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The central mathematical problem of this thesis is to find an effective method for solving a nonlinear Fredholm integral equation of the first kind. The primary objective in this problem is to obtain the kinetic temperature $T(h)$, as a function of altitude of the Earth's atmosphere, given that we have a discrete set of radiant energy measurements, the oxygen absorption-emission law, and the radiative transfer equation.

From basic physical principles, the differential equation of radiative transfer is derived. By inverting this differential equation and making some simplifying assumptions about our atmosphere, we arrive at the nonlinear Fredholm integral equation. By leaving the integrand in its nonlinear form, we are not able, by known analytic or numerical methods, to solve for the unknown temperature in the integrand. One method of overcoming this difficulty is to make two

simplifying assumptions--one concerning our atmosphere, and the other concerning the oxygen absorption coefficient α_ν. By assuming an exponential form for $\alpha_\nu(h)$ that satisfies boundary layer conditions, and by dividing the atmosphere up into small stratified layers subject to the condition of constant lapse rate, we linearize the integrand With the integrand thus linearized, we apply the method of least squares which helps to circumvent instability problems. A polynomial representation, with coefficients to be determined, is chosen for the temperature $T(h)$. These coefficients $\{t_n\}$ are solved from a set of matrix equations that are a result of certain known boundary conditions and the least squares method. With the values of the $\{t_n\}$ coefficients, we set up an iteration scheme by using an equation that relates pressure and temperature for any adjacent pair of atmospheric layers. New corrections are then made to the set of matrix equations and absorption coefficient values. We continue this process of solving and recorrecting the $\{t_n\}$ values until we converge on the solution, $T(h)$.

A summary statement appraising the whole iteration scheme cannot be made at this point since a computer is necessary in the final calculations and has not been available.

We have ignored statistical deviations in $T_b(\nu)$ (brightness temperature) and $T(h)$. Thus, unwanted oscillations in $T(h)$ might result since the Fredholm integral, being nonlinear, is usually quite unstable and sensitive to errors. If statistical errors should exceed a given threshold, depending on the kernel, then we must "extract" or "smooth" out these statistical errors; otherwise, in using the iteration scheme, we could not converge on the solution T(h). At this point, it is not known if statistical errors are significant or not.

From the chain of simplifying assumptions leading to our Fredholm integral, a price is paid on accuracy desired. With a computer, an error analysis on the iteration scheme might yield some useful information on cumulative error effect and error limits. All error sources contribute a residual of error. These need investigating in order to establish error threshold limits, otherwise, unwanted oscillations could dominate the solution.

A Nonlinear Fredholm Integral Equation as Encountered in Radiation Through an Atmosphere

by

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The central mathematical problem is to find an effective method in solving a nonlinear Fredholm integral equation of the first kind. The primary objective in this problem is to obtain the unknown quantity, T(h) kinetic temperature, as a function of altitude, of the earth's atmosphere, given that we have a discrete set of radiant energy measurements, the oxygen absorption-emission law, and the radiative transfer equation. Thus, we are dealing with the equation,

$$T_b(\nu) = \int_0^{H_\nu} T(h)K(\nu, h, T(h))dh, \qquad (1.1)$$

where

$$K(\nu, h, T(h)) = \alpha_\nu(h, T(h)) \cdot \exp\left[-\int_0^h \alpha_\nu(h', T(h'))dh'\right], \qquad (1.2)$$

is the kernel expressed as a product of the oxygen absorption coefficient α_ν, and the atmospheric attenuation factor, $\exp\left[-\int_0^h \alpha_\nu dh'\right]$. Brightness temperature $T_b(\nu)$, a function of frequency ν, is a known measured quantity which we define as intensity of radiant energy divided by $2k\frac{\nu^2}{c^2}$. That is,

$$T_b(\nu) = I_\nu / 2k \frac{\nu^2}{c^2} , \qquad (1.3)$$

where k is Boltzmann's constant, the speed of light c, and frequency ν.

T(h), occupying two places in the integrand, is our unknown. The functional form of the kernel is known. We say this since the absorption coefficient α_ν is known and derivable from quantum mechanics. If the integrand of Equation (1. 1) is linear, we may solve for the unknown T(h), but this is not so with the integrand in its present form. Under appropriate assumptions, we may linearize the integrand which means T(h) may be extracted from K, thus permitting Equation (1. 1) to be solvable by known numerical methods.

We also consider what experimental conditions are necessary in order to simplify our atmospheric model which will permit a simplifying mathematical model that is not too unwieldly and will at least permit a "numerical methods" solution, and stay within the frame work of existing laboratory equipment.

II. A PHYSICAL DESCRIPTION OF THE EXPERIMENT

Measurements will be taken on "brightness" temperature and kinetic temperature both on the ground and at the outer atmospheric limits. This will require an airplane or satellite as well as the antenna and various electronic equipment. The ground observer will have his antenna pointing vertically upward while the airplane's antenna will be pointing vertically downward as measurements are being taken at the same geographic point and time. The reason for antenna orientation being vertical (either down or up), is that this will eliminate the added complication of the cosine factor in the differential equation of radiative transfer. No generality in theory is lost by doing this. By dividing radiant intensity I_ν by $2k \frac{\nu^2}{c^2}$, the instruments may be calibrated to read intensity in terms of "brightness" temperature in degrees Kelvin. A brightness temperature measurement is observed and stored for twelve to thirteen different frequencies. By taking the $T_b(\nu)$ measurements in the frequency interval of 50 to 56 GHz (Gigahertz), we are able to construct a non-degenerate system of equations from the nonlinear integral Equation (1.1). Then, by using the method of least squares, we may solve for $T(h)$.

The antenna, being a parabolic disc, is a highly directionalized passive receiver of radiant energy from its surroundings. The antenna is highly directionalized in the sense that the radiometer attached to it will respond only to energy being funneled into a very narrow cylindrical beam whose axis concides with the antenna's axis. Since no antenna can be ideally designed, there will exist low energy side lobes which means the radiometer will register a reading on all in-coming off-axis energy. This un-wanted feature is minimized greatly by choosing the parabolic shape for the disc. An additional design feature is incorporated into the antenna by providing a shielding for the small energy side lobes so that we may ignore the in-coming off-axis energy with negligible error.

III. INTRODUCTION AND HISTORICAL BACKGROUND

Before coming to grips with the central problem, an introduction and historical background will be given. Also, the subsidiary problems that arise when dealing with nonlinear integral equations have to be discussed, and fundamental definitions and relationships clearly stated.

The problem of specifying the radiation field in an atmosphere which scatters light in accordance with well-defined physical laws originated in Lord Rayleigh's investigations in 1871 on the illumination and polarization of the sunlit sky. But the fundamental equations governing Rayleigh's particular problem had to wait 75 years for their formulation and solution. However, the subject was given a fresh start under more tractable conditions, when Arthur Schuster formulated in 1905 a problem in Radiative Transfer in an attempt to explain the appearance of absorption and emission lines in stellar spectra, and Karl Schwarzschild introduced in 1906 the concept of radiative equilibrium in stellar atmospheres. Since that time the subject of Radiative Transfer has been investigated principally by astrophysists, though in recent years the subject has attracted the attention of nuclear physists also, since essentially the same problems arise in the theory of the diffusion of neutrons [1, 2].

Meteorologists and physicists at The Boeing Airplane Company for the last seven years have expressed a marked interest in 'Transfer Theory' problems in the field of Atmospheric Sciences under the division of Applied Physics, Nuclear Weapons Environment. Research and environmental testing is still being carried on both at the Boeing Scientific Research Laboratories, and at the Environmental Sciences Laboratories, National Bureau of Standards, Boulder, Colorado. Because of its role as a supplier of aerospace and aircraft systems, Boeing is interested in encouraging such atmospheric research studies. Successful demonstration of the radiometer method with eventual application to weather satellites for obtaining global pressure topography would have far-reaching civil and military significance. Accurate determination of pressure topography and surface pressure particularly applies to military problems of reentry vehicle inertial fusing and burst height and yield determination. Also, by having accurate and instantaneous atmospheric information, an important step preceding nuclear testings would be realized. The laboratories at Boulder, Colorado express their interests as being purely theoretic and oriented toward basic research. Their contributions exceed those of Boeing, especially on new mathematical methods of solving a non-linear Fredholm integral equation. They have carried the analysis to the point where statistical errors are considered in the variables on brightness temperature and kinetic temperature. One of the main by-products derived from Boulder's activities is that they are able to establish general atmospheric temperature profiles, and from these, predict weather trends.

IV. RADIATION THROUGH AN ATMOSPHERE

Fundamental Concepts, Definitions, and Relationships

1. Specific Radiant Intensity, I_ν

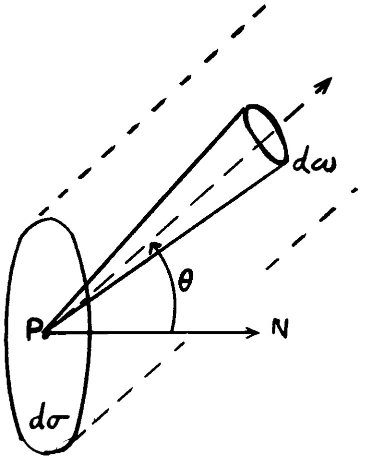


Figure 1.
Radiation Through $d\sigma$ For Arbitrary θ

Let P be a fixed point and let L be a fixed line through P (see Figure 1). Let $d\sigma$ be a small element of area containing P and let θ be the angle between L and the normal N to $d\sigma$. Through each point P' of $d\sigma$ draw a line parallel to L and, with this line as axis and P' as vertex, construct an elementary cone of solid angle $d\omega$. The aggregate of all such cones defines a semi-infinite truncated cone with $d\sigma$ as the finite end [1, 2].

