AN ABSTRACT OF THE THESIS OF

Michelle J. Zandieh for the degree of Doctor of Philosophy in Mathematics presented on June 13, 1997. Title: The Evolution of Student Understanding of the Concept of Derivative.

Signature redacted for privacy.

Abstract approved: ______________________________

Thomas P. Dick

This research seeks to answer the question, "What does it mean for a student to understand the concept of derivative?" A structured way to describe an individual student's understanding of derivative is developed and applied to analyzing the evolution of that understanding for each of nine high school seniors during their year-long calculus course. The methodology is a multiple case study. Interviews, including both task-based and open-ended questions, are the primary instruments for collecting data on each student's understanding. Other data collected include tests, written questions, and classroom observations.

Several theoretical frameworks contribute to the research: concept image (Tall, Vinner, and Dreyfus), process-object (Sfard; Dubinsky and colleagues), and notions of multiple representations for function, limit, and derivative. I describe the concept of derivative as three layers of process-objects: the ratio or difference quotient, the limit, and the function layers. Each layer may be observed in multiple contexts: graphical (slope), verbal description (rate of change), kinematic (e.g. velocity or acceleration), and symbolic (the symbolic difference quotient definition of derivative). A description of the connections between the various aspects of the concept of derivative comes from the work of Fischbein on paradigmatic, analogic, and diagrammatic models and the work of Lakoff on metaphor and metonymy.

The major theoretical result of the dissertation is the development of a structured way of describing the concept of derivative including a diagrammatic methodology for displaying which aspects of the derivative concept a student has demonstrated. This methodology may be applied to other studies and other concepts. The major result of this study of nine students is the realization that the layers and representations of the concept of derivative do not appear to be hierarchical in that none of the nine students learn the aspects in the same order.

The Evolution of Student Understanding of the Concept of Derivative

by

Michelle J. Zandieh

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Michelle J. Zandieh, Author

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The Evolution of Student Understanding of the Concept of Derivative

Chapter 1 — Introduction

Two separate but sometimes interrelated lines of thought influenced my decision to study the evolution of a student's understanding of derivative. One was the growing body of research on advanced mathematical thinking, and the other was the discussions of those teachers and curriculum developers in the United States who are part of what is called the calculus reform movement.

The research on advanced mathematical thinking seeks to explore ways in which students learn mathematics at the upper secondary or tertiary levels. The concepts that have been explored most in-depth so far are functions and limits. Some of the concepts of calculus and abstract algebra have been explored as well. The researchers in this area describe models for knowledge or its acquisition that seek to provide insight into how advanced mathematical topics are understood by students or experts. Examples of these models are described in detail in the literature review and theoretical framework chapters of this dissertation. Some of the researchers such as David Tall and Ed Dubinsky have also developed curricula based on their research models.

On the other hand, curriculum development is not always based strictly on research into student understanding. In recent years, there has been a considerable amount of curriculum development for the first year college calculus course. This curriculum development has come from the perception that calculus has been used as a filter to weed out weak students, that calculus courses have come to be about mindless drills, and that perhaps the graphing, numeric, and symbolic capabilities of new technologies can be used to enhance student understanding of calculus by performing "mindless" operations so that students can focus on concepts and by allowing students to more easily see calculus concepts in more than one representation.

One issue that calculus reform proponents and skeptics continue to discuss is whether or not the reform curricula that have been developed actually leave students with a better understanding of calculus concepts, or a more conceptual understanding of calculus concepts, than other curricula. Another issue is the role of symbolic manipulation skills in understanding particular calculus concepts. Implicit in both of these concerns is an unanswered question: "What does it mean to understand a particular calculus concept?"

As stated previously, there is some research on student understanding of function and limit. There is less research on student understanding of derivative or integral. The available research on student understanding of function, limit, and derivative is described in detail in Chapter 2 of this dissertation. I chose to study student understanding of the derivative concept because of the relative lack of research in that area and the importance of this research in terms of examining the purported goals and achievements of the calculus reform movement.

Therefore, the first question that I must answer in my research is "What does it mean to understand the concept of derivative?" Answering this question in a complete, concise, and well-organized way is a necessary foundation for discussions of curriculum and teaching issues as well as for continued research into student understanding. Chapter 3 of this dissertation is my answer to this question.

Having a structured definition for student understanding of derivative is useful for many tasks including teaching, curriculum development, curriculum evaluation, and research. These possibilities are described in the implications sections of Chapter 5. The most basic reason for any of these activities is to improve student learning. Researchers need to describe students' understanding in a way that is useful for making predictions about how students learn. Curriculum developers and teachers need to know how students learn in order to develop meaningful and effective explanations, problems, and activities.

Therefore, it is important to see if this framework for describing student understanding may be used effectively to study how students learn the concept of derivative: in other words, how each student's understanding of derivative evolves.

This research chooses to take an in-depth look at nine students so that each student's understanding may be evaluated by this framework in as much detail as possible. We will use the framework to examine what aspects of a student's understanding of derivative are most prominent, which are more or less easily learned, and what connections a student sees between different parts of his or her understanding.

The students in this study were chosen because of their easy accessibility to the researcher. However, this group has some characteristics that are interesting for this study. The subjects for this study were academically gifted high school seniors. Six of the nine students were National Merit Finalists. Therefore, the difficulties that the students had with the material are likely to be shared by many other students. The students also had remarkably similar academic backgrounds. Eight of the nine students had been in the same mathematics and science classes since their freshman year of high school. Because of this any striking differences in student understanding may be more readily attributed to individual differences in the students and generalizations may be due to their similar backgrounds.

This study uses a multiple case study methodology. Each student is evaluated separately for the evolution of his or her understanding of derivative. Chapter 4 explains the method of data collection and analysis for this study. The principal data for this study is a set of three interviews in each of which the student is asked many different questions to give the student a chance to discuss as many aspects of his or her understanding as possible. Other data collection methods such as additional interview questions, written tests, individual questions given in class, and classroom observations and discussions with the classroom teacher put the data in a context, provide information about the students' abilities on routine skills, and provide a redundancy for comparison with the primary interview data.

Given the structure for the concept of derivative that we will use in our evaluation, we must devise a method for analyzing the data that will be as concise and well-organized as the knowledge structure itself. For the analysis I have defined a set of diagrams that may be used both to summarize the total structure of the concept of derivative and to record what aspects of the concept of derivative are present in a particular student's response or in a written paragraph of text or a test question. These diagrams and their use for analyzing student understanding are presented in Chapter 4.

Chapter 5 provides a summary analysis of the evolution of each of the nine students understanding of derivative. Appendix A provides a more detailed analysis including interview transcripts and responses to written questions. Since our nine students are not a random sample, we cannot make implications from these results for all calculus students. However, in the discussion sections of Chapter 5, we will explore trends in the data that may have implications for teachers, curriculum developers, or researchers. This discussion includes an exploration of whether certain characteristics of a student's understanding are common to most students in our study or highly individualistic, and whether there appears to be a hierarchy in the way that a student's understanding develops.

Chapter 2 — Literature Review

Since the concept of derivative is built using the notions of function and limit, it is appropriate to review the literature on student understanding of function and limit as well as the literature on student understanding of derivative. The present review is not comprehensive. Many of the articles in this review describe student understanding of function, limit, and derivative in the context of one or more of the following theoretical frameworks: multiple representations, concept image and concept definition, and the process-object duality. Multiple representations refers to the notion that functions, limits, and derivatives may be thought of in a variety of representations, particularly graphic, numeric, and symbolic representations. A person's concept image of function, limit, and derivative is everything a person associates with that concept whereas the person's concept definition is made up of the words a person would use to define the concept. The process-object duality refers to the notion that a mathematical concept may be viewed both as a process and as an object. These terms are described more fully in the literature review below.

Understanding the Concept of Function

There is a rich body of research on student understanding of function. Leinhardt, Zaslavsky, and Stein (1990) have reviewed studies on 9-14 year-olds' understanding of functions, graphs, and graphing. There are also less comprehensive reviews covering older students by David Tall (1992) and Patrick Thompson (1995b). In addition, two books have been published with a collection of articles of this topic: *Integrating research on the graphical representation of function*, edited by Romberg, Fennema, and Carpenter (1993), and *The concept of function: Aspects of epistemology and pedagogy*, edited by Harel and Dubinsky (1992).

All the articles in this review use a variety of representations for functions, particularly graphical and algebraic representations. Other representations mentioned include arrow diagrams (Markovits, Eylon, and Bruckheimer, 1986), tables (Janvier, 1987), verbal descriptions (Janvier, 1987; Vinner, 1983; Vinner and Dreyfus, 1989; Breidenbach et al., 1992) and physical models (Monk, 1992).

Janvier (1987) emphasizes the importance of translations between representations. Restricting himself to verbal descriptions of situations, tables, graphs and formulae, he constructed a 4x4 table of the translation processes possible between these representations (see Table 2.1). Janvier reports that processes are best developed in symmetric pairs. For example, translation from a verbal description to a graph is best taught when combined with instruction on translating from the graph to the verbal description.

Table 2.1. Translation Processes (After Janvier, 1987)

To: / From:	Situations, Verbal Descriptions	Tables	Graphs	Formulae
Situations, Verbal Descriptions		Measuring	Sketching	Modeling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

Markovits, Eylon, and Bruckheimer (1986) also describe student understanding of function in terms of multiple representations. For Markovits, et al. understanding of function at the level of 9th grade (14-15 year-old) students consists of three subconcepts: domain, range, and a rule or correspondence. Each may be represented through various representations: arrow diagrams, verbal, graphic, and algebraic representations. This study presented a number of problems that checked for the ability of students to recognize and give examples of functions and nonfunction relations, identify images and preimages, translate between representations of functions, and identify or give examples of functions satisfying given constraints. For all these problems only the graphical and algebraic representations were used.

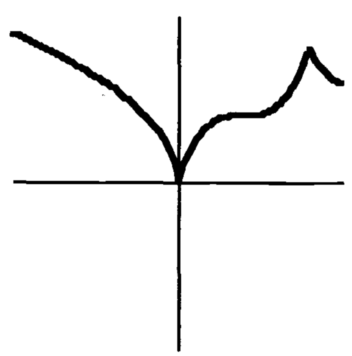
The study concluded that students have particular difficulties with constant functions, piecewise functions, and functions defined as a discrete set of points. Students were found to be weak on issues of domain, range, image and preimage. When asked to give an example of a function, students most often gave linear examples. Students could give a larger variety of examples graphically than algebraically, and could transfer from the algebraic representation to the graphical more easily than the reverse.

Vinner (1983), and Vinner and Dreyfus (1989) discuss understanding of function in terms of the concept image framework. Vinner (1983) describes in detail the notions of concept image and concept definition. A person's *concept image* consists of everything he or she person associates with the concept -- images, symbols, words, examples, properties. A *concept definition* is a relatively brief verbal definition. This may be a definition given by a teacher or textbook that the student has learned or it may be something the student has developed or develops on the spot as a way of describing his or her concept image. Vinner explains that a student will usually refer to his or her concept image instead of his or her concept definition when dealing with the concept. Additionally, that person's concept definition may be unrelated to or even contradict elements of the concept image.

In his 1983 study, Vinner examined the concept definition and concept image of tenth and eleventh graders who had studied functions in some form in ninth grade and had been introduced to the function concept formally in the tenth grade. The textbook referenced by these students in the tenth grade defines function as a correspondence between two sets (a domain and a range) such that every element in the domain has exactly one element in the range that corresponds to it. Vinner examines two aspects of student understanding of function: 1) how well the concept definitions of the students matched the textbook's definition, and 2) whether students with a concept definition that matched the book's had a concept image that did not contradict this definition.

For the study 146 students completed a written questionnaire by answering five questions. For each of the first four questions the students were asked to choose between "yes "and "no" and to explain their choice in words. Only answers with explanations are used for the evaluation. The five questions are listed in Figure 2.1.

1. Is there a function that corresponds to each number different from 0 its square and to 0 it corresponds -1?
2. Is there a function that corresponds 1 to each positive number, corresponds -1 to each negative number, and corresponds 0 to 0?
3. Is there a function that admits integral values for nonintegral numbers and admits nonintegral values for integral numbers?
4. Is there a function the graph of which is the following? [The graph of a nondifferentiable, nowhere linear, continuous function is given.]



5. In your opinion what is a function?

Figure 2.1. Vinner's function questionnaire. (After Vinner, 1983.)

Question 5 asks for the student's concept definition, while the other four questions are aimed at the student's concept image. For the analysis Vinner divided the answers to question 5 into the following four categories.

Category 1 (57% of the students): The textbook definition sometimes mixed with other elements from the concept image.

Category 2 (14%): The function is a rule of correspondence. This definition eliminates the possibility of an arbitrary correspondence.

Category 3 (14%): The function is an algebraic term, a formula, an equation, an arithmetical manipulation, etc.

Category 4 (7%): Some elements in the mental picture are taken as a definition for the concept. This category included graphs, descriptions of graphs, arrow diagrams, and the symbols $y = f(x)$.

Eight percent of the students gave no answer. In addition, the responses of some students may have fit more than one category, but are placed in only one category for the percentages reported above.

Vinner examined whether the students whose concept definitions fit into category 1 (the textbook definition) had answers to the first four questions matching that definition. Only 34% of these 57 students met such criteria. Of the students not in category 1, none answered all of the first 4 questions correctly. Vinner listed six ways in which the concept image of some students did not match their concept definitions (i.e. the textbook's definition) .

1. Some students thought that a function could have only one rule or correspondence for the entire domain.

2. Other students believed that a function could have different correspondences for different domains provided these are intervals, but that it could not have a different correspondence relation at just one point, such as in question 1.

3. Some students believed that if a function is not given by an algebraic expression, then it only exists if mathematicians have a specific name for it, (e.g. the signum function.)

4. Some students thought the graph in question 4 was not a function for a variety of reasons related to its unusual shape. One student complained that there was no "regularity" in the graph. Another stated that all graphs of functions must increase or decrease, but not both.

5. Some students thought that a function must have the property that for each y in the range there is only one x in the domain corresponding to it. Vinner characterizes this as a failure to recall the textbook definition correctly.

6. Some students assume that a function must be a one-to-one correspondence.

A later paper by Vinner and Dreyfus (1989) refines these ideas. Vinner and Dreyfus investigated the concept images and concept definitions of 271 college students and 36 junior high school teachers. In addition to using the notions of concept image and concept definition, they describe the notion of *compartmentalization.* "This phenomenon occurs when a person has two different, potentially conflicting schemes in his or her cognitive structure. Certain situations stimulate one scheme, and other situations stimulate the other" (p. 357). A person may not stimulate the most appropriate scheme in a given situation.

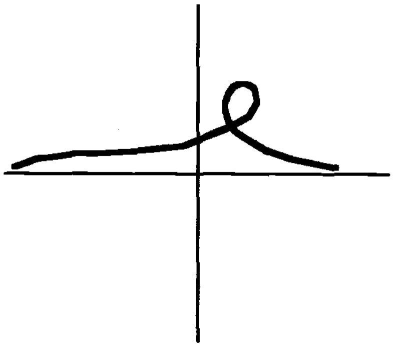
This study investigated the common definitions of the function concept given by this group of respondents, the principal aspects of their concept images that they used in identification and construction problems, and the frequency with which respondents compartmentalized their formal definition of function and their images of the function concept.

A questionnaire with 7 questions was administered to 511 respondents, but only 307 contained enough information for the desired analysis. For each of the first six questions the respondents circled "Yes", "No" or "I do not know" and wrote an explanation for their answers. The questions are listed in Figure 2.2.

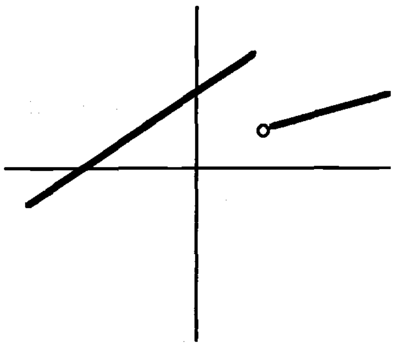
The responses are categorized in a refinement of those categories used by Vinner (1983). The number of respondents, out of 307, in each category has been added in parentheses. Vinner and Dreyfus's (1989) categories are listed below. Note that 43 students did not fall into any of the categories.

Does there exist a function whose graph is:

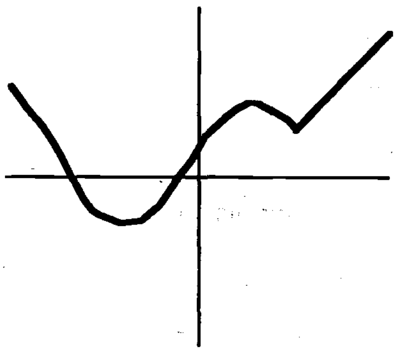
1.



2.



3.



4. Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

5. Does there exist a function whose values are equal to each other?

6. Does there exist a function whose values for integral numbers are non-integral and whose values for non-integral numbers are integral?

7. What is a function in your opinion?

Figure 2.2. Vinner and Dreyfus's function questionnaire. (After Vinner and Dreyfus, 1989.)

I. *Correspondence* (82 respondents): A function is any correspondence between two sets that assigns to every element in the first set exactly one element in the second set (the Dirichlet-Bourbaki definition).

'A correspondence between two sets of elements.'

'For every element in A these is one and only one element in B.'

II. *Dependence Relation* (81): A function is a dependence relation between two variables (y depends on x).

'One factor depending on the other one.'
'A dependence between two variables.'
'A connection between two magnitudes.'

III. *Rule*: A function is a rule (32). A rule is expected to have some regularity, whereas a correspondence may be 'arbitrary'. The domain and the codomain are usually not mentioned here, contrary to Category I, where they are.

'Something that connects the value of x with the value of y.'
'The result of a certain rule applied to a varying number.'
'A relation between x and y is a function.'

IV. *Operation* (14): A function is an operation or a manipulation (one acts on a given number, generally by means of algebraic operations, in order to get its image).

'An operation.'
'An operation done on certain values of x that assigns to every value of x a value of $y = f(x)$.'
'Transmitting values to other values according to certain conditions.'

V. *Formula* (30): A function is a formula, an algebraic expression, or an equation.

'It is an equation expressing a certain relation between two objects.'
'A mathematical expression that gives a connection between two factors.'
'An equation connecting two factors.'

VI. *Representation* (25): The function is identified, in a possibly meaningless way, with one of its graphical or symbolic representations.

'A graph that can be described mathematically.'
'A collection of numbers in a certain order which can be expressed in a graph.'
'$y = f(x)$'
'$y(F) = x$' (p. 360)

Four aspects of the function concept are used frequently by respondents in explaining their answers to questions 1-6. These included:

1. *One-valuedness*: If a correspondence assigns exactly one value to every element in its domain, then it's a function. If not, then it's not a function.

2. *Discontinuity*: The graph has a gap. The correspondence is discontinuous at one point in its domain.

> 3. *Split Domain*: The domain of the correspondence splits into two subdomains, in each of which a different rule of correspondence holds. As a consequence, the graph may change its character from one subdomain to the other.
>
> 4. *Exceptional Point*: There is a point of exception for a given correspondence, that is, a point for which the general rule of correspondence does not hold. (p. 361)

Interestingly, some students used discontinuities, split domain, and exceptional point to explain why something is a function whereas other respondents used the same three characteristics to explain why something is *not* a function. For example, a respondent might say that the graph in question 2 is a function because it is a piecewise defined function, whereas another student might say that it is not a function because it has a different correspondence for different parts of its domain.

Other errors made by respondents were the creation of incorrect algebraic rules. For question 4 some respondents answered yes, since the correspondence is $y = x^2 + 1$. On question 6 a typical error was to answer yes, since the correspondence is $y = \frac{1}{x}$.

Vinner and Dreyfus also noted how frequently respondents exhibited compartmentalization. In particular, of the 82 respondents who gave the Dirichlet-Bourbaki textbook definition for the function concept, only 46 of them use this reasoning when answering questions 1-6.

Monk (1989) discusses student understanding of function in terms of their ability to answer six different types of questions about what he calls functional situations. He uses the words "functional situation" to emphasize concrete situations and a diversity of functional characteristics. Each type of question may be asked in any context and using any representation of the given function. Questions which pose a question (using any functional representation) without a physical context Monk calls "naked questions."

1) *Forward Questions* give an input value of a function and ask for its output.

2) *Backward Questions* give an output value and ask for an input value.

3) *Across-Time Questions* ask what happens to an output or other related value as the input values of the function change in a certain way.

4) *Articulation Questions* ask the student to coordinate quantities that are related to or derived from the function values. His examples in this category usually involve in some form the rate of change of the function.

5) *Multiple Representation Questions* ask a student to represent the function situation using a different representation than the one given.

6) *Change of Context Questions* ask a student to transfer his or her understanding of a function in one context to a new context. The function and its representation remain the same, but the context and hence the wording of the questions change. Monk has found that very few students can answer change of context questions.

In a later study Monk (1992) reported the results of interviews with 12 first-term calculus students and 8 upper class mathematics students on a series of across-time questions dealing with rate of change. The students were given a model of a ladder that can be moved to simulate the top of the ladder sliding down a wall as the bottom of the ladder slides across the ground. The students were then asked to use the model to determine whether the ladder slides down at a constant, increasing, or decreasing speed when the ladder is pulled out at a constant speed. The first two questions referred only to the displacement of the top of the ladder when the bottom of the ladder is displaced by a constant amount. This allowed the students to take measurements from the ladder in order to answer the questions. The third question asked directly about speed. Monk reported that by using the model, all but 3 of the 20 students could distinguish the concepts involved and found their own method to correctly answer the questions.

Monk distinguishes students as having either a *pointwise* or *across-time* view of functions. The students with a pointwise view of function concentrate on particular input and output pairs and are reluctant to discuss the qualitative change in the function over time. Students who do not have this difficulty are said to have an across-time view of function. In a follow-up question, 88% of students with an across-time view of function were able to draw an appropriate graph of the function situation while only 22% of students with a pointwise view are able to do so.

Several other researchers describe in more detail another dichotomy in function understanding (Sfard, 1992; Dubinsky and Harel, 1992; Breidenbach, Dubinsky, Hawks and Nichols, 1992). Sfard (1992) labels this duality with the terms *operational* and *structural*. She sees understanding of not only the function concept, but all mathematical concepts as being composed of these two complementary facets (also Sfard, 1991).

Sfard maintains that historically and psychologically, mathematical concepts such as function develop first operationally, and are only later understood structurally. In particular, functions are historically thought of in terms of computational processes, a description of how one quantity changes in relation to the change in a dependent quantity. It is not until much later that the structural definition of a function as a collection of ordered pairs with certain properties is devised.

Sfard sees the development from an operational to a structural understanding as proceeding through three stages -- *interiorization*, *condensation* and *reification*. Interiorization occurs when a person can take an input value to a function and use a rule or formula to generate an output value. A person is said to have a condensed understanding of function when the person can think of the input-output pairs without having to work through the rule. This person can thus use the function as a subprocess of a more complex process. Sfard explains in her own words, "Reification is the next step: in the mind of the learner, it converts the already condensed process into an object-like entity. In other words, while condensation is a gradual quantitative change, reification should be understood as a sudden qualitative jump in the way of looking at things -- an ontological shift comparable to a transition from one scientific paradigm to another" (Sfard, 1992, p. 64). Sfard contends that the person has only an operational view of function unless reification has occurred. Thus, a condensed understanding of function is still an operational understanding.

Based on her theoretical framework, Sfard recommends that new concepts should be introduced to students first from an operational viewpoint. Further, a structural conception should not be introduced unless the course material absolutely requires it. She suggests that a condensed operational conception of function should be sufficient for dealing with differentiation and integration. "Indeed, it is only natural to explain these two operators using the process-interpretation of the concept of function: the former may be described as finding velocity from change and the latter as reconstructing the process of change from its velocity" (p. 69).

To conduct her study, Sfard administered a questionnaire to a group of 22-25 year old students who had completed a course on set theory, algebra, and calculus. Her two primary conclusions were that despite having been taught a structural definition, the students had conceptions that were closer to operational than to structural, and that many students developed *pseudostructural* conceptions. A student has a pseudostructural conception if he or she treats the function as an object that has no underlying process associated with it. One example of a pseudostructural conception would be to say that a function is simply a formula without thinking of that formula as rule for generating input-output pairs. Another example would be to treat a function as as a graph, thought of as a shape or picture, without considering it as representing a collection of input-output pairs.

Breidenbach, et al. (1992) use the words *process* and *object* to describe the dual nature of functions. Following Piaget, processes and objects are said to be constructed by *reflective abstraction* (see also Dubinsky, 1991). Breidenbach, et al. describe this theoretical view for any arbitrary mathematical object and then relate it specifically to functions. Their theory is strikingly similar to that of Sfard's, except for slight differences in language that the reader should be careful to note in the following description.

Breidenbach, et al. describe an *action* as "any repeatable physical or mental manipulation that transforms objects (e.g., numbers, geometric figures, sets) to obtain objects" (p. 249). When the action can be thought of as a whole without running through all of the steps, the action is said to be *interiorized* to become a *process*. The subject can then use the process to obtain new processes. When the person can imagine the process as transformed by some action, then it has become *encapsulated* as an *object*.. (Note that the interiorization of Breidenbach, et al. is most similar to Sfard's description of condensation, but is different from her use of the term interiorization.)

Breidenbach, et al. apply their theory to functions by considering three ways of thinking about functions (see also Dubinsky and Harel, 1992). The *prefunction* stage is used to describe a student's understanding which involves a minimal conception of function. Additionally, that student is characterized as being unable to perform basic tasks involving functions. Thinking of a function as an *action* includes being able to input a number into an algebraic expression for a function and calculate the corresponding output. However, a person with this understanding will think of this for only one value at a time. A *process conception* of function involves a dynamic transformation understood for all values at once. The subject can therefore imagine combining this function process with other function processes and can also imagine reversing the process. The subject is said to have an *object* conception of function if it is possible for him or her to perform actions on it, in particular actions that transform it.

In conclusion, it is clear that understanding functions consists of a number of interrelated cognitive aspects. An attempt will be made here to emphasize connections between those aspects described in the preceding studies.

A function consists of a domain, a range and a description for associating each element of the range to each element of the domain. This may be described in a variety of representational environments. It also may be described dynamically as a rule or mapping that takes an element of the domain to an element of the range or staticly as a collection of ordered pairs. A complete understanding of function consists of seeing the parallel nature of the representational environments, as well as the duality of the process-object distinction. A reified notion of function includes all of the above as well as being able to use the function as an object in other higher level processes. In contrast, a pseudostructural understanding of function consists of treating a function as an object with no internal structure.

A person's concept image of function consists of all aspects of the function concept available to that person -- representations, properties, and operational or structural aspects. A person's concept definition is whatever subset of these aspects a person uses when asked to define the concept. Compartmentalization occurs when connections between the various aspects of the concept image are not well formed, and the person uses one part of his or her concept image when use of a different part would be more appropriate. Compartmentalization also occurs when a person does not realize that he or she has conflicting elements in his or her concept image.

Monk's description of pointwise understanding of function is similar to the action conception of function described by Breidenbach, et al. and the interiorization stage presented by Sfard. It is simply the ability to take an input value for a function and find its associated output value, and is therefore less than a full process conception. In the latter, the action is condensed so that a person may imagine the evaluation of one or more output values without having to actually go through the process. Monk's across-time view of function is different than the process conception of function. Monk's interest in across-time questions and articulation questions involving rate of change suggests the importance of understanding function in terms of the covariation of two quantities. The importance of covariation in student understanding of function is also reported by Confrey and Smith (1994) and Thompson (1995b).

Thompson (1995b) relates the idea of covariation to the process-object duality in the following way. Once a person has a process conception of function, then that person "can begin to imagine 'running through' a continuum of numbers, letting an expression evaluate itself (very rapidly!) at each number" (p. 26). "Once students are adept at imagining expressions being evaluated continually as they 'run rapidly' over a continuum, the groundwork has been laid for them to reflect on a *set* of possible inputs in relation to the *set* of corresponding outputs" (p. 27). The correspondence between the sets would then be a view of function as object. Thus, Thompson's work suggests that the view of function as covariation depends on an understanding of function as process, but is less comprehensive than a view of function as object.

Understanding the Concept of Limit

Student understanding of the concept of limit can also be discussed using the language of concept image and concept definition, and by contrasting dynamic and static aspects of the concept. The studies reviewed in this section cover both of these theoretical frameworks. However, only Krussel (1995) intentionally makes use of the process-object distinction or the notion of reification. In the other studies the contrast of dynamic and static notions of limit comes out of observations of student understanding, and not from an a priori choice to focus on a process-object distinction.

Tall and Vinner (1981) reported that first year university students who had been introduced to limits in high school were more likely to describe limits in terms of a dynamic process rather than by using the formal definition. Even when a student can state a correct dynamic or static definition of limit, his or her concept image of limit might contain elements conflicting with the usual mathematical notion of limit. In particular, the language of approaching or getting close to a limit implies to some students that the limit value can never be reached.

Davis and Vinner (1986) studied the understanding of the limit of a sequence held by 15 twelfth grade students who had been enrolled in a special calculus class during their eleventh-grade year. All the students had shown mastery of the formal definition of the limit of a sequence during their work the previous year. The students were given a written quiz on the first day of their twelfth-grade year to probe for naive conceptions that might be part of their concept image along with the formal concept definition. Students were asked to explain the concept of a limit of a sequence both formally and intuitively. The authors reported that none of the 15 student responses showed "the depth of understanding and precision of expression that had been expected" (p. 297). Observed errors included the following:

1. A sequence must never reach its limit.

2. The limit is the number that a sequence is going toward. (Note that .9, .99, .999, ... goes "toward" an infinite number of values.)

3. The limit must be an upper or lower bound to the sequence, something which the terms cannot go past.

4. There is a last term of the sequence or one can actually go through all the terms of sequence to get to the end.

Davis and Vinner explain that these errors are probably the result of a number of factors which include:.

1. Natural language meanings for the word limit and pre-mathematical ideas about limits can influence a student's mathematical understanding of limit. For example, a speed limit is a limit which one should not go past.

2. Specific mathematical examples shape the concept image held by a student. For example, if a student has seen mostly monotonically increasing sequences, he or she might assume all sequences have that characteristic.

3. Mathematical ideas must be built gradually over time. It is not possible to instill a fully developed understanding of limit, much less a wealth of relevant examples to a student all at once. In the process of learning a student will at times have an incomplete understanding of limit, one that may not well represent the concept as a whole.

A French researcher, Cornu (1981, 1983a, as reported in Tall, 1986, and 1981, 1983b as reported in Williams, 1989), also relates his work on student understanding of limit to the framework of concept image and concept definition. He reports several major obstacles to student understanding of limit. These include students thinking of the limit as a bound which can not be crossed over, or similarly as something not attainable. He found students using the word "approaches" for a limit that is attainable versus "tends to approach" for a limit that is not attainable. Additionally, other obstacles are the notions of an "infinitely small number," a number smaller than all numbers but not zero, and an "infinitely large number," a number larger than all the others, but not infinite.

Cornu also discusses the obstacle involved in passage from the finite to the infinite. He describes students who can envision finite approximations without any idea of a limiting process. The static images are an obstacle to a more dynamic perception. In the dynamic perception the finite approximations are used to determine the infinite result. However, Cornu states that this passage from static to dynamic must be followed by a transition on to the static formal definition of limit in which nothing moves.

Another French researcher, Robert (1982a, as reported in Tall, 1986, and 1982b as reported in Williams, 1989), analyzed the responses of 1253 university students to a single questionnaire. In classifying their definitions of limit she reported five basic models of student understanding:

1. Primitive models that include the notion of a limit as a barrier which can not be passed or describes sequences which are bounded and monotonic.

2. Dynamic models that include any description which implies motion, such as the use of the words "approaches" or "tends to."

3. Static models that include responses that describe the formal definition in the student's own words.

4. Mixed models that combine static and dynamic elements in the same response.

5. Nonexplicit models or tautologies that provide no useful information about limits in general.

Robert recommends that students should reflect on their mental images related to limit and compare them with a wide variety of mathematical examples in order to improve their understanding.

Williams (1989, 1991) devised a study based on the work of Cornu and Robert. Students in 18 discussion sections of a second semester calculus were given a questionnaire to determine their formal and informal notions of limit. From the students who volunteered for the second part of the study, 12 were chosen based on the diversity of their responses. Of these, 10 students completed a series of 5 interviews with the researcher. The second, third, and fourth interviews consisted of a discussion of two conflicting statements about limits and examples of limits designed to cause cognitive conflict in students in order to alter one of three views of limit: those views being limit as unreachable, limit as a boundary, and limit as motion.

Williams indicated that students' notions of limit are resistant to change, because their knowledge is compartmentalized. They preferred to treat conflict-producing problems as special cases rather than use them to restructure their understanding of limit. Students suggested that limit problems can be divided up into cases with different notions of limit for each case without having one notion of limit that is appropriate for all cases. Williams also reported that while most of the students participating in this study were eventually willing to relinquish the notions of limit as boundary or of limit as unreachable, the students were not willing to give up the idea of limit as motion in favor of the static formal definition. The students preferred notions of limit that were simple and practical over more formal and more general notions.

Using the terminology of Sfard (1991, 1992), it is appropriate to say that these students had developed an operational understanding of limit. Their understanding was condensed in that it became more organized and less compartmentalized. However, these students never gained a structural understanding of limit, and reification did not occur.

Krussel (1995) studied visualization and reification in the mathematical conceptions held by advanced mathematical thinkers. She interviewed 3 advanced undergraduate mathematics majors, 4 mathematics graduate students and 2 mathematics professors. Limit was one of many concepts she investigated. For each concept the participant was asked to state the first thing he or she thought of, and any visual images or theorems associated with that concept. The participants were also asked when they first learned the concept and whether they remembered any changes or milestones in their understanding of that concept. Krussel did not probe for any further aspects of the concept image not evoked by these questions. Therefore, only each person's primary conceptions and any remembered significant changes in their understanding are recorded. She analyzed the data for each person's major visual images and whether reification had occurred, but she did not examine the person's understanding of any one concept, including limit, in depth.

Krussel discovered that even these advanced mathematical thinkers talk about limit primarily in terms of a dynamic definition. Each of the subjects mentioned the formal definition at some point, but none of them mention it first, and four of them only mention it when asked about the related concept of continuity, as opposed to when asked about limit. The two advanced undergraduate students even made the misstatement that a limit is unreachable.

The most sophisticated and interesting example of a primarily dynamic understanding was held by one of the mathematics professors. He initially described his understanding dynamically and proceeded to give several sophisticated examples of the use of limits, again employing dynamic language. When questioned as to whether any changes had occurred in his understanding, he described how he understood and enjoyed using epsilon's and delta's in working with limits, but then later came to realize that this is not a very intuitive definition. He described that thereafter he made an effort to gain a more intuitive understanding.

When the literature on understanding limit is viewed as a whole, it is clear that there are two main misconceptions that students have regarding the limit concept. These are limit as a boundary, and limit as unreachable. Additionally, most students, and even some mathematicians, hold the dynamic notion of limit as primary. Issues of multiple representations are not dealt with explicitly in any of these studies, but tables of values, graphs, and symbolic expressions are all used as representational environments. As with the studies on functions, students have concept images which contain conflicting elements. Few of the students in these studies exhibit an understanding of limit that recognize the duality of the operational and structural aspects of the concept. None of the studies examine whether the concept of limit can be viewed as an object used by higher level processes.

Understanding the Concept of Derivative

The studies reviewed here regarding understanding of derivative are more varied than the studies in the previous two sections. This is largely due to the greater complexity of the concept of derivative, since it incorporates both the concept of function and the concept of limit. In addition, it involves the notions of rate, slope, or however one chooses to describe the ratio of the changes in the output and input values of the function whose derivative is being discussed.

The theoretical framework of concept image is used in a number of the studies (Vinner, 1982; Tall, 1986; Hart, 1991; Vinner, 1992; Ellison, 1993). Thompson (1995b) uses the notion of reflective abstraction in a way related to the development of the action, process, object trio described by Breidenbach, et al. (1992). Krussel uses the notions of a operational-structural duality and the reification framework of Sfard (1992). All of the studies presented here discuss a wealth of representational environments.

Perhaps due to the complexity of the concept of derivative, there is only one study (Ellison, 1993) that seeks to describe comprehensively what is meant by understanding of derivative and then uses that meaning to analyze student responses . Two studies (Confrey and Smith, 1994; Thompson, 1995a) discuss what it means to understand the concept of rate at a level considerably prior to calculus. A collection of articles (Rubin and Nemirovsky, 1991; Nemirovsky and Rubin, 1992; Monk and Nemirovsky, 1995) describe the ability of high school algebra students to work with the notion of rate in various physical settings and to relate this understanding to graphs of a function and its derivative, labeled not as such but according to the physical model being examined.

Several articles describe specific aspects of a calculus student's understanding of derivative. In examining the development of student understanding of the fundamental theorem of calculus, Thompson (1994) discusses student understanding of rate of change and average rate of change. Vinner (1982) looks at student understanding of tangent lines. Orton asks a broader range of questions, but his main finding involves student difficulty with the idea of secant lines approaching a tangent line. Amit and Vinner (1990) consider one student's confusion about the relationship between the derivative and the tangent line at a point. Vinner (1992) asks students to answer the question, "what is a derivative?"

Heid (1984), Tall (1986), Crocker (1991), Hart (1991), and Ellison (1993) wrote dissertations whose main purposes were to examine what happens when calculus students use an experimental curriculum that involves computers or graphing calculators. As part of the process of documenting the understanding of calculus held by the students involved in these experimental courses, each includes some discussion of the students' understanding of derivative. As mentioned above, only Ellison sets out specifically to document student understanding of derivative in a comprehensive way.

In her dissertation Krussel (1995) sought to describe visual imagery and the reification processes in advanced mathematical thinkers. As part of her study she provided some data on the understanding of derivative held by these advanced mathematical thinkers.

Children's Understanding of Rate

To begin, it is appropriate to discuss the student understanding of rate prior to calculus. Confrey and Smith's (1994) discussion of rate comes out of their work on understanding of function in terms of covariation. This approach emphasizes how the y values change as the x values change. Thus, rate of change is a natural entry point to a covariational approach to functions. Confrey and Smith argue that "even young children can use rate of change as a way to explore functional understanding" (p. 33).

Confrey and Smith define rate as a unit per unit comparison. Their definition includes multiplicative units as well as additive units since they found both used by children in understanding functions. In particular, exponential functions are often seen by children as having a multiplicative rate. For example, in a function taken from biology where the number of cells doubles every hour, the multiplicative rate is this notion of twice per hour, a multiplicative unit per an additive unit. However, the use of rate of change in the derivative concept is restricted to an additive unit per additive unit construction.

Confrey and Smith note that children can distinguish a variation across rates of change. "A child recognizes a change of speed in an automobile; a gust of wind is heard 'blowing harder', both implicit rate concepts" (p. 156). Confrey and Smith also suggest, "that students form a direct connection between slope and rate of change which is not mediated by numerical analysis. This second rate concept is more holistic than the analytic 'unit per unit', and connects the experiential basis of slope with rate variation in contextual problems" (p. 157).

Thus, Confrey and Smith envision an integrated understanding of rate that includes not only the unit per unit comparison, but also the physical experience of comparing varying rates in real-life situations and the graphical interpretation of rate as (varying) slope. "A coordination of these multiple representations of rate will be necessary for a more robust concept of rate" (p. 158). They see this integrated understanding as building a strong foundation for calculus.

Thompson (1995a) examined the concept of rate held by a fifth-grader called JJ. Thompson met with JJ eight times for approximately 55 minutes each over a three week period. JJ initially thought of speed as a distance. When asked to determine the amount of time it would take to go 100 feet at 30 ft/sec, she reasoned based on the number of "speed-lengths" in 100 feet to find the time $3\frac{1}{3}$. JJ explained, "because there are three 30's in 100, and 10 is $\frac{1}{3}$ of 30, so $3\frac{1}{3}$ seconds." Thompson (1995a) noted that, "for JJ it is the case that *speed is a distance* (how far in one second) and *time is a ratio* (how many speed-lengths in some distance)." (Thompson's emphasis, p. 198) JJ appeared to measure the distance in speed-lengths to obtain an associated time. When given the distance and the time and asked for the speed, JJ tried to guess the correct speed and then checked to see if it gave the desired time. For JJ time was a ratio of the distance over the speed. It is not until later in the teaching experiment that JJ was able to construct speed as a ratio of distance over time.

When JJ was able to think of speed as a constant ratio, something invariant over changes in distance and time, Thompson says she understood speed as a rate. "A rate as a reflectively abstracted constant ratio symbolizes that structure as a whole, but gives prominence to the constancy of the result of the multiplicative comparison" (Thompson, 1995a, p. 192). Using the language of Piaget so that it can be interpreted as parallel to the use of Breidenbach, et al. (1992) on the understanding of function, Thompson calls rate an *interiorized ratio*. Even though Thompson speaks of reflective abstraction and interiorization, he does not use the terms "process" or "object." Based on what he has said, it is useful to describe the ratio as the process, the rate as the reflectively abstracted object, and the speed (as in speed-lengths) as a pseudostructural construct, an object with no internal process structure.

Algebra Students' Understanding of Rate of Change through Physical Models

Rubin and Nemirovsky (1991) conducted teaching experiments with six high school students who had taken algebra but not calculus. Two of the students worked with each of three environments in three-hour structured interviews broken into one and a half-hour sessions. The environments were designed to embody the calculus constructs of rate of change and accumulation.

Rubin and Nemirovsky describe the three environments as follows:

> --a motion environment, in which the student manipulates a small Lego car in front of a motion detector that can record the car's relative position many times a second. The computer which is connected to the motion detector displays graphs of either position or velocity vs. time. The software provides students with the capability of finding out the value of points of the two graphs and of comparing two different 'runs' of the car.
>
> --an air pump environment, which uses a hand-driven air pump instead of a car and motion detector as the physical world analog, with very similar supporting software. Students control the flow of air into and out of a transparent, calibrated bag using a hand pump and a series of valves; the computer records air flow several times a second and can display both volume and air flow over time.
>
> --a spreadsheet environment, which allows students to define functions in terms of first and second differences and initial values. A spreadsheet representation of the functions' values is derived and the corresponding graphs are drawn. The function is labeled A, the first difference B and the second difference C (p.169).

Their paper is organized with the purpose of demonstrating how students interacted with the three environments in different ways. The most persistent conceptual problem for the students was the idea of negative velocity in the motion environment. In this environment position zero was arbitrarily set to be at the location of the motion detector. Distance from the motion detector was considered positive so that the distance function could only have positive values. However, velocity can change signs with positive velocity representing motion away from the motion detector and negative velocity representing motion toward the motion detector. The two students who used this environment struggled with the notion of negative velocity. Student S eventually mastered the notion, and considered it one of the main things he learned in that day's lesson. Student N never quite believed that there could be negative velocity since she figured that no matter which way the car moved its distance traveled (or odometer reading) would still be increasing.

In the air pump environment, there was much less confusion. With minimal experimentation, each of the two students recognized that a positive air flow meant that air is flowing into the bag whereas a negative air flow meant that air is flowing out of the bag.

Confusion about a first derivative being negative also occurred in the spreadsheet environment. The situation being modeled was, "The cost of home computers is still decreasing, but more slowly than it is last year." The students used first differences in column B that are decreasing but positive, i.e. 50, 48, 46, etc.

One nice feature of the motion environment was its familiarity for the student, and the variety and amount of natural language available for its discussion. The familiarity of the motion environment allowed students to use their own experiences for reference, and to more easily perform thought experiments to predict results. Having a natural language expression for the second derivative, acceleration, was also helpful. However, the variety of natural language also caused confusion in the case of negative acceleration . Consider a car moving with negative acceleration. If the velocity is positive we say the car is slowing down, but if the car has a negative velocity we say that it is speeding up (but in the opposite direction).

In the motion environments the students usually described the car as speeding up or slowing down without a directional specification. In the air flow problems the students always use language that indicate the directionality. Since as the bag fills it moves up, the verbal expressions are of the form, "it goes up slowly" or it "goes down quickly." This clarified issues of negative velocity or acceleration.

In another article based on the interviews described above, Nemirovsky and Rubin (1992) note that, in all three environments, the students tended to assume certain resemblances between the graph of a function and its derivative. Again, these students had not taken calculus, and the word derivative is not used in the interviews. The graph of the derivative was labeled as car speed, rate of air flow, or the values in column B, respectively, and the student was asked to predict the graph of the function, stated as the car distance, the volume of air, or the values for column A. The types of resemblances include having the graphs start at the same initial value, be of the same sign (both positive or both negative), be of the same shape (both straight or both curved) or have the same direction of change (both increasing or both decreasing). The resemblances chosen by the students do not seem to be based on a confusion between volume and air flow or speed and position. Nemirovsky and Rubin (1992) describe the students as distinguishing the two quantities, but assuming the graphs must have some similarities since the quantities are closely related. The students look at the global features of each graph separately and do not examine how the change in the function graph is represented by the derivative graph.

Both Nemirovsky and Rubin (1992), and Monk and Nemirovsky (1995) described in detail an interview with "Dan," one of the students working with the air flow device. Dan was an eleventh-grade student who had not taken calculus, but was taking physics at the time of the study. During the first interview with Dan, he was given flow rate graphs that were positive and constant or positive and increasing, and then Dan was asked to predict the volume graph. During the second interview, analyzed in detail in these two studies, Dan was first shown a decreasing positive air flow graph and asked to predict the associated volume graph. Dan drew a graph almost identical to the given graph except that Dan's graph had a different slope. The initial starting value was the same, and both graphs were positive, decreasing, and straight. Dan then attempted to use the air flow device to create a volume graph of this type, forgetting that it was the air flow graph that was given. In this process Dan realized that the flow rate must be negative for the volume to decrease. He then used his predicted volume graph as a given and drew an air flow graph that started at 0 and was a negative, decreasing, straight line. Using the device, Dan recognized that not all of the similarities he initially advanced between the two graphs held true. The resemblances between his new volume and airflow graphs were only that the line is straight and decreasing. It no longer had the same initial value or the same sign. So, in the course of using the air flow device, Dan was able to further distinguish the two graphs.

The next task presented to Dan was to predict the volume graph when the air flow graph was increasing and then decreasing, but remaining positive. Dan had a hard time trying to predict what would happen. Finally he decided to try to make the graph. He pushed air into the bag to represent the upward movement of the air flow graph and then pulled air out of the bag with a bellows, to represent the downward movement of the air flow graph. He was surprised when this produced an air flow graph that went below the x -axis. Through using the device he recognized that the air flow graph goes to zero when he starts to pull the air out of the bag. However, Dan could not figure out how to keep the air flow graph positive but decreasing. When Dan decides it was not possible to create such a graph, the interviewer volunteered to give it a try. When the interviewer successfully created the desired graph by pushing air in quickly at first, and then more and more slowly, Dan responds, "Well, it's just the amount of increase is less and less. I see ... Yeah ... That's, that's different. I didn't think of that" (Nemirovsky and Rubin, 1992, p. 25). Here Dan has started to examine the situation using what Nemirovsky and Rubin (1992) call a variational approach. He recognizes that the volume does not only increase, but that it increases by "less and less."

Here Dan recognized an aspect of the problem that he previously had not examined, the idea that the volume not only increases or decreases, but that it may do so slowly or quickly. However, his refinement of his understanding still was not a completely variational approach. His next prediction for the volume graph was an increasing, positive graph with two components: a steep, increasing straight line starting at zero connected by a sharp corner to a less steep, increasing straight line. Dan recognized that the volume always increases and that it increases quickly and then more slowly, but he saw these as two separate events and not a gradual change across the whole time. When Dan sees that the volume graph does not consist of two separate events, he assumed it was an experimental error. In other words, he thought that the first quick increase occurred too fast to be recognizable on the graph.

Nemirovsky and Rubin (1992) end their analysis of Dan's interview at this point. The emphasis of their article is on the process of Dan moving from predictions based on resemblances to a more variational approach. In a similar manner, Monk and Nemirovsky (1995) emphasize Dan's refinement of his understanding as he becomes aware of additional aspects of the situation. Both studies point out that in the context of the interview interaction, the use of the air flow device served as a catalyst to the development of Dan's understanding.

Difficulties with Rate of Change

Thompson (1994) studied the understanding of the fundamental theorem of calculus as held by senior college students and graduate mathematics students in a class designed for mathematics education majors. He found that the difficulties these students had with the fundamental theorem stemmed from a weak understanding of function, functional covariation, and rate of change.

The students had a primary image of function as "a short expression on the left and a long expression on the right, separated by an equals sign" (p. 268). This was not the only image of function available to the students, but it seems to be the one most easily elicited. This image led students away from thinking about the meaning behind the symbols. The students often acted as if only the expression for the function or the output values of the function needed to be considered, without taking into account how these values vary depending upon the input variables.

During classroom discussions, students confused "changing" with "rate of change," and confused "amount" with "change in amount" Students also had difficulties, both in classroom discussions and on a follow-up exam, interpreting the details of the notation for average rate of change. On an exam given two weeks after the end of the teaching experiment, 17 of the 19 students treated an expression for the average rate of change of the function as if it were an expression for the derivative without considering the difference between the two. One test question was stated as follows:

> The volume in cubic meters of a cooling object t hours after removing a heat source is given by the function $v(t)$. Suppose a function $x(t)$ is defined as
>
> $$x(t) = \frac{v(t+0.1) - v(t)}{0.1}.$$
>
> State precisely what information $x(t)$ gives about this object. (That is, don't tell me what $x(t)$ approximates. Tell me what it actually gives.) (Thompson, 1994. p. 256).

Only four of the 19 students correctly identified $x(t)$ as average rate of change of volume. Six students said it was the derivative, and five others said it was the rate of change of cooling. When asked, only seven students correctly stated the units of $x(t)$ as cubic meters per hour.

Thompson tried to have the students understand the fundamental theorem of calculus by coordinating several processes. First, the students should recognize the accumulation of a function over time as a new function. Thompson instructed the students to write this as approximated by a Riemann sum with a variable ending point that their computer program could evaluate. Then the students were to approximate the rate of change of the accumulation function using an average rate of change function (thought of as the slope of a sliding secant line). Again, this average rate of change function would be calculated using the computer program. The students could then see that this was (approximately) the same as the original function. The purpose of Thompson's emphasis on the approximations, Riemann sums, and average rate of change was to force the students to deal with the underlying processes. This proved to be quite difficult for the students.

As indicated above for average rate of change, each process itself was not well understood by the students. The Riemann sum difficulties were the result of the students thinking of the sum as occurring only for a fixed input interval. The students had difficulties explaining the coordination of the two variables involved in the Riemann sum function -- the counter that delineates each partial sum and the variable that tells what interval the sum is calculated over. Even when the Riemann sums were left out of the discussion the lack of understanding of average rate of change was enough to derail insight into the fundamental theorem.

Difficulties with Tangent Lines

Vinner (1982) questioned 278 first-year college students enrolled in calculus courses in Chemistry, Biology, Earth Science, and Statistics about tangent lines to curves. The written questionnaire asked students to tell whether there are 0, 1, 2 or more tangent lines to a given curve at a point and to draw in any tangent lines they predicted. Vinner saw students as using the prototype of a tangent line to a circle. In other words, students considered tangent lines to be lines that touched the curve at one and only one point, and that did not cross the curve at the point of tangency.

Item 1 was a curve similar to the graph of $y = x^3$ at $x = 0$ so that the correct answer would be to draw one tangent line with slope 0. However this line cuts through the curve, which is contrary to the tangent line to a circle prototype. The correct response was given by 18% of students. The favorite response of students (38%) was to draw a line tangent just off of zero so that the line did not cut through the curve at the point of tangency (see Figure 2.3.).

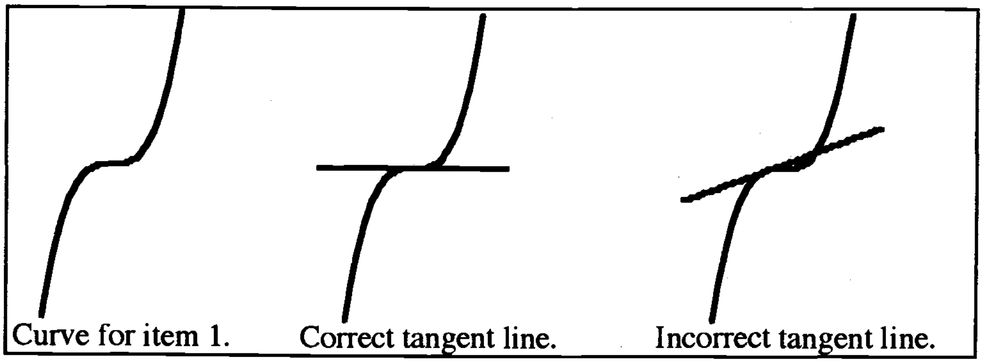


Figure 2.3. Cubic polynomial with a correct and an incorrect tangent line for the inflection point. (After Vinner, 1982.)

Item 2 was the graph of a function similar to $x^{2/3}$ at $x = 0$. Vinner considered this curve to have a vertical tangent (with infinite slope) that also cuts through the curve at the point of tangency. Note that most texts would consider this curve to have no tangent line at $x = 0$ since the limit of the derivative values from the left and right, $-\infty$ and ∞ respectively, are not equivalent. The responce considered by Vinner to be correct was given by only 8% of students. The most common response (42% of the students) was to abstain from drawing a graph. Students who did answer, though incorrectly, drew two tangents on either side of $x = 0$ (18%), mulitiple tangents all under the curve touching the corner point (18%), or one tangent under the curve touching the corner point or the latter plus the correct tangent (14%) (see Figure 2.4).

Item 3 was a function similar to the graph of $y = \begin{cases} 0 & x \le 0 \\ x^3 & x \ge 0 \end{cases}$. The correct tangent line for $x = 0$ is a horizontal tangent line that lies on top of the left half of the graph. This correct response was given by 12% of students. The favorite response, given by

33% of the students, was a line tangent at a point with x value greater than 0 so that the line touched only one point of the curve (see Figure 2.5).

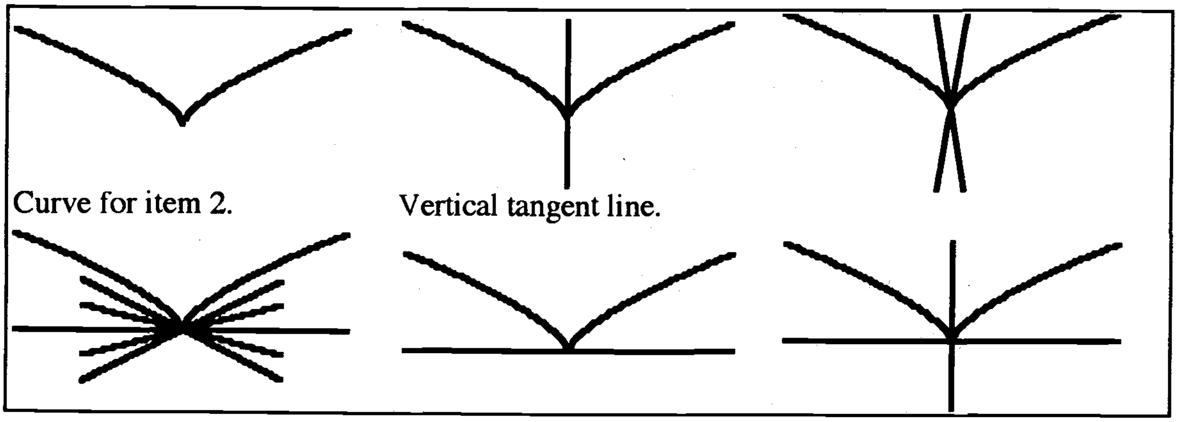


Figure 2.4. Curve with a correct vertical tangent line at the cusp and numerous other incorrect tangent lines for the cusp.

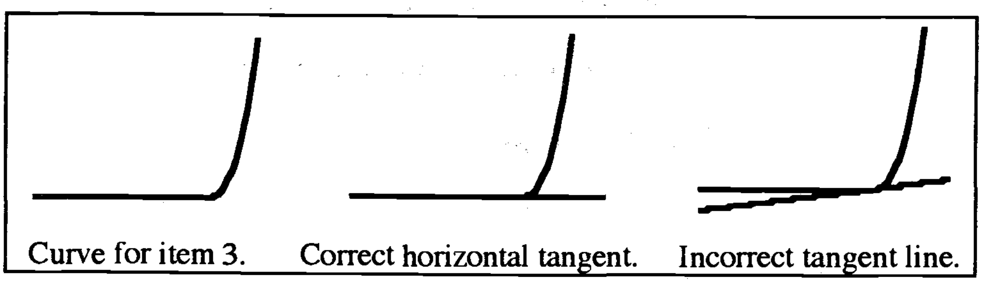


Figure 2.5. Curve with correct horizontal tangent and incorrect slant tangent for the point where the curve changes from flat to concave up.

Orton's (1983) study of students' understanding of differentiation involves individual, task-based interviews with 110 students. Sixty of the students were in sixth-form schools, ages 16-18, and the other 50 were college students ages 18-22 who were preparing to become teachers of mathematics. The tasks included symbolic differentiation, interpretation of secant lines moving toward a tangent, graphical interpretation of both average and instantaneous rates of change, and graphical interpretation of a symbolic limit definition of derivative at a point.

The students scored well on problems involving symbolic differentiation or finding the rate of change or slope of the tangent line from the symbolic derivative (mean scores of 3.32 or higher out of 4). Students had significantly more trouble finding average or instantaneous rates of change from a graph (mean scores of 1.92 to 2..22 from 4). Orton describes these errors as structural rather than "executive." In other words, the errors were conceptual rather than calculation mistakes. Orton does not describe further the nature of the conceptual errors.

One particular error, involving secant lines moving to a tangent line, was particularly enlightening with regard to student difficulties in understanding graphical interpretations of the limit definition of derivative. Orton showed students a diagram of a circle with several secant lines drawn through it, all of which passed through the point P. The other intersection point of a secant line with the circle was labeled Q_n , where $n = 1,2,3,4$ .... The question was: "As Q gets closer and closer to P what happens to the secant?" (p. 245). Orton noted that out of 110 students "43 students were unable to state that the secant eventually became a tangent, despite considerable encouragement, through further questioning, to say more about what happened to the secant until they ran out of things to say about it" (p. 237). Students seemed to focus on the chord as shortening or the area between the chord and the circle as becoming smaller instead of noting what happened to the slope of the secant line.

Amit and Vinner (1990) analyzed in detail the written answers of one student, referred to as Ron, to two three-part questions. Ron was a first-year economics student at an Israeli university who had taken calculus courses both in high school and at the university. Ron's questionnaire was mostly correct; out of six question parts, five were correct. In the first question, given a clearly marked graph of a function with a tangent line drawn in at one point, Ron was able to read off the value of the function at the point of tangency and use the slope of the tangent line to determine the value of the derivative at the point of tangency.

In the second question, in answer to the question, "what is a derivative?", Ron wrote, "The derivative is the slope of the tangent to the graph at a certain point" (p. 7). He went on to explain that the derivative function tells the slope of the tangent to the function at any point, and he correctly stated the definition of derivative as the limit of difference quotients.

Ron's one mistake occurred when he was calculating the equation of the tangent line at the point of tangency. He used this equation as if it were the equation for the derivative function. This usage was not only wrong, but it contradicted the answer for the value of the derivative Ron had previously calculated, and it contradicted his explanations about the nature of derivatives. Amit and Vinner explain this as an instance of compartmentalization. There were both correct and incorrect ideas present in Ron's concept image of derivative. He used both ideas on the same questionnaire, simply without noticing the contradictions.

Amit and Vinner also point to Ron's statement in answer to "What is a derivative?" as a partial clue to Ron's conceptions and as an example of errors they see in other students' answers to this question. Ron's answer was correct except that he referred to only one tangent, "the tangent." Perhaps the best explanation for this is that Ron was speaking of the derivative at a point and not the derivative function. However, by thinking of only one tangent line, it was easier for Ron to assume that this tangent line was in fact the derivative itself.

Amit and Vinner turn to a linguistic analysis of the standard statement of the geometric definition of derivative, "The derivative of a function at a certain point is the slope of the tangent to the graph of the function at this point" (p. 9). They state that since this definition is hard to memorize students make certain *omission-transformations*. Thus, a typical student version of this statement is, "The derivative is the tangent to the function at a certain point" (p. 10). If students proceed to use this formulation to reconstruct their understanding of derivative, they likely will assume that the derivative is the tangent line itself instead of the slope of that line.

Even mathematicians who clearly recognize the distinction between the tangent line and the derivative function may not always state that precisely. William P. Thurston (1994), a noted differential geometer, makes a list of different ways of understanding the concept of derivative. This list includes the formal definition, "the derivative is the slope of a line tangent to the graph of the function," "the instantaneous speed of $f(t)$, when t is time," "the derivative of a function is the best linear approximation to the function near a point," and "the derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power" (p.163). The last two descriptions listed are clearly important to a complete understanding of derivative, but each of them describes the tangent line itself and *not* the slope of that line. If a noted mathematician is not always careful to distinguish between the tangent line and its slope, then it is not surprising that a student like Ron would sometimes fail to make the distinction. The difference is probably that mathematician's knowledge is not as compartmentalized as Ron's knowledge. A mathematician would not make the mistake Ron does of using the equation for a tangent line at a particular point as if it were the equation for the derivative function at any point.

What is a Derivative?

Vinner (1992) questioned 119 students at the beginning of a university calculus course for physical science majors, all of whom obtained a grade of A on their mathematics matriculation exam. These students had all taken calculus in high school but then spent 2 or 3 years in compulsory military service before entering the university. Vinner's purpose was to determine what the students had retained about the notion of derivative from their high school coursework. He asked the students to respond in writing to the question, "what is a derivative?" His categorization of their answers, along with examples from each category, is presented below in the following quoted excerpt. The number in parentheses tells what percentage of the 119 students were determined to be in that category.

I. A correct conception of the derivative as a limit (6%).

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

The derivative is a limit of the function change in a very small time period when the change is very small and tends to zero.

II. A correct conception of the derivative in its visual sense (25%).

The derivative is a function which indicates the slope of the original function at each point.
The derivative is the slope of the ascent or descent of a function at a certain moment.

III. An instrumental conception of the derivative which relates to the methods of obtaining it or to its application but ignores its meaning. (23%).

The derivative is a function obtained from a given function by means of fixed mathematical rules.
The derivative is a subfunction of the original function, for instance:
$y = 2x^3 + 3x,\ y' = 6x^2 + 3.$
A derivative is a means to investigate the domains of increase or decrease of a given function.
It is a mathematical formula.

IV. A vague unacceptable reference to the concept of limit (8%).

It is a function tending to infinity.
It is $\lim \frac{f(x)}{d(x)},\quad dx \to 0.$

V. A vague unacceptable reference to the visual aspect of the derivative (26%).

It is a function which is a tangent to another function.
It is the equation of the tangent to a given function.
The derivative is a function whose graphical representation is a tangent.
The angle that the function forms with the x -axis.

VI. Answers which seem totally irrelevant or no answer (12%).
(Vinner, 1992, p. 210-211)

The most common correct response was to refer to the graphical interpretation of derivative. However almost as many students evoked the derivative in category III: merely something that is calculated, without referring to the structure or nature of the derivative itself or as a tool useful for applications.

Vinner was interested in whether the students mentioned the limiting process as seen in categories I and IV. However, as he mentioned, very few students had such a concern (6% correctly, 8% incorrectly). On the other hand, Vinner was not concerned with whether the student mentioned that the derivative is a function or simply described it at a point (see the examples in I and II). He also had no category or examples for interpretations of the derivative such as speed or velocity or for statements that the derivative refers to a rate of change. Perhaps these interpretations are not taught or at least not emphasized in the high school calculus courses taken by these students. It is important to note that students in these examples who had fragmented knowledge of the visual interpretation of derivative often mentioned the tangent, whereas the students with a correct visual interpretation thought of the derivative as referring to the slope of the function graph at a point, not the slope of the tangent line.

Understanding Derivative in a Technology-Intensive Calculus Course

For her dissertation research Heid (1984) taught two small experimental sections of applied calculus. She also collected data from students in a large, lecture-style, traditional, applied calculus course to use as a backdrop for her study. The experimental sections used the program *MuMath* to compute limits, derivatives and integrals and several programs written in BASIC to draw graphs of functions, make tables, and demonstrate Riemann sums.

The curriculum for the experimental sections emphasized concepts, postponing the teaching of symbolic differentiation and integration until the last 3 weeks of the semester. Heid's (1984) data consisted of interviews, written work such as quizzes, tests and assignments, and class transcripts. The data seemed to indicate that students in the experimental class scored as well as the students in the large lecture on a final exam that was based on traditional skills. Her data illustrated a richer conceptual understanding held by many of the experimental students.

Of particular interest in this review of research on student understanding of derivative is the data Heid collected in a series of interviews with a sample of students. Twenty students interviewed were chosen from volunteers in the three classes: seven from the experimental class, eight from the other experimental class, and five from a large lecture-style, traditional class. The sample was stratified based on the student's expressed opinion on the need for algorithms and rules versus a need for creativity in solving math problems. Seventeen students were interviewed four or five times, spread throughout the semester and the other three, one from each class, were interviewed only two or three times.

Heid (1984) asked the 17 students to explain what a derivative is. Most students responded to this question during the first interview the week after the first exam. The students from the lecture-style class had already studied derivative formulas for some functions. Such formulas had been mentioned briefly, but not emphasized in the first experimental class, and not mentioned at all in the second experimental class. The experimental classes instead could use *MuMath* to calculate any derivative formulas needed. The students in the experimental class had also studied and had been tested on more nontraditional problems and more problems involving real world models than had the large lecture class.

In answer to "what is a derivative?", nine out of twelve of the experimental students mentioned slope, but only one of these students mentioned a tangent line. Most referred to the derivative as the slope of a curve at a particular point. Four experimental students mentioned rate of change, two others mentioned rate only, and two more mentioned change only. The two students who did not mention slope or rate or change discussed derivative as useful for finding local extrema. Seven of the twelve experimental students mentioned both slope and one other interpretation of derivative. Of the four students from the lecture class interviewed, none correctly stated two different interpretations. One student said the derivative was the rate of change or rate of increase, a second student referred to derivative as how fast the curve is going, a third student said it was useful to find maxima or minima, and the fourth student incorrectly referred to derivative as change in slope or change in rate.

It is interesting to note that multiple students referred to derivative as "rate" or "change." In Heid's study the students were enrolled in a course called Applied Calculus and there was some emphasis, especially in the experimental section, on problems involving real world models.

One example of a real world problem occurred on the first exam for both the experimental and large lecture classes. (The experimental classes had worked with problems of this nature, but the large lecture class had not.) The problem was asked of both classes for comparison purposes:

> Suppose transportation specialists have determined that $G(v)$, the number of miles per gallon that a vehicle gets, is a function of the vehicle's speed, v, in miles per hour.
>
> (a) Interpret, in terms of mileage and speed, the fact that $G'(55) = .4$.
>
> (b) How might that fact be used in a debate about setting an appropriate national speed limit? (p. 156)

The percentage of students answering this question correctly was 69.6% for the first experimental class, 65.1% for the second experimental class, and 41.4% for the lecture-style class. Heid suggests that the difference between the experimental and large lecture classes might be due to the skill of students in the experimental classes at interpreting this information in terms of marginal values. "43.5% and 39.1% of the experimental students, as opposed to only 5.0% of the large lecture students, interpreted

$G'(55)$ as being approximately equal to the change in mileage experienced when the speed of the car increased from 54 to 55 or from 55 to 56" (p. 164). The error made by most students who missed the problem was to say that $G'(55)$ is the number of miles per gallon the car gets when driving at 55 miles per hour, i.e., the function value instead of the derivative value.

This problem was on the first major exam for all three classes. During the interviews following this exam when Heid asked students, "what is a derivative?", students expressed confusion about the notion of derivative as rate of change. Some students confused "rate of change" with "change" or "rate of change" with "change of rate". Other students thought only of "marginal change," a change in y for a unit change in x. Some students could state that derivative is "rate of change," but could not describe rate of change in other words. This suggested that the phrase might only have been memorized. Some students thought the phrase "rate of change" was appropriate only for narrowly defined situations, such as applied problems or word problems. Difficulties with the phrase "rate of change" occurred in both the experimental and the large lecture students.

Here is another type of problem that appears on the first major exam that was difficult for students in both the experimental and large-lecture sections: "Thus far in the course you've learned no rule for finding the derivative of a function like $f(x) = 3^x$. Explain how you can find $f'(4)$" (p. 157).

In the two experimental classes 13.0% and 8.7% of the students answered correctly, while in the lecture class only 0.9% of the students were correct. Although only a few students across all the classes recognized the derivative as the instantaneous rate of change, many experimental class students did describe the derivative as being approximated by the slopes of secant lines or by a marginal change in function values, i.e. $f'(4)$ is approximately $f(4) - f(3)$. The total percentage of students who explained, in some form, the notion that the derivative is approximated by the slopes of secant lines was 47.8% and 56.5% for the experimental classes and 9.0% for the large lecture class. In addition to this failure to describe the limiting process, the major error made by students, particularly in the large lecture class, was to make up an incorrect symbolic differentiation rule for a function, with 87.8% of the large lecture students and 34.8% and 17.3% of the experimental students made this type of error.

During the fourth interview Heid asked eight students from the experimental class and three students from the lecture class a similar problem: Explain why, "If $f(x) = x^2$ then $f'(x) = 2x$" (p. 100). By this point in the semester the eight experimental students were able to describe a derivative as a process of finding the limit of the slope of secant lines or the limit of a specific difference quotient, though two needed a hint to think about the graph. Heid noted that the wording differed from student to student and did not appear to be memorized. The only confusion was that a few students thought that the limiting process still only gave an approximate answer. In other words, they thought that the values get closer and closer but never reach the actual slope of the tangent line. None of the three students from the lecture class were able to describe this process accurately, although two of them knew the limiting process had something to do with the slope of a secant or tangent line.

Tall (1986) compared the understanding of the gradient (slope) of a curve at a point, tangents, and differential functions of sixth-form students (ages 16-18) in five control classes and three experimental classes. Students beginning an A-level university calculus course were also used for comparison. The university students had studied calculus in sixth-form and received at least one grade of A on an A-level mathematics examination.

The experimental, sixth-form classes used the same traditional curriculum as the sixth-form, control classes except that at certain points their teachers used a computer program and accompanying teaching suggestions to help students develop their understanding of the graphical interpretation of derivative. Each class had either one or two computers so students used the computers themselves infrequently. The experimental students were shown that for a differentiable function, the graph can be magnified to look straight at any point, and also that a series of secant lines approach a tangent line to the curve at a point with the slopes of the secant lines approaching the slope of the tangent. In addition, the computer program could graph an approximate derivative function from the graph of a given function. (For a choice of a small fixed value for Δx, the program repeatedly drew a secant line, displayed the slope of the secant line, and simultaneously plotted a point on the same axes with the slope value as its y value.)

Tall noticed several similarities and differences in the experimental and control groups. Both groups had equal ability at symbolic differentiation. Although students in all groups had various misconceptions about the tangent, including that it touches at only one point or that there may be infinitely many tangents at a point, the experimental students were less likely to make these types of errors.

Students in the experimental class also performed better than the control students on three other questions that were related to their work with the software program. About half of the experimental students and only one of the control students could produce a function that was not differentiable at $x = 1$. On the second of these questions, the students were given the graph of four functions and then asked to sketch the graph of the derivative of each function. The experimental classes averaged 15.83 to 17.86 out of 20 possible, while the control classes averaged 3.00 to 12.73 points. In another problem, Tall gave a graph of the derivative function and asked the students to choose the graph of the original function from three choices. The students were also asked to explain their choice. Of the experimental students, 67% chose the correct graph for an appropriate reason, while only 8% of the control students did so. The experimental students scored similarly on these problems to university students, who were the most well prepared for the study of mathematics.

Tall described in more detail the open-ended responses of the students when asked to "Explain what is meant by the derivative of a function at a point" (p. 318). Of the students, 71 (34%) mention "gradient" and 33 (16%) also referred to it as a function, while only 9 (4%) mentioned "rate of change" and 3 (1%) mentioned a limiting process. Only 1 student mentioned tangent, 40 (20%) of the students mentioned differentiation, and 78 (38%) made no response at all.

Tall noted that the textbook used at the sixth-form school referred to the derivative as the "gradient function" as did the software used in the experimental classes. Tall had no category for applications of differentiation. The sixth-form students, even those with previous polynomial calculus, had only studied the application of maxima and minima of a graph. They had not discussed the interpretation of derivative as speed or velocity. It is unclear whether the university students may have seen other applications and just not mentioned them.

In the comparison of matched pairs of experimental and control sixth-form students, Tall reported that of those students who have not taken calculus previously, the experimental students were much more likely to mention gradient than the control students (9 of 12 versus 1 of 12). For students with previous calculus experience there were more experimental students who mentioned gradient, but the difference was not statistically significant (17 of 27 versus 10 of 27).

Tall also noted the number of students referring to the derivative as a function (16%) and those mentioning a limiting process (1%). Tall asked these same students three other questions on the same post-test, with the derivative question that gave them the opportunity to mention the limiting process. He asked them to explain how one might calculate the gradient of the tangent line (of a specific example) "from first principles." In the sixth-form classrooms "from first principles" usually refers to calculating the derivative from the symbolic limit definition. Of the 204 students given this question, 31 students (15%) mentioned a limiting process. For the other two questions -- to explain what is meant by the tangent to a curve or the gradient of a nonlinear graph -- 8 (4%) and 10 (5%) students respectively mentioned a limiting process. Experimental students tended to make the error of stating that a gradient is the slope of a line connecting two very close points on the graph. Recall that this was the method used by the computer program to sketch an approximate gradient function. The program specified that this was not the actual gradient function, but apparently some experimental students did not make this distinction.

Crocker (1991) concentrated on the students in two experimental university calculus courses covering differential and integral calculus. The students used the computer software *Mathematica*, and their text consisted of *Mathematica* notebooks written by Brown, Porta, and Uhl (1991) at the University of Illinois and revised by the course instructor, Davis, for use at Ohio State University. The students were not required to meet together except for once a week for a quiz. Otherwise they could use the lab during any of its open hours. The lab was usually staffed by one or more teaching assistants or by the instructor.

To collect data, Crocker (1991) observed student interaction in the labs and interviewed 9 students, four times each, over the course of the two quarters. The students interviewed were chosen from volunteers (who were requested during the fifth weekly quiz time). Of the 38 students who initially registered for the two courses, 36 were present that day for the quiz and 27 of those volunteered. The volunteers were ranked by the teaching assistants as performing in the class at a high, medium or low level. Twelve students were chosen from the volunteers to stratify the group based on this criterion. Subsequently, three of the students did not take the second term of the experimental course and were dropped from the analysis. The interviews involved questions on differentiation, using the derivative to give information about the graph of the original function, comparing the graphs of a function and its derivative, integration, and some standard applications of integration.

Crocker was also interested in students' interactions with each other, the instructor, the teaching assistants, and the computer software. This review only discusses her findings related to student understanding of derivative.

The first interview was held late in the first quarter. The students were given a function, $y = xe^{-x}$, and then asked to use its derivative to give information about the graph of the original function. The students did not have access to a computer during the first interview. Only two of the nine students were able to correctly differentiate this function, so the discussion mostly focused on what the students would do if they had access to Mathematica. Most students said they would graph the function and its derivative. When asked what they could tell from the graph of the derivative without seeing the graph of the function, seven of the nine gave some answer. Two of the students could accurately use the idea that the derivative gives the slope to analyze where the function would be increasing and decreasing, and that a maximum or minimum would occur when the derivative equals zero. Another student could also state this information from the graph of the derivative but was unsure how to proceed just from the algebraic representation. The other four students knew that the derivative is zero at a maximum or minimum or that the derivative being positive means the function is increasing, but were unsure why. This led them to contradict themselves in other statements.

The second interview occurred during the first and second weeks of the second quarter. The students were asked the same question, but were allowed to use *Mathematica*. All students used *Mathematica* to graph the derivative of $y = xe^{-x}$, and eight of the nine could use this graph to predict the graph of the function. The one student who missed this problem thought that the graph of the function should be identical to the graph of the derivative.

The third interview occurred during the third and fourth weeks of the second quarter. The only question that related to derivatives in this interview was a discussion of the relationship between distance, velocity, and acceleration. Only four of the nine students were able to correctly state this relationship. Two students were partially correct and the other three did not state any relationship. The follow-up question gave a formula for acceleration with initial conditions for velocity and position and asked the student for the formulas for velocity and position. No students were able to completely solve this problem. Crocker stated that students became uneasy during this interview because many of the problems were unlike the ones they had seen in class. She did not state specifically if this problem fell into that category.

The fourth interview occurred during the fifth and sixth weeks of the second quarter. The first question Crocker asked was, "what is a derivative?" Seven of the students mentioned slope. Four of them said that the derivative is the "slope of the function". One said that it is the "slope of the tangent line." One said that it is the "slope of a line" without being more specific. One other student said that it is the "average slope." This student also tried incorrectly to state a difference quotient. The two students who did not mention slope referred to derivative as the process of differentiation. A few students mention other meanings for derivative as well. One mentioned "instantaneous growth rate," another related derivative to velocity, one states its use in finding maxima and minima, and one said derivative is the "opposite of integral."

During this same interview students were asked to do a related rates problem. One student was unable to begin, and the other eight students were able to give partial, but incomplete answers. Crocker noted that this class had not studied related rate problems as such, but that the relevant ideas had been covered.

The last part of the fourth interview concerned the relationship between the graphs of the function and its derivatives. All nine students could determine which graph was the function and which was the derivative when they were given in pairs. When given the graph of the derivative and asked for the graph of the function, all nine could do the first of these, eight could do the second and seven could do the third.

Crocker summarized the students' development of the concept of derivative by saying that by the middle of the second quarter the students have a "strong understanding" of derivative. The understanding of derivative continues to develop as the students study integration. She noted that students could "approach and solve problems much sooner than they were able to verbalize, accurately, the meaning of derivative" (p. 82), and that derivative as slope was a strong connection for these students.

