setminus F = \phi.$$

If $x \neq 0$, there are three possibilities. If x is even, then $x-1 \in F$. Choose $G = \{0\}$. Then $x + G = \{x\} \subseteq F$ and

$$F \setminus (x+G) = F \setminus \{x\} = (F \setminus \{x-1, x\}) \cup \{0, x-1\} \in \mathfrak{F}_H.$$

If x is odd and $x + 1 \in F$, choose $G = \{0, 1\}$. Then $x + G = \{x, x+1\} \subseteq F$ and

$$F \setminus (x+G) = F \setminus \{x, x+1\} \in \mathfrak{F}_H \cup \{\{0\}\}.$$

If x is odd and $x + 1 \notin F$, choose $G = \{0\}$. Then $x + G = \{x\} \subseteq F$ and

$$F \setminus (x+G) = F \setminus \{x\} \in \mathfrak{F}_H \cup \{\{0\}\}.$$

Therefore, $\mathfrak{F}_H$ is trans-3.

This completes the proof.

For the next two theorems let $(S, \mathcal{F})$ be any density space and let A, B, and $C = A + B$ be subsets of S with $0 \in A \cap B$ and hence in C.

<u>Theorem 11.4.3 (Freedman)</u>. Suppose $\mathcal{F}$ is trans-3. Then for each $F \in \mathcal{F}$ we have

$$C(F) \geq A(F) + \beta(S \setminus A)(F).$$

Proof: If $A \cap F = \{0\}$, then $A(F) = 0$ and $(S \setminus A)(F) = S(F)$. Therefore, since

$$\frac{C(F)}{S(F)} \geq \gamma \geq \beta$$

by Theorem 2.3.7, we have

$$C(F) \geq \beta S(F) = A(F) + \beta(S \setminus A)(F).$$

Suppose $(A \cap F) \setminus \{0\} \neq \phi$. Let the set

$$A \cap F = \{0, a_1, \ldots, a_n\}$$

be indexed in such a way that

(1) $$a_i \succ a_j \quad \text{implies} \quad i < j.$$

We can satisfy statement (1) by taking $a_n \in \operatorname{Min}(F \setminus \{0\})$,

$a_{n-1} \in \mathrm{Min}\,(F \setminus \{0, a_n\})$, $a_{n-2} \in \mathrm{Min}\,(F \setminus \{0, a_{n-1}, a_n\}$, and so on.

Since $\mathcal{F}$ is trans-3 there exists a set $G_1 \in \mathcal{F} \cup \{\{0\}\}$ such that $a_1 + G_1 \subseteq F$ and

$$F_1 = F \setminus (a_1 + G_1) \in \mathcal{F} \cup \{\{0\}\} \cup \{\phi\}.$$

Now if $a_2 \in a_1 + G_1$ then $a_2 \succ a_1$ which contradicts statement (1). Therefore, $a_2 \in F_1$ and since $a_2 \neq 0$ we have $F_1 \in \mathcal{F}$. Again since $\mathcal{F}$ is trans-3 there exists a set $G_2 \in \mathcal{F} \cup \{\{0\}\}$ such that $a_2 + G_2 \subseteq F_1$ and

$$F_2 = F_1 \setminus (a_2 + G_2) \in \mathcal{F} \cup \{\{0\}\} \cup \{\phi\}.$$

We can continue this process until finally we obtain a set $G_n \in \mathcal{F} \cup \{\{0\}\}$ such that $a_n + G_n \subseteq F_{n-1}$ and

$$F_n = F_{n-1} \setminus (a_n + G_n) \in \mathcal{F} \cup \{\{0\}\} \cup \{\phi\}.$$