Radiation is being transmitted through $d\sigma$. Let the amount of energy in the frequency interval $(\nu, \nu + d\nu)$, which is transmitted through $d\sigma$ in time dt in directions lying within the truncated cone, be dE_ν. Then it is found that, for radiation fields occurring in nature, the ratio

$$\frac{dE_\nu}{\cos\theta \, d\sigma d\omega d\nu dt} \qquad (4.1)$$

tends to a definite limit as dt, $d\omega$, $d\sigma$, $d\nu \to 0$ in any manner. This limit is essentially positive and it is a function of P and L. It is independent of the angle θ. We denote it by I_ν and call it the specific intensity of radiation at P in the direction L or, more briefly, the intensity at P along L. With this definition, if I_ν is given, the amount of energy flowing through an element $d\sigma$, in a specified frequency interval $(\nu, \nu + d\nu)$, in a direction making an angle θ with the normal to $d\sigma$, within an elementary solid angle $d\omega$, in time dt, is

$$dE_\nu = I_\nu \cos\theta \, d\sigma d\omega d\nu dt. \qquad (4.2)$$

The appearance of $\cos\theta$ in the definition of I_ν deserves a word of explanation. It is clear that the amount of energy flowing through the truncated cone, defined by $d\sigma$ and $d\omega$ at P, will grow smaller in the same field as the normal N moves towards the position of the perpendicular to L through P, i.e., as $d\sigma$ rotates towards the position parallel to L. What is really fundamental is not the surface element $d\sigma$ but its projection, $d\sigma' = d\sigma \cos\theta$, on the plane through P normal to L. It follows that the intensity I_ν can also be defined as the energy flowing at the point P in the direction L, per unit time, of frequency interval, of solid angle, and of surface area normal to L.

2. The Absorption Coefficient, κ_ν

The interaction between radiation and matter is usually expressed in terms of an absorption coefficient and an emission coefficient [1, 2].

Definitions of these fundamental quantities can be somewhat clarified by borrowing from hydrodynamics the distinction between 'the Lagrangian point of view' and 'the Eulerian point of view'. In the Lagrangian point of view, the movement of individual particles is followed. Here such particles are the photons of the radiation field. In the Eulerian point of view, local variations in the field itself are considered, without any reference to the individual history of each particle.

The 'transfer theory', as described in terms of absorption and emission coefficients, makes exclusive use of the Eulerian point of view, in as much as we never follow the evolution of single photons (as one does, more or less, in the study of the interaction between radiation and matter in quantum mechanics). Nevertheless, it is extremely convenient, in transfer problems, to use the concept of 'Lagrangian particles' with a more loose and general meaning than in hydrodynamics, namely that of a group of particles, the movement and history of which are followed as a whole. This is in opposition to the Eulerian point of view, in which no distinction is made between different groups of particles in the description of the field.

Let us consider an element of matter, of volume dV and mass dm, in the radiation field. Then two kinds of photons, 'incident' and 'emergent', can be distinguished. The incident photons are those which enter into the volume dV, and the emergent ones are those which come out of dV. This distinction is essentially Eulerian since, if several groups of photons enter dV, we do not (unless the contrary is explicitly specified) distinguish between the different groups emerging from dV.

The situation becomes a little more complicated as soon as we try to distinguish, not only between incident and emergent photons as described by the scheme of Figure 2, but, more precisely, between those photons which are incident and emergent in a given direction. Indeed, if we neglect for a moment complications arising from the fact that most of our photons refer to a given frequency interval $d\nu$, a given time interval dt, and so on, it is obvious that photons emergent along the path $\vec{\ell}'$ (Figure 3) are not necessarily those which are incident along $\vec{\ell}$ (in the same direction as $\vec{\ell}'$), even in the loose Lagrangian sense defined above.

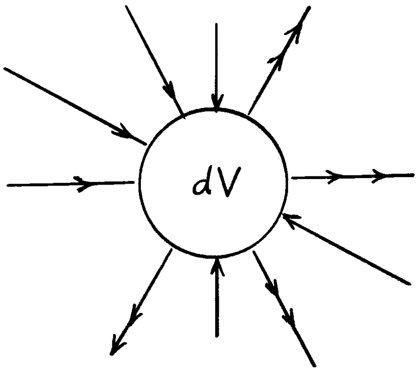
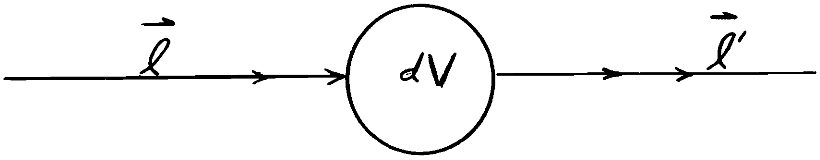


Figure 2.
Incident-Emergent
Photons in dV



Lagrangian Residue in Specific Direction, $\vec{\ell}$

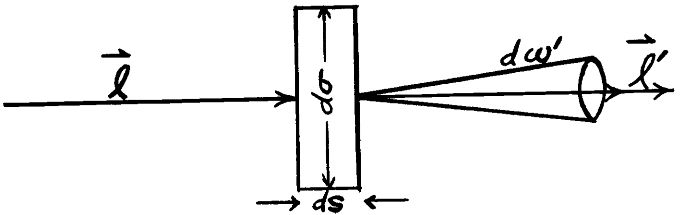
Figure 3.

For not only may some of the $\vec{\ell}$ photons be truly absorbed, i. e., annihilated as photons of frequency ν by interaction with the matter of dV, or scattered in directions other than $\vec{\ell}'$, but, reciprocally, the $\vec{\ell}'$ photons may contain photons truly emitted, i. e., created as photons of frequency ν by the matter of dV, or they may contain photons coming from directions other than $\vec{\ell}$ which have been scattered into the direction $\vec{\ell}'$.

It is, however, obvious that the $\vec{\ell}'$ photons will usually contain a large number of $\vec{\ell}$ photons, made up of those which dm (i. e., those which have 'passed through the mesh of the net'), and of those $\vec{\ell}$ photons which, after different kinds of interaction, are nevertheless re-emitted as $\vec{\ell}'$ photons. We shall call all of these the <u>Lagrangian residue</u> of $\vec{\ell}$ photons, and the difference between the number of the incident $\vec{\ell}$ photons and the residue of these in the emergent $\vec{\ell}'$ photons can then be called the <u>Lagrangian loss</u> of $\vec{\ell}$ photons.

It is important to stress that the <u>absorption coefficient</u>, which we are about to define, does not refer in any way to a relation between all the $\vec{\ell}'$ photons and all the $\vec{\ell}$ photons (Figure 3), but only to the Lagrangian loss of $\vec{\ell}$ photons.

We imagine a (rather difficult) physical experiment in which the energy of dE_ν, in the frequency interval $(\nu, \nu + d\nu)$, is incident normally on a slab of surface area $d\sigma$ and thickness ds in a solid angle $d\omega$ and in the time dt. The element of matter does not receive any other incident radiation and it is isolated, so that it does not receive back any of the radiation truly emitted or scattered outside the solid angle $d\omega'$ (equal numerically to $d\omega$) around $\vec{\ell}'$ (Figure 4). Usually only a certain fraction of dE_ν will be received in the solid angle $d\omega'$, in the frequency interval $d\nu$ and during the time interval dt.



Lagrangian Loss of $\vec{\ell}$ Photons on the element, $\rho \cdot d\sigma \cdot ds$

Figure 4.

We can avoid second order differentials and simplify our notation, if we describe this experiment in terms of the intensity defined by Equation (4.2). The intensity I_ν in the direction $\vec{\ell}$ becomes $I_\nu + \delta I_\nu$ along $\vec{\ell}'$ by interaction between the radiation and the matter dm contained in the volume $d\sigma ds$. Physical experiment and theory show that δI_ν is proportional to the density ρ of the matter, to the thickness ds of the slab and to the incident intensity I_ν. We put

$$\delta I_\nu = -\kappa_\nu \rho I_\nu ds \tag{4.3}$$

and we call the quantity κ_ν thus introduced the mass absorption coefficient for radiation of frequency ν. This quantity allows us to specify, in a real radiation field, the Lagrangian loss of a given incident pencil of radiation. It is generally assumed that κ_ν is independent of the direction $\vec{\ell}$ of the incident beam.

3. The Emission Coefficient, j_ν

Let us return to the scheme of Figure 3. Besides the Lagrangian residue of $\vec{\ell}$ photons, the $\vec{\ell}'$ photons contain some which are truly emitted, i.e., created as photons of frequency ν by the matter in dV, and some which have come from directions other than $\vec{\ell}$ and which have been scattered into the solid angle $d\omega'$ around $\vec{\ell}'$. It is usual to designate as photons emitted by the element dV, all the photons in the $\vec{\ell}'$ pencil other than the Lagrangian residue of $\vec{\ell}$ photons. Thus both the photons created in dm and the photons scattered by dm are considered as emitted photons. These are specified by an emission coefficient j_ν.

The emission coefficient j_ν represents the energy emitted, in the sense explained above, per unit frequency interval near the frequency ν, per unit solid angle, per unit interval of time and per unit mass, in a given direction $\vec{\ell}\,'$ [1, 2].

In other words, an element of mass dm sends in directions confined to an elementary solid angle $d\omega'$, in the frequency interval $(\nu, \nu + d\nu)$ and in time dt, an amount of radiant energy equal to

$$j_\nu \, dm d\nu dt d\omega'$$

and composed of that truly emitted by (created in) dm and that scattered into $d\omega'$ (from all directions) by dm.

4. The Source Function, F_ν

The ratio of the emission coefficient to the absorption coefficient at a given point of the radiation field is an important quantity defined as the source function. Thus, by definition, (from [1]),

$$F_\nu = j_\nu / \kappa_\nu = \text{source function}. \tag{4.4}$$

5. Isotropy and Homogeneity

A radiation field is said to be isotropic at a point if the intensity is independent of direction at that point. If the intensity is the same at all points and in all directions, the radiation field is said to be homogeneous and isotropic [1].

6. Local Thermodynamic Equilibrium

Consider an evacuated enclosure whose walls are maintained at a uniform temperature T, and suppose that the walls are capable of absorbing and emitting radiation of all frequencies. When the radiation within the enclosure has reached a steady state, it is of the kind known as black body radiation of temperature T. The intensity of black radiation of frequency ν and temperature T will be denoted by $B_\nu(T)$.

7. Radiative Equilibrium

The atmosphere is supposed to be in strict radiative equilibrium, i. e., it is assumed that the heat interchanged by convection and conduction is negligible compared with that interchanged by radiation. Moreover, no nuclear energy is being liberated. Then the total radiant energy incident on an element of volume is equal to the total energy emergent from the same element of volume.

8. Kirchoff's Laws

Now consider an enclosure containing different media in which discrete particles of matter may be embedded. Suppose that the walls are again maintained at the temperature T and that the radiation within the enclosure is in a steady state. Then the following laws, established by Kirchoff, hold for radiation of frequency ν in such an enclosure:

(a) The radiation is everywhere isotropic and the intensity is the same throughout any one medium, where we have our atmosphere stratified into layers.

(b) If I_ν is the intensity at any point of a medium whose refractive index is μ_ν, then I_ν/μ_ν^2 is constant throughout the enclosure.