Hart (1991) analyzed interview transcripts of 24 students, one 30-60 minute interview per student. These interviews were chosen for analysis from a larger pool of interviews by a random stratified sampling. Twelve of the interviews were from students near the end of the study of differential calculus and covered material on limits, continuity, and derivatives. The other twelve interviews were from students near the end of the study of integral calculus and covered integrals and antiderivatives. In each sample of twelve, half of the students came from experimental classes and the other half came from traditional large lecture classes. Each subsample of six was stratified across the grading scale so that two of the students were "A" students, two were "B" students two were "C" students.

Hart (1991) also analyzed the answers to written questions on functions, limits, continuity, derivatives, and integrals. Students answering these questions come from 12 different experimental classes (324 students total in those classes), 2 traditional classes and volunteers from four large-lecture traditional classes. The experimental classes came from 2 high schools, 6 two-year colleges and 4 four-year colleges or universities. In each of these settings the students used HP48 graphing calculators and material from an early version of the text by Dick and Patton (1992). Most of the classes participated by answering a subset of the questions.

Hart's analysis of the interviews focused on the choices students made while problem solving. In particular, she noted whether and when a student chooses to use a calculator and whether and when a student chooses to use a graphical, symbolic, or numeric representation. In the first interview two questions touched on student understanding of derivative. The first questions asked students to differentiate a polynomial. All students could correctly accomplish this task and only one student used a calculator. Even the student who used a calculator stated that he would normally do such a problem by hand.

The second question gave the students a function and its derivative symbolically and asked the students to find all critical points and classify these points as a local maximum, a local minimum, or neither. The derivative is given in factored form so that the critical points can be read off easily. Four of the six experimental students and two of the six traditional students were able to use information from the graphs of these functions to monitor their work.

During the second interview one question was related to student understanding of derivative. This question gave the students a sketch of the graph of the derivative function and asked the students to sketch the graph of the original function. Three of the six experimental students and only one of the six traditional students were able to accomplish this task.

Three written questions were relevant to student understanding of derivative. The first and second tasks were completed by 147 students (130 experimental and 17 traditional) and the third task was completed by 200 students (154 experimental and 46 traditional). It should be noted that the traditional students were not considered a control group but simply provided a backdrop of additional information.

For the first of these tasks the students were given a table of values for the distance traveled by a spider after a given number of seconds (in increments by .1). The question asked the student to "Estimate the instantaneous speed of the spider at 1.3 seconds." Approximately half of the experimental students, and only 2 of the 17 traditional students, could find an appropriate estimate.

The second written question asked the students to use the graph of the derivative function to determine where the original function was increasing, decreasing, concave up, and concave down. Approximately half of the experimental students and a third of the traditional students were able to complete each of these tasks.

The third written question on derivatives had two parts. In the first part, a graph was provided on a grid with a tangent line drawn at $x = 2.5$, and the students were asked to find $f'(2.5)$. Over 60 percent of both the traditional and experimental students were able to complete this task. The second part asked the students to sketch the graph of the derivative of the function whose graph is given in the first part, and 73% of the experimental students while only 18% of the traditional students were able to accomplish this task. This was the largest discrepancy between the traditional and experimental students on any of the tasks in Hart's study. Hart explained that the experimental students had seen problems similar to the type in both parts of the problem whereas the traditional students had not seen problems of either type.

In her dissertation, Ellison (1993) described the development of the concept image of derivative for 10 students during a semester-long university calculus course. The students came from two classes, one of which was taught by the researcher. Both classes used a traditional text, but incorporated activities involving TI-81 graphing calculators and the computer software, *A Graphic Approach to Calculus* (Tall, 1991). Each student was interviewed three times. Interview data was supplemented with pretests, posttests, unit exams, homework assignments, and an exit survey. Each student was treated as a separate case study and analyzed individually. Ellison also wrote a summary that compared the students to each other and put them into the context of the class as a whole.

Ellison focused her study around what she calls "five characteristics of a mature concept image of the derivative" (Ellison, 1993).

1. The first of these characteristics includes ideas of differentiability and its relationship to "local straightness" and tangent lines. If a function is differentiable at a point, magnification of the graph of that function at the point will cause the curve to appear increasingly straight. As long as this straight line is not vertical, its slope is the value of the derivative at that point. This characteristic also includes examples of nondifferentiable functions and the effect of magnification of the graph of these functions at a point of nondifferentiability. Finally, this characteristic includes knowledge of the equation of the tangent line and the idea that the tangent line is the best linear approximation to the curve at the point of tangency.

2. The second characteristic is the parallel nature of gradient (i.e. slope), tangent, instantaneous rate of change, and derivative. A function is differentiable at a point if it has a gradient at that point, a nonvertical tangent line at that point, and a value for instantaneous rate of change at that point. The value of the derivative is equal to the gradient, the slope of the tangent line, and the value of the instantaneous rate of change at that point.

3. The third characteristic emphasizes that the derivative is itself a function and that this derivative function may be represented by a graph that has as output values the gradients of the parent function. This characteristic includes being able to sketch the derivative graph given the graph of the parent function and vice versa. The understanding of the relationship between these graphs should be linked with the notions of derivative as the slope of a tangent line and as instantaneous rate of change.

4. The fourth characteristic is knowledge of the formal definition of derivative as a limit of difference quotients and an understanding of its relationship to a graphical understanding of derivative as slope of the tangent line. This includes understanding the role of limits, especially the visual picture of secant lines approaching the tangent line.

5. The fifth characteristic is knowledge of symbolic differentiation and its use in problem solving.

Ellison's characteristics emphasize a graphical approach. Four of the five characteristics include the notion of derivative as slope or gradient. On the other hand physical models such as velocity are not emphasized. Only two of the five characteristics mention instantaneous rate of change, and none mention a specific physical model. Ellison acknowledges that the wider mathematics community might not agree that these are the only or even the most important aspects of a concept image of derivative. However, this is the lens through which her results were analyzed, and her emphasis was in keeping with her choice of the software, *A Graphic Approach to Calculus,* for teaching the course.

At the beginning of the course, three of the ten subjects had already studied a year of high school calculus. However, the understanding they brought to this course consisted primarily of symbolic differentiation and the knowledge that the derivative gives the slope. A knowledge that derivative gives the slope does not, however, extend to the ability to apply that information. Three other students had been taught differentiation of polynomials at the end of their precalculus course but nothing more. The remaining four students had no exposure to calculus at all.

The most robust aspect of all of the students' concept images appeared to be the fifth characteristic, symbolic differentiation. By the end of the semester all students were skilled in this and most could use the symbolic derivative to produce graphs or analyze graphs. Additionally, most could use symbolic derivatives to solve standard optimization problems.

The majority of the students had gained a mental image of differentiable functions as those that are "locally straight," and they could produce and discuss examples of nondifferentiable functions. Students understood that differentiability at a point was equivalent to having a slope at that point, and the existence of the instantaneous rate of change at that point. Students knew that the derivative at a point is also the gradient at that point and the slope of the tangent line and the instantaneous rate of change. Thus, most students had mastered the basic ideas listed in characteristics 1 and 2.

Characteristic 3 was a little more difficult for students. By the end of the semester most could sketch the graph of the derivative given the graph of the "parent" function. Three students who had particular trouble with this made assumptions that the graph of the derivative must be some type of mirror image of the graph of the parent function. The students used different reflections for different problems. Students found graphing a parent function given the graph of the derivative function more difficult. By the end of the semester most students still had trouble using the graph of the derivative to draw conclusions about the slope of tangent lines or to reason about the graph of the parent function.

All students initially had misconceptions about tangent lines, assuming that they must only touch the function's graph once and must not pass through it. By the end of the semester most students were free of these misconceptions involving tangent lines. Vertical tangent lines were the most lasting source of cognitive conflict for all ten students. It was not until the third interview that students worked through their vertical tangent line misconceptions. No further data was collected after the third interview to determine if the students retained this understanding of vertical tangents.

Students also had difficulties imagining a tangent line as the limit of secant lines. No student could do this on the pretest, although 7 of the 10 could by the posttest. Four students had trouble with this idea because they concentrated on the secant segments becoming shorter or smaller and closer to the curve and not on the rotation of the secant lines into the tangent line.

Students were able to memorize the formal definition of derivative and evaluate the limit of appropriate difference quotients. However, students had much more difficulty linking the formal definition to its graphical interpretation.

Ellison also noted two difficulties with functions that may keep students from developing a stronger concept image of derivative. One student had a tendency to confuse the values of the function with the values of the derivative. She stated on the pretest that "Derivatives are used to find different points on the graph" (p. 435), and even in the third interview gave the value of $f(-2)$ when asked for $f'(-2)$. Several other students had problems on the pretest with functional notation. These were corrected early in the semester except for one student who as late as the third interview was not able to read the value of $f'(3)$ from a graph of the derivative function.

Advanced Mathematical Thinkers

Krussel (1995) studied visualization and reification in the mathematical conceptions held by advanced mathematical thinkers. She interviewed 3 advanced undergraduate mathematics majors, 4 mathematics graduate students, and 2 mathematics professors. The derivative was one of the concepts she investigated. For each concept the participant was asked to state the first thing he or she thought of and any visual images or theorems associated with the concept. The participants were also asked when they first learned the concept and whether they remembered any changes or milestones in their understanding of that concept. Krussel did not probe for any further aspects of the concept image not evoked by these questions. Therefore only each person's primary conceptions and any remembered changes in his or her understanding were reported. She analyzed the data for each person's major visual images and whether reification occurred, but she did not examine the person's understanding of any one concept, including derivative, in depth.

Given these restrictions, there are still results in Krussel's work relevant to this study. A summary of the responses regarding derivative reveals that:

1. All participants, except one undergraduate student, mentioned the symbolic process of taking derivatives.

2. All participants, except one undergraduate student, mentioned slope or tangent That undergraduate stated this as the "tangent of the line" (p. 102).

3. Five participants (two undergraduates, one graduate student and both faculty members) mentioned rate or rate of change with one additional graduate student mentioning velocity.

4. Two participants (an undergraduate and a graduate student) mentioned the formal limit definition.

5. Three participants (both faculty members and one graduate student) discussed linear approximation.

Again it should be emphasized that Krussel did not probe to determine what other understandings the participant might also possess. Those listed are what was evoked by the very general questions she asked. Krussel described a participant as having an operational understanding when his or her language focused on the process of taking derivatives and using the derivative as a tool. She noted that the participants often first evoked an image or a prototypical example from their first experiences with the concept and then used this as a hook to connect to more sophisticated parts of their concept image.

Five participants described major changes in their understanding of derivative. Two mentioned a change from derivative as a tool to understanding the definition. One undergraduate student described how he learned the formal limit definition of derivative at the same time he learned how to take derivatives and use them as a tool. However, he forgot what he had learned about the limit definition until he was reintroduced to it in a real analysis class. During that analysis class he was able to connect those two earlier understandings. A graduate student said that he could now "think of all the different things associated with derivatives" (p. 162), and that he understood what the definition means whereas before he could "crank the formula out" (p. 162).

Two others, one graduate student and one faculty member, described their understanding changing from thinking of derivative as a tool or rote operation to understanding the importance of the derivative to linear approximation. This graduate student and faculty member both mentioned learning the formal power of the derivative in abstract algebra as a major change in their understanding.

An extension of Krussel's analysis relates to the study by Amit and Vinner (1990). Amit and Vinner focused on one student who used the equation for the tangent line as the equation for the derivative. Krussel points out the importance of visual images as an entry point to other aspects of a person's concept image. Many of the participants in her study mentioned a visual image of a tangent line to a curve when asked about derivatives. Perhaps for less sophisticated students of mathematics the association of this visual image with derivative is equally as strong, but these students sometimes misinterpret this image as being the derivative itself.

One of the graduate students in Krussel's study stated both rate of change and slope of the tangent line as things that first came to his mind when asked about derivatives. Next he was asked about any visual images he has. He described that he had a visual image of the tangent line, but that he had no visual image for rate of change. Perhaps the lack of visual image for rate of change was one of the reasons that rate of change was only mentioned by five participants whereas eight mentioned slope. Another possibility is that mathematicians may have a more theoretical rather than applied focus. Four of the five participants who mentioned rate also mentioned being a physics major at one time or taking physics when first learning calculus.

It is interesting that only two participants, and neither of the faculty members, mentioned the formal limit definition of derivative. Krussel did not ask them to define derivative or to explain what a derivative is. She asked for what first came to mind when questioned about the concept and what images, examples, or theorems that they had associated with the concept. She also asked for changes in their understanding. It is possible that for those participants who had long ago grasped the formal definition, this was no longer memorable. Other participants may not have considered the definition important because they had a very operational understanding of derivative. One graduate student specifically stated that she did not consider the derivative as a concept, but only considered it a tool.

One graduate student and both faculty members emphasized the importance of the derivative to linear approximation, but no other participant mentioned it. Although linear approximation is covered in many freshman level calculus texts, it is not usually emphasized. None of the three who mentioned linear approximation described what caused this idea to gain in importance in their understanding.

Summary and Conclusions

Even young children have some notion of rate or speed and a notion of steepness. The studies by Nemirovsky, Rubin, and Monk show that students can examine graphically notions of rate of change and the relationship of graphs of the derivative and the original function without studying calculus.

Intuitive notions of rate as speed and slope as steepness are beginning points for a more detailed understanding of rate and slope in terms of the covariation of the input and output variables of a function. Thompson (1995a) and Nemirovsky and Rubin (1992) describe some improvement in student understanding of rate prior to studying calculus. However, the details of this covariation idea are still not clear to senior and graduate student mathematics education majors (Thompson, 1994). Note that in studying the concept of derivative, the covariation of both the original function and the derivative function must be considered. To produce a constant rate or an average rate over an interval one must examine the covariation of the input and output variables of the original function. To create a derivative or rate function, one must consider the covariation of the input values (the same inputs as for the original function) and the output values (rate values). The precalculus students in the study by Nemirovsky and Rubin (1992) were primarily struggling with the former while the advanced students in the study by Thompson (1994) struggled to coordinate the former with the latter.

Rate of change is a difficult concept for calculus students as well. Heid (1984) found students confusing change and rate of change or unable to explain rate of change in other words. The students in Crocker's (1991) study had difficulties relating distance, velocity and acceleration and in completing related rate problems. Hart (1991) found only half of experimental students and 2 of 17 traditional students could estimate the instantaneous speed of a spider given a table of values for its distance traveled.

Heid (1984) found that students who were instructed to think of instantaneous rate as approximated by the marginal value (average rate over one unit interval) can use this to successfully analyze real world problems involving the concept of rate. Over half of the students in her study (in both experimental and large-lecture sections) mentioned rate or change when asked what a derivative is. By contrast, fewer than half of the students in other studies of calculus students (Tall, 1986; Crocker, 1991; Vinner, 1992) mentioned rate or change under similar circumstances.

Slope is the response given by most students in each of the four studies to the question, "what is a derivative?" In fact, many of the students in the experimental studies describe derivative as the slope of the function or the slope of the curve at a point. Fewer students describe the derivative as the slope of the tangent line to the curve. None of these studies attempted to explain the infrequent mention of a tangent line.

Other studies, however, describe difficulties students have with tangent lines in the study of calculus. Vinner (1982), Tall (1986), and Ellison (1993) all documented students' difficulties with the notion of a tangent line. However, both Tall and Ellison found that these difficulties could be mostly remedied by the examination, through a graphing software package, of the most troublesome examples of this concept. Orton (1983), Tall (1986), and Ellison (1993) also noted student difficulties with the notion of secant lines approaching a tangent line. Again Ellison found this could be remedied through the discussion of appropriate examples illustrated with a graphing software package.

Ellison (1993), Tall (1986), and Heid (1984) noted student difficulties in explaining the meaning of the definition of derivative by describing the limit of the slope of secant lines approaching a tangent. Tall noted that some of the experimental students described the gradient at a point on the curve as the slope of the line connecting two nearby points. Heid also described experimental students as initially thinking of the derivative as the slope of one secant line very close to the tangent. However, by the end of the term these students could describe the limiting process.

One other type of problem was frequently posed in the studies of calculus students in technology-intensive curricula: Given the graph of the function sketch the graph of the derivative or vice-versa. Given the graph of the function, the majority of students in experimental sections could sketch the graph of the derivative (Tall; 1986, Hart, 1991; Ellison, 1993). In fact, both Tall and Hart mentioned that this ability was one of the most distinguishing differences between experimental and non-experimental students. As Ellison points out, sketching the graph of the function or giving information about the function from the graph of the derivative proves more difficult. However, the students in Crocker's (1991) study were able to score particularly well on this task by the middle of their second quarter of calculus study.

In general, the use of experimental (graphing) technology-intensive curricula tends to focus student understanding on the notion of derivative as slope and to build up graphical understandings of tangent lines, a tangent line as the limit of secant lines, and the relationship between the graph of the derivative and the graph of the original function. Except for Heid's (1984) study, these experimental curricula do not focus on the notion of derivative as rate of change. Even in Heid's study, difficulties with the notion of rate of change were apparent.

Chapter 3 — Theoretical Framework

In order to describe the understanding of derivative of individual students and its evolution over the course of a school year, I need to answer two questions:

1. What will I mean by *understanding the concept of derivative*?
2. Can I find a structured way to describe that understanding for an individual student?

This chapter describes my attempts to answer these two questions, and the research that influenced my answers. This synthesis brings together several theoretical frameworks highlighted in the literature review of chapter 2: the concept image framework of Tall, Vinner, and Dreyfus, the process-object framework of Sfard or Dubinsky and colleagues, and notions of multiple representations for function, limit, and derivative. Additional theoretical structures from the work of Fischbein (1987) and Lakoff (1987) described below provide more detail on the connections between the various conceptions of derivative.

The initial premise of my investigation into understanding is that for a concept as multifaceted as derivative it is not appropriate to ask simply whether or not a student understands the concept. Rather one should ask for a description of a student's understanding of the concept of derivative -- what aspects of the concept a student knows and the relationships a student sees between these aspects.

Concept Image

As described by Tall and Vinner (1981) a person's *concept image* for a particular concept is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). The research on students' concept image of function (e.g. Vinner and Dreyfus, 1989) shows that students have many different associations for this concept and that students do not always evoke the associations most useful for solving a given task. In fact, a student's concept image for function may contain notions that are mutually contradictory. The word *compartmentalization* is used to refer to either of these errors. A part of a student's concept image is considered to be compartmentalized, or separated from other parts of the concept image, when the student fails to connect this idea to other aspects of the concept image.

One aspect of the concept image that has been singled out by researchers in this area is a student's *concept definition.* The concept definition is the statement a student makes when asked to define the concept under discussion. This statement may or may not coincide with a definition acceptable to the mathematical community. As with other aspects of the concept image, the concept definition may or may not contradict other aspects of the student's concept image.

The notion of concept image as "the total cognitive structure that is associated with the concept" is broad enough to describe what I will mean by a student's understanding of derivative. The notions of concept definition and compartmentalization suggest that the concept image is structured, but they are not adequate to provide a detailed description of a student's understanding of derivative. I will use the phrase "a student's concept image of derivative" synonymously with "a student's understanding of derivative." What follows provides further means for describing the structure for this understanding.

The Role of Multiple Representations

As seen in the studies in the literature review, researchers using the notion of concept image often find that a person's concept image includes a number of different representations of the concept. For functions these include analytic or symbolic, graphic, numeric, verbal, and physical representations. Many calculus reform texts, including the one used in this study, have emphasized the use of multiple representations as a way to develop student understanding (e.g. Dick and Patton, 1992; Dick and Patton, 1994;

Hughes-Hallett, et al., 1994; Ostebee and Zorn, 1995). Just as functions have many representational environments, concepts that involve functions, such as the limit of a function or the derivative of a function, may be described in terms of the same variety of representations as functions.

The concept of derivative can be seen (a) graphically as the slope of the tangent line to a curve at a point or as the slope of the line a curve seems to approach under magnification, (b) verbally as the instantaneous rate of change, (c) physically as speed or velocity, and (d) symbolically as the limit of the difference quotient.

Many other physical examples are possible, and there are variations possible in the graphical, verbal, and symbolic descriptions. What do each of these descriptions of derivative have in common that allow us to say they represent the same thing? What are the relationships between the different representations? The next section on the process-object framework describes the similar structure underlying each representation. Later sections on analogies, salient examples, and diagrams explore the types of connections between the representations.

Process-Object Framework

The underlying structure of any representation of the concept of derivative can be seen as a function whose value at any point is the limit of the ratio of differences. This is most easily seen in the formal symbolic definition, e.g.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

As discussed in the literature review, each of these aspects of the concept of derivative -- function, limit and ratio -- can be viewed both as dynamic *processes* and as static *objects*. For example, functions may be seen as the process of taking an element in the domain and acting on it to produce an element in the range. Functions may also be viewed statically as a set of ordered pairs. Limits may be viewed dynamically as a process of approaching the limiting value or statically through the epsilon-delta definition. A ratio or a rational number may be thought of operationally as division, but also structurally as a pair of integers.

Sfard (1992) concentrates her research on the historical and psychological transition of mathematical concepts from a process or *operational* conception to a static *structural* conception. She describes three stages in the transition from an operational to a structural conception -- *interiorization*, *condensation*, and *reification*. Interiorization occurs when a person can step through the relevant process. Condensation occurs when the person can view that process as a whole and use it as a subprocess in other processes. Reification occurs when the process may be viewed structurally as an object. In Sfard's theory processes are operations on previously established objects. Each process is reified into an object to be acted on by other processes. This forms a chain of process-object transitions.

The concept of derivative contains three such transitions. The ratio process takes two objects (two differences, two lengths, a distance and a time, etc.) and acts by division. The reified object (the ratio, slope, velocity, etc.) is used by the next process, that of taking a limit. The limiting process "passes through" infinitely many of the ratios approaching a particular value (the limiting value, the slope at a point on a curve, the instantaneous velocity). The reified object, the limit, is used to define each value of the derivative function. The derivative function acts as a process of passing through (possibly) infinitely many input values and for each determining an output value given by the limit of the difference quotient at that point. The derivative function may also be viewed as a reified object, just as any function may. (Although the definition of the derivative in freshman calculus texts usually stops at this point, we might continue by stating that the derivative function may be thought of as an object that is the output of another process, the derivative operator.)

I will refer to each of these process-object entities -- ratio, limit, and function -- as a layer of the derivative concept. Suppose a student has not developed a structural conception of one of the layers. How can that student consider the next process in the derivative structure without an object to operate on? One simple solution is to use what Sfard (1992) calls a *pseudostructural* conception. A pseudostructural conception may be thought of as an object with no internal structure. In fact, even for a person who can conceptualize each layer as both a process and an object, it is often simpler to describe a process by having it operate on a pseudostructural "object."

Pseudostructural examples for the concept of function include viewing a graph or symbolic expression as an object to be manipulated without recognizing the domain, range and relationship (either dynamic or static) between the input and output values. A pseudostructural conception of limit refers exclusively to the value of the limit, the end result of the process, without recognition of the process that leads to that result or the epsilon-delta criterion that requires that result. A pseudostructural conception of a ratio would be to see a common fraction (e.g. $\frac{1}{2}$) as a single value or a place on the number line without recognizing that the ratio can also represent a division process.

For an example of a process operating on a pseudostructural object consider the derivative function as a process that gives us the speed at each point, like a car's speedometer. For this description a student can concentrate on the function process and it's output without, for the time being, working with the complications of the underlying limit or ratio processes.

Pseudostructural conceptions usually have the form of a gestalt. By this I mean that the conception is thought of as a whole without parts, a single entity without any underlying structure. Sometimes, as in the speedometer example above, a gestalt may be used to simplify a thought process . The details underlying the gestalt may be known, but not emphasized in that context. In other cases a student may not be aware of any underlying structure or may have compartmentalized any knowledge about the underlying structure so that the student does not evoke this information in an appropriate context. The recognition of underlying structure is the transition from a pseudostructural conception to an operational or process conception. This transition will be examined here as closely as the other transitions emphasized by Sfard.

Derivative Concept

We have seen that the derivative concept consists of a progression of three process-object layers -- ratio, limit, and function -- and that these layers can be described in several representations. This section describes the layers in each representation in more detail in preparation for developing a system for describing aspects of a student's understanding of derivative.

A difference quotient can be used to measure the average rate of change of the dependent variable with respect to change in the independent variable. The calculation of this ratio of differences is a process. We might represent this in Leibniz notation as $\frac{\Delta y}{\Delta x}$.

The consolidated process, the average rate, may be used as an object in the second process, the limiting process. The limiting process consists of analyzing a sequence of average rates of change as the difference in the denominator of the ratios goes to zero. We can represent this in Leibniz notation by $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. The limiting process is consolidated to an instantaneous rate of change, represented by $\frac{dy}{dx}$.

This consolidated process, the instantaneous rate of change at each input value, is used as an object in the construction of the derivative function. The function value at each point has already been described by the limiting process. The function process we will stress here is the covariation of the input values with the output or instantaneous rate of change values. The function as a process and object is not easily represented by the Leibniz notation. Below we describe a more complete symbolic representation, followed by an interpretation of these layers in terms of a graphic representation and a particularly important physical example: the velocity function as the derivative of the position function.

Symbolic

The first layer, the symbolic difference quotient, is often written as

$$\frac{f(x)-f(x_0)}{x-x_0} \quad \text{or} \quad \frac{f(x_0+h)-f(x_0)}{h}$$

where x and x_0 are values in the domain of the function and h is the distance between x and x_0. This quotient may be thought of as a process or an object. As an object it is acted on in the second layer by a limiting process:

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \quad \text{or} \quad f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

These expressions give the value of the derivative function at x_0. This limiting process must be thought of as consolidated and repeated for every value in the domain of f' to progress to the third layer, the derivative function. The formula

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is now considered as one that applies to a domain of possible infinitely many values of x.

The second layer object, the derivative at a point, and the third layer object, the derivative function, are conceptually very different. However, symbolically the difference is extremely subtle, the use of a subscript.

Slope -- A Graphic Interpretation of Derivative

Now we examine these three layers in a graphic representation. The first layer is the slope, specifically the rise over run, of a line connecting two points on the curve described by the graph of the function in question. The line connecting these two points is often referred to as a secant line. The process is the calculation of the rise over run. The object is the slope itself. For the purpose of building the layers we will think of one of the two points as being fixed; however, this layer may exist without reference to further layers, as the slope between any two points.

For the second layer we look at the limit of a sequence of slope values. These slope values may be thought of as the slopes of secant lines all going through one common point. We see that as the second point on the curve which determines the secant line approaches the common point, the secant lines approach the tangent line at the similar point. This is the limiting process. The object is the slope of the tangent line at that point.

Another way to view this limiting process graphically is to think of zooming in on the curve at a point of interest. At each or any step in the process of zooming in one may find the slope of a line between two points close to the point being zoomed in on. As one zooms in, one finds the slope determined by two points closer and closer to the point of interest. The object here is again the slope of the tangent line at the point of interest, but in this case we think of the tangent line as becoming a better and better approximation to the curve itself under magnification as opposed to a sequence of secant lines rotating to the position of the tangent.

In either case the third layer is the same. The limiting process, viewed through magnification or rotating secant lines, becomes consolidated so that this process may be thought to have occurred for every point on the curve of the original function's graph. The function process is the notion of running through every point on the original curve and extracting the instantaneous slope value. The object is the derivative function, whose graph can also be thought of as a curve itself.

Motion -- Interpreting the Derivative Function as a "Speedometer"

The context of motion actually gives us two models for derivative -- velocity if the function is displacement and acceleration if the function is velocity. I will concentrate on the former as a physical representation in which to examine the three layers.

The first layer process is the ratio of the change in distance (displacement) to the change in time. The first layer object is this average velocity. The second layer process consists of looking at average velocities over shorter and shorter intervals of time. This limiting process culminates in an instantaneous velocity. The third layer process consists of imagining the consolidated limiting process occurring for every moment in time so that the final result is a function that has associated with each moment in time an instantaneous velocity. The derivative function serves as a speedometer (or more accurately, a velocimeter) in this context.

Other interpretations or contexts are, of course, possible. In each case a parallel three-layered structure could be described. The totality of the three-layered structure paralleled in different environments and contexts and the links between these environments and contexts will be what we refer to as the structure of the concept of derivative.

Using the Framework for Describing the Structure of the Concept of Derivative

The structures developed above may be used to describe what the mathematical community means by the concept of derivative at the freshman calculus level. The same structures may also be used to describe the parts of an individual's concept image that coincide with the concept of derivative of the mathematical community. Missing in either case are the rules for taking derivatives symbolically such as the power, product and chain rules. These rules by-pass the limit of the difference quotient definition of derivative or any other description of the layers of the concept of derivative. This study will note an individual's knowledge of these rules, but the theoretical framework focuses on the concept of derivative as a three-layered structure that may be characterized in many different contexts or representations.

Each individual's understanding may be described in relation to the three-layered concept of derivative. Such a description will highlight the following:

1. What layers of the structure are available to the person?
2. In what representations or contexts are these layers available?
3. Does the person understand both the process and object nature of each layer?
4. Can the person coordinate all three layers simultaneously?
5. Does the person recognize the parallel nature of each of the layers in the symbolic, graphic, kinematic, and other settings?
6. Does the person prefer to use a particular representation or context as a model or prototype for the derivative concept when no representation or context is specified?
7. Does the person's understanding of derivative include ideas that do not fall into the three-layered structure of the concept of derivative? Does the student's concept image include understandings considered incorrect by the mathematical community?

It is not necessary for a person to think of the layers as "layers" or to name the dual nature of the layers with the terms "process" and "object." This terminology is that of the researcher. However, we suggest that a mathematician (or any person with a robust understanding of derivative) is aware of each of the layers even without naming them as such and is able to recreate that structure in any relevant context.

An additional note should be made about the process-object duality of these layers. As discussed above, the notion of a process becoming consolidated or even reified to form an object for a higher-level process follows the model developed by Sfard. At the same time each layer may be looked at as a gestalt without internal structure, what Sfard refers to as a pseudostructural conception. For example, the notion of slope may be thought of as steepness without recognition of the ratio process that gives the concept structure. Velocity may be thought of as speed without recognition of the ratio of distance over time. An instantaneous velocity may be simply thought of as a speedometer reading at an instant in time without considering a limiting process.

Any of these gestalts may be used as the objects for the higher level processes with or without the user having knowledge of the underlying structure. For this reason, in our analysis we will note first whether the gestalt object, the pseudostructural conception, is available, and second, whether the underlying process is known. If both are known to a person we can assume that the gestalt object also acts as the consolidated (or possibly reified) object. If only the gestalt object is known to the person and the underlying process is not, we will consider this to be a pseudostructural conception.

In the following chapter I will suggest a way to diagram the layers of the concept of derivative that a student mentions in answering interview questions or uses in working test items. The diagrams will provide both a summary of the concept of derivative as a three-layered structure characterized in many different contexts or representations, and a methodology for describing which of these aspects of the concept of derivative a student discusses. The methodology chapter also describes three interviews, each of which are used to determine a reasonable approximation to the student's concept image of derivative at the point in time that the interview is given. Each of these interviews provides enough opportunity for a student to describe each of the layers in several primary contexts and representations that a summary of the student's responses to one of those interview may be considered an approximate concept image.

Missing from the description of the structure of the concept of derivative in terms of layers and contexts is a discussion of the relationships a student sees between the derivative structure in two different representations or contexts. For example, does a student realize what the symbolic definition of derivative in terms of the limit of difference quotients has to do with stating that the derivative gives the slope of the function at any point or the speedometer reading for a car at any time? The following sections describe some of the relationships between the different representations or contexts of the derivative concept, and outlines the relationship of each of these to the concept as a whole.

Lakoff's Knowledge Structures

George Lakoff in his 1987 book, *Women, Fire and Dangerous Things,* provides a comprehensive view of the nature and organization of human knowledge. As an application of his description, he uses two case studies to demonstrate that all the meanings of a single word or all the ways of viewing a particular concept may be connected using the types of knowledge structures described in his book.

My description of the concept of derivative as a three-layered structure which recurs in a variety of representations and contexts is a first attempt at applying Lakoff's idea to the concept of derivative. To further this development I will describe four additional knowledge structures relevant to the concept of derivative that are also common in natural language or in intuitive mathematical thinking. These structures influence both how the mathematical community discusses the concept of derivative and how naive students come to make misstatements concerning the same concept. The four knowledge structures are 1) analogical models, 2) paradigmatic models, 3) diagrammatic models and 4) individual metonymy.

Lakoff is a linguist, and his emphasis is on structures that occur in natural language. For some similar and related structures with an emphasis on intuitive mathematical thinking I will refer to Efrain Fischbein's 1987 book *Intuition in Science and Mathematics*. My discussion here will combine Lakoff and Fischbein's approaches. Of the four knowledge structures I will discuss, the first two are described by both Lakoff and Fischbein, the third by Fischbein only and the fourth by Lakoff only. Since Fischbein's approach is more mathematical I will start this discussion from Fischbein's point of view and refer to Lakoff as appropriate.

Fischbein's Models

Fischbein (1987) describes 3 types of models that occur frequently in mathematics and science. The models are the analogical model, the paradigmatic model, and the diagrammatic model. Each of the representations and contexts in the structure of the derivative concept may function as a model of one or more of these three types.

"Generally speaking a system B represents a model of system A if, on the basis of a certain isomorphism, a description or solution produced in terms of A may be reflected consistently in terms of B and vice versa" (Fischbein, 1987, p. 121). The relationship between a model and the original entity has the following characteristics:

1. The model is faithful to the original based on a structured isomorphism between them. Thus the model may be used as a substitute for the original in reasoning processes involving the isomorphic structures.

2. The model is autonomous with respect to the original. In other words, one must be able to determine the relationship between two characteristics of the model or the outcome of some action in the system of the model without any reference to the entity or system being modeled. Without this autonomy a model would be useless as a substitute for the original.

3. The model is easier for a person to use than the original. The model may be more familiar, more concrete, or more easily manipulated than the original. The model is

valuable as a substitute since it allows a person to generate hypotheses or make judgments that were difficult or unclear with the original alone.

The above properties hold for all three types of models. The distinctions between analogical, paradigmatic, and diagrammatic models are based on the relationship between the model and the original. Each of the three models will be described briefly first, followed by more detailed examples related to the concept of derivative.

Analogical Models

"Two entities are considered to be in the relation of an analogy if there are some systematic similarities between them, which would entitle a person to assume the existence of other similarities as well" (Fischbein, 1987, p. 122). In an analogy, the model and the original are entities in two different environments. The environments may be closely related such as two different symbolic mathematical structures or very different such as a mathematical structure and something extramathematical. For a symbolic-symbolic analogy, Fischbein suggests the case of operations on imaginary numbers being defined by analogy with operations on real numbers. For a mathematical-extramathematical example, Krussel (1995) suggests the use of the image of dominoes falling as a metaphor for proof by induction.

Paradigmatic Models

Paradigmatic models describe our tendency to see a whole class of objects or an entire concept through the knowledge of particular examples or a submodel that exemplifies the concept or class. Not all examples are paradigmatic models, only those that provide enough variety of features to be representative of the entire group, yet are simple enough to be easy to use in reasoning. These representative examples are sometimes called exemplars or prototypes. Fischbein suggests that an irregular pentagon or hexagon might be an exemplar, a paradigmatic model, for the class of all polygons.

Simpler or more regular polygons such as squares and equilateral triangles are not general enough to function effectively as a sufficiently representative example for the entire class, while other examples such as irregular nonagons or dodecagons would be less effective as models since they are more complicated and less familiar.

Diagrammatic Models

Diagrammatic models are constructed specifically to serve as a synoptic, global representation of the original. "Generally speaking, diagrams are graphical representations of phenomena and relationships amongst them. Venn diagrams, tree diagrams, and histograms used for statistical representations, belong to this category" (Fischbein, 1987, p. 154). Diagrammatic models are not immediately interpretable models of a physical phenomena but must be interpreted using the conceptual structure that underlines their construction. Fischbein uses the example of a Cartesian graph as a diagrammatic model of the relationship between time and space in the case of falling bodies. He states that "there is no direct, sensorial similarity between the phenomenon of falling and the form of the graph. The graph represents, rather, a function (a conceptual structure) representing in turn, the constant relationship $s = s_0 - \frac{1}{2}gt^2$. No direct interpretation of the graph is possible (in terms of the real phenomenon) without an understanding of the intervening structure (the mathematical function)" (Fischbein, 1987, p. 159-160).

Fischbein sees the mathematical notion of function, especially in its symbolic or numeric form, as serving as an intermediary between the physical functional situation and the graphical representation. He says that the graph is in an analogical relationship to the symbolic or numeric function, but in a diagrammatic relationship to the physical phenomenon.

Relationship of Fischbein's Models to Lakoff's Knowledge Structures

Lakoff uses the word metaphor to describe the type of relationship that Fischbein calls an analogy. If a person experiences two structures in different domains as being isomorphic (at least in some limited way) then the person may set up a metaphorical mapping from one domain to the other preserving the appropriate structure.

Lakoff uses the word metonymy to refer to a type of cognitive structure based on the use of a part to stand in for the whole. A metonymic model consists of two entities A and B that are in the same conceptual structure. B is either part of A or closely related to it. "Compared to A, B is either easier to understand, easier to remember, easier to recognize, or more immediately useful for the given purpose in the given context" (p. 84). The metonymic model describes how B is used to represent A in the conceptual structure.

Metonymic models may model individuals or categories. Metonymic models of categories take an example or submodel and use it to stand for the entire category. Metonymic models of categories describe the relationship between a prototype and the whole category. These are what Fischbein calls paradigmatic models. Metonymic models of individuals do not completely fit the characterization of models given by Fischbein and are not described by him. These individual metonymic models will be discussed further below.

Lakoff does not discuss diagrams as a separate type of model. As a linguist, Lakoff concentrates on the verbal rather than the diagrammatic. At one point when he does discuss Venn diagrams he describes them in terms of metaphoric mappings.

Models Present in the Concept of Derivative Structure

In the structure of the concept of derivative described in the first part of this chapter the three layers of process-objects occur in symbolic, verbal, and graphical representations, as well as in contexts such as velocity, acceleration, the change in any entity over time or the change in any entity as it relates to the change in a second entity.

What are the relationships between the representations, contexts, and the overall concept of derivative?

Each of the contexts and representations are examples of the overall derivative structure. Velocity is a particularly important example, a paradigmatic model of the concept of derivative in a physical context. Velocity is an exemplar because it is an extremely familiar phenomenon for which we have additional natural language structure. For example, increasing velocity is called acceleration and decreasing velocity is called deceleration.

Velocity also has an analogical relationship to interpretations in other physical contexts. If f is a function that tells the outside temperature in degrees at a given time in hours past noon, what is meant by $f'(3)=4$? One may reason by analogy with position and velocity. If f was a function representing position in miles at a given time in hours past noon, it would mean that the speed is 4 miles per hour at 3 PM. So in this situation, it must mean that the temperature is increasing at 4 degrees per hour at 3 PM. Similarly, if we add the condition that $f''(3)=-2$, one might interpret that as a deceleration of 2 miles per hour per hour in the velocity context. Hence, by analogy, the temperature must be slowing its increase at a rate of 2 degrees per hour per hour. For some individuals, this analogy to velocity may be unnecessary or even cumbersome to use. For others it may be the easiest way for them to make sense of the information given.

This analogical model of velocity to other rates is present in our natural language as well. Words related to speed, such as fast and slow, are often used metaphorically in situations where there is no change in position over time. For example, he enjoyed a speedy recovery, or the economy slowed in the fourth quarter.

Just as velocity may be used as a metaphor for other types of rates, any context of the derivative that is familiar to a person may be used to model a less familiar context. The symbolic, numeric, and graphic representations provide additional analogical and diagrammatic models. As discussed in the section on diagrammatic models, Fischbein sees the graphical representation of function as being in an analogic relationship to the symbolic or numeric representation of a function, but in a diagrammatic relationship to the physical contexts in which the derivative may occur. Thus the symbolic derivative as given by the limit of the difference quotient has an analogical relationship with the graphical representation. Each symbol has a counterpart in terms of the graphical setting and the relationship between the symbols tells one how the parts of the graphical setting must be related in order to describe the derivative in that setting. Similarly there is an analogical relationship between the limit of the difference quotient and each of the physical contexts for the derivative. Each part of the symbolic description has a counterpart in the physical context, and the relationship between the symbols has an analogy in the physical context.

Since each of the contexts or representations of the derivative concept may act as a model for one of the others, it is relevant to consider when each type of model is most useful. Analogic relationships between the derivative in any two contexts are valuable when a context that is more familiar or more easily manipulated may be used to reason about a less familiar situation. The value of the analogic relationships between the symbolic and the others is that the symbolic allows for a simplicity of calculations whereas the other contexts are what give the symbolic representation meaning and relevance. The value of the graph as a diagrammatic model is its easily interpreted global characteristics -- positive, negative, increasing, decreasing, concave up or concave down -- and its ability to display the so many attributes of a function in a single image.

Individual Metonymy

After the three models of Fischbein, the fourth and final knowledge structure discussed in this chapter is individual metonymy. Recall that Lakoff defines metonymy as a type of cognitive structure based on the use of a part to stand in for the whole. A metonymic model consists of two entities A and B that are in the same conceptual structure. B is either part of A or closely related to it. "Compared to A, B is either easier to understand, easier to remember, easier to recognize, or more immediately useful for the given purpose in the given context" (p. 84). The metonymic model describes how B is used to represent A in the conceptual structure.

If the model, B, is an example of the original category or concept, A, then we may call B a paradigmatic model. If the model, B, is a part of A that is not an example, I will call this individual metonymy. For individual metonymy the part-whole relationship is between a part of an individual entity and the entity itself.

Lakoff uses the example of going to a party. The trip consists of a precondition that you have a way to get to the party, embarkation, the travel itself, arrival and an end point. If someone asks you how you got to the party, you would not recount the entire scenario. You might say, "I drove", letting the center stand for the whole. Alternatively you might say, "I have a car", letting the precondition stand for the whole.

Functions (including the derivative function) have a structure similar to a trip. There is a domain of starting values, a rule or correspondence, the calculation of which is analogous to traveling, and an end point or value of the function for each starting value. We sometimes name a function by referring only to its rule or correspondence without reference to its domain or range. This short hand is a type of individual metonymy, letting the part stand for the whole. The short hand also provides an emphasis on one aspect of the whole over other aspects.

Individual metonymy does not have all the characteristics of models specified by Fischbein. In particular, individual metonymy is not useful for making generalizations or understanding the structure of the original because the metonymic model is not a faithful representation of the entire concept. The value of individual metonymy in natural language is for brevity of expression or emphasis on a particular aspect of a concept.

For example, we use the word derivative to refer to both the derivative at a point (instantaneous velocity, slope at a point) and the derivative function (velocity function, slope function). What is the relationship between these two notions that leads us to call both by the same name? I will use a direct parallel to two nonmathematical examples from Lakoff to argue that the above mathematical relationship is a type of individual metonymy.

A different example of individual metonymy from Lakoff concerns the use of the word "over" to refer both to a trip, "He walked over the hill," and to its destination, "He lives over the hill." Similarly, "She traveled through the woods" and "She lives through the woods". In both cases the same word is used to refer to the path and the endpoint of that path. This theme also occurs in our mathematical language. Consider our use of the word derivative. We use derivative both to refer to the whole function, the derivative function, and to the output (or end point) values of the derivative function. In this case we are not letting the output stand for the whole, but we are giving the same name to the part and to the whole. The relationship between the two entities with the same name is the relationship of part to whole.

It must be remembered that the existence of a metonymic (part-whole) relationship between two entities is not enough to guarantee that these entities may be properly referred to using the same word or that the part may properly be used to stand in for the whole. The use of the same word for both the part and the whole is motivated by the part-whole relationship, but is not implied by it. Lakoff's theory does not address why some part-whole pairs are called by the same name and others are not, or why some parts may be used to represent the whole and others may not, except that it is a cultural artifact.

Now let's look more closely at the concept of derivative. The concept of derivative has a multipart structure described by 3-layers of process-objects, the representations and contexts in which the layers occur, and the models that connect the representations and contexts. There are numerous part-whole relationships evident in this structure. By mathematical convention, the only parts of this whole concept of derivative which are properly called by the name derivative are the derivative function and the value of that derivative function for a specific point. Just thought of in terms of the layers, we are giving the same name to the object that is the result of the second layer (limit) process and to the object that is the result of the third layer (function) process.

Potential Misconceptions

Just as each of the four knowledge structures has potential benefits, each has potential drawbacks as well. For the three models of Fischbein, a potential hazard is that features of the model that are not part of the isomorphic relationship between the model and the original may be interpreted as such by the naive student.

Two types of analogical models provide clear examples of faulty analogies. As an example of a faulty mathematical-extramathematical analogy, Fischbein discusses the interpretation of the equals sign as relating the inputs and outputs of a process. With this interpretation a student may consider $3+7=10$ to have the meaning that 3 and 7 combine to make 10, whereas $10=3+7$ is meaningless. This leads to other mathematical writing errors such as $3+7=10-2=8$. I have seen a similar phenomenon where calculus students may write $\ln x = \frac{1}{x}$ where $\ln x$ is the input to the derivative operator and $\frac{1}{x}$ is the output.

For an example of a faulty symbolic-symbolic analogy I would suggest the common student error $(a+b)^2 = a^2 + b^2$ or $\cos(a+b) = \cos a + \cos b$. Students try to use an analogy based on similarity of structure to generalize the distributive property of multiplication over addition to other possible distributions of a function over addition. For a calculus example, consider the generalization of the linearity of the derivative operator to distributing the derivative operator over quotients, e.g. $f(x) = \frac{\sin x}{x^2}$ yields $f'(x) = \frac{\cos x}{2x}$.

Both confusions with the symbolic come not only from faulty analogies, but also from a lack of proper analogies to give meaning to the symbols and a way of checking for errors. For example, a better analogy for the equals sign is that of the fulcrum of a balance with equal weights on either side. The left and right expressions are the weights.

Difficulties with the analogies between the various representations or contexts of the structure of derivative are usually the lack of realization of a complete isomorphism rather than the inappropriate extension of unrelated characteristics.

The paradigmatic model has its own set of potential downfalls. Fischbein points out that Vinner's (1982) study on student understanding of tangents provides an example of this phenomenon. In that study, students used the example of a tangent to a circle as their model for tangents to more general functions. Every point on a circle has a tangent and that tangent will touch the circle at one and only one point. This will not always be the case for points on more general curves. Hence the use of this paradigmatic model was problematic for these students.

I suggest that another example of this phenomenon occurs in the work of Vinner and Dreyfus (1989) on student understanding of function. This example is best understood in light of two examples of metonymic (i.e. paradigmatic) reasoning from Lakoff.

Lakoff discusses an example based on the work of Rips (1975) to show how typical examples are used in reasoning. Subjects considered robins to be typical birds and ducks to be nontypical birds. They "inferred that if the robins on a certain island got a disease, then the ducks would, but not the converse. Such examples are common. It is normal for us to make inferences from typical to nontypical examples" (Lakoff, 1987, p. 86). Lakoff continues with a second example, "If a typical man has hair on his head, we infer that atypical men (all other things being equal) will have hair on their heads. Moreover, a man may be considered atypical by virtue of not having hair on his head" (Lakoff, 1987, p. 86).

Some of the results of Vinner and Dreyfus (1989) on student understanding of function may be described by this type of metonymic reasoning. A typical example of a function is a continuous function. Since typical functions are continuous, a student may expect all functions to be continuous. Even if a discontinuous function is recognized as a function, it may be considered an atypical function and given a special designation. Recall that some students in Vinner and Dreyfus's study stated that a discontinuous function was not a function because it wasn't continuous, whereas others categorized a discontinuous function as a function for the same reason, stating specifically that the function was of that atypical subcategory, a discontinuous function.

Fallacies in using diagrammatic models, particularly Cartesian graphs, come from viewing the diagram as a pictorial or immediately interpretable image. (Note: Monk calls this "iconic translation.") Depending on what phenomenon the graph is recording, the graph may be shaped considerably differently than the physical action. Suppose a bicyclist travels over a hill. Her speed slows as she reaches the top and speeds up as she comes down the other side. In the physical situation, the bicyclist goes up and then down. On the other hand, the graph of the velocity function curves down and then up.

With individual metonymy there is no implied isomorphism between the two entities that have a part-whole relationship. Two entities may be given the same name by convention or they may not. However, students may not possess the knowledge to either recognize which metonymic connections are conventional and which are not, or recall distinctions in the metonymically connected entities when solving problems. Experienced users of mathematical language know which uses of the word derivative are acceptable by the mathematical community and which are not. Naive students may not be aware of the distinctions and this may lead to error.

One type of error involves equating two items that have the same name because of the metonymic connection, but do not share other aspects of their structure. Consider a student who is asked to find the equation of the tangent line to the curve $y = x^2$ at the origin. The student correctly calculates the derivative function as $y' = 2x$ and then writes the equation of the tangent line as $y = (2x)x$. Here the student substitutes the derivative function for the derivative value at a point.

Further problems occur when a student equates two parts of the derivative concept that have a metonymic connection but are not given the same name by the mathematical community. Here the metonymic relationship may be between a part of the concept of derivative structure and the whole structure or by transitivity between two different parts of the concept of derivative structure. One recurring example is that of equating the tangent line and the derivative concept itself, or the tangent line and some other part of the derivative concept such as the derivative at a point or the derivative function.

Amit and Vinner (1990) analyzed in detail the written answers of one student called Ron. Given a clearly marked graph of a function with a tangent line drawn in at one point, Ron was able to read off the value of the function at the point of tangency and use the slope of the tangent line to determine the value of the derivative at the point of tangency. In answer to "what is a derivative?", Ron wrote, "The derivative is the slope of the tangent to the graph at a certain point" (p. 7). He went on to explain that the derivative function tells the slope of the tangent to the function at any point, and was able to correctly state the definition of derivative as the limit of a difference quotient.

Ron seemed to have an idea of the relationship of tangent lines to the concept of derivative. However, in a different problem Ron made the mistake of identifying the tangent line with the derivative. Ron calculated the equation for the tangent line at the point of tangency and in the next problem used this equation as if it were the derivative function. He integrated the tangent line equation to find the equation for the original function. This usage was not only wrong, but it contradicted his other correct answers. Amit and Vinner explain this as an instance of compartmentalization. I would agree and also point out that this is an example of individual metonymy. Ron equated a part of the derivative concept, the tangent line, with another part of the derivative concept, the derivative function.

Even mathematicians who clearly recognize the distinction between the tangent line and the derivative function may not always state that precisely. William P. Thurston (1994), the noted differential geometer, makes a list of different ways of understanding the concept of derivative. This list includes the formal definition, "the derivative is the slope of a line tangent to the graph of the function", "the instantaneous speed of $f(t)$, when t is time," "the derivative of a function is the best linear approximation to the function near a point," and "the derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power" (p. 163). The last two descriptions listed are clearly important to a complete understanding of derivative, but each of them describes the tangent line itself and *not* the slope of that line. If a noted mathematician is not always careful to distinguish between the two concepts, it is not surprising that a student like Ron would sometimes fail to make the distinction. The difference, I suspect, is that a mathematician's knowledge is not as compartmentalized as Ron's knowledge. A mathematician would not make the mistake Ron did of using the equation for a tangent line at a particular point as if it were the equation for the derivative function at any point.

The individual metonymic mechanism is a linguistic short-hand motivated by the part-whole connection. It gives a certain brevity to our speech or allows us to emphasize a certain aspect of the whole. Unlike analogies or paradigmatic models, no further extensions of meaning are implied. Individual metonymy is not particularly useful in reasoning or developing theories.

Individual metonymy provides a source of possible confusion to the inexperienced user of the terminology. Errors are caused by the assumption that the part and the whole (or that two parts of the whole -- both considered equivalent to the whole) are the same or may be used in the same way. Important distinctions are ignored.

Summary

This chapter gives a description of the structure of the concept of derivative that I will use to analyze the understanding of the concept of derivative of each of the students in this study and how this understanding evolves over time.

The structure includes three layers of process-objects -- ratio, limit and function -- each of which are present in a variety of representations and contexts. These include symbolic, numeric, graphic, and verbal representations as well as physical contexts such as velocity or acceleration. The connections between the representations or contexts are described in terms of analogical, paradigmatic and diagrammatic models as well as individual metonymy. The next chapter discusses the methodology used to collect data and analyze it according to these layers, representations and models.

Chapter 4 — Methodology

Principal Research Question

How does a student's understanding of derivative evolve? The first step in answering this question was to determine a framework for discussing how a student's understanding is structured and how this structure could be most clearly described. The researcher's framework is described in Chapter 3 as the structure of the derivative concept. This structure consists of models for the derivative concept including slope, velocity, rate of change and the limit of the difference quotient definition. For each of these models the structure notes three aspects of the derivative concept: a ratio or rate, a limiting process which gives an instantaneous value, and the notion of derivative as a function.

Each individual's understanding may be described in relation to the three-layered concept of derivative. Such a description will highlight the following:

1. What layers of the structure are available to the person?
2. In what representations or contexts are these layers available?
3. Does the person understand both the process and object nature of each layer?
4. Can the person coordinate all three layers simultaneously?
5. Does the person recognize the parallel nature of each of the layers in the symbolic, graphic, kinematic, and other settings?
6. Does the person prefer to use a particular representation or context as a model or prototype for the derivative concept when no representation or context is specified?
7. Does the person's understanding of derivative include ideas that do not fall into the three-layered structure of the concept of derivative? Does the student's concept image include understandings considered incorrect by the mathematical community?

The structure of the derivative concept was developed from a review of the literature as well as being refined through observing the variety of notions of derivative stated by students in a pilot study. The details of a pilot study including its influence on both the theoretical framework and the data collected in the main study form the first part of this chapter. In the second part of the chapter we describe a diagrammatic scheme that provides both a summary of the concept of derivative as described in the previous chapter and a methodology for recording which of these aspects of the concept of derivative a student mentions in answering interview questions or uses in working test items. The third part of the chapter describes the main study -- the setting and subjects, what data was collected and how the data was analyzed.

Advanced Placement (AP) Calculus

Both the pilot study and the main study involve subjects who were high school students taking an Advanced Placement (AP) Calculus course. Since high school calculus curricula are often influenced by the AP syllabus and exam (Simonsen, 1995), we will describe the AP program briefly before continuing with the description of the pilot study.

The College Board gives Advanced Placement exams in many subjects to high school students each spring. Grades on all AP exams are given as an integer between 1 and 5 with 5 being the highest: 5 -- extremely well qualified, 4 -- well qualified, 3 -- qualified, 2 -- possibly qualified, 1 -- no recommendation. Many colleges award credit or advanced standing for grades of 4 or 5 and in some cases for a 3. These awards vary from college to college.

AP Calculus is divided into two courses, the AB course and the BC course. Each course has its own exam. The syllabi for both the AB and BC Advanced Placement courses for the 1993-1994 school year are given in Appendix A. The principal difference is that the AB exam covers approximately two-thirds of the usual university freshman calculus sequence, whereas the BC exam covers the full year.

The AP calculus syllabi recently have been in transition. The year this study was conducted, the 1993-1994 school year, was the last year that students were *not* allowed to use graphing calculators on either AP calculus exam. Scientific calculators were allowed. The test format consisted of a multiple-choice section with 45 questions and a 90-minute time allotment and a free-response section with six questions and a 90-minute time allotment. The free-response questions required the students to show how they arrived at their answers.

Pilot Study

During the spring of 1993, the spring prior to the beginning of the main study, the researcher conducted a pilot study. The purposes of the pilot study were to test written and interview questions for their effectiveness in gaining information about a student's conception of derivative, for the researcher to gain experience in interviewing and asking appropriate follow-up questions, and to use the breadth of student responses on the concept of derivative to help refine the theoretical framework, i.e., what is meant by a student's understanding of the concept of derivative.

At the time of the pilot study the researcher intended to probe for different aspects of a student's conception of the derivative and its uses. The researcher was also interested in whether a student would refer to various notation systems (symbols, graphs, tables) or media for calculation (calculators, computers, paper and pencil). The researcher developed a list of written questions with these goals in mind and intended to use some of the same questions for the interviews. The questions are listed in Table 4.1.

The written questions were designed to start with very general questions and develop into more specific questions. Thus, students were asked to write and turn in their answer to "What is a derivative?" before seeing any of the other questions.

Table 4.1. Pilot Study Written Questions

Page	Questions
1	1. What is a derivative?
2	2. What can derivatives be useful for? In other words, what types of problems can derivatives be used to help solve? List as many different types as you can. 3. Explain what a derivative is in a way that a person with very little mathematics background could understand. 4. List a few real world situations which involve the concept of derivative. 5. How can you tell if a function is differentiable?
3	The next set of questions ask you to "Give an example of a problem" that fits certain constraints. For these exercises it is not necessary to give a detailed description of a problem. Simply give enough information so that the reader will recognize the type of problem. Make sure it is clear why your problem fits the given constraints. 6. Give an example of a problem involving the concept of derivative in which you would use a calculator or computer to help you solve the problem. 7. Give an example of a problem involving the concept of derivative in which you would not use a calculator or computer to help you solve the problem. 8. Give an example of a problem involving the concept of derivative in which you would use a graph to help you solve the problem. 9. Give an example of a problem involving the concept of derivative in which you would use a table of values to help you solve the problem.
	10. Give an example of a problem involving the concept of derivative in which you would use a symbolic calculation to help you solve the problem.
	11. Give an example of a problem involving the concept of derivative in which you would not use any of the above, i.e. a graph, a table of values or a symbolic calculation to help you solve the problem. *HP user only* 12.a. Give an example of a problem involving the concept of derivative in which you would use a calculator's graphical capabilities to help you solve the problem. 12.b. Give an example of a problem involving the concept of derivative in which you would use a calculator's solver capabilities to help you solve the problem. 12.c. Give an example of a problem involving the concept of derivative in which you would use a calculator's symbolic differentiator to help you solve the problem.

Table 4.1. Pilot Study Written Questions (continued)

4	For each of the following terms state whether or not its mathematical meaning is related to the concept of a derivative in any way. If it is related, explain in what way it is related. If it is not related, explain why not. 13. slope 14. speed (velocity) 15. change (rate of change) 16. line or linear 17. measurement 18. prediction (approximation) 19. optimization 20. continuity 21. limit 22. integral 23. function 24. differential equations
5	For each of the following state whether the theorem or method involves the concept of derivative. If the concept of derivative is involved explain how. If not, explain why not. 25. Newton's Method 26. l'Hôpital's Rule 27. Intermediate Value Theorem 28. Mean Value Theorem 29. Fundamental Theorem of Calculus
6	30. Give a formal mathematical definition for a derivative as used in calculus. 31. Compare your answer to question 3, an explanation of derivative to someone with very little mathematics background, with your answer to question 29, a formal definition for derivative. What is the relationship, if any, between your answers to these two questions? If there is no relationship, explain why there does not need to be one.
7	L1. Is there anything you know about the concept of derivative or its applications which you have not had a chance to express in any of the above questions? If so, give a brief description or outline of what is missing. L2. Which question, of those numbered 1-30, was hardest for you to answer? Why? L3. Which question, of 1-30, was easiest for you to answer? Why?
8	*Demographic Information* D1. Are you male or female? D2. How old are you? D3. Do you plan on attending college next fall? If so, what college will you attend and what is your intended major? If not, what will you be doing instead? D4. Do you own a graphing calculator? If so, what model? D5. How often do you use a graphing calculator for calculus?

The written questions were administered to a total of 66 students from four different high schools. Table 4.2 contains more detail about the written data collected. In addition, some of these same questions were asked in an interview setting to 7 students, at three of the four schools. Due to time restrictions, no student interviewed was able to answer all of the questions which appeared in the written survey. The students who were interviewed during the first half of the class period were asked questions from the beginning of the survey (answering on average the first 14 questions). The students interviewed during the second half of class began with question 13 and answered through survey question 29.

Table 4.2. Data Collected from the Subjects of the Pilot Study

School	Subjects for this study	# of subjects interviewed	Average # written questions answered by non interview subjects (of 30)	Time subjects spent answering written questions	Amount of detail subjects provided on written questions
A	17 students: 16 AB, 1 BC	3 students: 3 AB	21.4	one class period (about 50 minutes)	little detail
B	23 students: all from one class	3 students	15.1	one class period (about 50 minutes)	much detail
C	12 students: all from one class 10 AB, 2 BC	2 students: both AB	20.7	one class period (about 50 minutes)	little detail
D	12 students: all from AB class; only 5 took exam	0	27.1	some time in each of 7 class periods (70-90 minutes)	fair detail

Each of the four schools were located in the southern part of the United States. Information about each of the four schools and the calculus classes of the subjects for the pilot study may be found in Table 4.3.

The researcher visited Schools A, B, and C near the end of the school year, after students had taken the AP exam. On the day of her visit the students in attendance spent their 50-minute class period answering as many of the written questions as they had time for. Concurrently the researcher interviewed students for 15-20 minutes each. At Schools A and C the students were interviewed individually. At School B one student was interviewed individually and the other two students were interviewed together. The subjects were generally not able to complete the questionnaire in the time allowed. The subjects at School B seemed to put more effort into answering the written questions than did the students at School A and School C.

The researcher did not visit School D. At that school, the AB calculus teacher had the students answer the questions a few at a time over seven class days. Students were given points on their final exam for a reasonable effort on each question.

Table 4.3. Settings for the Subjects of the Pilot Study

School	Description of school	# and level of calculus classes	Subjects for this study	Technology used by subjects	Text or Curriculum used by subjects
A	non affluent, rural	1 class: combination AB & BC	17 students: 16 AB, 1 BC	1 HP-48 1 TI-81 1 *Mathcad*	Leithold (traditional)
B	school for gifted students	6-8 classes: advised not to take AP exam	23 students: all from one class	all *Mathcad*, Derive 13 TI-81 2 Casio 7700	innovative curriculum emphasizing real world data
C	fairly affluent, suburban	2 classes: both combination AB & BC	12 students: all from one class 10 AB, 2 BC	all *Mathcad* 1 TI-81	same as school B with additions so students could take AP exam
D	fairly affluent, urban	2 classes: 1 AB class and 1 BC class	12 students: all from AB class; only 5 took exam	all HP-48	Dick and Patton (emphasizes multiple representations)

The written data from each student was analyzed for: the model of the concept of derivative referred to by the student in the answer to each of the questions, whether the applications stated were physical or theoretical, whether the students mentioned characteristics of the derivative (function, instantaneous), and what notation systems or calculation devises were mentioned. The researcher made a table for each student's responses as illustrated in the sample table (Table 4.4) for student 2 from School D (D2). Student D2 had time to complete all 30 questions. If she referred to the concept in the left column in her response to a question, that question is listed in the right column. Additional comments deemed interesting by the researcher but not fitting in the columns are listed at the bottom of the table.

Other information, such as the frequency with which various models of derivative or characteristics of derivative were referred to, was examined for each class as a whole. What follows in an analysis of D2's writings as well as their relationship to the writings of the students in her class and in the other classes.

D2 saw the derivative as the rate of change (Q1) and as slope (Q3). She recognized the instantaneous nature of the derivative on three occasions (Q1, Q3, Q15). She did not take any opportunity to refer to the derivative as a function. She recognized the derivative as being useful for problems involving related rates, velocity, minimal path, max/min, inflection points, Newton's Method and l'Hôpital's rule. Note that most of her applications refer to theoretical constructs as opposed to real world situations. This is not to say that she doesn't recognize max/min problems or minimal path problems as involving real world situations, but she did not mention any real world situations in her answers.

When she was asked to give an example of a problem involving the concept of derivative which satisfied certain constraints (Q6-11c), she repeatedly suggested finding the derivative of a function given as an equation. These responses were interpreted as her having a somewhat computational notion of the derivative. Her earlier responses stated that derivative is rate of change or slope and that derivatives can be used to solve a number of different types of problems, but only symbolic derivative computations and the application of velocity were incorporated in the problems she designed.

Table 4.4. Summary of Student D2's Answers to the Pilot Study Questionnaire

D2	answered Q1-31
rate of change	Q1(is), Q2(related rate), Q15
slope	Q3(is), Q13, Q28
velocity, speed, acceleration	Q8, Q12a, Q14
symbolic operation	Q6(find the deriv of $y = ...$), Q7(find the deriv of $y = ...$), Q10(find the deriv of $y = ...$), Q11(find the deriv of $y = ...$), Q11b(find the deriv of $y = ...$), Q11c(find the deriv of $y = ...$)
limit definition	Q30(correct limit definition)
applications -- physical	related rate(Q2) • velocity, acceleration(Q8) • velocity, displacement(Q11a) • velocity(Q14)
applications -- nonphysical	minimal path problem, max/min, inflection points(Q2) • max/min the area of an object, minimal path from point A to point B(Q4) • find the limit by approximating values of x (Q9) • max/min area or volume(Q17) • Newton's method(Q18) • minimal path(Q19) • l'Hopital's rule(Q21) • max/min, inflection points(Q23) • Newton's method(Q25) • l'Hopital's rule(Q26)
characteristics--function	
characteristics--instantaneous	instantaneous(Q1) • at a point(Q3) • instantaneous(Q15)
characteristics--differentiability, continuity	To be differentiable, a function must be continuous.(Q5) • continuity is a requirement for a function to be differentiable(Q20)
graph	Q3(slope), Q8(x -axis), Q11a(x -axis), Q13(slope), Q28(slope)
table of values	
symbolic expression	Q6($y = ...$), Q7($y = ...$), Q8($x^3 - 2x^2 + 4$), Q9($2x^2 - 4x + 5$), Q10($y = ...$), Q11($y = ...$), Q11a($v(t)...$), Q11b($y = ...$), Q11c($y = ...$), Q30($\frac{dy}{dx} = \lim_{h \to 0} ...$)
media -- paper, calculator or computer	Q6(calculator or computer: differentiator), Q6,7(same problem, differ in complexity of problem to be differentiated), Q10, 11b(solver), Q11c(calculator's differentiator)

Q9: find the $\lim_{x \to 0}$ of ... by approximating values of x .
Q16: derivative of a linear function is a constant
Q22: the integral of a derivative of a function is the original function.
Q24: a differential equation is based on the concept of antiderivative
Q27: the IVT is an application of continuity of a function not of a derivative.
Q28: the MVT is used to prove that functions with equal slopes have equal derivatives.
Q29: The FTC is used to find the area under a curve which uses antiderivatives.
Q31: there doesn't need to be any direct relationship because the above definition is purely mathematical but the definition I gave in Q3 is purely conceptual.
L1: I didn't discuss the antiderivative and its applications ...

In terms of notation systems, she only directly mentioned a graphical representation five times out of 33 questions answered. In three instances, the word slope was the only graphical reference. In the other two cases, the x-axis was the graphical reference. For example, her answer to Q8 was, "a particle is moving along the x-axis with a velocity that is equal to $x^3 - 2x^2 + 4$. What is this particle's acceleration when $v(t) = 0$?" On the other hand, she used a symbolic notation system on ten of the 33 questions. This would seem to indicate a somewhat more symbolic than graphical way of looking at derivatives. She referred directly to a particular medium (calculator mostly) on five questions.

How do these responses compare to what others wrote? That her first reaction to "what is a derivative?" was rate of change was slightly unusual for those students not using the innovative materials used by Schools B and C. However, she did not follow up on this model in her later examples or statements as students using the other materials did. This makes one suspect that she knew this was a correct answer, but did not have a strong connection between this idea and other ideas she had about the derivative. That another strong model for her was slope was typical for her class that used the text by Dick & Patton. Her applications were typical for someone in her class and yet more theoretical (and less "real world") than those given by students from Schools B and C.

D2 did not mention that the derivative is a function. There were only two vague references to this idea from the 12 students in her class. On the other hand, many students from Schools B or C stated that the derivative is the "rate of change function." D2's three references to the instantaneous nature of the derivative was above average for her class and for the students in all classes.

D2's discussion of continuity and differentiability, "to be differentiable, a function must be continuous," states a relevant fact but is not a complete description. Each of the other students in her class provided either a less accurate or less complete response. Students in other classes provided both more and less complete responses than D2. Some students had no idea how to answer Q5 (How can you tell if a function is differentiable?), while others blatantly stated that if a function is continuous it must be differentiable. On the other extreme, some students gave graphical descriptions of how they would tell if a function is differentiable or described the concept of local linearity. Some students said that a function is differentiable if you can take the derivative of it.

Although students from the same classroom varied in their responses, there were some responses that were similar between students from the same class and different from students using different curriculum materials. Those characteristics are summarized here.

All of the students from Schools B and C at some point referred to the derivative as the rate of change. In addition most of these students described problems involving the rate something was changing. These problems were usually real world situations involving contexts such as population or disease models, blood permeability, or temperature or pressure changes in air.

Most students at School D had slope as their primary model of derivative and were more likely to list theoretical applications of the derivative such as its uses in finding out more about the function, its maximum value, or inflection points. Of course, these uses apply to real world situations, but the students in School D did not state those connections. In terms of non-theoretical applications, most made reference to both velocity and related rate problems.

The students at School A had a very operational notion of derivative. Most could not say what a derivative is. They were only able to state how to calculate the derivative using derivative formulas and to list a few standard applications. For many students at School D the only real world application they listed was velocity. About a third listed some sort of related rate problem.

Very few students in any of the classes made a point to emphasize the instantaneous nature of the derivative in a consistently correct manner. Students in all classes had trouble sorting out the relationship between differentiability and continuity as well as the distinction between necessary and sufficient conditions in discussing these matters.

Although it had not been the original intention of the interviews, the interviews were particularly helpful in the pilot study for gathering more information about the curriculum, particular examples that had been emphasized in class, or the philosophy of instruction. Students' written responses often involved a shorthand that was understandable to the instructor of the class and the other students in the class, but not to the researcher. This discrepancy caused the researcher to realize the importance of knowledge of the classroom and curriculum of the class in analyzing student understanding. Another factor in this realization was the difficulty the researcher had in asking appropriate follow-up questions to students using a curriculum with which she was unfamiliar, such as that used at Schools B and C.

Because of the possibility for follow-up and clarifying questions, the researcher found that the interviews allowed for a deeper exploration of a student's understanding than the written questions. The need for follow-up and clarifying questions was especially noticeable on the question asking students to make connections between a model of the concept of derivative mentioned in answer to "what is a derivative?" and the formal definition of derivative. Some students did not know what was meant by "the formal definition." Others did not understand what they were supposed to be comparing.

Other students did not see that any comparison was relevant. For example, D2 stated, "There doesn't need to be any direct relationship because the above definition is purely mathematical [the formal definition], but the definition I gave in Q3 [slope at a point] is purely conceptual." This result led the researcher to realize that it would be important to see if students made connections between other models of derivative as well.

The analysis of both the written and interview questions led the researcher to think more carefully about the role of the instantaneous nature of the derivative concept and the notion of the derivative as a function. These two characteristics were later incorporated into the structure of the derivative concept described below.

The researcher determined that the written questions were generally effective. The questions to give an example of a problem given a constraint of notation system or media were dropped in the subsequent study since the researcher chose to focus more directly on the concept of derivative itself. The rest of the written questions were incorporated into the questions used in the first two interviews of the main study.

Circle Diagrams -- Summarizing the Structure of the Concept of Derivative

One further effect of the pilot study was my desire to have a more concise way to record each student's written and oral comments for analysis. This coincided with the notion of Lakoff (1987) from my theoretical framework that one may describe the structure of a concept in an organized and consistent manner. What follows is a description of a diagrammatic way to summarize the concept of derivative in terms of the three-layer process-object structure described in the theoretical framework. These circle diagrams may then also be used to record which aspects of this structure a student discusses in an interview or uses in answering a written item.

Think of the three layers of the derivative concept as a set of three concentric circles. The first layer, the ratio or rate, is represented by the smallest, inner-most circle. The middle circle represents the second layer, the limit, the instantaneous rate. The largest, outer-most circle represents the third layer, the derivative function. The circles themselves will represent the gestalt object. If the process aspect of a layer is described as well, then that layer will be shaded in (see Figure 4.1). Note that the embedding of the circles within each other is not meant to represent a subset or a hierarchy. Rather each process layer is seen as acting on the enclosed object. For example the second layer, limiting process, is seen as acting on the difference quotient object.

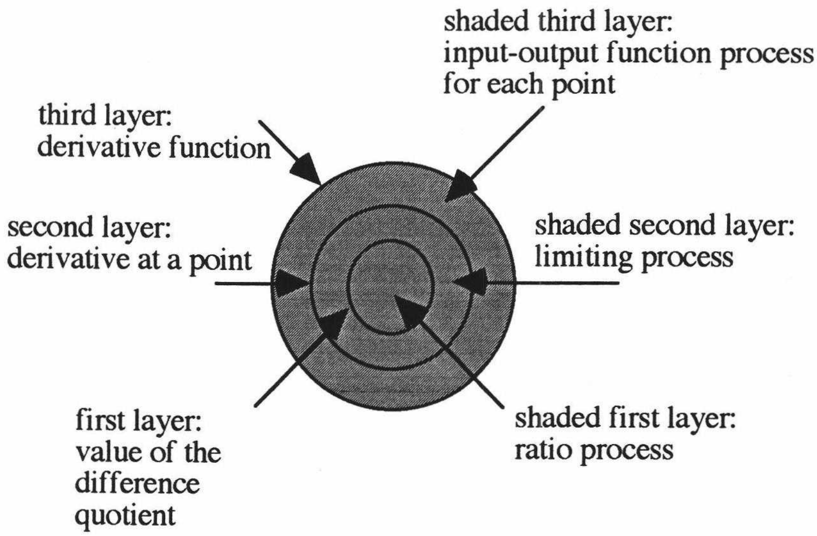


Figure 4.1. The meaning for each of the three layers of the circle diagrams.

The examples in Table 4.5 and Table 4.6 include all possible icons using the smallest two circles and most icons including the largest circle. For each icon a verbal description gives one or more scenarios that would be represented by that icon. Not all possible scenarios are listed.

Each of the first eight icons in Table 4.5 that do not use the largest circle may be modified in one of two ways to include the largest circle representing the notion of the derivative as a function. If the nature of the derivative as a function is mentioned only in passing while describing any of the above 8 situations, then the third circle will not be shaded. If the derivative is described as a function with inputs and outputs while describing any of the above 8 situations, then the outside circle will be shaded. In practice, if someone mentions one of the 8 scenarios above in the same sentence as the fact that the derivative is a function, he or she will probably be using the scenario to describe the output of the function (and hence given the shaded circle), instead of just mentioning that the derivative is a function in passing (the unshaded circle). Therefore, with one exception, only the icons with shaded large circles are listed in Table 4.6.

Table 4.5. Circle Diagram Examples with One or Two Circles

Circle Diagrams	Meaning of the circle diagrams in at least one context.
	Slope or rate or velocity.
	Slope given as rise over run or velocity described as distance over time or $\frac{\Delta y}{\Delta x}$.
	A description of the instantaneous nature of the derivative without a further description of what the instantaneous value represents (e.g. slope, rate, velocity).
	A description of a limiting process where the values in the process are not made clear; e.g. "It's when you get closer and closer."
	Slope at a point or instantaneous rate or velocity at a given time.
	The limit of the slopes of secant lines approaching the slope of a tangent line or the limit of average velocities converging on an instantaneous velocity.
	Slope at a point where the slope has been specifically identified as the rise over run or instantaneous rate where the rate has been identified as the change in y over the change in x.
	The situation described for the immediately preceding icon with the addition that the limiting process has been described as well, or symbolically $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$.
	A pseudostructural conception of a derivative function as a symbolic expression or a curve with no further structure described, or simply a statement that the derivative is a function without elaboration.
	The derivative as a function where each input value has a certain output value, but the nature of these output values is left unspecified.

Table 4.6. Circle Diagram Examples with Two or Three Circles

Circle diagram	Meaning of the circle diagram in at least one context.
	Slope function, velocity function, or rate of change function where the nature of the descriptors as the output values has not been made explicit.
	The covariation of input values to a function with slope values as outputs. The slope values are not specified to be instantaneous slope values. They may be specifically described as slopes of particular secant lines at each point.
	The covariation of input values to a function with slope values as outputs where the nature of the slope as rise over run is specified. Here again the slope values are not specified to be instantaneous slope values. For a symbolic example, $\frac{f(x+.001)-f(x)}{.001}$.
	The covariation of input values to a function with instantaneous slope values as outputs.
	The covariation of input values to a function with instantaneous slope values as outputs where the limiting process for obtaining the instantaneous slope values is described.
	The covariation of input values to a function with instantaneous slope values as outputs where the nature of the slope as rise over run is described.
	The covariation of input values to a function with instantaneous slope values as outputs where both the limiting process and the nature of slope as rise over run are described. For a symbolic example, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Each circle diagram gives information about the three layers but does not distinguish representations or contexts. To show this information as well, I will use a table with headings for each of the models used. Examples of this presentation are demonstrated in Table 4.7 below.

The total coding scheme in table form adds several additional pieces of information for clarity. The symbol $\mapsto$ is used when derivative is equated with "taking the derivative," i.e. finding the derivative symbolically using only formulas such as the power rule or the chain rule. These processes by-pass a description of derivative using any of the three layers. Another instance where the three layers are not used is a reference to the derivative as a tool in application problems. Further questioning about the nature of this tool may reveal insight into the person's understanding of the derivative concept that correlates to the three-layered structure. If no further information is available, references to derivative as a tool without any comment related to the three layer structure will be labeled simply as "tool" in the final column. Misstatements are labeled by the most appropriate diagram and then labeled as "misstatement" in the final column.

An Illustration of the Coding Scheme Applied to a Study of Crocker

Crocker's (1991) study provides student responses to the question, "What is a derivative?" The nine students were asked this question during their second quarter of a university calculus course using the software *Mathematica.* Table 4.7 shows how I would apply the coding scheme to the responses of Crocker's subjects. Note that the comments are coded based only on a student's response to one question. A student may have further understandings that are not stated and hence not coded.

To save space in Table 4.7 some of the headings have been abbreviated. Sym. is an abbreviation for a symbolic representation, and Vel. is an abbreviation for velocity.