Now the sets $F_1, \ldots, F_n$ have been constructed so that $a_1 + G_1$, $a_2 + G_2, \ldots, a_n + G_n$, and F_n are pairwise disjoint and

$$F = F_n \cup \left(\bigcup_{i=1}^{n} (a_i + G_i) \right).$$

Therefore, since $0 \in G_i$, for each $i = 1, 2, \ldots, n$, we have

$$(2) \qquad C(F) = C(F_n) + \sum_{i=1}^{n} C(a_i+G_i)$$

$$= C(F_n) + C(A \cap F) + \sum_{i=1}^{n} C(a_i+G_i \setminus \{0\}).$$

Now since $B \subseteq C$, $A \subseteq C$, and $a_i \in A$, which implies $C(a_i+G_i \setminus \{0\}) \geq B(G_i)$, equality (2) yields

$$C(F) \geq B(F_n) + A(F) + \sum_{i=1}^{n} B(G_i)$$

$$\geq A(F) + \beta S(F_n) + \beta \sum_{i=1}^{n} S(G_i)$$

$$= A(F) + \beta(S(F_n) + \sum_{i=1}^{n} S(G_i))$$

$$= A(F) + \beta(S(F_n) + \sum_{i=1}^{n} S(a_i+G_i \setminus \{0\}))$$

$$= A(F) + \beta(S(F)-n)$$

$$= A(F) + \beta(S(F)-S(F \cap A))$$

$$= A(F) + \beta(S(F)-A(F))$$

$$= A(F) + \beta(S \setminus A)(F).$$

This completes the proof.

<u>Theorem 11.4.4 (Freedman)</u>. If $\mathfrak{F}$ is trans-3, then $\gamma \geq \alpha + \beta - \alpha\beta$.

Proof: By Theorem 11.4.3, we have

$$\begin{aligned} C(F) &\geq A(F) + \beta(S \setminus A)(F) \\ &= A(F) + \beta S(F) - \beta A(F) \\ &= A(F)(1-\beta) + \beta S(F) \\ &\geq \alpha\,(1-\beta)S(F) + \beta S(F) \\ &= (\alpha+\beta-\alpha\beta)S(F). \end{aligned}$$

Dividing by $S(F)$ we obtain

$$q(C, F) \geq \alpha + \beta - \alpha\beta$$

for each $F \in \mathfrak{F}$ and, therefore, $\gamma \geq \alpha + \beta - \alpha\beta$. This completes the proof.

We now show, by example, that $\gamma \geq \alpha + \beta - \alpha\beta$ does not imply trans-3.

Let $(I, \mathfrak{F}_H)$ be the density space determined by

$$H(x) = \begin{cases} \{0, 1, 5\} & \text{if } x = 5, \\ \{0, 1, 5, 6\} & \text{if } x = 6, \\ \{0, x\} & \text{otherwise,} \end{cases}$$

for all $x \in I \setminus \{0\}$. Then $(I, \mathfrak{F}_H)$ is nested singularly discrete of order 3 and so, by Theorem 6.2.1, $\gamma \geq \alpha + \beta - \alpha\beta$. We proceed to show that $\mathfrak{F}_H$ is not trans-3.