(c) If κ_ν and j_ν are the coefficients of absorption and emission at a point where the intensity is I_ν, then $j_\nu/\kappa_\nu = I_\nu$.

(d) The value of $j_\nu/(\mu_\nu^2\kappa_\nu)$, which is constant throughout the enclosure, is the same for any two enclosures at the same temperature. It is equal to the intensity $B_\nu(T)$ of black body radiation of frequency ν and temperature T.

By the last law, (d), we see that the coefficient of emission j_ν of any matter in the enclosure is connected with its coefficient of absorption κ_ν by the relation

$$j_\nu = \mu_\nu^2\kappa_\nu B_\nu(T). \tag{4.6}$$

It is the content of the above equation which is called Kirchoff's Law.

Usually it can be assumed that $\mu_\nu = 1$, though recent work in radio-astronomy has been dealing with media for which, in radio frequencies μ_ν varies between 0 and 1. In our case, for the atmospheric media and for the narrow microwave frequency interval of 50 to 56 GHz (1 Gigahertz = 10^9 cycles per sec.), $\mu_\nu = 1$.

The explicit form for the Planck function $B_\nu(T)$ is derived in quantum statistics, and this theory predicts the expression,

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}, \tag{4.7}$$

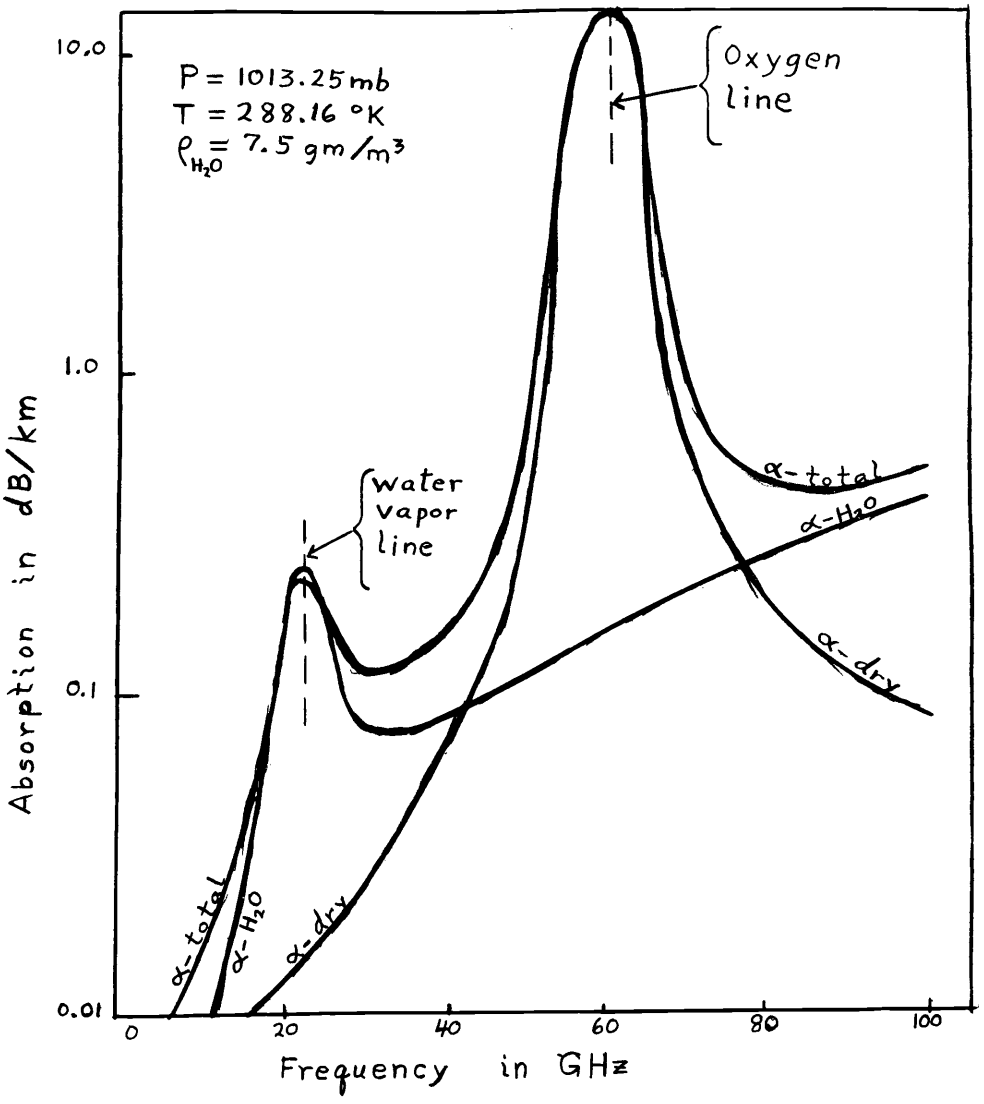
where h is Planck's constant, k is Boltzmann's constant, ν the frequency, c the speed of light and T the temperature [12, 13].

V. PHYSICAL PROPERTIES OF EARTH'S ATMOSPHERE

Absorption, scattering, and emission takes place as the Earth's atmosphere is subjected to radiant energy of all frequencies. The radiant energy is equivalent to a photon field emanating from some primary energy source such as the sun. The primary atmospheric constituents causing absorption, scattering, and emission are the oxygen molecules, solid particles, rain drops snow, nitrogen, etc. We will simplify this phenomena if we conduct the experiment under the conditions of a sunny, cloudless day in a geographic region that has a predominantly dry atmosphere, and we keep our $T_b(\nu)$ readings within the 50 to 56 GHz interval. Our analysis is simplified since our differential equation of radiative transfer is simplified. This is done by reducing Radiative Transfer Theory (for this experiment) down to two separate processes of oxygen molecular absorption and emission--the other atmospheric constituents, we may ignore [6]. Scattering may be ignored by imposing several restrictions, namely, the frequency interval that is chosen, and no snow, rain, water vapor, or smog. We need not be bothered with polarization effects because of the frequency interval chosen [7]. No polarization effects will mean an isotropic medium. Also, local thermodynamic equilibrium will permit us to equate the source function, j_ν/κ_ν to the Planck function, $B_\nu(T)$. The ionosphere is highly polarized by the Earth's magnetic field, but this effect may be ignored since energy in the 50 to 56 GHz range passes through unpolarized.

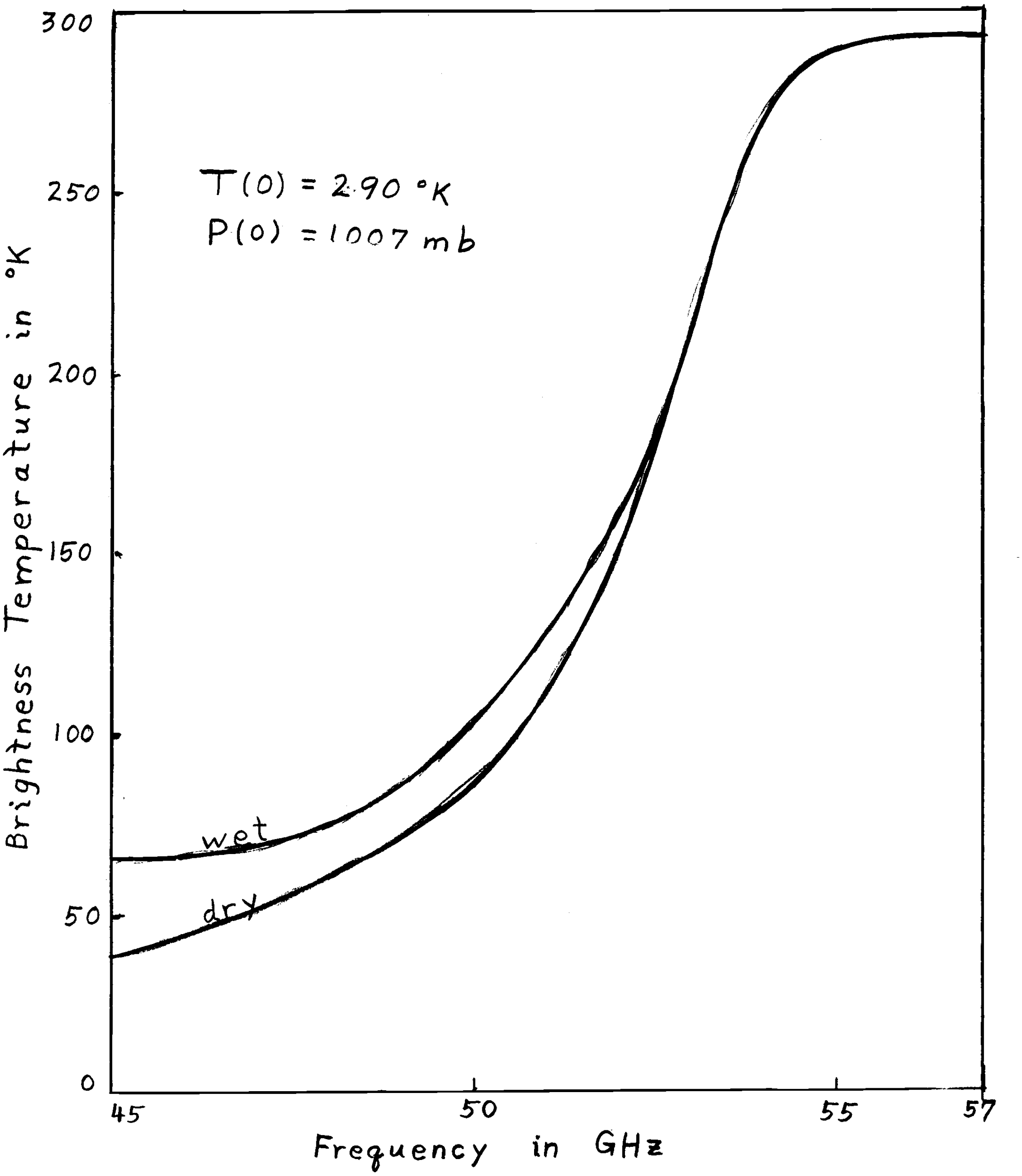
In the frequency range (10-100 GHz), the primary atmospheric gaseous absorbers and, hence emitters of radiation, are oxygen and uncondensed water vapor [9]. Water vapor absorption is due to a pure rotational transition at 22. 2 GHz and non-resonant contributions of other rotational lines starting around 180 GHz and extending well into infra-red. Oxygen absorption is due to a band of rotational transitions starting at about 53 GHz and extending to about 66 GHz. Figure 5 shows the calculated wet, dry, and total absorption as a function of frequency. It is evident from this figure that oxygen absorption dominates over that of water, except in the immediate region of the water vapor line [9].