Table 4.7. Examples of the Circle Diagrams Using Data from Crocker's (1991) study

ID	Statement	Slope	Rate	Vel.	Sym.	
A	A measure of the slope of a function.	○				
B	The derivative of a function is when you take the function whatever it is -- say if it was x squared. Take x squared and take the exponent and move it down to the coefficient and you subtract 1 to it so it would be $2x$. And you would use that to determine zero points of the function. So wherever the function crosses zero usually that's maximum and minimum point.				↦	tool misstatement
C	... I know what it is but it's hard to explain. You take the function and you just take the derivative of it. What is it? I don't know. I can't think - I just know what it is.				↦	
D	It's the slope of a function.	●				
E	A derivative is - if you're going from A to B let's say it would be your function at F of (A+B) minus F of A all over B as B is going to zero ... It's the average slope.	●			◉	misstatement misstatement
F	... a way to find the instantaneous growth rate of a function or a point of a line or it's the slope of a line at a certain point.	●	◎			misstatement
G	Opposite of integral ... First derivative defines the slope of the equation at whatever point. It's the opposite of an integral.	◎			↦	
H	A derivative is the slope of the tangent line to the curve. Or its the velocity. If you have the distance you can find the velocity.	◎		○		
J	...Well I guess the first thing that comes to mind is the slope function meaning that if you have a function f of x and you take the derivative using the power rule, chain rule, product rule and so forth you end up with the function f prime of x from which if you plug in a value of x you get the slope of the original function at that point.	◎ ◉			↦ ◯	

A glance at the Table 4.7 yields the following observations:

1. Students were more likely to mention slope than any other representation or context. (seven of the nine have circles in the graph column while there are only a few circles in the other columns.)

2. Four of the nine students described the derivative as differentiation, one exclusively. (The symbol, $\mapsto$, appears for four of the students; for one student's row this is the only symbol.)

3. About half the students mentioned the instantaneous nature of the derivative; only one mentioned the limiting process and that was misstated. (The medium circle appears in 5 of the 9 rows, but is only shaded in one case. Even here it is involved in a misstatement.)

4. Only one student described the derivative as a function. (The largest circle appears in only one row.)

5. Only one student mentioned the details of the ratio involved in the derivative concept and this was part of a misstatement. (Student E is the only one with a shaded small circle.)

Again, these observations are about the statements made in response to one question. They do not give a complete picture of the understanding of any one of the students. They do, however, provide an overview of the aspects of the derivative concept that were most easily accessible for these students. The table of circle diagrams provides a way to note at a glance what layers of the derivative concept and what models were used and to what extent they were used.

Methodology for the Main Study

The main study examines in detail the understanding of derivative held by nine high school calculus students and its development during a two-semester calculus course. A multiple case study methodology was chosen. By studying a small number of students, the researcher could examine each student's understanding in depth. By using a multiple case study instead of a single case study the results of the case studies may be compared for striking similarities or differences that give more information about the generalizability of the results.

The strength of this set of case studies is that these students have remarkably similar and rich academic backgrounds, are using the same technology, with the same curriculum, and have the same teacher. The contrasts in the evolution of these individuals' understanding of derivative can give us some ideas about the dynamics of personal construction of knowledge. The similarities may suggest some researchable hypotheses of generalizable phenomena.

Environment and Participants

Setting. The setting for this study was a reasonably affluent school district in a suburb of a large midwestern city. We will call it Suburban High School. Suburban High School has approximately 1400 students in grades 9-12. Two calculus classes are offered at Suburban High School. One class prepares for the AB version of the College Board's Advanced Placement (AP) exam. The higher level class prepares for the BC version of the AP exam and is the source of subjects for this study.

The Teacher. The BC calculus teacher for our nine subjects will be referred to as Mr. Forrest. Mr. Forrest (not real name) started the 1993-1994 school year with six years of teaching experience including three years teaching the AB calculus course and five years teaching three sections per year of honors physics. The year of the study was Mr. Forrest's first year to teach the BC calculus course instead of the AB calculus course.

Mr. Forrest has a bachelor's degree in both mathematics education and physics and a masters degree in mathematics education. During the 1993-1994 school year, Mr. Forrest had an eight-period schedule consisting of one section of BC calculus, two sections of honors physics, three periods as computer coordinator, one period of preparation time and one period for lunch. (Note: Suburban High School is unusually well-equipped with computers, and Mr. Forrest shares the computer coordinator responsibilities with one other teacher.)

Mr. Forrest's interest in technology included extensive use of *Mathematica* in his calculus class the year before. He had written *Mathematica* Notebooks entitled Inverse Functions, Limits, Extrema of Functions, An Analysis of Motion, Newton's Method, Numerical Approximations of Definite Integrals, as well as a calculus "toolbox." The toolbox summarized important *Mathematica* commands relevant to calculus as well as some calculus commands created by Mr. Forrest. Each command was explained and then a working example was provided. The purpose of the toolbox was to enable students to execute commands without having to learn much *Mathematica* syntax.

Calculus Connections Project / Hewlett-Packard Award. During the previous year Mr. Forrest had applied for and been awarded a Hewlett-Packard (HP) equipment grant. The terms of the award were that Suburban High School would receive 30 HP-48 graphing calculators, a printer, and an overhead device. The award required that Mr. Forrest (as the designated teacher from his school) teach calculus using the text *Calculus*, by Dick and Patton (then in its final preliminary edition) and that Mr. Forrest attend a two-week summer training institute on the teaching of calculus using the HP calculators and the text.

The workshop was given by the staff of the Calculus Connections Project, the project that developed the textbook. The purpose of the workshop was not only to provide instruction to the high school teachers in the use of the graphing calculators in the teaching of calculus, but also to give them some insight into the intentions of the authors and ways in which the text might differ from a more traditional text. These differences include an emphasis on multiple representations of the function concept -- symbolic, numerical and graphical -- and how viewing the core calculus concepts through those representations enhances students' mental images of those concepts. One example is thinking of differential functions as functions that are approximately locally linear. The text also emphasizes the intelligent use of technology, visualization, and approximation, as well as problem solving and mathematical modeling.

As part of the award the students were expected to check out the HP calculators to use during the year. All the students were allowed to check the calculators out during the first week of class and kept them until a few weeks before the AP exam. For Mr. Forrest, it was the first time teaching with this text or with the HP graphing calculators. His previous teaching experience with graphing calculators was limited to one community college class in college algebra that he had taught the previous spring in which the TI-81 graphing calculator was a required part of the course.

Subjects. The nine subjects of this study consisted of five males and four females. To protect their identities we will use the names Alex, Brad, Carl, Derick, and Ernest for the five males and the names Frances, Grace, Helen, and Ingrid for the four females. The nine subjects of our study represented the entire enrollment of the BC calculus class at Suburban High School for the 1993-1994 school year. This enrollment was relatively low, with the average yearly enrollment in BC calculus for the four years preceding the 1993-94 school year being 18.5 students.

All nine students in the BC calculus class were previously enrolled in honors precalculus during their junior year at Suburban High School. In fact, all students except Alex had been enrolled in the same math and science classes during their 9th through 11th grade years. The math courses were honors advanced algebra, honors geometry, and honors precalculus, while the science courses were honors biology, honors chemistry, and honors physics (see Table 4.8). During their senior year, eight of the nine (all but Ingrid) also enrolled in AP Chemistry. Neither AP Biology nor AP Physics were offered during the 1993-94 school year. Mr. Forrest thought that many of the BC students would have taken AP Physics had it been offered, but not enough seniors wanted to take it for the course to run (seventeen students would have been required.)

Not all students planned to continue with their mathematics course work in college. Brad, Grace, and Ingrid all indicated that their main reason for taking AP calculus was to fulfill their college mathematics requirements before college.

Alex differed the most from the other students. Alex had immigrated to the United States from an eastern European nation. He studied math and science in his home country before completing his sophomore, junior, and senior years at Suburban High school. Alex placed out of geometry and took honors advanced algebra as a sophomore instead. Alex also took a regular physics class his sophomore year and both honors chemistry and AP Physics (Physics C: Mechanics) as a junior. Alex received a 5 on the AP Physics exam.

Table 4.8. Mathematics and Science Backgrounds of the Nine Subjects

Student	**9th grade**	**10th grade**	**11th grade**	**12th grade**
Alex	(not yet immigrated to the United States)	honors adv. alg. regular physics	honors precalc. AP physics (5) honors chemistry	BC calculus AP chemistry
Brad	honors adv. alg. honors biology	honors geometry honors chemistry	honors precalc. honors physics	BC calculus AP chemistry
Carl	honors adv. alg. honors biology	honors geometry honors chemistry	honors precalc. honors physics	BC calculus AP chemistry
Derick	honors adv. alg. honors biology	honors geometry honors chemistry	honors precalc. honors physics	BC calculus AP chemistry
Ernest	honors adv. alg. honors biology	honors geometry honors chemistry	honors precalc. honors physics	BC calculus AP chemistry
Frances	honors adv. alg. honors biology	honors geometry honors chemistry	honors precalc. honors physics	BC calculus AP chemistry
Grace	honors adv. alg. honors biology	honors geometry honors chemistry	honors precalc. honors physics	BC calculus AP chemistry
Helen	honors adv. alg. honors biology	honors geometry honors chemistry	honors precalc. honors physics	BC calculus AP chemistry
Ingrid	honors adv. alg. honors biology	honors geometry honors chemistry	honors precalc. honors physics	BC calculus (no science)

The students also excelled academically in areas other than math and science. Six of the nine students, Carl, Derick, Grace, Frances, Helen, and Ingrid, were National Merit Finalists. National Merit finalists are chosen from students with the highest scores on the Preliminary Scholastic Aptitude Test (PSAT). (Note: In determining finalists the verbal score is weighted twice as heavily as the score on the math portion of the test.) In addition, eight of the nine students completed the AP History course during their junior year. On the AP History exam Derick scored a 5, Ernest scored a 3, and the rest of the students each scored 4's. Alex did not enroll in the AP History course. At the beginning of their senior year, the students had signed up for a variety of AP classes (see Tables 4.8 and 4.9.) although both Ernest and Carl did not take the AP Chemistry and AP European History exams.

Table 4.9. Non-Mathematics or Science AP Course Enrollment for the Nine Subjects (AP score indicated in parentheses for AP history)

Student	**11th grade**	**12th grade**
Alex		
Brad	AP history (4)	
Carl	AP history (4)	AP English AP European history
Derick	AP history (5)	AP English
Ernest	AP history (3)	AP European history
Frances	AP history (4)	AP Spanish
Grace	AP history (4)	AP English AP Spanish
Helen	AP history (4)	AP English AP Spanish
Ingrid	AP history (4)	AP English AP Spanish

Student's Prior Experiences with Technology, Calculus and Physics . The students had used TI-81 graphing calculators previously in their junior year precalculus class. The students did not check out the calculators but had used them during some class periods and were required to use them on one test. Ingrid reported this in one of her interviews and said she did poorly on this test because of the required use of the calculators as did others who were not that interested in the calculators. Carl, Derick, and Ernest had purchased TI-81's the previous year, although Carl and Ernest each reported that their calculator was stolen from their locker at some point during that school year. The AB teacher (who I will call Ms. Sands) reported that the class used TI-81 calculators on occasion during class with activities from *Using the Graphing Calculator* by Charlene Beckman.

In addition to some prior experience with graphing calculators, the students had considerable previous experience with differential calculus topics. At the end of the precalculus course, from March through May of their junior year, the students studied differential calculus covering the first four chapters of a traditional calculus text by Larson and Hostetler (2nd edition) with additional worksheets and problems assigned from a second traditional calculus text by Swokowski (4th edition). This material included limits, derivatives, and applications of derivatives. Over the summer, Mr. Forrest assigned the students enrolled in his BC class the following year problems to complete from the review sections at the end of the first four chapters in the Swokowski text. These problems covered functions, limits, derivatives, and applications of derivatives.

Mr. Forrest also reported that he had covered several topics in the junior year honors physics with an eye toward preparing the students for calculus. Mr. Forrest described how he talks in physics about distance over time for smaller and smaller intervals, that these are the average velocities, and that when the interval is very small you may find a "stretch" where the average velocity is equal to the instantaneous velocity on that interval. He also gave the researcher a copy of the worksheet that he had used last year with these students in physics and was using with his physics students again this year. It included two graphs, one of displacement versus time and the other of velocity versus time. Both graphs were piecewise linear connecting data points with the title of the each graph given as "Data from Moving Object2." Questions for the displacement graph asked for the velocity over an interval. Questions for the velocity graph asked for the acceleration or displacement over an interval.

Data Collection

The research methods used in this study were aimed at collecting the following information:

• Descriptions of each student's emerging understanding that reflected the structure of the concept of derivative (as described in the theoretical framework.)

• Notation of differences in these descriptions of student understanding as they evolve over time.

• Notation of any moments of insight or descriptions of such moments of insight given by each student.

• An examination of the factors that may be influencing this evolution. These include internal factors such as preexisting structure of knowledge related to the concept of derivative and external factors such as the calculus class's activities, assignments, and tests.

The data collected were of three principal types:

• Data gathered as a participant observer in the environment of the classroom.

• Data gathered from student writings including tests.

• Data gathered from students' responses to questions in a series of five structured interviews.

Multiple ways of viewing the same event give a more complete picture of the phenomenon, and each source of data may be used to validate the results obtained from the other sources of data. The strengths and weaknesses of one type of data may be balanced by a second, different type of data. For example, data from interviews may provide more information about talkative students than about shy students. Collecting written data from the same students may provide a balance for this effect.

Observation Data. For this study, the researcher was a participant observer with definite bounds on the level of participation and on what could be observed. The researcher participated in the daily calculus class strictly as an observer. This role was negotiated with Mr. Forrest in discussions preceding the first class meeting and made more specific in subsequent private discussions as necessary for clarification. The researcher did not lecture and rarely spoke during the class period.

On the first day of class, Mr. Forrest introduced the researcher and briefly explained her purpose. The researcher then described this role in more detail explaining that the students' participation was voluntary and based on informed consent. Students were directed to speak with Mr. Forrest if they did not want to participate in the research although no students did so.

The extent of the researcher's participation in class was typically to set up the audio tape in front of the class. On some occasions, the researcher asked the students a single question while writing it on the board. The students wrote their answers to this question in a few minutes and handed their papers to the researcher. (The content and frequency of these questions is listed in Table 4.7 and discussed below under the heading "Questions of the day.") Otherwise, the researcher sat quietly in the same desk, and generally refrained from engaging in any class discussions, although Alex had a tendency to make side comments to her that she might briefly respond to.

Since Mr. Forrest did not teach a class before or after calculus he was often available to talk with the researcher for part of one or both of those time periods. Discussions included what Mr. Forrest intended to cover, his opinions on what the students were learning, his insights about the school in general, and a wide-ranging exchange of ideas about mathematics, physics, computers, calculators, and pedagogy.

The discussions were usually general in nature or a chance for the researcher to ask for Mr. Forrest's reflections on his pedagogical or mathematical choices. The researcher's intent was not to steer what was covered in class. On the other hand, the understanding between the researcher and Mr. Forrest from before the school year started was that Mr. Forrest would open up his classroom to the research with the benefit that the researcher was available to Mr. Forrest for assistance with the calculators and the implementation of the new project. As it turned out, the researcher's help was more in the form of a knowledgeable peer with whom to exchange ideas.

The objective of the participant observation was both to obtain an appreciation for the environment in which these students were learning calculus as well as to document more specifically the external influences on their understanding of derivatives and any clues the students might divulge in their comments or questions in class regarding the nature of their understanding or its development.

It should also be noted that the principal instrument for collecting observation data was the researcher. Since her background experiences may have influenced her view of observed events, it is appropriate to describe her background relevant to the environment of this study. The researcher had taught several traditional calculus courses at the college level, but had no experience teaching any course at the high school level. The researcher was very familiar with the reform calculus text used by the students in this study. Specifically, she had written answers to problems in the text, taught lab activities based on material from the text to college students using the HP-48 graphing calculators, and had participated in in-service workshops for high school teachers on the use of the text and the HP graphing calculators.

The specific observation data collected included:

• audio tapes of class discussion.

• handwritten notes made by the researcher during class.

• typed field notes written after the researcher had left the school and returned home.

• electronic mail messages from (and a few phone conversations with) Mr. Forrest regarding classes not observed by the researcher.

Table 4.10 summarizes the number of class days observed and the number of days for which each type of observation data was collected.

The class actually met for 152 meetings. The school calendar included 174 days for seniors, but two days were missed by the entire school due to extreme cold. The other missing days from the calculus class's schedule were used for the semester exam schedule (2 days), field trips (4 days), or AP exams (4 days) that took over half of the nine-member class, and a "relaxed" period (12 days) after the AP exam. During the field trips or AP exams, the students present worked individually, sometimes on calculus and sometimes not on calculus. After the AP calculus exam the class met for two days during which the exam was discussed. No other days after the AP exam were considered for the in-class-observation aspect of this research.

Table 4.10. Number of Days over Which Observation Data was Collected

	1st	**2nd**	**3rd**	**4th**	**5th**	**totals**
Dates	Aug. 24- Sept. 14	Sept. 17- Oct. 29	Nov. 2- Dec. 17	Jan. 3- April 8	April 11- May 13	Aug. 24- May 13
class days	14	30	31	55	22	152
observed	14	24	21	7	9	75
audio taped	9	21	15	6	8	59
hand notes	14	24	21	7	9	75
typed notes	14	24	22	5	9	74
email				7 messages for 41 days		

The researcher visited the calculus class on 75 of its 152 meetings. On an additional 10 days the researcher interviewed one or two students during the class period. These interviews were scheduled on days when the class was working in groups on an assignment outside of the usual lecture/discussion format. The researcher's class visits were concentrated in the beginning of the year. The intention was to observe the class during the time it covered the material on functions and limits, leading up to derivatives, the content of derivatives and their applications through the introduction of antiderivatives, and the fundamental theorem of calculus. After this period the class was visited later in the year when derivatives were again discussed -- Newton's method, l'Hôpital's rule, Taylor series, and the review for the AP exam. This was largely accomplished, although a few relevant classes were missed due to illness or attendance at a professional meeting. The researcher discussed all missed classes with Mr. Forrest to determine what was covered and which students were in attendance.

Of the observed sessions, 59 were recorded on audio tape. (While the original intent was to videotape the classes, at the beginning of the school year the audiovisual department at Suburban High School was not able to loan a video recorder. Mr. Forrest also expressed an aversion to being videotaped. Later in the year when a video recording was attempted, Mr. Forrest and the researcher encountered further technical and logistical problems, and this aim was abandoned.) The observed classes that were not audio taped were due either to technical problems or because the class was taking a test or working individually on problems so that an audio tape would not have been very effective. Days when the class worked on problems were sometimes calculator lab days. Other times there were problems from the main text for review or from other texts for extended practice. Students were allowed to work in groups, but more often worked individually and then consulted Mr. Forrest and/or their neighbor at intervals.

The audio tapes clearly recorded nearly all the words spoken by Mr. Forrest but often did not clearly record student comments. Obviously, the audio tapes would also not record what was written on the board or presented on an overhead. To supplement the record handwritten notes were taken during all 75 observed classes. The notes emphasized student comments and what Mr. Forrest wrote on the board. Students did not write on the board. An overhead was used only occasionally to project Mr. Forrest's calculator screen. (The overhead mechanism for this purpose was not available until the middle of the school year.)

In addition, for 74 sessions the interviewer typed field notes. Usually these were typed immediately upon returning home from her observations. The typed field notes emphasized discussions with Mr. Forrest before or after class and highlighted some aspects of the day's class. They also provided an opportunity for the researcher to record personal reflections on her observations.

Electronic mail messages from Mr. Forrest consisted of a brief description of the topics covered on a daily basis as well as the attendance and homework assignment for that day. Occasionally an interesting comment about a particular student's understanding would be included as well. There were 7 messages covering a total of 41 class days during the months of January, February, and March. Five of those 41 days were days that the researcher was also in attendance.

Three days of class in January and 17 more in March and April were not covered by any of the preceding data collection methods. The researcher was able to reconstruct most of what was covered on those days through her field notes on discussions with Mr. Forrest at school, some handwritten notes on a few phone conversations with Mr. Forrest, and from incomplete notes Mr. Forrest had made in his grade book.

Students' Written Work. Data on students' written work took the following forms:

- answers to questions of the day (QOTD), written principally by the researcher.
- answers to problems on chapter tests, written by Mr. Forrest.

Questions of the day (QOTD). The questions of the day (QOTD) were designed to be short questions asked at the beginning of the class by the researcher to which the students could write an answer in about five minutes. QOTD were asked on 28 different classroom days. However, some of those questions were initiated by Mr. Forrest and covered material not pertinent to this study. Other questions asked the students about their interests to help the researcher gain a report with the students. Only the QOTD directly relevant to determining a student's concept of image of derivative are listed in Table 4.11.

The first four QOTD explored a student's understanding of function. Since the derivative as a function is the third layer of the concept of derivative as described above, an understanding of function is part of an understanding of derivative. The primary aspects of an understanding of function that these questions explored are 1) whether a student used the definition of function in describing a function or in determining whether or not a relation was a function, and 2) which representations of function a student was most familiar with. The representations of function that a student has as most prominent may be related to the representations of derivative that a student has as most prominent. The definition of function came into play in the interviews most clearly when a student was asked whether a derivative is a function.

The fifth QOTD explored a student's understanding of limit. Since the derivative is defined as a limit, and the second layer of the concept of derivative described above includes a limiting process, a student's understanding of limits is part of an understanding of derivative.

Since the students were familiar with functions and limits from their precalculus class, the first five questions served to ask students questions about functions and limits before they were covered in the class to provide the researcher with some data on the students' concept image of functions and limits from their studies as juniors.

The next three questions were about derivatives and were asked before the relevant material had been covered in this class. The sixth question and to a certain extent the next two served to examine a student's skills at computing derivatives. Questions #12 and #13 also looked at a students computational skills. Since a student's concept image of derivative includes everything that a student associates with derivatives, this skill is part of a student's concept image. However, this skill is not represented in the concept of derivative depicted in three-layers in the circle diagrams. This process is demonstrated in the charts by the symbol $\mapsto$ to indicate a process that by-passes the three-layers.

The seventh and eighth questions provided an opportunity for a student to demonstrate a preferred interpretation of the derivative other than the process of taking derivatives. For example, a student might have answered either question by comparing the slopes of the function with the values of the potential derivative functions.

Questions #9, #10, #11, and #12 asked the students about material that they had just covered. The most general of these questions gave the students an opportunity to discuss aspects of the concept of derivative that were most prominent to them at that time. Question #11 specifically attempted to have students give a context for the concept of derivative that was outside of their typical answers to "what is a derivative?"

QOTD's were designed on a daily basis by the researcher, sometimes in consultation with Mr. Forrest, in accordance with what was being covered in class or what was soon to be covered in class. The QOTD's were not graded. This fact, combined with the students' becoming more comfortable with the researcher, seemed to result in a lower quality of responses to the questions in terms of correctness, relevance, and detail, over time. The students also complained about the QOTD's, feeling that though they were not graded, QOTD's added unwanted pressure to the class period. After the students had been interviewed for the first time, and particularly after the first chapter on derivatives had been covered, the researcher was sufficiently comfortable with the information on student understanding that she had gathered or was gathering from other sources to feel the QOTD's were no longer vital. In addition, questions about derivatives began to seem redundant, and continual repetition of similar questions raised the concern that the evolution of each student's understanding of derivative might be more strongly influenced by having seen the questions many times than by the factors that were "normally" a part of this class. All of the factors listed above combined with the planned absences of the researcher later in the school year led to the sparse occurrence of the QOTD's after mid October of that school year.

Table 4.11. Questions of the Day

#	Date	Absent	Question
1	930824	Ernest	What is a function?
2	930825	Ernest	a) Give an example of two functions that are very different from each other. In what way are they very different? b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?
3	930826	Ernest	Give an example of a function <u>without</u> using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.
4	930827	Ernest	a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1? b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) nonintegers are integers?
5	930830	none	What is a limit? What is a limit of a function f at a point $x = a$?
6	930920	Derick	Find the derivatives of the following functions: $f(x) = (x-1)^2(x^2-4)$ $g(x) = \frac{x-1}{\sqrt{5-x^3}}$ $h(x) = \sin x$ $j(x) = \ln x$
7	930921	Carl, Frances, Ingrid	The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be. $y = \log(x^3)$ $y = \frac{x}{\lvert x \rvert}$ $y = x^e$ $y = e$ (using your knowledge of derivative)
8	930922	Brad	a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$? Why <u>not</u>? b) What is the derivative of $y = \tan x$?
9	930928	Ernest	What do you understand about derivatives now that you didn't know at the end of last year?

Table 4.11. Questions of the Day (continued)

10	931001	none	a) Mathematical Highlights of yesterday's class. b) Any insight you gained from the class. (The researcher explained this orally also telling them that since she had been absent she wanted to know what she had missed.)
11	931014	Ernest	Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.
12	931202	Brad	What's the most important idea that we have studied so far in this class?
13	940105	Alex, Grace	Find the derivative of $f(x) = \ln(x^2)$.
14	940106	Grace, Carl	Find the derivative of $f(x) = \sec(x^2)$.
15	940201	none	Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

In-class Tests. The test dates and topics are listed in Table 4.12 below. All tests were written by Mr. Forrest with the intention of assessing student progress in his course. Not all tests were given equal weight in grading for the course. In particular, if a test covered significantly less than a chapter it was weighted less than a chapter test.

The researcher had little to no influence on the content of the tests and often saw the tests only moments before or after the test was given. In other words, none of the tests were designed specifically as research instruments. They do, however, serve to provide evidence of student skills and understandings at various points in the course. The tests also served as data regarding the external influences on student learning in this course. The only tests discussed in the analysis of each student's understanding are those tests or parts of tests concerning information on a student's concept image of derivative, including the layers and contexts of the concept of derivative and the skill of taking the derivative that by-passes the layers.

Table 4.12. In-Class Tests

#	Date	Sections	Topics
1	930910	1.3-2.3	Functions, Limits, Continuity.
2	930924	2.4-3.5	Discontinuities, Asymptotic behavior, Linear functions, What is a derivative, Derivative formulas and properties.
3	931012	4.1-4.4	Derivative as rate of change, Linear approximations, Using the derivative to analyze function behavior.
4	931026	5.1-5.3	Optimization problems, Implicit differentiation, Related rates.
5	931104	5.4	Parametric and polar equations.
6	931124	6.1-6.5	Definite and indefinite integrals, Fundamental theorem of calculus, Numerical integration techniques.
7	931214	7.1-7.4	Differential equations, Logarithmic and exponential functions, Methods of integration.
8	940112	8.1-8.2	Integrals to measure area and volume.
9	940126	1.3-8.2	Semester Final -- comprehensive to date.
10	940292	8.3-8.4	Arc length, Area in polar coordinates.
11	940221	9.1-9.2	Definite integrals to measure averages and make physical measurements.
12	940301	9.3	Improper integrals.
13	940314	10.1-10.3	Sequences, l'Hôpital's rule, Newton's method.
14	940414	11.1-11.4	Series, Convergent tests, Taylor polynomials and series.
15	940427	---	Techniques of integration.

Interviews. The students were each interviewed five times. The timing, purpose, and topics for each interview are described briefly in Table 4.13, and in more detail in individual sections for each of the five interviews. However, it is important to note here that there is a dichotomy in the interviews. The first, second and fifth interviews provide the most fundamental information for this study. Each of these interviews asks the students a diverse enough and complete enough set of questions about the concept of derivative so that a student's responses may be taken as an approximate snapshot of his or her concept image of derivative at that point in the course. The third and fourth interviews serve more limited purposes. The third interview examines what aspects of the concept of derivative come into the discussion of the relationship of derivatives to integrals. The fourth interview focuses on open-ended questions and problems that are related specifically to rate of change.

Table 4.13. Summary of the Timing, Purposes, and Topics of the Five Interviews

#	Date / Timing	Purpose	Topics
1	September 13-15, 1993 After covering functions and limits, but before covering derivatives. (This was a review. Students had studied this material during the previous spring.)	Concept image of derivative.	•What is a derivative? •Uses of the derivative, differentiability, formal definition, relationship between models of derivative.
2	October 27-29, 1993 After covering derivatives and their applications. (This was also a review. Students had studied this material during the previous spring.)	Concept image of derivative for comparison with the results of Interview 1.	•What is a derivative? •Uses of the derivative, differentiability, formal definition, relationship between models of derivative. •Theorems involving derivatives. •Sketch the graph of a function from the graph of its derivative.
3	December 15-17, 1993 After covering the fundamental theorem of calculus and an introduction to differential equations.	Understanding of the relationship between derivatives and integrals. Not a snapshot of a student's concept image.	•Sketch the graph of a function from the graph of its derivative. •Slope fields, area functions, the fundamental theorem.
4	April 5-8, 1994 After covering almost all of the textbook including applications of integration, some integration techniques, polar coordinates, sequences and some material on series.	Focus on rate of change. Not enough diversity of questions to be considered a snapshot of a student's concept image.	•What is a derivative? •What is meant by rate of change? •The relationship between distance, velocity and acceleration. •Related rates.
5	May 16-19, 23-24, 31, 1994 At the end of the course, after the AP exam.	Concept image of derivative for comparison with Interviews 1 and 2. Variations based on a refined theoretical framework and new questions for the possibility of revealing an aspect of the concept image not evoked by previous questions.	•What is a derivative? •What is meant by rate of change? •Uses of the derivative, differentiability, formal definition, relationship between models of derivative. •Mean Value theorem. •Average rate of an area function. •Meaning of the first and second derivative when the function yields the temperature given the time.

The interviews were structured by a protocol of principal questions and follow-up questions designed by the researcher prior to the start of each round of interviews. The researcher attempted to ask the same questions to each student; however, there were occasional differences. In a few instances, the interviewer failed to ask a student a question from the protocol or asked a question that had not been part of the original protocol during an early interview and then used that question in all subsequent interviews of that round. Varying student responses called for different follow-up questions. The researcher became more sophisticated in asking appropriate follow-up during the course of the year, as she became more familiar with possible student responses to a given principal question.

Interviews lasted from 15 to 50 minutes depending on the interview round and the student being interviewed. The longest interviews were those in the fifth set. Each of these interviews lasted 35-50 minutes. All interviews were audio taped with students writing on a blank page in the researcher's notebook. The researcher occasionally recorded a student's answer or sketched a graph pertaining to a question on the same notebook page. Otherwise the researcher did not take written notes during the interviews. On some occasions the interviewer wrote field notes after completing the interview.

All of the first round interviews took place outside of class during a period that a student had open, during his or her lunch period, or before school. For later interviews some students were reluctant to take time from their out-of-class schedule for the interviews. To accommodate this, Mr. Forrest allowed students to be interviewed during his class period. This tended to put a time constraint on the interviews. The interviews in each set needed to be conducted with all the students within a relatively short period of time so that they could be considered to occur for the students at the same point in the course.

To finish nine interviews in a three- or four-day period with most students interviewing in class, in-class interviews could only last half a class period, i.e. no more than 25 minutes. The 25-minute constraint affected the third and fourth interviews more than the other interviews. For the first and second interviews, a substantial number of students agreed to be interviewed outside of class (see Table 4.14), allowing for longer interviews. The fifth interview was conducted after the AP exam and thus could occur over a time period greater than four days. With a longer time period, most students could be interviewed during class at a rate of one student per class.

Table 4.14. Students Interviewed During Versus Outside of Class

Interviews:	**Interviewed During Class**	**Interviewed Outside of Class**
Interview 1	0 none	9 all
Interview 2	4 Brad, Ernest, Grace, Helen	5 Alex, Carl, Derick, Frances, Ingrid
Interview 3	6 Alex, Brad, Ernest, Grace, Helen, Ingrid	3 Carl, Derick, Frances
Interview 4	7 Brad, Carl, Ernest, Frances, Grace, Helen, Ingrid	2 Alex, Derick
Interview 5	7 Brad, Carl, Derick, Ernest, Frances, Grace, Helen	2 Alex, Ingrid

Some students seemed to find the interviews stressful. Partially as a response to that stress, the third and fourth interviews were designed with fewer questions. Since the fifth interview was the last interview and occurred after AP exams, the students were more tolerant of its being longer. Alex and Derick did not seem to find being interviewed stressful and did not mind long interviews or being interviewed outside of class. In one case during the fourth interview, these two students were asked an additional problem that there was no time to ask in the interviews of the other students. Helen exhibited the most stress related to the interviews and her interview transcripts contained the fewest lines of text.

Since the order in which the students were interviewed sometimes affected what questions they were asked, this order is recorded in Table 4.15. The order was not predetermined but based on who was willing to volunteer to go first for in-class interviews and on available times for scheduling out-of-class interviews. Grace and Brad tended to volunteer early for in-class interviews. Alex and Derick were often scheduled later for out-of-class interviews. This occurred because the researcher was confident that those two would be easy to schedule and concentrated on scheduling the other students first.

Table 4.15. Order of Student Interviews

Interviews:	**Order:**
Interview 1	Grace, Helen, Derick, Frances, Carl, Ernest, Brad, Ingrid, Alex
Interview 2	Grace, Brad, Frances, Ernest, Ingrid, Derick, Helen, Carl, Alex
Interview 3	Ingrid, Grace, Carl, Helen, Alex, Brad, Frances, Ernest, Derick
Interview 4	Brad, Ingrid, Carl, Frances, Helen, Grace, Derick, Ernest, Alex
Interview 5	Brad, Frances, Ingrid, Grace, Derick, Helen, Ernest, Alex, Carl

Interview 1. The first set of interviews was timed to occur before the BC class started covering derivatives, but after they had reviewed functions and limits. Waiting until a few weeks into the school year allowed the students to become readjusted to the classroom setting and to become accustomed to the presence of the researcher. All interviews took place in the three days immediately before the instruction on derivatives began. The interview protocol for the first interview is contained in Table 4.16.

The intention of the interview was to learn as much as possible about a student's concept image of derivative based on what they remembered from their three-month study of differential calculus at the end of their junior year. The questions began broad and open-ended to allow students the opportunity to show what aspects of the derivative concept they held most strongly. Later questions were more pointed, checking to see if a particular interpretation of the derivative concept was held. These questions were a refinement of questions the researcher had used for interviews and a written questionnaire during a pilot study the previous spring.

Table 4.16. Protocol for Interview 1

• How do you like using the calculator? • Do you think it's hard to use? • Have you used a graphing calculator before? What kind? Did you own it? • Do you use it much on your homework? Do you ever use it on homework that doesn't specifically ask you to use it? • What do you use it most for? • What problems on the exam (test on chapters 1 and 2) did you use the calculator for?
1. What is a derivative? 2. What can derivatives be useful for? In other words, what types of problems can derivatives be used to help solve? List as many different types as you can. 3. Explain what a derivative is in a way that a person with very little mathematics background could understand. 4. List a few real world situations which involve the concept of derivative. 5. How can you tell if a function is differentiable?
For each of the following terms state whether or not its mathematical meaning is related to the concept of a derivative in any way. If it is related, explain in what way it is related. If it is not related, explain why not. 6. slope 7. speed (velocity) 8. change (rate of change) 9. line or linear 10. measurement 11. prediction (approximation) 12. optimization 13. continuity 14. limit 15. integral 16. function 17. differential equations
18. Give a formal mathematical definition for a derivative as used in calculus. 19. Compare your answer to question 3, an explanation of derivative to someone with very little mathematics background, with your answer to question 19, a formal definition for derivative. What is the relationship, if any, between your answers to these two questions? If there is no relationship, explain why there does not need to be one.

Before asking the questions about derivatives, the researcher asked each student some more general questions to "break the ice" and to find out their background with and impressions regarding graphing calculators. The researcher also asked specifically about the student's use of the HP graphing calculator on the first test on functions and limits, which had occurred immediately prior to the beginning of this set of interviews.

Interview 2. The second set of interviews occurred about 6 weeks after the first interview. Between the first and second interviews the class had covered 3 chapters (chapters 3-5) in the text on derivatives and their applications. Immediately prior to the first day of interviews, the students had taken a test covering optimization problems and related rates. The second interview protocol is contained in Table 4.17. During the three days of interviews the students worked in class individually or in groups on activities on polar and parametric equations.

The purpose of the second interview was to provide a direct comparison with the first interview, allowing for a before-and-after perspective on covering the material on derivatives. With this in mind, the second interview used nearly identical questions to the first interview to determine whether or how students responses had changed.

Table 4.17. Protocol for Interview 2

• How was the test (test on sections 5.1-5.3)? • Did you use the calculator on this test? Did you use the grapher for any of the problems? the SOLVR (i.e. LOAD and PLUG)? (Note: SOLVR refers to the solver application that the students used on the HP-48. LOAD and PLUG were programs the students used to access the solver feature more easily.) • Do you feel like you know how to use the calculator any better than you did a month ago when we last talked? What do you know now that you didn't then? • What do you think of the book? What book did you use last year? • What do think of how this book covered derivatives as compared with the book you used last year? Were there any differences you noticed in content or presentation?
1. What is a derivative? 2. What can derivatives be useful for? In other words, what types of problems can derivatives be used to help solve? [Can you think of any others? How does the derivative help in that situation? OR How does the derivative fit into that situation? What role does the derivative play?] 3. Explain what a derivative is in a way that a person with very little mathematics background could understand [someone in precalculus, someone with minimal math background]. 4. List a few real world situations which involve the concept of derivative. 5. How can you tell if a function is differentiable? [What does it mean to say a function is differentiable?] If the student mentions it not being differentiable at "sharp corners", what is it about the sharp corner that makes the function not differentiable there?]

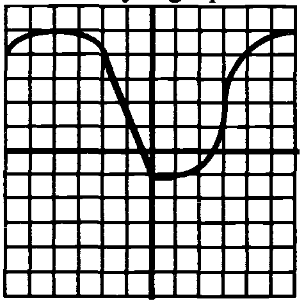
Table 4.17. Protocol for Interview 2 (continued)

For each of the following terms state whether or not its mathematical meaning is related to the concept of a derivative in any way. If it is related, explain in what way it is related. If it is not related, explain why not.
6. slope
7. speed (velocity)
8. change (rate of change)
9. line or linear
10. measurement
11. prediction (approximation)
12. optimization
[Here or later when the student talks about max/min ideas, ask: How do those problems work? If the student mentions that the derivative is equal to 0 at a max or min, ask: Why is it that the derivative is equal to zero at a max or min?]
13. continuity
14. limit
15. integral -- Do you know what is meant by an antiderivative?
16. function -- Is the derivative a function?

17. Give a formal mathematical definition for a derivative as used in calculus.
How does this definition relate to other ways to describe derivative which you have mentioned [i.e. slope, rate of change, velocity]?
What does this definition have to do with some of your earlier definitions like slope of the tangent line or rate of change?

For each of the following state whether the theorem or method involves the concept of derivative. If the concept of derivative is involved explain how. If not, explain why not. You may not have studied some of these theorems or methods. If you don't think you have studied one or more of them, let me know.
18. Newton's Method
19. l'Hôpital's Rule
20. Intermediate Value Theorem
21. Mean Value Theorem
22. Fundamental Theorem of Calculus

23. a. If I give you a derivative of some function, what can you tell me about the original function?
b. Did you think I was going to give you the formula for the function or the graph of the function?
c. Let's try a graphical example:



Here's the graph of the derivative of some function. What can you tell me about the original function? Could you sketch the graph of a function that has this derivative? Give it a try. [If the student needs prodding, ask: What do you know about the graph of the original function? What is its shape? Where does it have a local max or min? Why? Does it have an inflection point? How do you know? What can you tell me about the original function's slope, concavity?]

The second interview protocol was, in effect, an extended version of the first interview protocol and served three purposes. First, a set of questions nearly identical to those used in the first interview were included to explore specific changes as a result of the intervening instruction on derivatives and applications. Second, five additional questions probed students' knowledge of some classical methods and theorems. Third, a final task-based question probed student understanding of antidifferentiation before they had received instruction on that topic. In addition, six new questions were asked. The first five of these had been part of the pilot study and were originally intended to be asked during the first interview. The last question was added to anticipate changes in student understanding that might occur in the next few months regarding the relationship between derivatives and integrals. Not all students had time to answer this question completely.

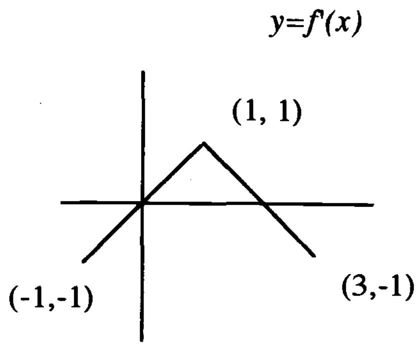
Interview 3. The third set of interviews occurred during the last three days before the two week winter break. Between the second and third interviews the class had covered two chapters in the text (chapters 6 and 7) on integrals, the fundamental theorem of calculus, and differential equations. Both slope fields and the notion of an integral such as $\int_a^x t\,dt$ as an area function had been covered, but not emphasized. The class had taken the test on chapter 7 the day prior to the first day of interviews. The interview protocol is contained in Table 4.18. During the three days of interviews the class worked individually or in groups on material that reviewed aspects of chapter 7, particularly exponential functions and differential equations.

The purpose of the third interview was to examine student understanding of the relationship between derivatives and integrals. Question 1 was similar in nature to the last question on the second interview. It was also a variation on a 1993 AP calculus free response item.

Table 4.18. Protocol for Interview 3

• Did you use the HP on the last test (the test on chapter 7)? What did you use it for?

1. [The interviewer sketches and labels the graph of the derivative as shown below and asks the student to sketch the graph of the original function.]



Follow up questions: How confident do you feel about the graph you've drawn, about the values of the function? Is there any way you could get a more accurate graph? Where does your graph have a maximum or minimum? How did you determine where the max or min occurred? Does your graph have an inflection point? How do you know?

2. What is a definite integral?
3. What is an indefinite integral? How is this integral the same or different than the definite integral?
4. What's the relationship between derivatives and integrals?
5. What does the fundamental theorem of calculus tell us? (If the student asks which one, ask: Which one do you think is easier? Ask for both eventually.)
6. (If this does not come up in the discussion of the fundamental theorem, ask:) Do you remember studying an area function? $A(x) =$ ...?
7. Find the area from 0 to 1 under the curve $y = x^2$. [The interviewer sketches a parabola with the appropriate area marked.]
8. What are slope fields? Do you think they have more to do with derivatives or integrals? Why?

Interview 4. The fourth set of interviews occurred in early April when the class had almost finished covering the material in the text. Between the third and fourth interviews the class covered three and a half chapters in the text (chapters 8, 9, 10 and 11.1-11.2). These chapters discussed applications of integrals, sequences, and series. Derivatives occurred in these chapters primarily in the development of the arc length formula, a discussion of the relationship between displacement, velocity, and acceleration, and in the development of l'Hopital's rule and Newton's method. Of course, antiderivatives occurred repeatedly in solving problems involving applications of integrals.

The researcher originally intended to have the fourth interview occur after all material related to derivatives had been taught and before the students started to review for the AP exam. However, the class did not finish covering new material until later in the school year than the researcher had first expected, and the researcher was concerned that putting the fourth and fifth interviews close together would negatively affect student attitude and would provide less useful data.

The questions in the fourth interview (and the fifth) were developed as the theoretical framework became refined during the winter and spring of that year. For the first two interviews, the goal was to determine a student's concept image of derivative in as much detail as possible. This goal became refined as the researcher probed for patterns and described a structure of understanding the derivative for the many images and ideas in a student's concept image. The questions in the fourth interview look for particular aspects of that structure. The interview protocol is contained in Table 4.19.

Question 1 in effect served as a warm-up question. The question was very familiar to the students (almost to the point of being an inside joke) and the students' initial responses to it were somewhat automated.

Questions 2 and 4 were motivated by the work of Patrick Thompson (1994, 1995a). Question 4 was similar to a question used by Thompson in his research (Thompson, 1994). Question 2 came from the realization (seemingly obvious but made more clear to me by his work) that we must examine the meaning behind student's words. Question 3 was developed based on questions asked by Steve Monk (1992).

Table 4.19. Protocol for Interview 4

1. What is a derivative? Does anything else come to mind?

2. What do you mean by instantaneous rate of change? (If the student does not mention rate of change in answer to #1, ask: Have you heard people say that the derivative is instantaneous rate of change? What do you think they mean by that?)

3. Given the following table of values for a function:

x	$f(x)$
1.0	0.00
1.1	-0.22
1.2	-0.46
1.3	-0.70
1.4	-0.90
1.5	-1.03
1.6	-1.02
1.7	-0.81
1.8	-0.30
1.9	0.60
2.0	2.00
2.1	4.04
2.2	6.89
2.3	10.70
2.4	15.67
2.5	22.03
2.6	30.02
2.7	39.90
2.8	51.97
2.9	66.56
3.0	84.00

Assume f is a differentiable function.
Estimate $f'(2)$, the derivative of the function, f, at $x = 2$.

[Follow up question: Is there any information I could give you that would help you get a better estimate?]

4. There is a car that's stopped. It then moves forward, increasing speed at a constant rate until it reaches 60 miles per hour. Then, it continues moving forward, but its speed decreases at a constant rate back down to 0 miles per hour. The car took 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far did the car travel in the 2 hour period?

5. This is a wall. [Draw a vertical line.] This is the ground. [Draw a connecting perpendicular line.] Assume they are perpendicular to each other. This is a 14-foot ladder leaning against the wall. [Draw a hypotenuse to the triangle and label it 14'.] The ladder is pulled out horizontally at a constant rate. [Draw an arrow pointing horizontally to the right underneath the base of the triangle.] This causes it to slide down the wall. Now, is the rate that it's sliding down the wall constant? If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate? [If the student has trouble determining whether the rate changes suggest that the student compute the rate (in terms of the constant rate of pulling out) for two values.]

Interview 5. The fifth set of interviews occurred about six or seven weeks after the fourth interview, during the two weeks following the AP calculus exam. The first two class days after the exam were devoted to a discussion of the exam. All but one of the students were interviewed during the third through eighth class days following the exam. Carl was interviewed on the eleventh day following the exam. Between the fourth and fifth interviews the students completed their study of series, including Taylor series. They also covered some integration techniques and spent about two weeks working problems from old AP calculus exams. The protocol for the fifth interview is contained in Table 4.20. Brad, having decided to take the AB exam instead of the BC exam, attended the AB class for several weeks before the exam, but returned to the BC class after the exam for the discussion of the exam and the days following. Carl and Ernest also decided to take the AB exam, but their decisions were made later than Brad's, and they stayed with the BC class throughout the school year.

The purpose of the fifth interview was to obtain as detailed a picture as possible of each student's understanding of derivative at the end of the school year. The questions in part 1 were similar to those asked in the first and second interviews. These were for comparison with those earlier interviews, but developed based on the refined theoretical framework.

In part 2, Question 3 was asked because Taylor series was the only new derivative topic that students had covered since the fourth interview. The question served mainly to see if Taylor polynomials were part of a student's understanding of derivatives. Question 2 was exactly the same as the sixth question on the free response portion of the 1994 AB exam. The BC students would not have seen this question. It also provided an opportunity for comparison with some questions from the third interview. The general focus of the questions in part 2 was the meaning students had for symbols used to denote derivatives and theorems involving derivatives. These questions also provided a different

Table 4.20. Protocol for Interview 5

Part 1

1. What is a derivative? OR Explain what is meant by the derivative of a function. Give several different explanations if possible.

2. Explain what is meant by rate of change (instantaneous rate of change, or the rate of change of a function).

3. Does the derivative involve a limiting process? Explain.

4. Is the derivative of a function a function? Explain why or why not.

5. Explain what is meant by a differentiable function. Give an example of a differentiable function and give an example of a nondifferentiable function. Why is the nondifferentiable function not differentiable?

6. State a formal, symbolic definition for the derivative of a function. Explain the relationship between the formal definition and your answer(s) to "What is a derivative?" in question 1.

7. Explain what a derivative is without mentioning a symbolic definition, slope, rate of change, velocity or acceleration.

8. What are derivatives useful for?

Part 2

1. The Mean Value Theorem states that for a function, f, that is continuous on $[a,b]$ and differentiable on (a,b), there exists a value c in (a,b) such that
$f'(c)=\frac{f(b)-f(a)}{b-a}$. What does this mean?
[If necessary, prompt: What does it mean in terms of the graph of $y=f(x)$?
Suppose f is a function telling the distance traveled by a walker at a time x. What does it mean in terms of this scenario?]

2. If the student took the AB exam: Did you do #6 [show it to them]? How did you do it?
If the student took the BC exam, ask them to look at the problem and talk about how they would solve it.

AB #6: Let $F(x)=\int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.

(a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.
(b) On what intervals is F increasing?
(c) If the average rate of change of F on the closed interval $[1,3]$ is k, find $\int_1^3 \sin(t^2)\,dt$ in terms of k.

Table 4.20. Protocol for Interview 5 (continued)

3. If I have a function, $y = f(x)$, and its 2nd degree Taylor polynomial around $x = a$, $y = p_2(x)$, what must f and p_2 have in common and what may be different about f and p_2? [If necessary discuss a particular example such as $f(x) = e^x$ about $x = 0$. If necessary prompt: Do they have any output values (derivative values, 2nd derivative values) in common? Are their graphs similar in any ways?]
4. Let f be a function that for any time x, given in hours, will tell you the outside temperature in degrees Fahrenheit. a. What do each of the following tell us about the outside temperature? $f'(3) = 4$ $f''(3) = -2$ $f'(x) = 4$ for $0 \le x \le 3$ $f''(x) = -2$ for $3 \le x \le 6$ b. When (during this 6 hours) is the temperature the highest (the lowest)? c. Approximately what time of day is associated with $x = 0$? [The student may have been given $f(0) = 50$ previously if needed as a prompt for earlier questions.] d. How long before the temperature is back to the temperature at time 0? e. After the student answers the above by their own method, if not done, prompt for the graph of the function on [0,6].

kind of opportunity than had questions in previous interviews for the students to describe their understanding of the concept of derivative.

Analysis of the Data

This study uses a multiple case study methodology. Each student was analyzed separately. The objective of the data analysis was to describe the evolution of each of the nine student's understanding of derivative. What is meant by understanding of derivative is explained in the third chapter of this dissertation, and an example of how this knowledge is diagrammed is provided earlier in this chapter. (See Tables 4.5-4.7.) Below we will discuss in more detail how this data was analyzed.

The most important pieces of data for answering the research question are the first, second, and fifth interviews. Each of these interviews allowed the researcher to create a snapshot of a student's understanding of derivative at an important point in the school year. The first interview occurred before the students had reviewed derivatives, so it was a snapshot based on what they learned during their junior year study of derivatives. The second interview occurred in the fall, immediately after the students studied derivatives in their senior year class. The fifth interview occurred in the late spring, after the students had completed their year of calculus study and taken the AP Calculus exam. The other data is supporting data for these three main interviews.

Analysis of the supporting data

The supporting data includes two additional interviews, QOTD, student tests, and the researcher's observations and discussions with the instructor. The researcher's observations, discussions with the instructor, and some of the tests served only as a means for the researcher to understand the context in which the students learned the concept of derivative. As noted with the pilot study, student responses to interview questions were more easily interpreted when the researcher knew the way the material had been discussed in class, including any phrases that served as an informal short-hand for this particular class. In addition, this information was put to use in determining whether or not a student had the opportunity to learn a certain aspect of the concept of derivative.

The QOTD, the third and fourth interviews, and the test questions related to the concept of derivative were used for several purposes. In some cases, when the questions asked were very similar to questions in the first, second, and fifth interview, these questions were used for comparison with those interviews. Thus student progress on understanding a particular aspect of the concept of derivative could be assessed at various points in the school year. These questions were also compared with the three main interviews to see if a student was able to express an idea on a written question that the student was not able to express in an interview. This served as a check on the information gathered in the interviews so that we could recognize any bias the interviews showed against shyer students who were not as comfortable talking about their ideas as writing them.

In other cases, the questions asked were not central to the three-layered structure of the concept of derivative, but were still relevant to exploring a student's concept image of derivative, since this is everything a student associates with derivative. Such questions include QOTD and test questions that assessed computational skills such as taking the derivative using various rules such as the product or chain rules. There were also test questions that asked a student to solve very standard, computational, application problems. Mr. Forrest used the test questions as the primary determination of student grades, with homework as a much weaker secondary source. Student grades in the course and student computational skills are discussed briefly in the evaluation of each student's understanding of derivative. This type of data provides a contrast to the interview data in two ways. First, with the grades one sees the relationship (or lack thereof) between a student's diligence and his or her performance on the extremely conceptual interviews. Second, with the standard computational problems one gets a glimpse at the relationship (or lack thereof) between a student's conceptual and procedural knowledge. This study was not designed to make a complete report on this relationship, but there is enough data (particularly with Mr. Forrest's test questions, which the researcher did not design) for some meaningful discussion of this area.

Other questions not directly related to the three-layered structure of the concept of derivative were the questions in the fourth interview that asked the student about definite and indefinite integrals. These were asked so that when a student was asked about the relationship between derivatives and integrals, the researcher had some knowledge of what a student meant when he or she referred to integrals. A student's understanding of the relationship between derivatives and integrals includes some aspects of the three-layered structure.

Analysis of the first, second, and fifth interviews

The first, second, and fifth interviews examined understanding of derivative for each student more directly than any of the other data. What is meant by understanding of derivative is explained in the third chapter of this dissertation. A student's concept image of derivative may include everything a student associates with the derivative, including computational skills. However, I have focused on the concept of derivative as three process-object layers each of which may be viewed in multiple contexts. The layers are the derivative as (approximately) a ratio or value of a difference quotient, the derivative as a limit or instantaneous value, and the derivative as a function. The four main contexts are graphical (slope), verbal (rate of change), a very common physical setting (velocity or acceleration), and the symbolic limit of the difference quotient. A student's understanding of derivative consists not only of his or her awareness of each layer and context but his or her knowledge of the relationship between the contexts.

For the analysis, each of the interview transcripts was marked using a color-coding scheme for each of the main contexts: graphical/slope, rate of change, velocity/acceleration, and symbolic. In addition there was a color for the mention of the instantaneous nature of the derivative and for mention that the derivative is a function. Connections, misstatements and comments that showed the student's attitude toward mathematics or the interview setting were also noted.

For the first, second, and fifth interviews, I used the color-coded interview transcripts to create a circle diagram chart for each student. Table A.1 shows a sample chart. The nature of these charts is described in detail earlier in this chapter of this dissertation.

In the first, second, and fifth interviews, questions about the concept of derivative were asked in many different ways so that a student had multiple opportunities to express each of the layers of the concept of derivative in each of the main contexts coded in the circle diagram charts. Thus the summary of all of a student's responses may be taken as an estimate of his or her understanding of derivative at that point in the school year. In the third and fourth interviews, since the questions were not designed for this purpose, the interviews are not directly comparable to the other three interviews, so the responses were not coded using this scheme.

In the circle diagram charts (see Table A.1 for an example), the questions in the first column are listed in the order in which they were asked in the interview, and the circles in the next four columns are listed in the order in which the student mentioned them so that the reader can tell what aspects of the understanding of derivative a student evoked first. A detailed description of what each circle refers to is explained earlier in this chapter of the dissertation. In particular, Figure 4.1 and Tables 4.5 and 4.6 should be helpful to the reader. The column headings for the middle four columns represent the major representations for the concept of derivative mentioned by students:

1. The Slope column is used for references to derivative as slope or any graphical reference to derivative.

2. The Rate column is used for references to derivative as a rate or a rate of change.

3. The Vel. column is used for references to derivative as a velocity or speed or an acceleration; to distinguish velocity references from acceleration references the word "acceleration" will appear in the final column whenever acceleration was mentioned.

4. The Sym. column is used for references to derivatives symbolically; this may include mention of the formal limit definition, the use of the phrase $\frac{dy}{dx}$ or a reference to taking the derivative using the power rule. The latter is noted not with circles but with the symbol $\mapsto$ to indicate a process that by-passes the structure of the derivative concept.

The final column is for additional remarks. As mentioned above, the word "acceleration" in the final column distinguishes an acceleration reference from a velocity reference. Also the phrase "related rate" is used to notate a circle in the Rate column that comes only from a student's mention of related rate problems (as opposed to the student stating that the derivative is a rate). The column is also used to notate the mention of optimization problems, denoted by the phrase "max/min" or the discussion of aspects of a graph such as of inflection points (inflection pts), increasing or decreasing behavior (in/decreasing), or concave up and concave down behavior. However, the principal use of the final column is to note misstatements. For example, if a student referred to the derivative as slope but incorrectly the circle for slope appears in the Slope column, but in addition, the word "misstatement" appears in the final column. Sometimes the type of misstatement is explained in parentheses. The following abbreviations are used for certain misstatements that recur in the analysis:

d=tl	derivative is the tangent line (as opposed to the slope of the tangent line)
d=change	derivative is change (as opposed to rate of change)
d=lim	all limits are derivatives or derivatives and limits are the same
d=approx	the derivative gives an approximate value

As mentioned earlier, the interviews form the core of the data analyzed. Question of the day (QOTD) responses and work on tests provide additional information on how a student's ideas developed between interviews. A summary of the evolution of each student's understanding may be found in Chapter 5. Appendix A provides a more detailed chronological history of each student's evolution of understanding of derivative including interview transcripts, responses to QOTD, and complete circle diagram charts for the first, second and fifth interviews. Throughout each case study, the aspects of the concept of derivative as described in the circle diagrams are emphasized. Whenever possible a student's work at a certain point in the year is compared to his or her work at previous times in the year.

Chapter 5 — Summaries, Discussion, and Implications

The primary research question for this dissertation is "What is the evolution of each student's understanding of the concept of derivative?" In Chapter 3, I define what I will mean by "understanding the concept of derivative?" This includes three-layers of process-objects: the ratio or difference quotient, the limit, and the function layers. In addition, each of these layers may be observed in multiple contexts: graphical (slope), verbal description (rate of change), kinematic (e.g. velocity or acceleration), and symbolic (the symbolic difference quotient definition of derivative). Each individual's understanding may be described in relation to the three-layered concept of derivative in multiple contexts. Such a description will highlight the following:

1. What layers of the structure are available to the person?
2. In what representations or contexts are these layers available?
3. Does the person understand both the process and object nature of each layer?
4. Can the person coordinate all three layers simultaneously?
5. Does the person recognize the parallel nature of each of the layers in the symbolic, graphic, kinematic, and other settings?
6. Does the person prefer to use a particular representation or context as a model or prototype for the derivative concept when no representation or context is specified?
7. Does the person's understanding of derivative include ideas that do not fall into the three-layered structure of the concept of derivative? Does the student's concept image include understandings considered incorrect by the mathematical community?

The first part of this chapter provides a summary for the evolution of each of the nine student's understanding of derivative. The summaries are followed by a discussion of the trends in the students' understandings. After a section on the limitations of the study, the final sections discuss the implications of this study for teachers and curriculum developers and for future research.

Summaries

Summary — Alex

Alex's understanding of derivative is very detailed even in his first interview, and therefore does not change very much during the school year (see Table 5.1). Throughout the interviews Alex's most frequently mentioned representation for derivative is slope followed closely by rate of change. A focus on slope and rate is held by only two other students in this class during the first interview. However, all but one of the students evolves to this focus by the fifth interview.

Table 5.1. Summary Circles for Alex

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1	◉	◎	◎	◉	d=line
Interview 2	◉	◎	◎	◉	d=tl d=lim d=approx
Interview 5	◉	◎	•	◉	(other)

In each of the principal interviews (first, second and fifth), Alex correctly states the formal definition of derivative and relates this understanding to the graphical representation of derivative. In the first interview he uses the graphical relationship to help him remember that the limit is taken as change in x goes to 0 and not as x goes to 0. The following excerpt from the first interview gives an additional example of Alex's understanding:

27 MZ: OK. How would you explain what a derivative is to someone who's like an AB student or a precalc student that hasn't studied it yet?
28 Alex: When you have two points on any graph and you get a line through them, and the slope of that line would be the average rate of change between those two points. If the derivative is instantaneous rate of change because those two points are really close to each other. I mean they are so close that you can even have just one point. So what you do is like-- OK, you can leave one of these points and you can move the other one closer and closer to the other one, the one you left. And that will give you more accurate rate of change between-- If you take two points far apart that will give you just average rate of change, through out the whole function But as you start moving closer to that point you're going to get more accurate average rate of change between those two points because they are really close. And when you get those points to be so close together that you could even consider them to be one point, that's going to be instantaneous rate of change of that function at that point.
29 Alex: So that's just the slope of the function between the two points.

Note that Alex discusses both the first and second layer of the derivative concept in graphical terms and that he understands that this is also a description of average and instantaneous rate of change.

Alex's understanding of derivative probably evolved most rapidly during the previous spring when the junior year math class studied derivatives. Alex gives a description of how his conception of derivative evolved in the following exchange from the first interview:

30 MZ: What if you had to explain [derivative] to somebody who just doesn't know anything about math, like a relative of yours that didn't have much math in school.
31 Alex: Wait, he doesn't know what slope is and--
32 MZ: Yeah, that doesn't know what a slope is or doesn't understand how a graph--
33 Alex: Well, how my uncle explained it to me when I didn't know what derivative is. Derivative of x^2 equals to $2x$ and derivative of x is equal to 1.
34 MZ: Was that helpful to you?
35 Alex: No. Even if he told me the power rule, how you subtract the power and bring it, it still wouldn't help me. I still didn't understand what the derivative is, even after I took the test. I understood how to take it, but it was kind of hard to get into my mind how could the slope of this line be here and what's happening there. I mean-- But if you don't really know what slope is-- If you don't know what the function is, you can't like say if it's derivative.
36 MZ: How did you learn it?

37 Alex: In physics we did some of this stuff with acceleration, velocity, distance because they are derivatives one of each other. And then in precalculus we drew the graphs by using the graphs of derivatives.
38 Alex: So, first I was confused, but then I understand that this graph represents the equation of derivative so each point, OK each x would be the same in the function as in the derivative, but y would be the slope of the function at that point.
39 Alex: So what I did is like I took the most accurate point that I could find. You know like $(1,2),(2,1)$ something like that, a whole number. Then plot the x 's that had. Then I plot the slopes, that's the y of the derivative graph. And I plotted like over a bunch of these lines.
40 MZ: Like drew little short lines with the slope.
41 Alex: Then it started to make sense to me, when I started to convert from derivative to function from function to derivative. Then just by looking at it I could find the, after awhile I could find just matching them without making the little the slope things. You know, the max and min points. After a while it's just started making sense to me.

Alex explains that he first learned to take the derivative using the power rule but that he still did not "understand what the derivative is." He considers himself to understand the derivative when he has a graphical understanding of the derivative function and how its y-values are the slopes of the original function. He continues by emphasizing his ability to use the derivative function to construct an original function and vice versa.

Note that Alex's work in both his physics and his precalculus classes is helpful to him. Alex is the only student in this class who enrolled in AP Physics during his junior year. All the other students in this class had a non-AP physics course during their junior year.

Alex's second interview shows similar insights but he makes a few additional misstatements. Twice Alex states that the "derivative is the tangent line." In the second instance he says the derivative is both slope and tangent line. "The slope of a linear function will be equal to derivative, cause derivative is the tangent line."

Alex confuses limit and derivative when discussing how to tell if a function is differentiable at a point. First he describes checking the slope from each side, but then he thinks maybe he should be checking the limit. It is possible he means the limit of the difference quotient, but he seems to agree with the interviewer when she asks if he means the "limit of the regular function." He explains further, "Cause limit will give you the slopes." Later when asked whether limit is related to derivative, he says, "Limit is one of the forms of derivative." Perhaps he means that the limit of the difference quotient (and not just any limit) is one of the forms of derivative. He does go on to state the limit of the difference quotient a few statements later.

The misstatements that derivative is the tangent line and that derivative is the limit of the function (as opposed to the limit of the difference quotient) are likely to be slips of the tongue and not misunderstandings. This hypothesis is based on Alex's much more frequent correct statements that the derivative is the slope of the tangent line and the limit of the difference quotient. In addition, Alex makes no errors in any of the QOTD or exams that would corroborate either misunderstanding.

Alex's other misstatement in the second interview is more subtle and is foreshadowed by one of his statements from the first interview: "When you have two points on any graph and you get a line through them, and the slope of that line would be the average rate of change between those two points. If the derivative is instantaneous rate of change because those two points are really close to each other. I mean they are so close that you can even have just one point." This is not a misstatement, but there is a hint that Alex considers two points that are "really close to each other" to be equivalent to one point. In the second interview Alex's statement is more problematic. "When you take the derivative, you think that it's at this point, but actually it's like really, really close to that point. ... At that point the slope will be equal to the slope of the points really close to it. I mean like infinitely close." Taken as a whole Alex's statement seems to posit the existence of infinitesimals which is not necessarily a problem, however Alex should be clear that the derivative is an exact value and for many functions no nearby point will have the same slope value as the slope at the point of interest.

It is not until the fifth interview that Alex explains the details of the ratio in the context of velocity. Considering the depth of Alex's knowledge in the first and second interviews, this change may not so much be an evolution of his understanding as a reaction to two specific questions posed only on the fifth interview to which he provides these more detailed answers. Alex is asked to interpret the mean value theorem specifically in terms of a distance function, and then is asked to interpret the average rate of change of a function defined as an integral.

Alex's interviews and participation in class discussions are characterized by his curiosity and his willingness to discuss mathematics. His talkativeness in the interviews leads to more misstatements than in quieter students such as Frances, but his general tendency to see connections and contradictions allows him to correct many of his misstatements immediately. The misstatements that Alex did not correct immediately are the only ones discussed in this section.

Alex's understanding of derivative is such that he can describe the derivative in many settings, in each of its layers, and can state many connections between these interpretations. His understanding is both holistic and detailed. From the first interview he is able to use the holistic tag of slope to help him reconstruct the ratio involved in the formal definition. He also uses an image of a graph including parallel tangent and secant lines to help himself reconstruct the symbols involved in the Mean Value Theorem during the fifth interview.

Throughout the interviews Alex shows insights that other students do not. He is the only student to state the formal definition completely correctly in the first interview and the only student to correctly interpret his symbolic calculation in the ladder problem of the fourth interview.

The ladder problem involves the traditional scenario of a ladder sliding down a wall. Alex is told that the base of a ladder is being pulled away from the wall, horizontally, at a constant rate. He is asked if the top of the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Alex first thinks that the rate is constant but not the same as the given rate. When asked to check his assertion, he labels the distance between the wall and the base of the ladder as a and the distance between the floor and the top of the ladder as b and completes the following sequence of calculations:

$$a^2 + b^2 = 14^2$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$b\frac{db}{dt} = -\frac{ada}{dt}$$

$$\frac{db}{dt} = -\frac{a}{b}\frac{da}{dt}$$

He notes that since $\frac{da}{dt}$ is constant, the answer depends only on a and b. He reasons that a is increasing and b is decreasing all the time so $\frac{a}{b}$ is increasing, and hence the ladder is falling faster and faster all the time. Alex is the only student in the class to come to this conclusion without hints from the interviewer. Other students with a correct symbolic calculation suggest that a and b must be specific numbers and do not consider how the linked variability in these values affects the solution.

Alex's diligence with his class work did not always match his excellent insight. In the first semester his grade was a midrange B, a step below the top group. By the second semester this improved to an A-, the third highest grade in the class. He was also one of the four students in his class to earn a 5 on the BC version of the AP Calculus exam.

Summary — Brad

Brad's understanding of derivative evolves from a very incomplete and disconnected understanding early in the school year to a more detailed but still far from comprehensive understanding at the end of the school year. Brad's preferred interpretation evolves from a focus on velocity and acceleration to a more complete view with rate of change and slope as most prominent. In the second interview rate is slightly more prominent than slope and in the fifth interview the two are reversed. What follows are a few quotes from Brad's interviews to give the flavor of his understanding in several different interviews. The first interview:

1 MZ: Do you remember what a derivative is?
2 Brad: [laughs] Isn't it like the opposite of a-- No, what's the opposite of a limit or something. I don't know. I have no idea how to say it. It's been a while. [pause] Isn't it like all those formulas for velocity and acceleration and something like that?
3 MZ: Let me see if I can jog your memory. I have a list of words here, and you're suppose to tell me if you think they're related to derivatives or not. OK, slope. Do you think slope has anything to do with derivatives?
4 Brad: Yeah.
5 MZ: Are you going to tell me why you think it has to do with it?
6 Brad: Because I think I remember in physics class something about--. I know slope has something to do with-- We were doing acceleration or something.
7 Brad: So I know slope is acceleration or I mean, I don't know. Some of it depends on how much it goes up; it accelerates. So if there's a straight line, it's zero. So it's not accelerating. Something like that.

In contrast with his focus on velocity and acceleration in the first interview, Brad speaks more of slope and change (not rate of change) in the second interview:

1 MZ: What is a derivative?
2 Brad: It's the ch-- See, I can do it. I just don't know what-- I think it's like the change in something or something. It's almost like the-- It's like the slope basically. I think.
3 MZ: The change in what?
4 Brad: Well, like-- Like $\frac{dy}{dx}$ is like the slope. If you find $\frac{dy}{dx}$ of a circle or something, at that point that's the slope of a tangent line It's just like the-- I think it's like the change.
5 MZ: What else were you saying?
6 Brad: That it's the slope of a tangent line. I don't know. I just think it's like-- I don't know the exact definition. I haven't really studied it.

By the fifth interview Brad can state the correct formal definition but does not know what the parts of the definition represent. Brad also discusses velocity, acceleration, change and slope.

1 MZ: OK. You were just about to tell me what a derivative was.
2 Brad: The definition?

3 MZ: I don't know. What do you think is the most important thing about what a derivative is?

4 Brad: Give you some points on the AP test. [laughs] I don't know. It's helpful. That's important. Help you find things faster. Help you also to check things.

5 MZ: Like what?

6 Brad: Like you can find velocity from an acceleration equation or the change, whatever. The things you get from the derivative thingy.

7 MZ: You were going to tell me the definition. You said you knew that.

8 Brad: [writes: $\dfrac{f(x+h)-f(x)}{h}$] Is that right?

9 Brad: Yeah, it is. Oh, wait a second. Sorry. [writing]

10 Brad: Yeah, I think that's right. I don't know. [it now reads: $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$]

11 MZ: You think that's right. What caused you to add this on there [referring to $\lim\limits_{h \to 0}$]

12 Brad: Yeah. 'Cause I think derivative deals with slopes of graphs.

13 MZ: OK.

14 Brad: And so you're taking a point, say 0. [short pause] Wait, I'm thinking. I don't know. That's just what I learned. [laughs]

15 MZ: Does this formula thing that you just wrote down have anything to do with slope?

16 Brad: I think so. Yeah, maybe.

17 MZ: Like what does it have to do with slope?

18 Brad: It'll help you tell what the slope is.

19 MZ: OK.

20 Brad: I think. As it approaches a point. I don't know. h doesn't always have to be 0. I think it could be a variable. I don't know. Maybe it does.

21 MZ: [short pause] Is it possible for you to draw a graph that would show what like $f(x)$ was and $f(x+h)$ and the difference?

22 Brad: No. I don't know.

23 MZ: You don't know. OK.

24 Brad: That's just something I learned from the book. That's basically why I know it. I think that's the derivative.

The layers that Brad mentions during the first, second and fifth interviews are summarized in Table 5.2. In the first interview Brad only states that the derivative is slope, rate of change, or velocity without discussing the ideas of slope at a point, instantaneous rate of change, or instantaneous velocity, and without describing the ratio involved. In terms of the formal symbolic definition he states the correct ratio but omits the limit. He gets the middle circle in the first interview symbolic slot for knowing that limit is somehow involved in derivative even though he misstates the connection. By the second interview

Brad's understanding of derivative has evolved so that he describes the instantaneous nature of the derivative in terms of slope and rate of change, describes the ratio involved in the phrase rate of change, explains a graphical limiting process, and states a formal definition including a ratio and a limit (unfortunately the ratio is now incorrect). In the fifth interview Brad states the correct formal definition and adds a description of the ratio involved in the slope concept.

Table 5.2. Summary Circles for Brad

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1	o	o	o	◉	d=opposite of limit d=limit
Interview 2	◉	◎	o	◉	d=change error in defn
Interview 5	◉	◎	o	◉	d=change

The connections Brad makes evolve only slightly. In the first interview Brad states only that rate of change and slope are the same and rate of change and velocity are the same without further detail. In the second interview he adds to this an understanding that the symbolic expression $\frac{dy}{dx}$ is slope and is rate of change. In the fifth interview Brad does not discuss $\frac{dy}{dx}$, but he is able to state that the ratio described in the Mean Value Theorem is "an average slope".

Brad's misstatements evolve into a repeated comment that the derivative is change, showing a lack of distinction between change and rate of change. This misstatement is not present in the first interview in which Brad mentions few layers or connections and concentrates on derivative as velocity or acceleration. In the first interview his misstatements are that the derivative is the "opposite of a limit" and later in the interview that "[limit] is the derivative". The misstatements that the derivative is change or that change and rate of change are equivalent occur five different times in the second interview, and by the fifth interview there are still three such statements.

In the second interview Brad is asked whether change and rate of change are the same or different. He says, "The rate of change is the amount it changes per interval." He uses the example of a change from 7 to 8 in 1 second. He then explains that the change is "the same because the change is how much it did change between the two." He is correct that for a unit interval the numeric values for change and rate of change are the same. The units, however, would be different. His choice of example helps him avoid seeing the distinction in the two terms.

Brad's interviews are characterized by his repeatedly saying, "I don't know." Brad is not afraid to talk, but he is not confident in his answers. He hedges his statements frequently with phrases such as "I don't know," "maybe," and "I guess". His lack of confidence is not without reason in that his understanding is weaker than most (if not all) of the other students in the class, and he refers to this at one point when he is struggling to answer a question in the first interview, "See they all retain this. I don't retain anything."

By the fifth interview Brad's understanding has evolved to the point where he focuses on rate of change and slope instead of just velocity, but he still sees only weak connections between the various interpretations of derivative and continues to err in thinking that rate of change and change are equivalent. He is, however, able to solve some problems. In the third interview he is able to graph the original function given the graph of the derivative, and in the fourth interview he estimates $f'(2)$ from a table of values and goes through the correct calculations on a related rate problem.

Brad was not a particularly diligent student. He scored a C and a C- as semester grades in his calculus class, and dropped down to take the AB version of the AP calculus exam, on which he earned a 3.

Summary — Carl

Carl's understanding of derivative evolves most in his preferred interpretations and in the connections he makes between interpretations. The layers of the derivative concept that Carl describes and his misstatements change somewhat but do not show much improvement. In the first interview Carl focuses on taking the derivative by means such as the power rule, but also emphasizes velocity and acceleration. He evolves to focus on rate of change and slope in the final interview. An example from Carl's first interview shows his initial focus:

1 MZ: What is a derivative?
2 Carl: Oh, you know it's like the thing you take. Well, you can use a derivative to find the velocity of something.
3 MZ: OK.
4 Carl: I think of it like that. It's a-- It's like-- I don't know exactly. I could say what I use it for. I couldn't tell you the definition of it.
5 MZ: Well, tell me some other things you use it for.
6 Carl: Like on a test when they say, "Take the derivative of this function." You say, "OK." and you move the exponents down and you subtract 1 and then you do that. I know all the rules, the quotient rules and the chain rules and all those funny little things.
7 MZ: Well, what other kinds of either word problems or real world situations would it be useful?
8 Carl: Oh, you mean like-- Like the boats going away from the dock, and you got the pulley on the boat, and then you got to find the acceleration of the boat and the tension and all that weird stuff.
9 MZ: So which part does the derivative fit in?
10 Carl: Well, the first derivative for the velocity, the second derivative for the acceleration.

The layers that Carl mentions during the first, second and fifth interviews are summarized in Table 5.3. The evolution of Carl's statements about the derivative is somewhat haphazard. Throughout the interviews he describes the derivative as a rate of change and recognizes this as the change in one quantity over the change in another quantity as in the symbolic expression, $\frac{dx}{dt}$. In the second and fifth interviews he describes the ratio for slope as well. However, he is never able to correctly state the formal definition of derivative as the limit of the difference quotient. During the first interview

Carl explains, "If you have to go with formal definitions, I don't know those things. I know my own definition in my head of what they are, what they do, and I can do problems like that, but when a teacher's asking for a formal definition, I go crazy."

Table 5.3. Summary Circles for Carl

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1					d=tl d=change (other)
Interview 2					d=tl error in defn (other)
Interview 5					d=change error in defn d=ave roc

Carl mentions the instantaneous nature of the derivative in terms of rate of change in the first interview, not at all the second interview, and in terms of slope, rate of change and velocity in the fifth interview. In the first interview he describes a graphical limiting process (sans slopes), but is unsure how it relates to derivatives. In none of the other interviews does he describe a limiting process except to note that the formal definition (which he misstates) contains a limit.

The connections Carl makes evolve as his understanding of the layers develops. In the first interview Carl connects instantaneous rate of change to the symbolic expression $\frac{dy}{dx}$ and mentions that velocity may be thought of as a rate and a slope. By the second interview Carl connects both slope and rate of change to a symbolic expression for change in y over change in x, but he does not see that this is related to the limit definition of derivative, which he continues to misstate. In the fifth interview he adds only a connection between the slope and its symbolic ratio in terms of the Mean Value Theorem.

Carl's misstatements evolve in that by the fifth interview he has stopped making two errors: he no longer identifies the derivative with the tangent line and does not repeat his first interview misstatement about the role of limits. However, his misstatements confusing change and rate of change continue from the first to the fifth interviews.

In the first interview, when Carl is asked whether derivative is related to slope, he says, "Derivative is the tangent line to the function." He is unsure of his answer and without prompting changes it to "derivative is the slope of the tangent line to the graph." When asked if derivative is related to line or linear, Carl is again unsure whether to say tangent line, slope of the tangent line, or both: "The derivative of a function is always one power less so if we had a parabola, the derivative is a line. And it's the line, the tangent line. The slope of the tangent line is the derivative. The tangent line to the graph is the derivative as well. They're connected somehow." On another occasion, when explaining optimization problems, Carl emphasizes tangent when he should say slope. He explains that the derivative equation is set equal to 0 and solved, "and that's going to be the maximum or minimum points because it's the tangent line. The tangent's going to be equal to 0 at a turning point."

Carl's knowledge of the relationship of limit and derivative at the time of the first interview is captured by the following quote. When asked if the two are related, Carl states, "The limit is a real separate thing from derivative 'cause I never seem to use derivatives when I'm doing limits".

Carl's most persistent misstatement is that derivative is change. In the first interview he states that $\frac{dx}{dt}$ and Δx are equivalent expressions and that the derivative is both change and rate of change:

47	Carl:	Like if you have a beaker full of water with a hole in the bottom, and it's leaking out, and you have water being poured, you can take the delta of the amount of water in there and take the derivative to find out how much is leaving and how much is coming in. Stuff like that.
48	MZ:	So what's the delta suppose to represent?
49	Carl:	The change. The rate of change.