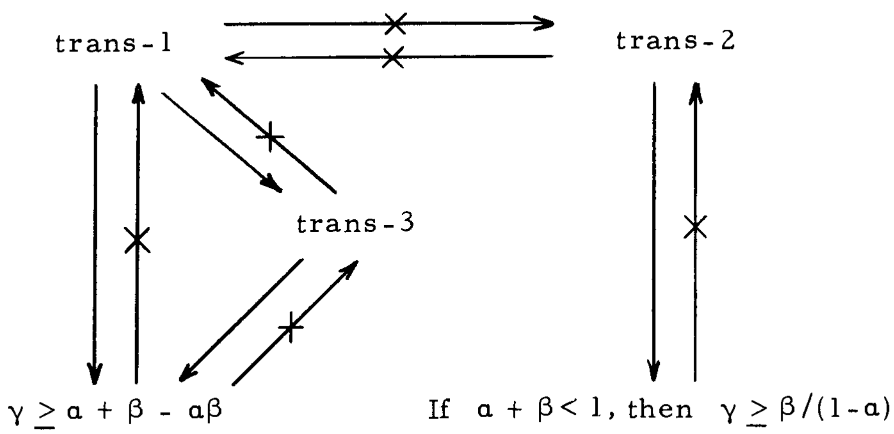
Let $F = \{0,1,5,6\}$ and $x = 1$. We now look for all sets $G \in \mathfrak{F}_H \cup \{\{0\}\}$ for which $(x+G) \subseteq F$. Since $x = 1$, we must have $G \subseteq \{-1,0,4,5\}$, and since $G \in \mathfrak{F}_H \cup \{\{0\}\}$, we know that $0 \in G$ and $-1 \notin G$. Also $5 \notin G$ because, by the way $\mathfrak{F}_H$ is defined, $5 \in G$ only if $1 \in G$. Therefore, the only two possibilities for G are $G = \{0\}$ and $G = \{0,4\}$. If $G = \{0\}$, then $F \setminus (x+G) = \{0,5,6\}$, and if $G = \{0,4\}$, then $F \setminus (x+G) = \{0,6\}$. In either case we have $H(6) \not\subseteq F \setminus (x+G)$ and so

$$F \setminus (x+G) \notin \mathfrak{F}_H \cup \{\{0\}\} \cup \{\phi\}.$$

Therefore, $\mathfrak{F}_H$ is not trans-3.

11.5. Summary

The following chart summarizes the relationships discussed in this chapter. No transformation property is known which is weaker than the trans-2 property but stronger than the Schur inequality.



BIBLIOGRAPHY

1. Cheo, Luther. On the density of sets of Gaussian integers. American Mathematical Monthly 58:618-620. 1951.

2. Dyson, F. A theorem on the densities of sets of integers. Journal of the London Mathematical Society 20:8-14. 1945.

3. Freedman, Allen R. A general theory of density in additive number theory. Ph.D. thesis. Corvallis, Oregon State University, 1965. 125 numb. leaves.

4. ________________ Some generalizations in additive number theory. Journal für die reine und angewandte Mathematik 235:1-19. 1969.

5. ________________ Unpublished research on the transformation properties of Freedman. Burnaby, Simon Fraser University, Mathematics Department, 1966.

6. Garrison, Betty Kvarda. A nontransformation proof of Mann's density theorem. Journal für die reine und angewandte Mathematik (In press).

7. Kasch, Friedrich. Wesentliche Komponenten bei Gitterpunktmengen. Journal für die reine und angewandte Mathematik 197:208-215. 1957.

8. Kvarda, Betty. On densities of sets of lattice points. Pacific Journal of Mathematics 13:611-615. 1963.

9. Landau, Edmund. Die Goldbachsche Vermutung und der Schnirelmannsche Satz. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, Fachgruppe II, 1930, p. 255-276.

10. Mann, Henry B. A proof of the fundamental theorem on the density of sums of sets of positive integers. Annals of Mathematics, Ser. 2, 43:523-527. 1942.

11. ________________ On the number of integers in the sum of two sets of positive integers. Pacific Journal of Mathematics 1:249-253. 1951.

12. Olson, Arthur E. New theorems and examples for Freedman's density spaces. M.A. thesis. Corvallis, Oregon State University, 1967. 70 numb. leaves.

13. Schnirelmann, L. Über additive Eigenschaften von Zahlen. Annales d'Institut polytechnique, Novočerkarsk 14:3-28. 1930.

14. ______________ Über additive Eigenschaften von Zahlen. Mathematische Annalen 107:649-690. 1933.

15. Schur, I. Über den Begriff den Dichte in der additiven Zahlentheorie. Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klass, 1936, p. 269-297.

16. Stalley, Robert D. On the fundamental metric theorems of additive number theory. Journal of Number Theory (In press).