For frequencies in the vicinity of the oxygen complex, and for low antenna elevation angles, the total integrated absorption (optical depth) is so large that the atmosphere is essentially a black body radiating at a temperature nearly equal to that at the Earth's surface. For frequencies near the center of the oxygen band, the atmosphere is again a black body radiating at nearly the surface temperature for all elevation angles. In the frequency region from 50-56 GHz and for a vertical antenna, however, a large change in brightness temperature with frequency occurs, as seen in Figure 6. Over this small frequency interval, a brightness temperature change of about 150°K



Microwave Absorption Coefficients at Standard Surface Conditions

Figure 5



Brightness Temperature vs Frequency for a Vertical Antenna

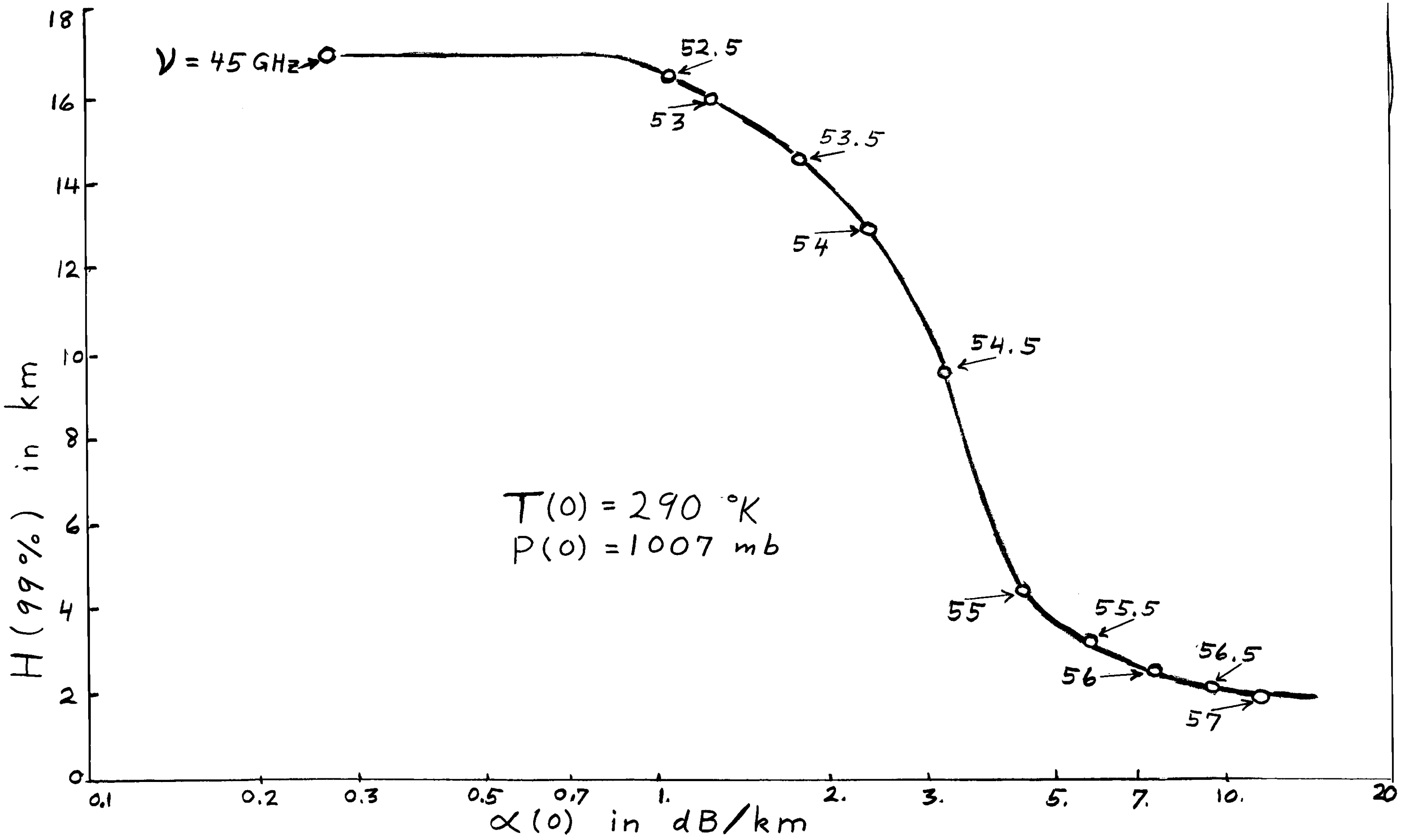
Figure 6

takes place and in the steepest portion of the curve, the slope is about 60°K per GHz.

It is also observed for the same 50-56 frequency interval that if the height above which the brightness temperature contribution may be neglected (99% level) is plotted against the surface absorption coefficient, as presented in Figure 7, the layers of the atmosphere which contribute most strongly to the brightness temperature at a given frequency can be determined [6]. In this figure it is evident that the relative contribution from various altitudes is strongly frequency dependent between 52.5 and 55 GHz. For frequencies below 52 GHz, the radiation contribution is averaged over a fairly large height interval (approximately 17 km); above 55 GHz through the center of the oxygen band, the contributions to the brightness temperature are from a narrow height interval close to the ground.

The large slope of the $T_b(\nu)$ versus ν curve and the pronounced changes of the radiation cutoff height, as pointed out above, arise because of the large variations in both the absorption coefficient and the optical depth.

It should also be pointed out that both the radiation cutoff height and, the brightness temperature, depend on the manner in which the line width parameter $\Delta\nu$ varies with temperature and pressure [6, 7, 8, 9, 10].



Height Above Which Thermal Noise May be Neglected (99% level)
vs Surface Absorption Coefficient

Figure 7

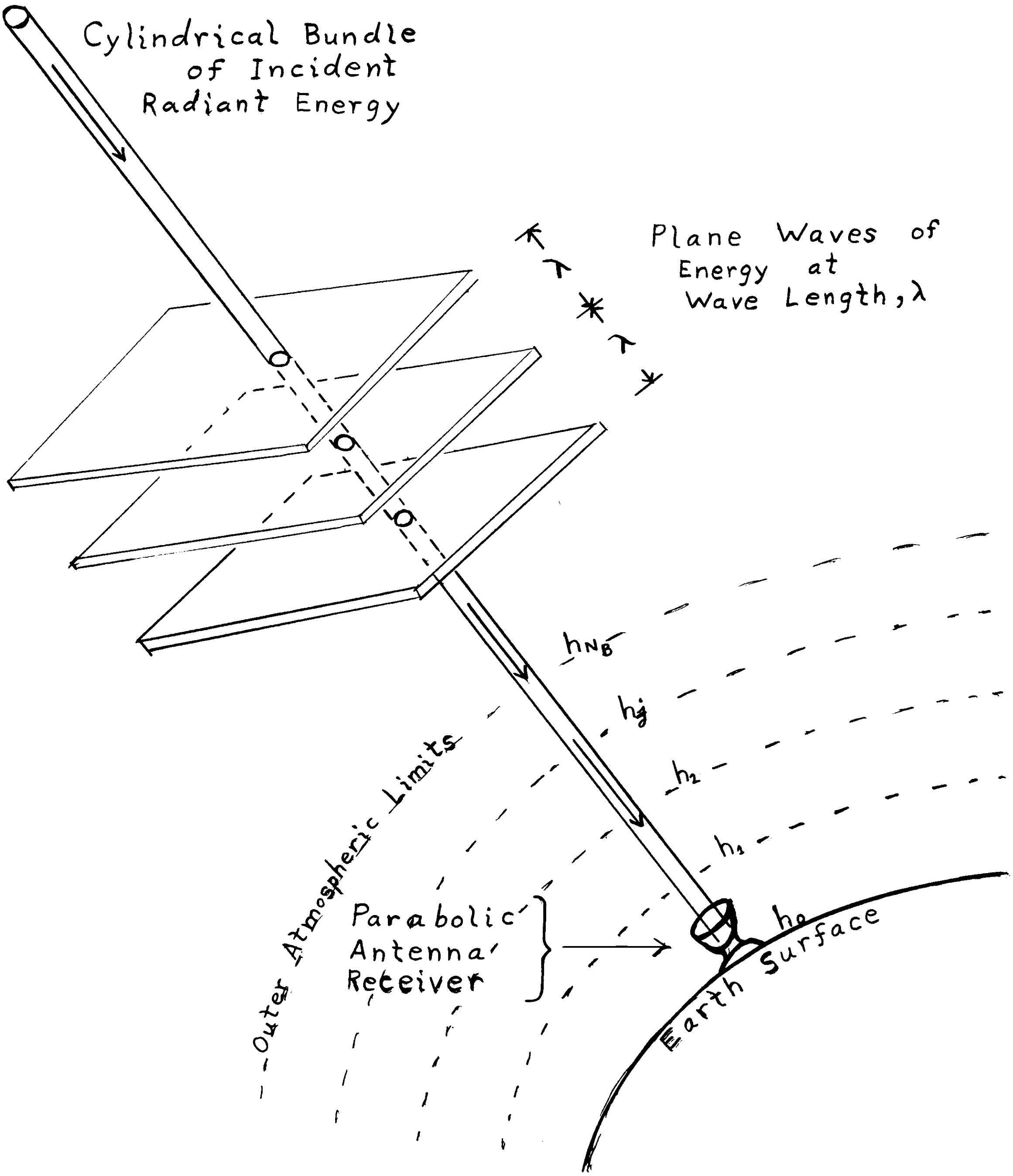
VI. A NONLINEAR INTEGRAL EQUATION OF FREDHOLM TYPE AND FIRST KIND

From the previous chapter, Radiative Transfer Theory was simplified to two separate processes of oxygen molecular absorption and emission as we consider energy passing through a long narrow vertical cone into our antenna (see Figure 8). With the basic concepts and definitions laid down, we may show the derivation of the Radiative Transfer equation.