50 MZ: So when you're thinking of delta are you thinking of this triangle guy [draws a Δ]?
51 Carl: Yeah, the triangle guy.
52 MZ: The change, OK. So how does this delta change relate to derivative? How does that fit together?
53 Carl: If you were to take the derivative, like x , you'd end up with like just $\frac{dx}{dt}$ or delta x basically, the change of x over time. So that's basically like Δx.
54 MZ: So this $\frac{dx}{dt}$--
55 Carl: --is like delta x.
56 MZ: Is like, is sort of like equal to-- [writing $\frac{dx}{dt} = \Delta x$]
57 Carl: kind of equal to
58 MZ: And how does the derivative fit in?
59 Carl: When you take the derivative of something you find the change in that.
60 Carl: So, you end up with an equation without an x in it, unless you take x^2, but without an x in it with a Δx instead. And it changes the equation from how much water is in it to how much it's changing at that instant, how much is leaving or going in at that instant. It's an instantaneous rate of change.

In the fifth interview he says the derivative is change in rate, and he confuses change with rate of change in a discussion of a temperature function f. Carl claims $f'(3) = 4$ represents "the rate of change equals 4 degrees Fahrenheit." Note that his units are change and not rate of change units. When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Carl replies that the temperature is changing 4 degrees from 0 to 3. When asked to clarify his statement he says, "At any instant in between that interval it's changing 4, but that doesn't make any sense because then you get really small intervals and it becomes a trillion degrees." Carl realizes that his two statements are contradictory and guesses that his first answer, 4 degrees for the whole interval, is correct. It is not until after the interviewer asks Carl to use an analogy with speed and distance that Carl realizes that the units should have been degrees per hour and that the change was 12 from 0 to 3.

Carl is a talkative person who has a reasonable knowledge of calculus jumbled together with numerous misstatements. His interests in the subject are purely practical.

This may be seen from his concentration on rules for taking the derivative and on velocity applications in the first interview as well as from his refusal throughout the school year to learn the formal definition of derivative.

Carl was not studious, and he missed many days of class especially in the second semester. He earned a midrange B in the first semester and an F in the second semester. He took the lower level, AB version of the AP Calculus exam on which he earned a 4.

Summary — Derick

Derick's understanding of derivative is detailed and well connected even at the beginning of the school year, but it does evolve to include even more details and connections.

Derick's preferred interpretations of rate of change and slope remain constant throughout the interviews. In the first two interviews slope is slightly more prominent than rate and in the fifth interview this emphasis is reversed.

The layers that Derick mentions during the first, second and fifth interviews are summarized in Table 5.4.

Table 5.4. Summary Circles for Derick

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1					d=tl
Interview 2					d=change confuses two averages
Interview 5					d=change change=roc

Derick mentions many layers for the derivative in the first interview, although he describes the details of the first two layers only in a graphical context. The following transcript from the first interview contrasts his descriptions in the graphical and velocity/acceleration contexts. Also note that, as described in the transcript, his descriptions are intended for different audiences:

12	MZ:	How would you explain what a derivative is to someone who doesn't have much math background?
13	Derick:	OK, would this person be assumed to know what a slope is?
14	MZ:	Let's do two different ones. First let's do somebody who's maybe like an AB or a precalculus student.
15	Derick:	What I would say is a derivative is -- instead of taking a slope, which is you take two separate points on the graph and find the change in y and the change in x, and that'll give you sometimes a decent approximation of what the slope is at a particular point. Instead you take the points and squeeze their x values together until the change in the y values becomes smaller and smaller and smaller and eventually you're assumed to take it so that the two points are lying on top of one another and it's the slope of the tangent line at that one point. The tangent line to the graph is the derivative.
16	MZ:	Yeah. So what if you had to explain it to somebody like -- say like you're grandmother or somebody like that?
17	Derick:	Basically you're trying to find out how fast something is changing at a particular moment. So, like for instance, say you are driving down the street and you are accelerating and you're in a stick shift. You're in first gear and you accelerate up and then you switch to second gear and you stop accelerating for a second and you start accelerating again. So the graph of your velocity in relation to time would not go straight up. It would start going up and then even off for a second and then start going up again. If you were to try and find your acceleration over a time, and it weren't, you would get an inaccurate reading because it would be an average. So instead you want to find out, just at one of the points where you were accelerating, how fast you were accelerating. And to do that you would need the derivative. Which is basically, if you were to draw a line so that it just touched the graph at one point and found the slope of that line, that's the derivative.

Derick's statements evolve so that in the fifth interview he describes the details of the ratio and the limiting process not only in terms of the graphical setting by also in terms of rate of change and velocity. His statement of the formal definition evolves from omitting the limit in the first interview, to a correct statement for the derivative at a point, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, in the second interview, to the statement most often used in his calculus course, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, for the final interview.

The connections Derick makes between the representations evolve as his understanding of the layers evolves. In each interview Derick relates each interpretation to at least one other interpretation. By the final interview he connects the details of the ratio and limiting process in the rate of change interpretation to each of the other three interpretations listed; he connects the graphical slope ratio and limiting process to the formal definition, and the formal difference quotient to velocity.

Derick makes a few misstatements along the way. These evolve from a first interview statement that the derivative is the tangent line to later interview misstatements that involve rate of change.

In the second interview Derick struggles with explaining the phrase "average rate of change" in a physical situation. He first correctly explains the difference between average and instantaneous values graphically and symbolically. When he attempts to explain the difference in another context he chooses a discrete example where the data is how much money is gained or lost for each day over a one month period. In this context he realizes that the average rate of change may be calculated by adding up the daily rate of change and dividing by the number of days in the month, i.e. by calculating an arithmetic average. This confuses Derick who had intended to mimic the ratio of differences explanation he has given in the graphical and symbolic contexts. He later changes the data to so that it provides the amount of money a business has on each day of the month. He starts to calculate an average by adding the first and last amounts and dividing by two or by subtracting the first and last amounts and divide by two. He then decides that this is a bad example, and does not attempt to improve it.

The other rate of change misstatement for Derick involves the difference between change and rate of change. He makes a couple of statements in the second and fifth interview in which he says "change" when he should more correctly say "rate of change". His confusion is most noticeable in the fifth interview when he is trying to explain what $f'(3)=4$ means for a function f that at any time x , given in hours, tells the outside temperature in degrees Fahrenheit. Derick says, "That's implying that at exactly 3 o'clock the temperature increased exactly four degrees Fahrenheit. That's kind of an extreme value don't you think?" Derick does not recognize that the change is 4 degrees *per hour*. Derick continues his explanation with an analogy to speed, "It's like the speed of the temperature is 4 degrees in the same way that you take f' of a car function. At that particular point, that's how fast it's moving. ... So that tells you that it's heating up quite rapidly, but just at that moment." The interviewer asks Derick to apply his argument to the car situation for a distance function f and $f'(x) = 40$. Derick recognizes that the car is traveling at 40 miles per hour and states, "So yeah, it didn't go up 4 degrees, but it's increasing that fast at that particular point. ... If it keeps going up at that constant rate, in an hour it will have gone up 4 degrees. ... It's like the instantaneous speed of the thing."

Derick's interviews are characterized by the many, many connections that he makes between the various interpretations of derivative. Occasionally he tries to make connections that are inappropriate but more often his multiple connections help him catch his misstatements, such as the change versus rate of change error above. Another benefit of strong connections is his ability to derive formulas that he does not have memorized such as his (unique for this class) statement of the definition of derivative, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ in the second interview. Similarly, he is able to use a graph to help him remember detailed formulas as when he recalls a graph with two parallel lines from which he reconstructs the formulas involved in the Mean Value Theorem. His connections include those to the practical applications as when he notices that the car problem in the fourth interview gives an unrealistically slow acceleration. Derick also shows an insight unique to his class when he uses slope fields to help him graph a function whose derivative graph is given in the third interview.

Derick was (with the possible exception of Alex) the top student in the class when evaluated in terms of his clear and detailed interview explanations. In his class work he scored a bit lower, receiving the second highest grade in the class, an A-, for the first semester and the fourth highest grade in the class, a B, for the second semester. He was one of the four students in the class to score a 5 on the BC version of the AP Calculus exam.

Summary – Ernest

Ernest's understanding of derivative does not give the appearance of evolving from the first interview to the fifth interview. If anything, his responses devolve from a detailed description of the layers in the graphical context in the first interview to a lack of any detailed descriptions (except an incorrect statement of the symbolic ratio) in the fifth interview.

Ernest's preferred interpretations evolve from a focus on slope and velocity in the first two interviews to a focus on slope and rate of change in the final interview. For Ernest this is not a dramatic change, just a slight shift in focus.

The layers that Ernest mentions during the first, second and fifth interviews are summarized in Table 5.5.

Table 5.5. Summary Circles for Ernest

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1	◉	∘	∘	○	(other)
Interview 2	◎	∘	∘	○	d=tl
Interview 5	◎	∘	∘	⊙	d=rate slope is changing error in defn

Ernest gives the most detailed discussion of derivative in the first interview. His first comment refers to the derivative as a graphical limiting process, although it is somewhat misstated and falls back into a focus on taking derivatives symbolically.

Another time he describes slope as a ratio and in a third instance he describes the process of building a derivative function by taking the slope of the original function at each x value. The relevant quotes are listed below. The first one is :

1 MZ: First what you remember about what a derivative is?
2 Ernest: I remember it's the slope of the tangent lines leading up to the point, leading up to the limit? I don't know if that makes any sense.
3 MZ: OK, well [short pause] you're making some sense. The slope of the tangent line-- and I didn't really get this part.
4 Ernest: --leading up to the limit, sort of, of x .
5 Ernest: Unfortunately it got kind of lost in translation. I just know how to find it and know how to take the derivative of something just by looking at the equation and making this-- putting this derivative down, but I kind of lost exactly what it means.

Later in the interview Ernest uses a function with a vertical tangent as his example of a nondifferentiable function. His explanation shows that he knows the details of the ratio in the graphical context, i.e. he knows that the derivative is the slope and the slope is rise over run.

20 MZ: And why isn't it differentiable at that point?
21 Ernest: Because the slope would be-- [short pause] It would be undefined. Cause you go up one and over none. So it would be 1 over 0 would be the slope.
22 MZ: So it would be undefined slope?
23 Ernest: Yeah.

Towards the end of the interview, Ernest shows that he knows the details of the third (function) layer of the derivative in the graphical context. When asked for a formal definition for derivative, Ernest responds as follows.

98 Ernest: You mean like how to find the derivative?
99 MZ: In a way. The one I was thinking of is how to find it, but not how to find it by like the power rule or something.
100 Ernest: Oh.
101 MZ: Like the first thing you learned, this is a derivative.
102 Ernest: I suppose you could find it by slopes--
103 MZ: OK.

104 Ernest: -- from the graph. You find the tangent line, the slope of like 1, find the tangent line, and on another graph put that answer. And you keep doing that.

In the first interview, Ernest also mentions that derivative is a rate of change and is related to velocity and acceleration. In subsequent interviews he gives less detail about the graphical interpretation saying only that derivative is the slope at a point. Symbolically he knows from the first interview on that the derivative is a limit, but he never states a correct formal definition. In the first two interviews he declines to state one at all, whereas he states an incorrect ratio with no limit in the final interview.

There is a contradiction in Ernest's attitude toward the role of limit in understanding the derivative. This is seen in the following transcript from the second interview:

16 MZ: How would you explain a derivative to someone who hasn't had much math?
17 Ernest: Well, I'd tell them to learn limits first. Then-- [pause] I'd explain it as what you use when-- [long pause]
18 MZ: Can you think of some way to explain it to someone who is so small in math background that limits is just not going to get them very far?
19 Ernest: Well, I'd tell them to wait until they had a proper math background to know what I'm talking about.
20 MZ: So there's just no way without having more math background?
21 Ernest: Not really, not for them to understand it. You couldn't just tell someone in third grade how to do it.
22 MZ: OK, then what if somebody had just finished doing limits, but hadn't studied derivatives yet. They were saying, "We're about to learn derivatives. What are derivatives anyway?"
23 Ernest: This. [He points to the paper where MZ has written "slope of the tangent line".]
24 MZ: You would say this, the slope of the tangent line? So what does the slope of the tangent line have to do with limits?
25 Ernest: What does it have to do with limits? I was never good at textbook definitions and stuff.
26 MZ: Somehow it does have to do with limits in your mind though?
27 Ernest: Yeah, kind of. Again, I can't give textbook definitions.
28 MZ: OK. Is there anything that occurs to you that's associated with limit and with derivative of why they're somehow related?
29 Ernest: I guess today wasn't a very good day for me.

Ernest has some recollection that limit is connected with the textbook definition of derivative, but he can not state this definition or describe the role of limit in this definition.

On the other hand, a few lines earlier he states that someone must have studied limits before he could explain to them what derivatives are.

The connections Ernest makes between the representations devolve just as his statement of the layers does. In the first interview Ernest knows that slope, rate of change and velocity are all the same thing for a graph of position versus time. He even explains that the rate of change of a function is increasing when its graph is concave up and decreasing when its graph is concave down. In the second interview Ernest describes no connections between the representations. In the fifth interview he only mentions the relationship between velocity and rate of change.

Other than omissions or calculation errors Ernest makes only a few misstatements. In the first interview he misstates his initial answer to "What is a derivative?" a bit by saying, "It's the slope of the tangent lines leading up to the point, leading up to the limit." In the second interview he says, "The derivative is the tangent line at a point," although he continues with a discussion of the slope of that tangent line. In the fifth interview he misstates the formal definition and makes a possible misstatement by saying that rate of change means "the rate the slope is changing". He does not clarify this answer under follow up questions.

Ernest's interviews are characterized by their lack of development through the school year. Ernest makes the most detailed description of derivative in the first interview. Except for his lack of knowledge of the formal definition, his first interview is one of the strongest of the first interviews in terms of layers mentioned and connections made. However, his fifth interview is the weakest in the class in both regards. In the interviews he is somewhat shy and concerned about whether his answers are correct, thereby making relatively few misstatements, but also giving very little detail about his understanding of derivative. The answers he does give are characterized by an emphasis in the early interviews on velocity and acceleration and a refusal to learn the symbolic formal definition.

Ernest's first semester grade was a midrange B and his second semester grade was a C-. He took the lower level, AB version, of the AP Calculus exam and scored a 4. He was absent from class frequently in the first semester when derivatives were covered, including one third of the class days between the first and second interviews. He also found that he was not as focused a student in his senior as in previous years. At one point in the third interview when Ernest sees that he is having trouble answering the questions, he says to the interviewer, "I don't know if you remember back to your senior year when you were just kind of lazy for a month's stretch." He said that he hoped to catch up with his studies over the winter break, but this did not seem to occur.

Summary -- Frances

Frances' somewhat shallow understanding of derivative in the first interview evolves to a deep, well-connected understanding. Frances consistently emphasizes slope as her preferred interpretation of derivative throughout all the interviews. An evolution of her understanding shows in her second favored interpretation, which in the first interview is the process of "taking the derivative," i.e. calculating the derivative function symbolically, and in later interviews is rate of change. The following excerpts contrast her first, second, and fifth interview comments. From the beginning of the first interview:

1 MZ: What is a derivative?
2 Frances: I don't know. Well, I know it's the slope of the tangent line, you know to the -- Like if you're taking the derivative of x^2 so it would be $2x$.
3 MZ: OK.
4 Frances: I mean, I know how to take the derivative. The slope of the tangent line at a certain point.

From the beginning of the second interview:

1 MZ: What is a derivative?
2 Frances: The slope of the tangent line to a point.
3 MZ: OK. Does anything else come to mind in terms of describing what a derivative is?

4 Frances: The rate of change.
5 MZ: OK.
6 Frances: Instantaneous, I mean you know, instead of the average, it's the instantaneous rate of change.

From the beginning of the fifth interview:

1 MZ: What is a derivative?
2 Frances: The instantaneous slope at a point. There's the formal definition,--
3 Frances: [writes: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$]
4 MZ: OK. And what else?
5 Frances: It's how fast the function is changing.

The layers that Frances mentions during the first, second and fifth interviews are summarized in Table 5.6.

Table 5.6. Summary Circles for Frances

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1	◎	◎	○	◎	(none)
Interview 2	●	●	○	●	(none)
Interview 5	●	◉	◉	●	d=change

In the first interview Frances discusses slope at a point, instantaneous rate of change, and velocity without explaining the details of the any of these layers. She is also unable to state the formal definition of derivative even though she knows that there is a ratio and a limit involved. She guesses that the derivative is a function because of its symbolic form, i.e. " because when you take the derivative you're just like decreasing the exponent for it."

In the second interview Frances can state the correct formal definition of derivative and explains in detail the ratio and limit layers of slope and rate of change. The final interview is similar, the only change being that she does not discuss the limiting process in terms of rate of change, but does discuss velocity as a change in position over a change in time.

Frances' ability to describe the connections between the representations evolves as her knowledge of the layers evolves. In the first interview she knows that the slope has something to do with the formal definition, but she can not remember what the connection is. She also knows that slope and rate of change are equivalent:

36	MZ:	Is derivative related to change or rate of change?
37	Frances:	Well like the slope at a certain point -- the average slope is like -- Well, say you had a curve. If you took the slope between two points on that curve that would be like the average -- you know, the average rate of change. But if you took the derivative, that would be the instantaneous rate of change there.

By the second interview she can explain the details of the ratio and the limiting process of the symbolic formal definition in terms of both slope and rate of change.

58	MZ:	How does the formal definition relate to the idea of slope or rate of change that you were talking about earlier?
59	Frances:	'Cause if you had like a function and you wanted to know like where-- You make the interval, the Δx, like the change in a really small interval, then it's not going to change the-- Cause, let's see-- So if Δx was approaching 0 it would be changing-- The instantaneous slope-- The average slope gets closer and closer to the instantaneous slope when Δx goes to 0. [has written: $m = \frac{\Delta y}{\Delta x}$]
60	MZ:	So this, what you wrote-- [referring to $m = \frac{\Delta y}{\Delta x}$]
61	Frances:	Yeah, that's the average change-- rate of change.
62	MZ:	OK, so that's the average rate of change, and that relates to this, I guess, how?
63	Frances:	Because-- The function doesn't have-- The closer you are to the point you want, the smaller that the-- There's not a lot of room for the function to change in between there. [has sketched a smooth curve on a pair of axes and marked one point and put a pair of parentheses around it]

64 MZ: OK. So is this exactly the same? I mean Δx here is the same as the Δx here?
65 Frances: This is the change in y.
66 MZ: That's the change in y on top?
67 Frances: Yeah.
68 MZ: Where would these different things be on this picture?
69 Frances: [pause] Uhm-- I don't know. I have no clue.
70 MZ: No idea. Somehow it's related to that picture?
71 Frances: Yeah.

In the fifth interview Frances describes the symbolic ratio not only in terms of slope and rate of change, but also in terms of velocity. Outside the context of relating them to the formal definition of derivative, she also sees that the details of the ratio for slope and for rate of change are the same. She discusses the limiting process a bit less frequently, only in terms of slope and not in terms of rate of change this time. Her fifth interview descriptions are generally clearer than her second interview descriptions. This can be seen when comparing the second interview transcript above with the following fifth interview transcript:

10 MZ: OK. This is what I'm kind of more interested in, how this statement, instantaneous slope at a point, relates to this formal definition.
11 Frances: Because if you say, as h gets really small, then it's kind of like you're taking the slope of the secant line, but the slope of the secant line-- The end points of the secant line are getting closer and closer together. So it's almost like you're taking the slope at that exact point.
12 MZ: OK. So how do the different little symbols here fit into that thing you just described?
13 Frances: This is like $f(b)$-- If this was x, then that would be $x+h$, if that was a really small distance [sketches a pair of axes and marks x and $x+h$ on the horizontal axis].
14 MZ: OK.
15 Frances: So then this would be $f(b)-f(a)$ over $b-a$. But $x+h-x$, you just get h [writes: $x+h-x$ and then crosses out the x 's].
16 MZ: OK. And then that describes the slope right?
17 Frances: Yeah.
18 MZ: Then the limit does what you were saying about--
19 Frances: Because you're making it a tiny distance between x and $x+h$.
20 MZ: OK. What does this formal definition have to do with this statement that you just gave me, how fast the function is changing?

21 Frances: [short pause] I don't know. 'Cause if the derivative is 0, then-- If you have a line like-- If you have a function like that [draws a flat curve], the slope of that is small because it's not changing very fast. But if you have a function like that [draws a steep curve], then the derivative is big because it's changing a lot.

Not counting times when she did not know an answer or omitted a relevant detail, Frances makes only one actual misstatement. In the fifth interview she says at one point that the derivative is the instantaneous change instead of the instantaneous rate of change.

Frances' interviews are characterized in part by her lack of misstatements and her shyness. She seems to reserve her comments for when she is fairly sure that her answers are correct. Her understanding of derivative improves to a point where she can describe the derivative in many settings, in each of its layers, and can state many connections between these interpretations. Her understanding is both holistic and detailed. As seen in the fourth and fifth interviews she can use a holistic tag such as slope to recall the details of a procedure that will estimate the derivative at a point given a table of input-output pairs (fourth interview) or find the average rate of change of a function defined as an integral (fifth interview).

In the interviews she occasionally shows insights that other students do not as in the third interview when she was the only student to use calculations of the area under the derivative graph to plot the graph of the original function. Her tendency to show original insights was even more pervasive in class discussions, as noted in the researchers daily field notes. Her insight was complemented by her diligence with her class work. Frances earned the highest grade in her calculus class, and was one of the four students in her class to earn a 5 on the BC version of the AP Calculus exam.

Summary – Grace

Grace's understanding of derivative becomes more detailed as she progresses through the school year. She is able to describe the layers of the derivative in more contexts and to make more connections between the layers. However, her interview comments remain plagued with misstatements and uncertainties.

Grace's preferred interpretation does not change significantly over the course of the interviews. In the first interview she mentions slope most often, with rate of change as a strong second. In the second interview she mentions rate of change a bit less prominently, but by the fifth interview rate of change is her most frequently mentioned interpretation, with slope as a strong second.

The layers that Grace mentions during the first, second and fifth interviews are summarized in Table 5.7.

Table 5.7. Summary Circles for Grace

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1	◉	◎	○	●	d=tl d is opposite to your slope
Interview 2	◉	◎	◎	◉	d=tl d=line (other)
Interview 5	◉	◉	●	◉	change in rate d=tl d=ave roc

The layers that Grace uses to describe the derivative grow more numerous as she progresses through the school year. In the first interview she only describes the ratio in terms of the symbolic formal definition and here she forgets the limit. By the fifth interview she states the correct formal definition (including the limit) and describes the ratio in terms of slope, rate of change, and velocity. She mentions a graphical limiting process in all three interviews, but adds a limiting process description in the context of velocity in the final interview.

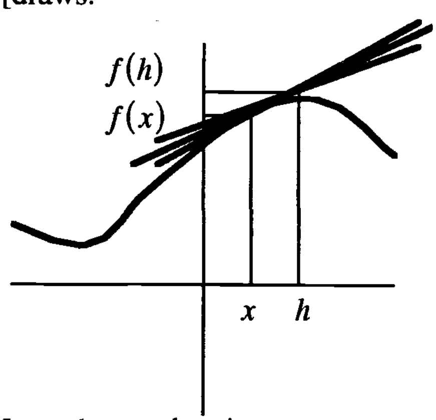
Grace is unique in this class for the frequency with which she describes the derivative by mentioning a graphical limiting process. In the first, second and fifth interviews this includes a description of two points moving closer together to give a more accurate slope. However, in the second and fifth interviews, she also describes the process of zooming in on a curve and seeing that it becomes a straight line in connection with differentiability. No other student in the class describes the latter process.

The connections that Grace makes between the representations evolve as her statements about the layers evolve. In the first interview, she notes that the ratio she has written for the formal definition is meant to indicate slope and rate of change but she does not accurately explain the details of this relationship:

56 Grace: I think it works like this. If you have a graph and you pick two points, and you want to find the -- what do you call it -- the change between the two of them. Right? I guess this is more accurate.

57 MZ: OK.

58 Grace: The closer these two points move together the more accurate it is. So what they're doing here I bet is they're moving their points together. So I guess this is supposed to be x and it probably isn't but I'm going to say that is h. [pause] I don't really know what I'm doing here so-- [pause] OK.
[draws:



59 MZ: Just take a stab at it.

60 Grace: All I'm going to say is that you have two points and the closer they move together the more accurate your slope is going to measure and the rate of change.

In the second interview she recognizes that $\frac{f(x+h)-f(x)}{h}$ is the slope between $(x, f(x))$ and $(x+h, f(x+h))$, but does not describe what the limit represents in this context. In the fifth interview she relate the details of the difference quotient not only to slope, but also to rate of change and velocity. In addition she clearly relates slope to rate of change and rate of change to velocity.

Grace's misstatements do not evolve as much as her understanding of the layers of the concept of derivative. Throughout the interviews she makes statements that the derivative is the tangent line instead of its slope. At the beginning of the second interview she describes a derivative as a tangent line and as the process of zooming in on a curve:

1 MZ: What is a derivative?
2 Grace: It's a slope. Tangent of the slope at a point, at a certain time.
3 MZ: What does that mean?
4 Grace: That means the line that's tangent of the slope [laughs].
5 MZ: It means the line that's tangent, OK. And the derivative is what? How does the derivative relate to this line that's tangent?
6 Grace: I know that if you have a curve-- We'll make it more curvy [draws a smooth curve].
7 Grace: You have your point here [marks a point on the curve]. And if you have your tangent line [draws a short tangent line at that point].
8 MZ: Yes.
9 Grace: What it gives you I guess is like-- Well, what you're seeing, I guess, is every time you [draws a box around the tangent line point and then a smaller box inside that one], that's a zoom box-- every time you move in--
10 MZ: OK
11 Grace: --the tangent line is going to become that thing.
12 MZ: The curve?
13 Grace: I mean it's not going to become, but it's going to get really really close. The derivative is what tells you that I guess. I'm not good at explaining things.

At the beginning of the fifth interview Grace says that the derivative is "the slope of the line tangent to a curve at a certain point," and goes on to explain how the ratio of the formal definition represents a slope. She also discusses slope when asked to explain $f'(3) = 4$ where f is a function that tells the temperature at a given time. However, even in the fifth interview, Grace has moments when she wants to say the derivative is a line. When asked to explain the equation she has given for the Mean Value Theorem, $f'(c) = \frac{f(b) - f(a)}{b - a}$, she explains clearly that the right-hand side of the equation is a slope between two points but is unsure how to describe the left-hand side. Her only attempt is, "Somewhere in here is a point c where $f'(c)$ will be the line that's-- that uh-- for those two points. I can't think of the word. Do you know what I'm saying?" At another point in the same interview is the following exchange:

103	MZ:	Explain what a derivative is without using the symbolic definition or mentioning slope or rate of change.
104	Grace:	[pause] Hmm. I don't know. [pause] Can I say line?
105	MZ:	What about the line?
106	Grace:	Just a line.
107	MZ:	The line itself is the derivative?
108	Grace:	No, the point. I don't know. I don't know how to explain that. Cause, it's like-- You don't really think about it except in terms of math. It's always math. Cause it's math.

Grace never correctly graphs a function given the graph of its derivative. However, she is not asked such a problem in the final interview. Throughout the year Grace's difficulties with this type problem are lessened when she has explicit point-wise information about the graph she is to draw. When the derivative graph is her focus, she does not notice the implicit pointwise information and often transfers global features incorrectly from the derivative graph to the function graph. She does not extract the derivative value at each point from the derivative function as a whole.

Grace's other misstatements are related to rate of change. In the second interview she correctly says that the derivative is instantaneous rather than average rate of change. However, a few lines later in the interview she gives an example that sounds like she is thinking of an arithmetic average instead of an average rate.

57	MZ:	Does measurement have to do with the derivatives?
58	Grace:	Well yeah because derivative gives you a really accurate measurement. Like I was saying before. When you have just like an average measurement-- OK, like if you had an 80% average for a class.
59	MZ:	OK.
60	Grace:	Like you could have had a 70, a 90 and an 80, but you'd still come out with an 80. But if you have your derivative, they'd be telling you-- Like if you have an 80, it can't be anything higher or lower.

In the fifth interview Grace makes a misstatement not found in her previous interviews when she describes derivative as change in rate instead of rate of change. Her initial statement might be taken as a simple transposition of words, but later she emphasizes this word choice with the description "change in y over change in x, which is the change in rate". The use of "change in" is mirrored in all three phrases. Even though Grace does not make this error in earlier interviews, in those interviews she does not connect rate to other interpretations of derivative as clearly as she does here. The addition of the rate of change misstatements coincides with her increased mention of the rate of change interpretation of derivative in the fifth interview.

Grace's interviews are characterized by her willingness to discuss her ideas, even when she is unsure of the correct answer. This allows her to express the many connections she sees between the various layers of the derivative concept even though she never overcomes some of her misstatements. The fact that she continues to make misstatements throughout the school year suggests a certain instability in her understanding.

Grace was not particularly studious. She earned a C for the first semester and a C+ for the second semester as well as a 3 on the higher level BC version of the AP Calculus exam.

Summary — Helen

Helen's understanding of derivative evolves from a shallow description in the first interview that centers on the statement of the formal definition (including an error in what the limit is approaching) to a much deeper and better connected understanding in the fifth interview.

Helen consistently emphasizes the formal definition of derivative in all the interviews. Her understanding evolves in that she allows the interpretation of derivative as slope to become more dominant than the formal definition in her answer to questions in the second and fifth interviews. The following is from the beginning of the first interview:

1 MZ: What is a derivative, if you remember?
2 Helen: It's -- I can tell you the equation.
3 MZ: OK, we'll go with that.
4 Helen [writes: $\frac{f(x+\Delta x)-f(x)}{\Delta x}$]
5 MZ: What else comes to your mind when you think of derivative?
6 Helen: [pause] Lots of messy equations.

Later in the first interview when asked if derivative and limit are related, Helen adds the limit as x approaches Δx to her definition. Contrast that with the following excerpt from the beginning of Helen's fifth interview:

1 MZ: What are the things that you know about what a derivative is?
2 Helen: It's the slope of an equation or it can give you the slope of an equation at a certain point. I don't know. There's all kinds of stuff.
3 MZ: Well let's see, let's just list a couple different things that you can think of for what a derivative is.
4 Helen: Well, I'll just right it down. [writes: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$]
5 MZ: What else could we put on a list of what it is?
6 Helen: I don't know.
7 Helen: You can use it to relate speed and acceleration and position. On a graph, that's the same as the-- The slope of one graph being the points on another graph.

The layers that Helen mentions during the first, second and fifth interviews are summarized in Table 5.8. In the first interview Helen mentions that the derivative is slope or velocity only after being asked about these interpretations. She concentrates her discussion on the details of the formal definition but even there she misstates the limit. She even uses the formal definition as her answer to how she would explain the derivative to other students:

81 MZ: If you were going to explain to somebody in the AB class who hasn't quite gotten to derivatives yet what a derivative is, what would you tell them?
82 Helen: I would probably tell them that it was like -- use that equation.
83 MZ: Use this equation? [indicating the limit of the difference quotient]

84 Helen: Yeah. I don't know how I would say what it does, but -- I don't know. [pause] I wouldn't want to say like you take the exponent and you multiply it times --

85 Helen: Because they wouldn't understand what they're doing.

Table 5.8. Summary Circles for Helen

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1	○		○	◉	error in limit in formal defn
Interview 2	◉	○	○	◉	(none)
Interview 5	◉	◎	◉	◉	(none)

In the second interview Helen can state the correct formal definition of derivative, and she explains in detail the ratio and limit layers of slope. She also knows that derivative is rate of change and velocity although she does not state its instantaneous nature. In the final interview Helen describes the instantaneous nature of the derivative in all four interpretations, and she describes the details of the ratio in terms of slope, velocity, and the formal definition.

Helen's ability to describe the connections between the representations evolves as her knowledge of the layers evolves. In the first interview she does not state any connections between the representations. By the second interview she can explain the details of the ratio and the limiting process of the symbolic formal definition in terms of slope, and she knows that the formal definition is a rate of change (although she does not explain this latter in any detail). In the fifth interview Helen describes the symbolic ratio not only in terms of slope, but also in terms of velocity. She also states that velocity is a rate of change, and that it is the slope of a graph of position. She relates the slope of the tangent line, or the velocity at one instant, to the formal definition with limit included, but she does not describe a limiting process in either of these contexts.

Not counting times when she did not know an answer or omitted a relevant detail, Helen makes only two actual misstatements. In the first interview she misstates the limit in stating the formal definition of derivative. In the fourth interview, in solving a related rate problem, she uses the notation dx and dy instead of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Helen's interviews are characterized in part by her lack of misstatements and her shyness. She seems to reserve her comments for when she is fairly sure that her answer is correct. Otherwise she states, "I don't know," and seems uncomfortable in the interview setting. Nonetheless her stated understanding of derivative improves to a point where she can describe the derivative in many settings, in each of its layers, and can state many connections between these interpretations. Her understanding of derivative is unique in this class in her focus on the formal definition and her failure to incorporate rate of change as a key descriptor for derivative. She uses velocity more often than rate.

Helen's knowledge of the layers of the concept of derivative and the connections between the various interpretations of derivative put her in the top students in this calculus class. Another characteristic that puts her in this group is an ability to use a holistic tag to remember a more detailed calculation or explanation. More than once she states specifically that she is finding the slope before doing a rise over run calculation. She also reconstructs the full statement of the Mean Value Theorem from her memory of a graph with a pair of parallel tangent and secant lines.

Helen was diligent with her class work. She earned the second highest grades in her calculus class (after Frances) an A- first semester and an A second semester, and was one of the four students in her class to earn a 5 on the BC version of the AP Calculus exam.

Summary — Ingrid

Ingrid's understanding of derivative evolves in the sense that it broadens from a focus on slope in the first interview to a focus on both rate of change and slope in the fifth interview. However, her understanding remains shallow in that she is the only student in this class never to describe the details of the ratio in an interview.

Ingrid's preferred interpretation changes quite a bit through the course of the school year. In the first interview slope is the only interpretation she mentions frequently, and slope is her only answer to the first question of the interview:

1 MZ: What do you remember, what is a derivative?
2 Ingrid: A derivative is the slope of line tangent to a certain point on a location.

In the second interview she mentions both slope and symbolic statements frequently:

1 MZ: What is a derivative?
2 Ingrid: The slope of a line, the slope of a tangent line, the slope of a line tangent to a function at a certain point.
3 MZ: Does anything else come to your mind for what a derivative is?
4 Ingrid: The limit-- That little picture that [Mr. Forrest] wants us to have in our mind of the graph. [pause] The limit as the change in x approaches 0-- You know, that one equation.
5 Ingrid: [writes: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$]
6 MZ: And do you happen to remember that picture that goes with it?
7 Ingrid: I'm thinking about it. I don't know. Maybe it'll come to me later.

By the fifth interview rate of change is her most common response, followed closely by slope. However, she still mentions slope and the formal definition as her initial answer to "what is a derivative?"

The layers that Ingrid mentions during the first, second and fifth interviews are summarized in Table 5.9. In the first interview Ingrid discusses slope at a point, rate of change, and velocity without explaining the details of the any of these layers, and she misstates the limit as $\lim_{x \to \Delta x}$ when giving the formal definition. In the second interview Ingrid can state the correct formal definition of derivative. She also explains the graphical limiting process and that the derivative is a function since it is a graph. In the final interview she actually mentions fewer layers, since she does not mention velocity or the function layer in the graphical context.

Table 5.9. Summary Circles for Ingrid

Interview	Slope	Rate	Vel.	Sym.	misstatements
Interview 1	◎	o	o	◉	error in limit in formal defn d=line
Interview 2	◉	o	o	◉	d=tl d at a point = d function
Interview 5	◉	o		◉	d=ave roc d=change

Ingrid's ability to describe the connections between the representations only evolves to incorporate her addition of the graphical limiting process layer and her eventual preference for the rate of change terminology. In the first two interviews she states that there is a relationship between the formal definition and slope but cannot explain the details of this connection. In the fifth interview she explains a graphical limiting process in connection to the symbolic limit in the formal definition but she never states that slopes are involved in this process. She also mentions that slope and rate of change are the same.

Ingrid's frequent misstatements most commonly occur because her understanding that "the derivative is the slope of the tangent line" does not extend to using the notion of slope to solve problems or gain insight into questions related to the derivative. Her understanding of the statement does not include a notion that the formal definition for derivative and the Mean Value Theorem include the rise over run notation for calculating slope. Further, her understanding of the statement sometimes gets misconstrued, especially in earlier interviews, to a notion that the derivative is the tangent line itself and not just its slope.

In the first interview, Ingrid shows confusion about the relationship of the tangent line and the derivative function. A minor foreshadowing of this occurs when Ingrid says she would explain derivative to another math student by drawing a tangent line and explaining that "you can find the equation of any line that's tangent to the graph by taking the derivative." This statement is perfectly accurate, but it does not mention the role of slope. Later when Ingrid is asked whether a derivative is a function she says that the derivative is a function because it is a line. This statement is false since the derivative function is only a line in the case that the original function was quadratic or linear. Finally, Ingrid is asked to explain the relationship between the limit of the difference quotient and the slope of the tangent line. She chooses to focus on the example of $y = x^2 + 1$. She finds its derivative to be $y = x$, later corrected to $y = 2x$, and tries to "figure out how that fits in." Although the interviewer's questions may be misleading her, Ingrid seems to want the derivative function to be a tangent line to the original curve. However, she never mentions anything about slope or the difference quotient in her attempted explanation:

88	MZ:	We have this formula [indicating the limit of the difference quotient], and we know this is supposed to be the same thing as the slope of the tangent line. Why should this be the slope of a tangent line?
89	Ingrid:	[long pause]
90	MZ:	What are you thinking?
91	Ingrid:	I'm trying to do an example. [pause]
92	MZ:	What kind of example?
93	Ingrid:	I'm thinking of like an equation when you-- I'm trying to think of the graph, like say $x^2 + 1$. So you find the equation of the derivative or something was $y = x$. I'm trying figure out how that fits in.
94	MZ:	How it fits into this thing.
95	Ingrid:	[long pause]
96	MZ:	Is that true that if you have $x^2 + 1$, you have a tangent line $y = x$.
97	Ingrid:	I think so.

MZ sketches the graphs to show that it is not a tangent line. Ingrid widens the parabola so that it may be a tangent line, but then realizes that she has taken the wrong derivative. She corrects it to $y = 2x$ and then draws the parabola with this new line.

102 MZ: [long pause during sketching] So is that a tangent line?
103 Ingrid: Yeah. It would be.

It turns out that $y = 2x$ is a tangent line of $y = x^2 + 1$, but not because it is the derivative. The interview ends before this matter can be clarified.

In the second interview, Ingrid also attempts to relate her statement of the formal definition to a graphical interpretation. She recognizes that the limiting process means that two x values are getting closer together, but she has the x values listed as x and Δx. She believes that the result of the limit is a line, as opposed to a slope, and that the resulting line goes through one point, but on her sketch this is not a tangent line.

129 MZ: So the [line] that goes through both [points] is turning into the [line] that just goes through the x point? Is that what you were motioning?
130 Ingrid: Yeah. It's like becoming-- As change in x approaches x it's going to become more like the same line.
131 MZ: Is it possible now to fit in some of the other stuff from this equation into the picture?
132 Ingrid: As change in x approaches 0, so it's closer. This, f of this, x, plus change in x, is going to become more like just regular $f(x)$, and this is going to be really small.
133 MZ: The change in x?
134 Ingrid: Right. So it's just going to be like the line.
135 MZ: So, so-- What part of this equation is the line?
136 Ingrid: $f(x)$? Oh, the line.
137 MZ: Yeah.
138 Ingrid: The result. This line. [refers to the tangent line]
139 Ingrid: The result of the equation.

Even though Ingrid mentions slope frequently during the second interview, she consistently misses opportunities to relate slope to other aspects of her understanding of derivative. When asked how she determines whether a zero of the derivative is a maximum or minimum of the original function, she describes a mnemonic involving where the derivative is positive and negative. When asked for clarification, she makes it clear that her mnemonic has nothing to do with slopes. In another instance when she is asked whether

$\frac{dV}{dt}$ and $\frac{dx}{dt}$ are related to slope, she says, "Probably not. I don't think so. Are all derivatives that, slope?"

During a discussion of differentiability, Ingrid makes two statements that sound like she thinks that the derivative is the tangent line itself and not the slope of that line. She says that if the function is differentiable at a sharp corner, then the derivative would be a horizontal line because it touches the graph at one point. She even seems to use the knowledge that derivative is slope to argue that any horizontal line would be the derivative.

90	Ingrid:	Well, the derivative is the slope so it's going to be just a line anywhere on here with the same slope is going to be 0.
91	MZ:	OK. So I could draw any horizontal line.
92	Ingrid:	And that would be the derivative.

This misstatement also indicates a confusion between the derivative at a point which is a single value like a slope and the derivative function which in some instances will be a line. This confusion also occurs in her first answer to whether the derivative is a function. She replies that it is not a function since "you can't graph a slope" and "you can't graph a limit." Both a single limit value and a slope value would represent the derivative at a point, not a derivative function.

In the third interview Ingrid fails to use slope to help her graph the original function given the graph of the derivative. She tries to look holistically at the graph of the derivative and make the original function the same or "opposite," without seeing the underlying detail that the y-values represented the slope of the original function. Interestingly, she does a better job on a similar problem in the previous interview. There, without referring to slope, she remembers how to use the derivative to find the maximum and minimum points on the original, and that when the derivative is positive, the original function is increasing.

Ingrid misses another opportunity to use the fact that the derivative is a slope ratio in the fourth interview. Given a table of values with x varying by .1, Ingrid is asked to estimate $f'(2)$, the derivative of the function at $x = 2$. Ingrid's first reaction is to try to find an equation for the function. When she is unable to find an appropriate equation, she says, "I feel like I need an equation to find it." If she had an equation for the function, she would "take the derivative and plug in 2." She is unable to state how she would find an estimation without an equation. When prompted with the suggestion of sketching a graph of the points, she does so but does not think of calculating the slope at a point.

Even though Ingrid states earlier in this interview that the derivative is the slope of the tangent line at a point, she does not think of using that information in this problem solving situation. Neither does she try to use the notion of rate of change, also mentioned earlier this interview, nor the difference quotient of the formal definition which she states correctly in Interview 2. These aspects of her understanding of derivative seem compartmentalized from the problem solving strategies that she evokes in this situation.

In the fifth interview, Ingrid again has trouble explaining the relationship between the formal definition of derivative and slope. She knows that the notion that her statement that "a derivative is the slope of a line tangent to a curve ... at a certain point" is related to the limit of the difference quotient by a drawing that she has seen before. When she tries to recreate it, she labels x and $x + h$ on the horizontal axis and denotes h as the distance between them. She indicates that "the smaller h gets, the more accurate you get," but she does not seem to know what gets more accurate. After some prompting by the interviewer for her to show what the ratio represents in her drawing or to indicate how the slope of the tangent line fits, she marks a point whose x-value is between x and $x + h$, and draws a tangent line through that point, but she does not explain further.

Ingrid has similar difficulties when asked about the Mean Value Theorem. When asked if she remembers what it says, her first reaction is to write $\frac{f(b) - f(a)}{b - a} = f'(c)$. She also remembers that c is between a and b, but thinks that any c between a and b will satisfy the equation. The interviewer gives her the actual statement including the fact that not any c works and asks Ingrid what the theorem means "in terms of something else?" Ingrid's reaction is to draw a sketch with a curve and the points $(a, f(a))$ and $(b, f(b))$ marked. She also marks c between a and b on the horizontal axis, but she does not draw any secant or tangent lines. She is not able to explain what the ratio or $f'(c)$ represent in terms of her picture, "I don't know. I never understood those theorems."

Later in the fifth interview, when Ingrid is asked to interpret the average rate of change of a function defined as an integral, she asks, "If they say that the average rate of change is k, does that mean the derivative is k?" Ingrid does not decide whether or not this is true, but she says that if it is true, then she would take the derivative of F and set it equal to k. Ingrid then changes the subject and goes on to the next problem.

Ingrid's failure to make a distinction between average rate of change and instantaneous rate of change, the derivative, is consistent with her answers to the Mean Value Theorem question and other questions in previous interviews. To understand the distinction Ingrid needs to know that the derivative at one point may be approximated by a ratio, a difference quotient for two points. However, nowhere in any of the interviews does Ingrid describe the details of the ratio in any context except for the symbolic one. When asked to explain the symbolic ratio in any other context, such as in the Mean Value Theorem question or in explaining the formal definition of derivative, she is unable to do so.

Ingrid's interviews and coursework are characterized by an ability to memorize formulas and learn algorithms without understanding the meaning behind what she is learning. Ingrid was diligent in her class work but she struggled with it, occasionally getting extra help from Mr. Forrest after class or after school. Her grades were midrange grades for this class, B first semester and B- second semester, and her score on the BC version of the AP Calculus exam was a 3. However, her understanding of the derivative concept was perhaps the lowest of any of the students in this class.

Discussion

Counterexamples and existence proofs

Since this study uses a multiple case study methodology, the focus of the data analysis is on each student as an individual. From individual results we can not prove any generalizations, but we can find counterexamples and existence proofs. By this I mean that a student's understanding of derivative may act as a counterexample to show that some possible generalizations about student understanding of derivative that one might expect or wish to be true are in fact not true in all cases. Similarly, any one student's understanding may serve as an existence proof that a particular type of understanding is indeed possible.

Layers and contexts of the concept of derivative. The concept of derivative is described in this research using three process-object layers in a variety of contexts. One might expect there to be a hierarchy in terms of the order in which these layers or contexts must be learned. However, this does not appear to be the case. In the first interview the interpretations known and preferred by each student are widely divergent. No one interpretation or representation seems to be the starting point for all students. As a student's understanding evolves the student may learn one interpretation first and use it to connect to another interpretation. The choice of first interpretation does not seem to be set for all learners but depends on the preferences of the individual student.

The contexts are related but do not appear to be hierarchical because the understandings of the concept of derivative held by each student in this study evolve so uniquely. A student may know of two interpretations without seeing the connection and then the relationship must be learned. Alternatively the student may use the relationship to build knowledge of another interpretation. The former occurs when students memorize the formal definition and know the concept of derivative as slope and then later connect the two. The latter occurs with both Carl and Derick on the temperature problem in which each of them needs the analogy of velocity to see the derivative of the temperature function as a ratio of temperature over time.

The layers of an approximate derivative at a point (difference quotient), the limit of difference quotients and the derivative function seem to be independent in terms of which one must be learned first. In addition, the gestalt or object form of each layer may be learned without seeing the detail or process form of the layer and the details may be learned without seeing the gestalt. In summary, any one aspect of the derivative concept may be learned without prior knowledge of the other aspects.

Relationship of student understanding to learning environment. The relationship between what a student learns and what a student is taught is not as straight-forward as we would like. Each of the nine students in this study has taken the same precalculus class, and all but one of the students has taken the same math and science classes for each of the preceding three years of their high school studies. However, each student's understanding of derivative is unique. Even though each of the students are introduced to derivatives in the same classroom the previous spring, what they remember from that experience is very different. These differences continue during the course of the study as students remember different aspects of the lecture and make different misstatements from each other, misstatements that are not made by their teacher or by their text.

A second variation on this theme is that a bright, dedicated student may fail to properly understand an aspect of the concept of derivative even though this aspect has been presented to the student clearly, repeatedly. A particularly striking example of this is Ingrid's failure to learn the meaning of the formal definition of derivative even though she is a National Merit Finalist and a dedicated student. Other students in the class also refrain from learning the connection between the limit of the difference quotient and its graphical interpretation as the slope of a tangent line approximated by the slope of secant lines that Mr. Forrest seeks to emphasize. This research gives further evidence to the realization that learning does not consist of knowledge being poured from a teacher, through his or her lecture, to the students.

During a two week period toward the end of September, Mr. Forrest presents the relationship between the formal definition and its graphical interpretation on the board on three different class days. On one other day the limit of the difference quotient is discussed without the secant line picture. The students are assigned problems involving computing and estimating derivative values using the full formal definition and just the ratio. The students take a test on the material and the questions on the test are discussed in class the following day.

Certainly all this is not done in vain. One day after the discussion of the test questions the students are asked to write what they understand about derivatives now that they did not understand previously. Five of the students mention the formal definition in some sense. During the next interview several weeks later, six students are able to state the formal definition correctly as compared to only one student on the first interview. (Five others stated the difference quotient correctly during the first interview but do not include the correct limit.) However, only three students in the second interview are able to correctly relate the formal definition of derivative to the secant line picture. Two others can explain that the ratio is the slope without connecting the limiting process. However, three students (Brad, Carl and Ernest) still cannot state the formal definition correctly and one student, Ingrid, who can state it correctly and even includes it in her answer to "What is a derivative?" cannot relate the two.

Trends in the Data

As discussed above, the focus of the data analysis has been on each student as an individual. This group of nine students is by no means a random sample of calculus students or even of high school AP Calculus students. Therefore, any generalizations made based on the amalgam of student responses should be taken only as a trend in this particular set of data. Whether or not these trends will hold for the general population of calculus students is a question for future research. However, given that the principal distinguishing characteristic of these nine calculus students is that they are extremely bright, I would suspect that difficulties that these students have will be apparent in other students as well. Thus it is appropriate for us to consider trends in the data as a source for future research questions and as a source of ideas that may well be relevant for teachers or curriculum developers.

In the third chapter of this dissertation I discuss four knowledge structures: 1) paradigmatic models, 2) analogical models, 3) diagrammatic models and 4) individual metonymy. These four structures may be used to organize the trends in the data.

Paradigmatic models. Each of the contexts or interpretations for the concept of derivative may be used as a model for the whole concept. To be a paradigmatic model the context must be an exemplar, an example that provides enough variety of features to be representative of the entire group, yet is simple enough to be used in reasoning. When we discuss a student's preferred interpretation for the derivative, we are discussing a paradigmatic model. Which context or representation did students use most often, i.e. which did they prefer as their paradigmatic model for derivative?

Table 5.10 shows the one or two most frequently mentioned derivative interpretations for each student for the first, second and fifth interviews. The first interpretation listed is the one a student mentions most often. A second interpretation is listed as well if the student mentions that interpretation as an answer to at least one-third of the questions for which the student mentions an interpretation.

In the first interview, the students have a wide variety of preferences, although slope is the most frequently mentioned interpretation with six students mentioning it most

Table 5.10. Student Preferences in Interpreting the Concept of Derivative

Student	Interview 1	Interview 2	Interview 5
Alex	SR	SR	SR
Brad	V	RS	SR
Carl	TV	S	RS
Derick	SR	SR	RS
Ernest	SV	SV	SR
Frances	ST	SR	SR
Grace	SR	S	RS
Helen	F	SF	SF
Ingrid	S	SF	RS

S=slope; R=rate / rate of change; V=velocity or acceleration;
T=taking the derivative symbolically using rules
F=formal definition / (limit of) the difference quotient

often. Only three students have rate as a preferred interpretation, and in each case it is second to slope. By the fifth interview, all but one student has rate as his or her first or second most prominent interpretation. Slope and rate (or rate then slope) become the most mentioned interpretations for all students except Helen who still mentions the formal definition as her second most prominent interpretation.

Why are slope and rate the predominate paradigmatic models for derivative in the fifth interview? This result may be influenced in this study by the questions that the students were asked and by the instruction the students were given. However, given that the questions were broad and wide ranging and that the instruction was fairly standard, I believe that there is also a more fundamental reason for this result.

The best paradigmatic models are familiar, well-understood and easily applied to understanding the derivative as a whole. In the first interview, students do not have a very complete understanding of derivative so they mention models that are most familiar. Depending on the student, the contexts mentioned are slope, velocity, rate and taking the derivative symbolically using rules. As the students broaden and deepen their knowledge of derivative, the students become more familiar with other possible models and applicability becomes more of an issue. Students see that rate of change is a general phrase that may be used in many different situations without as much explanation or interpretation as a specific context such as velocity. Slope continues to be a model that is particularly effective for interpreting other contexts because of the special characteristics that graphs provide. In particular, graphs, as diagrammatic models, are useful as synoptic, global representations.

Analogical models. When a student uses one representation or context for the derivative to help understand another context, the first is being used as an analogical model for the second. Rate of change is a useful phrase in terms of it's generalizability. It is useful as a paradigmatic model for the entire concept of derivative. However, it is not always the most useful interpretation for analogies between interpretations because it is not as well-understood by students as a more concrete setting such as velocity.

One example of the effectiveness of the velocity analogy occurs in the fifth interview. Each student is told that f is a function that gives the outside temperature at a given time. The student is then asked to interpret $f'(3) = 4$ and later $f'(x) = 4$ for $0 \le x \le 3$.

Both Carl and Derick initially interpret $f'(3) = 4$ to mean that the temperature changes instantaneously by 4 degrees at the 3 hour mark. Derick states, "That's implying that at exactly 3 o'clock the temperature increased exactly four degrees Fahrenheit. That's kind of an extreme value don't you think?" Derick does not recognize that the change is 4 degrees *per hour*. Derick continues his explanation with an analogy to speed, "It's like the speed of the temperature is 4 degrees in the same way that you take f' of a car function. At that particular point, that's how fast it's moving. ... So that tells you that it's heating up quite rapidly, but just at that moment." The interviewer asks Derick to apply his argument to the car situation for a distance function f and $f'(x) = 40$. Derick recognizes that the car is traveling at 40 miles per hour and states, "So yeah, it didn't go up 4 degrees, but it's increasing that fast at that particular point. ... If it keeps going up at that constant rate, in an hour it will have gone up 4 degrees. ... It's like the instantaneous speed of the thing." After using the analogy to speed to help him clarify his understanding, Derick makes no other misstatements in interpreting the other statements in this problem.

Carl also needs the analogy of speed to correct his misstatement. He begins by interpreting $f'(3) = 4$ to mean "the rate of change equals 4 degrees Fahrenheit". When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Carl replies that the temperature is changing 4 degrees from 0 to 3. When asked to clarify his statement he says, "At any instant in between that interval it's changing 4, but that doesn't make any sense because then you get really small intervals and it becomes a trillion degrees." Carl realizes that his two statements are contradictory and guesses that his first answer, 4 degrees for the whole interval, is correct. Carl presses the interviewer for the correct answer. She responds by giving him an example where the derivative is known to be the speed of a car and $f'(x) = 55$ for $0 < x < 3$. Carl easily calculates that the distance traveled is 165 miles, and he can state that the units on f' are miles per hour. The interviewer points out that Carl has used units of degrees Fahrenheit for the other derivative units. Carl then realizes that the units should have been degrees per hour and that the change was 12 from 0 to 3.

There are two other issues about the use of velocity as a model that should be mentioned here. First, each of the nine students in this study has taken a physics course during the preceding school year. It is possible that the context of velocity would not be as familiar to these students otherwise. In particular, two students interviewed for the pilot study mention (without being asked) that they have not taken physics and that the velocity context is more difficult for them than it is for their fellow students who have taken or are concurrently taking physics.

The second issue is gender related. In this class, only male students (Brad, Carl and Ernest) have velocity as one of their predominant interpretations. Only male students (Carl and Derick) use velocity to help in interpreting the temperature problem described above. On a fourth interview question in which the changing speed of a car is given, only male students (Alex and Derick) state that the acceleration of the car is unrealistic. This data gives some indication that males are more likely to use a velocity interpretation than females, perhaps because they are more familiar with or have a better understanding of the notion of velocity than females. Of course, this may be an anomaly of the students in this study, and certainly would need to be studied further.

A similar gender issue occurs with the use of the formal definition as a paradigmatic model. Only female students (Helen and Ingrid) use the formal definition as a primary or secondary model in these interviews. Helen is the only student to list the formal definition when asked, "What is a derivative?" in the first interview. Helen and Ingrid are the only two to do so in the second interview. Helen even suggests the formal definition for explaining derivative to someone who does not know much math or is in the class below her and has not learned about derivatives yet. For the latter case Helen explains, "I don't know how I would say what it does, but I wouldn't want to say you take the exponent and you multiply it times-- because they wouldn't understand what they're doing."

On the other hand, the three students who cannot state the formal definition during the second interview are all male (Brad, Carl, and Ernest), and two of these (Carl and Ernest) still cannot state the formal definition at the final interview. Carl's comment during the first interview about formal definitions explains his point of view. "If you have to go with formal definitions, I don't know those things. I know my own definition in my head of what they are, what they do, and I can do problems like that, but when a teacher's asking for a formal definition, I go crazy." Ernest says more simply during the second interview, "I never was good at textbook definitions and stuff. ... I can't give textbook definitions."

The two males who are able to state the formal definition by the second interview (Alex and Derick) both use unique symbolic statements that they state are constructed from their knowledge that derivative is an instantaneous slope value. Alex's statement is only unique in that he uses L instead of h, in the standard expression, i.e. $\lim_{L \to 0} \frac{f(x+L)-f(x)}{L}$, but it is significant because it occurs in the first interview when he is the only student to state the formal definition correctly including the correct limit. Derick's unique (for this class) statement of the formal definition is $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

The four female students all learn to state the formal definition by the second interview. Three of them also can relate the symbolic formal definition to slope during this interview. Only Ingrid is unable to do this throughout all the interviews.

The role of the formal definition. Even though most of the students in this class eventually learn to state the formal definition and relate it to slope, the formal definition does not function as a predominant paradigmatic model for any student except Helen. In addition, two students refuse to learn the formal definition (Carl and Ernest). Another student who does learn the formal definition and even states it often in the second interview (Ingrid) does not ever relate this formal definition or any symbolic expression to slope or rate.

These circumstances bring into question the role of the formal definition of derivative in a student's understanding of the concept of derivative. Vinner and Dreyfus's (1989) research on student understanding of functions indicates that students often do not use their concept definition of function when deciding whether or not something is a function. They rely on other aspects of their concept image (i.e. understanding) of function. Often these are global characteristics such as that a function must be continuous or one-to-one. Vinner (1981) finds a similar phenomenon when students are labeling tangent lines. Students tend to look for lines that lie on one side of a curve like the tangent to a circle, whether or not these lines have the same slope as the function at the point of intersection. These are examples of students using inappropriate paradigmatic models instead of a formal definition or a more appropriate paradigmatic model. In either case, students do not seem to use a formal definition as a paradigmatic model because it is neither familiar to nor well-understood by students. The benefits of a formal definition, that it is precise and concise, are not valued strongly by beginning students.

The phenomenon of a beginning student relying on global characteristics instead of a formal definition when classifying an object, and of devaluing a formal definition, also occurs in the work on Van Hiele levels (see for example, Burger and Shaughnessy, 1986). Van Hiele levels describe a model for the development of an understanding of geometry. The levels are hierarchical in that students progress from one level to the next. Burger and Shaughnessy (1986) describe indicators for the various Van Hiele levels. Level 1 includes comparing shapes explicitly by means of properties of their components and rejection of textbook definitions of shapes in favor of personal characterizations. The next higher level, Level 2, includes explicit references to definitions and ability to sort shapes according to a variety of mathematically precise attributes. A student must proceed from Level 1 to Level 2.

Although the Van Hiele level's do not directly apply to understanding the concept of derivative we may make the following analogy. Carl (and to a lesser extent Ernest) can describe some representations of the concept of derivative in detail and can use these to interpret other representations or contexts. The fact that he does not use the formal definition makes him a student who is at a state analogous to Van Hiele Level 1. He is building an understanding of derivative that has the potential to lead to a state analogous to Van Hiele Level 2. Ingrid, on the other hand, is a student who memorizes the formal definition and claims to value that definition, but cannot explain what the formal definition means. Her understanding is actually weaker than Carl's in that she never explains the ratio involved in the derivative concept in any context except to state the formal definition. Without these other contexts her statement of the formal definition has no meaning. A teacher or curriculum developer should be careful to differentiate between a student who has the definition memorized and one who understands what the definition means in other contexts. In this sense, understanding the derivative concept in more than one context may be a prerequisite to learning (although not to memorizing) the formal definition.

Even if a student progresses so that he or she values and uses the formal definition in many contexts, this does not mean that the use of other paradigmatic models will be abandoned. Krussel (1995) reports many instances of mathematicians using intuitive models as a metaphor for a concept. Graphical and physical contexts may provide insights that are not as obvious in a formal context.

We see that the formal definition is not central to these students' understanding in the sense that by the fifth interview it is not the primary paradigmatic model for any student's understanding, and only Helen has it as a secondary paradigmatic model. The formal definition is also not a beginning point for any student's understanding (with the possible exception of Helen), but rather a statement in precise, symbolic form for a concept that is already understood by the student in at least one other context.

Taking the derivative. At this point it is appropriate to address the role of symbolic rules for taking the derivative such as the product rule and chain rule in a student's understanding of the concept of derivative. Note that a student may learn these rules without any knowledge of the three layers of the derivative concept in any of its representations. In addition a student need not know any of these rules to understand the concept of derivative in all of its layers and contexts. Heid (1984) demonstrated this possibility in her dissertation in which she had students use computers to take the derivative whenever necessary until the last three weeks of the term when they learned the computations for a joint final exam with a traditional class. For knowledge of *when* to take the derivative it is necessary that a student understand the contexts of the derivative, but even here it is often only necessary for a student to know that the derivative is the rate of change or slope of the tangent line or velocity at a point without understanding the ratio underlying these statements. Understanding of the ratio is necessary in situations for which one does not take the derivative. For example, when one has discrete data and needs to estimate the rate of change or when one has information about the derivative at one point and wants to estimate the value of the function at a nearby point.

Standard problem types - paradigmatic model for a related rate problem. Students learn to recognize types of problems from a few words and then expect the problem to follow type. A significant variant in the problem type causes confusion. The ladder problem in the fourth interview is a prime example. Each student is told that a ladder is leaning against a wall and is being pulled away from the wall, horizontally, at a constant rate. The student is asked if the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Each of the nine students set up a correct Pythagorean relationship and take the derivative (all but Carl take the derivative correctly). However only Alex is able to readily use his calculation to answer the questions asked. Many of the other students request the values of the rates and lengths involved so that a single value for the remaining rate may be calculated. This scenario fits the standard related rate problems where a single rate at a given instant is requested given the values at that instant of any other rates or quantities involved. Related rate problems with similar solution methods are done by the instructor in class and the students in homework in both their junior and senior year classes. Ernest's comments are typical of student reactions. Upon completing his symbolic calculations Ernest states, "And then we need to know some values." When reminded of the question and that the y and x values will be different at different times, Ernest responds, "Right, but in these kind of problems, they tell you what x is usually."

To get beyond this type of response the interviewer has to have the student calculate the rate for different specific values so that the student can answer the question and see that the rate the ladder is sliding down the wall is not constant.

Expert students: Lack of compartmentalization and diagrammatic models. Alex, Derick, Frances and Helen are the students in this class to score a 5 on the BC version of the AP Calculus exam. In many ways, not just by AP exam score, these four students set themselves apart as the "experts" in this class. Several characteristics seem to separate the experts from the other students in the class. These include the number and strength of the connections these students have between the various interpretations of derivative, their ability to use a graphical gestalt as a hook to more detailed information, and their lack of compartmentalization.

These expert students are also characterized by their ability to remember a concept in the form of a gestalt (often graphical) and then recall the details from the gestalt. The less successful students recall information in pieces which are not well connected to other aspects of their knowledge. Two examples come to mind involving the concept of slope. When students are asked in the fourth interview to estimate $f'(2)$ from a table of values most of the students explain that they are finding the slope. Four students (Alex, Derick, Frances and Helen) state first that they will find the slope and then do the calculation. Four other students (Brad, Carl, Ernest and Grace) complete a correct calculation, and then when asked, explain that they have calculated a slope. Only Ingrid is unable to solve this problem.

The other example is the Mean Value Theorem. During the fifth interview most students (all but Frances) are asked if they remember the theorem before being presented with the theorem for further interpretation. Three students (Alex, Derick and Helen) recall the graph involved first and are able to construct the symbolics from the graph and the idea that the slopes are parallel. Two other students (Grace and Ingrid) recall the symbolic equality correctly but have trouble interpreting it. Grace has a bit of trouble and never draws two parallel lines; Ingrid is not able to interpret the expression at all. The other three students, all males (Brad, Carl and Ernest) are not able to remember the theorem when asked, although Carl is able to give a graphical interpretation once the theorem is presented.

Experts have less compartmentalization so they fix their mistakes more quickly. This certainly characterizes the interviews of Alex and Derick who talked through their misstatements and considered other possibilities throughout. Frances and Helen make almost no misstatements. Their shyness may mask an internal process of weeding out misstatements before they are spoken. The other five students make many more misstatements that are not corrected and give the appearance of guessing more often at answers. Guessing is especially characteristic of Brad and Carl. Ingrid's principle characteristic is the memorization of theorems or algorithms without knowing the mathematics behind them.

Individual Metonymy

In analyzing the words students use to describe the derivative, one notices a persistence of the linguistic phenomenon metonymy. Metonymy is the use of a part to stand for a whole. One type of metonymy, which I call group metonymy, is the use of an exemplar to stand for a whole. This is the same as the paradigmatic examples discussed above. On the other hand, individual metonymy is the use of a part which is not an example to stand for a whole. Since the derivative concept may be thought of as three-layers, each in multiple contexts, there are many parts of this whole concept of derivative. Some parts of this whole, such as the paradigmatic models, may be used to accurately describe this whole. Other parts of this whole are unacceptable descriptions of the concept of derivative and these latter lead to student misstatements. In fact, most student misstatements in the interviews may be classified as examples of individual metonymy.

An interesting question is the extent to which student misstatements are just accidental slips of the tongue rather than actual misunderstandings. This section shows how several common student misstatements are examples of individual metonymy and discusses the issue of misstatement versus misunderstanding in a few specific cases.

There are two principal ways that individual metonymy occurs in the students' discussions of derivative. One is purely syntactic. The student needs to say a long phrase to correctly describe the derivative, but instead shortens the phrase. For example the derivative at a point is the slope of the tangent line to the curve at that point. If the student says instead that the derivative is the slope at that point there is no problem, but if the student shortens the phrase to say the derivative is the tangent line at that point then the statement is incorrect. A second example is the phrase instantaneous rate of change. If a student shortens this phrase to say that derivative is a rate, then the statement is acceptable. However, if the student says the derivative is a change, then the statement is not true.

There is no evidence from the students' interviews that the students consciously choose to shorten the phrases. Their statements are spontaneous attempts to explain their ideas to the interviewer. One wonders, however, whether or not a student who says the derivative is the tangent line or the derivative is change means these statements literally or whether the student has just used the statement as a subconscious short-hand for their true understanding that derivative is the slope of the tangent line or the rate of change. There may be a different answer for each student. It is also the case that students may hold both views. Contradictory ideas may be compartmentalized by a student so that the student does not notice the contradiction.

Brad, Carl, and Derick each make the misstatement that the derivative is change (instead of rate of change) in more than one interview. For each of these students there is at least one exchange with the interviewer in which the student shows that he means change literally, and he is not just shortening the phrase "rate of change." On the other hand, each of these students also say that the derivative is a rate or rate of change at other points during the same interviews.

In the first interview, Carl makes several misstatements involving rate of change. Although he does state that the derivative is the instantaneous rate of change, his explanations focus more on the derivative as change instead of rate of change. Carl says, "If you were to take the derivative, you'd end up with $\frac{dx}{dt}$ or Δx. The change in x over time. So that's basically like Δx." When asked for clarification, Carl explains further, "When you take the derivative of something you find the change in that. ... And it changes the equation from how much water is in it to how much it's changing at that instant, how much is leaving or going in at that instant. It's the instantaneous rate of change."

In the second interview, Derick and Brad speak of the derivative as change instead of rate of change. Derick's first answer to "What is a derivative?" is that it is "the change of a function at a particular point". Later when giving an example involving money he says that the derivative is "how much was gained this exact second" speaking of an amount gained as opposed to the rate at which the business was gaining money. Brad's first answer to "What is a derivative?" is that it is "the change in something". He repeats this statement often and uses it in relationship to slope stating, "The change is the slope. The slope is the change." When asked whether change and rate of change are the same or different, Brad says, "Change is like the actual change, where rate of change is the process of that at a certain-- Rate of change is at a certain interval. That's how much it changes at each point. And the change, I would interpret it as being the end product that you calculated with rate of change." He then constructs an example where he says the change is from 8 to 7 meters in 1 second. He says the rate of change is 1 meter per second, but then decides that this is the change as well so the two are the same. Later he continues to confuse rate and change in terms of the symbol Δx. He says, "Δx means the change in x. I mean that's the rate. That's like a change."

In the fourth interview Brad, Carl, and Derick make a symbolic error that is essentially a derivative equals change misstatement. In solving a related rate problem each of these students uses dx instead of $\frac{dx}{dt}$ and dy instead of $\frac{dy}{dt}$. This symbolic error does not impede any of the students in their attempts to solve the related rate problem, but it may be indicative of their focus on change rather than rate of change.

In the fifth interview each student is told that f is a function that gives the outside temperature at a given time. The student is then asked to interpret $f'(3) = 4$ and later $f'(x) = 4$ for $0 \leq x \leq 3$. Both Carl and Derick initially interpret $f'(3) = 4$ to mean that the temperature changes instantaneously by 4 degrees at the 3 hour mark as in the previous section on analogical models. Brad's initial misstatement, "3 hours later the change in temperature would be 4 degrees," is corrected to "It's changing at a rate at that time of 4 degrees per hour." He continues to speak of the change in temperature instead of the rate of change, but this does not prevent him from stating reasonable numeric estimates showing that he knows that the derivative is the change per hour.

Note that whenever these students discuss the derivative as change in a substantive way, they evidence a different kind of individual metonymy. Instead of taking a part of the phrase for the whole phrase, they take a part of the concept, the change in output values, for the whole concept, the change in output values divided by the change in input values. The denominator involving change in input values may be implicit for these students, especially when it is change with respect to one unit of time. However, keeping this aspect of the ratio implicit may lead to difficulties for these students as it does for Carl and Derick in the fifth interview. A teacher or text should use examples that force a student to make the denominator of a rate of change explicit. The consistent use of one unit intervals when considering difference quotients or of functions whose input is time may lead a student to make (possibly subconscious) assumptions that these are always the same and may be ignored.

Another type of conceptual individual metonymy occurs when discussing slope of the tangent line. Consider the complex process by which we describe the derivative at a point graphically. Whether we describe a sequence of secant lines approaching the tangent line at a point or we describe zooming in on a point until the curve appears to be a line, in each case the derivative is the slope of that line. However, a student who has focused on this image may say that the derivative is this whole process or that the derivative is the most obvious image or endpoint of this graphical process, the tangent line. The slope is implicit in both graphical images, but the tangent line is explicit, visible and thus more easily remembered. Even without the idea of a limiting process a student may remember a single image, a curve with a line tangent to it, when asked what a derivative is. Again one might pick up on the tangent line as the explicit, visual representation of the derivative instead of remembering that the derivative is the slope of that line.

At least five students in each of the first two interviews refer to derivative as a line or tangent line instead of the slope of the tangent line. However, in most cases these seem to be simple cases of shortening the phrase.

In the first interview, Derick gives a detailed and accurate description of a slope between two points being an approximation for derivative and the limiting process necessary to discuss the slope of the tangent line. He then concludes with the sentence, "The tangent line to the graph is the derivative."

Carl makes two similar slips. When asked if derivative is related to slope, Carl says, "Derivative is the tangent line to the function," before correcting this a few lines later to "The derivative is the slope of the tangent line to the graph." When asked if derivative is related to line or linear, Carl again confuses the two, "It's the line, the tangent line. It's the slope of the tangent line is the derivative, so the tangent line to the graph is the derivative as well."

On the other hand, Ingrid's statements are more problematic. When asked if the derivative is a function, she says she thinks it is because "It has to be a line. A line is a function." The second example occurs when Ingrid is asked to explain the relationship between the limit of the difference quotient and the slope of the tangent line. She chooses to focus on the example of $y = x^2 + 1$. She finds its derivative to be $y = x$, later corrected to $y = 2x$, and tries to "figure out how that fits in". Ingrid seems to want the derivative function to be a tangent line to the original curve. She does not consider the difference quotient or slope in her explanation.

Grace's misstatements are subtle. She says that the derivative is "tangent to the slope at a given point", "tangent of your slope" and "tangent of slope". At first these statements sound like a simple transposition of words. That these statements refer to the tangent line to the curve (slope) is made more clear in Grace's second interview. In the second interview she says that the derivative is "tangent of the slope" and when questioned clarifies by saying "the line that's tangent of the slope". She also says that the derivative is a line when asked to relate derivative to line or linear. "I mean your derivative is like a line. I guess that's kind of the context, isn't it, of derivative. When you have a point and you're finding tangent lines getting closer to that point, approaching that point."

Is it the student's (mis)understanding which causes the metonymic statements or the metonymies which influence student understanding? I suspect the answer is both and neither.

Certainly whatever conceptions a student has will provide fodder for possible metonymic occurrences, but it seems that metonymy is a natural linguistic phenomenon that occurs no matter what a person is discussing. A very knowledgeable person may even make metonymic misstatements if the person is not making an effort to be careful in his or her use of language. On the other hand metonymic misstatements made by a less knowledgeable student may be what influences that student to make more serious errors.

A student who says that the derivative is change (instead of rate of change) may in the next moment reason from this statement without considering other aspects of his or her understanding. Contradictory ideas may be compartmentalized by a student so that the student does not notice the contradiction. This may lead to further errors. Prime examples of such errors are Carl and Derick's misinterpretation of $f'(3) = 4$ or Ingrid's failed attempts to explain the formal definition of derivative.

Even without compartmentalization, a student may think that it is legitimate to use derivative to mean both slope of the tangent line and the tangent line itself. Many words in English have two meanings that are not synonymous but that are related in some way. Using the word derivative to refer to both a value (the derivative at a point) and a function (the derivative function) is an example. A student might consider that using derivative to mean both slope and tangent line is also acceptable.

On the other hand when a word does legitimately have two meanings that are not synonymous, this may cause problems as well. Ingrid does not recognize the distinction between the slope at a point and the derivative function and this plays into her confusion between slope and tangent line in the second interview.

When asked whether the derivative is a function, Ingrid initially says that it is not and then convinces herself that it must be. Her reasoning is as follows, "It's just a slope. It's not like y equals something. I just keep thinking, cause you can't graph just a slope or you can't graph a limit. But then if you say, like on a test, 'this is the graph of the derivative'. I guess it has to be. It could be a function." This lack of distinction between the derivative at a point and the derivative function is inherent in her confusion as to whether the derivative is the tangent line or the slope of that line. In a discussion of differentiability Ingrid is asked for the derivative at a point, she draws a line and says it is the derivative. A few moments later she explains, "The derivative is the slope. So it's going to be just a line anywhere on here with the same slope. ... [so any horizontal line] would be the derivative."

Another example of Ingrid confusing the derivative and a line or tangent line occurs as she attempts to relate her statement of the formal definition of derivative to a graphical interpretation. She believes that the result of the limit of the difference quotient is a line, as opposed to a slope and that the resulting line is going through one point, but on her sketch this is not a tangent line.

Another word that has two related but different mathematical meanings is the word *average*. Two students (Derick and Grace) make comments that confuse *average rate of change* and an *arithmetic average*. In the second interview Grace describes the difference between average and instantaneous as follows, "It's like average velocity when we studied that in physics last year. There's a difference between average velocity and instantaneous velocity because when you have an average velocity that just means it could be really high and really low at one point, but if it's like kind of in the middle for the rest of the time, then it's just like middle. Know what I mean?"

Her description of average velocity conveys the idea that an average is in between a range of possible values that may occur, but it does not seem to imply a ratio of distance over time. She may be thinking of an arithmetic average as seems the case in a later excerpt from the second interview. When asked if measurement is related to derivatives she says, "Yeah because derivative gives you a really accurate measurement. When you have an average measurement-- OK, like if you had an 80% average for a class. You could have had a 70, a 90 and an 80, but you'd still come out with an 80. But if you have your derivative, that'd be telling you-- Like if you have an 80, it can't be anything higher or lower."

Derick's confusion involves an example he is using to describe the difference between average and instantaneous rate of change. He says that we have a table of how much money was gained or lost everyday for one month. To find the average gain or loss we add up these numbers and divided by the number of days in the month. His statement is true, but it does not fit what we usually think of in calculus as average rate of change and does not lead to describing instantaneous rate of change for this setting. He tries anyway discussing how to determine exactly how much was gained or lost on the 16th. "You could take smaller and smaller intervals [around the 16th] until you have an exact amount of how much you gained this exact second". When the interviewer points out that in his example both the original function and the derivative tell how much was gained or lost, Derick changes the table to list how much money the business had on a given day. However, he is still confused about how to find the average rate of change. First he wants to add the amounts for the 1st and 31st and divide by 2. Then he wants to subtract the amounts for the 1st and 31st and divide by 2. He finally decides to switch to a velocity example where he no longer tries to explain the relationship of average to instantaneous rate of change.

Derick can explain the relationship between average and instantaneous values graphically and symbolically but struggles when applying them to a physical situation. One source of confusion is the distinction between an average rate of change, where one finds the arithmetic average of rate of change values, and an average rate of change, where one calculates a differences quotient using no rate of change values. Perhaps the discrete nature of Derick's example leads him to think of an arithmetic average.

Limitations of the study

The method of analyzing a student's understanding of derivative by means of the structure iconized in the circle diagrams leads to an emphasis on those characteristics of the concept of derivative. Other aspects of the concept of derivative, such as the derivative as a symbolic manipulation that converts one function into another function or as a tool for certain purposes, are de-emphasized in the interview follow-up questions. This may lead students to de-emphasize these types of replies in later interviews.

One particular aspect of the derivative concept that is de-emphasized in this analysis is the use of the derivative at a point to extend knowledge of the function in question. The circle diagrams concentrate on the notion that the derivative is the limit of the ratio. The determination of this ratio, using information from the function, becomes all important. However, given only a point, the derivative value at the point, and the knowledge that the derivative is a ratio, one may predict other output values of a function for input values near the point that is known. This method of prediction based on turning a single value for the derivative into a ratio, may also be used to help in interpreting expressions such as $f'(3) = 4$ for a particular real world situation in which the function is defined.

Collecting data in an interview format has limitations. Some students feel more comfortable discussing their ideas than others. This level of comfort is not in a one-to-one correspondence with their level of knowledge. Therefore, shyer students may not show as much of their understanding in an interview as less shy students. Some students are careful not to express their ideas unless they have a high level of confidence in their answer. This makes it harder for the interviewer to know the student's thought process.

Another concern with any one day snapshot of a student's understanding is that a student may not perform as well on one day as on another because of mental or physical fatigue or emotional stress having nothing to do with calculus. Ernest alludes to this type of problem in one of his interviews when he says, "I guess today wasn't a very good day for me."

Both a strength and a limitation of the study is its qualitative methodology. The in-depth classroom experience with the students, the interaction with their instructor and the variety of data collected for each student -- tests, questions of the day and especially the interviews -- led to a rich source for seeing patterns in student understanding, the development of that understanding and possible causes for that development. However, since this is not an experimental statistical study, the extent to which the conclusions for these students will hold for other students is unknown.

An additional concern with the interviews is the short time period allowed for some of the interviews (at times as little as 25 minutes) considering the large number of questions asked. This may have affected the data in that in 25 minutes a student may not have had time to remember and express all of his or her knowledge regarding a long series of problems or questions.

Implications

Reflections on the Study

This study is qualitative in nature and includes many different types of data collection techniques. In this section I discuss what I learned from the study in terms of what use I was able to make of the multitude of data collected. As the study progressed and the theoretical framework came more clearly into focus, some of the data collected could be seen to be more valuable to the study while other data did not contribute as much to the study in its final form.

I was unable to make much use of the audiotaped class sessions. If I had had easy access to transcripts of this data, I would have been more likely to make use of it. I did make use of my typed field notes. Primarily I used them to keep track of the order of events; i.e. what material was covered each day in class and when did interviews, tests or questions of the day occur with respect to what the students had covered. I also kept track of student attendance in my written field notes.

Attending classes allowed me to understand the culture of this particular class, the terminology that they used and the examples that they were familiar with. It also allowed me to observe expressions of the personalities of the students and to note strengths that would not have been easily captured by other assessment instruments, such as an ability to ask good questions. However, because I had not devised a systematic way to collect data on such information, this data did not have any direct influence on my conclusions.

Knowing the culture, however, did allow me to better understand some interview responses because I could put them into the context of what the student had been experiencing. Knowing the culture and understanding interview responses helped me ask better follow-up questions.

The circle diagram coding scheme was only used for the first, second and fifth interviews. These were the only interviews that were comprehensive enough to serve as a snapshot of a student's understanding of derivative. Other data that was collected could have been coded using the circle diagrams, but it would have only reflected a small portion of a student's understanding. I chose to use the circle diagrams only for data in which the student had multiple opportunities to express each of the layers (ratio, limit, function) in each of the four main contexts (slope, rate of change, velocity and symbolic) used in the circle diagrams. Because of this choice, the students' question of the day (QOTD) responses and written test responses were not coded, and hence the students had only one format, the interviews, in which to show their knowledge for the circle diagrams.

The tests were designed by the instructor of the course. If I had been that instructor, I may have been able to devise written tests that would have been comprehensive enough to allow for the use of the circle diagrams. This would have provided a comparison with the interview data. A written format would have allowed students to reflect more on questions before answering. The students who were shy or nervous in the interviews may have performed better in writing. On the other hand, the written tests would not have allowed for follow-up questions or clarifications. Having both written and interview formats for determining the circle diagrams may have provided a more complete picture of each student's understanding.

The QOTD by their nature were one or two brief questions that did not allow for a comprehensive picture of a student's understanding. These questions were used mostly for information that was related to but not directly covered by the circle diagrams. Some of the QOTD not listed specifically in this dissertation were questions that helped me get to know the students and the culture of the class. For example, I asked the students questions about their extracurricular activities and their use of calculators. The QOTD were more useful for these purposes than for determining student understanding of the concept of derivative. The most effective QOTD for student understanding of derivative was the question that asked the students what they had learned about derivatives this year that they had not learned in their previous year's class. For QOTD to have been more effective, they should probably have been part of a student's grade, because as the students became more familiar with the researcher, they took the QOTD less seriously, and provided less useful responses.

Some interview questions asked students to solve tasks instead of answering a more general question such as "what is a derivative?" These occurred in both the fourth and fifth interviews. Those in the fifth interviews were coded with the circle diagrams as part of the total picture of a student's understanding for that interview. Those in the fourth interview could have been coded but were not because that interview did not provide enough opportunity for a student to mention all the layers and contexts. In other words, it was possible to code task-based interview questions as well as more general questions using the circle diagrams.

If I had designed a comprehensive enough task-based interview, I could have created a circle diagram for a student's concept image based on that interview. However, one issue with such an interview would have been the time involved in problem solving.

For a comprehensive picture of a student's concept image, several different types of questions would have been needed. In my rather short interviews (usually less than 45 minutes), students did not always have enough time to completely solve the number of problems that I had given them. I observed their initial responses to setting up the problem when there was not time for complete solutions. One way to improve this situation would have been to allow students to work on the problem alone before asking the student to discuss the problem with the interviewer. This might also have helped the students to communicate more of their knowledge.

Implications for Teachers and Curriculum Developers

The goal of both teachers and professional curriculum developers is to design explanations, problems, and activities to help students develop as complete as possible understanding for each concept covered in a particular course or curriculum. In this dissertation I describe the concept of derivative as three layers of process-objects — the ratio or difference quotient, the limit, and the function layers — each of which may be observed in multiple contexts — graphical (slope), verbal description (rate of change), kinematic (e.g. velocity or acceleration), and symbolic (the symbolic difference quotient definition of derivative). This description may be used to evaluate a curriculum for completeness. Not only should all aspects of the concept of derivative be covered in the text or in the lecture, but the curriculum should include activities and problem sets that require students to work with each aspect of the concept directly.

Every calculus textbook covers each of the three layers of the concept of derivative in at least one representation (symbolic) and usually more than one (symbolic, graphic, and velocity). However, not all layers are emphasized equally in all contexts, particularly in physical contexts other than velocity or acceleration. Calculus reform texts (e.g. Hughes-Hallet et al., 1994; Dick and Patton, 1994; Ostebee and Zorn, 1997) have tended to be better than traditional texts at emphasizing the ratio layer and in emphasizing all the layers in multiple representations and contexts.

For a student to fully understand the concept of derivative, he or she should be able to identify the three-layers of the concept of derivative in any context, i.e. for any function that describes a relationship between two covariants. As discussed above, these students have a wide variety of preferred contexts for understanding the concept of derivative at the beginning of their senior year, based on their three month study of derivatives as juniors. This means it is particularly important for teachers to emphasize multiple representations and the connections between these representations. Each student in a class, no matter which part of the concept of derivative he or she understands or prefers, must have the opportunity to connect what they already understand to the parts that they do not.

Another main result is that students do not learn in a linear fashion. The layers and contexts of the derivative are learned by the students in this study in many different orders. Therefore, a teacher will need to revisit ideas that he or she has presented before. In this way, students who are not able to connect a particular aspect of the derivative to his or her understanding when seeing it for the first time may at a later time have an understanding that can incorporate this aspect.

Teachers should be particularly careful to ask questions that force students to work with the ratio layer. After an initial discussion of this layer in defining the derivative, many texts never return to the ratio, allowing a student like Ingrid to avoid learning this layer. This is extremely problematic in that a student who does not understand the ratio layer will not understand what the symbolic formal definition represents and will not be able to estimate a rate of change when a function is given by a table of data and not a symbolic expression (as often occurs in real life scientific experiments).

The formal definition must be a series of symbols that represents for the student the three-layered structure. As we have seen in this study, only the very best students (out of a class of good students) are able to give a complete and accurate explanation of the symbolic definition in another context. This mediocre result occurs in a classroom where the instructor has painstakingly described the relationship between the symbolic and graphic representations on several different occasions.

Other than asking a student directly to explain the symbols of the formal definition, one may ask students to explain the meaning of the symbols involved in the statement of the Mean Value Theorem. In other words, ask students to interpret $f'(c) = \frac{f(b) - f(a)}{b - a}$ where $a < c < b$ in terms of a graph, in terms of a position function, and in terms of other functions. The calculus text by Hughes-Hallett et. al (1994) provides additional exercises for students to explain or interpret the symbols of the difference quotient graphically or numerically, but switches to asking students only to interpret symbols such as $f'(c)$ in various physical contexts. Students should be made aware of the value of the definition as a precise and concise way to express ideas that may have more meaning for them in other interpretations.

Teachers and curriculum developers should make use of velocity and acceleration as a context of derivative that may illuminate the meaning of the three-layers in other physical contexts. At the same time, student knowledge of velocity and acceleration should not be overestimated. Students who have not studied physics or have not had the opportunity to experiment with the relationships between position, velocity and acceleration may not be able to take full advantage of velocity as a paradigmatic model.

In these cases teachers may want to involve the students in physical experiments that are interpreted by the class in words, graphs, tables of values and symbolically. One type of physical experiment would be to have students use a motion detector that graphs a student's position or velocity over time. The student would then be asked to move with the appropriate changes in speed and direction to create particular graphs. Several different environments are described by Rubin and Nemirovsky (1991) in their work with precalculus students' understanding of rate of change. The environments include a motion environment, an air pump environment and a spread sheet environment.

Teachers and curriculum developers should be aware of the dangers in only asking students to solve standard traditional problems. As we have seen above, students become familiar with certain types of problems, for example the related rate problem involving a ladder sliding down a wall. Students then solve these problems by modeling their previous solutions (or worse, the book or instructor's template solution) and then are frustrated or confused if a problem does not conform to type. Since most real world problems will not conform to a particular type that students have seen before, students should have some experience with unusual problems or variations on standard problems.

Teachers and curriculum developers should also be aware of tricks of language that may cause confusions for students. When there are two or more different meanings for a word such as derivative (the function or the value at a point) or average (average rate of change or arithmetic average), it is important to make students aware of the distinction between the two meanings, and the relationship (if any) between the two meanings. Students should be asked to write or talk about their evolving understanding of calculus. Simply finding symbolic or numeric answers to exercises will not cause students to confront the disparities in their understandings that are related to language. Carefully evaluating student writing is important so that students may be made aware of the care with which mathematicians use language.

Directions for Future Research

The methodology of this study should be used with other students to see if the trends found with this group of students are indicative of more general trends. Students with weaker or stronger backgrounds or using different textbooks or technologies may emphasize different interpretations of the derivative or may make different misstatements. In particular, the students in this study do not refer to the derivative as a function very often and many students are uncertain when asked directly if the derivative is a function. However, for one of the classes involved in the pilot study, almost every student says that the derivative is a rate of change function because it has been emphasized in their course materials. It would be interesting to compare the circle diagrams for two parallel classes using different course materials.

Another important continuation of this research would be to use these methods to examine the understanding of derivative held by beginning students. The students in this study have covered derivatives during a three month period in their junior year prior to their senior year involvement in this study. Therefore, we do not see the initial development of their understanding of derivative that leads to students having such a wide variety of preferred representations during the first interview of this study. Tracking students during their first experiences with the concept of derivative would provide valuable information about how students form their initial representation preferences and about which layers and connections are initially harder or easier for students to develop.

The tools used to describe understanding of derivative in the theoretical framework and to determine a methodology for describing it using icons are not tools which are unique to the concept of derivative. Other concepts with multiple process-object layers and multiple interpretations could be described by a similar methodology. The definite and indefinite integrals would be an interesting place to start. The three layers involved in the concept of derivative (ratio of differences, limit of the ratio, generalization to a function that gives the limit of the ratio for each input) have close parallels with Riemann sums, the definite integral and an accumulation function. The definite integral may be approximated by a sum of products, the limit of this sum produces the single value of a definite integral, and we may generalize this to a function by letting the upper limit of integration in the definite integral be a variable.

A different type of source for parallel structures occurs in linear algebra. Here we do not see the multiple layers of process-objects that are involved with the derivative or integral, but we do see that there are parallel ideas that may be represented in multiple settings. For example, the following five statements are equivalent:

1. System of Linear Equations: A solution to a system of m equations in n unknowns exists for any set of numbers that the equations are set equal to.

2. Vector Equations: A set of n column vectors in R^m span R^m.

3. Matrix Equations: An $m \times n$ matrix has a pivot in every row.

4. Linear Transformations: A linear transformation from R^n to R^m maps onto R^m.

5. Subspaces: An $m \times n$ matrix has the property that its column space in equivalent to R^m.

Research in this area would involve analyzing whether a student knows the structure of each context and whether a student makes the connections necessary to recognizes the parallel nature of the contexts.

A number of trends in student understanding of derivative are mentioned earlier in this chapter. Some of them are supported with only a few examples. Further research should determine to what extent these trends hold and for what populations. In addition to whether these trends hold, one wonders why some of the trends hold. Some questions:

1. Why do some students prefer one representation while a student in the same class prefers a different representation? Individual differences to be sure, but what causes these individual differences?

2. What makes a particular representation memorable to a particular student?

3. Are there particular physical experiences that might make some representations more memorable to students? These might include work with motion detectors that translate a student's movement into a graph of velocity or position versus time.

4. Are males more interested in the velocity interpretation than females? If so, why?

5. Are females more likely to learn the formal definition than males? If so, why?

6. What causes some students to easily make connections between representations while others do not?

7. Why does Ingrid (or a student like Ingrid) never express the ratio layer?

8. How do particular deficits in a student's understanding of derivative (e.g. Carl and Ernest's inability to state the formal definition or connect it with other contexts or

Ingrid's failure to express the ratio layer) affect these student's ability to solve problems involving the concept of derivative?

A future research project should attempt to code a student's concept image of derivative strictly from problem solving scenarios. As a student solves a problem involving the concept of derivative, the student makes choices about what context or representation will be helpful in solving the problem. The student also makes decisions to work with the derivative as a function or at a point, and a student may use a difference quotient to estimate a derivative value. In these ways a students concept of derivative is observable through his or her problem solving choices.

Although a student's concept image is observable through his or her problem solving decisions, one wonders whether solving derivative problems includes an additional type of knowledge different from the structure of the concept of derivative in terms of objects and processes. Certainly there are general problem solving skills, but are there specific skills involved in solving *derivative* problems that are unique to those sets of problems (i.e. part of the concept of derivative) but different from understanding the concept of derivative as three process-object layers in multiple contexts?

My reaction is to say that there are not. There are of course the rules for finding the symbolic derivative (product rule, chain rule, etc.). However, I would argue that this skill along with general problem solving skills and a truly connected and complete knowledge of the layers and contexts should allow a student to solve any problems that involve only the concept of derivative at the freshman calculus level. With this, let me state two caveats. Problems involving specific real world contexts might involve knowledge of those contexts that a calculus student might not a priori be familiar with. Problems involving both the concept of derivative and another concept would require knowledge of that other concept and its relationship to derivative.

The notion of individual metonymy deserves more examination. Most of the misstatements made by students in this study can be characterized as the student using a part to represent a whole. The primary examples of this are students saying that the derivative is the tangent line or that the derivative is a change instead of a rate of change. A related problem is when a student confuses two different meanings for the same word — the derivative at a point and the derivative function or average rate of change versus average value of a function.

The confusion with the word average is a particularly rich source of study because of the multiple connections (both valid and invalid) that students may make when examining the relationships involved. The question in the fifth interview that asks a student to find the average rate of change of a function defined as an integral points to some of these connections. In that problem a student may use a difference quotient with a difference of integrals in the numerator to find this average rate of change, or the student may find the rate of change function, the function involved in the integral, and find its average value. A study of student understanding of the relationship between these averages would examine the relationship a student sees between arithmetic averages, the average value of a function computed by an integral and the average rate of change. The relationships could be examined in multiple contexts — graphic, symbolic, kinematic. Both the Mean Value Theorem and the Fundamental Theorem of Calculus would be involved in detailing the relationships between the various averages. This study might best be considered after a study on student understanding of the layers involved in the definite integral and the accumulation function as mentioned earlier in this section.

Researchers should be aware of the extent to which natural language phenomena such as individual metonymy influence a student's statements. A direction for future research is the relationship between the language a student uses and the students understanding of a mathematical concept. As discussed in the section above on individual metonymy, a student may sometimes use a metonymic, but inappropriate statement such as stating that the derivative is the tangent line instead of the slope of the tangent line at a point on the curve. Can the language a student uses unreflectively (such as the metonymic statement above) influence the student's understanding? For example, is it possible that a student who does not have a strong belief that the derivative is the tangent line could convince himself to believe this more strongly because of his unreflective, metonymic use of the statement, "the derivative is a tangent line," when he is trying to repeat the phrase, "the derivative is the slope of the tangent line?" A similar question may be asked about the phrases "the derivative is change" and "the derivative is rate of change."

There are other questions involved in using a student's verbal (written or oral) statements to evaluate his or her understanding. Sometimes a student's language may be an echo of classroom utterances rather than a true indicator of understanding. For example, a student may have memorized the often recurring phrase "instantaneous rate of change" or the statement of the formal definition, but not actually connect these to any meaning other than a verbal association with the term "derivative". Is it also possible that a student may understand a part of the concept of derivative but not be able to express in words? I have found that students in my classes sometimes like to say, "I understand it, but I can't explain it." Is such a student inaccurate in saying he or she understands, accurate but too lazy to find the words, or is there such a thing as an inexpressible understanding? Is understanding that is not expressible in words (or symbols or diagrams) a true understanding? Does it have any value for the student in terms of being usable by the student? If so, is there a way for a researcher to be able to observe or evaluate an understanding that a student can not express?

Bibliography

Amit, M., & Vinner, S. (1990). Some misconceptions in calculus - Anecdotes or the tip of an iceberg? In G. Booker, P. Cobb, T. deMendicuti (Eds.) *Proceedings of the Fourteenth International Conference for the Psychology of Mathematics Education* (Vol. I, pp. 3-10), Mexico City: Organizing Committee of the 14th Annual PME Meeting.

Breidenbach, D., Dubinsky, E., Hawkes, J. & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23,* 247-285.

Brown, D., Porta, H.A., and Uhl, J.J. (1991). Calculus and Mathematica. Reading, MA: Addison-Wesley.

Confrey, J. & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26,* 135-164.

Cornu, B. (1981). Apprentissage de la notion de limite: Modeles spontanes et modeles propres. *Proceedings of the Fifth Conference of the International Group for the Psychology of Mathematics Education,* 322-326.

Cornu, B. (1983a). *Apprentissage de la notion de limite: Conceptions et Obstacles.* These de Doctorat, Grenoble.

Cornu, B. (1983b). Quelques obstacles a l'apprentissage de la notion de limite. *Recherches en Didactique Mathematiques, 4,* 236-268.

Crocker, D. A. (1991). A qualitative study of interactions, concept development and problem-solving in a calculus class immersed in the computer. (Doctoral dissertation, The Ohio State University, 1991). *Dissertation Abstracts International, 52A* , 2850-2851.

Davis, R. B. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior, 5,* 281-303.

Dick, T. P. & Patton, C. M. (1992). *Calculus.* Boston: PWS.

Dick, T.P. & Patton, C.M. (1994). Calculus of a Single Variable. Boston: PWS.

Dubinsky, E. (1991). Constructive Aspects of Reflective Abstraction in Advanced Mathematics. In L.P. Steffe (Ed.), *Epistemological Foundations of Mathematical Experience.* New York: Springer-Verlag.

Dubinsky, E. & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy.* Washington, DC: Mathematical Association of America.

Ellison, M. (1993). The effect of computer and calculator graphics on students' ability to mentally construct calculus concepts. (University of Minnesota, 1993.) *Dissertation Abstracts International, 54A,* 4020.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach.* Boston: Reidel.

Harel, G., & Dubinsky, E. (Eds.). (1992). *The concept of function: Aspects of epistemology and pedagogy.* Washington, DC: Mathematical Association of America.

Hart, D. K. (1991). Building concept images: Supercalculators and students' use of multiple representations in calculus. (Oregon State University, 1991.) *Dissertation Abstracts International, 52A,* 4254.

Heid, M. K. (1984). An exploratory study to examine the effects of resequencing skills and concepts in an applied calculus curriculum through the use of the microcomputer. (Doctoral dissertation, University of Maryland, 1984). *Dissertation Abstracts International, 46*, 1548A.

Hughes-Hallet, D. & Gleason, A.M. (1994). Calculus. New York: John Wiles & Sons, Inc.

Janvier, C. (1987). Translation processes in mathematics education. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathemtaics* (pp. 27-32). Hillsdale, NJ: Erlbaum.

Krussel, C. E. (1995). Visualization and reification of concepts in advanced mathematical thinking. (Doctoral dissertation, Oregon State University, 1995). *Dissertation Abstracts International, 56A,* 127.

Lakoff, G. (1987). *Women, fire and dangerous things.* Chicago: The University of Chicago Press.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*, 1-64.

Markovits, Z., Eylon, B. & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics, 6*, 18-24.

Monk, G. (1989, March). *A framework for describing student understanding of functions.* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

Monk, G. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy. MAA Notes, Volume 25.* Washington, DC: Mathematical Association of America.

Monk, G. & Nemirovsky, R. (1995). The case of Dan: Construction of a functional situation through visual attributes. *Research in Collegiate Mathematics Education. I. Issues in Mathematics Education, Volume 4*, 139-168.

Nemirovsky, R. & Rubin, A. (1992). *Students' tendency to assume resemblances between a function and its derivative.* TERC Working Papers.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics, 14,* 235-250.

Ostebee, A. & Zorn, P. (1997). Calculus: From Graphical, Numerical, and Symbolic Points of View. Fort Worth: Saunders College Publishing.

Rips, L. (1975). Inductive judgements about natural categories. *Journal of Verbal Learning and Verbal Behavior, 14*, 665-681.

Robert, A. (1982a). L'acquisition de la notion de convergence des suites numeriques dans l'enseignement superieur. These de Doctorat d'Etat, Paris VII.

Robert, A. (1982b). L'acquisition de la notion de convergence des suites numeriques dans l'enseignement superieur. *Recherches en Didactique Mathematiques, 3*, 307-341.

Romberg, T. A., Fennema, E. & Carpenter, T. P. (Eds.). (1993). *Integrating research on the graphical representation of functions*. Hillsdale, NJ: Erlbaum.

Rubin, A. & Nemirovsky, R. (1991). Cars, computers, and air pumps: Thoughts on the roles of physical and computer models in learning the central concepts of calculus. In R. G. Underhill (Ed.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education -- North American Chapter Conference* (pp. 168-174), Virginia.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification -- The case of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy. MAA Notes, Volume 25.* Washington, DC: Mathematical Association of America.

Simonsen, L.M. (1995). Teachers' Perceptions of the Concept of Limit, the Role of Limits, and the Teaching of Limits in Advanced Placement Calculus. *Dissertation Abstracts International*, 56-06, 2158. (University Microfilms No. 9535977).

Tall, D. O. (1986). *Building and testing a cognitive approach to the calculus using interactive computer graphics.* Unpublished doctoral disseration, University of Warwick, Coventry, England.

Tall, D.O. (1991). A Graphic Approach to Calculus [Computer Program]. Pleasantville, NY: SunBurst Communications, Inc.

Tall, D.O. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In Grouws, D. (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York: Macmillan.

Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics., 26*, 229-274.

Thompson, P. W. (1995a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics.* (pp. 179-234) Albany, NY: SUNY Press.

Thompson, P. W. (1995b). Students, functions, and the undergraduate curriculum. *Research in Collegiate Mathematics Education. I. Issues in Mathematics Education, Volume 4*, 21-44.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society, 30*, 161-177.

Vinner, S. (1982). Conflicts between definitions and intuition -- the case of tangent. In A. Vermandel (Ed.), *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education* (pp. 24-28). Universitaire Instelling, Antwerpen.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*, 293-305.

Vinner, S. (1992). The function concept as a prototype for problems in mathematics learning. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. Washington, DC: Mathematical Association of America.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*, 356-366.

Williams, S. R. (1989). Understanding of the limit concept in college calculus students. (Doctoral dissertation, University of Wisconsin at Madison, 1989). *Dissertation Abstracts International, 50*, 3174A.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22*, 219-236.

APPENDICES

Appendix A — Data Analysis

Case Study 1 — Alex

Academic record

*Other AP courses: Physics (junior year), Chemistry.

*Immigrated to the U.S. from an eastern European nation between his freshman and sophomore years of high school.

*Plans to major in engineering in college.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: "The function is a dependence between two variables where one variable depends on another."

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different?

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: "a) 1) $y = x^3 - x^2 + 10x - 18$

2) $y = |x|$

The 1st function is polynomial so it doesn't have corners or sharp curves. $y = |x|$ is an absolute value function which doesn't have smooth curves, but has straight lines."

QOTD #3

Give an example of a function <u>without</u> using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Date: August 26, 1993. This question occurs while the class is doing a quick review of functions.

Response: Alex draws a function, on a standard x-y axis, which is continuous but at several points not differentiable.

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) non integers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: "a) Only a piece wise function can be like that. Otherwise there is no such function because $y = x^2$ wouldn't assign 1 to 0.

b) No. Since between two x integers there are infinite number of nonintegers the best way to assign integer y to every noninteger x to try to make it a function is to draw a straight line like $y = c$ where c is an integer. But that wouldn't fulfill the other condition about every integer x is to have noninteger y ."

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to class discussion on limits.

Response: "A limit of a function, f, at a point $x = a$ a slope of the function at this point."

Comment: Alex's confusion may stem from his knowledge that the derivative is defined as a limit.

Test 1

On the first test covering limits a week later, Alex is able to correctly find limits by reading values from a graph, by substituting into a piecewise function, and by using algebra to simplify a limit calculation. However, for $f(x) = \frac{3x}{|x|}$ Alex finds $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$. Given Alex's other correct responses, I suggest that he is answering this question based on his knowledge of the function $f(x) = \frac{x}{|x|}$ without considering that the two are different.

Alex also has problems working with the formal definition of limit. When asked to prove a limit for a simple linear function using the ε-δ definition, Alex chooses ε to be .01 not realizing that he must show that it's true for all ε. He does correctly find δ to be .03 for this function and ε. However, on the following problem where epsilon is given as .01 and Alex is asked to find delta graphically, he assumes incorrectly that $\delta = .03$ would work here as well.

Interview 1

This interview occurs after the test on limits but prior the class discussing derivatives. For this reason, Alex's answers are presumed to be based on what he remembers from his junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table 5.1 which codes these responses. A summary discussion follows.

1 MZ: So first I was just going to ask you, what is a derivative?

2 Alex: Derivative is the instantaneous rate of change of the slope of the function at some point.

3 MZ: What can derivatives be used for?

4 Alex: You can find maximum and minimum value of the function. Like for example, you have some kind of equation for industry and you need to do like, to find the maximum of something and you have this equation. So you take the derivative. Find the critical points. You find where the slope is zero, and that's your maximum or minimum. There's two ways to check it. One you use second derivative test or you use first derivative test, and you find your maximum.

5 MZ: And how do I find the critical points?

6 Alex: You make the equation equal to zero.

7 MZ: The original equation that you started with?

8 Alex: Uh, wait a second. Yeah, the original equation. Yeah.

9 MZ: So you have the equation of something you're trying to find the maximum of.

10 Alex: Yeah.

11 MZ: And then so you set that equal to zero.

12 Alex: Wait a second. When you take the derivative, then you get another equation and then-- Oh, I know. You have to find critical points of the equation you get from the derivative of this. That's where you have to find the critical points.

13 MZ: So you take the derivative of that--

14 Alex: And you set it equal to zero.

15 Alex: And you find the critical points. Then you use like first derivative test. You put this on your number line and you find the intervals between them--

16 MZ: You mean the critical points?

17 Alex: Yeah. --to be positive, negative. Then where it goes from negative to positive that's the critical point.

18 Alex: I mean if it goes from positive to positive it's a critical point too, but it's not going to be a maximum or minimum.

19 MZ: If it goes from negative to positive is that a max or a min?

20 Alex: It's a min.

21 MZ: So what is this doing?

22 Alex: It's the slope. Negative slope goes to positive slope. So negative goes down and positive goes up.

23 MZ: Yeah.

24 Alex: So it's going to be on the bottom. So it's a minimum.

25 MZ: That's how I remember it also. OK. Can you think of anything else that the derivative would be useful for besides finding a max or a min?

26 Alex: You can find the concavity which somehow can help in physics when you find like terminal velocity. If you put your graph like this which would perfectly be OK if you use just first derivative test. If we go this way or if we go this way. Like for terminal velocity it's going to have an asymptote. So it's going to go like that. When you plot velocity over the y value. It's going to reach terminal velocity so it's not going to go up. It's going to go along the line. But the first derivative wouldn't show that, but the second derivative will. Because it's going to be concave down instead of being concave up.

27 MZ: OK. How would you explain what a derivative is to someone who's like an AB student or a precalc student that hasn't studied it yet?

28 Alex: When you have two points on any graph and you get a line through them, and the slope of that line would be the average rate of change between those two points. If the derivative is instantaneous rate of change because those two points are really close to each other. I mean they are so close that you can even have just one point. So what you do is like-- OK, you can leave one of these points and you can move the other one closer and closer to the other one, the one you left. And that will give you more accurate rate of change between-- If you take two points far apart that will give you just average rate of change, throughout the whole function. But as you start moving closer to that point you're going to get more accurate average rate of change between those two points because they are really close. And when you get those points to be so close together that you could even consider them to be one point, that's going to be instantaneous rate of change of that function at that point.

29 Alex: So that's just the slope of the function between the two points.

30 MZ: What if you had to explain it to somebody who just doesn't know anything about math, like a relative of yours that didn't have much math in school?

31 Alex: Wait, he doesn't know what slope is and--

32 MZ: Yeah, that doesn't know what a slope is or doesn't understand how a graph--

33 Alex: Well, how my uncle explained it to me when I didn't know what derivative is. Derivative of x^2 equals to $2x$ and derivative of x is equal to 1.

34 MZ: Was that helpful to you?

35 Alex: No. Even if he told me the power rule, how you subtract the power and bring it, it still wouldn't help me. I still didn't understand what the derivative is, even after I took the test. I understood how to take it, but it was kind of hard to get into my mind how could the slope of this line be here and what's happening there. I mean-- But if you don't really know what slope is-- If you don't know what the function is, you can't like say if it's derivative.

36 MZ: How did you learn it?

37 Alex: In physics we did some of this stuff with acceleration, velocity, distance because they are derivatives one of each other. And then in precalculus we drew the graphs by using the graphs of derivatives.

38 Alex: So, first I was confused, but then I understand that this graph represents the equation of derivative so each point, OK each x would be the same in the function as in the derivative, but y would be the slope of the function at that point.

39 Alex: So what I did is like I took the most accurate point that I could find. You know like $(1,2),(2,1)$ something like that, a whole number. Then plot the x 's that had. Then I plot the slopes, that's the y of the derivative graph. And I plotted like over a bunch of these lines.

40 MZ: Like drew little short lines with the slope.

41 Alex: Then it started to make sense to me, when I started to convert from derivative to function from function to derivative. Then just by looking at it I could find the, after awhile I could find just matching them without making the little the slope things. You know, the max and min points. After a while it's just started making sense to me.

42 MZ: Yeah, that's good. Here's a different question, say if I give you a function, how can you tell if it's differentiable or not?

43 Alex: If it's continuous [pause] If a function is continuous it's going to be differentiable.

44 Alex: No, not necessarily.

45 MZ: Why not necessarily? What did you think of?

46 Alex: When the-- It's a sharp turn.

47 Alex: It's still continuous, but it's not differentiable at that point.

48 MZ: OK.

49 Alex: So if you don't have-- If the function is not a piecewise function and it doesn't have sharp turns, it's differentiable. If it's piecewise, it still can be differentiable if it has like a hole in it because the hole doesn't matter. Since the derivative at the point-- [pause] If it has a hole in it, it has a limit, but a function is not defined at the point. You can't find the slope at the point. So it's not differentiable.

50 MZ: Did you say you can not find the slope?

51 Alex: Yeah, if it's not defined at the point.

52 MZ: OK.

53 Alex: Because when you find the derivative, your equation like $x^2 + x + 1$ or whatever. You have to plug in something for x. So you can find the slope right by this point, as close as you can get on your calculator, from both sides, but you can't really find the slope right at this point. So if the function is not continuous, it's not differentiable.

54 MZ: OK.

55 Alex: But if it's differentiable, it is continuous.

56 MZ: Here's a list of words. The idea is that for each word you tell me if it has anything to do with derivative, but some of them you've kind of already explained so you don't have to say a lot more. The first one is slope.

57 MZ: You already said really. Speed or velocity.

58 Alex: Well, the speed-- the velocity is the derivative of distance. So if you know the equation of the distance you can find the instantaneous velocity at any point.

59 Alex: I'm not talking about the average velocity.

60 Alex: I'm talking about instantaneous at any point because in the real world, there isn't much things that move at the same constant velocity through all the distance. It's fairly easy to find the equation for the distance. If something moves with acceleration, increasing decreasing acceleration you know. If you have distance, you take derivative. You plug in time and you get velocity at that point.

61 MZ: Change or rate of change. You kind of already talked about this too.

62 Alex: Instantaneous rate of change. Same thing.

63 MZ: Line or linear?

64 Alex: Well, if you have quadratic equation the line would represent the derivative. Because the highest power for that equation is two, the degree is two. So if you take the derivative, it would be one degree less. And x to the first power is a line.

65 MZ: What about linear? Do you think of anything of linear, just that word, associated with derivative?

66 Alex: Slope. Slope should be linear. It's a line.

67 MZ: Measurement?

68 Alex: Since a derivative [puts?] units between variable space [between?] points, so we ignore measurements all kinds of measurements. Well, except finding the point or something. You take-- you count the units up or you measure. But really you don't measure distance between points because derivative let us do it without it.

69 MZ: Prediction or approximation.

70 Alex: [pause] Uh, yeah. When you work with derivative, you don't always find exact numbers. So you have to approximate. And since you get this one point where you-- You still have some kind of approximation when you find derivative at this point.

71 MZ: So, what is it that you're approximating? You're approximating the derivative?

72 Alex: I don't really know what. You kind of-- You're getting really close. You're approximating maybe the slope at that point because-- Like when you have-- I don't know.

73 MZ: Optimization?

74 Alex: When you find your extremas like maximum and minimum. Your optimum something.

75 MZ: OK.

76 Alex: Wait. When you have to-- You have so much, a lot of fence and you have to cover the most area. You have an equation for the area, whatever you want it to be already. And then you optimize your area.

77 Alex: You can use derivative. You can also use quadratic formula, but it kind of involves derivative. You just don't see that.

78 MZ: How would you do it if you just were using quadratic formula?

79 Alex: You find this m and n things. I mean the vertex.

80 MZ: So you would find the points of the--

81 Alex: Yeah, but vertex is what? The derivative equals zero.

82 MZ: Continuity. I guess you already explained this one.

83 Alex: Yeah.

84 MZ: Limit?

85 Alex: Well, derivative is a limit. Not all the limits are the same as derivative, but derivative is a limit because you have x approaching some number just like in the derivative. You have one x moving down this line approaching another x. So that's called the limit.

86 MZ: OK.

87 Alex: You're still finding slope at the limit too.

88 MZ: Integral? Did you guys study those yet?

89 Alex: It's like opposite of derivative.

90 MZ: Yeah.

91 Alex: We didn't study them. Somebody told me.

92 MZ: Function. I was just going to ask you a more specific question. Is the derivative a function.

93 Alex: Yes.

94 MZ: And how do you know that? Why do you say yes?

95 Alex: Because if you take the derivative of a function, you get a function. You can't-- I mean, what could be example of taking derivative-- You're not making the inverse of this function. I mean the derivative is going to be always a function because you take derivative of a function.. So you just have a bunch of slopes moving down and when you connect them in the line it's still going to be a function. Oh! Because for every x there's going to be one y from the function, right?

96 Alex: So when you take the derivative, for every x in the derivative there's going to be one y in the function.

97 MZ: OK.

98 MZ: Did you learn a formal definition of derivative?

99 Alex: No.

100 MZ: What would be a definition a book would have for derivative?

101 Alex: Instantaneous rate of change. That's what I remember.
102 MZ: Do you remember a formula for derivative?
103 Alex: Like when you have the limit as x approaches to a then $\frac{f(x+L)-f(x)}{L}$. The L would be difference in x.
104 Alex: $f(x)$, like you're finding slope, over L.
105 MZ: [has written: $\frac{f(x+L)-f(x)}{L}$] This was the limit? Did I get this part right?
106 Alex: x approaches to 0.
107 MZ: To 0? [adds before quotient: $\lim_{x\to 0}$]
108 Alex: I think so. No, L approaches to 0, the difference between x 's approaches to 0.
109 MZ: Uhm, OK. Let me mark this out. [marks out: $x \to 0$]
110 Alex: Sorry.
111 MZ: So, say that again.
112 Alex: The limit of L approaches to 0.
113 MZ: L approaches to 0. [writes: $L \to 0$]
114 Alex: Yeah, because L is the difference in x 's and x 's are moving closer to each other so it approaches 0. So you can actually find the derivative by using that.
115 MZ: So you said that L is the difference in the x 's?
116 Alex: Mm hmm.
117 MZ: And then what's this top part?
118 Alex: The difference in-- Δy. That's how you find slope, Δy over Δx.

Table A.1 summarizes Alex's first interview transcript. Alex recalls that the derivative is related to rate of change, slope, and velocity. He states the instantaneous nature of derivative in each of these models. He describes a graphical limiting process, and he is aware that the derivative is symbolically a limit. When asked about a formal definition having to do with limit, he correctly states the limit of the difference quotient definition using a nonstandard L for the change in x, i.e. $\lim_{L\to 0}\frac{f(x+L)-f(x)}{L}$. He describes the derivative as a function by telling that for each x value there will be only one slope value.

Alex discusses the relationship of the graphical model to both rate of change and the symbolic difference quotient. When asked how he would explain derivative to another math student, he describes how slope is related to average and instantaneous rate

Table A.1. Alex: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎	◎			misstatement
What can derivatives be used for?	○		○		max/min
Explain what a derivative is to someone who's an AB student or precalc student who hasn't studied it yet.	○ ◉	○ ◎			
Explain what a derivative is to someone who doesn't know anything about math.	○			↦	
How did you learn it?	◎		○		acceleration max/min
How can you tell if a function is differentiable?	◎			↦	
Is derivative related to speed or velocity?			◎ ○	↦	acceleration
Is derivative related to change or rate of change?		◎			
Is derivative related to line or linear?	○			↦	possible misst. (derivative is a line)
Is derivative related to prediction or approximation?	◉				
Is derivative related to limit?	◉			●	
Is the derivative a function?	◉			○	
Did you learn a formal definition of derivative?		◎			
Do you remember a formal definition for derivative having to do with limit?	◉			◉	
Summary	◉	◎	◎	◉	

of change [ln 28]. Later he explains that the symbolic difference quotient is a calculation of slope, Δy over Δx, and that the limit indicates that the difference in the x values approaches 0.

Alex makes a few minor misstatements in this interview. His first answer to "What is a derivative?" is "the instantaneous rate of change of the slope of a function at some point" [ln 2]. However, later in the interview he clearly explains that the derivative is rate of change of the function, not its slope, and that instantaneous rate of change of a function is equivalent to the slope of the function at a point. On several other occasions Alex makes a misstatement that he corrects immediately without prompting. Only one strange statement passes by without further discussion. When asked if derivative is related to the word linear, Alex replies, "Slope should be linear. It's a line." It is unclear what Alex thinks is "a line" [ln 66].

QOTD #6

Find the derivatives of the following four functions:

$$f(x) = (x-1)^2(x^2-4)$$

$$g(x) = \frac{x-1}{\sqrt{5-x^3}}$$

$$h(x) = \sin x$$

$$j(x) = \ln x$$

Date: September 20, 1993. This question occurs prior to the class learning about short-cut rules for taking derivatives of various forms.

Response:

$$f'(x) = 2(x-1)(x^2-4) + 2x(x-1)^2$$

$$g'(x) = \frac{\left(\sqrt{5-x^3}\right)(1) - (x-1)\frac{1}{2}\left(5-x^3\right)^{-\frac{1}{2}}\left(-3x^2\right)}{5-x^3}$$

$$h'(x) = \cos x$$

$$j'(x) = \frac{x}{|x|}$$

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be (using your knowledge of derivative).

$y = \log(x^3)$ $\quad y = \dfrac{x}{|x|}$ $\quad y = x^e$ $\quad y = e$

Date: September 21, 1993. This question also occurs before the class studies short-cut rules for taking derivatives but after they have studied the limit definition of derivative.

Response: "$y = e$ $\quad \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \dfrac{\ln(x+h)-\ln x}{h} \neq e$

It is so obvious that doesn't need any explanation."

Comment: This question is presented to the students since no student correctly stated the derivative of $y = \ln x$ in the previous Question of the Day. Alex may or may not be kidding about not explaining his answer. The second quotient certainly does not look like e, but Alex has dropped the limit and does not explain what effect taking the limit would have on the quotient.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$? Why <u>not</u>?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: "a) Because $y = \dfrac{\sin x}{\cos x} \Rightarrow y = \tan$

$y' = \dfrac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \csc^2 x$ "

Comment: Originally, he does the correct calculation using the quotient rule and then incorrectly simplifies it to $\csc^2 x$. Later, after we discuss it, he changes it to $\sec^2 x$.

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. Alex writes the limit of the difference quotient definition of derivative as $\lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$. He correctly uses the difference quotient numerically to estimate the value of the derivative at a point. When asked to use the definition to symbolically solve for the derivative at a point, Alex substitutes correctly for the expressions in the difference quotient and then performs an algebraic error that allows him to get the correct answer without noticing the problem with its limit. The error of writing $\lim_{x \to 0}$ is the same error that Alex initially made in stating the definition of the derivative in the first interview. However, in the first interview, as Alex continues to explain his answer, he catches his mistake and corrects it.

On the first test on derivatives Alex also knows to estimate the derivative at a particular point by finding two nearby points and calculating a difference quotient for those two points. Given the graph of a position function for a car he correctly answers questions about the speed and direction of the car. Given the graph of a function, he is able to sketch a correct graph for the derivative function.

Alex has difficulties working through two complex chain rule derivatives. In one case Alex correctly finds the derivative of $f(x) = \cos(3x^2 + 4)^5$ to be $f'(x) = -\sin(3x^2 + 4)^5 5(3x^2 + 4)^4 (6x)$, but then simplifies his answer to $f'(x) = -\sin 30x(3x^2 + 4)^9$ treating all adjacent placement as multiplication. He makes the same error in the second chain rule problem. Having correctly performed the chain rule, Alex should recognized that the cosine involves a composition of functions. It is unclear what composition of functions Alex recognizes in his answer for the derivative. Perhaps in his simplification, the habit of multiplying adjacent elements temporarily overrides Alex's knowledge about the compositions and multiplications involved.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: "Now I understand more what derivative really is, when I was introduced to δ & ε it really started to make sense."

Comment: Alex's comment may be a joke for several reasons. First, it does not really make sense because the class did not study δ & ε with regard to derivatives, but only with regard to limits. Second, Alex likes to make jokes; he already understood the limiting idea of derivative well in his interview. Further, he and Mr. Forrest were joking about squirrels named δ & ε. On the other hand, Alex has in the past shown confusion between limits and derivatives so it is possible that he is just getting these concepts confused again.

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "a) We discussed homework - related rates problems. Mr. Forrest liked them a lot. Math highlight: NONE

b) Now I feel more comfortable with related rate problems."

Comment: Since the researcher had not been present the day prior, this question is presented both as a means for the researcher to see the material covered and to ascertain the students' understanding of it.

Test 3

About two weeks later the class is tested on Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Alex correctly calculates a third degree Taylor polynomial. He is able to use the first and second derivatives of the position function to find the speed and acceleration of an object at a given time. He correctly uses the graph of a derivative function to estimate when the original function is increasing or decreasing, concave up or concave down and where it has extrema and inflection points.

QOTD #11

Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: "Find the rate of change of the growth of ... population."

Test 4

Two weeks later the class has a test on the applications of derivatives. Alex correctly uses derivatives to solve three traditional max/min problems and three traditional related rate problems. He also correctly calculates the derivative of an implicitly defined function.

Interview 2

The second interview occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by a summary table and a discussion.

1 MZ: What is a derivative?

2 Alex: It's instantaneous rate of change in a function at a given point. It is slope. The equation of the derivative can give you the slope of the tangent line to this point. At a certain point the equation of derivative can give you the slope of the tangent line at any point of the function if the function is differentiable at that point.

3 MZ: OK. What can derivatives be useful for?

4 Alex: Optimization problems. You can also see the rate of change. Well like if it's a business thing. You have a graph of ups and downs, and you can see the rate of change in them. Like this year you made this much a month, this much an hour or whatever.

5 MZ: In the max/min problems, how does the derivative fit into those problems?

6 Alex: OK. Whenever a function has maximum or minimum, it means it goes that much high-- I'm talking about local mins or max.

7 Alex: It goes that much high or that much low.

8 Alex: Like a turn.

9 Alex: So when it has like a turn, it reaches its maximum. And at one point during that turn the tangent line, the graph of the tangent line is $y = 0$. So the slope is 0. The graph of the function that represents the tangent line is the derivative. But at this point on the top or on the bottom [pointing to a max and min on his graph], the derivative is equal to 0. So when we have a function, the equation of a function, and we take the derivative. And then we make it equal to 0 and find the x value in which the derivative will be equal to 0. And at the same x value this function has a local max or min.

10 Alex: OK. So this turn and this turn, they seem to be local maxima. I mean, this point is like hard to determine just by looking at the graph. This is going to happen somewhere. [draws a horizontal tangent at the local max]

11 MZ: OK. I believe you. We'll keep going.

12 Alex: No, what's happening-- The slope of the function from being positive goes to being negative. So at some point it's got to be 0. You can't go from positive to negative without going through 0.

13 MZ: OK.

14 Alex: So the slope of this tangent line at some point will be 0. So when derivative-- Set equal to 0, solve for x. Then plug in x in the original equation and you get your y point. [writes:

$f'(x) = 0$

solve x]

$f(x) = y$

Well yeah, because you don't know how your function goes. So that would be your x value and that would be your y value. That's how you find local max at that particular point. Here would be local min.

15 MZ: OK. How would you explain what a derivative is to someone without very much math background?

16 Alex: Even a simple function, a linear function. The slope of the linear function will be equal to derivative. 'Cause derivative is the tangent line. Like tangent to the circle, tangent to this. Just show an example of tangent at a point. It just passes through one point on the graph and is tangent to that point. So the derivative, when you know the derivative, if

you plug in x value for the point, you find y value which will be the slope.

17 MZ: OK. I'm going to keep going. Give an example of a real world situation that involves the concept of derivative.

18 Alex: Real world-- [Mr. Forrest] assigns homework to us and makes us do the derivative stuff.

19 MZ: Let's just go on. Say if I give you a function, how can you tell if it's differentiable?

20 Alex: First of all, I'd find if it's defined at this point. So I'd find the domain, and if x is in the domain, then it's defined. Then I'll find the derivative of the limit of the function-- The derivative on the left and on the right of this point. Not the limit, the derivative. And if they're equal, then it's differentiable.

21 MZ: So this slope on the left has to equal that slope on the right?

22 Alex: That's what we do when it's given as a piecewise function. That's what we did for $f(x)$ like for one piece and then $f(x)$ for the other piece. But if the function-- Maybe you should just find limit as it's coming in from the left and from the right.

23 MZ: Just the limit of the regular function?

24 Alex: Yeah, and see they're equal.

25 MZ: So take limit from the left and from the right of the function and see if they're equal?

26 Alex: Yeah. 'Cause limit will give you the slopes and when you're taking limit it doesn't care about the point. It cares what's going around it.

27 Alex: So when you're taking limit, you're not assuming it's differentiable. So if you take the limit, and it shows that from the left it's $-b$ and from the right it's $-b$, then it's differentiable. [writes: $-b \quad -b$]

28 MZ: Then it's differentiable?

29 Alex: And for the piecewise we use something like that. For one piece and for the other, if they're equal.

30 MZ: You would use something like that quotient. [referring to what Alex has previously written: $\dfrac{f(x)-f(a)}{x-a}$]

31 Alex: Yeah. That's a limit too, to find the slope. For this on the top, the initial would be a different expression, because piecewise is given in different expressions. But at the end they could come out the same, when you simplify.

32 MZ: Can you give me an example of a function that's not differentiable?

33 Alex: Function that's not differentiable. All the absolute value functions. I think. At one point.

34 MZ: So why is it not differentiable?

35 Alex: Because the slope from the left is negative and then it's positive. And it's not negative getting closer to 0. It's just negative all the time. It could be close to 0, but if you say it goes like this [sketches a concave down curve with a cusp], it has a big value here. Absolute value. The slope has-- It's negative but the absolute value is big. It's really small.

36 MZ: Yes, I understand. The value of the slope is--

37 Alex: --a big negative. And then after the point it's all of a sudden a big positive. So it's not going to the small positive and going through 0. So it never goes through 0.

38 MZ: OK.

39 Alex: It's going from negative to big positive so it never goes to 0. So it's undifferentiable at that point.

40 Alex: And also the functions like, square root, third root of x [writes: $\sqrt[3]{x}$]

41 MZ: And why isn't that differentiable?

42 Alex: Because if you look at the original function, it has a derivative equal to 0 at that point. [Alex is referring to $y = x^3$ which he sketches.]

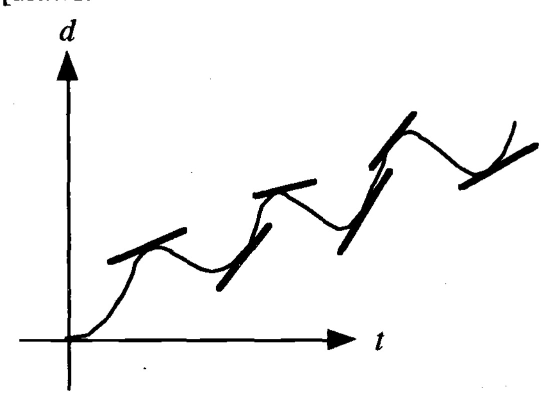
43 Alex: And if you have cube root of x you kind of have the same thing if you turn this way [sketches $y = \sqrt[3]{x}$ on the same graph as $y = x^3$].

44 Alex: So you have a function-- The tangent to that point (0,0) is x equals 0 [writes: $x = 0$].

45 Alex: And this has undefined slope, so it doesn't have a derivative.

46 MZ: Does velocity have to do with derivative?

47 Alex: In some way. Like if you have a function d in terms of t, then instantaneous rate of change at any point will give you instantaneous velocity.
[draws:



48 MZ: Fine. Line or linear?

49 Alex: Well, the tangent line is linear.

50 MZ: OK. Measurement?

51 Alex: [short pause] No.

52 MZ: Why not?

53 Alex: Because in the derivative you're not really measuring anything. You are assuming that you are really close to that point that you differentiate. So you're not measuring anything. You can't measure the derivative, if you just measure it with the ruler.

54 MZ: Prediction or approximation?

55 Alex: Yeah. When you take the derivative, you think that it's at this point, but actually it's like really, really close to that point. And it doesn't matter. Because if you have at that point the slope will be equal to the slope of the points really close to it. I mean like infinitely close.

56 MZ: Continuity? How is continuity related?

57 Alex: If the function is differentiable at that point, then it's continuous at that point. But not vice versa.

58 MZ: OK. Limit?

59 Alex: Limit is one of the forms of derivative.

60 MZ: How do you mean?

61 Alex: It gives you the slope too.

62 MZ: Limit gives you the slope.

63 Alex: Limit as h approaches to 0.
[writes: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$]
That's the derivative. That's the definition of the derivative, one of the definitions of derivative at a point x y .

64 MZ: Very good. Do you know what an integral is?

65 Alex: Opposite of derivative. Like integral of $3x^2$ will be x^3 [writes: $3x^2 = x^3$].

66 MZ: What about an antiderivative? Same thing?

67 Alex: An antiderivative will just mean like this, just with this stuff. But an integral can be used for finding an area under the curve and stuff like that.

68 MZ: Is the derivative a function?

69 Alex: Yes.

70 MZ: How do you know or what makes you say yes?

71 Alex: Because it's like finding slopes to the function.

72 Alex: It just can't be not function. How can it be not function if it's following this function all the way along?

73 MZ: OK. Now, you gave a formal mathematical definition. How does this definition relate to some of the other things you said like slope or rate of change?

74 Alex: Well, this is a slope. This is y_1 y_2, and this is delta x [writes y_1 y_2 above the numerator and $= \Delta x$ next to the h in the denominator].

75 MZ: OK, so that is a slope, this fraction. And then how does this part fit in, the limit as h goes to 0?

76 Alex: Between the points that's really close together. This close [circles $1 \times 10^{-\infty}$].

77 MZ: Have you ever heard of these things and if so, do you think it has to do with derivative. Newton's method?

78 Alex: Is it like when you have this slope of the function, when it crosses the x axis. OK, when it crosses the x axis like this, and then what. You like go in closer and closer to the slope. Between two points maybe.

79 MZ: That's OK.

80 Alex: I forgot. You're moving closer and closer until you find the slope. [short pause] Oh, OK. I remember. Whenever it crosses x axis you're finding the y value. And then you go like this. The slope between--OK, I'll make a better sketch.

81 MZ: Go on to the next page so it's clearer.

82 Alex: I'll make it like that. And Newton's method doesn't always work. I can't give you an example, but it doesn't always work.

83 MZ: Do you happen to know, just in general, why it doesn't always work?

84 Alex: Because sometimes the function can do really strange things. It can change really dramatically over a small x interval, and then Newton's would work.

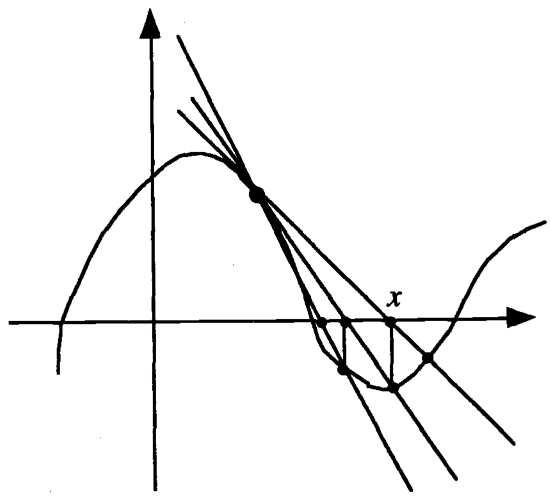
85 MZ: OK.

86 Alex: You start with like two points, and you want to find the slope of the tangent line at one of them. Draw a tangent line. The secant line between these two points. Connect them.

87 MZ: I connect those two points.

88 Alex: Yeah, first.

[Alex draws:



]

89 MZ: So it's not a tangent line. It's just a secant line.

90 Alex: Close. You're like moving closer to this real tangent line.

91 MZ: OK.

92 Alex: I mean in some way it's close because that's the average rate of change between these two points. So you're trying to move this point closer to this. Whenever it crosses the x -axis, you take this x . You put it in y . You go to the function, and then you draw a secant line between these two points. And then it crosses here. And you go again, and you move like that. See even now you are going away from that point. If you notice, the rate of change here is, like if you draw this, the rate of change is pretty close to the rate of change here. So it becomes really messy when you do it. Then you cross the x -axis here. Then you draw it again.

93 MZ: And so the whole point is to try to approximate or come close to the slope of the tangent line at that point.

94 Alex: And it's not-- You never come equal to it.

95 MZ: Intermediate Value Theorem?

96 Alex: I think so. If the function-- [short pause] Two y values. It has a value-- If it's continuous-- It's something to do with, if it has y value between those points, it's going to have x value between those points. If these are two point A and B. Has y value between A and B here. It's going to have x in between those. [Alex drew a smooth curve on a pair of axes. He marked A and B on the curve in the 1st quadrant. He then drew horizontal lines from A and B to the vertical axis and vertical lines from A and B to the horizontal axis.]

97 MZ: OK. Mean Value Theorem?

98 MZ: Mean Value Theorem?

99 Alex: I think it's when you have two points that have the same y value. You have a graph in between them. There should be at least one point where the derivative's equal to 0, if the function's continuous.

100 Alex: --and differential.

101 Alex: No.

102 MZ: Say I give you a derivative. What kinds of things can you tell me about the original function?

103 Alex: I can tell you where it's increasing and decreasing. Concave up, down because I can find the second derivative from that. Critical points. Where max and min could occur, or where they occur. If I have like two critical points I can tell you the min and max. If I have more than that I can't tell which one is more minimum or more maximum.

104 MZ: Oh, right. But you could tell me the relative min and max.

105 Alex: Yeah.

106 MZ: OK, thanks. That's it. You're done.

107 MZ: Were you thinking that I was going to give you like an equation, when I said this is the derivative, or that I was going to give you the graph?

108 Alex: Equation.

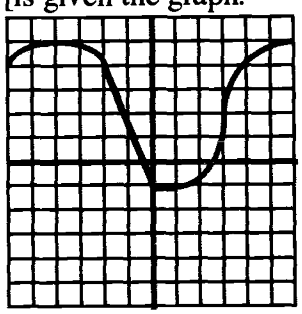
109 MZ: You were thinking equation.

110 Alex: Well, from the graph it's hard to find concavity from the graph. You can find critical points.

111 MZ: OK. So, here's a graph.

112 Alex: You don't trust me.

113 MZ: Number 24. This is the derivative of some function. [bell] I know you got to go. Just, tell me about the max and min of the original. [is given the graph:



114 Alex: When ever it crosses over here-- Oh, you can tell about concavity. OK, whenever it crosses 0, at that x value the original function will have a max or a min.

115 MZ: OK, cool.

116 Alex: And at that thing, the derivative going from positive to negative so it's going to be max. And here it's going to be a min.

117 MZ: How do you remember that going from positive to negative is going to be a max not a min?

118 Alex: Positive, negative.

119 MZ: OK, he drew-- He went up and then he went down. Points of inflection.

120 Alex: Over here where the second derivative is going to be equal to 0. It's going to be $-.5$ and somewhere in here.

121 MZ: Can you tell where the original function is concave up?

122 Alex: Whenever the slope of the derivative is positive, from negative infinity to here and here.

123 MZ: OK, thanks. That's it. You're done.

From Table A.2, it is evident that Alex mentions a graphical interpretation of derivative more often than any other interpretation. He does also mention rate of change, velocity and a symbolic formulation of derivative, recognizing the instantaneous nature

Table A.2. Alex: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎	◎			
What can derivatives be useful for?		○			optimization
In max/min problems, how does the derivative fit into those problems?	○ ◯ ○ ◎			↦	misstatement (d=tl)
How would you explain a derivative to someone without very much math background?	◎				misstatement (d=tl)
How can you tell if a function is differentiable?	• ○ ○			● • ● ↦ • ●	misstatement (d=lim)
Is derivative related to speed or velocity?		◎	◎		
Is derivative related to prediction or approximation?	◎ ●				misstatement (d=approx)
Is derivative related to limit?	○			● ◉	possible misstatement (d=lim)
Is the derivative a function?	◉				
How does the formal definition of derivative relate to slope or rate of change?	○			◉	misstatement (d=approx)
Summary	◉	◎	◎	◉	

of the derivative in each of these forms. Alex mentions the ratio, limit and function processes in only the graphical and symbolic interpretations of derivative. His symbolic interpretation includes correctly stating the formal definition as $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

Alex makes several connections between different aspects of the derivative. When asked, "What is a derivative?", he mentions instantaneous rate of change and slope of the tangent line at a point in rapid succession [ln 2]. Later, when discussing a secant line, Alex says that it is the average rate of change. He should have said that the slope of the secant line is the average rate of change, but there seems to be some connection for him between rate of change and the graphical interpretation [ln 92]. Twice Alex describes correctly how a difference quotient is a slope [ln 61, 73]. He also states that instantaneous rate of change and instantaneous velocity are the same for a distance function [47].

Alex makes three types of misstatements: those involving the relationship between derivative and tangent line, those involving the relationship between derivative and limits and those involving whether or not the derivative is an approximation. Twice he states that the "derivative is the tangent line" [9, 16]. In the second instance he says the derivative is both slope and tangent line. "The slope of a linear function will be equal to derivative, cause derivative is the tangent line" [16].

Alex confuses limit and derivative when discussing how to tell if a function is differentiable at a point [ln 19-45]. First he describes checking the slope from each side, but then he thinks maybe he should be checking the limit. It is possible he means the limit of the difference quotient, but he seems to agree with the interviewer when she asks if he means the "limit of the regular function" [ln 23]. He explains further, "Cause limit will give you the slopes" [ln 26]. Later, when asked whether limit is related to derivative, he says, "Limit is one of the forms of derivative" [ln 59]. Perhaps he means that the limit of the difference quotient (and not just any limit) is one of the forms of derivative. He does go on to state the limit of the difference quotient a few statements later [ln 63].

Alex makes a subtle error with regard to whether the derivative is an approximation or an exact value. "When you take the derivative, you think that it's at this point, but actually it's like really, really close to that point. ... At that point the slope will be equal to the slope of the points really close to it. I mean like infinitely close" [ln 55].

For the last part of the second interview Alex is given the graph of the derivative function and asked to give information about the original function. He correctly notes the locations of the local maximum and minimum and where the original function will be concave up. There is no time in the interview for Alex to sketch a graph of the original function.

Because Alex already has a very complete understanding of derivative at the time of the first interview, his performance on the second interview does not show any major changes. In the second interview he does state two connections that he does not state in the first interview. One is between velocity and rate of change; the other is between the function aspect of the symbolic and graphical interpretations. He relates the ratio and limit aspects of the graphical interpretation in both interviews. On the other hand, Alex makes more misstatements during the second interview. Most of the additional misstatements occur in his descriptions of the relationship between derivatives and limits and approximations.

QOTD #12

What is the most important idea that we have studied so far in this class?

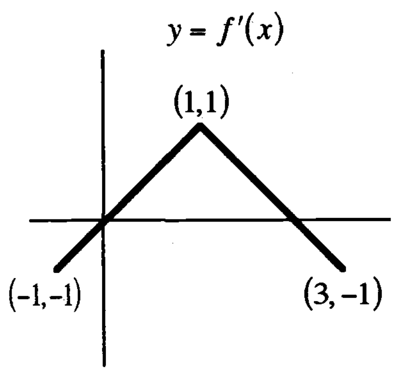
Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

Response: "Fundamental Theorem"

Interview 3

The third interview occurs during the three days after the test on differential equations, and antiderivatives by substitution and by parts. The first part of the interview is a summary of Alex's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike in the second interview, the point $(1,0)$ on the original function is given so that only one solution is possible.

1 MZ: I'm going to make a graph here. This is going to be the graph of the derivative. [starts drawing axes] This is $(1,1),(-1,-1),(3,-1)$ [plots these points and line segments connecting $(1,1)$ to each of the other two] and I don't care what happens outside of that domain. So this is $y = f'(x)$ [writes: $y = f'(x)$] and you're going to graph $f(x)$, the original-- the antiderivative. And I'm even going to give you the initial condition that $f(1) = 0$ [writes: $f(1) = 0$] And here's a green pen.



2 Alex: [40 second pause in dialog; Alex thinking and beginning a sketch] I forgot about the initial condition. I have to use it.

3 Alex: This is maximum. [referring to a max on his sketch at $x = 2$]

4 MZ: OK.

5 Alex: --that goes 0 at 2. And now-- It's concave down because derivative is negative, the second derivative. So it's like that. Now here is point of inflection.

6 MZ: At $x = 1$?

7 Alex: Yeah, because it's kind of max here [on f'] although the second derivative might not exist.

8 MZ: Oh, why might it not exist?

9 Alex: Well, because of the sharp turn there. There isn't-- It's going to equal to-- The slope here is going to be $-a$. The slope here is going to be like a [draws a small diagram indicating $-a$ to the left of 0 and a to the right of 0].and between there's too much space and zero somewhere in between. It's got to go smooth for a derivative to exist.

10 MZ: And so, did you say you're not sure if it goes smooth or not?

11 Alex: Yeah, well the way it looks it doesn't.

12 MZ: OK.

13 Alex: OK, now $x = 0$ again here so this is a minimum point and point-- [at $x = 0$]

14 MZ: So how do you know it's a minimum and not a maximum for example?

15 Alex: Oh, because it goes from negative slope to positive slope. See this thing is below y and that thing is above y. I mean x -axis. [referring to the graph of $y = f'(x)$ being above or below the x -axis]

16 MZ: Right, yes.

17 Alex: So the value of derivative is the slope.

18 Alex: So that's how it goes up to -1.

19 MZ: OK.

20 Alex: Oh, this thing could go like this too.

21 MZ: Go way far down?

22 Alex: Not very far. No. Nah, it won't go that far.

23 MZ: How come?

24 Alex: Because it starts with a slope which is about like 1 --

25 MZ: Yeah, it's suppose to be that.

26 Alex: --so it comes back here, a slope of -1, and it goes down to 0. And rate of change changes. OK. But no, smoothly.

27 MZ: OK.

28 Alex: Well, point is, if this were to go very far down, the slope in here would have to be a very big number, but from here you can tell it's one or something.

29 Alex: So it wouldn't go that far down.

As in the second interview, Alex has no difficulty in discussing how the graph of the derivative is related to the graph of the function. He correctly notes the locations of the maximum and minimum and uses slope to explain his choices. He also notes where the original function is concave down explaining that the slope of the derivative, which is the second derivative, is negative there [ln 5]. He notes that the inflection point occurs where the first derivative has a maximum and that the second derivative will be undefined at that point, since the slope of the first derivative changes so suddenly at that point [ln 9].

Alex's only imperfection is that he does not know exactly how high or low to make the extrema. He marks the y values of the extrema as ± 1 instead of the correct answer of $\pm\frac{1}{2}$. He does not use any techniques such as slope fields and areas that have been covered in class since the second interview that could be helpful in solving this problem.

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

30 MZ: Next question. What's a definite integral?
31 Alex: It's an area under the curve bounded by these two points. And it's also antiderivative.
32 MZ: OK. Anything else?
33 Alex: Well, definite integral has like boundaries on it, x values, so it's used to find the area under the curve, which is like-- Well, I know how to use it from physics.
34 Alex: To find like uhm-- When you have a graph of velocity you take definite integral to find distance over the time.
35 MZ: OK.
36 Alex: You have v versus t, something like that [sketches first quadrant graph with axes labeled v and t] You find this area and it gives you distance. So it gives you the value of the derivative-- of the antiderivative, yeah.
37 MZ: What about indefinite integral?
38 Alex: Uh, those are just an antiderivative function.
39 MZ: So how are they different from the definite integral?
40 Alex: They don't have bounds.
41 Alex: Well, you can find expression for antiderivative and from that your points. But you only can find the formula for antiderivative and not like exact values. How are they different? I mean they are kind of close friends. But you know what I mean. The definite has like boundaries so it is used to find area under the curve. But the indefinite integral finds the antiderivative, if you take it. Basically they are the same. They both are for finding antiderivatives, but it's a different way.
42 MZ: OK. How is the derivative related to the integral?
43 Alex: It's opposite of integral. If you take integral of the derivative, you get original function.
44 MZ: OK. Uhm, do you remember what any of the fundamental theorems say?
45 Alex: The second one, yes.
46 Alex: It says that if you have $f(x)dx$, values a to x, it equals to-- [talking quietly while writing]
47 Alex: [when he's done it reads: $\int_a^x f'(x)\,dx = f(x)\Big|_a^x$] And the first says that area under the curve is equal to integral-- of well, whatever function. [writes: $A = \int_a^x f'(x)\,dx$].
48 MZ: OK. So now the first one has area equal to integral from a to x of $f'(x)dx$?
49 Alex: Well, whatever function is here.

50 MZ: OK, whatever function is there. It doesn't have to be f'.

51 Alex: Well, it is a derivative of some function.

52 MZ: OK.

53 Alex: I mean, I think there is always a function that has a derivative, an antiderivative. I don't know. Well, if it is under the integral somebody at some point took derivative and found it. He was very bored.

54 MZ: What's the difference between these two that you said that, the first and the second? I mean, it looks like they are almost the same because this integral part from a to x is exactly the same.

55 Alex: [short pause] This one finds area exactly. This one finds area exactly.

56 MZ: Oh, they both find the area exactly?

57 Alex: Well, yeah if you use this to [inaudible]. If you just-- Oh, this trapezoid rule and Simpson's rule. Then it's not exact.

58 MZ: OK. But does that trapezoid and Simpson's rule have to do with one of these two or is that just something separate?

59 Alex: Because sometimes it's hard to take antiderivative so you would use like Simpson's rule which gives-- If you choose n to be like 100 [writes: $n = 100$], it gives a pretty close approximation.

60 MZ: OK. Let me ask you a problem. Say, this is going to be a find the area problem, but an easy one. $y = x^2$ from 0 to 1. Area under the curve between the curve and the x axis. [sketches the curve on axes, shades the appropriate region and labels as $y = x^2$]

61 Alex: Yeah.

62 MZ: So how would you find the area?

63 Alex: I would use the triangle. [sketches a line segment from $(0,0)$ to $(1,1)$] So it's going to be about one times one half times one half. It's going to be one fourth. [writes: $= \frac{1}{4}$] That's approximate.

64 MZ: Now how did you get one fourth? You said--

65 Alex: Oh, this is the point $(1,1)$ right? So it's one half. [changes $\frac{1}{4}$ to $\frac{1}{2}$]

66 MZ: So it's approximately one half by the triangle rule.

67 Alex: But now I'm going to take the integral. [writes: $\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right| = \frac{1}{3}$] So it's one-- Ooh it's $\frac{1}{3}$.

68 MZ: So you were close. The estimate's not too bad.

69 Alex: Simpson's rule I think will give a better estimation.

70 MZ: Did you already tell me about slope fields?

71 Alex: No. I can draw you one. [draws a series of parallel small line segments]

72 MZ: So, do these slope fields have more to do with derivatives or integrals?

73 Alex: Integrals.

74 MZ: Why is that?

75 Alex: Because you look at the function of derivative or the table of values or just the function. You pick like x values and you find y 's. And y values give you the slope at the x value. [Sketches a curve on an axes to the right of the line segments.]

76 MZ: OK.

77 Alex: For example, this is equal to 3 here and x equals 0 [referring to the point on the curve he just drew] so the slope will be 3. So you draw a line like through some-- whatever periods you want. And you go on like this. [draws axes near line segments and then puts more segments with slope approximately 3 vertically at $x = 0$] And you get the slope field.

Then you find your initial value. And you draw a line through the slope field how ever it goes. [chooses an initial point and draws a curve through the slope field]

78 MZ: OK.

79 Alex: And you find the derivative with it.

80 MZ: And this is the derivative you just found?

81 Alex: Well, the antiderivative.

82 MZ: The antiderivative.

83 Alex: The original function.

84 MZ: So that's how it's related to integrals then?

85 Alex: Yeah.

86 MZ: OK. Are slope fields related to finding areas?

87 Alex: No.

88 MZ: No. Well, slope fields are related to finding antiderivatives, right?

89 Alex: Yes.

90 MZ: And you can use antiderivatives to help you find areas right?

91 Alex: You can use integrals, definite integrals to find the area.

92 MZ: Right.

93 Alex: So you have to look at this function to find the area not this. [look at the function on right not one drawn through slope field]

94 MZ: Oh, this uhm-- what we're calling the derivative.

95 Alex: But if you know the formula for this function that's how you can find the antiderivative like here. You basically-- using the second fundamental, the second part. Take $f(x)$ here minus $f(x)$ say here. [marks two points on the curve through the slope field] That gives you a value which will be equal to the area, x here to x here. [marks two x values on the right side graph, draws vertical lines from them up to the curve and shades in the area; the x values chosen for the slope field points seem close to those on the right side graph]

96 MZ: Oh, how come taking these values should be the same thing as finding this area?

97 Alex: Because if you find $f(x)$ original from $f'(x)$ which is this. If you take integral you find something like that, from whatever values. [above the right side curve writes: $\int f'(x)\,dx = f(x)\Big|_a^x$ and draws an arrow from $f'(x)$ to the right side curve]

98 MZ: Why is this is a true statement, the one you just wrote down?

99 Alex: Because it's second part of the First Fundamental Theorem.

100 MZ: [MZ laughs] OK.

101 Alex: Oh, why does the fundamental theorem work?

102 MZ: Yeah, why does the fundamental theorem work?

103 Alex: OK. Area under the curve is an antiderivative so you're finding like opposite of the rate of change? I don't know. Value of the original function when it changes from one x to another x. So that's why it works.

104 MZ: Hmm.

105 Alex: I don't know. Why wouldn't it work?

106 MZ: You have an idea? [pause] You don't have to answer the question.

107 Alex: I know, but I have to answer it sometime. So you want me to prove the theorem, don't you?

108 MZ: Well, kind of.

109 Alex: Don't you? Newton spent like years finding it out.

110 MZ: What were you thinking of?

111 Alex: When you're finding $f(x)$ minus $f(a)$ [here or previously labeled the two designated points on the slope field graph as $F(a)$ and $f(x)$;writes $f(x)-F(a)$] , it gives you the difference between two values of the function. [10 second pause] And when you find derivatives in the [inaudible], how much derivative changes. I don't know. I can't think.

112 MZ: OK, we'll quit.

Alex associates definite integral with area and antiderivatives. Because of his junior year study of AP Physics, Alex also knows that definite integrals are useful for finding the distance traveled, since the area under a velocity curve gives this distance. Alex associates indefinite integral only with "an antiderivative function" [ln 38]. When asked how integrals and derivatives are related, Alex focuses on the notion of integral as antiderivative explaining that derivative is "opposite of integral. If you take the integral of the derivative, you get the original function" [ln 43]. Here he is using derivative in the sense of the derivative operator, not the derivative at a point or the derivative function.

Alex's statement of the Second Fundamental Theorem of Calculus, $\int_a^x f'(x)dx = f(x)\Big|_a^x$, is nontraditional and probably reflects his partial memory of the First Fundamental Theorem. Note that his statement of the First Fundamental Theorem, $A = \int_a^x f'(x)dx$, gives only the new information that the integral is an area. He seems to assume without clearly stating the main idea of the First Fundamental Theorem; i.e. if $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

Alex shows he can correctly execute the Second Fundamental Theorem for a simple area calculation. He also states that he is familiar with methods such as the trapezoid rule and Simpson's method for approximating areas.

Alex states that slope fields are related more to integrals than derivatives since they are used, like antiderivatives, to find the original function [ln 75-77]. When asked to relate area to slope fields, Alex states that one uses antiderivatives and the Second

Fundamental Theorem to find areas. Trying to ponder the relationship he states, "Area under the curve is an antiderivative so you're finding like opposite to the rate of change?" Alex believes that understanding why the Second Fundamental Theorem works is important, "I have to answer it sometime" [ln 107] and he draws an appropriate diagram for an area function and a slope field to see how they are related. He is not able to make any further statements about their relationship.

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: While it is recorded that Alex answered this question incorrectly, his exact response is not recorded.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Alex's exact answer is not available. It is known though that Alex answers the question correctly.

Test 9: Semester final

This test, which is a cumulative semester exam, covers all of material on functions, limits, derivatives, areas, and volumes. The test questions are largely computational. On the semester final, Alex correctly solves an optimization problem as well as questions about domain, range, inverse functions, and continuity. He also computes several limits, derivatives and integrals correctly. Alex only makes four errors. Two are on derivative computations, a minus sign error and a chain rule error. The third is the statement that $\lim_{x\to 0} \frac{\sin x}{x} = 0$. The fourth is due to his inability to find the inverse of the function $f(x) = \ln\left(\frac{x}{x-1}\right)$. He correctly computes the inverse of $y = \frac{x}{x-1}$, but ignores the log function.

QOTD #15

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: "continuous everywhere and diff. $(-\infty, 0) \cup (0, +\infty)$ "

Interview 4

The fourth interview discussion is broken into four parts. The first section includes general questions about derivatives. The second part asks for the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity, and acceleration, given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the fourth interview.

1 MZ: What is a derivative?
2 Alex: It's an instantaneous rate of change of the function at a point.
3 MZ: So why did you give me that definition instead of some other definition?
4 Alex: Because I like that one better. I know it by heart.
5 MZ: Do you know any other ones?
6 Alex: OK. Well, it's like the slope of a tangent line to any function at a given point.
7 MZ: OK. Know any more?
8 Alex: There's only two. Well you can say-- [writes: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$]
9 MZ: OK. Do you know any more?
10 Alex: There's no more.
11 MZ: What does somebody mean if they say it's the instantaneous rate of change?

12 Alex: Well maybe how fast the function like turns. Maybe it even tells you how fast it's increasing or decreasing or if it has positive or negative slope. You know what I mean?
13 MZ: And all of this you're getting out of the words rate of change?
14 Alex: Right. So, how fast it's changing. Let's say it was going-- It was sloping up a little bit, but then you get derivative to go 25 and you know it's going like, very fast it's increasing.
15 MZ: Mm hmm.
16 Alex: So derivative can tell you that it's increasing very fast.
17 MZ: OK. What do those words, instantaneous rate of change, have to do with this symbolic thing that you just wrote on the page?
18 Alex: Well, since h is equal to 0 that means we're trying to make this tangent line go through two points that are very, very close together, 'cause h is the difference between those two points.
19 MZ: OK.
20 Alex: So this tangent line basically gives slope at a given point. Rather than having one point with $x = 1$ and the other with $x = 5$, which are very far from each other. You have two points which are really, really close are actually like one point as h approaches 0. It's like different. h is like Δx [writes: $h = \Delta x$], which is like-- Delta x is very, very small.
21 MZ: What do the words rate of change have to do with this symbolic thing?
22 Alex: Rate of change. OK. Well, when you find-- If you plug in numbers for x -- Well, let's say you have a function here. We can find the function of the derivative-- the equation for the derivative just out of that. Sometimes it's not easy, but it's possible.

As in previous interviews, when asked "what is a derivative?" Alex first mentions instantaneous rate of change and second mentions the slope of the tangent line at a point. Here, for the first time, he volunteers a third description, the formal limit definition. When asked to explain what is meant by instantaneous rate of change, Alex responds, "how fast the function turns ... how fast it's increasing or decreasing or if it has a positive or negative slope." Note that Alex uses both the metaphor of speed and the graphical interpretation in tandem. He associates rate of change and slope so closely that when asked to connect rate of change to the limit of the difference quotient, Alex replies in purely graphical terms. He connects the limit of the difference quotient to a slope without mentioning the words rate of change at all. When pressed to talk about rate of change specifically, Alex only talks about taking the derivative of a function symbolically to get an equation that gives the rate of change. He seems to have no independent description for rate of change.