Let us consider the long vertical beam slicing through an infinitesimal slab of atmosphere of thickness dh. As energy enters at face A and leaves at face B, a difference in intensity per unit change in path length is observed (see Figure 9). This difference in energy comes from an excess of emission over absorption. As photons enter face A, a small fraction of the incident bundle is absorbed. The amount absorbed is proportional to the incident intensity on face A, and is

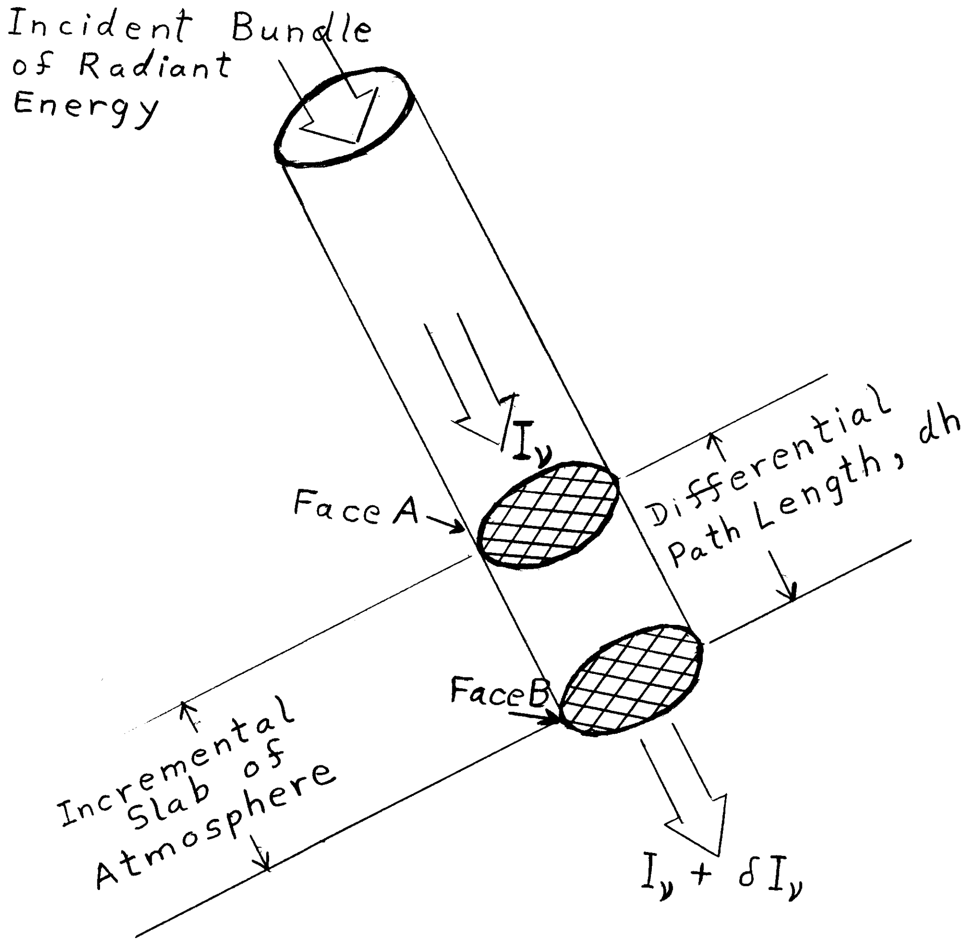
$$\kappa_\nu \rho I_\nu .$$

Those photons that do interact, raise the molecular energy levels of oxygen. The cylinder of molecules in their "excited" state will emit energy at the same frequency ν that the incident photons coming in have, assuming local thermodynamic equilibrium. Also,



Radiant Energy as Seen and Measured by a Ground Based Antenna

Figure 8



I_ν, Intensity at frequency, ν
δI_ν, Intensity differential at a specific frequency, ν

Differential Element of Radiative Transfer

Figure 9

photons from other directions are scattered into our cylindrical beam and exit in the dh direction. Thus, these two processes are lumped together as that portion emitted by dm, and is equal to

$$j_\nu \rho .$$

Then the gain over the loss is equal to the difference in energy transferred across the incremental cylinder in the dh direction. That is,

$$\frac{dI_\nu}{dh} = -\kappa_\nu \rho I_\nu + j_\nu \rho \, , \tag{6.1}$$

where the minus sign denotes that amount absorbed or taken away. We define the linear absorption α_ν, in the dh direction, as

$$\alpha_\nu = \kappa_\nu \rho . \tag{6.2}$$

Also,

$$j_\nu = \kappa_\nu B_\nu(T), \tag{6.3}$$

from Kirchoff's law. Therefore, Equation (6.1) can be written as,

$$\frac{dI_\nu}{dh} = -\alpha_\nu I_\nu + (\kappa_\nu B_\nu(T)) \frac{\alpha_\nu}{\kappa_\nu} \, ,$$

or

$$\frac{dI_\nu}{dh} = -\alpha_\nu I_\nu + \alpha_\nu B_\nu(T). \tag{6.4}$$

Equation (6.4) is a simple linear first order differential equation that may be readily transformed into an equivalent integral equation by finding the appropriate integration factor which may be found in any differential equations text. Thus, the integral form of Equation (6.4) evaluated at the ground is

$$I_\nu = I_{\nu 0}\, e^{-\int_0^\infty \alpha_\nu(h')dh'} + \int_0^\infty B_\nu(h)\alpha_\nu(h)\, e^{-\int_0^h \alpha_\nu(h')dh'}\, dh, \qquad (6.5)$$

where $I_{\nu 0}$ is the unattenuated intensity of discrete sources lying outside the Earth's atmosphere. The first term represents contributions to the received energy from external noise sources attenuated by the intervening medium. The second term in (6.5) is interpreted physically by considering an incremental element along the antenna path emitting energy toward earth at a rate equaling $B_\nu \alpha_\nu dh$, and this emitted energy is, in turn, attenuated by the intervening medium by the factor $\exp\left[-\int_0^h \alpha_\nu(h')dh'\right]$. The total contribution is then obtained by integrating (or summing) all contributions along the ray path. The absorption coefficient α_ν is an explicit function of temperature and pressure while temperature and pressure are, in turn, functions of position along the ray path, thus making the absorption an implicit function of position. It is this dependence only which had been indicated in Equation (6.5). Similarly, the Planck function B_ν, which is an explicit function of temperature alone, can be written as a function of position. The intensity I_ν, received at ground level, thus depends on the meteorological profile at every point on the ray path.

Two conditions, along with one experimental fact, permit a simplification of Equation (6.5) into "Fredholm" form. We also use the definition,

$$I_\nu = 2k \frac{\nu^2}{c^2} T_b(\nu), \tag{6.6}$$

thus allowing all terms in (6.5) to be expressed in degrees Kelvin upon dividing through by $2k \frac{\nu^2}{c^2}$. Those conditions necessary for simplification are:

(1) By selecting the microwave frequency interval, $\nu_{50} \leq \nu \leq \nu_{56}$, we have $h_\nu \ll kT$.

(2) The selection of a set $\{H_\nu\}$ of finite upper limits or cut-off heights from a standard model atmosphere such that any energy contributions above H_ν are negligible and beyond the sensitivity of measurement. That is,

$$T_b(\nu)\Big|_{h=\infty} - T_b(\nu)\Big|_{h=H_\nu} < \frac{1}{2}\delta,$$

where δ is the instrument sensitivity.

The experimental observation is that the first term in (6.5) may be dropped since its measured value is beyond instrument sensitivity. That is,

$$I_{\nu 0} \exp[-\int_0^{\infty} a_\nu(h')dh'] < \delta .$$

From condition (1), if $h\nu \ll kT$, we may expand $B_\nu(T)$ in a MacLaurin series, and drop terms of second order or higher. Thus, we can say,

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT}-1} \cong 2k \frac{\nu^2}{c^2} T(h) . \qquad (6.7)$$

Using (6.7) and (6.6), Equation (6.5) then becomes,

$$T_b(\nu) = T_{b0}(\nu)e^{-\int_0^{\infty} a_\nu(h')dh'} + \int_0^{\infty} T(h)a_\nu(h, T(h))\, e^{-\int_0^{h} a_\nu(h', T(h'))dh'} dh \qquad (6.8)$$

The first term in (6.8) may be dropped since

$$T_{b0}(\nu)\cdot\exp\left[-\int_0^{\infty} a_\nu(h')dh'\right] < \frac{1}{2}\delta .$$

Thus, we may write,

$$T_b(\nu) \cong \int_0^{\infty} T(h)a_\nu(h, T(h))\, e^{-\int_0^{h} a_\nu(h', T(h'))dh'} dh, \qquad (6.9)$$

where $0 \leq h' \leq h \leq \infty$. The second condition allows us to replace the infinite upper limits on the integral with finite limits, and there will be a discrete set of these cut-off heights $\{H_\nu\}$ --one H_ν for each frequency ν. Thus we can say,

$$T_b(\nu) - \int_0^{H_\nu} T(h)\alpha_\nu(h, T(h))\, e^{-\int_0^h \alpha_\nu(h', T(h')dh'} dh < \frac{1}{2}\delta. \qquad (6.10)$$

Consequently, keeping in mind the approximations and assumptions made, we may write,

$$T_b(\nu) = \int_0^{H_\nu} T(h)K(\nu, h, T(h))dh, \qquad (6.11)$$

where the kernel is

$$K(\nu, h, T(h)) = \alpha(\nu, h, T(h))\cdot\exp\left[-\int_0^h \alpha(\nu, h', T(h'))dh'\right]. \qquad (6.12)$$

Again, we say $T_b(\nu)$ is the known measurable parameter, $T(h)$ is unknown, and the kernel K will be known if we can legitimately extract from it the unknown $T(h)$. Also, $T(h)$, $T_b(\nu)$, and K are well behaved-smooth continuous functions and bounded within the interval $[0, H_\nu]$ for $\nu_{45} \leq \nu \leq \nu_{60}$.