The next part of the fourth interview is a summary of Alex's solution to the first of three problems of the interview. Given a table of values with x varying by .1, Alex is asked to find $f'(2)$, the derivative of the function at $x = 2$. Alex's first reaction is to suggest that he can find a slope. He calculates the slope between the points with x values 1.9 and 2.1. When asked for a better estimate, he suggests finding the slope between the points with x values 2.0 and 2.1 or 2.0 and 1.9, although he admits that these may not be better. He knows that if he had x values closer to 2.0 then he could get a better estimate. His statement hints at, but does not explicitly state, a limiting process for finding a more accurate estimate.

The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward, increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car took 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far did the car travel in the 2 hour period?

Alex's first reaction is that taking 1 hour to get from 0 to 60 miles per hour is rather slow. He notes that the acceleration is 60 miles an hour per hour, that the problem will have to be in parts since it is accelerating and then decelerating and that the distance traveled in the two parts will be the same. He uses a formula that he knows from physics, $d = \frac{at^2}{2}$, with $a = 60$ and $t = 1$ to find that the distance traveled in the first hour is 30 miles or 60 miles for two hours.

When asked to explain the formula, Alex first sketches a graph of the velocity function emphasizing that the rate is constant so the slope does not change. He notes that the area under this curve gives the distance and tries to use this fact to explain his physics formula. He writes that the area under the curve is $\frac{vt}{2}$ and that $v = at$ to get $\frac{at^2}{2}$. The first formula, $\frac{vt}{2}$, seems to come from the formula for a triangle applied to the straight-line graph of velocity versus time. The second formula, $v = at$, is correct whenever the initial velocity is 0. When asked if $v = at$ always holds, Alex explains that it would be more accurate to say v final minus v initial equals acceleration times time, written $v_f - v_i = at$. In other words, the change in velocity equals acceleration times time, written $\Delta v = at$.

Next Alex is asked to sketch a graph for distance versus time for the car scenario. He correctly sketches an increasing concave up graph for the first hour. For the second hour he initially draws a decreasing concave up curve and then a decreasing concave down curve before realizing that the curve should be an increasing concave down curve. He uses his knowledge that the slope of this curve is the velocity to help him catch his own errors without prompting. Alex also points out that the inflection point is where the acceleration changes from positive to negative. He initially believes that the derivative is undefined at that point, assuming that it is the location of a vertical tangent line. After hints to note the value of the derivative function at that x value, he realizes that it is the location of the maximum velocity, but that this velocity is not infinite.

Alex reasons symbolically, graphically, and in terms of the physical situation. He uses his knowledge of the connections between these representations to explain his answers and catch his errors. His physics knowledge is helpful in finding a correct answer quickly, but the physics formula seems somewhat disconnected from his understanding of the derivative relationships involved in this problems.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Alex is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. He is asked if the top of the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Alex first thinks that the rate is constant but not the same as the given rate. When asked to check his assertion, he labels the distance between the wall and the base of the ladder as a and the distance between the floor and the top of the ladder as b and completes the following sequence of calculations:

$$a^2 + b^2 = 14^2$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$b\frac{db}{dt} = -\frac{ada}{dt}$$

$$\frac{db}{dt} = -\frac{a}{b}\frac{da}{dt}$$

He notes that since $\frac{da}{dt}$ is constant, the answer depends only on a and b. He reasons that a is increasing and b is decreasing all the time so $\frac{a}{b}$ is increasing, and hence the ladder is falling faster and faster all the time. Alex is the only student in the class to come to this conclusion without hints from the interviewer.

Interview 5

Alex's fifth interview occurs almost two weeks after he takes the BC version of the AP exam. During that week the class discusses the written questions from the BC version. Between the fourth and fifth interviews, the class studies series and integration techniques and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Alex's response to a set of questions on a particular topic, and provide an analysis of those responses.

1 MZ: What is a derivative?
2 Alex: It's the instantaneous rate of change of a function.
3 MZ: And it is--?
4 Alex: The slope of a function at a given point.
5 MZ: OK.
6 Alex: Well, it's slope-- That's fine; that's fine.
7 MZ: What else?
8 Alex: You want the definition?

9 MZ: Sure, why not. You can write it.

10 Alex: [writes: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$]

11 MZ: Can you define a derivative without using these three things? Or just explain what it is somehow?

12 Alex: No.

13 MZ: Why not?

14 Alex: Because that's what it is. The simplest is the slope of the function.

15 MZ: It's the easiest one you think?

16 Alex: Yeah, because you know what the derivative is since fifth grade.

17 MZ: Oh, OK. Can you describe what a derivative is in terms of a particular situation?

18 Alex: For example, velocity is a derivative of distance and acceleration is the derivative of velocity which makes it the second derivative of the displacement function, the distance function.

19 MZ: OK. Well, what do these all have to do with each other? How can they all be what a derivative is?

20 Alex: 'Cause this is the slope.

21 MZ: What's the slope?

22 Alex: The slope of a line that goes through points that are very near each other. Because the difference between them goes to zero. So it comes to the point where the slope at that point-- Do you know what I mean?

23 MZ: Sort of. Why don't you draw me a little picture?

24 Alex: [drawing a picture] Say these are the two points. They're really close to each other. The line goes through there. If this differs, it ends. [Has drawn a curve with two points marked on it and a line through the two points.]

25 MZ: So what do all these symbols have to do with these pictures?

26 Alex: This is h [marks an interval on the horizontal axis and labels it h]. If this h goes to 0, these points come closer and closer to each other which finally becomes one point as far as we're considered, and the slope through this point is the derivative.

27 MZ: OK. What does instantaneous rate of change have to do with any of this?

28 Alex: You know how a function turns as it's increasing, decreasing, not increasing at all.

29 MZ: That's what instantaneous rate of change means?

30 Alex: Yeah, like at this point, what's the function doing.

31 MZ: OK. What's the derivative useful for besides velocity, acceleration you already told me?

32 Alex: Derivatives are useful for-- [short pause] To find max and min of a function. So like when they have a production plant and they have a cost function, we can find the optimum of things they have to produce in order to get the most profit for less product.

33 MZ: What does the derivative have to do with that?

34 Alex: Well, when you find the derivative, whenever it's equal to zero, at that point the function is at max or min. Probably. There's a critical point. So, it lets you find your max or min, whatever you're looking for.

35 MZ: Why would the derivative be equal to 0 at a max or a min?

36 Alex: Because the slope is equal to 0. Because the slope is changing from positive to negative.

37 MZ: OK, so what else are they useful for?

38 Alex: I know what integrals do. Besides all that stuff that I said.

39 MZ: What about, not in real life?
40 Alex: Well, second derivative might be useful.
41 MZ: Is it?
42 Alex: To find instantaneous rate of change of first derivative.
43 MZ: Let's go on. OK. Does the derivative involve a limiting process? Explain.
44 MZ: What does that mean if I say limiting process?
45 Alex: That you're getting closer to some number, but you don't really care if it gets to that number as long as it's really close.
46 MZ: OK.
47 MZ: Is the derivative of a function a function?
48 Alex: No, not always.
49 MZ: Why not.
50 Alex: Because a function-- A function could not be differentiable at some point.
51 MZ: OK.
52 Alex: But then the derivative would be-- It would be a not continuous function or something? But it would still be a function. Like a defined piecewise function or something.
53 MZ: OK, so--
54 Alex: I mean it's not going to be the derivative of a function is not a function. No, I guess it's always a function. It's got to be a function. It is a function.
55 MZ: Well, what caused you to say that all of a sudden, well yeah, it does have to be a function.
56 Alex: Oh. Well because, if it's a function, it has to have slopes through each of the points. You know what I mean.
57 MZ: OK.
58 Alex: It has to somehow be related to the function that you take the derivative of. So if it's not a function, for each value of y you're going to have to be two values of x. So let's say the derivative is not a function. It means-- Wait. For one x, two values of y. You can't have two slopes for one x value.
59 MZ: You cannot.
60 Alex: No. A function can only change in one direction.
61 MZ: Explain what is meant by a differentiable function. Give an example of a differentiable function and a nondifferentiable function.
62 Alex: Differentiable function, I guess, is a function that has a derivative at any point. Like the expression for derivative-- The function that represents the derivative of some function has to be also continuous and-- You know what I mean?
63 MZ: Mmm. Not really.
64 Alex: Like at every point-- Well, a function can be not differentiable at one or several points. Not like it's not differentiable at all.
65 MZ: OK.
66 Alex: Like-- Well, a function like this-- [writes: $x = 5$]
67 MZ: What about $x = 5$?
68 Alex: It doesn't have a slope. It's one over zero always [writes: $\frac{1}{0}$]. The change in x is 0 so this function wouldn't have a derivative.
69 MZ: OK. Is $x = 5$ a function?
70 Alex: No.
71 MZ: OK.

72 Alex: Oh. Duh. But still-- Oh, that's right. That's why it doesn't have a derivative.
73 MZ: How did you know it wasn't a function?
74 Alex: For one x there's millions of y 's, an infinite amount.
75 MZ: Fine. Explain what is meant by a differentiable function.
76 Alex: A function that is differentiable everywhere maybe?
77 MZ: OK. Well, how about an example of a differentiable and a nondifferentiable function?
78 Alex: OK. [sketches a pair of axes with a line segment of positive slope] This is a differentiable function.
79 MZ: OK. That's a straight line?
80 Alex: No. [attaches a wavy curve to either end of his line segment]
81 MZ: How do you know that it's a differentiable function?
82 Alex: How do I know?
83 MZ: Yeah, like what about right here where it changes from being straight and starts being curved? Is it differentiable there?
84 Alex: It's very smooth here [goes over the curve to smooth it out].
85 MZ: Well, what's an example of nondifferentiable function?
86 Alex: [sketches a cusp pointing up]
87 MZ: What's wrong with that? What makes it nondifferentiable?
88 Alex: The slopes over here. They change abruptly. Say it was some constant positive number and then all of a sudden it changes to negative. It doesn't come close to zero and then become zero. It becomes zero, but it never comes closer and closer and closer and closer. Even if you had slope really close to zero, but it's just constant. It doesn't change.
89 MZ: OK. Can you think of the equation of a function that's not differentiable?
90 Alex: Sure. [writes: $|x|$]
91 MZ: A lot of people used for their example of a nondifferentiable function something that has a cusp, something like this [draws a cusp pointing up].
92 Alex: Well, that's the same thing isn't it?
93 MZ: Yeah. Do you happen to know an equation of a function that has a cusp? Not necessarily this one, but any kind of cusp as opposed to a--
94 Alex: [pause] I don't know. Do you?
95 MZ: Yeah, I think something like $x^{\frac{2}{3}}$ will have a cusp going the other way.
96 Alex: I don't remember.
97 MZ: OK, fine.

As in previous interviews, Alex mentions instantaneous rate of change as his first answer to "what is a derivative?" and the slope of a function at a given point as his second answer. But in the fifth interview, for the first time, Alex gives a third answer to the question, stating the limit of the difference quotient. As usual Alex mentions a graphical interpretation of derivative more frequently than other interpretations (see Table A.3). Even though the portion of the fifth interview focusing on general questions about

Table A.3. Alex: Interview 5 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎	◎			
What else?				◉	
Can you describe derivative in terms of a particular situation?			○ ○		acceleration
How does the formal definition of derivative relate to slope or rate of change?	◉			◉	
What is meant by instantaneous rate of change?					increasing/ decreasing
What are derivatives useful for?		◎			max/min
Does the derivative involve a limiting process?				●	
Is the derivative of a function a function?	◎				
What is meant by a differentiable function?	○ ●				
Asked to interpret the Mean Value Theorem.	◎ ●		●	● ●	
Asked to find the average rate of change of a function defined as an integral.		◉		●	misapplication of the MVT
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.	○	◎		↦	in/decreasing maximum
Summary	◉	◉	●	◉	

the derivative has fewer questions than similar sections of the first and second interviews, Alex gives almost as complete answers here as in the first two interviews. There are only two omissions compared to his previous responses. In the fifth interview Alex does not mention the instantaneous nature of the derivative when discussing velocity and acceleration. He also does not describe the graphical function process.

These two omissions carry over into the connections he makes between the various interpretations. Alex describes the relationship between the graphical and symbolic interpretations of derivative in terms of the details of the ratio and the limiting process, but not in terms of the function process as he does in the second interview. He mentions that rate of change and slope are both ways of describing the derivative but does not compare them explicitly.

The major improvement from the second interview is that in the fifth interview Alex does not make any of his previous misstatements. In the second interview Alex makes misstatements involving the relationship between derivative and tangent line, the relationship between derivative and limit, and whether or not the derivative is an approximation. In the fifth interview Alex makes no such misstatements.

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answer. But in the fifth interview, for the first time, Alex gives a third answer to the question, stating the limit of the difference quotient. As usual Alex mentions a graphical interpretation of derivative more frequently than other interpretations (see Table A.3). Even though the portion of the fifth interview focusing on general questions about the derivative has fewer questions than similar sections of the first and second interviews, Alex gives almost as complete answers here as in the first two interviews. There are only two omissions compared to his previous responses. In the fifth interview Alex does not mention the instantaneous nature of the derivative when discussing velocity and acceleration. He also does not describe the graphical function process.

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For the second part of the fifth interview Alex is asked, "what the Mean Value Theorem says." He replies, "If a function is continuous from a to b, then there is a point c where the slope, the derivative, is going to equal the slope through a and b." When asked to explain a bit further, Alex draws a sketch of a curve with points labeled as a and b and a line connecting them. He also draws a line tangent to a point between a and b that is parallel to the first line and explains that the two lines have the same slope. When asked to write his ideas symbolically, he writes, "If $f(x)$ is continuous from a to b, then $f'(c) = \frac{f(a) - f(b)}{a - b}$ where $a \le c \le b$." When asked to explain the theorem in terms of a position and velocity function, Alex explains that distance divided by time gives you average velocity and that the theorem says that, "at some point you went at that speed."

Alex demonstrates his knowledge of graphic, symbolic and kinematic descriptions of the Mean Value Theorem, but he seems to remember these through the graphical interpretation. He mentions slope in his first answer to the question. When asked, he can then give other interpretations.

The next question on the fifth interview involves a problem from the AB version of the AP exam which Alex has not taken. He reports that he has seen the AB exam but not worked through the questions. The question is as follows:

> Let $F(x)=\int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.
> (a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.
> (b) On what intervals is F increasing?
> (c) If the average rate of change of F on the closed interval $[1,3]$ is k, find $\int_1^3 \sin(t^2)\,dt$ in terms of k.

The interviewer asks Alex to discuss his methods of solution for parts (a) and (b), but does not require him to complete the solution of either part. For part (a) Alex recognizes that he must find the definite integral $\int_0^1 \sin(t^2)\,dt$ and correctly applies the trapezoid rule to do so. For part (b) Alex finds $F'(x)=\sin(x^2)$ and notes that where F' is positive the function is increasing.

For part (c), Alex first thinks that average rate of change is the derivative, $F'(x)=\sin(x^2)$. He immediately remembers that the derivative is instantaneous, not average, rate of change but he thinks he can get around this problem, "At some point, $\sin(x^2)=k$ by using the Mean Value Theorem." Alex continues with this reasoning writing, "$\int_1^3 k\,dt = kt\big|_1^3 = 3k-k=2k$". This reasoning yields the correct answer, but since the Mean Value Theorem holds only at one point, it is inappropriate to replace $\sin(t^2)$ with k in the integral.

The next section of the interview concerns Taylor polynomials. Alex is asked what a function and its second degree Taylor polynomial have in common and how they differ. Alex responds that at one point they have the same function value, first derivative and second derivative. When asked how the graphs would compare, Alex sketches a curve similar to $f(x)=e^x$ and adds the graph of what he calls a "Taylor function" that intersects the graph at one point where he also draws a tangent line and states that both functions are concave up. When asked, Alex says that Taylor polynomials are useful for approximating function values.

The final section of the fifth interview concerns a function, f, that at any time x, given in hours, tells the outside temperature in degrees Fahrenheit. Alex is shown a series of symbolic expressions and asked what information each one provided about the outside temperature.

For $f'(3) = 4$ Alex first considers that the function is linear with its derivative constant over a range. When the problem is clarified he says, "Oh, you mean it could be like the whole expression; when we plug in 3, we get 4.... Well, then at that point the function changes 4 degrees per hour." For $f''(3) = -2$ Alex explains that, "the rate the function changes is slowing down. It means the function is negative 2 degrees per hour slower. It's going to slow down the rate of increase. It might still be increasing ... but not that fast."

When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Alex replies that there will be a straight line from 0 to 3 with the temperature increasing at four degrees per hour during that period for a total increase of 12 degrees. When asked to interpret $f''(x) = -2$ for $3 \le x \le 6$, Alex says, "It's a slowing down increasing temperature, and after 2 hours it's going to be a 0 increase. When it reaches 6, it's going to start decreasing again."

When asked for the location of maximum and minimum values, Alex responds that the maximum would occur at $x = 5$ since that the where the derivative is 0. He also thinks that the minimum temperature would occur at $x = 0$ since the function is increasing much longer and steeper than it is decreasing. When asked to sketch a graph he sketches a reasonable graph that is an increasing straight line from 0 to 3 and a concave down curve with maximum at $x = 5$ for 3 to 6. When asked for an equation for temperature, Alex correctly describes how to use antiderivatives and initial conditions to find the first derivative, and the original function using the second derivative.

Overall Alex seems comfortable describing the physical situation in terms of temperature or a graph or symbols. Even though he does not draw a graph until he is asked to do so, he seems have a mental graph that allows him to easily answer where the function has a maximum or minimum. He mentions a symbolic approach early in terms of $f'(3) = 4$, but does not need a symbolic approach to answer the interview questions, and does not do any symbolic calculations until specifically asked to do so.

Case Study 2 — Brad

Academic record

*Other AP courses: US. History (junior year), Chemistry.

*Writing tutor at the high school writing center.

*Plans to become a cardiologist.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: "It's a problem in which you derive an answer."

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different.

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: "a) $\begin{matrix} y = 2x + 4 \\ y = 8x^9 - x^4 + 3x^2 - 2 \end{matrix}$

They are different because of the degree of the polynomial and # of zeros.

b) No clue!"

QOTD #3

Give an example of a function without using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Date: August 26, 1993. This question occurs while the class is doing a quick review of functions.

Response: Brad sketches two ovals with three points in each. The first oval's points are labeled x_1, x_2, and x_3. Similarly, the other oval has points labeled y_1, y_2, and y_3. There are lines with arrows to indicate a one-to-one mapping between the points across the ovals.

Comment: Notice that in his answers to the QOTD #1-4 on functions, Brad never mentions the property that each input must have only one output, nor does he draw a standard Cartesian graph of a function.

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) nonintegers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: "a) No. b) No."

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to class discussion on limits.

Response: "A limit is a point approached by both sides."

Test 1

On a test on limits, Brad is able to correctly find limits by reading values from a graph, by substituting into a piecewise function and by using algebra to simplify a limit calculation. He is able to find a δ for a given ε in a graphical setting. However, he is unable to do an ε–δ proof in a symbolic setting. Other than the formal proof, whatever Brad did not remember from his junior year study of limits seems to come back (or be learned) by test time.

Interview 1

This interview occurs after the test on limits but prior the class discussing derivatives. Therefore, Brad's answers are presumed to be based on what he remembers from his junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table A.4, which codes these responses. A summary discussion follows.

1 MZ: Do you remember what a derivative is?

2 Brad: [laughs] Isn't it like the opposite of a-- No, what's the opposite of a limit or something. I don't know. I have no idea how to say it. It's been a while. [pause] Isn't it like all those formulas for velocity and acceleration and something like that?

3 MZ: Let me see if I can jog your memory. I have a list of words here, and you're suppose to tell me if you think they're related to derivatives or not. OK, slope. Do you think slope has anything to do with derivatives?

4 Brad: Yeah.

5 MZ: Are you going to tell me why you think it has to do with it?

6 Brad: Because I think I remember in physics class something about--. I know slope has something to do with-- We were doing acceleration or something.

7	Brad:	So I know slope is acceleration or I mean, I don't know. Some of it depends on how much it goes up; it accelerates. So if there's a straight line, it's zero. So it's not accelerating. Something like that.
8	MZ:	Right. Then what would that be a graph of?
9	Brad:	That would be like velocity, wouldn't it? A graph of the velocity.
10	MZ:	So that line is a graph of velocity?
11	Brad:	Mm.
12	MZ:	For example, in this case [draws an axes with a straight line at about $y = 2$]. What's the x and y ?
13	Brad:	Time, velocity [points to x then y axis].
14	MZ:	This would be like time [labels "x"-axis with t] and this would be velocity [labels "y"-axis with v].
15	Brad:	Yeah.
16	MZ:	And then the slope of this line is like acceleration. [short pause]
17	MZ:	[shows the next word -- speed or velocity] Sort of we're talking about this already. Speed or velocity.
18	Brad:	Yes, I think it's related.
19	MZ:	So, in this example [line drawn on axes], the slope-- does the slope-- Wait, how did derivative fit into this picture?
20	Brad:	The same way I think with like-- It had to do with the slope which was the acceleration. Or like if you have an equation and you keep going down and down and take the derivative, you can find the acceleration, then displacement, speed.
21	MZ:	Mm hmm.
22	Brad:	It's like the first one's the v and then the a.
23	MZ:	Which one was the displacement?
24	Brad:	It was either like the first one or like the third one. I don't know. [laughs]
25	MZ:	Rate of change? Change?
26	Brad:	Acceleration. That'd be the slope, wouldn't it?
27	MZ:	Which is-- The slope is the acceleration is the rate of change? Is that what you just said?
28	Brad:	Yeah. [half-hearted]
29	MZ:	What does that make you think of, rate of change?
30	Brad:	How fast the change. Like the slope. 'Cause rate of change, that's the slope. It's a way of finding the change.
31	MZ:	How much the function changes?
32	Brad:	Or acceleration. Something like that.
33	MZ:	Could velocity be a rate of change? Or does it just make more sense to say acceleration is a rate of change?
34	Brad:	'Cause velocity stay the same and it won't really change.
35	MZ:	It could stay the same. Could acceleration stay the same?
36	Brad:	Yeah. Then there is no change.
37	MZ:	OK. Line or linear?
38	Brad:	Yeah. Like-- I don't know. I like the first subject.
39	MZ:	Measurement?
40	Brad:	For derivative? I don't think so.
41	MZ:	Why would you think so?
42	Brad:	You gotta measure something. If you measure something you usually get a measurement, so--
43	MZ:	What would you measure?
44	Brad:	Velocity, acceleration, something on the graph, displacement.

45 MZ: Ever heard of derivatives in association with anything else, like being useful for finding other kinds of things besides velocity or acceleration?

46 Brad: I think so, but I don't remember [laughs].

47 MZ: Prediction.

48 Brad: I don't know. I don't think so.

49 Brad: [reading next word on list] Uhm, opt-- What do you mean like optimization, like maximum?

50 MZ: Yeah.

51 Brad: 'Cause you find like local max and mins, don't ya?

52 Brad: You have the focus point. Is that it? I don't know. And there's extrema. Oh, yeah 'cause you find-- Oh. [snaps figures].

53 MZ: You remember something.

54 Brad: 'Cause you find the cont-, the con-- the curvature of the thing.

55 MZ: The curvature?

56 Brad: Where you are talking about concave up or down. You use that if it's derivatives, I think. The first derivative's the max and mins and the second one's if it's concave, isn't it? I don't know.

57 MZ: Yeah, it is.

58 Brad: You find max and mins to find the point like if it changes. Like if it's a max-- Like where the curve changes. It goes from-- No, where the slope changes, where it goes from positive to negative. That's usually where there's a change; I think.

59 MZ: Now, which one is it?

60 Brad: I don't know.

61 Brad: OK, uhm. It's usually a root. No, it's a zero. I think it's a zero. It's a zero or something. It's a root or something.

62 MZ: Is a root different from a zero?

63 Brad: [pause] No. Not really. I don't think so. Cause if you find a root, you usually get a zero if you plug it in, don't you? I don't know. I think it is different, not zero.

64 MZ: What is a root? What do you mean by a root?

65 Brad: Like with a polynomial, a polynomial root, which actually if you put the root in there would actually equal zero. It is kind of--

66 MZ: So it's a zero.

67 Brad: Yeah. A zero. Which is usually like-- [pause] I think it's the focal point.

68 MZ: What do you mean by the focal point?

69 Brad: Well, like where the-- I'm not sure what it's called, but it's like the change. It would be like right here [sketches local max and points to the maximum point].

70 MZ: OK.

71 Brad: That's where your change-- The slope changes from positive to negative. Which like on the first derivative line, when you're finding it, at that one point. I think it's the zero.

72 MZ: So the zero of the first derivative something. You said line, first derivative line?

73 Brad: Yeah, it's the f' line or whatever.

74 MZ: So the zero of that is going to be where that's--

75 Brad: Well, it depends. If it's-- When you do a derivative line test. It's like if it's positive There's a point.

76 MZ: I'm not sure what a derivative line test is.

77 Brad: You want me to show you?

78 MZ: Yeah.

79 Brad: So like $f'(x)$. [Writes: $f'(x)$] And you have your zero here that's one [draws a number line and marks 1]. And you plug in numbers less than one--

80 MZ: Oh, OK.

81 Brad: --and that would be positive and that's negative. [Sketches a wavy vertical line at 1 and puts a plus above the number line on the left and a minus on the right.] So here that would mean like at one you might have a maximum because, I think, because the slope changes from positive to negative so that usually means it's more like a curve.

82 MZ: OK, what if I've got a negative on this side and a positive on this side.

83 Brad: It would be the opposite. It would be a minimum I think. I think.

84 MZ: What if I get a positive on both sides?

85 Brad: Then it's nothing. [laughs]

86 Brad: All polynomials are continuous.

87 MZ: All polynomials are continuous. True. Does this have to do with derivatives?

88 Brad: Yeah, cause I guess you have to be-- I think derivatives have to be continuous.

89 MZ: Derivatives have to be continuous.

90 Brad: I don't know. Maybe they don't.

91 MZ: Is a derivative a function?

92 Brad: Yes.

93 MZ: Why?

94 Brad: 'Cause I guessed. [laughs] I don't know.

95 MZ: Limit? Does the limit have anything to do with derivative?

96 Brad: Yes.

97 MZ: What?

98 Brad: It's the derivative.

99 MZ: The limit is the derivative?

100 Brad: No the derivative is li-- It's closely related. I don't know how.

101 MZ: You feel sure it's closely related.

102 Brad: I think it is. Because we're talking about them. Yeah, it's related.

103 MZ: Do you remember a formal definition of derivative?

104 Brad: [Writes: $\frac{f(x+h)-f(x)}{h}$] Is this right?

105 MZ: Yeah.

106 Brad: That's it.

107 MZ: Do you know how this relates to--

108 Brad: It's a function.

109 MZ: This is a function.

110 Brad: That's a rational-- I don't know.

111 MZ: A rational function? Do you know how this relates to earlier when you told me like the derivative is like the slope?

112 Brad: [pause] No. I don't remember.

Table A.4 summarizes Brad's first interview transcript. Brad's primary model of derivative is that of velocity or acceleration. This is the only model, except for the misstatement "opposite of a limit" [ln 2], that Brad mentions without being specifically

Table A.4. Brad: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?			○ ○	○	misstatement (derivative is opposite of a limit) acceleration
Is derivative related to slope?	○		○		acceleration
Is derivative related to speed or velocity?		○	○ ○	$\mapsto$	acceleration misstatement
Is derivative related to change or rate of change?	○	○	○		acceleration
Is derivative related to measurement?			○ ○		acceleration
Is derivative related to optimization?	○				max/min curvature concavity
Is the derivative a function?				○	guess
Does the limit have anything to do with derivative?				○	misstatement (d=limit)
Do remember a formal definition of derivative?				◉ ○	
Summary	○	○	○	◎	

asked. Even when the discussion turns to slope or rate of change, Brad immediately mentions acceleration. In fact, Brad mentions acceleration as derivative more often than velocity. He recognizes acceleration as the slope of a velocity graph, but then confuses the derivative relationships between the equations for acceleration, displacement and speed.

Brad mentions slope, acceleration, and rate of change in rapid succession as if they are all the same thing, but is unable to articulate what the connection is. For these three models, Brad does not mention a ratio structure or the second or third layers of the derivative concept. He describes the ratio when stating the formal definition and does not know how the symbolic ratio relates to the other models of the derivative concept.

Brad connects the derivative to limit via two misstatements, "derivative is opposite of a limit" and "It's [the limit] is the derivative" [ln 98] which he backs down from saying only that the limit are derivative are "closely related" [ln 100] because "we're talking about them" [ln 102]. "We're" probably refers to the class, but it may refer to the interview. Brad omits the limit in his statement of the formal definition. Brad guesses that the derivative is a function and states explicitly that the difference quotient describes a rational function.

QOTD #6

Find the derivatives of the following four functions:

$f(x) = (x-1)^2(x^2-4)$

$g(x) = \dfrac{x-1}{\sqrt{5-x^3}}$

$h(x) = \sin x$

$j(x) = \ln x$

Date: September 20, 1993. This question occurs prior to the class learning about short-cut rules for taking derivatives of various forms.

Response:

$$f'(x) = 2(x-1)^1(x^2-4) + (x-1)^2 1(x^2-4)2x$$

$$g'(x) = \frac{1\left(\sqrt{5-x^3}\right) - (x-1)\frac{1}{2}(5-x^3)^{-\frac{1}{2}} - 3x^2}{\left(\sqrt{5-x^3}\right)^2}$$

$$h'(x) = -\cos x$$
$$j'(x) = \text{e}$$

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be using your knowledge of derivative.

$y = \log(x^3)$ $\quad y = \frac{x}{|x|}$ $\quad y = x^e$ $\quad y = e$

Date: September 21, 1993. This question also occurs before the class studies short-cut rules for taking derivatives but after they have studied the limit definition of derivative.

Response: "$y = e$, because $y = e$ is a constant funct, and would equal a horizontal line. The derivative is an graph like so" Brad includes the sketch of a graph that looks similar to $y = x$.

Comment: This question is presented to the students since no student correctly stated the derivative of $y = \ln x$ in the previous Question of the Day. Brad apparently recognizes his mistake from the day before but still does not know what the derivative of $j(x) = \ln x$ is.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$? Why <u>not</u>?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: Brad does not attend class on this day, and so there is no answer to this question from him.

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. Brad shows that he can correctly state the limit of the difference quotient definition of derivative and substitute a function into it correctly. Algebraic errors keep him from correctly using it to find the derivative. Brad knows to estimate the derivative at a particular point by finding two nearby points and calculating a difference quotient for those two points. Given the graph of a position function for a car he correctly answers questions about the speed and direction of the car. Given the graph of a function, he is able to sketch a correct graph for the derivative function. Brad has some difficulties working through a complex chain rule derivative. He seems to omit one or two steps in the chain rule process by finding the derivative of $f(x) = \tan^{\frac{3}{2}}(2x+1)$ to be $f'(x) = \frac{3}{2}\sec^2(2x+1)2 = 3\sec^2(2x+1)$ instead of $f'(x) = \frac{3}{2}\tan^{\frac{1}{2}}(2x+1)\sec^2(2x+1)2$.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: "Yes, I've relearned how to do them, and have a better understanding of the pts of inflection and other aspects of the derivative."

Comment: Notice that Brad does not mention a definition of derivative or what it means only "how to do them" and a graphical application of derivatives.

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "A. Don't Remember -- behavior from derivatives. B. Nope"

Test 3

About two weeks later the class is tested on Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Brad correctly calculates a third degree Taylor polynomial, and he is able to use the first and second derivatives of the position function to find the speed and acceleration of an object at a given time. However, when given the graph of the derivative function and asked about the original function, Brad answers the questions as if they are being asked about the derivative graph itself.

QOTD #11

Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: "the rate of profit of interest on a loan"

Test 4

Two weeks later the class has a test on the applications of derivatives. Brad uses derivatives to solve two traditional max/min problems. On a third max/min problem Brad sets up an appropriate equation for the quantity that he is trying to optimize, but never takes the derivative of this or any other equation in the problem. Instead he sets the quantity itself equal to 0 and solves that equation for the dependent variable. In doing this he simply skips a step in the algorithm without noticing that anything is missing. This is somewhat similar to his error of skipping a step in the algorithm for complex chain rule derivative on the previous test.

On the current test Brad correctly calculates the derivative of an implicitly defined function. He also correctly solves two traditional related rate problems. On a third related rate problem he takes the derivative with respect to time of $V = \frac{\pi}{3} r^2 h$ and gets $\frac{dV}{dt} = \frac{2\pi}{3} r \frac{dh}{dt}$, failing to apply the product rule.

Interview 2

The second interview occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by a summary table and a discussion.

1 MZ: What is a derivative?
2 Brad: It's the ch-- See, I can do it. I just don't know what-- I think it's like the change in something or something. It's almost like the-- It's like the slope basically. I think.
3 MZ: The change in what?
4 Brad: Well, like-- Like $\frac{dy}{dx}$ is like the slope. If you find $\frac{dy}{dx}$ of a circle or something, at that point that's the slope of a tangent line It's just like the-- I think it's like the change.
5 MZ: What else were you saying?
6 Brad: That it's the slope of a tangent line. I don't know. I just think it's like-- I don't know the exact definition. I haven't really studied it.
7 MZ: OK. We'll keep going. What are derivatives useful for?
8 Brad: To find velocity or displacement or if you want to find the uh-- You could use them for cost functions. I think. I don't know.
9 MZ: How would they related to cost functions?
10 Brad: You know, to find maximums or how much-- Related rate problems where you can find like how much length and dimension and stuff that will give you a certain price I think. I don't know.
11 MZ: How does the derivative relate to a maximum?
12 Brad: Uhm, I'm not sure. I'm not sure it does now that I think about it.
13 MZ: Uhm, did you use derivatives at all on the test you just took?
14 Brad: Yeah.
15 MZ: So what was it useful for on that test?

16 Brad: Finding dimensions. Yeah, because you found like the maximum. I think I used it for the maximum area of the rectangle. That's where I found the derivative to find one of the sides of the rectangle. I think.

17 MZ: What causes the derivative to be useful in those kind of problems?

18 Brad: 'Cause you set it equal to zero I think you get the maximum. Like in my case I got the maximum x value that would, I think, that would equal to zero.

29 MZ: OK. So when you set the derivative equal to zero and you solve for that x value that told you something about the maximum?

20 Brad: Not the-- The maximum for the side, I think. I'm not sure.

21 MZ: The maximum for the side?

22 Brad: I'm not sure.

23 MZ: How would you explain what a derivative is to someone without very much math background?

24 Brad: I'd say, go see Mr. Forrest [laughs].

25 MZ: What it is?

26 Brad: I'd say, go see Mr. Forrest. I don't know. I'd just say basically it's something that you can use to find-- Or if you had a problem with-- It's like the change. I guess I would tell them it's like the change in-- It's like almost the same thing as the slope, if they knew what the slope was, I guess. The derivative would equal the slope at a certain instant and that equals like the change in the function. I think. Yeah, the derivative does equal the slope because if you graph it-- Oooh. [Makes a face.]

27 MZ: Yeah, it does. OK. So if I graph what?

28 Brad: Well, if you have a graph and you graph the derivative you're graphing the slope. So the derivative does equal the slope. And I'd just tell them that the derivatives the change in a certain function.

29 MZ: OK. So is that the same thing, to talk about the change in the certain function and to talk about the slope of the same certain function?

30 Brad: I think so. 'Cause the change is the slope. The slope is the change. I think. I'm not sure.

31 MZ: OK. Real world situation involving the concept of derivative?

32 Brad: Like cost functions or dimensions.

33 MZ: OK.

34 Brad: Like filling up the cone.

35 MZ: In the filling up the cone problem how does the derivative relate to that?

36 Brad: To find the rates. 'Cause you find the rates by taking the derivative. Say I had the volume of a cone which is like one third pi something r squared or something, r to the third. You take the derivative. You get a factor that's dr over dt or something. Either the change in height or the change in radius, whatever you know. You just solve it. You get--

37 MZ: OK. So that change in radius, for example, that's the derivative part then in that problem?

38 Brad: Well, the derivative is the whole equation. It's a product of the derivative, I think. 'Cause you wouldn't have it if you didn't take the derivative.

39 MZ: OK.

40 Brad: So by taking the derivative you get that and you solve for that.

41 MZ: And then you solve for the dr dt part?

42 Brad: Yeah.

43 MZ: OK. How can you tell if a function's differentiable?

44 Brad: I'd uh-- I'd graph it.

45 MZ: OK.

46 Brad: Then if it's continuous and it didn't have a certain point where the slope changed dramatically, like a cusp or like a 45 degree angle.
47 MZ: So why is it that the derivative doesn't exist at a point like that?
48 Brad: Because-- I know this, but I don't know it. It's because the slope is, I think, changing too fast.
49 MZ: Do you want to draw something?
50 Brad: You have like a-- [Brad draws a smooth curve on a pair of axes.]
51 MZ: OK. I have a regular function.
52 Brad: You're finding the slope as it comes through here [maximum point on curve]. They both start reaching-- You know how they come to where it's almost equal. But if you go like that [draws a cusp pointing upward], they're like really-- See as you reach a point [max on smooth curve] the slope becomes smaller and smaller, I think. But it seems like when you go here [the cusp] it just gets larger and larger. So that's why.
53 MZ: Is derivative related to velocity?
54 Brad: Yeah, because you can take the derivative of displacement and find the velocity.
55 MZ: OK.
56 Brad: And take that and find the acceleration.
57 MZ: Derivative--
58 Brad: Derivative of velocity.
59 MZ: Rate of change or change?
60 Brad: That's the derivative, I think. Maybe like rate of change.
61 MZ: Yeah. I was going to ask you, do you see those as being different, the change versus the rate of change?
62 Brad: Well, change is like the actual change, where the rate of change is, I think, the process of that at a certain-- Rate of change is at a certain interval. That's how much it changes at each point. And the change, I would interpret it as being the end product that you calculated with rate of change.
63 MZ: Which one is the end product?
64 Brad: The change. The rate of change is the amount it changes per interval. Like if you had a meter per second, every second that would be-- every second-- Since it changes one meter every second, that'd be the rate of change. Like if it started at 8 and every second went one less. It'd go 7, 8 and that would be the change. Well, the change would be-- Well, it's almost actually I guess the same because the change is how much it did change between the two. And since it's 1 meter per second, then the change-- I think it's from 8 to 7. I think it would be the same. Yeah.
65 MZ: OK, and is this related to the derivative?
66 Brad: Yeah.
67 MZ: OK. Line or linear?
68 Brad: Yeah, because-- I think so. When you find the slope of a function and it's a $y = mx + b$ function, then if you find the derivative it's going to be a straight line. Or if you take it of a quadratic function, it'd also be a straight line. But if you take it of more than that, it's going to be a polynomial. I believe.
69 MZ: Measurement?
70 Brad: Do you mean like the measurements, what you're measuring or--
71 MZ: Do any kind of measurements have to do with derivative?
72 Brad: Yeah. Like the problem on the test with the airplane, 2 airplanes. Remember that?
73 MZ: Yeah.

74 Brad: You have a measurement like in miles. There's like a certain distance you have to know when you find the derivative.
75 MZ: So what part of that problem on the test was the derivative?
76 Brad: Well you-- Since you had a triangle[sketches a triangle], 'cause they intersected perpendicularly.
77 MZ: Yeah.
78 Brad: And there was rate going this way and rate going that way [along the legs of the triangle]. You want to find the rate of change here [along the hypotenuse]. And they give you the distance here and here [on the legs]. And then you knew the rate of change cause they were going like 520, I think, each way [writes 520 next to each of the legs]. You made a Pythagorean theorem. Then you equal that. [writes: $x^2 + y^2 = z^2$] Then you take the derivative of that which is uh-- [writes: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$]
79 MZ: OK.
80 Brad: The $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the change. That's the rate of change and that's at the distance right there. So that's a measurement.
81 MZ: OK. So the measurement is more the, I don't know, the x part you were pointing to.
82 Brad: Yeah. Well I mean it's sort of a measurement too because you have to know the velocity.
83 MZ: The rate of change. So if I said point to the part or parts of this problem that is the derivative, what would you point to?
84 Brad: Probably the dx, dy.
85 MZ: Each of those?
86 Brad: Yeah. But since we already know these, probably just that one.
87 MZ: Oh, that's the only one we didn't know for that problem.
88 Brad: Because you plug in values for those.
89 MZ: Right. Prediction or approximation?
90 Brad: No, I don't know. I don't know.
91 MZ: Continuity? Did you say it has to do with derivatives?
92 Brad: You can always keep your pencil on it. And it would be differentiable which has to do with derivatives.
93 Brad: So it's related.
94 MZ: So, how did that work? Did you say, if it's continuous, then it has to be differentiable?
95 Brad: Well, not necessarily.
96 MZ: Not necessarily.
97 Brad: It could have a cusp, but it needs to be continuous.
98 MZ: You have to at least have that.
99 Brad: Yeah, as a factor of that. It's required. It's a prerequisite of that.
100 MZ: It's a prerequisite, but it's not the only thing?
101 Brad: Yes.
102 MZ: OK. Limit. Does limit have to do with derivatives?
103 Brad: [pause] Yeah, because when you use the derivative-- er, the limit, I remember using the Δx which would be the change in x. And since the derivative, I've already said this, is like the change in something, so then it would probably be related to the derivative.
104 MZ: So how does the limit fit in there?

105 Brad: Well, like if you had an equation where like y -- or the limit of something equals $x + \Delta x$, or you know, plus h.

106 MZ: Why don't you jot something like you just said down so I can have something to look at.

107 Brad: [writes: $\lim_{h \to 0} ? \dfrac{x + \Delta x - x}{h}$] So you have like the limit as h approaches 0 of something. Since you have a Δx which means the change in x -- I mean that's the rate-- That's like a change. And since I said earlier the derivative has to do with change, it would probably be--

108 MZ: OK. So in this thing that you just jotted down, am I taking the limit of something else and then this [the quotient] is kind of later down the line?

109 Brad: Yeah. I think so. It's like a function. It would be like f of-- [short pause]

110 MZ: OK. We'll go on. Is the derivative a function?

111 Brad: Sometimes.

112 MZ: Sometimes, not always?

113 Brad: Wait. [Bell]

114 MZ: Let me ask you quick. Do you know a formal definition of derivative?

115 Brad: No.

116 MZ: You sure?

117 Brad: No, I do not.

118 MZ: If I give you a function and I say, this is a derivative, what things can you tell me about the original function?

119 Brad: I could probably give you the original function because you can-- I know there's a way. I know how to do it. I can't explain, but I know how to--

120 MZ: OK. What if I give you the graph and say this is the derivative of some function?

121 Brad: Then I could tell you about the original graph because that's the slope. So I could tell you where it's increasing and decreasing. Tell you where it changes concavity.

122 MZ: OK. So you could probably graph exactly the original function.

123 Brad: Probably. Or a good estimation.

124 MZ: A good estimation?

125 Brad: --approximation.

126 MZ: OK. Well, thank you.

As Table A.5 illustrates, Brad mentions derivative in terms of slope, rate of change, velocity and a symbolic interpretation. In each of these except for velocity he mentions the instantaneous nature of the derivative. Brad discusses a graphical limiting process but it is in terms of slopes becoming "smaller and smaller". He describes the limit of slope (i.e. derivative) values in describing whether a function is differential at a point. He does not describe the limit of average slopes becoming the instantaneous slope. Brad describes the details of the ratio in terms of rate of change and symbolically. When

Table A.5. Brad: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	○ ◎	○		◉	misstatement (d=change)
What can derivatives be useful for?		○	○		misstatement maximums related rate
How would you explain a derivative to someone without very much math background?	◎	○		$\mapsto$	misstatement (d=change)
Give an example of a real world situation involving derivatives.		○		$\mapsto$ ◉	
How can you tell if a function is differentiable?	◎ ●				
Is derivative related to speed or velocity?			○ ○	$\mapsto$	acceleration
Is derivative related to change or rate of change?		○ ◎ ●			misstatement (d=change)
Is derivative related to line or linear?	○			$\mapsto$	
Is derivative related to measurement?		○ ○	○	◉	misstatement (d=change)
Is derivative related to limit?		○		◉	misstatement (d=change) misstated the ratio
Summary	◉	◉	○	◉	

asked whether the derivative of a function is a function, Brad says, "Sometimes." Because of time constraints, he is not asked to elaborate on his answer.

Brad makes several connections between different aspects of derivative, but many of these include the error that derivative is change in something instead of rate of change. Brad recognizes that the symbolic expression $\frac{dy}{dx}$ represents a slope and states that $\frac{dx}{dt}$ represents a change or rate of change [ln 80]. He states that the slope is the change in a function. He also states that the limit expression $\lim_{h \to 0} ? \frac{x + \Delta x - x}{h}$, which he knows he is not remembering correctly, is related to the derivative since derivative is the change in something and his limit expression has a Δx in it [ln 107].

When asked whether change and rate of change are different, Brad says, "The rate of change is the amount it changes per interval" [ln 64]. He uses the example of a change from 7 to 8 in 1 second. He then explains that the change is "the same because the change is how much it did change between the two" [ln 64]. He is correct that for a unit interval the numeric answers are the same. The units, however, would be different. His choice of example helps him avoid seeing the distinction in the two terms.

For the last part of the second interview Brad is asked what he could tell about the original function if he is given the graph of its derivative. Brad says he could tell where the original is increasing or decreasing and where it changes concavity. There is no time left in the interview for him to demonstrate this competence. Brad's work on Test 3 taken two weeks previously, shows incorrect answers to questions asking for these same skills.

In comparing Brad's responses during the first two interviews it is clear that he states a much more complete understanding during the second interview . In the first interview his focus is on velocity and acceleration. However, in the second interview, slope, rate of change, and the symbolic representation are all more prominent than velocity and acceleration. Brad's first interview contains no mention of the instantaneous nature of the derivative or descriptions of the details of the ratio or limit processes except in misstatements involving the symbolic context. In the second interview he states the instantaneous nature of the derivative in terms of slope and rate of change, and he describes a graphical limiting process and the details of the ratio for rate of change.

In stating the formal symbolic definition, Brad changes errors between interviews. In the first interview he states the ratio correctly as $\frac{f(x+h)-f(x)}{h}$, but forgets the limit and when asked directly, has no idea how limit and derivative are related. In the second interview he remembers that a limit is involved but misstates the ratio writing $\lim_{h\to 0} ? \frac{x+\Delta x - x}{h}$ which he thinks needs an f of something.

QOTD #12

What is the most important idea that we have studied so far in this class?

Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

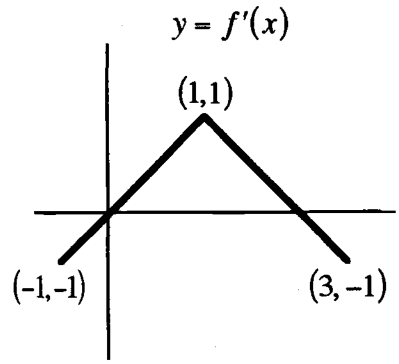
Response: Brad is absent on this day of class.

Interview 3

The third interview occurs during the three days after the test on differential equations and antiderivatives by substitution and by parts. The first part of the interview is a summary of Brad's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike the second interview, the point $(1,0)$ on the original function is given so that only one solution is possible.

1 MZ: This first question is actually kind of a problem. So here's the graph. This is going to be the graph of the derivative of some function. So this is the point $(1,1)$. This is the point $(-1,-1)$and this is the line in between those. And then this is the point $(3,-1)$ and this is the line in between those. [puts dots at those three points and connects $(1,1)$ to the other two

with line segments] And so consider that this is a function with just an abbreviated domain, so we don't care what happens-- So this is $y = f'(x)$ and we want to draw $y = f(x)$, so it's the original function. It's all yours. [hands Brad the notebook]



2 Brad: OK, I know how to do this. I know this is the slope of the actual function. [short pause] These are critical points, aren't they, where it passes the x axis.

3 MZ: Where it crosses the x axis? OK.

4 Brad: [short pause; laughs] I forgot.

5 MZ: Well, what parts do you remember?

6 Brad: I know that this is the slope and that if I-- The only part I'm having trouble with right now is knowing where the points are. That's--

7 MZ: Oh, you know what. I didn't give you enough information. Thanks for that.

8 MZ: We need an initial condition.

9 Brad: Yeah.

10 MZ: So the initial condition is that $f(1) = 0$. Good thing you told me that.

11 Brad: This is--

12 MZ: So he just marked $(1,0)$.

13 Brad: OK. This is decreasing now so that's-- See I know this is an increasing slope and that's decreasing. And that's negative. No wait, it's not actually. Positive slope, negative slope. Oh, actually it would be like-- I think it would be like that. I forgot.

14 MZ: He was just motioning a-- what was that a concave up little curve thing.

15 Brad: Yeah. Concave up little thing. [stated directly into the tape recorder; both laugh]

16 MZ: -- on the left side of that point.

17 Brad: OK, uhm. I don't know. That's what I think it is. I'm not exactly sure. Oh, actually wouldn't it be more like that, I think. Cause it would be positive, decreasing or something like that. I don't know. [has sketched in a curve which is concave up from $x = -1$ to 1 and concave down from $x = 1$ to 3 with intercepts at $x = -1, 1$ and 3, a min at about $(0, -1)$ and a max at about $(2, .5)$]

18 MZ: Let's just talk about that left side. That first thing you drew.

19 Brad: Mm hmm.

20 MZ: How come you decided that it was that shape?

21 Brad: 'Cause I know uhm that the slope is negative and so--

22 MZ: The slope of--

23 Brad: slope of f

24 MZ: of just f, regular f?
25 Brad: Yeah, because f' is negative, a negative number there.
26 MZ: OK
27 Brad: so I know that it's negative--
28 MZ: It's negative.
29 Brad: --so it's decreasing. Then it's positive so it would be increasing.
30 MZ: OK.
31 Brad: Then it's positive again so I went on up and then it's decreasing so I went down. I think that's right. I don't know.
32 MZ: OK. That sounds pretty reasonable. How confident do you feel about how far you went down and how far you went up?
33 Brad: [laughs] Guess. I mean, I don't know, somewhat I guess.
34 MZ: I mean like is it possible this could have gone down to like say -3 [at the min]?
35 Brad: Naw, because it only went down to a -1.
36 MZ: How do you know it only goes down to a -1?
37 Brad: Cause that's the-- Yeah, because the slope-- Well, actually-- Uhm. Well, cause I guess I can tell here because the slope is one to one.
38 MZ: The slope--
39 Brad: The slope of f is one to one because f' is like one square for one up. That's the slope, I guess. I guess since we started here at 0 [initial condition point $(1,0)$] we're not going to be able to go down to 3 if we have a slope of--
40 MZ: So you have a slope of what?
41 Brad: I think 1.
42 MZ: You have a slope of 1, at 0 are you talking about?
43 Brad: At 0 you got uhm--
44 MZ: I mean not--
45 Brad: 0 [both laugh]
46 MZ: OK, we'll just you know-- Hold that thought. [short pause] Oh, does the original function have an inflection point?
47 Brad: Yes. I think. Uhm, Inflection. Is that when it's changing concavity or something?
48 MZ: Mm hmm.
49 Brad: Well, it's not really concave. It's a straight line. I mean, I guess it would but it wouldn't. I think. You know what I mean.
50 MZ: Uhm, are you talking about the f', the pointed one?
51 Brad: Is that what you were talking about?
52 MZ: I was asking you about the original function.
53 Brad: Oh f?
54 MZ: Yeah.
55 Brad: Yeah it does. It has two actually 'cause concavity ch-- Oh, actually there's one. Right here at $(1,0)$.
56 MZ: At $(1,0)$?
57 Brad: Cause it's concave up from -1 to 1. Then it's concave down from 1 to 3.
58 MZ: OK. Uhm, what would be the value of the second derivative at that point?
59 Brad: At where?
60 MZ: At the inflection point.
61 Brad: Uhm, 0?
62 MZ: How can you tell?

63 Brad: I'm a good guesser. [both laugh] Naw. I don't know. I think it's cause the slope, I think. It has something to do with that. 'Cause f' is like a maxima at the second derivative point. It tells you the change in slope doesn't it?
64 MZ: Mm hmm.
65 Brad: And since-- there's a slope there is -- 0, I guess-- [both laugh]
66 MZ: Uh, you didn't sound like you believed that.
67 Brad: No, I didn't.

Brad concentrates on his knowledge that the derivative function tells the slope of the original function. He knows that when the derivative is negative, the function is decreasing, and when the derivative is positive, the function is increasing [ln 27-29]. Using this information he can sketch a reasonable graph of the original function. Brad makes two misstatements. One is to say that the value of the second derivative at $x = 1$ is 0 instead of noting that it is undefined [ln 58-61]. The other is that on his graph of the original function he marks the y values of the extrema as ± 1 instead of the correct answer of $\pm \frac{1}{2}$. He does not use any techniques such as slope fields and areas that have been covered in class since the second interview and could be helpful in solving this problem.

Brad's overall performance on this problem is an improvement from his earlier work on the chapter 4 test. His knowledge may have improved in the two weeks between the chapter 4 test and the second interview where he claims that he could tell some information about the original function, but then did not have time to show this competence. His performance may also be enhanced by the *Mathematica* lab that the students completed after the second interview. As already noted, at the time of the lab Brad writes, "The activity on Friday gave me a better understanding of the way the function, the 1st and 2nd derivative compare. It showed me how the slope, and the max and min are related to one another."

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

68 MZ: OK. Well, we're just going to go on to the next question. What's a definite integral?

69 Brad: Isn't that when you find the antiderivative? It's like, that's what I think of it as. It's like-- I don't know.

70 MZ: Well, let me ask you this way. There's the indefinite integrals and the definite integrals?

71 Brad: Mm hmm.

72 MZ: What's the difference between the two? Or how do you describe what one is versus what the other is?

73 Brad: Maybe a larger uh-- Well, one I guess-- I don't know how to explain it. I don't know. I think maybe one's more of an abstract maybe, I guess. I think it would be the indefinite.

74 MZ: The indefinite would be the more abstract one?

75 Brad: I think. I'm not positive about that, but I think so because I guess definite would seem more-- You get more of an exact answer. Maybe not exact but-- I don't know. Maybe I'm--

76 MZ: Well, do you get-- What do the answers look like for a definite integral versus an indefinite integral?

77 Brad: See I forgot what an indefinite integral was.

78 MZ: OK.

79 Brad: So I know indefinite there's always a C which is a constant which is--

80 MZ: Oh, on indefinite, mm hmm.

81 Brad: Or is that definite, isn't it?

82 MZ: No, indefinite.

83 Brad: Is definite when you plug in values? Where you have like-- You take the antiderivative from something to something?

84 MZ: Yeah, that's the definite integral.

85 Brad: Oh, OK. So in that case you get more-- You do get more because you know the boundaries of the-- I think. --the boundaries of the thing that you're trying to find. Where in indefinite you don't exactly know so then you have to add C because you always have to add a certain margin of error. Got that?

86 Brad: Mm I can tell you. Just that the $\int$ has the things.

87 MZ: It just has the--

88 Brad: Yeah.

89 MZ: -- the upper and lower bounds.

90 Brad: Yeah. the upper and lower boundaries.

91 MZ: Yeah, OK. So then graphically what would be the difference between the two of them?

92 Brad: You would know I guess where to graph the definite because you have those boundaries where as indefinite you have the C. So all you know is maybe-- You don't know exactly where to put it. I guess.

91 MZ: You don't know--

92 Brad: You don't know where to start out.

93 MZ: OK. What's the definition-- I mean, not the definition. --the relationship of the derivative to the integral?

94 Brad: To integral. Integral, OK. Well, you use derivatives to find integrals. I guess, you use-- from what I understand. Not that I understand that well. I showed that. I guess, cause uhm-- Because you have a u and then a du and you put everything together, and it's kind of like you are using derivatives to find your answer.

95 MZ: OK. Do you remember what the Fundamental Theorem says?

96 Brad: [laughs]
97 MZ: At least sort of, I mean. Not exact wording.
98 Brad: I mean, I know there's like $F(b)$ minus $F(a)$. Isn't that it?
99 MZ: OK, so what does that one say, the one that has the $F(b)$?--
100 Brad: Oh, that's telling you the area of a graph, isn't it?
101 MZ: Yeah.
102 Brad: Doesn't it tell you the area under the curve of the graph which has to do with the stupid little box square things that I hated.
103 MZ: [laughs] the box square things--
104 Brad: Like the trapezoidal regions and all, the higher and lower boundaries.
105 MZ: OK. So which one is this?
106 Brad: This is the second one I guess.
107 MZ: OK, this is the Second Fundamental Theorem, and what does it kind of say in general?
108 Brad: This is $f(b)$, $F(b)$minus $F(a)$ which just tells you the area under the curve without having to plug in all the little rectangles to estimate. I think it's is an estimate. Isn't it an estimate? I think it's an estimation. I don't know.
109 MZ: With the rectangles you mean?
110 Brad: The rectangle's I think an estimation where the-- the other one's more exact.
111 MZ: the $F(b)$minus $F(a)$ part?
112 Brad: Yeah.
113 MZ: So that capital F, what's that supposed to represent, the fact that it's a capital F?
114 Brad: I think it's the-- Well, I know cause you use two f's, a little f and a big F. I think one's-- One's slope-- and one's the function. I think it's the ant-- Aw no, I don't know. I know it's-- Uhm--
115 MZ: Well, let me ask you a slightly different question. You have capital $F(b)$minus $F(a)$. What's that suppose to be equal to?
116 Brad: The area under the curve.
117 MZ: OK. How does the integral come into play there?
118 Brad: Because-- You have to find the-- Oh, you got-- I think cause you have to find the slope, but you take the derivative, I think, and you times that by something I know cause it's like-- It's x -- It's the distance times the height times like slope or times the function. So it's like $F(x)$ times x something. I think. I don't know.
119 MZ: OK. Do you remember what the First Fundamental Theorem says?
120 Brad: [short pause] No.
121 MZ: No? Do you remember anything about uhm-- You said that the integral had to do with area under the curve. Do you remember anything about an area function-- that had to do with integrals?
122 Brad: Like A_x equals--
123 MZ: Yeah.
124 Brad: Yeah, I think that is basically what I think I was explaining, but I had it the wrong way. It's like $A_x$equals-- Think of the distance, you know. I think, times the height. It has to do with like the rectangles. It's only an estimation.
125 MZ: Mm hmm.
126 Brad: So you got to like-- Something with slope, the height, the distance.

127 MZ: How would you write, just like as an equation or symbolically, what $A(x)$ is.
128 Brad: You want me to write it?
129 MZ: Sure.
130 Brad: I mean, I think it would be something like the distance which would be x --
131 MZ: Mm hmm.
132 Brad: --times the height which is I think-- I think it's-- I don't know what the height is actually, to tell you the truth. I don't know how you would calculate it. I think it's like h or something. I'm not sure. And then I think you take like $f(x)$ or something like that. I don't know. I'm lost. I forgot. It's something like that though. I should probably study it. [Somewhere in here writes: $A(x) = xf(x)$]
133 MZ: Uh, maybe. So does this area business have anything to do with derivatives or it just has to do with integrals?
134 Brad: Well, yeah because you have to use derivatives to find certain parts. 'Cause I know you got to find slope which would be the derivative of the actual function which you can use to find the area of something like that.
135 MZ: Do slope fields have anything to do with derivatives?
136 Brad: [short pause] Yes.
137 MZ: [laughs] Why?
138 Brad: Don't know. [laughs] I think, well isn't it chance or something. I figured it's got to have something to do with it. I was lost by slope fields actually. I know you use that-- Slope! 'Cause slope's the derivative of the function. I think--
139 MZ: Yes. [laughs]
140 Brad: Yeah, I think we've established that, I think. So basically you're telling the slope of that function at certain points and then by having your C value you know where to start out and you go to-- then not trace, but kind of trace around along that line.
141 MZ: OK.
142 Brad: Basically.
143 MZ: So, slope fields we've established have to do with derivatives because of the idea of slope. Do they have anything to do with integrals?
144 Brad: [short pause] Yes.
145 MZ: What? [laughs]
146 Brad: Uhm, well I guess because-- I know this one. Because in integrals you have that C.
147 MZ: That plus C part.
148 Brad: Yeah, indefinite integrals you have the C part.
149 MZ: OK
150 Brad: And so, the other part of that integral is basically telling you, I think kind of, the structure of a graph, I think. And then by adding the C you know-- So see so the slope fields would kind of like show you, I think
151 MZ: So when you said the other part of the integral, I wasn't sure what other part you were referring to.
152 Brad: Well there-- I mean, the C is just an end part where it's like added for error where to start out there's an actual equations part. You know what I mean.
153 MZ: OK. When you're describing that to me do you see an integral sign anywhere in what you're describing or is that sort of like--

154 Brad: Yeah I see one of those.
156 MZ: But I mean how does that fit in to what you just said?
157 Brad: I have no idea. I don't know. Like one of the equations, like that. You figure that out and you get like a plus C at the end. [writes:
$\int xe^x \quad + C$]
And so this would tell you like the structure, I think, of the graph and this would just tell you how to start out
158 MZ: For example in this, are you saying like the xe^x part would tell you the structure of the graph?
159 Brad: Yeah, I think. I'm pretty sure because then this-- this is only like a number, a numeric value.
160 MZ: The C?
161 Brad: So that only tells you where to start out.
162 MZ: OK. So I think you're done.

Brad associates definite integral with finding the antiderivative and plugging in values whereas indefinite integral is associated with having a $+C$ [ln 85]. His answer is computational. Similarly, his response about the relationship between derivatives and integrals, "you use derivatives to find integrals" [ln 94] is computational. Brad does not mention area until he is asked specifically about the Fundamental Theorem of Calculus [ln 100]. He knows that the Second Fundamental Theorem of Calculus involves the expression $F(b) - F(a)$ and that this expression is the area under the curve [ln 116]. Even though he is asked about the relationship of this expression to integrals, he never states that this is equal to an integral or that the function, F, represents an antiderivative. Instead Brad mentions slope and taking the derivative [ln 126].

Brad does know that areas can be approximated by using rectangles, and he seems to want to relate this idea to slope [ln 124-126]. However, when asked about slope fields, Brad concentrates his discussion on the $+C$, a value that affects where to start drawing a graph on a slope field, and relates slope fields to integrals by discussing the $+C$ involved in indefinite integrals [ln 143-152].

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: While it is recorded that Brad answered this question incorrectly, his exact response is not recorded.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Brad's exact answer is not available. It is known though that Brad answers the question correctly.

Test 9: Semester final

This test, which is a cumulative semester exam, covers all of the material on functions, limits, derivatives, areas, and volumes. The test questions are largely computational. Brad's semester final score is the lowest in the class, 42%. On the multiple choice portion of the final, he solves an optimization problem, a related rate problem, and two problems relating distance, velocity, and acceleration. He misses two derivative calculations, two integral calculations, and a problem asking for the equation of the tangent line at the point of inflection. On the portion of the exam that is not multiple choice, Brad fails to complete an optimization problem, misses another derivative calculation and can not find the inverse of $f(x) = \ln\left(\frac{x}{x-1}\right)$.

One error on the multiple choice portion indicates Brad's continued confusion with the relationship between limits and derivatives. The test question asks, "If $\lim_{x \to a} f(x) = L$, where L is a real number, which of the following must be true?" Brad chooses the only answer that mentions the derivative, "$f'(x)$ exists". This is reminiscent of his first interview misstatement, when asked how limit is related to derivative, "It's the derivative." In the second interview, Brad only refers to a limit when specifically asked and he is not sure what the relationship is. He writes, $\lim_{h \to 0} ? \frac{x + \Delta x - x}{h}$, noting that it's the Δx that is related to derivative. He is not sure whether the limit of something equals the expression with the Δx, or whether he should be taking the limit of the Δx expression.

QOTD #26

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: "is continuous everywhere except along $(0, \infty)$ Is differentiable along $(0, \infty)$" He includes a sketch of a graph which is increasing, concave down, and starting at the origin.

Interview 4

The discussion for the fourth interview is broken into four parts. The first section includes general questions about derivatives. The second part asks the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity, and acceleration, given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the fourth interview.

1 MZ: What is a derivative?

2 B2: You want like the definition where it's like $f(x + h)$ over $f(x)$ over h or do you want actually what it is? Do you want me to write it down?

3 MZ: First write that down, what you said, and then you can tell me actually what it is, what you just said.

4 B2: [writes: $\frac{f(x+h) - f(x)}{\Delta h}$] --over h. [marks out the Δ] That's the derivative of change. That just telling you like the change. Like if you had a graph that'd just tell you like the change in the slope of the graph. I think. I'm pretty sure.

5 MZ: OK. Anything else come to mind in terms of what a derivative is?
6 B2: In terms of what? Velocity, speed type stuff.
7 MZ: OK.
8 B2: Like for speed. Take the derivative of velocity and that gives you acceleration. No, displacement. Displacement, velocity, acceleration. I think.
9 MZ: What about displacement, velocity, acceleration?
10 B2: Well, you take the derivative of displacement. It gives you velocity. You take the derivative of velocity, it gives you acceleration. You write down an equation for them.
11 MZ: OK. Well, have you ever heard when people say the derivative is instantaneous rate of change?
12 B2: Yeah.
13 MZ: What do they mean by that?
14 B2: The idea of a point, say on a graph which I'll show you. [starts sketching a pair of axes with a smooth curve, inverted parabola type, and one point marked]
15 B2: They mean-- You find the derivative here, you're basically finding the slope for that point right there which is the change in the graph. Say if you had an equation like-- I don't know, negative, plus like x or something. I don't know if that's right. [writes: $-x^2 + x$] You take the derivative. Say this is displacement. You take the derivative of that, and that's the equation for the acceleration and the slope of the displacement. [has written: $-2x+1$] I think. I don't know. It's been a while.

Unlike previous interviews, during the fourth interview, when asked what a derivative is, Brad wants to know if the interviewer wants the definition or "actually what it is" [ln 2]. His definition includes the correct ratio, an improvement from the second interview, but the limit is missing. His description emphasizes change instead of rate of change, which is similar to the second interview. He also says that the ratio is the change in the slope instead of just the slope. His next answer to "what is a derivative?" is reminiscent of his first interview emphasis on velocity and acceleration. Brad's response to what is meant by instantaneous rate of change shows a similar range of answers as his responses to "what is a derivative?". He mentions slope, says the derivative is change instead of rate of change, and says (incorrectly after saying it correctly before) that acceleration is the derivative of the displacement function [ln 8]. There is no indication that he is explaining the rate of change interpretation of derivative as opposed to just talking about derivative in general.

In the fourth interview, Brad's understanding of derivative shows breadth, but it is also somewhat erratic with the misstatements that jump from one idea to another. This could be said of Brad's second interview statements as well.

The next part of the fourth interview is a summary of Brad's solution to the first of three problems involved in this interview. Given a table of values with x varying by .1, Brad is asked to find $f'(2)$, the derivative of the function at $x = 2$. Brad's first reaction is to graph the function given by the values in the table. From the graph he states that the derivative at $x = 2$ is "very large." He explains that the function "changes more" at that point and is "increasing more and more rapidly" so the "slope is larger." When pressed for a numeric estimate, he calculates that its value is approximately 14. When asked, he explains that he has calculated the slope, the change in y over change in x, for the points with x values 1.9 and 2.0. Brad says that he could find a different estimation using the points with x values 1.9 and 2.1. He says that the former would probably be more accurate since derivative is "instantaneous change" and "it would be a smaller area."

Throughout this discussion Brad uses a graphical viewpoint but discusses derivative in terms of both slope and change. His statement about a smaller area hints at, but does not explicitly state, a limiting process for finding a more accurate estimate. The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car took 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far did the car travel in the 2 hour period?

Brad's first reaction is that the distance traveled is 120 miles since the average speed is 60 miles an hour and the time traveled is 2 hours. To get the "average speed" he calculates the "change in speed over the change in time", 60 minus 0 over 1. Note that this is the average rate of change of the speed and not the average speed itself. When Brad is asked how the average speed could be 60 miles per hour when the car travels less than 60 except at one instant, Brad guesses that the average speed is actually 30 miles per hour making the distance 60 miles. He tries to explain his guess by sketching a graph of the velocity function. Even though he states that the velocity is increasing at a constant rate, he draws the graph as a concave down parabola instead of using linear pieces. He is not sure that the graph explains his guess, but he does say that the average velocity is the change in distance over the change in time.

Although Brad's statement about average velocity may be considered a statement related to calculus, Brad is asked if he can now use calculus to solve this problem. Brad suggests using an integral or a summation using the graph and the idea that velocity multiplied by time equals distance. He knows he wants to find the area under the curve. When he is unable to explain exactly what values should be involved in the sum, he decides to write an integral. He writes $\int_0^{120} f(x) \cdot t\,\Delta t$ explaining that 0 to 120 is the number of minutes involved, $f(x)$ describes the curve, and "to find the area it's $f(x)$ times $t\Delta t$". Perhaps he uses the redundant $t\Delta t$ because he sees $f(x) \cdot t$ as the area calculation and Δt as a symbol that one adds at the end of an integral.

The last part of the discussion of this problem concerns the interviewer leading Brad to realize that since the velocity is changing at a constant rate, the graph of velocity should be straight, not curved. Once Brad realizes this he states that he can find the area without using sums or integrals. One other error concerns his statement that the constant rate is "a mile per minute" instead of one mile per hour per minute.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Brad is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. He is asked if the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

As the interviewer begins describing the problem, Brad indicates that he is familiar with the problem type, "Oh, rate problems." Brad guesses that the rate the ladder is sliding down would be a constant rate. When asked to use math to solve the problem, Brad labels the wall as a and the floor as b and the ladder as c. He writes $a^2 + b^2 = c^2$ and says that he is taking the derivative to get $2ada + 2bdb = 2cdc$. Then he marks out $2cdc$ indicating that there is "no rate to change with that". Note that he does not use a ratio, e.g. $\frac{dc}{dt}$, for rate or derivative. Brad's previous work on related rate problems does not show this notational error. However, since the related rate test five months previous, the class has covered integrals which include the dx notation as a common element.

Brad continues by ignoring the actual question posed and suggesting what the interviewer may want to know. "You probably want to know the distance. I don't know if that'll come up later. Somebody might know the distance there or might be calculating the distance." He proceeds to explain how he could plug in numbers to $2bdb = -2cda$ and asks if he will be given a numeric value for db or for a and b. Even when the original question is restated, Brad asks to be given specific values for the distances. When the interviewer relents, Brad can calculate da for a given value of db and two sets of values for a and b. When asked, he acknowledges that da has a larger magnitude when the ladder is closer to the ground, so the rate is increasing.

Brad's problem solving attempt does not focus on answering the interviewer's question about the relationship of the two rates, but seems to focus on a familiar procedure for solving problems with scenarios of this type. Related rate problems with the sides of a right triangle as the principle construction were likely common in Brad's junior year study of calculus and are definitely assigned and tested in his senior year study of calculus. In these traditional problems the student is usually asked to find a numerical value for one rate given numerical values for the other quantities involved. Brad seems to be operating under the assumption that all quantities in this problem will have a single numerical value.

Interview 5

Brad's fifth interview occurs almost one week after he takes the BC version of the AP exam. During that week the class discusses the written questions from the BC version. Between the fourth and fifth interviews the class studies series and integration techniques and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Brad's response to a set of questions on a particular topic, and provides an analysis of those responses.

1 MZ: OK. You were just about to tell me what a derivative was.
2 Brad: The definition?
3 MZ: I don't know. What do you think is the most important thing about what a derivative is?
4 Brad: Give you some points on the AP test. [laughs] I don't know. It's helpful. That's important. Help you find things faster. Help you also to check things.
5 MZ: Like what?
6 Brad: Like you can find velocity from an acceleration equation or the change, whatever. The things you get from the derivative thingy.
7 MZ: You were going to tell me the definition. You said you knew that.
8 Brad: [writes: $\frac{f(x+h)-f(x)}{h}$] Is that right?
9 Brad: Yeah, it is. Oh, wait a second. Sorry. [writing]
10 Brad: Yeah, I think that's right. I don't know. [it now reads: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$]
11 MZ: You think that's right. What caused you to add this on there [referring to $\lim_{h\to 0}$]
12 Brad: Yeah. 'Cause I think derivative deals with slopes of graphs.
13 MZ: OK.
14 Brad: And so you're taking a point, say 0. [short pause] Wait, I'm thinking. I don't know. That's just what I learned. [laughs]
15 MZ: Does this formula thing that you just wrote down have anything to do with slope?
16 Brad: I think so. Yeah, maybe.
17 MZ: Like what does it have to do with slope?
18 Brad: It'll help you tell what the slope is.
19 MZ: OK.
20 Brad: I think. As it approaches a point. I don't know. h doesn't always have to be 0. I think it could be a variable. I don't know. Maybe it does.

21 MZ: [short pause] Is it possible for you to draw a graph that would show what like $f(x)$ was and $f(x+h)$ and the difference?
22 Brad: No. I don't know.
23 MZ: You don't know. OK.
24 Brad: That's just something I learned from the book. That's basically why I know it. I think that's the derivative.
25 MZ: Does derivative have anything to do with rate of change?
26 Brad: Yeah.
27 Brad: It tells you what it is.
28 MZ: It tells you what the rate of change is?
29 Brad: Yeah, 'cause uh-- Like the rate of change in a graph. Say you had a displacement graph or showing the increasing or whatever. Then you took the derivative of that. That gave you the velocity or the m or the slope is the velocity. So that's the rate of change. OK.
30 MZ: Does the derivative involve a limiting process?
31 Brad: Yeah, 'cause as it approaches a certain-- 'Cause when you deal with the slopes, sometimes you have things in the graph like cusps or something where the derivative won't exist because the-- As you go to a certain point say like 0, something's at zero. There's a cusp at 0.
32 MZ: He's hand-motioning a cusp.
33 Brad: Yeah. It's a cusp. The derivatives or the slopes of the graph, they get larger instead of getting smaller, going to zero.
34 Brad: Or being, reaching one point where they reach it too fast and it doesn't exist. I don't know why I said that.
35 MZ: OK. Can you explain-- You said derivative has to do with a limiting process--
36 Brad: Yeah, as it reaches a certain point.
37 MZ: Can you explain the limiting process in terms of derivative being velocity?
38 MZ: Does that have anything to do with a limiting process at all?
39 Brad: Uhm. Not that I can say, I guess. I don't know.
40 MZ: But it does with the slopes because you were describing how it--?
41 Brad: Well, I guess it might if that's the derivative of displacement. I don't know how. But I would assume that since it works for something else that deals with derivative that it would probably, somehow.
42 MZ: OK. Let me ask you a different question.
43 Brad: Yes.
44 MZ: Why?
45 Brad: The derivative of a function is a function. It can be. It's not necessarily. Well, it should be. Let me think. Well it'd have to be because you couldn't have a graph. I think.
46 MZ: Because you can have a graph of the derivative? Is that what you meant when you said that?
47 Brad: Yeah. By taking the derivative you can graph that. And if you can't have a graph it won't be a function. I think. See I'm not too good with all this math stuff.
48 MZ: OK. You've used this formal definition and you've mentioned slope and velocity and we talked rate of change. Can you explain what a derivative is in a different way than those ways or in a different context?
49 Brad: I don't know. I just think derivative's basically the rate of change. That's how I look at it. I don't think I can say it in another, describe it in another way.

50 MZ: Can you say something else that derivatives are useful for or a different kind of situation besides velocity where derivatives come into play.
51 Brad: Max/min problems, change in rate like volume decreasing. [short pause] I said max/min, right? Yeah. Integration. It's more of an anti-- Are we dealing with antiderivatives or just derivatives?
52 MZ: Oh. Well, I was more thinking of derivatives, but you're talking about antiderivatives with integration?
53 Brad: Yeah, we can use that. Sums. That's all I can basically recall.
54 MZ: OK. [pause] What is meant by a differentiable function?
55 Brad: [pause] If there's-- I know this is basically what we said before about the cusp.
56 MZ: Right.
57 Brad: I think the limit or the derivative-- It deals with something-- It's either the limit or the derivative exist throughout the whole function. I think continuous. It deals with continuous. [short pause] Let me think. There's a process I know that it goes with. That's what I remember for being differentiable, with the limit. I don't know. I forgot everything since Wednesday or whenever we took the test.
58 MZ: Yeah, right.
59 Brad: Has, I mean-- Nah.
60 MZ: How about this, can you give an example of a differential function and give an example of a non differentiable function and just say what the difference is between them? What makes one differentiable--
61 Brad: Well, I think it's whether the derivative exists throughout the whole thing.
62 MZ: OK, so what would be an example of a nondifferential function?
63 Brad: Where there might be a cusp.
64 MZ: Do you happen to know an equation for something like that?
65 Brad: For the cusp. [pause]
66 MZ: Or just in general an equation for something that's not differentiable at some point?
67 Brad: I think you can have one with a hole too, can't you, where it's not differentiable.
68 MZ: Yeah
69 Brad: I don't know about a cusp.
70 MZ: OK. What's an example of a function that is differentiable?
71 Brad: Like x 's-- Well, no. x^2, I guess, or x , $y = x$ Not too hard, huh. I guess.

In the fifth interview, as in the fourth, when Brad is asked, "What is a derivative?", he responds with a question about what kind of answer the interviewer wants. The interviewer responds by asking him what he thinks is most important, and he discusses velocity and acceleration, postponing his statement of the formal definition (see Table A.6). In the fourth interview he states the two in the opposite order because the

Table A.6. Brad: Interview 5 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?		∘	∘ ∘		acceleration misstatement (d=change)
Did you learn a formal definition of derivative?	∘			◉	
How does the formal definition of derivative relate to slope?	∘				
Is derivative related to rate of change?	∘	∘	∘		
Does the derivative involve a limiting process?	● ∘				
Is the derivative of a function a function?	◯			↦	
Explain what a derivative is without mentioning slope, velocity, rate of change or the formal definition.		∘			
What are derivatives useful for?		∘			max/min misstatement (change in rate)
What is meant by a differentiable function?				◯	
Asked to interpret the Mean Value Theorem.	●			●	
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.	◎	◎		↦	misstatement (d=change) in/decreasing max/min incorrect calc
Summary	◉	◎	∘	◉	

interviewer said he could state the definition first. In the initial discussion Brad also mentions that the derivative is change, not rate of change, and that the derivative is slope.