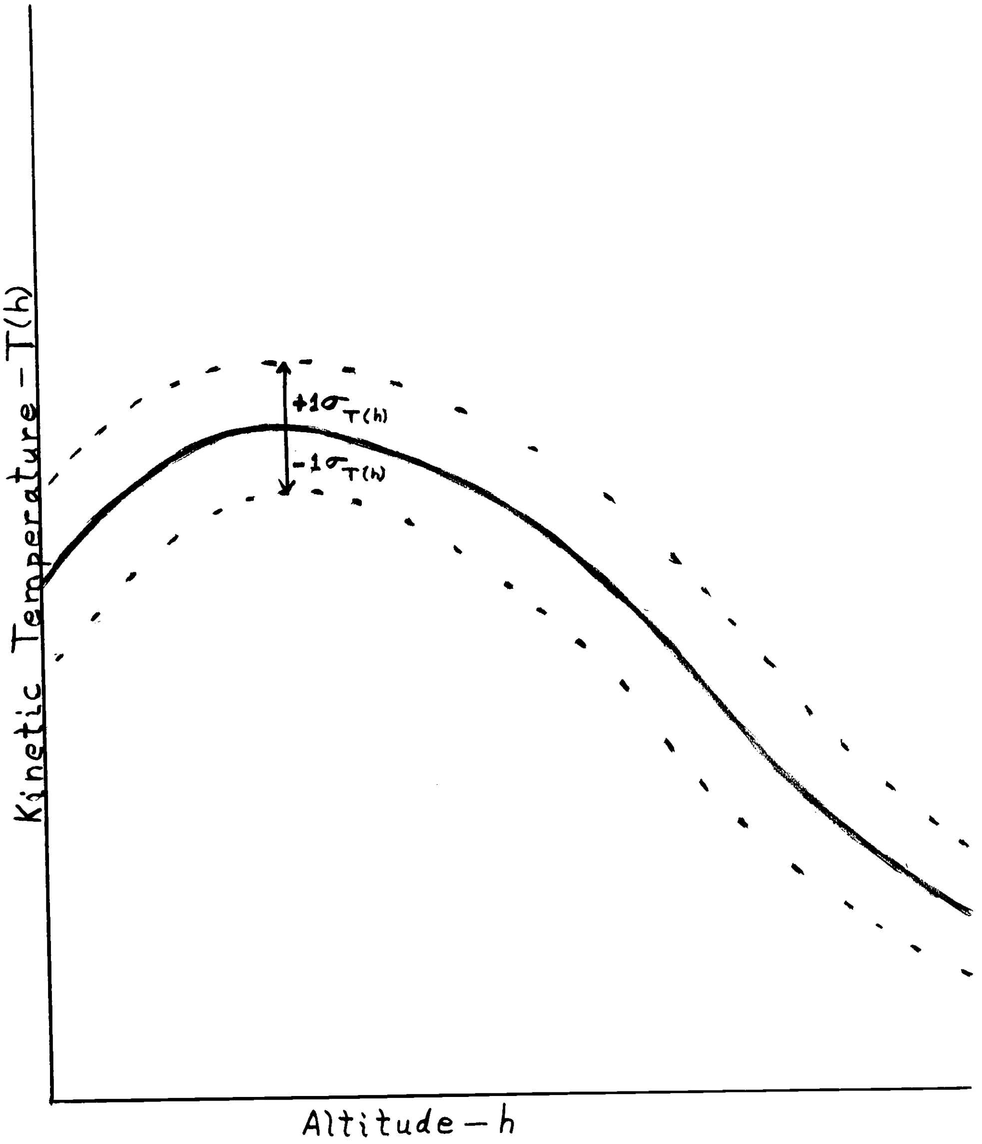
With the above derivation leading to our nonlinear Fredholm integral equation, we come to the central mathematical problem stated at the beginning, namely, presenting an effective method of solution to Equation (6.11). The central physical objective, again is to obtain the kinetic temperature $T(h)$. The following two chapters discuss the approach and the ensuing difficulties.

VII. LINEARIZATION OF THE INTEGRAL EQUATION

Before any inversion technique can be applied to Equation (6.11), the nonlinear property must be solved. We must extract the unknown T(h) from the kernel, and yet leave it comparatively unmolested. The newly linearized kernel must not have unreasonable errors, and still present a realistic model for the entire atmospheric profile. Indeed, this can be done by dividing the atmosphere into small stratified layers where any layer at a given height above ground is governed by the condition of constant lapse rate, where

$$\ell_{j-1,j}(\text{lapse rate}) = \frac{T(h_j)-T(h_{j-1})}{h_j-h_{j-1}} = \frac{\Delta_j T(\text{degrees}\,{}^\circ\text{K})}{\Delta_j h(\text{km})} = \text{constant}. \tag{7.1}$$

By means independent of our experiment and measurements, we obtain the set of numbers, $\{\ell_{j-1,j};\ j = 1, 2, \ldots, N_B\}$ for all layers of the atmosphere. We do this by gathering all data on the atmosphere's temperature profile at different geographical locations for the past five or ten years. From the gatherings, we can establish a "mean" or "model" temperature profile with statistical and error limits drawn on either side of the profile curve. See Figure 10. From the "mean" profile curve, measure off a set of lapse rate numbers subject to the condition,



Computed Kinetic Temperature
vs Altitude

Figure 10

$$\frac{\delta \ell_{j-1,j}}{\ell_{j-1,j}} \leq 0.01. \tag{7.2}$$

Then for the conditions of stratified layers--constant lapse rates, we can assume an exponential variation with height for the absorption coefficient, and be quite accurate. Thus, holding ν temporarily constant, we assume,

$$\alpha_\nu(h) = \alpha_\nu(h_{i-1}) e^{-\frac{h-h_{i-1}}{H_i}}, \quad (h_{i-1} \leq h \leq h_i), \tag{7.3}$$

$$i = 0, 1, 2, \ldots, N_B,$$

and $h_0 = 0$, $h_{N_B} = H_\nu$; where, now the absorption, $\alpha_\nu(h)$ is strictly a function of height. The constant, H_i is determined for each adjacent pair of boundary layers,

$$\frac{1}{H_i} = -\frac{\ln\left(\frac{\alpha_\nu(h_i)}{\alpha_\nu(h_{i-1})}\right)}{h_i - h_{i-1}}, \tag{7.4}$$

where the coefficient, $\alpha_\nu(h_{i-1})$ in (7.3), also a constant, must be evaluated at each boundary interface. From basic quantum mechanical principles, Van Vleck derived an explicit expression for $\alpha_\nu(h_{i-1})$ in terms of temperature T, pressure P, frequency ν, and line width $\Delta\nu$ [10]. We can, therefore, write $\alpha_\nu(h_{i-1})$ as an implicit function of height h_{i-1} since $\Delta\nu$ is a function of P and T, which are, in turn, functions of height. Thus,

$$a_\nu(h_{i-1}) = a(\nu, P, T, \Delta\nu), \tag{7.5}$$

and

$$a(\nu, P, T, \Delta\nu) = \frac{c_1 P \nu^2}{T^3} \sum_N S_N e^{-E_N/kT}, \tag{7.6}$$

where

$$S_N = F_{N+}\mu_{N+}^2 + F_{N-}\mu_{N-}^2 + F_0\mu_{N_0}^2, \tag{7.7}$$

with

$$F_{N\pm} = \frac{\Delta\nu}{(\nu_{N\pm}-\nu)^2+(\Delta\nu)^2} + \frac{\Delta\nu}{(\nu_{N\pm}+\nu)^2+(\Delta\nu)^2}, \tag{7.8}$$

$$F_0 = \frac{\Delta\nu}{\nu^2+(\Delta\nu)^2}, \tag{7.9}$$

$$\mu_{N+}^2 = \frac{N(2N+3)}{N+1}, \tag{7.10}$$

$$\mu_{N-}^2 = \frac{(N+1)(2N-1)}{N}, \tag{7.11}$$

and

$$\mu_{N_0}^2 = \frac{2(N^2+N+1)(2N+1)}{N(N+1)}. \tag{7.12}$$

The exponent in the Boltzmann factor of Equation (7.6) is,

$$E_N/kT = (2.06844)N(N+1)/T. \tag{7.13}$$

And for the normal concentration of O_2 in air, the constant C_1 is defined as

$C_1 = 2.6742$, (for $\alpha_\nu(h_{i-1})$ in decibels per kilometer, or

$C_1 = 0.61576$, (for $\alpha_\nu(h_{i-1})$ in nepers per kilometer).

Finally, the empirical formula for the line width parameter, $\Delta\nu$ is,

$$\Delta\nu(P,T) = \alpha P[0.21 + 0.78\beta](300/T)^{0.85}, \tag{7.14}$$

where α and β are constants which take on certain values for different height intervals in the atmosphere. Thus,

$\alpha = 1.95$ Mc/s (mm of Hg)$^{-1}$, and

$\beta = 0.25$ for $h < 8$ km, and $P > 267$ mm of Hg.

$$\beta = 0.25 + (0.5)\frac{(h-H_1)}{(H_2-H_1)} \text{ for } H_1 \leq h \leq H_2,$$

where $H_1 = 8$ km, $H_2 = 25$ km

$\beta = 0.75$ for $h > H_2 = 25$ km, and $P < 19$ mm of Hg.