Even though the portion of the fifth interview focusing on general questions about the derivative has fewer questions than similar sections of the first and second interviews, Brad gives almost as complete answers here as in the second interview. The principal omission is that in the fifth interview Brad does not discussed the instantaneous nature of rate of change and the details of the rate of change ratio as he does in the second interview. Improvements from the second interview include a correct statement of the formal definition, which is also correct in the fourth interview, a mention that the derivative is a function because you can graph it, and a brief statement that slope, velocity, and rate of change are all equivalent. One carry-over is the misstatement that the derivative is a change, as opposed to a rate of change.

For the second part of the fifth interview Brad is asked about the Mean Value Theorem. Initially he thinks of the average value calculation that involves integration. He then remembers that the Mean Value Theorem involves a c between a and b but must be reminded of the full statement including $f'(c) = \frac{f(b) - f(a)}{b - a}$. When Brad is asked to interpret these symbols, he initially restates them without giving any graphical or contextual interpretation. When pressed for a graphical interpretation, Brad admits that derivative tells the slope and that the ratio looks like it could be an average slope, but he never states that the slope of the line connecting a and b is the slope of the function or tangent line at c. Brad is not asked for any further interpretations.

The next question on the fifth interview involves a problem from the AB version of the AP exam which Brad has taken. The question is as follows:

Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.
(a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0, 1]$ to approximate $F(1)$.
(b) On what intervals is F increasing?

(c) If the average rate of change of F on the closed interval $[1,3]$ is k, find $\int_1^3 \sin(t^2)\,dt$ in terms of k.

Brad first comments, "That's one of the two [problems on the AP exam] I couldn't do." He says that he only tried part (b), but he doesn't remember whether he tried to find where $\sin(t^2)$ is negative or positive or whether he tried to find where the antiderivative of $\sin(t^2)$ is negative or positive. He thinks that he should have done it for the antiderivative. His error is surprising since he has in the past correctly used the fact that a function is increasing when its derivative is positive. In this case Brad may be confused by a function defined as an integral, thinking that the only way to work with an integral is to calculate the antiderivative. Brad does not attempt to solve parts (a) and (c).

The next section of the interview concerns Taylor polynomials. Brad is not able to answer any of the interviewers questions on this topic. Brad says that the only thing he remembers is that the class did Taylor polynomials, "on the graphing calculator a long time ago." For Brad it is true that his last exposure was "a long time ago"; it was seven months previous to the fifth interview. The rest of the class revisited Taylor polynomials with Taylor series for one week approximately five weeks prior to the AP exam. However, Brad spent the last five weeks reviewing with the AB calculus class to help him prepare for taking the AB version of the exam, which does not require knowledge of sequences and series.

The final section of the fifth interview concerns a function, f, that at any time x, given in hours, that tells the outside temperature in degrees Fahrenheit. Brad is shown a series of symbolic expressions and asked what information each one provides about the outside temperature.

For $f'(3) = 4$ Brad first mentions slope, but then says, "3 hours later the change in temperature would be 4 degrees." When asked about the units he clarifies his answer to say, "It's changing at a rate at that time of 4 degrees per hour." He also asks if the function is the equation of a line. For $f''(3) = -2$ Brad explains that the second derivative means concavity so maybe the graph is concave down. He continues by saying that the first derivative is decreasing while the function is increasing, "So it's very complex."

When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Brad replies that the equation is $f(x) = 4x + C$. When asked what this means in terms of temperature, Brad says, "The average change in the temperature is 4 degrees, I guess, because that would be the slope of the function." When given the initial condition of $f(0) = 50$, he knows that $C = 50$ and he can state that the temperature at 3 hours is 62 and the temperature at 2 hours is 58. When asked to interpret $f''(x) = -2$ for $3 \le x \le 6$, Brad says, "The equation changes." He writes $f'(x) = -2x + 4$ and is unsure of his answer, but does not recognize that he should have used $f'(3) = 4$ to get $f'(x) = -2x + 10$. When asked to discuss the temperature, he at first says that it is decreasing but then changes his answer to, "The change in temperature is decreasing so maybe it's not increasing at the same rate."

Brad uses $f(3) = 62$ and the second derivative information to estimate $f(4) = 64$. When asked for the location of maximum and minimum values, Brad initially thinks that the maximum occurs at $x = 6$ since he believes that the function will continue increasing at a decreasing rate. Upon further questioning, Brad thinks that the maximum may occur at $x = 4$ or $x = 5$ since the change in values becomes 0 at $x = 5$. Brad thinks that the minimum would occur at $x = 0$ since the function is 50 at $x = 0$, but in the 60's for 3 to 6.

Brad talks about each of the symbolic expressions he is given in terms of temperature, but his first reaction to the expressions seems to initially be graphic, such as slope and concavity, or symbolic in terms of finding antiderivatives. Brad speaks of the change in temperature instead of the rate of change, but this does not prevent him from stating reasonable numeric estimates showing that he knows that the derivative is the change per hour.

Case Study 3 — Carl

Academic record

*National Merit Scholar.

*Other AP courses: US. History (junior year), European History, Chemistry.

*Writing tutor at the high school writing center.

*Math team participant.

*Plans to major in accounting in college.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: "It's an equation with like variables that you can evaluate and use to do calculus type stuff."

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different?

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: "a) $y = x$ This is easy to graph.

$y = \sqrt{9x - 4\left(\sqrt{2x^{\frac{2}{3}}}\right)\left(\frac{92}{x^2\sqrt{3y}}\right) - 1}$ This is nearly impossible.

b) $y = 0$ I don't know, it's a guess."

QOTD #3

Give an example of a function <u>without</u> using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Date: August 26, 1993. This question occurs while the class is doing a quick review of functions.

Response: Carl provides two sketches of graphs on a standard x-y axis. The first sketch is a parabola which is approximately $y = x^2+.5$ and the second is approximately $y = -x^2-.5$. He also writes, "Or you could say "hey" that's a function if the lines you draw vertically don't cross more than one point on the graph type thing."

Comment: On this question, he shows that he knows to express a function as a graph and that he is aware of the vertical line test for functions.

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) non integers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: For both parts of the question Carl responds, "It's too early to answer these. Yes?"

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to class discussion on limits.

Response: "A limit of a function, f, at a point $x = a$ a slope of the function at this point."

Test 1

On the test on limits, Carl is able to find limits correctly by reading values from a graph, by substituting into a piecewise function and by using algebra to simplify a limit calculation. However, he is not able to work with the formal definition of limit to find a δ for a given ε in either a symbolic or graphical setting.

Interview 1

This interview occurs after the test on limits but prior the class discussing derivatives. Therefore Carl's answers are presumed to be based on what he remembers from his junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table A.7, which codes these responses. A summary discussion follows.

1	MZ:	What is a derivative?
2	Carl:	Oh, you know it's like the thing you take. Well, you can use a derivative to find the velocity of something.
3	MZ:	OK.
4	Carl:	I think of it like that. It's a-- It's like-- I don't know exactly. I could say what I use it for. I couldn't tell you the definition of it.
5	MZ:	Well, tell me some other things you use it for.
6	Carl:	Like on a test when they say, "Take the derivative of this function." You say, "OK." and you move the exponents down and you subtract 1 and then you do that. I know all the rules, the quotient rules and the chain rules and all those funny little things.
7	MZ:	Well, what other kinds of either word problems or real world situations would it be useful?
8	Carl:	Oh, you mean like-- Like the boats going away from the dock, and you got the pulley on the boat, and then you got to find the acceleration of the boat and the tension and all that weird stuff.
9	MZ:	So which part does the derivative fit in?
10	Carl:	Well, the first derivative for the velocity, the second derivative for the acceleration.
11	MZ:	What about an example of its being useful for something that's not velocity or acceleration?
12	Carl:	Could you give me an example?

13 MZ: Well, maybe some of these later ones will jog your memory. Say I give you a function. How can you tell if it's differentiable?
14 Carl: I knew this last-- Before last summer. I know, but I forget.
15 MZ: What pops into your mind as being related to that?
16 Carl: Something that's not differentiable. [pause] Well, I know a constant's not differentiable, right?
17 Carl: Like $y = 0$.
18 MZ: So why wouldn't that be differentiable?
19 Carl: Because you can't take the derivative. Or isn't it 0? You see, it's weird. It's like a constant always goes to 0 when you take the derivative of it.
20 MZ: OK, the derivative's 0.
21 Carl: I can't think of anything that's not differentiable.
22 Carl: Uh, like $x^2 + 1 = x$ Really easy things like that.
23 MZ: OK, if I give you a function like, just-- $x^3 + 5x^2$. Is it differentiable?
24 Carl: Yeah.
25 MZ: And how do you know? Why is it so obvious?
26 Carl: Because you take the 3 and--
27 MZ: Because you know a method or a way--
28 Carl: To do it. There have been some that I for the life of me I can't get because they use 3 quotient rules and 2 chain rules all on the same thing.
29 MZ: Here's another question. Say you needed to explain what a derivative is to someone who is like a precalc or AB student.
30 Carl: I actually did this. There's someone in precalc who came in for help. And they actually wanted to know how to do this, and I said, "I know how to do this. I can show you how to do the problem, but if you want to know why. I can't help you."
31 MZ: And you mean take the derivative problems.
32 Carl: Yeah, and like if they wanted to know why you do this, I don't know. They just tell you to do it. Honestly I've never actually gotten into math that much to find out a lot why they do it.
33 MZ: What if someone who doesn't know much about math wants to know, "Well, how are you doing in math? What's a derivative? I see it here in your book."
34 Carl: OK, you have a function. If they didn't know what a function is, I'd have to say, "OK, you have a line, right." OK, you have a line on a graph. If they don't know-- I would actually have a real hard time telling people just because to me, like I said, I personally don't even know the correct definition or why they work. I just know they do work, and I know what you can use them for. So if I had to actually explain this to somebody, I would just say it's helpful in solving problems, to find different velocities and accelerations or to find-- [pause] other things.
35 MZ: Well, let's do this next section. There's a word and then you say if it's related to derivative and if so how. Slope?
36 Carl: Sure. Yeah. Let's see. Slope. Uhm. OK, the derivative is the--
37 Carl: Derivative is the tangent line to the function, isn't it? Isn't it?
38 Carl: It has to do with the tangent and the slope to the graph.
39 MZ: Make a sentence.
40 Carl: The derivative is the slope of the tangent line to the graph. Something like that. Is that close?
41 MZ: Yeah, that's close. OK.
42 Carl: But what is it really? Tell me, please.
43 MZ: No, actually that sounds good to me.
44 MZ: Fine. Speed or velocity?

45 Carl: The derivative of something. If you have an equation, to find, when you plug in, the x in the new equation, find the velocity of the thing in the old equation, the distance type thing.

46 MZ: Change or rate of change?

47 Carl: Like if you have a beaker full of water with a hole in the bottom, and it's leaking out, and you have water being poured, you can take the delta of the amount of water in there and take the derivative to find out how much is leaving and how much is coming in. Stuff like that.

48 MZ: So what's the delta supposed to represent?

49 Carl: The change. The rate of change.

50 MZ: So when you're thinking of delta are you thinking of this triangle guy [draws a Δ]?

51 Carl: Yeah, the triangle guy.

52 MZ: The change, OK. So how does this delta change relate to derivative? How does that fit together?

53 Carl: If you were to take the derivative, like x , you'd end up with like just $\frac{dx}{dt}$ or delta x basically, the change of x over time. So that's basically like Δx.

54 MZ: So this $\frac{dx}{dt}$ –

55 Carl: --is like delta x.

56 MZ: Is like, is sort of like equal to-- [writing $\frac{dx}{dt} = \Delta x$]

57 Carl: kind of equal to

58 MZ: And how does the derivative fit in?

59 Carl: When you take the derivative of something you find the change in that.

60 Carl: So, you end up with an equation without an x in it, unless you take x^2, but without an x in it with a Δx instead. And it changes the equation from how much water is in it to how much it's changing at that instant, how much is leaving or going in at that instant. It's an instantaneous rate of change.

61 MZ: The derivative is?

62 Carl: Yeah, I think.

63 MZ: Which of these little symbols is suppose to be the derivative, or all of them are?

64 Carl: The d. [laughs]

65 MZ: The d, it means derivative? But the delta is not the derivative, that's something different?

66 Carl: Kind of-- [pause]

67 MZ: Let's go on. Line or linear?

68 Carl: The derivative of a function is always one power less so if we had parabola the derivative's a line. And it's the line, the tangent line, it's the slope of the tangent line is the derivative, so the tangent line to the graph is the derivative as well, they're connected somehow--

69 MZ: OK, next one. Measurement?

70 Carl: To measure the rate of change of something, the acceleration or velocity is the derivative. It's just like an application to measure something. Other than just it being useless it has a measurement value.

71 MZ: Prediction or approximation.

72 Carl: Well, I kind of think of like limits and stuff.

73 MZ: OK. How are you thinking?
74 Carl: I'm kind of thinking about how you take, let's say you take the derivative of something. Anything with an asymptote basically. It never really gets there, but you're saying, well it's about there. It's approximately 0, if it's going to 0. Or if you-- [short pause] I really don't know what it has to do with derivative.
75 MZ: OK. Optimization. [short pause]
76 Carl: The maximum.
77 MZ: Could be like the maximum.
78 Carl: OK, yeah. [pause] Well, you can find the maximum and minimum points of the graph, the local max, local mins, with the derivative, setting it equal to zero and stuff.
79 MZ: OK.
80 Carl: And you can find the-- like the greatest number of things produced to test things with those derivative problems.
81 MZ: OK. Why does setting the derivative equal to zero help give you the--
82 Carl: 'Cause it's the root of the equation.
83 MZ: It's the root of what equation?
84 Carl: Well, no, wait. Derivative equals zero. See this is another why question. I know you're suppose to. Now let's see if I can remember why. 'Cause you take-- Yeah, you find the roots of the equations, where it crosses the graph.
85 MZ: OK. And this is of which equation?
86 Carl: The derivative equation.
87 MZ: OK.
88 Carl: And that's going to be the maximum or minimum points because it's the tangent line. The tangent's going to be equal to zero at a turning point. So those could be the maximum and minimum points of the graph.
89 MZ: Why does it turn out to be the case that the derivative is equal to zero at a maximum point?
90 Carl: Because-- OK. Like uh, if the maximum points like at 4--
91 MZ: $x = 4$ you mean? Or $y = 4$?
92 Carl: $y = 4$. The maximum point's at 4. It's like a parabola. The maximum point's at 4.
93 MZ: $y = 4$, yes.
94 Carl: The tangent line is $y = 4$ if it turns, the maximum point's there.
95 MZ: OK.
96 Carl: So, that's the maximum point on the graph, and when you take the derivative, the derivative's root, cause that's 0, is going to be-- [motioning with his hands]
97 MZ: Because what's 0?
98 Carl: This slope of that line. [motioning with his hands]
99 MZ: --horizontally.
100 Carl: the slope of the derivative is 0. So when the derivative is 0, the slope is 0 and that's the maximum or minimum point..
101 MZ: OK. I think I got it.
102 MZ: Continuity?
103 Carl: Well, they're continuous.
104 MZ: What's continuous?
105 Carl: Derivatives?
106 MZ: Derivatives are continuous.
107 Carl: Maybe? [pause] I don't know. That's a tough one.

108 MZ: Uh, do you think it has something to with derivatives or do you think that it probably doesn't have that much to do with it?
109 Carl: [pause] [shrugs]
110 MZ: OK. Next question. Limit?
111 Carl: [pause] I don't know if-- The limit is a real separate thing from derivative 'cause I never seem to use derivatives when I'm doing limits.
112 MZ: OK. Integral. You probably didn't study that yet.
113 Carl: No, we haven't studied that.
114 MZ: Function?
115 Carl: Like a function. Take the derivative of a function. Everything I've basically been saying for the last 10 minutes.
116 MZ: OK, let me ask a different way.
117 Carl: OK.
118 MZ: Is a derivative a function?
119 Carl: Yeah. If you graph the derivative of a function, of any function it's going to be there. Yeah it's another function. It makes another function out of a function. I guess.
120 MZ: Did you learn a formal definition of derivative?
121 Carl: I'm sure we did, but I have no clue. I couldn't even start.
122 Carl: I mean, I know those things aren't that hard. That's why I was really upset. I don't know those things. I know my own definition in my head of what they are, what they do, and I can do problems like that, but when a teacher's asking for a formal definition I go crazy.
123 MZ: So what do you consider your definition for derivative? What did you just point to, the 'derivative is the slope of the tangent line'? [pointing to first line of text MZ had written on the notebook page]
124 Carl: Mm hmm. That's all I can think of. It's more. I mean, it is more, but that's the basic, what it's used for. It's the slope of a tangent line to a function.
125 MZ: OK.
126 Carl: That's all I can think of. I can think of that, and I can think of velocity and acceleration with the second derivative.
127 MZ: Yeah. OK.

Table A.7 summarizes Carl's first interview transcript. Carl's two principal models for the concept of derivative during the first interview are velocity and the symbolic rules (e.g. power, quotient, product) for taking derivatives. It is not until he is asked specifically that he also remembers that derivative is related to slope and rate of change.

Carl does not remember the symbolic difference quotient definition of derivative. He only states the instantaneous nature of derivative in terms of instantaneous rate of change and writing the symbols $\frac{dy}{dx}$. He never describes the details of a ratio process. He only mentions a limiting process in terms of predicting values graphically, admitting

Table A.7. Carl: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?			○	↦	
What can derivatives be used for?			○ ○	↦ ↦	acceleration
How can you tell if a function is differentiable?				↦	
Explain what a derivative is to someone who doesn't know anything about math.			○ ○		acceleration
Is derivative related to slope?	○				misstatement (d=tl)
Is derivative related to speed or velocity?			○		
Is derivative related to change or rate of change?		○ ◎		↦ ◎	misstatement (d=change)
Is derivative related to line or linear?	○			↦	misstatement (d=tl)
Is derivative related to measurement?		○	○ ○		acceleration
Is derivative related to prediction or approximation?	●				misstatement
Why does setting the derivative equal to zero give you the location of a maximum?	•				misstatement (d=tl)
Is derivative related to limit?				↦	misstatement (not related)
Is the derivative a function?				↦ ◯	
Did you learn a formal definition of derivative?	○		○ ○		acceleration
Summary	◉	◎	○	⦾	

that he is not sure how this relates to derivatives. He guesses that a derivative is a function based on the example that $y = x^2 + 1$ has a derivative that is a line, and lines are functions.

Carl makes only a few weak connections between different models for derivative. He knows rate of change is related to the symbolic notation $\frac{dx}{dt}$ for "change of x over time" [ln 53]. He mentions "rate of change" and "acceleration or velocity" [ln 70] in rapid succession indicating some unstated relationship. Similarly he mentions slope of the tangent line and velocity and acceleration together as being related to derivative but does not describe how or why they are related [ln 45].

Carl makes two major types of misstatements regarding the derivative, one regarding the tangent line and the other regarding rate of change. When Carl is asked whether derivative is related to slope he says, "Derivative is the tangent line to the function" [ln 37]. He is unsure of his answer and without prompting changes it to "derivative is the slope of the tangent line to the graph" [ln 40]. When asked if derivative is related to line or linear, Carl is again unsure whether to say tangent line, slope of the tangent line, or both: "The derivative of a function is always one power less so if we had a parabola, the derivative is a line. And it's the line, the tangent line. The slope of the tangent line is the derivative. The tangent line to the graph is the derivative as well. They're connected somehow" [ln 68] On another occasion, when explaining optimization problems, Carl emphasizes tangent when he should say slope. He explains that the derivative equation is set equal to 0 and solved, "and that's going to be the maximum or minimum points because it's the tangent line. The tangent's going to be equal to 0 at a turning point" [ln 88]

Carl is also confused about whether the derivative is the change or the rate of change or both. He states that a capital delta represents change and that taking the derivative gives you $\frac{dx}{dt}$, change in x over time, but then seems to equate Δx and $\frac{dx}{dt}$ [ln 53]. He also states that "when you take the derivative of something, you find the change in that" [ln 59]. He both corrects this misstatement and reiterates it with the following: "[taking the derivative] changes the equation from how much water is in it to how much it's changing at that instant, how much is leaving or going in at that instant" [ln 60].

QOTD #6

Find the derivatives of the following four functions:

$f(x) = (x-1)^2(x^2-4)$

$g(x) = \dfrac{x-1}{\sqrt{5-x^3}}$

$h(x) = \sin x$

$j(x) = \ln x$

Date: September 20, 1993. This question occurs prior to the class learning about short-cut rules for taking derivatives of various forms.

Response:

$f'(x) = (2)(x-1)(1)(x^2-4) + (x-1)^2(2x)$

$g'(x) = (5-x^3)^{+\frac{1}{2}}(1) - (x-1)\frac{1}{2}(5-x^3)^{-\frac{1}{2}}(-3x^2)$

$h'(x) = \cos x$

$j'(x) = \mathrm{e}$

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be using your knowledge of derivative.

$y = \log(x^3) \qquad y = \dfrac{x}{|x|} \qquad y = x^e \qquad y = e$

Date: September 21, 1993. This question also occurs before the class studies short-cut rules for taking derivatives but after they have studied the limit definition of derivative.

Response: Carl is absent on the day the class answers this question.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$?

Why not?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: "a) No Because

b) $\csc x$ "

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. Carl incorrectly states the definition of the derivative as $\lim\limits_{x \to 0} \frac{f(x + x_0) - f(x)}{x - x_0}$ and is unable to correctly interpret $f(x + x_0)$. However, his performance improves on questions where he is given a graph. Given the graph of a position function for a car he correctly answers questions about the speed and direction of the car. Given the graph of a function, he is able to sketch a correct graph for the derivative function.

Carl has more difficulties working through two complex chain rule derivatives. For $f(x) = \cos(3x^2 + 4)^5$ he finds the derivative to be $-5\sin(3x^2 + 4)^4 \cdot 6x$. For $f(x) = \tan^{\frac{3}{2}}(2x + 1)$ he finds the derivative to be $f'(x) = \frac{3}{2}(\tan(2x + 1))^{\frac{1}{2}}(\sec^2 x)(2)$.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: "The formal definitions are alot more clear and using derivatives in respect to trig function derivatives. It's overall a great class."

Comment: Carl's mention of "formal definitions" is curious. On the most recent test, Carl is still unable to state the definition of derivative correctly or use the difference quotient to estimate values. Perhaps in using the plural "definitions" Carl is referring to the rules for calculating symbolic derivatives. However, Carl has difficulties with these as well.

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "a) I was totally spaced out yesterday and I forgot what we did. I think I knew how to do the stuff though." After referring to his notes, "b) We learned about instantaneous rate of change which, as I stated previously, have learned before."

Comment: Since the researcher had not been present the day prior, this question is presented both as a means for the researcher to see the material covered and to ascertain the students' understanding of it.

Test 3

This test covers Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Carl correctly calculates a third degree Taylor polynomial, and he is able to use the first and second derivatives of the position function to find the speed and acceleration of an object at a given time. However, when given the graph of the derivative function and asked about the original function, Carl has more difficulty. For the critical numbers of the original function he lists not only where the derivative is 0 but where the derivative has local extrema. He correctly states where the original function is increasing and decreasing, where it has local maxima or minima and where the original function will have inflection points, but his answers for where the original function is concave up or concave down seem to be at the locations where the second derivative is concave up or concave down respectively. His answer includes a sketch of a graph that appears to be the derivative of the derivative graph given.

QOTD #11

Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: "People having kids and the rate at which the population is increasing."

Test 4

Two weeks later the class has a test on the applications of derivatives. Carl correctly uses derivatives to solve three traditional max/min problems. He also correctly calculates the derivative of an implicitly defined function. He correctly solves two traditional related rate problems, but on a third problem he takes the derivative with respect to time of $V = \frac{\pi}{3} r^2 h$ and gets $\frac{dV}{dt} = \frac{2\pi r}{3}\frac{dr}{dt}\frac{dh}{dt}$ failing to apply the product rule.

Interview 2

The second interview occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by Table A.8, which summarizes the interview, and a discussion.

1 MZ: What is a derivative?
2 Carl: This time I know. It is the slope of the tangent line to the curve.
3 MZ: Anything else come to your mind for what is a derivative in addition to that?

4 Carl: The maximum or minimum values of a profit or anything. Basically it helps to find maximums because the peaks of graphs become the zeros of the derivative. So setting the derivative equal to 0 you can find a lot of neato stuff.

5 MZ: How come the peaks of the graph are the zeros of the derivative?

6 Carl: Because they say so. I'm not joking. I have no idea. Because when you take the derivative and set it equal to 0, its the maximum or minimum value.

7 MZ: OK. Well, you already started on this but-- What can derivatives be useful for?

8 Carl: Maximum or minimum. Velocity. You know, real life problems.

9 MZ: How would you explain a derivative to someone without much math background?

10 Carl: I wouldn't. That's why I'm not the teacher.

11 MZ: What's an example of a real world concept that involves the concept of derivative?

12 Carl: Related rate stuff. You know, if a ladder's falling over your head, you want to know how fast it's going to hit you. You can sit down and work it out and find out when it's going to hit you. Stuff like that. Well, like in business, what I'm going to go into, finance and stuff. Maximize profit, minimize cost and stuff like that.

13 MZ: In the related rate example, what part of that example is the derivative or how does the derivative fit into that?

14 Carl: Oh. 'Cause there's a ruler on the floor and you see how far it's moving every second on the bottom and you want to see how fast it's moving down at you. You can tell the relationship of the change in y to the change in x .

15 MZ: So that was the derivative part, the change in y with the change in x ?

16 Carl: [acknowledgment]

17 MZ: Say I give you a function. How can do you know if it's differentiable?

18 Carl: You try a lot of different things, and you see if it works.

19 MZ: What kind of things were you thinking of?

20 Carl: Like x^3. You take the 3 and you put it in front and you have a 2.

21 MZ: x^3 is a function and you're saying you can tell if it's differentiable or not because you can take the derivative of it.

22 Carl: [acknowledgment]

23 MZ: What if I gave you the graph of a function, is there a way you could tell if it's differentiable?

24 Carl: No, I don't think I could do it. I know it has to do with all those-- You gave me the graph of a function?

25 MZ: I give you the graph of a function and I say, is this function differentiable?

26 Carl: If it is a-- I don't know.

27 MZ: OK. Could you give me an example of a function that's not differentiable?

28 Carl: [Negative.]

29 MZ: All functions are differentiable.

30 Carl: I can't think of any off hand that aren't. Oh. [short pause]

31 MZ: What were you thinking when you said, Oh?

32 Carl: I was thinking like this-- 'Cause to be a function-- I can't think of any. I keep thinking of a vertical line, but it's not a function.

33 MZ: What is it you were thinking about a vertical line that it wouldn't be differentiable?

34 Carl: Because the slope of the tangent line is undefined.

35 MZ: Yeah. OK. Here's this list of words and you're supposed to say if each of the words has to do with derivative.

36 MZ: Speed or velocity? And how is velocity related to derivative?

37 Carl: That when you take the derivative, uh, equation that represents velocity at any given time. So if you use time in your position equation and the same time in your velocity equation, at that position the velocity is whatever the velocity equation says.

38 MZ: OK. Change or rate of change?

39 Carl: You want to know when the ladder coming down on your head, how fast it's coming.

40 MZ: OK, so that's the rate of change. Line or linear?

41 Carl: [short pause] The derivative of a-- at a local curve type thing. You know like on a graph there's always a line to the curve. It's basically like if you sectioned off all of your pieces of curve on your graph, have a whole bunch of lines together. Like all those little x^2 things, when you take the derivative becomes just an x. It's a line.

42 MZ: So even though it's not an x^2 curve that you're working with--

43 Carl: You can like define it in a certain way to make it that. You know a certain derivative at the line.

44 MZ: OK. Measurement?

45 Carl: You can measure the velocity. You can measure the profit, minimize the profit is the same.

46 MZ: OK. Prediction or approximation?

47 Carl: [pause] When you take the derivative of something, you're approximating the greatest value at like profit and stuff.

48 MZ: [short pause] Were you thinking of-- when you described before the max/min problems? [Carl acknowledges] And you're thinking of the derivative helping you to find the approximate max?

49 Carl: Right, the approximate max. Cause, well, nothing in this world is exact, always have to approximate.

50 MZ: Continuity?

51 Carl: That if a graph is discontinuous, it doesn't have a derivative at that point.

52 MZ: OK.

53 Carl: And-- [pause]

54 MZ: If it is continuous will it definitely have a derivative at that point?

55 Carl: No, because it could be a sharp turn or something like that.

56 MZ: You're saying that at a sharp turn it wouldn't be--

57 Carl: --differentiable.

58 MZ: --differentiable? So what is it about that sharp turn that causes it to not be differentiable there?

59 Carl: The rapid change in the slope.

60 MZ: Does limit have anything to do with derivative?

61 Carl: Yeah.

62 MZ: What?

63 Carl: Like the definition of a derivative has a limit.

64 MZ: OK. Do you remember what that definition is?

65 Carl: No.

66 MZ: But you're pretty sure it has a limit in it?

67 Carl: I know it's like, the limit as x approaches to h of $f(x)$ plus $h-$

68 Carl: [writes: $\lim_{x \to h} \frac{f(x+h) - f(h)}{}$] Something else on the bottom. Something to do with the change of h or--

69 MZ: So what's an antiderivative?

70 Carl: You know when you have like a derivative of something. You work backwards. You do the same thing as derivative and you end up with a C at the end because you don't know-- The constant could be anything because when you take the derivative it becomes 0. So it's what ever the original function is plus C. Then if you have a point, you can find the C by plugging back in.

71 MZ: OK. Is the derivative a function?

72 Carl: Why yes it is.

73 MZ: How do you know or why do you say yes?

74 Carl: Because a line is a function.

75 MZ: [short pause] A line is a function. True.

76 Carl: It's not always a function. It can be a function.

77 MZ: Oh, a line, you mean?

78 Carl: No-- Yeah. Like a straight line up [vertical line].

79 MZ: Oh, right. A straight line up is not a function. So--

80 Carl: And derivative can be a straight line up.

81 MZ: So the derivative is the same--

82 Carl: Is sometimes a function.

83 MZ: Is sometimes a function. So the derivative is the same thing as that line? Is that what you're thinking? [short pause] So the derivative is a function as long as it's not a vertical line?

84 Carl: Maybe. [short pause] I don't know.

85 MZ: OK. We were giving the formal definition here. Can you related this formal definition to the idea of slope or slope of a tangent line?

86 Carl: [pause] Well, we've got the minus in there.

87 MZ: OK.

88 Carl: Like if you had one point minus the other point over-- $y_2 - y_1$ over $x_2 - x_1$. That's y_2, y_1 [pointing to numerator]. Maybe $x - h$ or something on the bottom.

89 MZ: So this quotient in here is suppose to be the y minus bla bla bla, what you just said. [Carl agrees.] And how is that related to slope?

90 Carl: Like, you can find it the same way.

91 MZ: That calculation you just described? [Carl agrees.] What does this limit stuff have to do with it then? If this quotient is suppose to be some kind of slope calculation--

92 Carl: [pause]

93 MZ: No, idea? [Carl agrees.] OK. Tell me if you've heard of this, and if you have, if it relates to derivative, if you can remember. Newton's Method?

94 Carl: Yeah, I've heard of it.

95 MZ: Happen to remember anything about it?

96 Carl: We did it on our calculators at the very end of last year.

97 MZ: Oh, really. Do you think it has anything to do with derivatives?

98 Carl: Oh, yeah. I just don't remember.

99 MZ: Intermediate Value Theorem?

100 Carl: Well, there's something here and here. You're asked for something in between. To me-- That's the one that I said in class that it seems very simplistic, and he got mad at me. But it does. It sounds so stupid. OK.

You have to get from point a to b and you never stop. So therefore you go through the point in the middle. Well, of course.

101 MZ: Does it have anything to do with derivatives?
102 Carl: [short pause] I don't know.
103 MZ: Mean Value Theorem?
104 Carl: Isn't that with the average in the middle? At some point you have to go a certain speed or something like that?
105 MZ: Yeah, it could be like that.
106 Carl: Was that the example in class where you're going from 0 to 90 and somewhere in between you have to be going 60 miles per hour. [MZ says nothing.] Just a guess. I don't know.
107 MZ: OK. So does it have to do with derivative? [short pause] No idea? OK. I give you a derivative. What kinds of stuff could you tell me about the original function?
108 Carl: Max and mins. Where it's positive and negative.
109 MZ: OK, well, here's a graph.
110 MZ: You don't have to do a big deal here. Maybe you could tell me--
111 MZ: This is the derivative of some function. So for the original function you're suppose to tell me the max and min and stuff like that. And just approximate. You don't have to--
112 Carl: Right. As it goes from positive to negative it's is a-- I think so. I know that these are local maxs and mins, right there, at 2 and –.7. And I believe that the –.7 is a min and 2 is a max, but I could be mistaken. It's one of those.
113 MZ: How do you remember which one is which?
114 Carl: The positive to negative change in the derivative means something, and if I had a second to look it up, I could tell which one it was.
115 MZ: Can you tell me anything about the inflection points of the original function?
116 Carl: Oh. Uh, inflection points. Uh-- [short pause] Maybe. Kind of. I should be able to, right?
117 MZ: Yeah, it's possible.
118 MZ: OK. Well, I'd actually be more curious in if you could figure out which would be the max or the mins, because I'm kind of curious about how you're going to figure that out.
119 Carl: Let's see. [short pause] OK. This one's the max.
120 MZ: –.7 is the max?
121 Carl: Yeah.
122 MZ: And how come that's the max?
123 Carl: Because the slope is positive. So you're like this.
124 MZ: OK. I'm kind of going up towards the right.
125 Carl: And then as it becomes negative-- It goes from positive to negative so you have to have a point on here that's higher because it's going up and then you come down. So that's the max right there.
126 MZ: Thank you.

From Table A.8, it is evident that Carl mentions a graphical interpretation of derivative first and more often than any other interpretation. However, he does mention rate of change, velocity, and a symbolic formulation of derivative each without being

Table A.8. Carl: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	○				max/min
Why are the peaks of the graph at the zeros of the derivative?				↦	
What can derivatives be useful for?			○		max/min
What's a real world concept that involves the concept of derivative?		○			max/min
How does the derivative fit into your related rate example?		●		●	
How can you tell if a function is differentiable?	●			↦	
Is derivative related to slope?	○				
Is derivative related to speed or velocity?			○		
Is derivative related to change or rate of change?		●			
Is derivative related to line or linear?				↦	misstatement (possible d=tl)
Is derivative related to measurement?			○		max/min
Is derivative related to prediction or approximation?					misstatement (finding max using deriv is an approx.)
Is derivative related to continuity?	○				
Is derivative related to limit?				◉	misstated the ratio and the limit
Is the derivative a function?	◯				possible misstatement (d=tl)
Can you relate your statement of the formal definition of derivative to the derivative is the slope of the tangent line?	●			●	
Summary	⊙	●	○	◉	

specifically asked to do so. Carl states the details of the ratio only in the graphical and symbolic interpretations. Carl does not state the instantaneous nature of the derivative or a limiting process except for his weak attempt to state the formal definition of derivative by writing $\lim_{x \to h} \frac{f(x+h)-f(h)}{}$. When asked directly whether the derivative of a function is a function, Carl says that it is "because a line is a function" [ln 74]. This statement seems to assume that all derivatives are lines.

Carl makes another misstatement in the interview about the relationship of derivative to lines. When asked if derivative is related to line or linear, Carl seems to describe the notion of local linearity, "On a graph there's always a line to the curve. It's basically like if you sectioned off all of your pieces of curve on your graph, have a whole bunch of lines together." However, these lines are then described as the derivative function, "Like all those little x^2 things, when you take the derivative become just an x. It's a line" [ln 41]

One other error Carl makes involves approximate versus exact values. He states that the derivative helps find an "approximate max" [ln 49]. When asked for clarification he says, "Nothing in this world is exact, always have to approximate" [ln 49]

Although Carl does not state the formal definition completely he does recognize that the ratio is supposed to be a slope calculation. He points out that the numerator is the $y_2 - y_1$ of the slope calculation $\frac{y_2 - y_1}{x_2 - x_1}$ [ln 88]. However, he does not know what the denominator should be or how to explain the limit in graphical terms. Carl also seems to indicate a connection between rate of change and a symbolic formulation when explaining how derivative fits in to a related rate problems. He says that the derivative is how fast something is moving and this has to do with "the relationship of the change in y to the change in x" [ln 14].

For the last part of the second interview Carl is given the graph of the derivative function and asked to give information about the original function. He correctly notes the locations of the local maximum and minimum but does not remember how to find the inflection points of the original function. There is no time in the interview for Carl to answer further questions or to sketch a graph of the original function. Carl's brief answers on the interview are consistent with his answers to a similar problem on the chapter 4 test given two weeks prior to the interview. On that test Carl correctly notes the locations of the local maximum and minimum, but states where the original function is concave up or down incorrectly.

In comparing Carl's responses during the first two interviews, it is clear that his focus is different during the second interview . In the first interview he mentions velocity and acceleration and the rules for computing derivatives most frequently. However, in the second interview, slope is more prominent. In the first interview Carl mentions a graphical limiting process and the instantaneous nature of rate of change. In the second interview Carl does not mention those two aspects of the derivative, but he does describe the details of the ratio in terms of slope and rate of change and connects them to symbolic ratios such as $\frac{y_2 - y_1}{x_2 - x_1}$ which he does not do in the first interview. In the first interview Carl cannot state any formal symbolic definition for derivative. In the second interview he states the definition incorrectly and incompletely as $\lim_{x \to h} \frac{f(x+h) - f(h)}{}$.

The second interview has fewer misstatements than the first. Carl continues to make a possible misstatement that the derivative is the tangent line, but he no longer states that the derivative is the change in the function.

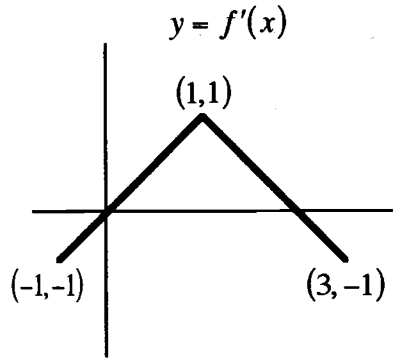
QOTD #12

What is the most important idea that we have studied so far in this class? Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

Response: "I'm tired. $\int$ is pretty important. Limits are important."

Interview 3

The third interview occurs during the three days after the test on differential equations, and antiderivatives by substitution and by parts. The first part of the interview is a summary of Carl's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike the second interview, the point $(1,0)$ on the original function is given so that only one solution is possible.



1 MZ: The first question is actually sort of like a problem to solve.

2 Carl: OK.

3 MZ: And I'm going to give you a graph, and this is the graph of the derivative. So this is the point $(1,1)$, and this is like the point $(-1,-1)$ and this is the line connecting them. And then over here you have $(3,-1)$ and this is the line connecting them. [draws an axes, the points listed and the line segments connecting the two outside points to $(1,1)$] And, OK. So you have this function and just don't worry about the domain outside of that, from -1 to 3. So this is $y = f'(x)$ and you're suppose to find a graph of the original function, f. [writes: $y = f'(x)$]

4 Carl: Mm hmm.

5 MZ: And you have an initial condition that $f(1) = 0$. [writes: $f(1) = 0$] Go for it. [hands him the notebook]

6 MZ: For the tape recorder he just drew an x , y -axis. OK. Marked $(1,0)$.
7 Carl: But I knew that already.
8 MZ: Yeah.
9 Carl: And then-- [short pause]
10 MZ: What are you thinking?
11 Carl: Nothing too much.
12 Carl: One's going up and one's going down. This one's going up. So this is the slope. The slope is increasing. So the slope is increasing. So it's going like that. Now it's positive. So the slope is positive versus negative. It's positive. So slope is negative. So it's coming down. Like this. It's coming down and then it's going up. Slope is-- coming down. So it's negative. I'm getting confused. Hold on. Uh. OK, so it starts out negative. So slope starts out negative. Negative slope it would be like that, right?. [draws a line with negative slope around $(-1,3)$]
13 MZ: OK.
14 Carl: That helps. So it's negative. It's big negative. So big negative is like that. Right?
15 MZ: OK.
16 Carl: Yeah, big negative is like that. That's big negative. That's little negative It starts out big negative like that and it goes to-- Yeah, when it hits this axis it goes to-- a positive slope, right. A positive slope 1.
17 MZ: OK. He's drawing little negative and positive slope line to the side [not on any axes].
18 Carl: Uhm, that's positive so it's negative on top and positive goes to there. So it could be like this. Positive, positive. This is-- [short pause while completes his sketch; the sketch goes through $(-1,0)$, $(1,0)$, $(3,0)$, has positive/negative slopes where it should, a min at $x = 0$ and a max at $x = 2$] I hope that's right.
19 MZ: OK. So he just finished drawing the-- I don't know what you want to call it.
20 Carl: I don't know. It's a squiggily line. I really don't know what I'm doing. Uh, wait. Negative, positive, positive, negative. Yeah, it's close enough, OK. It's my guess.
21 MZ: OK. How confident do you feel about how high you went on like these turn around points?
22 Carl: Not really confident at all.
23 MZ: Any ideas on how to check that or is it possible to know--
24 Carl: Uh. [15 second pause]
25 MZ: I mean, do you think that for example it could have gone down to say like -3 and turned around or is it more likely that it didn't go down that far?
26 Carl: Uh, I really don't think it would. It doesn't seem like it would.
27 MZ: Why doesn't it seem like it would?
28 Carl: This line is going up with a slope of one.
29 MZ: OK.
30 Carl: So it's not very changing. So the derivative here, the slope of these lines should be slowly changing, but that would make it-- I don't know.
31 MZ: OK. Let me ask you a different question. Does the graph of the original function that you just sketched--
32 MZ: Does the original function have a inflection point, an inflection point?
33 Carl: Uh, yeah.
34 MZ: OK, where?

35 Carl: At the maximum there.
36 MZ: At the maximum of the derivative function?
37 Carl: Yeah, that's not right is it? Yeah, it is. That should be right. Yeah, that's right. Yeah, that's what it is because it's a maximum here and on the second derivative function it's where it hits the axis and that's the inflection point. That's right. I remember that.
38 MZ: So the value of the second derivative at that point--
39 Carl: Is 0.
40 MZ: Is 0?
41 Carl: And it's the inflection point of the first one.[the original function]
42 MZ: OK.
43 MZ: It's the inflection point of the original function?
44 Carl: Yeah.
45 MZ: OK. So is it true that all inflection points have a second derivative equal to 0 or could the second derivative equal to something else at an inflection point?
46 Carl: I think it's just 0, but I could be wrong. It could be a maximum or a minimum. No, that's something else. I don't know. I'm getting confused now.
47 MZ: In this particular example how do you know that the second derivative is 0?
48 Carl: Because I always thought that a maximum here is on the axis there.
49 MZ: Is on the axis for the--
50 Carl: So the maximum here is-- That's why I turned it in at that point, at 0.
51 MZ: So the maximum of the original function is a zero of the original function.
52 Carl: Right. And the same thing at the next one. The derivative function is a zero of the second derivative function. That's just how I've thought of it.

Carl concentrates on his knowledge that the derivative function tells the slope of the original function. He does not actually draw a slope field, but off to the side of his graph he draws lines that have negative or positive slopes of the different steepnesses to help him determine how the original function should be drawn. Using this information he does sketch a reasonable graph of the original function.

Carl makes two misstatements. One is to say that the value of the second derivative at $x = 1$ is 0 instead of noting that it is undefined [ln 38-39]. He bases this statement on the idea that since the first derivative is a maximum then the second derivative must be 0 [ln 48-52]. After using the notion that the derivative represents the slope so effectively in drawing the graph, he does not consider slope during this discussion. The other misstatement is that on his graph of the original function, he marks the y values of the extrema as larger in magnitude than ± 1 instead of the correct answer of $\pm\frac{1}{2}$. He has uses slope to reasonably approximate these, but he does not think of using areas to find exact values.

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

53 MZ: OK. What is a definite integral?
54 Carl: The area.
55 MZ: OK, so it's the area--
56 Carl: Under the graph,
57 Carl: between there and anywhere and there and back
58 MZ: Back in this case was the x -axis. What's an indefinite integral?
59 Carl: It's a-- a graph. It's a graph-- with a bunch of slope field lines. It's a slope field basically and any other graphs could be on it like that. They're all the same graphs and they're just like up or down by C. [sketched a few slope field type line segments and then a series of parallel curves; many more curves than slope field lines]
60 MZ: OK. So, if you say what is the indefinite integral, is it one of those graphs or is it just all of those graphs or any of those graphs.
61 Carl: It's all of the graphs
62 MZ: So what's the relationship between the definite integral and the indefinite integral?
63 Carl: There isn't.
64 MZ: They're just two totally separate things.
65 Carl: Mm hmm.
66 MZ: Well, how come they're both called integral?
67 Carl: I don't know.
68 MZ: What's the relationship between derivatives and integrals then?
69 Carl: You take antiderivatives to get integrals and sometimes to derivatives, I guess.
70 MZ: So it seems like the derivative's related to the antiderivative so it's related to which one of these?
71 Carl: The-- This one.
72 MZ: The indefinite. The one with the slope fields and all that stuff.
73 Carl: Yeah.
74 MZ: Is the derivative related to the uhm-- the definite integral.
75 Carl: [12 second pause] Kind of.
76 MZ: What are you thinking when you say kind of?
77 Carl: Well, you gave me that and I can get that. And if you want to find the area under that, if you gave me that, then--
78 MZ: OK. I gave you the derivative and you were able to find the function.
79 Carl: Mm hmm.
80 MZ: And then you said--
81 Carl: If you want the area under the function which is the definite integral--
82 MZ: OK.
83 Carl: --you have to look under the function, but you couldn't do that unless you had the function. So you gave me the derivative. So with the derivative I got the function and got the definite integral.
84 MZ: OK. But if I had just given you the function and said find the--
85 Carl: Then you wouldn't need the--

86 MZ: Then you wouldn't need the derivative.
87 Carl: Exactly.
88 MZ: OK. Do remember what the fundamental theorem says?
89 Carl: Uh. No. The first or the second?
90 MZ: Pick one. We'll do them both.
91 Carl: Neither. I'm not going to be [inaudible] So. It says something pretty easy I think, the first one. The second one-- The first one's kind of-- It says that-- It has all the a's and b's in it. If $y(a)$ minus $y(b)$, something like that.
92 MZ: This is the--
93 Carl: That's all I remember.
94 MZ: OK. Which one does that seem to be related to, the $y(a)$-- ?
95 Carl: Either one. The first one, could be the second.
96 MZ: How about this, have you ever heard of an area function?
97 Carl: Yeah. Let's see, the definite integral.
98 MZ: The definite integral would be the area function? Is that what you're thinking?
99 Carl: That's what I'm thinking.
100 MZ: OK. Uhm. If you take the definite integral, you get an area number right? A value for that area?
101 Carl: Yeah.
102 MZ: OK. So then in some sense just a definite integral by itself isn't an area-- I mean, isn't a function because it's just a number?
103 Carl: In a sense.
104 MZ: So I was wondering if there would be an area function that was changing? Do remember when we did that capital $A(x)$. Here.
105 Carl: Just kind of vaguely. If you asked me if I've used it before I could tell you yeah.
106 MZ: OK. Well, let me ask you a different question. We were talking about finding the area from the definite integral. So, say if you knew the equation to this function, how would you find the area, say from -1 to 1, if I knew the equation for that?
107 Carl: Don't you take the antiderivative of it?
108 MZ: OK.
109 Carl: And then-- [short pause] plug in the values for y .
110 MZ: OK, so what values would I be plugging in?
111 Carl: The maximum ones. And then like this would be b and this would be a, and you subtract the a's from the b's and you get the areas. Kind of. Vaguely.
112 MZ: OK. So-- I mean, to do a maybe easier example say--
113 Carl: You can take this from here to here and then find, oh-- It's like those, uh, sum things.
114 MZ: Oh.
115 Carl: You know like the box from here to here and another box from here to here, estimating the--
116 MZ: Estimating the area?
117 Carl: Yeah, now I remember.
118 MZ: I know. Say if we even did an easier one Say this is $y = x^2$ and I want to know the area from 0 to 1 say. [sketches the graph of $y = x^2$ on an axes and labels it as such; marks a vertical line at $x = 1$ from the x -axis to the curve]

119 Carl: Mm hmm.

120 MZ: Uhm, between the curve and the x axis.

121 Carl: Mm hmm.

122 MZ: Uhm, how would I find that area?

123 Carl: See I'm still kind of foggy on the entire like integral concept.

124 MZ: What do you think makes them confusing?

125 Carl: I don't know. They're just weird.

126 MZ: They're weirder than derivatives though I mean--

127 Carl: This chapter-- This chapter doesn't make any sense though.

128 MZ: Chapter 7 you mean? [chapter 7 covers ordinary differential equations]

129 Carl: It doesn't make any sense. Like in the book, I was looking at it and like equations for the uhm differentiable equations. The y primes just like drop out, and they say like don't worry about them and take the antiderivative of y next to it.

130 MZ: Oh.

131 Carl: And I'm just like, why?

132 MZ: Hmm. [short pause] So let me go back to this. You gave two different ideas when you were talking about finding the area under this curve up here so I just gave you a specific example and though maybe you could recreate one or both of those ideas, you know with a specific equation.

133 Carl: But it confused me.

134 MZ: [laughs] But it seems harder now that I gave you a specific equation?

135 Carl: Yes.

136 MZ: Well, what was the first idea that you had?

137 Carl: I don't remember. I really don't remember. Hold on. Let me just think about this for a second.

138 MZ: OK.

139 Carl: So, what was the question?

140 MZ: So the question is, what's the area--

141 Carl: Oh, OK.

142 MZ: --under this curve right there?

143 Carl: So you take-- [10 second pause while writing: $y' = \frac{1}{3}x^3$] the antiderivative.

144 MZ: OK.

145 Carl: [15 second pause while writing under previous statement: $\begin{array}{l}\frac{1}{3}-0\\ \frac{1}{3}\end{array}$] One third.

146 MZ: OK, so what did you just do?

147 Carl: Plugged in 1 and 0.

148 MZ: OK.

149 Carl: And subtracted.

150 MZ: Subtracted and you got one third. So that was your-- So you are saying that one third is this area, and that was your first idea that you were explaining to me, right, was the antiderivative and you plug in. OK, that's cool.

151 MZ: And then you were explaining this other idea about the rectangles. Is that supposed to give the same answer then?

152 Carl: It's an estimation of it.

153 MZ: An estimation of it?

154 Carl: Mm hmm.

155 MZ: OK. [short pause] You already mentioned slope fields, do slope fields have anything to do with this stuff? [turns the tape over; while tape off

asks Carl whether slope fields are related more to derivatives or integrals]

156 Carl: I guess they would be more related to integrals.

157 MZ: How come?

158 Carl: So now I know how to think of why. The question was, are slope fields more related to derivatives or integrals.

159 MZ: Yeah, I mean, do you think they are more related to either one? I mean are slope fields related to derivatives?

160 Carl: Uh, not that I can see. I mean, no, not really. They're related to antiderivatives. Integrals. Yeah, they're related to integrals because when you take an indefinite integral you always have a C added on. And so you can have slope fields that are off by a factor of C which is the antiderivative of the function plus C. So it's integrals. The answer is integrals.

161 MZ: OK. I think that's all the questions I have.

Carl associates definite integral with area [ln 54]. Interestingly he associates indefinite integral with a graphical representation also, that of a slope field. He describes the graphs that can be drawn through the slope field lines, "They're all the same graphs and they're just up or down by C" [ln 59]. For him the indefinite integral is all the graphs not just one of them. When asked, Carl states that derivatives are not related to slope fields [ln 62-65]. He seems to know how to use a slope field without knowing how the slope field is constructed. On the previous test Carl uses the slope field functionality on his calculator to construct the slope field. He may not know how to construct one by hand.

With two very different graphical associations for definite and indefinite integrals, it is perhaps not surprising that Carl says definite integrals and indefinite integrals have nothing to do with each other. Later in the interview Carl almost relates them in that he associates indefinite integrals with antiderivatives, and he uses antiderivatives to find the area under a curve. Carl correctly uses the Second Fundamental Theorem of Calculus to find the area under a curve, even though he can not state it when asked. Carl is also unable to state the First Fundamental Theorem of Calculus, and he does not know what is meant by an area function.

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: While it is recorded that Carl answered this question incorrectly, his exact response is not recorded.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Carl's exact answer is not available. It is known though that Carl answers the question correctly.

Test 9: Semester final

This test, which is a cumulative semester exam, covers all of the material on functions, limits, derivatives, areas, and volumes. The test questions are largely computational. On the semester final, Carl solves an optimization problem, a related rate problem and problems involving limits and differentiation. He does make a minus sign error and a chain rule error on two differentiation problems and misses all problems involving integration. On tests later in this time period Carl has trouble setting up two out of five area and volume problems. He correctly completes average value problems, but struggles with work problems and improper integrals.

QOTD #15

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: "continuous at all points differentiable everywhere but zero"

Interview 4

The discussion for the fourth interview is broken into four parts. The first section includes general questions about derivatives. The second part asks for the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity, and acceleration given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the fourth interview.

1	MZ:	What is a derivative?
2	Carl:	A derivative is the slope of a tangent line to a curve or the change in-- the way to find like from the speed of something the velocity or the acceleration. Get it?
3	MZ:	I think so.
4	Carl:	The maximum or minimum values for things. That's about it.
5	MZ:	Ever heard when people say derivative is instantaneous rate of change?
6	Carl:	Oh, yeah.
7	MZ:	What do you think they mean by that?
8	Carl:	Well, since it's-- Like here's a curve or something, and you get a slope right here. It's like at that point so it's instantaneous. It's like right at that point.
9	MZ:	So that's the instantaneous part. What about the rate of change part?
10	Carl:	Oh, that's the way the graph-- the slope's changing as you go from one point to another along the curve.
11	MZ:	How the slope is changing?
12	Carl:	Yeah.

As in the second interview, Carl mentions slope of a tangent line to a curve as his first answer to what a derivative is. However here, unlike in previous interviews, he mentions more than one interpretation in answer to this question. He relates derivative to change, not rate of change, and to velocity and acceleration. His response to the question about rate of change is graphical. He uses the slope at a point to explain the word "instantaneous" [ln 8]. When pushed to explain rate of change itself, he stays within the graphical interpretation, although he focuses incorrectly on how the slope is changing instead of stating that the slope is the rate of change. He does not attempt to explain rate of change for itself without reference to another interpretation of derivative.

The next part of the fourth interview is a summary of Carl's solution to the first of three problems in the interview. Given a table of values with x varying by .1, Carl is asked to find $f'(2)$, the derivative of the function at $x=2$. Carl's first reaction is to graph the function given by the values in the table and sketch in a tangent line at $x=2$. He states that the slope of this tangent line is around 2. When asked to explain his estimate, he starts to calculate the change in x over the change in y for the points with x values 1.9 and 2.0. He is able to catch his error by remembering that slope is "rise over run". Without prompting he goes on to state that he could do the same calculation for the points with x values 2.0 and 2.1 and then average the two estimates. Carl also knows that if he were given points closer to $x=2$, he could give a better estimate.

Carl's focus from the beginning is graphical. He initially thinks only of steepness and then later discusses the details of the slope ratio. His statement about using points closer to $x=2$ hints at, but does not explicitly state, a limiting process for finding a more accurate estimate.

The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car took 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far did the car travel in the 2 hour period?

Carl's work on this problem contains many correct general ideas, but he often misses the details, especially those concerning units and dimensions. Carl states that the car is "accelerating a mile per minute" without realizing that he has given a unit of velocity and not acceleration. He then sketches a reasonably shaped graph for velocity, emphasizing appropriately that the function is "increasing at a constant rate and decreasing at a constant rate." He thinks that the area under the curve, the area of a triangle, would give the distance, but he can not calculate it because he does not know what the horizontal distance is. He describes the vertical and horizontal axes on his graph as y and x. "I guess the y is how fast it's going, and then the x is how far it's traveled."

Next he decides to calculate an integral, $\int_0^{60} x\,dx = \frac{1}{2}x^2\Big|_0^{60}$. Since his graph shows a straight line, he chooses the integrand to be x without considering that the slope of the line might not be 1. He decides that using 0 to 60 as the bounds is wrong since the bounds should be along the x-axis. When this idea stalls, the interviewer suggests that he calculate how far the car traveled in the first hour. Carl calculates $\frac{1}{60} + \frac{1}{30} + \frac{1}{20} +$ explaining that after the first minute the car is going 1 mile an hour so it is going $\frac{1}{60}$ of a mile in a minute. The next minute it is going 2 miles an hour so it is going $\frac{1}{30}$ of a mile in a minute, and this continues until at 60 minutes the car travels 1 mile in one minute. Carl does not, and is not asked to, relate this correct summation to any of his other solution attempts.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Carl is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. He is asked if the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Carl first guesses that the rate the ladder is sliding down is constant but different from the rate at which it is being pulled out. He also asks if any of the side lengths or rate values will be given. The interviewer points out that the side lengths are not given because they are changing and that he should assume that the rate the ladder is being pulled out is an unknown constant.

Carl states that he does not know how to do the problem, but he knows derivatives are involved because the problem is about rate of change. He starts by labeling the wall as y and the floor as x and setting up the equations $14^2 = x^2 + y^2$ and $x = \sqrt{196 - y^2}$. He thinks that he wants to involve the "rate of change", dx, but initially tries to do this in terms of "a proportion kind of thing" written as $\frac{x}{dx} = \frac{y}{dy}$. After a hint to find a rate from one of his previous equations, Carl abandons the proportions and writes $dx = \tfrac{1}{2}/\left(196 - y^2\right)^2(-2y)$. Note that there are several errors. To see them, compare Carl's calculation to the following: $\frac{dx}{dt} = \tfrac{1}{2}/\left(196 - y^2\right)^{\frac{1}{2}}(-2y)\frac{dy}{dt}$. Carl still can not answer the original question based on his calculation and there is no time left in the interview to give Carl more hints.

Note that Carl does not use a ratio, e.g. $\frac{dx}{dt}$, for rate. His previous work on related rate problems does not show this notational error. However, since the related rate test five months previous, the class has covered integrals which include the dx notation as a common element.

Interview 5

Carl's fifth interview occurs almost three weeks after he takes the BC version of the AP exam. During that week the class discusses the written questions from the BC version. Between the fourth and fifth interviews, the class studies series and integration techniques and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Carl's response to a set of questions on a particular topic and provides an analysis of those responses.

1 MZ: What is a derivative?
2 Carl: The slope of the tangent line to a curve at a given point, the rate of change of something and that's it.
3 MZ: [short pause] What does it mean the rate of change of something?

4 Carl: How the rate changes. Like pouring liquid in a container and it's leaking out the other end. It's the rate of--

5 MZ: What does that mean, rate?

6 Carl: You know, how something changes. Like the rate at which it changes. The amount over a given period of time.

7 MZ: OK. Do you remember that formal definition?

8 Carl: $f(x)$ minus. No I don't know. Like $f'(x)$. Something like that. It's close. Around there. I don't remember. [has written: $\lim_{x\to 0}\frac{f(x)-f'(x)}{h}$]

9 MZ: OK. What do these three things have to do with each other? They are all suppose to describe what a derivative is, but how are they related to each other?

10 Carl: I don't know.

11 MZ: Do you have any idea how slope and rate are suppose to be-- I mean, do they seem like the same thing to you?

12 Carl: Well, the rate is like something over like time and the slope is like distance over distance. It's kind of like the x axis was time, then the slope would be a rate.

13 MZ: OK. Does the derivative involve a limiting process?

14 Carl: [pause] The limit?

15 MZ: Does the derivative involve a limiting process?

16 Carl: Yeah.

17 MZ: Explain.

18 Carl: There's a limit in the definition.

19 MZ: OK. What does it mean by a limiting process?

20 Carl: [pause] I don't know.

21 MZ: OK. Is the derivative of a function a function?

22 Carl: Yeah.

23 MZ: Why?

24 Carl: [pause] 'Cause--

25 MZ: Well, what struck you to say yeah versus no.

26 Carl: I thought of examples.

27 MZ: OK.

28 Carl: This isn't one of those always, never or sometimes questions? [MZ laughs] Well then yeah because the examples work.

29 MZ: The examples work. Could it be that just those examples work or some examples work?

30 Carl: They all work.

31 MZ: You think they all work. Well let me ask you first, what examples did you think of?

32 Carl: $f(x)=x^2$, $f(x)=x$ [short pause] That's it.

33 MZ: Then how come you think that they all probably work.

34 Carl: It wasn't an always, sometimes or never question.

35 MZ: Oh, OK. 'Cause to me sometimes goes with no. Yes, would be always. Then no would be--

36 Carl: Right. Yes, I guess.

37 MZ: Fine. Explain what is meant be a differentiable function. Give an example of a differentiable function and a nondifferentiable function.

38 Carl: Differentiable function means you can take the derivative of the function. Give an example. [pause, writing:

diff	N - diff
$f(x) = x^2$	$f(x) = 1/x$]
$f(x) = 2x$	

Uh. I don't know. $1/x$ is $\ln x$.

39 MZ: Derivative of $1/x$?
40 Carl: Derivative of $\ln x$ is $1/x$.
41 MZ: Yeah, that way.
42 Carl: So what's the derivative of $1/x$?
43 MZ: Do you know?
44 Carl: No. I just don't know.
45 MZ: Would you know if that was rewritten as x^{-1}?
46 Carl: Zero?
47 MZ: Actually, you can use the power rule on that. Like x^{-1}. Then you go, well the derivative is -1 times x^{-2}.
48 Carl: Oh, yeah.
49 MZ: Just like you would use the power rule on x^3 or something.
50 Carl: Oh, yeah. [pause] I don't know. I can't think of one right now.
51 MZ: Can you sketch the graph of a nondifferentiable function?
52 Carl: [sketches a graph similar to the curve $x = y^2$ on a pair of axes] Is that one?
53 MZ: Uhm-- That is nondifferentiable. Well, I mean, it's not a function.
54 Carl: Right. I don't know.
55 MZ: OK. What are derivatives useful for?
56 Carl: Finding out slope of the tangent line or rate of change of something.
57 MZ: OK. Can you think of a specific example of a rate of change that you might be finding?
58 Carl: Pour water into a bucket. Let it leak out the other end. Find the rate at which the water goes out.
59 MZ: OK. Well, here's another. Explain what a derivative is without mentioning the symbolic definition, slope, rate of change. Can you think of another way to describe what it is in general?
60 Carl: No.

As in the second and fourth interviews, Carl mentions a graphical interpretation of derivative as his first response to "what is a derivative?" (see Table A.9). However, here for the first time, his initial graphical interpretation includes the instantaneous nature of the derivative, "the slope of the tangent line to a curve at a given point" [ln 2]. Even though the portion of the fifth interview focusing on general questions about the derivative has fewer questions than similar sections of the first and second interviews, Carl gives nearly as complete answers here in the fifth interview as in the second

Table A.9. Carl: Interview 5 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎	○			
What is meant by rate of change?		○			misstatement (change in rate)
Did you learn a formal definition of derivative?				◉	misstated the limit and the ratio
How are slope and rate of change related?	●	●			
Does the derivative involve a limiting process?				○	
Is the derivative of a function a function?				↦ ○	
What is meant by a differentiable function?				↦	
What are derivatives useful for?	○	○			
Asked to interpret the Mean Value Theorem.	◉	○	○	●	misstatement (roc of velocity) misstatement (change in the rate of distance)
Asked to find the average rate of change of a function defined as an integral.		○			misstatement (d=average roc)
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.		◎	◎ ○		(d=change) acceleration decreasing maximum incorrect calc
Summary	◉	◉	◎	◉	

interview. The main difference is that Carl does not mention velocity and acceleration in the fifth interview, even though he does in all previous interviews.

There is a slight difference in the connections Carl makes. In the second interview Carl describes the details of the rate of change ratio symbolically, and later describes the details of the slope ratio symbolically. In the fifth interview Carl does not quite describe slope and rate of change symbolically, but he compares them to each other saying, "The rate is like something over time, and the slope is like distance over distance. [If] the x-axis was time, then the slope would be a rate" [ln 12].

In the fifth interview Carl does not make any of the misstatements, as in the first and second interviews, regarding equating the derivative and the tangent line. He continues his tradition of misstating the limit and the ratio of the formal symbolic definition. Further, he makes the same misstatement, as made in the second and fourth interviews, that the derivative is a change or a change in rate instead of a rate of change.

For the second part of the fifth interview Carl is asked about the Mean Value Theorem. His first thought is to give a graphically interpretation of a theorem that turns out to be the Intermediate Value Theorem. When he is given the symbolic statement of the Mean Value Theorem including the expression $f'(c) = \frac{f(b) - f(a)}{b - a}$, Carl states that the ratio is a slope and that the derivative is a slope. His initial description of the relationship between the two slopes is muddled, "The slope at that value would equal to the slope at each end point, the average of those two points," but when asked for clarification, he correctly sketches a curve with a secant line and a parallel tangent line.

When asked how to interpret the Mean Value Theorem if f is a distance or position function, Carl replies that he does not know. However, when asked specifically what is the derivative of a distance function he says that it is the rate of change of the distance which is the velocity. When further asked what the ratio represents, he replies, "The rate of the distance. The change in the rate of the distance." In addition to his "change in the rate" error, Carl does not recognize that this is an average rate of change.

For Carl the Mean Value Theorem is tied to a graphical representation. His explanation of the theorem in terms of rate of change contains the same error, "change in rate" as in the earlier part of the interview. The explicit question about derivative of a distance function gives him a chance to show that he knows the derivative is velocity, something he does not mention in the first, open-ended part of the fifth interview.

The next question in the fifth interview involves a problem from the AB version of the AP exam which Carl has taken. He says of the problem, "This is the one I didn't do." The question is as follows:

> Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.
> (a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.
> (b) On what intervals is F increasing?
> (c) If the average rate of change of F on the closed interval $[1,3]$ is k, find $\int_1^3 \sin(t^2)\,dt$ in terms of k.

The interviewer asks Carl to explain how he would solve parts (a) and (b), but does not require him to complete the solution of either part. After trying to change the subject several times, Carl finally begins to set up the trapezoid rule correctly. He sketches the desired partition for a function that looks like $y = \sin x$, and then describes that the area is the two heights (sides of the trapezoid) divided by two times the width. However, when pressed for details, Carl loses confidence in his work and says that maybe he should be finding the antiderivative.

For part (b) Carl recognizes that he needs to find the derivative of F and that the derivative of an integral is what is inside the integral. However he interprets, "inside the integral" to be $\sin(t^2)dt$, including the dt. He knows that wherever the derivative is positive the function is increasing, but he does not attempt to find those values.

For part (b) Carl states that he needs to be able to find the antiderivative of $\sin(t^2)$, but he does not know how to find it. When he gets stuck, the interviewer asks him to interpret the phrase "average rate of change of F." Carl answers, "Well, is it like the derivative of F?" He is asked if he would have a different interpretation if the question had said just rate of change instead of average rate of change. Carl replies, "I'd be confused either way." Carl does not do any further work on the problem.

Although Carl uses the phrase rate of change in discussing the derivative in all of his interviews, in the first interview he confuses change with rate of change. Moreover, in the fourth and fifth interviews he calls the derivative the change in rate. Carl does not use the phrase average rate of change in any of the interviews.

Carl has some notion of the details of the ratio. In the second interview he states that a rate is the change in y over the change in x ,and on many occasions he explains the details of the ratio in terms of slope. However, he does not seem to connect these with the phrase average rate of change. Even in discussing the Mean Value Theorem in the fifth interview, Carl knows the difference between the ratio and the derivative in terms of a graph, but has trouble formulating this using the word average; "The slope at that value would equal to the slope at each end point, the average of those two points." In discussing the theorem in terms of velocity, Carl does not recognize the ratio as an average rate of change. He says the ratio is "the rate of the distance. The change in the rate of the distance."

The next section of the interview concerns Taylor polynomials. Carl is not able to answers any of the interviewer's questions on this topic. Carl says that the only way he knows how to find a Taylor polynomial is with a calculator. Carl misses class frequently during the class discussion on Taylor polynomials and series during the five weeks previous to the fifth interview and received 0 points on in-class test questions covering this material. He also does not need to know Taylor polynomials for the AP exam since he decided to take the lower level AB version.

The final section of the fifth interview concerns a function, f, that at any time x , given in hours, tells the outside temperature in degrees Fahrenheit. Carl is shown a series of symbolic expressions and asked what information each one provides about the outside temperature.

For $f'(3) = 4$ Carl claims that it represents "the rate of change equals 4 degrees Fahrenheit." For $f''(3) = -2$ he says, "The acceleration at that time is -2. The acceleration of the temperature which means it's getting warmer slower." When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Carl replies that the temperature is changing 4 degrees from 0 to 3. When asked to clarify his statement he says, "At any instant in between that interval it's changing 4, but that doesn't make any sense because then you get really small intervals and it becomes a trillion degrees." Carl realizes that his two statements are contradictory and guesses that his first answer, 4 degrees for the whole interval, is correct. Carl presses the interviewer for the correct answer. She responds by giving him an example where the derivative is known to be the speed of a car and $f'(x) = 55$ for $0 < x < 3$. Carl easily calculates that the distance traveled is 165 miles, and he can state that the units on f' are miles per hour. The interviewer points out that Carl had used units of degrees Fahrenheit for the other derivative units. Carl then realizes that the units should have been degrees per hour and that the change was 12 from 0 to 3.

Next Carl is asked to interpret $f''(x) = -2$ for $3 \le x \le 6$. He says, "That means it's slowing down, getting warmer or colder. ... So [the rate of change] is being slowed by 6." Carl is able to numerically explain how the temperature changes from 0 to 6. He says that it rises 12 degrees in the first three hours then by 2, by 0 and finally by -2 for each of the second three hours. He estimates that the maximum temperature would occur at time equals 5.

Carl is next asked to find an equation for the temperature given that $f(0) = 50$. He knows that $f(3) = 62$ and determines that $f(x) = 50 + 4x$ for the first time interval since it is, "4 times how many hours it's been out there." For the second time interval, he says $f'(x) = 4 - 2x$. Then he recognizes that the starting point has shifted to 3, so he changes his equation to $f'(x) = 4 - 2(x - 3)$. When asked for the equation for temperature on that interval, he tries to adjust his equation to fit the numeric data that he has already calculated suggesting $f(x) = 58 + 4 - 2(x - 3)$. Since Carl does not suggest finding an antiderivative for any of these calculations, the interviewer suggests a different problem; if the velocity function is given as $v(x) = \cos x$, how would you find the distance function? Carl knows immediately that he should find the antiderivative and that the analogy is parallel to what he has been working on. The time for the interview runs out before he has a chance to do any further calculations.

From the beginning of the discussion of this problem, Carl uses ideas of rate of change and acceleration to help him make statements about the symbols. When he gets stuck, initially in calculating the temperatures numerically, and later in finding a temperature equation, the interviewer's use of the example of speed or velocity gives Carl the connection he needs to understand the problem. Part of Carl's initial difficulties stem from a focus on temperature change in degrees instead of a rate of change in degrees per hour. His later difficulties with the equation were because he did not think of using antiderivatives.

Case Study 4 — Derick

Academic record

*Other AP courses: US. History (junior year), English, Chemistry.

*National Merit Scholar.

*Writing tutor at the high school writing center.

*Math team participant; chosen as the school's oral competitor.

*Plans to major in physics in college.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: "A function is a series of operations conducted on a variable such that the results of using any two values for the variable will be different."

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different?

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: "a) $f(x) = x$ $\quad f(x) = x^6 + 3x^5 - 2.4x^3 + 18x - 13.4$

The first is simply a straight line, while the other curves several times.

b) $f(x) = 2$ for $x \leq 4$

$f(x) = -2$ for $x \geq 4$

There are two values for the x value 4, and therefore it does not pass the stupendous, incredible, and otherwise really neat Vertical Line Test."

QOTD #3

Give an example of a function without using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Date: August 26, 1993. This question occurs while the class is doing a quick review of functions.

Response: Derick gives two answers. First he includes a sketch of a graph of the function $y = \frac{x^3}{9} + \frac{x^2}{4} + 2$. Second, he writes. "If you listen to one cricket on a hot night, and count its chirps in one minute, double the number and add ten, that is the Fahrenheit temperature. (OK, it doesn't really work, but it's a function."

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) non integers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: "a) Yes

$$f(x) = \begin{cases} x^2 & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

You can work miracles with piecework functions.

b) Yes

$$f(x) = \begin{matrix} \|x\| & \text{for all } x \text{ such that } x \neq |x| \\ x+.01 & \text{for all } x \text{ such that } x = |x| \end{matrix}$$

$\|x\|$ is the greatest integer function .

$x \neq \|x\|$

It's a stretch, but it does work."

Comment: In a later conversation, Derick explains that $x \neq \|x\|$ means x is not an element of the integers and similarly $x = \|x\|$ means x is an element of the integers.

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to class discussion on limits.

Response: "The limit as x approaches a of a function $\left(\lim_{x \to a} f(x)\right)$ is the value the function approaches as x nears a."

Test 1

On a test on limits a week later, Derick is able to find correctly limits by reading values from a graph, by substituting into a piecewise function, and by using algebra to simplify a limit calculation. He is able to work with the formal definition of limit to find a δ for a given ε in a graphical setting, but he is not able to complete an ε-δ proof for a linear function.

Interview 1

This interview occurs after the test on limits but prior the class' discussing derivatives. Therefore Derick's answers are presumed to be based on what he remembers from his junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table A.10, which codes these responses. A summary discussion follows.

1	MZ:	What is a derivative?
2	Derick:	Slope of a function at a point.
3	Derick:	Or the rate of change
4	MZ:	Is there a difference or is this the same or versions of the same thing?
5	Derick:	Well, it's really -- You can only say a slope if you have a graph. If you just have a function They're basically the same thing depending on how you express the function.
6	MZ:	What can derivatives be useful for?
7	Derick:	They can be used to find-- for instance if you have a function of position of a car, you can find out the acceleration and the velocity at certain points. You can find the rate of change at a given point instead of over an average. It gives you a more exact reading than to find the average from the secant line.
8	MZ:	Can you think of any other examples?
9	Derick:	You can also use them to find the shape of a slope, when you're graphing it to find where the maximum and minimum points are by when the derivative equals zero you can find the vertexes.
10	MZ:	Now you're doing your hands like this. So are you thinking of vertexes like this kind of vertex --
11	Derick:	No, like this rather. [His intent is curved max and min not pointed as MZ suggested.]
12	MZ:	How would you explain what a derivative is to someone who doesn't have much math background?
13	Derick:	OK, would this person be assumed to know what a slope is?
14	MZ:	Let's do two different ones. First let's do somebody who's maybe like an AB or a precalculus student.

15 Derick: What I would say is a derivative is -- instead of taking a slope, which is you take two separate points on the graph and find the change in y and the change in x, and that'll give you sometimes a decent approximation of what the slope is at a particular point. Instead you take the points and squeeze their x values together until the change in the y values becomes smaller and smaller and smaller and eventually you're assumed to take it so that the two points are lying on top of one another and it's the slope of the tangent line at that one point. The tangent line to the graph is the derivative.

16 MZ: Yeah. So what if you had to explain it to somebody like -- say like you're grandmother or somebody like that?

17 Derick: Basically you're trying to find out how fast something is changing at a particular moment. So, like for instance, say you are driving down the street and you are accelerating and you're in a stick shift. You're in first gear and you accelerate up and then you switch to second gear and you stop accelerating for a second and you start accelerating again. So the graph of your velocity in relation to time would not go straight up. It would start going up and then even off for a second and then start going up again. If you were to try and find your acceleration over a time, and it involved one part where you were accelerating and one part where you weren't, you would get an inaccurate reading because it would be an average. So instead you want to find out, just at one of the points where you were accelerating, how fast you were accelerating. And to do that you would need the derivative. Which is basically, if you were to draw a line so that it just touched the graph at one point and found the slope of that line, that's the derivative.

18 MZ: Say I give you a function. How can you tell if it's differentiable?

19 Derick: You mean differentiable at a point?

20 MZ: OK.

21 Derick: OK, you would have to look for any cusps or sharp turns like at the zero at the absolute value. Sharp turns like that are not differentiable. Or where ever it's discontinuous which is what we're finding now where the limits aren't equal -- the right and left limits. If there is a jump or discontinuity, it's not differentiable at that point.

22 MZ: Do you know of an equation that gives a cusp?

23 Derick: Well, $|x|$. That's not really a cusp. That's more of a sharp turn. Uhm, [pause] let's see here. I can't think of one. I wouldn't know the exact formula for that sort of thing.

24 MZ: Yeah, I guess you're thinking of cusp as when they come in more like that. [sketches a cusp]

25 Derick: Yeah, a stinger shape for lack of a better term.

26 MZ: OK, these next ones -- you sort of answered some of them but -- It's like here's a word. Does it have anything to do with derivative and if so what? You've already talked a lot about slope so we'll skip that one.

27 Derick: Yeah.

28 MZ: Velocity you've talked a little about. What would you say the relationship is?

29 Derick: The derivative of velocity is acceleration because acceleration is the rate of change of your velocity. And velocity is derivative of position or distance because the speed you're going is the rate of change of your position from a given point.

30 MZ: Rate of change you've kind of already talked about. I'll skip that one. Line or linear?

31 Derick: Well, if you have a linear graph it's going to have a constant derivative. [pause] And I suppose you could also say that when you have a graph, -- the tangent line, the slope of that determines the derivative.

32 MZ: What about linear?

33 Derick: Linear, OK. Well, I suppose -- Well, no that's not right. Linear. [Pause] That's kind of a blank there.

34 MZ: Measurement?

35 Derick: I guess you could say that the derivative is the measurement of the rate of change at that point.

36 MZ: Yeah, it works. We'll forget that one. Prediction or approximation.

37 Derick: I guess you could say for that that you can sort of predict the derivative without using derivative formulas if you take two point on a graph where the x values are very very close like .0001 away, and find the two y values and find the slope of that and that'll give a pretty good approximation for derivative. I mean, the closer together the two x values are together the better the derivative value is.