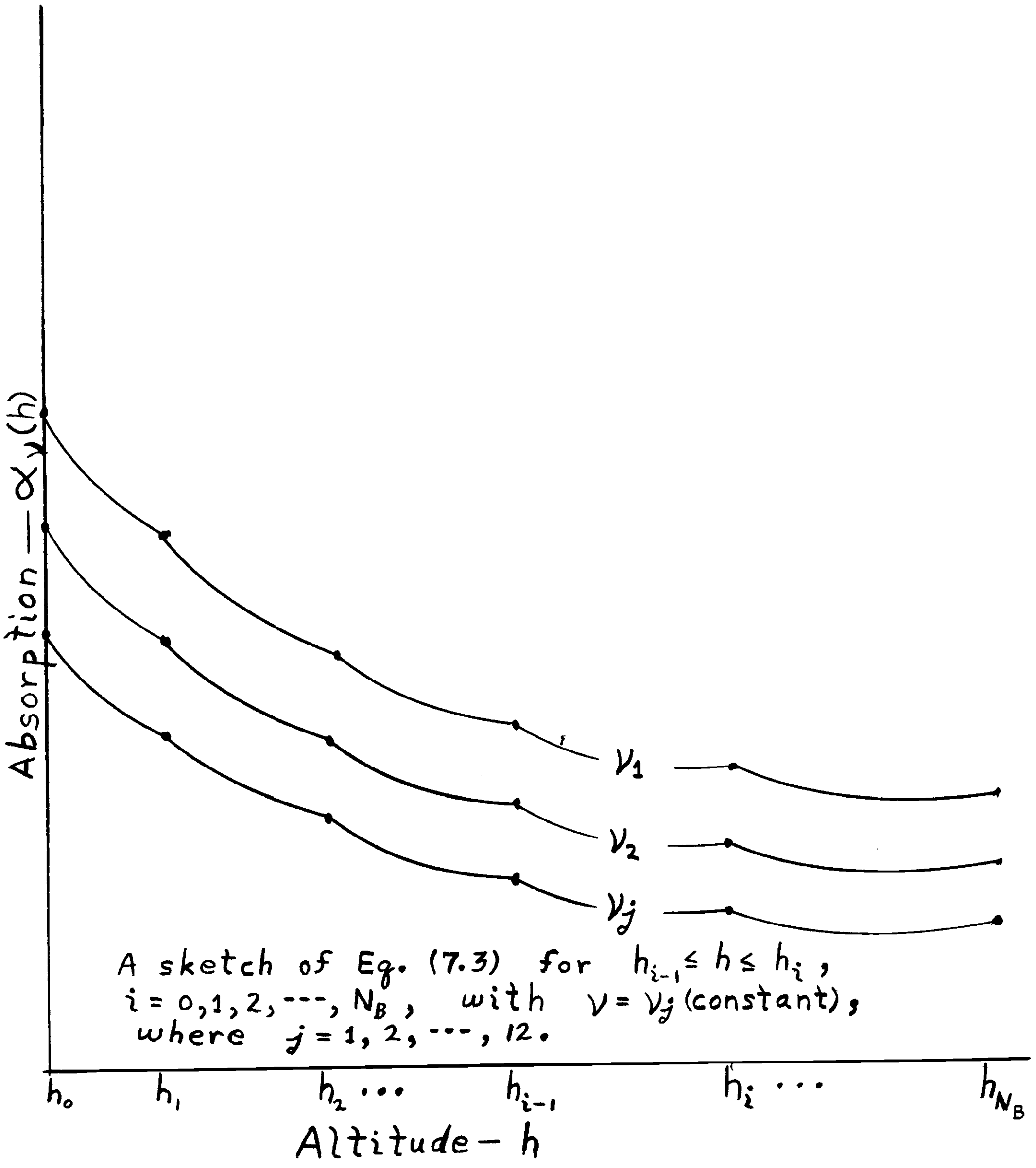
Now, a few more words need to be said about the remaining terms in Equation (7.6) and (7.7). E_N is the energy of the N-th state, k is Boltzmann's constant, $\mu_{N\pm}$ is the magnetic dipole moment of the N-th state, $\nu_{N\pm}$ is the resonant frequency of the N-th state, and N is a generic label for a particular type of rotational transition of the oxygen molecule. In evaluating Equation (7.6), we have to sum over all permissible rotational transition energy states. Thus, N can assume only odd values (from 1 to 45) because of the Pauli exclusion principle. We can now evaluate the coefficient, $\alpha_{\nu}(h_{i-1})$ in Equation (7.3) by the use of (7.6) with all of its terms defined. Thus, we have linearized the Fredholm integral, (6.11) by extracting the unknown, $T(h)$ from the kernel, $K(\nu, h, T(h))$. The absorption coefficient curve, represented by (7.3) for the entire stratified atmosphere, is a sequence of broken, connected line segments. The entire curve is piecewise smooth, continuous, and bounded for $(0 \leq h \leq H_{N_B})$, although the slope, $d\alpha_{\nu}(h)/dh$ is discontinuous at the boundary layers. See Figure 11.

After linearizing Equation (6.11), the use of an inversion technique leads to a treatment of a linear Fredholm integral. We state the problem by writing,

$$\int_a^b K(x, y)f(y)dy = g(x) + \epsilon(x), \quad a \leq x \leq b, \tag{7.15}$$

where $g(x)$, and $K(x, y)$ are known, $f(y)$ unknown. K, g, and f are smooth, continuous, and bounded for $a \leq x \leq b$. $\epsilon(x)$ is an arbitrary function, representing the measurement error in $g(x)$, with the restriction.

$$|\epsilon(x)| \leq M, \quad a \leq x \leq b,$$



A sketch of Eq. (7.3) for $h_{i-1} \le h \le h_i$, $i = 0, 1, 2, \cdots, N_B$, with $\nu = \nu_j$ (constant), where $j = 1, 2, \cdots, 12$.

Computed Absorption
vs Altitude

Figure 11

where M, depending on the shape characteristics of the kernel K, is an upper bound on the maximum allowable error. If $|\epsilon(x)| > M$, oscillations will plague the solution, $f(y)$, [3, 4, 5, 11].

The error function, $\epsilon(x)$ brings up the next difficulty--instability in the solution, $f(y)$. This is best illustrated by the following elegant argument given by Phillips [4]. Let $f(y)$ be the solution to

$$\int_a^b K(x, y)f(y)dy = g(x), \tag{7.16}$$

and add to it the function $f_m(y) = \sin(my)$. Thus,

$$\int_a^b K(x, y)[f(y)+\sin(my)]dy = g(x) + \epsilon(x), \tag{7.17}$$

where

$$\epsilon(x) = \int_a^b K(x, y)\sin(my)dy. \tag{7.18}$$

Now, for any integrable kernel $K(x, y)$, it is a well known theorem (Dirichlet's integral theorem) that,

$$g_m \equiv \int_a^b K(x, y)\sin(my)dy \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \tag{7.19}$$

Hence, only an infinitesimal change, g_m in g causes a large, finite change, f_m in f. Such oscillatory behavior can destroy all physical meaning of the solution, f if the threshold error in g is exceeded. Also, one would expect that $g_m \rightarrow 0$ as $m \rightarrow \infty$ faster for flat smooth kernels than for sharply peaked kernels. Indeed if $K(x, y)$ were the Dirac-delta function, $K(x, y) = \delta(x-y)$, then $g_m = f_m$ would not approach zero. Hence, we conclude that the success in solving Equation (7.15) depends to a large extent on the accuracy of $g(x)$ and the shape of $K(x, y)$.

Instability (or pronounced oscillations) in the solution, $f(y)$ still faces us if we resort to solving the integral Equation (7.15) by substituting a system of equations generated by the method of numerical quadratures and solved by matrix inversion since the original integral, (6.11) is very sensitive to measurement errors. Thus, numerical quadratures with matrix inversion is abandoned since $|\epsilon(x)| > M$.

VIII. THE SOLUTION BY NUMERICAL METHODS

By turning to the method of least squares, instability can be overcome [3, 8]. It is this method that allows convergence to the solution f(y), (or T(h) in (6.11)) by the setting up of an iteration scheme. We now develop and bring together the equations and bits of information that go into the chain of the iteration scheme [9].

Upon getting the absorption coefficient in the form of Equation (7.3), we can write (6.11) in the form,

$$T_b(\nu) = \int_0^{H_\nu} T(h)\,\mathcal{K}(\nu, h)dh, \tag{8.1}$$

since T(h) is extractable from the original kernel. The temperature T(h) can now be determined by inverting (8.1).

A polynomial expansion is chosen for T(h), i.e.,

$$T(h) = \sum_{n=0}^{N} t_n U_n(h), \tag{8.2}$$

where $\{t_n\}$ is a set of coefficients to be determined and $\{U_n(h)\}$ is some given set of basis functions. Substituting (8.2) into (8.1), we get,

$$T_b(\nu) = \sum_{n=0}^{N} t_n K(\nu, n), \tag{8.3}$$

where

$$K(\nu, n) = \int_0^{H_\nu} U_n(h)\alpha_\nu(h)\, e^{-\int_0^h \alpha_\nu(h')dh'} dh. \tag{8.4}$$

We delay the development of (8.4) to present, first the iteration scheme. Our first step is to determine $\{t_n\}$ and the Lagrangian multipliers $\{\gamma_k\}$ by minimizing,

$$Q = \sum_{j=1}^{N_\nu}\left[T_b(\nu_j) - \sum_{n=0}^{N} t_n K(\nu_j, n)\right]^2 + 2\sum_{k=N+1}^{N+N_b} \gamma_k\left(\sum_{n=0}^{N} t_n U_n(h_k) - T(h_k)\right), \tag{8.5}$$

or in the matrix notation,

$$Q(t, \gamma) = [T_b(\nu) - K^* t]^* [T_b(\nu) - K^* t] + 2\gamma^*[U^* t - T], \tag{8.6}$$

where * **means transpose.**

K, U, $T_b(\nu)$, t, γ, T are matrices, the last four being column matrices. $K(\nu_j, n)$ is the element of K in the n-th row and j-th column, where $n = 0, 1, \ldots, N$; $j = 1, 2, \ldots, N_\nu$. $U_n(h_k)$ is the element of U in the n-th row and k-th column, where $n = 0, 1, \ldots, N$; $k = N+1, \ldots, N+N_b$. The column matrices are,

$$T_b(\nu) = [T_b(\nu_1), T_b(\nu_2), \ldots, T_b(\nu_{N_\nu})]^*,$$

$$t = [t_0, t_1, \ldots, t_N]^*,$$

$$\gamma = [\gamma_{N+1}, \ldots, \gamma_{N+N_b}]^*,$$

$$T = [T(h_{N+1}), \ldots, T(h_{N+N_b})]^*.$$

We minimize Q by differentiating with respect to t_n and γ_k. The conditions,

$$\frac{\partial Q}{\partial t_n} = 0, \quad n = 0, 1, \ldots, N,$$

and

$$\frac{\partial Q}{\partial \gamma_k} = 0, \quad k = N+1, \ldots, N+N_b,$$

can be used to show that the following respective matrix equations hold:

$$KK^* t + U\gamma = KT_b(\nu),$$

and

$$U^* t - T = 0.$$

Then, in terms of partitioned matrices, we may write,

$$\begin{bmatrix} KK^* & U \\ U^* & 0 \end{bmatrix} \begin{bmatrix} t \\ \gamma \end{bmatrix} = \begin{bmatrix} KT_b(\nu) \\ T \end{bmatrix}. \qquad (8.7)$$

Upon solving for t and γ, we have,

$$\begin{bmatrix} t \\ \\ \gamma \end{bmatrix} = \begin{bmatrix} KK^* & U \\ \\ U^* & 0 \end{bmatrix}^{-1} \begin{bmatrix} KT_b(\nu) \\ \\ T \end{bmatrix} . \qquad (8.8)$$

With $\{t_n\}$ and $\{\gamma_{N+1}, \gamma_{N+2}\}$ determined, the computed temperature $T(h)$ is determined. We select,

$$\{U_n(h) = h^n\}, \quad n = 0, 1, \ldots, N, \qquad (8.9)$$

as our set of basis functions. Then we use Equation (8. 2) in the iteration scheme in the following manner. From the newly derived temperature curve,

$$T(h) = \sum_{n=0}^{N} t_n h^n , \qquad (8.10)$$

measure off a new set of lapse rate numbers, $\{\ell_{i-1,i}; \ i = 1, 2, \ldots, N_B\}$ subject to the conditions,

$$\ell_{i-1,i} = \text{constant}, \quad i = 1, 2, \ldots, N_B, \qquad (8.11)$$

with

$$\frac{\delta(\ell_{i-1,i})}{\ell_{i-1,i}} \leq 0.01 . \qquad (8.12)$$

Then we may determine the corresponding pressures for all layers from the relation (Hewson and Longley, 1944),

$$\left(\frac{P_2}{P_1}\right) = \left(\frac{T_2}{T_1}\right)^{\frac{Mg}{R\ell_{12}}} , \qquad (8.13)$$

where M is the molecular weight of air, g the acceleration of gravity, R the universal gas constant, and ℓ_{12} the lapse rate between layers 1 and 2. Thus, from (8.10), (8.13) and the set, $\{\ell_{i=1,i}\}$, we establish a set of $\{P_j, T_j;\ j = 0, 1, 2, \ldots, N_B\}$ which, in turn, are used to determine a new corrected set of coefficients, $\{\alpha_\nu(h_{i-1})\}$. With this corrected set, the exponential model (7.3) is corrected for all layers and boundaries of the model atmosphere. Thus, a new K matrix is calculated where we take into account the P and T dependence and corrections of the absorption coefficients and line widths $\Delta\nu$. Equation (8.8) is again used to determine t by utilizing the new K. The iteration process is then continued until a self-consistent set of temperature coefficients is obtained. By self-consistency, we mean that neither K nor t change from the N-th to the (N+1)-th iteration.

We return to finish the development of Equation (8.4). The quantity of interest, $K(\nu, n)$ may be expressed as a linear combination of functions of the form,

$$K(\nu, n) = \int_0^{H_\nu} h^n \alpha_\nu(h)\, e^{-\int_0^h \alpha_\nu(h')dh'}\, dh, \qquad (8.14)$$

where $K(\nu, n)$ is a typical element of the K matrix. The linear combination, expressing $K(\nu, n)$, is a result of dividing the atmosphere into layers. Now, we wish to evaluate (8.14) under the assumption that $a_\nu(h)$ has the exponential form of (7.3). Also, we choose (8.9) for our set of basis functions. Then we can write an equivalent expression for (8.14),

$$K(\nu, n) = \sum_{j=1}^{N_B} \int_{h_{j-1}}^{h_j} h^n a_\nu(h_{j-1})\, e^{-\frac{h-h_{j-1}}{H_j}}\, e^{-\Lambda_\nu(i,h)}\, dh, \tag{8.15}$$

where

$$\Lambda_\nu(i, h) = \int_0^h a_\nu(h_{i-1})\, e^{-\frac{h'-h_{i-1}}{H_i}}\, dh'.$$

Now the exponent, $\int_0^h a_\nu(h')dh'$ of the attenuation factor in (8.15) is easily evaluated. Thus, we may write,

$$\int_0^h a_\nu(h')dh' = \int_{h_0}^{h_1} a_\nu(h')dh' + \int_{h_1}^{h_2} a_\nu(h')dh' + \dots$$
$$+ \int_{h_{j-2}}^{h_{j-1}} a_\nu(h')dh' + \int_{h_{j-1}}^{h} a_\nu(h_{j-1})e^{-\frac{h'-h_{j-1}}{H_j}}\, dh',$$

or

$$\int_0^h \alpha_\nu(h')dh' = \sum_{i=1}^{j-1} \alpha_\nu(h_{i-1})H_i\left(1-e^{-\frac{h_i-h_{i-1}}{H_i}}\right)$$

$$+\ \alpha_\nu(h_{j-1})H_j\left(1-e^{-\frac{h-h_{j-1}}{H_j}}\right), \qquad (8.16)$$

where the integration for the i-th layer is the following,

$$\int_{h_{i-1}}^{h_i} \alpha_\nu(h')dh' = \alpha_\nu(h_{i-1}) \int_{h_{i-1}}^{h_i} e^{-\frac{h'-h_{i-1}}{H_i}} dh',$$

or

$$\int_{h_{i-1}}^{h_i} \alpha_\nu(h')dh' = \alpha_\nu(h_{i-1})\, e^{\frac{h_{i-1}}{H_i}} (-H_i) \int_{h_{i-1}}^{h_i} e^{-\frac{h'}{H_i}} \left(\frac{-1}{H_i}\right)dh',$$

$$= \alpha_\nu(h_{i-1})\, e^{\frac{h_{i-1}}{H_i}} (-H_i) \left[e^{-\frac{h'}{H_i}} \right]_{h_{i-1}}^{h_i}$$

$$= \alpha_\nu(h_{i-1})H_i\left(1-e^{-\frac{h_i-h_{i-1}}{H_i}}\right). \qquad (8.17)$$

Therefore, (8.16) has the form,

$$\int_0^h \alpha_\nu(h')dh' = \tau_{j-1} + \alpha_\nu(h_{j-1})H_j\left(1-e^{-\frac{h-h_{j-1}}{H_j}}\right), \qquad (8.18)$$

where

$$\tau_{j-1} = \sum_{i=1}^{j-1} \alpha_\nu(h_{i-1}) H_i \left(1 - e^{-\frac{h_i - h_{i-1}}{H_i}}\right) . \tag{8.19}$$

Coming back to Equation (8.15), we see that $K(\nu, n)$ has the form,

$$K(\nu, n) = \sum_{j=1}^{N_B} \int_{h_{j-1}}^{h_j} h^n \alpha_\nu(h_{j-1}) \, e^{-\frac{h-h_{j-1}}{H_j}} \, e^{-[\tau_{j-1} + \xi_\nu(j, h)]} dh, \tag{8.20}$$

where

$$\xi_\nu(j, h) = \alpha_\nu(h_{j-1}) H_j \left(1 - e^{-\frac{h-h_{j-1}}{H_j}}\right) . \tag{8.21}$$

And after regrouping,

$$K(\nu, n) = \sum_{j=1}^{N_B} e^{-[\tau_{j-1} + \alpha_\nu(h_{j-1}) H_j - \frac{h_{j-1}}{H_j}]} \, G(n, j), \tag{8.22}$$

where

$$G(n, j) = \int_{h_{j-1}}^{h_j} h^n \alpha_\nu(h_{j-1}) \, e^{-\frac{h}{H_j}} \, e^{\theta_\nu(j, h)} dh , \tag{8.23}$$

and

$$\theta_\nu(j, h) = \alpha_\nu(h_{j-1}) H_j \, e^{-\frac{h-h_{j-1}}{H_j}} . \tag{8.24}$$

Now the $G(n, j)$ functions cannot be expressed in closed form, but we can express them as an infinite sum of integrals, each of which can be evaluated in closed form. We can define $G(n, j)$ more succinctly by writing,

$$G(n, j) = A \int_{\alpha}^{\beta} h^{n} e^{ah} e^{b e^{ah}} dh, \tag{8.25}$$

where

$$A = a_{\nu}(h_{j-1}), \quad a = \frac{-1}{H_j},$$

$$b = a_{\nu}(h_{j-1}) H_j e^{\frac{h_{j-1}}{H_j}},$$

$$\alpha = h_{j-1}, \quad \beta = h_j .$$

By using a MacLaurin expansion for the second exponential term, $e^{b e^{ah}}$ in (8.25), we have the following:

$$G(n, j) = A \int_{\alpha}^{\beta} h^{n} e^{ah} \left(\sum_{i=0}^{\infty} \frac{b^{i} e^{ahi}}{(i)!} \right) dh,$$

$$= A \int_{\alpha}^{\beta} h^{n} \sum_{i=0}^{\infty} b^{i} \frac{e^{(1+i)ah}}{(i)!} dh,$$

$$= A \sum_{i=0}^{\infty} \frac{b^{i} B_{i,n}}{(i)!}, \tag{8.26}$$

where

$$B_{i,n} = \int_{\alpha}^{\beta} h^n \, e^{(1+i)ah} \, dh. \tag{8.27}$$

$B_{i,n}$ can be integrated, thus,

$$B_{i,n} = \left[\left\{ \sum_{s=0}^{n} (-1)^s \frac{\dfrac{d^{(s)} h^n}{dh^{(s)}}}{[(1+i)a]^{s+1}} \right\} e^{(1+i)ah} \right]_{h=\alpha}^{h=\beta} , \tag{8.28}$$

where $\dfrac{d^{(s)} h^n}{dh^{(s)}}$ has the following meaning:

$$\frac{d^{(0)} h^n}{dh^{(0)}} = h^n, \quad s = 0$$

$$\frac{dh^n}{dh} = nh^{n-1}, \quad s = 1$$

$$\frac{d^2 h^n}{dh^2} = n(n-1)h^{n-2}, \quad s = 2, \quad \text{etc.}$$

Thus, in evaluating the $K(\nu, n)$ elements of K, we are led to the $G(n, j)$ functions, which, in turn, have to be evaluated by a MacLaurin expansion technique. Enough terms must be carried in the series in evaluating $G(n, j)$ such that upon truncating at some point, the error will be less than instrument sensitivity.

IX. CONCLUSIONS

A summary statement appraising the whole iteration scheme cannot be made at this point since the computer is necessary in the final calculations, and has not been available. We have ignored statistical deviations in $T_b(\nu)$ and $T(h)$. Thus, unwanted oscillations in $T(h)$ might result since the Fredholm integral Equation (6.11), being nonlinear, is usually quite unstable and sensitive to errors. If statistical errors should exceed a given threshold, depending on the kernel, then we must "extract" or "smooth" out these statistical errors; otherwise, in using the iteration scheme, we could not converge on the solution $T(h)$. At this point, it is not known if statistical errors are significant or not.

From the simplifying assumptions leading to Equation (6.11), a price is paid on accuracy desired. With a computer, an error analysis on the iteration scheme might yield some useful information on cumulative error effect and error limits. Briefly reviewing, we begin with Equation (6.5). (6.5) was simplified by assuming the Reiley-Jeans approximation $B_\nu(T) \cong 2k \frac{\nu^2}{c^2} T(h)$. Then the first term in (6.8) was dropped, and finite upper limits set on (6.9). The absorption coefficient was simplified to fit an exponential approximation. Measurement error is made in $T_b(\nu)$. $T(h)$ and $T_b(\nu)$ have statistical errors. $\Delta\nu$, being empirical, is subject to error.

Acceleration, g and lapse rate $\ell_{i-1,i}$ are subject to error. The kernel is subject to error after T(h) is extracted. The G(n, j) functions are subject to error when truncated. And lastly, numerical integration is to be used in the final calculations. All of the above error sources contribute a residual of error. We say all this since all cumulative error effects must be kept below a certain threshold, otherwise, unwanted oscillations could dominate the solution.

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