38 MZ: Optimization?

39 Derick: To find the optimal points on the graph you have to find where the derivative equals zero. And then find whether that is an absolute max or min depending on like whether you're trying to optimize cost in which case you'd want to find the minimum and all those would be the points where derivative equals zero.

40 MZ: Continuity?

41 Derick: OK, well to be able to find a derivative the graph has to be continuous at that point.

42 MZ: OK, would all continuous graphs have derivatives?

43 Derick: Not necessarily, but all differentiable graphs are continuous.

44 MZ: Limit?

45 Derick: OK, well to find out if something is continuous you have to find out the limit from each side and that is the value, the y value, the $f(x)$ value, that the graph approaches as x nears whatever value you want. If the limit from the left and the right are equal to $f(x)$, then that graph is continuous.

46 MZ: OK. Can you think of a way that limit would be more directly related to derivative as opposed to limit related to continuous related to derivative?

47 Derick: OK, well I suppose you could say that -- As you get closer to your x point the limit of the derivative value from the right and from the left -- assuming that it's continuous and there're no bizarre changes -- the limit coming from the right and from the left of the derivative value should equal the derivative at that point.

48 MZ: Uh, integral? I don't know if you guys did this yet.

49 Derick: An integral is kind of a reverse derivative. Instead of finding -- like for instance, velocity is the rate of change in the derivative of location and location is the integral of velocity. It's finding -- [pause] It's the opposite of derivative.

50 MZ: Function?

51 Derick: That's an equation or an expression where each x value that you put in has only one y value.

52 MZ: OK. Is the derivative a function?

53 Derick: Well, not exactly. The expression for a derivative of a function will be a function. No, not even. Yeah, I guess it would have to be. Actually yes it is.

54 Derick: If you take any point on a function and find the derivative there'll be only one derivative at that point. So for every x value there's only one y value, one derivative value. So it would be a function.
55 MZ: OK, do remember a formal definition of limit? I mean, not epsilon delta formal, but just sort of--
56 Derick: The limit will be the value that the y approaches or the $f(x)$ approaches as you get very very close to a given x value. And the limit at x can only exist if the limit from the right and the limit from the left are equal and also equal to $f(x)$ at that point. I don't know if that's as formal as you wanted but --
57 MZ: OK. I'm going to ask you a different question. What about a formal definition of derivative?
58 Derick: That would be $\frac{f(x+h)-f(x)}{h}$
59 MZ: [writes this difference] How does this formal definition relate to the slope or rate of change idea?
60 Derick: --you have x and h is some given very very small amount.
61 MZ: OK.
62 Derick: So the $f(x+h)-f(x)$ is the change in y which -- [pause] How does that work?
63 Derick: I know how to use it. I just don't know how to explain it. That's the problem. So that is the change in y --
64 MZ: So this numerator is the change in y ?
65 Derick: Yeah, that'll be the change in the two y values. [pause] Can't assume that.
66 MZ: What can't I assume?
67 Derick: I'm trying to fit this into the slope. It's not fitting because on the bottom unless you assume that x has been to 0 -- [pause]
68 Derick: Yeah, I'm pretty much lost.

Table A.10 summarizes the transcript of Derick's first interview. Derick recalls that the derivative is related to slope, rate of change, acceleration, and velocity. He states the instantaneous nature of derivative in each of these models. He describes a graphical limiting process, and the details of the ratio both graphically and symbolically. When asked about a formal definition, he forgets the limit, writing $\frac{f(x+h)-f(x)}{h}$. He decides that a derivative must be a function because for every x value on a function there's only one derivative value so the derivative function will have only one y value for each x value.

Derick's understanding of derivative shows many connections. He states that slope and rate of change are the same [ln 5]. He knows that the slope of a velocity curve

Table A.10. Derick: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎	○			
What can derivatives be useful for?	○	◎	◎ ◎		acceleration max/min
Explain what a derivative is to someone who's an AB student or precalc student who hasn't studied it yet.	◉ ●			●	misstatement (d=tl)
Explain what a derivative is to someone who doesn't know anything about math?	◎	◎	○ ○ ◎		acceleration acceleration
How can you tell if a function is differentiable?	◯				
Is derivative related to speed or velocity?		○	○ ○ ○		acceleration
Is derivative related to line or linear?	○				
Is derivative related to measurement?		◎			
Is derivative related to prediction or approximation?	◉				
Is derivative related to optimization?					max/min
Is derivative related to limit?	●				
Is derivative related to integral?		○	○		
Is the derivative a function?	◯			◯	

Table A.10. Derick: Interview 1 Circle Diagrams (continued)

	Slope	Rate	Vel.	Sym.	
Did you learn a formal definition of derivative?				◉	
How does the formal definition of derivative relate to slope or rate of change?	○			•	
Summary	◉	◎	◎	◉	

is acceleration, that the rate of change of velocity is acceleration, and the rate of change of position is speed [ln 17]. He knows that slope has the form change in y over change in x, but he has trouble matching this to his difference quotient, $\frac{f(x+h)-f(x)}{h}$. He recognizes that the numerator is the change in y, but does not think the denominator looks like the change in x [58-67].

Derick makes only two misstatements and these seem more linguistic contrivances than actual misunderstandings. Early in the interview Derick says that derivatives can be used "to find the shape of a slope" [ln 9] when one is trying to find extrema. Here the word slope seems to refer to the curve rather than the steepness of that curve. In a later statement, Derick correctly explains the process of approximating the slope at a point by finding the slope of two points that are closer and closer together. He ends his description with the following two statements, "[the result of this process is] the slope of the tangent line at that one point. The tangent line to the graph is the derivative" [ln 15]. The interviewer does not ask Derick any further questions about this statement. Since Derick has a clear and detailed understanding of the derivative as slope and has just stated that the approximation process gives the slope of the tangent line, I suspect that he knows that the derivative is the slope of the tangent line and not the tangent line itself. If so, why would he make this misstatement? Perhaps this is an example of individual metonymy for the phrase "slope of the tangent line". Since the phrase is long it is natural for an individual (consciously or subconsciously) to want to shorten it in his speech by using only a part of it to represent the whole. His statement does just that.

QOTD #6

Find the derivatives of the following four functions:

$f(x) = (x-1)^2(x^2-4)$

$g(x) = \dfrac{x-1}{\sqrt{5-x^3}}$

$h(x) = \sin x$

$j(x) = \ln x$

Date: September 20, 1993. This question occurs prior to the class learning about short-cut rules for taking derivatives of various forms.

Response: Derick is absent on this day.

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be using your knowledge of derivative.

$y = \log(x^3) \qquad y = \dfrac{x}{|x|} \qquad y = x^e \qquad y = e$

Date: September 21, 1993. This question also occurs before the class studies short-cut rules for taking derivatives but after they have studied the limit definition of derivative.

Response: "$y = \dfrac{x}{|x|}$ This cannot be the derivative because it only gives values of 1 or -1, while the slope of $y = \ln x$ includes other values."

Comment: This question is presented to the students since no student correctly stated the derivative of $y = \ln x$ in the previous Question of the Day.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$? Why not?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: "a) No, because $\cot x$ can be negative. Derivative of $\tan x$ is always positive

b) The derivative of $\tan x$ is $\sec x$"

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. Derick shows that he can correctly state the limit of the difference quotient definition of derivative and use it both symbolically and numerically to calculate a derivative value. He also knows to estimate the derivative at a particular point by finding two nearby points and calculating a difference quotient for those two points. Given the graph of a position function for a car he correctly answers questions about the speed and direction of the car. Given the graph of a function, he is able to sketch a correct graph for the derivative function. Derick's only difficulty with two complex derivative problems is his use of $-\tan x \sec x$ for the derivative of the tangent function.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: "I didn't really learn any new stuff, but I do understand the Chain Rule and the trig derivatives better now. I did learn the Taylor form, but I still don't know why."

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "1) We talked about monotonic functions and their relation to the Intermediate Value Theorem and the definition of critical points.

2) Nothing was really new, but we did clarify a few points."

Comment: Since the researcher had not been present the previous day, this question is presented both as a means for the researcher to see the material covered and to ascertain the students' understanding of it.

Test 3

This test includes problems about Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Derick calculates the Taylor polynomial for $\tan(x)$ at $x = 0$ to be $x + \frac{2}{3}x^3$ instead of $x + \frac{2}{3!}x^3$. Otherwise, he correctly uses the first and second derivatives of the position function to find the speed and acceleration of an object at a given time. He is able to use the graph of a derivative function to estimate when the original function is increasing or decreasing and concave up or concave down. He also correctly notes the locations of the local maxima and minima of the original function. His only error on this section is to omit the x value where the derivative is undefined as a critical number.

QOTD #11

Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: "Finding the profit that a company made on a given day by looking at the derivative of the revenue graph."

Test 4

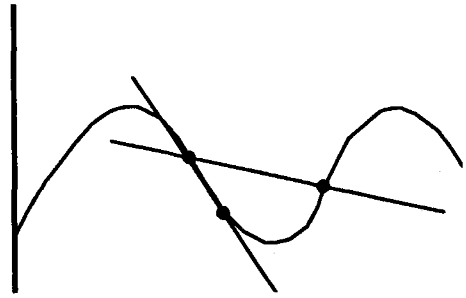
Two weeks later the class has a test on the applications of derivatives. Derick correctly uses derivatives to solve three traditional max/min problems and three traditional related rate problems. He also correctly calculates the derivative of an implicitly defined function.

Interview 2

The second interview occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by Table A.11 and a discussion.

1	MZ:	The first question is what is a derivative?
2	Derick:	A derivative is the change of a function at a particular point.
3	MZ:	Is there anything else that comes to your mind for what is a derivative?
4	Derick:	Rate of change. Slope of a tangent line. I know there should be more and I know [Mr. Forrest] would kill me if he heard that. There are like 18,000 applications but that's basically the main ones, rate of change and slopes of the tangent line.
5	MZ:	What are derivatives useful for?
6	Derick:	Well, you can use them to analyze how a function is going to act and how it can be graphed. You can also use it to determine accelerations and velocities and rates of profits and losses at given points in time or a given distance or whatever. Instead of just finding an average over a period, you can find it exactly at one point.
7	MZ:	And you're finding it exactly at one point. What was one of those examples?
8	Derick:	Acceleration or velocity.
9	MZ:	If you were going to explain this to someone who doesn't have much math background, what a derivative is, what would you tell him?
10	Derick:	And I assume he knows about slopes and functions?
11	MZ:	Well, OK. Do that one first. Assume somebody knows about slopes and functions.
12	Derick:	Well, for instance if you're looking at the function of a graph, and you take two points and draw a line between them--

[Derick draws:



]

13 Derick: OK, so you've got this function that's kind of a sine function.

14 MZ: OK, fine.

15 Derick: So you take two points on here and draw a line between them. That's going to be-- You can find the slope of that and that'll give you an average slope over this period.

16 MZ: OK.

17 Derick: But that slope won't be the same as the slope at any point in between this period. So say you wanted to find the point there which is between these two.

18 MZ: OK.

19 Derick: Finding the derivative there would be like taking these points and squeezing them together until they're very very very close to that point and finding the slope of the secant line between them. And we're imagining that the two points eventually meet at that point, and the slope that they would form if they were exactly together--

20 MZ: OK.

21 Derick: And that will end up being the slope of the line that's tangent to the graph at that point, just touching it at that one point. The slope of that line is the derivative of that graph at that point.

22 MZ: What if you were going to explain it to somebody who didn't know about slopes and functions?

23 Derick: OK. We'll take a real world example. Say you have a business, and you have a chart of how much money it made or lost on July 1st, July 2nd, July 3rd, all the way down, the whole month. Now you could find the average amount that it gained or loss each day over the month just by taking the first one and the last one-- No, you would have to take all of them and add it up and divide by 30 or 31, whichever.

24 MZ: For the month, right?

25 Derick: Exactly, how ever many days there are. But that won't tell you how much you gained or lost on the 16th. To do that you would have to take the profits and losses at smaller and smaller intervals around the 16th. And then assuming you could get it down to like each hourly or whatever, you could take smaller and smaller intervals until you have an exact amount of how much you gained this exact second theoretically. At any given point in time, how much money was gained or lost right then as opposed to an average amount.

26 MZ: OK. So, the part you are calculating at the end, the derivative, is how much you gained or lost--

27 Derick: The rate of change at that one point.

28 MZ: What was the original function that we had or the original data?

29 Derick: Yeah, that-- If you found like average change--

30 MZ: What was the output?

31 Derick: Well, I mean, if you turn that into a function, if that's what you mean--

32 MZ: Originally we had a list of for each day how much it gained or lost?

33 Derick: Right.

34 MZ: But we're also, the derivative is also how much it gained or lost.

35 Derick: Right. OK. Actually, I shouldn't have used that. The derivative is not how much it gained or lost, but how much money we had on that day. How much total assets. Then by taking the first one and the last one, adding them up and dividing by 2 we can find an average gain or loss over the month. [short pause] No wait. That isn't right. By subtracting them and then dividing by 2. Otherwise you just find average assets. Yeah. You have to find the change between those two and that would be-- [short pause] You'd have to divide it by 30 then wouldn't you. Now that I think about it this is a bad example. OK. So scratch that entire example and forget we ever talked about it.

36 MZ: [laughing at Derick's humorous tone]

37 Derick: So you are driving down the street.

38 MZ: OK.

39 Derick: Driving down the highway, and the speed limit is 55.

40 MZ: OK.

41 Derick: And you're speeding up and slowing down and speeding up and slowing down because you're on the Kennedy during rush hour or whatever.

42 MZ: Oh, no.

43 Derick: And the cop pulls you over and says you're back there speeding. And assuming you had graph paper and a computer that recorded your speed at any given point or whatever, you could show him that your-- A computer that picked up your position, how far you drove at each second or whatever, you could show him, by graphing out these points of your position, the derivative at the point where he said you were speeding. And you could prove to him that you really were or were not speeding at that point by seeing whether the slope was 55 or greater or not.

44 MZ: OK, so you're going to graph the position.

45 Derick: The position graph, right, and then find the slopes at given points. Find the slopes at given points and see what they are at the point where he said you were speeding. A slightly better example.

46 MZ: How do you tell if the function is differentiable?

47 Derick: In general, it cannot have any sharp cusps or sharp turns. It has to be continuous over the entire interval. There can't be any holes. There can't be any jumps, stuff like that. It cannot have any vertical areas that are-- Of course, if it did that it wouldn't be a function anyway so that's kind of redundant.

48 Derick: To be differentiable it has to be defined at that point. The limit coming in-- The limit of the function value itself coming in from one side and the other side have to be equal. And then the limits of the derivatives from the left and from the right have to be equal. The limit from each side has to be equal because otherwise it'll end up having the sharp spike or whatever.

49 MZ: Each of these words, does it have to do with derivative or not. Slope?

50 Derick: The derivative is the slope of the tangent line at a point.

51 MZ: Speed or velocity?

52 Derick: To find the velocity you take the derivative of the position graph.

53 MZ: Change or rate of change.

54 Derick: That's what a derivative is basically. The change or rate of change at a point is what the derivative is.

55 MZ: So, let me ask a question. Would you say, when somebody says slope, would you say that's what the derivative is or is it--

56 Derick: The slope of the tangent line to a graph at a particular point.

57 MZ: So in both of these cases you would say that's what the derivative is, both slope and rate of change.

58 Derick: Yeah, but at a certain point.

59 MZ: I have to make sure to say at a certain point.

60 Derick: Because otherwise it turns into a secant or average rate of change.

61 MZ: Line or linear?

62 Derick: The derivative of a linear graph is always going to be a constant. Let's see, what else? The derivative of a second degree equation is always going to be linear. x^2 derivative is $2x$. And then you could just work your way up from there. The second derivative of a third degree equation.

63 MZ: Measurement?

64 Derick: I guess you could say that the derivative is the measurement of the slope at a point. I don't know how else to fit that in.

65 MZ: Prediction or approximation?

66 Derick: If you're trying to find the derivative at a point, if you take the limit of the derivatives coming in from both sides and get very very close that'll give you a pretty good approximation of what the derivative at that point is going to be. So if you want to find the derivative at like $x = 1$, if you find the slope between the two points like $x = .9999$ and $x = 1.001$. If you find the slope between those two points, that'll give you a pretty good approximation of the derivative.

67 MZ: Optimization?

68 Derick: When you're trying to optimize something to find a minimum or maximum value, you have to find the point where the derivative equals zero.

69 MZ: OK, and why does it turn out that that's the point that you--

70 Derick: Because that will end up being a maximum or minimum point on the graph where it changes from positive to negative. On the graph of your original function.

71 MZ: On the graph of the original function, at a maximum, what changes from positive to negative?

72 Derick: The slope changes from positive to negative. The graph either stops increasing and starts decreasing or the other way around.

73 MZ: So that makes the derivative zero?

74 Derick: It ends up being zero at that point.

75 MZ: Continuity?

76 Derick: For something to be differentiable, it has to be continuous either over a given period or at a certain point.

77 MZ: If it's continuous will it necessarily be--

78 Derick: Not always. But it has to at least be continuous. That's one of the factors.

79 MZ: Limit?

80 Derick: Well the derivative is the limit coming from the-- The limits have to be equal coming in from each side of the derivative function. For it to be differentiable at a point the two limits coming in from the right and the left have to be equal. And the derivative at that point will end up being those limits.

81 MZ: The limit of--

82 Derick: The derivative coming in from the right and the left.

83 MZ: OK. Integral?

84 Derick: Integral is kind of the reverse of derivative. If you have a function for the derivative of a graph you can find the original graph from that. Or a guess at the original graph. You won't know what the y intercept is. It can move up or down.

85 MZ: How come it can move up or down?

86 Derick: 'Cause when you take a graph and move it up or down any amount the slope at a given point won't change. It always stays the same.

87 MZ: Do you know what's meant by an antiderivative?

88 Derick: I'm assuming that's the same thing as an integral, but I'm not sure. I always use those two words--

89 MZ: To mean the same thing?

90 Derick: Yeah, but I'm not sure if that's exactly right.

91 MZ: I think that in the context you're explaining it they are the same. Is the derivative a function?

92 Derick: It should be. Yeah, it would have to be. Assuming you're taking the derivative of a function to begin with, it'll end up having to be a function.

93 MZ: Why would it have to be a function?

94 Derick: Because if you take a function, your original function, at a point there's only one tangent line that'll match up there at a point, and so there's only one slope for that tangent line. If you don't start out with a function, then the derivative doesn't necessarily have to be a function either.

95 MZ: OK. Give a formal mathematical definition for derivative.

96 Derick: Oh, no. Not this. OK. limit as x approaches a-- a is the point that you're trying to find the derivative of.

97 Derick: [writes: $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$]

98 Derick: Yeah, I think that should work.

99 MZ: How does this equation fit into some of the other pictures you've been describing?

100 Derick: $\frac{f(x) - f(a)}{x - a}$ with a given point for x and a, that's going to be the secant line, the slope of the secant line between x and a, between the points x and a. If you take the limit as x approaches a, as gets very very near to a, that secant line, the slope of that secant line, will approximate the slope of the tangent line at $x = a$ which will be the derivative.

101 MZ: OK. Have you ever seen Newton's method?

102 Derick: We did this last year, and I know it's a method for finding b. I think it's a method for finding the derivative at b. No, it's a method for finding a function value at a point by using the derivatives.

103 MZ: Yeah--

104 Derick: I'm real iffy on it. I didn't understand it last year and don't understand it now because we haven't done it. I know it involved a lot of plugging and chugging though, because you had 8 or 9 steps to get it exact.

105 MZ: OK. So you remember kind of a lot about it, but just not--

106 Derick: Kind of vaguely, but not exactly.

107 MZ: Intermediate Value Theorem?

108 Derick: Well, that's the idea that if you have a continuous function and you have two points, a and b and $f(a)$ is not equal to $f(b)$. $f(c)$ is between

$f(a)$ and $f(b)$. [has written: a b
$f(a) \neq f(b)$
$f(c)$]

108 MZ: OK.

109 Derick: Then there has to be a point c between a and b that that works for, that $f(c)$ will be between $f(a)$ and $f(b)$.

110 MZ: And you said this function has to be continuous.

111 Derick: Over the interval (a, b).

112 MZ: Does this have to do with derivative at all, the Intermediate Value Theorem?

113 Derick: Not in itself. There is an intermediate value theorem for derivatives which is: if the function is continuous and differentiable over (a, b), there will be a point c such that the derivative at c is equal to the slope between a and b, the slope of the secant line between a and b.

114 MZ: That's the Mean Value Theorem which is next, what I was going to ask you.

115 Derick: No, we haven't gotten around to figuring that one out yet.

116 MZ: Let's see. Say I give you a function. What can you tell me about-- OK, I give you a function and I say, this is the derivative. What can you tell me about the original function?

117 Derick: By finding where the derivative equals zero I can find out where there is a local extrema or where there could be a local extrema. By finding where the derivative is undefined I can find out where the original function either has a sharp cusp or is undefined. You can't be sure about that until you plug it back into the original. And also by finding the derivative of the derivative I can use that to find the concavity and also points of inflection.

118 MZ: Thanks. [turns tape recorder off]

119 MZ: So you were thinking that I was going to give you an equation right?

120 Derick: And that'll be the derivative of the function, right?

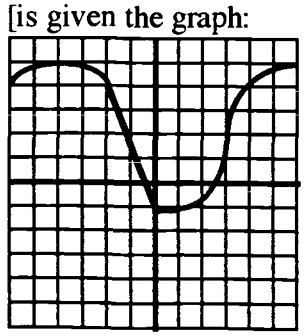
121 MZ: Right. Now, what if I give you a graph and I say this is the derivative?

122 Derick: Oh, OK.

123 MZ: Is that easier or harder to find stuff?

124 Derick: Well, it's pretty much the same thing. And again you can't know anything specific without knowing some other things, by plugging back into the original. But if you had the graph of the derivative equation or something, you can find out--

125 MZ: I'll give you a specific example. You can play with this one. You've seen this graph before, #24. Pretend this is the derivative and you want to--
[is given the graph:



]

126 Derick: Find something out about the original.

127 MZ: Right, sketch it or something.

128 Derick: You're going to know that there's going to be a local maximum at $x = -.8$ or whatever.

129 MZ: Whatever that is.

130 Derick: Yeah, and a local minimum at about $x = 2.1$.

131 MZ: Now, how did you know that this one was a max and that was a min?

132 Derick: Well, in both cases the derivative equals 0 at those points. For $x = -.8$ the derivative starts out positive and goes to negative. So on the original graph it's going to be increasing then decreasing so there's going to be a local max at that point, and just the opposite for $x = 2.1$ or whatever we've decided on, starts on negative and goes to positive.

133 MZ: OK. I'm going to have you do this on the back side of this page.

134 Derick: By taking the derivative of that derivative you can find out that there's going to be an inflection point at about $x = -4$ because that's where there's a local max on the derivative graph. There's also going to be an inflection point at $x = 1.5$, and I guess that's about 6.

135 MZ: And how come a max on the first derivative gives you an inflection point on the original function?

136 Derick: Because that's where the original graph starts increasing or decreasing less or more. The original graph will have been increasing rapidly and then slow down and increase less. It is still going up, just not as quickly at that point, because on the derivative graph it's still positive there, but it's going to be less and less positive. So the original graph is kind of flattening out at that point.

137 MZ: What you just drew here would be an inflection point that matches something like this first maximum here?

138 Derick: Uh, yeah. It's going up faster and faster and then at that inflection point it begins to slow down its increase. [Derick has drawn a curve fitting this description.]

139 MZ: What about at like this minimum point on the derivative?

140 Derick: That's something like that, where it's going down and down and down faster and then beginning to level off. Again it's still decreasing, but not as much. [Derick has drawn a curve fitting this description.]

141 MZ: Can you kind of put this together and maybe give a sketch the original function?

142 Derick: I can try.

143 MZ: Give it a shot. It doesn't have to be real accurate.

144 Derick: Well, I know there's going to be a local max at-- No, that's not right. A local max about $-.8$. We'll make that there and one at around there, a local min. And there's going to be an inflection point at about $x = -4$. And it's-- Put that down there then. So it's increasing more rapidly there and it begins to level off around $x = -4$ until you hit $x = -.8$ at which point there is a local max.

145 MZ: OK.

146 Derick: And then it begins decreasing at that point. Decreasing faster and faster until about .5 where it begins to level off and then starts increasing again faster and faster until x equals about .6

147 MZ: x equals?

148 Derick: No, 6. And there should be a local max somewhere around there, a little bit farther out. [Derick has drawn a curve fitting his description.]

149 MZ: Right, but we don't see what happens there. Well that's good. That's it. I ran out of questions.

Table A.11. Derick: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	○	◎ ○			misstatement (d=change) tool
What can derivatives be useful for?		◎	◎ ◎		acceleration
Explain what a derivative is to someone who's an AB student or precalc student who hasn't studied it yet.	○ ◉				
Explain what a derivative is to someone who doesn't know anything about math.	○ ◎	● ◎	○ ◎		misstatement (confusing two types of averages) misstatement (d=change)
How can you tell if a function is differentiable?	●			↦	
Is derivative related to slope?	◎				
Is derivative related to speed or velocity?			○		
Is derivative related to line or linear?				↦	
Is derivative related to measurement?	◎				
Is derivative related to prediction or approximation?	● ○				
Is derivative related to optimization?	○				max/min in/decreasing
Is derivative related to limit?	◉				

Table A.11. Derick: Interview 2 Circle Diagrams (continued)

	Slope	Rate	Vel.	Sym.	
Is derivative related to integral?	○ ◎			↦	
Is the derivative a function?	◉			○	
Did you learn a formal definition of derivative?		∘		◉	
Can you relate your statement of the formal definition of derivative to the derivative as slope?	◉			◉	
Is derivative related to the Mean Value Theorem?	•				
Given the derivative, what can you tell me about the original function?				↦	extrema concavity inflection pts
Given the graph of the derivative, can you sketch the original function?		∘			max/min in/decreasing inflection pt
Summary	◉	◉	◎	◉	

As the table illustrates, Derick mentions a graphical interpretation of derivative more often than any other interpretation. However, he also mentions rate of change, velocity, and a symbolic formulation of derivative, each without being specifically asked to do so. Derick recognizes the instantaneous nature of the derivative in each of these forms. He describes a limiting process in terms of slope, rate of change, and the symbolic interpretation. Derick describes the derivative as a function on several occasions, but not when stating the formal limit definition, which he writes as $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Derick makes several connections between the different interpretations of the derivative. When first asked, "What is a derivative?" he mentions both rate of change and slope of a tangent line [ln 4], but does not explicitly state that these are the same.

Similarly when asked what derivatives are useful for, he mentions instantaneous velocity and instantaneous rate without saying that they are the same [ln 6]. He specifically states that for a graph of position the speed at a point is the same as the slope at a point [ln 43]. He is also able to explain correctly a graphical interpretation for the limit and ratio of his formal definition for the derivative at a point, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ [ln 100].

When first asked "What is a derivative?", Derick makes the misstatement, "A derivative is the change in a function at a particular point" [ln 2]. His other confusion involves an example he is using to describe the difference between average and instantaneous rate of change. He says that we have a table of how much money was gained or lost every day for one month. To find the average gain or loss we add up these numbers and divide by the number of days in the month. His statement is true, but it does not fit what we usually think of in calculus as average rate of change and does not lead to describing instantaneous rate of change for this setting. He tries anyway discussing how to determine exactly how much was gained or lost on the 16th. "You could take smaller and smaller intervals [around the 16th] until you have an exact amount of how much you gained this exact second" [ln 25]. When the interviewer points out that in his example both the original function and the derivative tell how much was gained or lost, Derick changes the table to list how much money the business had on a given day. However, he is still confused about how to find the average rate of change. First he wants to add the amounts for the 1st and 31st and divide by 2. Then he wants to subtract the amounts for the 1st and 31st and divide by 2. He finally decides to switch to a velocity example, in which he no longer tries to explain the relationship of average to instantaneous rate of change.

Derick can explain the relationship between average and instantaneous values graphically and symbolically but struggles when applying them to a physical situation. One source of confusion is the distinction between calculating an average rate of change by finding the arithmetic average of rate of change values, and calculating an average rate of change by finding a difference quotient using no rate of change values. Perhaps the discrete nature of Derick's example leads him to think of an arithmetic average.

For the last part of the second interview Derick is asked to graph an original function when given the graph of the derivative function. He correctly answers questions about the behavior of the original function, a competence he also shows on the chapter 4 test given two weeks prior to the interview, and he goes on to sketch a reasonable graph for the original function.

Because Derick already has a very complete understanding of derivative at the time of the first interview, his performance on the second interview shows only a slight improvement in terms of completeness. The additions are that in the second interview he discusses the function process in terms of the slope interpretation, a limiting process in the context of rate of change, and the limit in the formal definition. He also adds a connection between slope and velocity and a connection between the graphical limiting process and the limit of the difference quotient. On the other hand, Derick makes more misstatements during the second interview. He states that the derivative is the change in a function and confuses two types of averages.

QOTD #12

What is the most important idea that we have studied so far in this class?

Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

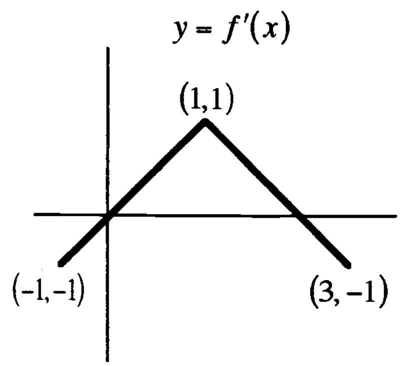
Response: "now we can go from slope formulas to original formulas as well as vice versa."

Interview 3

The third interview occurs during the three days after the test on differential equations, and antiderivatives by substitution and by parts. The first part of the interview is a summary of Derick's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike the second interview, the point $(1,0)$ on the original function is given so that only one solution is possible.

1 MZ: OK. It's going to be the graph of the derivative. [starts drawing and labeling x and y axes]

2 Derick: The graph of the derivative.



3 MZ: OK. So this is the point $(1,1)$. This is $(-1,-1)$. That's connecting. [draws the two points and the straight line connecting them] This is $(3,-1)$ and that's connecting. [draws the point and connects it to $(1,1)$ with a straight line] So this is your function. Your derivative function, $y = f'(x)$. [writes $y = f'(x)$] And we don't care what happens outside of -1 and 3. We just care about this domain.

4 Derick: OK.

5 MZ: And you have the initial condition that $f(1) = 0$ [writes: $f(1) = 0$] And you get to draw the original function. [hands notebook to Derick]

6 Derick: OK. Let me tackle this here. [10 second pause] I'm going to try to do it as a slope field. So that's a negative 1 slope. 0 slope. 1 slope. 0 and -1. [draws small lines with the indicated slope over the appropriate value of x] $f(1) = 0$ so it's going to go up there and down like that. [plots the point $(1,0)$ and then sketches in a curve going through that point and following the slope field] Something kind of sort of like that.

7 MZ: OK. Uhm, how confident do you feel about how high and low you went for these. [indicating max and min on his sketch]

8 Derick: Well, not really at all-- Well -- That's sort of hard to tell without a bit more information. If I had maybe more of these [indicates $f(1) = 0$].

9 MZ: More points?

10 Derick: Something like that would have helped, but -- The stuff in the middle you kind of have to guess at.

11 MZ: Do you feel like it could go down as low as say like –3 here [indicating his min at $(0,-1)$] ? I mean, do you have any feel for like approximately--

12 Derick: Well, because along here [indicating from $x=-1$ to $x=0$ on his graph]-- It starts out here [at $x=-1$] with a slope of -1. But because this slope keeps decreasing, it won't go down much farther than that I don't think. It won't go down very far because this slope is continually decreasing. So it's going to be going up, going to the right, it's got to be going up less and less. So it's got to be kind of -- flattening out along that area there.

13 MZ: Flattening out towards, what is that $x=0$?

14 Derick: $x=0$, yeah.

15 MZ: So it can't just --

16 Derick: It can't go very far down.

17 MZ: OK. Do you know of any other methods for doing this same problem?

18 Derick: Well, yeah because you know that where the slope is going to be equal to 0 that's where there's going to be a max or a min point. And like here the slope is equal to zero where $x=0$. And the other parts the slope is negative so that's going to have to be a local minimum. And then the same thing where $x=2$, where $f(x)=0$. And there the slope goes from positive to negative so it's going to have to be a local max.

19 MZ: Mm hmm.

20 Derick: And then once you have that kind of sort of idea where they fall, you can find the derivative of the derivative and that would give you a good idea for concavity and inflection. And then just use the $f(1)=0$ to find the initial point and sketch.

21 MZ: Where is the inflection point for the original function?

22 Derick: That's going to be where the derivative of that is 0 or undefined which would have to be where $x=1$. Yeah, it ends up being an inflection point on the graph. And then I assume -- if it's suppose to be following a sort of a set pattern, there should be one out around $x=-2$. Somewhere out around there -- but that's an assumption that I probably shouldn't be making. [MZ laughs] Yeah, so the only one that we can know for sure is that there is one at $x=1$.

23 MZ: OK. Why were you thinking like around $x=-2$ there might be one?

24 Derick: Because that looked like -- just from the part that I graphed out, it looks like it should be a periodic function.

25 MZ: Oh. OK, right.

26 Derick: But that's an assumption that can't really be made without more information.

27 MZ: What's the value of the second derivative at that inflection point?

28 Derick: The value of the second derivative in this case would have to be undefined because there is a sharp turn in the derivative graph.

29 MZ: OK.

30 Derick: If that were just a curve, it would be 0, but the way it is now, it would have to be undefined.

31 MZ: What does the second derivative being undefined say about the concavity of original function at that point?

32 Derick: It's going to change at that point or it could change rather. You'd have to plug in values into a second derivative checker, which I call that -- the second derivative equation and see whether it changes from positive to negative or stays the same or whatever -- but at that point it could change concavity. It could be a point of inflection. And by the look of things there should be one there.

Derick correctly sketches the graph of the original function by using a slope field. When asked if he knows of another method to do the problem, he describes using information about the slope of the original function to determine the location of its maximum and minimum. The slope field method is based on material covered by the class between the second and third interviews. The other method is the same method that Derick successfully uses in the second interview. Derick's only imperfection is that he does not know exactly how high or low to make the extrema. He marks the y values of the extrema as ± 1 instead of the correct answer of $\pm \frac{1}{2}$.

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

33 MZ: New question -- what is a definite integral?
34 Derick: A definite integral is the area under a -- bounded by the slope and the x -axis from a given point to a given point -- two given x values.
35 Derick: The area in between these two x values, the slope and the x -axis.
36 MZ: OK. So when you say the slope, you mean --
37 Derick: An $f(x)$ graph. A function graph.
38 MZ: OK. How bout an indefinite integral?
39 Derick: An indefinite integral. That is, uhm -- See this one I can never explain real well because it kind of stretches over into the antiderivative. Now I'm not sure if this is exactly right, but the indefinite integral of a function is the function whose derivative is that function.
40 MZ: OK.
41 MZ: Is that different than an antiderivative?
42 Derick: I know that they are. I just can't think of how. At least I think they are. They're linked in that manner, but I know that there's a difference there that I can't even think of right now between an indefinite and an antiderivative.
43 MZ: Hmm. [short pause] How about the definite and the indefinite integral -- how are they related?
44 Derick: Well, the definite integral you have two set points that you're finding in between there while the indefinite is just all x values.
45 MZ: So does the indefinite integral have anything to do with area?

46 Derick: I guess that you could say that the indefinite integral is the area -- except that would have to be infinite.

47 MZ: How are you thinking about it that it would be --

48 Derick: That it would be the area under the curve for the entire curve. But that would end up being probably either be infinite or zero depending on if its like a -- I mean if its like a sine curve that'll obviously end up being zero--

49 MZ: Oh, right.

50 Derick: But if it's basically -- Yeah, I suppose you could say that it's the limit of x approaches infinity of the area under the curve would end up being indefinite.

51 MZ: Hmm. OK. So it's just the area under the whole curve instead of a --

52 Derick: -- instead of a chunk of it, a section.

53 MZ: OK. What does the derivative have to do with integrals?

54 Derick: Well, the derivative and the integral are sort of opposite inverse sort of functions. If you take the derivative of an integral of a function, you should end up getting the original function.

55 MZ: OK.

56 Derick: Or vice versa.

57 MZ: Do you remember what either of the fundamental theorems say?

58 Derick: OK. I know that the second one says that the definite integral from -- we're going to call it two points a and b --

59 MZ: OK.

60 Derick: -- is going to be the antiderivative of b minus the antiderivative of a. I believe or it might-- I think -- yeah.

61 Derick: [writes: $\int_a^b f(x)\,dx = F(a) - F(b)$

$F'(x) = f(x)$]

62 MZ: OK.

63 Derick: Two variables would have been more useful but whatever.

64 MZ: But they never do it that way, do they?

65 Derick: Too confusing.

66 MZ: What about the First Fundamental Theorem?

67 Derick: OK, the First Fundamental Theorem -- and this one I was always a bit kind of twitchy on --

68 MZ: Yeah.

69 Derick: -- but I believe it's something along the lines of this -- [writes: $\int f'(x)\,dx = f(x)$].

70 MZ: OK.

71 MZ: Let me ask you this question. What is a graphical interpretation of the second fundamental theorem that you wrote down here?

72 Derick: Graphical representation. That's going to be like for instance -- [sketches a graph in the first quadrant that is a wavy horizontal line; labels it $F(x)$; labels a and b on the x-axis and draws a vertical line to the curve from each of those places] It should be this value -- whatever that's going to end up being. $F(b) - F(a)$ which in this case is probably ends up being somewhere around zero.

73 MZ: OK.

74 Derick: Should be the total area under the curve of the derivative of that which makes sense because the derivatives all going to be pretty close to 0. In

this particular case the derivatives going to always end up being kind of close to 0 and taking these two points will probably end up being 0, the total area underneath.

75 MZ: Mmm.

76 Derick: Just 'cause that's kind of sort of periodic.

77 MZ: OK. Is there any graphical interpretation for the first fundamental theorem?

78 Derick: Mmm. [15 second pause] Mm, well, that might work. [starts drawing graph, again just in the first quadrant, of $y = \frac{1}{x}$ and labels as $\frac{1}{x}$;graphs something like $y = \ln|x|$ and labels it as $\ln|x|$; also labels a and b on the x -axis with vertical lines up to the graph of $y = \frac{1}{x}$.] We're going to call that $\frac{1}{x}$.

79 MZ: OK.

80 Derick: It's not exactly, but it's useful. And that's going to be looking-- Uhm, now if the derivative of $\ln|x|$ is going to be $\frac{1}{x}$.

81 MZ: OK.

82 Derick: And -- so that -- that's for this requirement of it.

83 MZ: OK.

84 Derick: The area from -- we're also going to choose two points here, a and b, -- the total area under $\frac{1}{x}$ between a and b should end up being $\ln|x|$ at that point, at the point a I guess it would have to be. So that should have to be the definite integral shouldn't it?

85 MZ: Oh, you're thinking, for where you wrote down about the first fundamental theorem --

86 Derick: Yeah, I'm thinking that should have to be a definite integral of $f'(x)$.

87 MZ: Hmm. So then what -- [short pause] Hmm. Maybe I should ask you this. Do you remember when we did that area function, capital $A(x)$?

88 Derick: Vaguely. Uhm, what about it?

89 MZ: 'Cause -- It can be related to the first fundamental theorem so that's why I was going to ask you if you remembered that.

90 Derick: [15 second pause] See I'm thinking of one, but it's not the one you're thinking of.

91 MZ: Well, which one are you thinking of?

92 Derick: I've got like the Simpson's rule and the right and left endpoints and all that stuff.

93 MZ: Oh right.

94 Derick: I'm thinking all of them, but I know that's not what you're thinking of.

95 MZ: So you can kind of picture that capital A of x that I'm referring to, but it's not associated with anything else.

96 Derick: Yeah, I have a general idea of it, and I know that if I had to do it I could find the area, but I don't know that one exact function.

97 MZ: OK, uhm, let me give you-- [speaks while writing] This is an easy curve, $y = x^2$. [starts drawing axes, the curve, labels 0 and 1 on the x

-axis, shades in the appropriate region and labels as $y = x^2$; completes these steps as continues talking]

98 Derick: Yeah.

99 MZ: And all I'm curious for you to tell me is the area from 0 to 1 underneath this curve.

100 Derick: Area from 0 to 1 underneath that curve.

101 MZ: Yeah.

102 Derick: So [short pause] I'm going to call it big $Y(x)$ should be $\frac{1}{3}x^3$. [writes: $Y(x) = \frac{1}{3}x^3$]

103 MZ: OK.

104 Derick: So it should be big $Y(1)$ minus big $Y(0)$ which is $\frac{1}{3}$ minus 0 which is one third. [writes: $Y(1) - Y(0) = \frac{1}{3} - 0 = \frac{1}{3}$

105 MZ: OK.

106 Derick: Should be the area between 0 and 1. Now is it?

107 MZ: I don't know. What do you think?

108 Derick: Well, yeah that seems accurate because the value of x^2 at 1 is going to be equal to 1. So we know that the total area has to end up being less than 1 because it's in a square 1 a side, but it's less than one half of it. So yeah, that seems accurate.

109 MZ: Yeah, I agree. Let me ask you a question. You did this kind of by this method up here of the second fundamental theorem [flips the page back to refer to his written statement of the 2nd fundamental theorem].

110 Derick: Yeah.

111 MZ: Uhm, [short pause] How do you feel about the order of the a and the b versus the order of the a and the b here? Is that the same thing that you did?

112 Derick: Uhm, this should be b-- $y(x)dx$ from 0 to 1, since 0 is a and 1 is b, this should be $F(b) - F(a)$. [writes in front of area calculation: $\int_{0=a}^{1=b} y(x)\,dx$]

113 MZ: OK. So you think this is right what you did here [in calculating the area as $\frac{1}{3}$]?

114 Derick: Yeah, I think that this way is right because I know it has to be positive because it's all above the x -axis.

115 MZ: OK.

116 Derick: So yeah, switching those last two. [changes b to a and a to b is his original statement of the second fundamental theorem]

117 MZ: OK. [short pause] Slope fields -- Do you see them as being more related to derivatives or integrals?

118 Derick: I see them more related to derivatives because-- Well, I don't know. That's one of the ways you can solve a differential equation is by doing the slope field of it and then if you have an initial condition you can do it that way. So actually that's really finding the integral of that differential equation.

119 MZ: OK.

120 Derick: So yeah, they're really a way to find the integral.

121 MZ: They're a way to find the integral, but they are related to derivatives--

122 Derick: Yeah, they are related to both because the two are so close anyway.

123 MZ: Uhm, why don't you state real quick how they're related to derivatives.

124 Derick: Well, when you're doing the slope field for a function, it's really just taking the derivative at each x value. Like for instance we're going to say that $y' = 2x$ [writes: $y' = 2x$]. y prime of a function and we'll say $y(0) = 0$ just to make it easy. [writes: $y(0) = 0$]

125 MZ: Fine.

126 Derick: OK. So those y prime values are going to be the derivative of $y(0)$. So by finding those values at each x value-- so it's going to be 0 at x equals 0, 2 at x equals 1, etc., etc, etc. [sketches a few small lines on a slope field graph]

127 MZ: Right.

128 Derick: You can find the slope, the derivative of the original y at all those various points.

129 MZ: OK.

130 Derick: And then by taking the initial condition on y you can find the graph of y of x , by using the derivative.

131 MZ: You're using the derivative so obviously this is related to derivatives, but you're finding the--

132 Derick: You're finding the integral by using derivatives.

133 MZ: OK. Does this have anything to do with area? [short pause] Because sometimes we think about integrals having to do with area under the curve, and this somehow is related to integrals so is it related to area somehow.

134 Derick: Oh well. OK. [short pause] Well, we know that -- using this -- y of x is going to have to be equal to x^2 plus something and that's going to depend -- in this case it's going to be equal to 0. [writes: $y(x) = x^2$]

135 MZ: OK, cool.

136 Derick: So the area under this curve--

137 MZ: $y = x^2$?

138 Derick: Right. --is going to have to be-- end up being-- oh, that doesn't work, does it?

139 MZ: Oh, right because this is kind of the same example we have the picture. [referring to MZ's sketch of area under $y = x^2$ from 0 to 1 still at the top of the page]

140 Derick: Right, exactly. That's weird I didn't even try and do that.

141 Derick: I'm trying to figure out how to link that to that. [points to bottom of the page where the slope field stuff is and then to the top of the page where the area under the curve stuff is]

142 MZ: You mean the slope field to the one third.

143 Derick: Right and it's just not working out because I was thinking -- $y'(1)$ ends up being equal to 2 which doesn't work at all.

144 MZ: Right.

145 MZ: We know these slope fields are related to integrals.

146 Derick: Right.

147 MZ: We also know some of this area stuff is related to integrals.

148 Derick: Mm hmm.

149 MZ: It seems like somehow --

150 Derick: The slope field should be related.

151 MZ: -- they should be related to each other.

152 Derick: They should indeed. But I can't think of a way. I mean, obviously in terms that like when you draw the graph out onto the slope field the area-- That's obvious. But in terms of just the slope field itself-- [short pause] No, that's not going to work. I really can't think of anything.
153 MZ: Yeah. I don't know either.
154 MZ: Now let's see. Did I ask you about Riemann sums and stuff?
155 Derick: Not yet.
156 MZ: Say I ask you to find that area right there, the one we were just looking at, by not using the second fundamental theorem.
157 Derick: OK. Well what I would do is-- Did you want an estimate or the exact?
158 MZ: Well, can you get an exact answer? Without using the fundamental theorem, what would you have to do?
159 Derick: No, I guess you wouldn't without the fundamental theorem just using the sums like that you don't want to mess with it. Because eventually n has to approach infinity which means that the Δx has to approach 0 which you can't really do--
160 MZ: Well, I mean you can take the limit.
161 Derick: But it'll only really work up to an approximation for the most part unless you use the second fundamental -- Well actually, I guess it will turn exact if you take the limit.
162 MZ: Yeah, if it's a limit that you know what it is.
163 Derick: Yeah, but a lot of them don't work out that way. Instead you have to go to Reimann that.
164 MZ: OK, thanks.

Derick associates definite integral with area. He also mentions "slope" in this context, but here he means the graph of the function and not its steepness [ln 34]. Derick associates indefinite integral with antiderivative. When asked to relate definite and indefinite integrals, Derick imagines that an indefinite integral would represent "the area under the curve for the entire curve" instead of on a bounded interval like the definite integral [ln 48]. Even though this is incorrect, there is a logic in generalizing from an integral sign with limits denoted which represents a bounded area to an integral sign with no limits denoted representing an unbounded area.

Derick can state the Second Fundamental Theorem and correctly calculates an area by applying this method to a definite integral. His statement of the First Fundamental Theorem is that the integral of the derivative is the function, stated symbolically as $\int f'(x)dx = f(x)$. Derick's attempt, when asked, to give a graphical interpretation for the First Fundamental Theorem causes him to think that perhaps the integral should be a definite integral [ln 72]. When prodded about an "area function" Derick states that he is familiar with methods such as the trapezoid rule and Simpson's method for approximating areas, but he does not remember how to describe an area function that would help with the graphical interpretation of the First Fundamental Theorem.

When asked whether slope fields are related more to derivatives or integrals, Derick recognizes both relationships. He knows that one uses derivative values, and slopes to calculate the slope field, but that the end result is an antiderivative, an integral. When asked, Derick attempts to relate slope fields to area, but is unable to do so.

In a discussion not included in the interview transcript above, the interviewer explains area functions to Derick and uses them in a specific example to describe the relationship between slope fields, antiderivatives, area, and integrals more clearly, without proving any results. Derick finds the fact that the difference in antiderivative values gives an area to be "counterintuitive". He comments that in calculus "a lot" of the reasons end up being "cause it is". He explains that he has gotten the impression that some facts in calculus, like the Second Fundamental Theorem result, and some derivative rules, are to be accepted, not proven, or, he hedges, "it's a good way to cop out at any rate".

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: Derick is the only student in the class who answers this question correctly.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Derrick answers this question correctly.

Test 9: Semester final

On the semester final Derick correctly solves two optimization problems as well as problems on domain, range, inverse functions, and continuity. He also computes several limits, derivatives and integrals correctly. He only makes five errors: one chain rule error, one substitution error on an integration problem, the statement that $\lim_{x \to 0} \frac{\sin x}{x}$ does not exist, the statement that $y = x^{-2}$ is continuous, and the inability to find the inverse of the function $f(x) = \ln\left(\frac{x}{x-1}\right)$.

QOTD #15

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: " $f(x) = x^{2/3}$ is continuous everywhere. It is differentiable everywhere except $x = 0$ $\quad f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$."

Interview 4

The discussion for the fourth interview is broken into four parts. The first section includes general questions about derivatives. The second part asks the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity and acceleration given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the fourth interview.

1 MZ: What is a derivative?

2 Derick: It is instantaneous change of a function, slope of a tangent line to a function at a particular point, the limit as h approaches 0 of $f(x+h)$ minus $f(x)$ over h, and I think that's about it.

3 MZ: You said instantaneous change of a function?

4 Derick: Instantaneous rate of change of a function at a particular point.

5 MZ: What does the rate of change part refer to?

6 Derick: Oh, the change in the function output as opposed to the change in the function input at that point.

7 MZ: OK, then what about the instantaneous part?

8 Derick: That's the-- If you take the limit of those changes as you get closer and closer and closer to a particular point. When you get to that point, you get the rate of change of that particular instant.

9 MZ: It seemed like you were motioning as if you had a picture in your mind.

10 Derick: Yeah, I'm picturing a sloping curve, I don't know, probably like a $-x^2$ kind of thing. If you have a particular point like when you find, like the speed of a car at a particular point in time. It's a matter of taking the speeds over smaller and smaller intervals and so eventually you get to the speed at one particular instant.

As in the second interview, Derick mentions change of a function and slope of a tangent line as his first answers to what a derivative is. In addition he gives the formal limit definition of derivative immediately, whereas he says the definition only after prompting in previous interviews. When asked about his statement that the derivative is the change in the function, Derick corrects the statement to rate of change, but then explains it again as change. He says that rate of change refers to "the change in the function output" [ln 6] and that instantaneous rate of change refers to taking "the limit of those changes" [ln 8]. He also relates this to speed without noticing that speed is a ratio, not simply a change in the output values of a function.

The next part of the fourth interview is a summary of Derick's solution to the first of three problems involved in this interview. Given a table of values with x varying by .1, Derick is asked to find $f'(2)$, the derivative of the function at $x=2$. Derick first states that he will find the "change from $x=1.9$ to $x=2.1$" and calculates a reasonable estimate for $f'(2)$ by writing $\frac{4.04-.6}{2.1-1.9}=17.2$. It is interesting that Derick says he will calculate the "change", but then calculates the rate of change. When asked if he can find a different approximation, Derick says that he could take "any x value less than 2 and any x value greater than 2 and [find] the slope of the secant line between those two". He also says that if he had the points with x values of 1.99 and 2.01, he could find a better estimate.

Derick's analysis is based on his strong connections between the idea of change, the ratio of differences, and the slope of the secant line. His misstatement that derivative is change does not cause him any difficulties in solving this problem because he associates it immediately with a difference quotient. His statement about using points closer to $x = 2$ hints at, but does not explicitly state, a limiting process for finding a more accurate estimate.

The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car takes 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far does the car travel in the 2 hour period?

Derick's first reaction is that the problem describes a very slow acceleration. Next he sketches a graph of the "speed function" that is shaped like a concave down parabola. He uses what he calls a "physics method" to calculate $\frac{1}{2}ax^2 = \frac{1}{2}(60)(1) = 30$ for the first hour and hence 60 miles traveled for the 2 hours. When asked to use calculus, Derick notes that the velocity function is $v(t) = 60x$ and its antiderivative is the distance function, $d(t) = 30x^2$. This is correct except that he has used two different variables t and x which should be the same. The variable choices are not surprising in that t is typically used with the kinematic functions and x is the most common variable for a typical equation. When the variable discrepancy is pointed out, Derick changes the x's to t's commenting, "I keep mixing up the physics and the calculus."

Derick uses what he knows about the increase and decrease in velocity to sketch a correct graph of distance that he describes as increasing faster and faster and then increasing at a slower rate. When asked about the velocity curve, he recognizes that it should be straight instead of curved since the acceleration is constant. He also states that the area under the velocity curve should give the total distance. In comparing the two graphs Derick thinks that the distance graph should have an undefined, vertical slope at $x = 1$, but knows that this contradicts the fact that the velocity is defined at $x = 1$. He uses his knowledge of the physical situation and how this relates to the graph to determine that slope of the distance graph at $x = 1$ is 60.

Derick reasons symbolically, graphically, and in terms of the physical situation. He uses his knowledge of the connections between these representations to explain his answers and catch his errors. His physics knowledge is helpful in finding the a correct answer quickly, but the physics formula seems somewhat disconnected from his understanding of the derivative relationships involved in this problem.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Derick is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. He is asked if the top of the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Derick's initial guess is correct. He states that the rate will be changing and that it should be increasing (in magnitude) the whole time. When asked for his reasoning he says, "I'm trying to fool around with the equations." After a few false starts he labels the wall as y and the floor as x and completes the following sequence of calculations:

$$x^2 + y^2 = 196$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

He says that the $\frac{dy}{dx}$ represents, "the rate that it's moving down by the rate that it's being dragged across." He is concerned that the calculations are not right (even though they are accurate) because he thinks that the final equation does not take into account the length of the ladder. When asked if the rate the ladder is falling is strictly increasing or decreasing, he thinks it is neither because "it should be speeding up and then slowing down." He thinks that the point at which the distance along the wall is equal to the distance along the ground should be the change-over point.

To check his answer he decides to calculate y values for x values of 1, 2, 3, ... 12, and find the difference between the y for each 1 unit increment in x. In this way he determines that "the change in y is just going to keep increasing the whole time." Note that his calculations focus on the change in y instead of the rate of change or the derivative. This is effective because he keeps a constant 1 unit increment for x. Also note that his symbolic calculations focus on $\frac{dy}{dx}$ instead of $\frac{dy}{dt}$ and $\frac{dx}{dt}$. This works for him because he thinks of $\frac{dy}{dx}$ as the ratio of the other two rates, and the fact that $\frac{dx}{dt}$ is a constant simplifies the expression.

Interview 5

Derick's fifth interview occurs about one week after he takes the BC version of the AP exam. During that week the class discusses the written questions from the BC version. Between the fourth and fifth interviews, the class studies series and integration techniques and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Derick's response to a set of questions on a particular topic and provides an analysis of those responses.

1 MZ: Just throw out your what is a derivative answers so we can kind of get talking about it.

2 Derick: OK. Instantaneous rate of change.

3 Derick: Instantaneous rate of change, slope of the tangent line of a function at a particular point, the limit as h approaches 0 of $f(x+h)$ minus $f(x)$ --

4 Derick: [writes: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$]

5 MZ: OK. I was going to ask you how these are related to each other.

6 Derick: Well, the function $\frac{f(x+h)-f(x)}{h}$ that's sort of the secant, the slope of the secant line between two points on a function. As you take h approaches 0, the limit as h approaches 0, that gets closer and closer to a tangent line at one particular point which is-- It goes from the average rate of change to the instantaneous rate of change at a point.

7 MZ: What is meant by rate of change?

8 Derick: How fast the function output is changing, either increasing or decreasing, over a given change in the input or a given interval.

9 MZ: How fast the function is changing, is that what you said?

10 Derick: Mm hmm.

11 MZ: Can you describe what a derivative is without using these sort of standard phrases?

12 Derick: OK. If you have a graph of a function, and you take-- you want to find-- You have a function and you want to find out how much it is changing at a particular moment. For instance, you have the speed of a car. You want to find out exactly what speed it has at a particular moment. Now you can find the average value by taking any two points in time and two distances and finding the average rate of change, but that won't be totally accurate. But if you bring the two points closer and closer together in time, your value will get more and more accurate. And theoretically if you get the two points exactly the same, you'll have the speed of the car at that exact instant, just as an example.

13 MZ: OK. Can you think of an example that would be similar but would not be speed?

14 Derick: Well, you could do-- For instance, from what it looks like they're doing [interview is taking place in a biology classroom], for instance a bacteria sample. The multiplication of it-- If you want to find how many there were at one exact point, you can form a function dealing with how many there are at points of time. But if you want to find out how much they are increasing at one particular moment, you've got to use the derivative to find that.

15 MZ: OK. Does the derivative involve a limiting process? Explain.

16 Derick: Yeah, because you've got to take-- If you wanted to find the variable, h is usually used as the x -distance between your two points. You want to take the limit as h approaches 0.

17 Derick: And that represents going from an average rate over a smaller and smaller interval to an instantaneous rate.

18 MZ: OK. Is the derivative of a function a function? Explain why or why not.

19 Derick: It might be or it might not be. If you have a polynomial function-- Oh, even then it won't necessarily be. It depends on the function itself. [short pause]

20 MZ: So you're saying that there could be a function that if you take it's derivative it's not a function? Is that what you're saying?

21 Derick: Oh, I'm thinking continuous. That won't work.

22 MZ: What were you thinking about continuous?

23 Derick: I was thinking that if you have a function and you take the derivative of it, occasionally it won't be continuous everywhere, but it will be a function.

24 MZ: So the derivative of a function is always a function?

25 Derick: Though it might not necessarily be continuous everywhere.

26 MZ: And how do you know that it will always be a function?

27 Derick: Because of the fact that at a particular point-- If you have a function, at a particular point, there will only be one rate of change of that function at that point. You can't get two values for that.

28 MZ: What did you just do? You hand motioned this curve or something?

29 Derick: Well, if you've got a function, a generic function , and you take the rate of change at one point [sketches a smooth wavy curve on a pair of axes, marks a point and draws a tangent line at that point], there's only one rate of change that works at that particular point. There won't be any other value.

30 MZ: Explain what is meant by a differentiable function. Give an example of a differentiable and a nondifferentiable function.

31 Derick: A differentiable function is one for which you can find the derivative at every point. Like for instance $f(x) = x^2$ is differentiable because the derivative is $2x$, and that's defined for every point. [writes:

$f(x) = x^2$

$2x$]

32 MZ: OK.

33 Derick: A nondifferentiable function is one for which you can't find a derivative at every point. It might be differentiable over an interval, but it won't be at all x . And anything not continuous-- $\frac{1}{x}$ is not differentiable because it's not continuous.

34 MZ: OK. It's also not defined, I guess, at that point.

35 Derick: Right.

36 MZ: Can you think of one that is continuous, but it's still not differentiable?

37 Derick: If you have any function that comes to you, a cusp or a spike or any other sort of asymptote like that-- If it's not a smooth transition, it won't be differentiable.

38 MZ: What is it about it that makes it not differentiable?

39 Derick: Like for instance absolute value. [sketches an absolute value curve on a pair of axes] At $x = 0$ there's not a smooth limit, a smooth transition

from the negative to the positive. It changes over instantly at $x = 0$ so it's not differentiable at that point. At $x = 0$ there is no derivative.

40 MZ: And what is it that's changing over instantly?

41 Derick: The rate of change goes from -1-- From negative infinity to 0 the derivative is negative 1, from 0 to infinity it's 1, but there's not change over point. It switches over instantly at $x = 0$.

42 MZ: OK. And how does that apply to this case with the cusp?

43 Derick: [At some point Derick has drawn a cusp pointing up on a pair of axes.] It's sort of the same idea. The derivative is positive up to this point and it's negative past it, but there's no point where the derivative equals 0.

44 MZ: OK. Do you happen to know the equation for something that has a cusp?

45 Derick: Well, anything with an absolute value in it will end up having a cusp. If you change that around and twist it or whatever, it will always have a cusp.

46 MZ: OK. [short pause] What are derivatives useful for?

47 Derick: You can use them to find rates of change at a particular point in time. You can also use them to sketch the behavior of the function in a graphical form because you can use them to determine maximum and minimum points. You can use them to determine concavity. You can use them to determine maximum yields and minimum yields of a function defined as profit and stuff like that.

Even though the portion of the fifth interview focusing on general questions about the derivative has fewer questions than similar sections of the first and second interviews, Derick gives even more complete answers here than he does during previous interviews (see Table A.12). He states the details of the ratio and the limiting process in four different interpretations, and relates these two layers in each interpretation to at least one other interpretation. In previous interviews Derick only compares slope and the symbolic form in this much detail, with the comparisons for other interpretations not describing the details of the ratio or limit processes.

The one misstatement that Derick makes in this opening section of the fifth interview is to say that the derivative is the change in a function at an instant. Earlier in this section Derick correctly explains that the derivative is the rate of change, "how fast the output is changing" [ln 8], and seems to relate this to the symbolic ratio. However, just a few lines later, Derick says that the derivative is "how much [the function] is changing at a particular moment" [ln 12]. Derick makes similar misstatements in the

Table A.12. Derick: Interview 5 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎	◎		◉	
How does the formal definition of derivative relate to slope or rate of change?	◉	◉		◉	
What is meant by instantaneous rate of change?		◎			
Explain what a derivative is without using these standard phrases [slope, velocity, rate of change or the formal definition].		◉	◉		misstatement (d=change)
Does the derivative involve a limiting process?	●			●	
Is the derivative of a function a function?	◎	◎		○	
What is meant by a differentiable function?	●	○		↦	
What are derivatives useful for?		◎			max/min concavity
Asked to interpret the Mean Value Theorem.	•	◉ •	◉	• •	
Asked to find the average rate of change of a function defined as an integral.		•		•	misstatement (change=roc)
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.		○	◎		misstatement (d=change) increasing maximum concavity incorrect calc
Summary	◉	◉	◉	◉	

second and fourth interviews, and this idea causes him to make an error in problem solving later in the fifth interview.

For the second part of the fifth interview Derick is asked if he remembers the Mean Value Theorem. He does, and his initial description includes both the slope and rate of change interpretations of derivative, "If you have a continuous differential function and you have any two points a and b, there will be a point c in the closed interval $[a, b]$ where $f'(c)$, that is the rate of change at c, will equal the average rate of change over $[a, b]$, the slope of the secant line between a and b." His accompanying sketch shows a curve with a secant line through points labeled a and b that is parallel to a tangent line through c. The only problem is that he has not distinguished the names of the points from their x-values, and he has restricted the range of functions to which the theorem may be applied by stating that the function must be continuous and differentiable at any point.

When asked to write out the theorem symbolically he writes, "For any continuous, differential function, for any $x = a$ & $x = b$, there is a point c in $[a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$." When asked, he is also able to describe the theorem in terms of a physical situation. Derick states that the average speed of a car over an interval is equal to the exact speed of the car at some point in that interval, noting that the ratio is distance divided by time and the derivative is the instantaneous speed.

Derick demonstrates that he is comfortable with graphic, symbolic and velocity interpretation both for instantaneous and average rate of change. His first complete description of the theorem is in terms of the graph, but it does include the symbolic, $f'(c)$, and the phrase "rate of change."

The next question in the fifth interview involves a problem from the AB version of the AP exam which Derick has not taken. The question is as follows:

Let $F(x)=\int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.
(a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.
(b) On what intervals is F increasing?
(c) If the average rate of change of F on the closed interval $[1,3]$ is k , find $\int_1^3 \sin(t^2)\,dt$ in terms of k .

The interviewer asks Derick to discuss his methods for solving parts (a) and (b), but does not require him to complete the solution of either part. For part (a) Derick correctly explains how to apply the trapezoid rule. For part (b) Derick recognizes that $F'(x)=\sin(t^2)$ and states that he must find where the derivative is positive to know where the function is increasing.

For part (c) Derick first suggests setting $\frac{\int_1^3 \sin(t^2)}{3-1}=k$ so that the integral is $2k$. He describes this as the total of F over the interval divided by 2 to get the average rate of change. When prompted for clarification, he thinks that he should have found the integral of the integral instead. He says, "If you were to take the integral of the integral and find it over that interval from 1 to 3, divided by 2 to get the average, that'll equal k." He writes that if the antiderivative of F is G then "$\frac{G(3)-G(1)}{2}$ = the average change of $F(x)$ on $[1,3]$." After this he is asked what is the difference between the average rate of change of a function and the average value of a function. Derick replies that he has been finding the average value of the function when he should have been finding the average rate of change. He decides that he should take the derivative of the integral to get the rate of change function, $\sin(t^2)$, and find the average value of that. This leads to his original calculation that $\frac{\int_1^3 \sin(t^2)}{3-1}=k$ so that the integral is $2k$.

It is interesting that Derick ignores the phrase rate of change when he begins solving the problem and thinks primarily in terms of the average value. This is an example of individual metonymy, letting the part stand for the whole, even when it is not appropriate to do so.

Three related notions of average factor into Derick's solving of this problem. Each involve a division calculation. One is the elementary school average: find the sum or total of the items and divide by the number of items. A second is the continuous version of this for function values: find the sum or total using an integral and divide by the length of the interval. A third is the notion of average rate of change: take the difference in the endpoints of the function and divide by the length of the interval. The second and third are related in that a definite integral may be determined by finding the difference of antiderivative values.

Derick seems to refer to each of these at least partially in the comments he makes prior to his recognition that he should think explicitly about rate of change. Even though he never tries to add up discrete quantities and divide by the total number, he does focus on average being the "total" divided by the length of the interval. The second average listed above is the focus of Derick's comments. However his use of $\frac{G(3)-G(1)}{2}$ is more reminiscent of the average rate of change as a difference quotient.

Although no previous interviews discuss average value problems, during the second interview Derick confuses the elementary school average with the average rate of change. He chooses an example to explain the difference between average and instantaneous rate of change for a person who does not know about slopes and function, suppose, however, that which is to be an example average rate of change calculation is actually an average value calculation. He describes taking all the money made or lost on each day of the month and dividing by the number of months to calculate the average gain or loss. During the second interview, when further questioned about his example, Derick recognizes his mistake.

The next section of the interview concerns Taylor polynomials. Derick is asked what a function and its second degree Taylor polynomial have in common and how they differ. Derick responds that at a certain point they have the same function value, derivative and concavity. When asked, Derick says that Taylor polynomials are useful for approximating function values.

The final section of the fifth interview concerns a function, f, that at any time x, given in hours, tells the outside temperature in degrees Fahrenheit. Derick is shown a series of symbolic expressions and asked what information each one provides about the outside temperature.

For $f'(3) = 4$ Derick says, "That's implying that at exactly 3 o'clock the temperature increased exactly four degrees Fahrenheit. That's kind of an extreme value don't you think?" Derick does not recognize that the change is 4 degrees *per hour*. Derick continues his explanation with an analogy to speed, "It's like the speed of the temperature is 4 degrees in the same way that you take f' of a car function. At that particular point, that's how fast it's moving. ... So that tells you that it's heating up quite rapidly, but just at that moment." The interviewer asks Derick to apply his argument to the car situation for a distance function f and $f'(x) = 40$. Derick recognizes that the car is traveling at 40 miles per hour and states, "So yeah, it didn't go up 4 degrees, but it's increasing that fast at that particular point. ... If it keeps going up at that constant rate, in an hour it will have gone up 4 degrees. ... It's like the instantaneous speed of the thing."

Derick is next asked to explain the equation $f''(3) = -2$. He says, "That implies that at 3 the temperature is still going up, but it's not going up as quickly. ... It's slowing down its increase." When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Derick explains that the temperature increases 4 degrees per hour for 12 degrees total on that interval. Next Derick is asked to interpret $f''(x) = -2$ for $3 \le x \le 6$. He says, "It will either be still warming up but slower and slower, or cooling off at a faster and faster rate." Derick is asked to combine the information that $f'(3) = 4$ and $f''(x) = -2$ for $3 \le x \le 6$, but he says that he can not tell without knowing more about the derivative on the interval. He incorrectly guesses that the curve will remain increasing throughout the interval $[3,6]$.

He sketches a graph with a straight line, slope 4, for $[0,3]$ and an increasing, concave down curve for $[3,6]$.

Derick relies on the metaphor of speed to make statements about the first and second derivative and to correct his initial misstatement about $f'(3)=4$. He does not think to use numeric approximations or symbolic antiderivatives to get more detailed information about the temperature on $[3,6]$.

Case Study 5 — Ernest

Academic record

*Other AP courses: US. History (junior year), European History, Chemistry.

*Math team participant.

*Plans to major in business or marketing in college.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: Ernest is absent on this day.

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different?

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: Ernest is absent on this day.

QOTD #3

Give an example of a function <u>without</u> using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Response: Ernest is absent on this day.

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) non integers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: Ernest is absent on this day.

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to class discussion on limits.

Response: "When it approaches a #"

Test 1

On a test on limits a week later, Ernest is able to correctly find limits by reading values from a graph, by substituting into a piecewise function and by using algebra to simplify a limit calculation. He is able to work with the formal definition of limit to find a δ for a given ε in a graphical setting, but he is not able to complete an ε-δ proof for a linear function.

One other error occurs when Ernest is asked to substitute the values $x = \pm\ 1, .1, .01, .001$, and $.0001$ into $f(x) = (1+x)^{\frac{1}{x}}$ and to use these values to estimate $\lim_{x \to 0}(1+x)^{\frac{1}{x}}$. Although Ernest correctly finds $f(x)$ for each x value, he writes his limit estimate as 2.87 instead of 2.718. His answer is closest to $f(-.1) = 2.868$, the largest of the output values he calculates which sits on the page almost in the middle of the calculated values, right after his calculation that $f(-1)$ does not exist. Perhaps the placement of the $f(-.1) = 2.868$ in the middle of a list of values calculated from both sides of $x = 0$ causes Ernest to think that this is actually in the middle numerically and hence closest to the value of interest.

Interview 1

This interview occurs after the test on limits but prior the class' discussing derivatives. Therefore Ernest's answers are presumed to be based on what he remembers from his junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table A.13, which codes these responses. A summary discussion follows.

1	MZ:	First what you remember about what a derivative is?
2	Ernest:	I remember it's the slope of the tangent lines leading up to the point, leading up to the limit? I don't know if that makes any sense.
3	MZ:	OK, well [short pause] you're making some sense. The slope of the tangent line-- and I didn't really get this part.
4	Ernest:	--leading up to the limit, sort of, of x .
5	Ernest:	Unfortunately it got kind of lost in translation. I just know how to find it and know how to take the derivative of something just by looking at the equation and making this-- putting this derivative down, but I kind of lost exactly what it means.
6	MZ:	Well, do you remember what it's used for or is useful for?
7	Ernest:	Oh, acceleration and velocity, particles.
8	MZ:	Yeah.
9	Ernest:	And another application, I've either forgotten or never remembered.
10	MZ:	OK. Say I give you a function. And how can you tell if it's differentiable?
11	Ernest:	What do you mean?
12	MZ:	Mmm, what do think the word differentiable means?

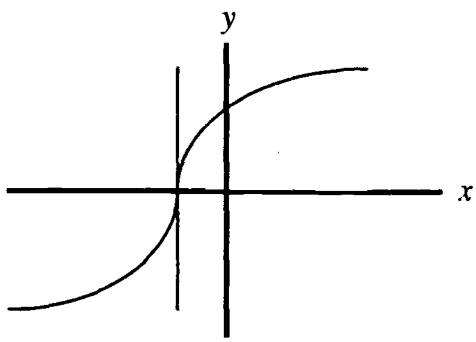
13 Ernest: Well, differentiable at a certain point or all around?

14 MZ: Yeah, I guess I mean all around, but the definition of all around is it works at every point. So you could just tell me for at a point.

15 Ernest: Continuous-- The only place I can think of where it's not differentiable is when it goes kind of like this [motioning with his hands].

16 MZ: Here. Why don't you draw it on here. [hands Ernest the notebook page where MZ had earlier written 'slope of tangent lines leading up to the limit of x ']

17 Ernest: I'll draw it. This-- you've got the x -axis, the y -axis. [draws and labels each] If it kind of goes like that [similar to $y = (x+1)^{\frac{1}{3}}$], this right here is not differentiable [draws vertical line through steep slope at $x = -1$]. [draws:



]

18 MZ: OK, so right where you drew that vertical line it's not differentiable.

19 Ernest: I think so.

20 MZ: That's what you're talking about. And why isn't it differentiable at that point?

21 Ernest: Because the slope would be-- [short pause] It would be undefined. Cause you go up one and over none. So it would be 1 over 0 would be the slope.

22 MZ: So it would be undefined slope?

23 Ernest: Yeah.

24 MZ: OK. Can you think of any other types of weird things that could happen to make it not differentiable at a point?

25 Ernest: If it's like an asymptote, is it, or if there's a hole in the graph. I think.

26 MZ: OK. Say you were going to explain to someone what a derivative is to someone who's like in precalc or AB and hasn't gotten to it yet. What would you tell them?

27 Ernest: I'd pretty much just explain to them that it's something that-- I guess it is this right here [points to what is written on the notebook, slope of tangent lines ...].

28 MZ: Slope of a tangent line, but what if they say, "what does that mean?" [pause] Would they already know what this part means, like slope of a tangent line?

29 Ernest: They probably should.

30 MZ: Did you guys study that before?

31 Ernest: Slopes.

32 MZ: OK. We'll do the next part. The next part is there's a list of words and for each just tell me if it has anything to do with derivatives or not and if it does, what it does and if it doesn't, why not. Slope you kind of already said.

33 MZ: Speed or velocity?

34 Ernest: If the slope is very high then the speed will be great and if it wasn't very high then it wouldn't be very great and if it-- that's acceleration.

35 MZ: So the way it sounded, what you said it sounded like the slope and the speed were pretty, about the same thing. I mean, like if this one was high, then that one was high. If this one was low-- Is that a good interpretation of what you said?

36 Ernest: Well, actually-- No, I'm sorry. That would be the rate of change, what I just said.

37 MZ: Rate of change. We're moving to the next one. OK.

38 Ernest: The speed would be how high it is above the x axis.

39 MZ: OK. So like if I had a graph of something--

40 Ernest: Yeah. If-- As it keeps going up the speed is increasing and as it keeps going down the speed is decreasing

41 MZ: So the speed would just be the height in this example.

42 Ernest: It would be the height, yeah. And the rate of change would just be the curve of the graph.

43 MZ: What do you mean by the curve of the graph?

44 Ernest: Well, like it would be-- The rate of change would be increasing if it was concave down-- No, or is it up? Concave up, I think, where it's going like this? [hand motions a concave up shape]

45 MZ: Yes.

46 Ernest: So if it's concave up the rate of change is increasing rapidly and if it's concave down the rate of change is getting smaller and smaller.

47 MZ: OK. Now so, let me relate some of this. Which part of this is the derivative?

48 Ernest: The rate of change.

49 MZ: The rate of change is the derivative. Now is the rate of change similar to the slope?

50 Ernest: Yes, it is.

51 MZ: Same thing?

52 Ernest: I think so.

53 MZ: But the velocity was not the same thing in this example. Is that a true statement?

54 Ernest: That's correct. I'd say it is.

55 MZ: OK. Line or linear? Do either of those words have anything to do with derivative?

56 Ernest: I don't really know.

57 MZ: OK. Measurement?

58 Ernest: Measurements. Well, I guess I think of the x -axis and the inputs and--. See I don't really know how to answer that one as well because measurement-- Well wait, input would be, I guess, time if you're talking about a vehicle and its speed, and the y -axis would be the output or how fast it's going.

59 MZ: Next. Prediction?

60 Ernest: Uhm. This is just derivatives right?

61 MZ: Yeah.

62 Ernest: Oh, OK. Cause if it was limits it's got to do with prediction.

63 MZ: Oh, so you think of limits as being predicting the value?

64 Ernest: Yeah, because you don't really go to the point. It's just around the point.

65 MZ: Near, close to the point.

66 Ernest: You're predicting what it is. Hmm, derivatives. Well, I don't know really know if it's a prediction, but you can predict the way a graph looks--

67 Ernest: --by taking the derivative of it to get like I said, concave down and to find maximum and minimum points. So you can sketch your graph.

68 MZ: So it gives you information about the graph. [pause] That's kind of like optimization. Do you know what I mean by optimization?

69 Ernest: Not really.

70 MZ: I was thinking of like you're getting the best possible result, could be a max or a min-- You're trying to do something so that your cost is minimized or profit is maximized.

71 Ernest: Yeah, you could use the derivative to find that, don't you?

72 MZ: Yeah. I mean you were kind of explaining about using the derivative to find maximums just in graphs.

73 Ernest: Yeah. And you can find local maxima I think with derivatives, can't you?

74 MZ: Yeah, do you happen to remember how that works?

75 Ernest: No, I usually remember something by example or like when Mr. Forest starts teaching us--

76 MZ: Continuity?

77 Ernest: Continuity. Well, if it's continuous, then there's a derivative there unless something like this occurs on a graph. [pointing to the sketch drawn earlier with a vertical tangent]

78 MZ: Like the vertical--

79 Ernest: Yeah, like asymptotes aren't continuous, and they don't have a derivative at that point

80 MZ: If you know it has a derivative at that point, does it have to be continuous? Will it always be continuous?

81 Ernest: I would think so.

82 MZ: Yeah, but if it's continuous it doesn't--

83 Ernest: It doesn't have to have a derivative.

84 MZ: Limit?

85 MZ: Do you think the limit has something to do with derivatives?

86 Ernest: Yeah, 'cause-- [pause] Well, if it doesn't have derivative-- [pause] I don't know.

87 MZ: Integral. You guys probably didn't study that yet.

88 Ernest: Not-- If we did, I don't recall it.

89 MZ: Function? Is the derivative a function?

90 Ernest: Derivative a function. You could make it a function, can't you?

91 MZ: I was wondering what you were thinking when you said you could make it a function.

92 Ernest: Well, like you have one graph, and you take the derivative of it, and you make it another graph. So in a way you could say that the derivative of that, the first graph you make your own function. Not your own function but a new function from that original graph. That's what I was thinking.

93 MZ: So the process of taking a derivative makes a new function? Is that--

94 Ernest: Yeah, to tell you a different sort of information.

95 MZ: Last question. Do you remember a formal definition of derivative?

96 Ernest: [pause]

97 MZ: I mean not like epsilon delta, that formal, but just sort of, a derivative is-- $y'(x)$ is--

98 Ernest: You mean like how to find the derivative?

99 MZ: In a way. The one I was thinking of is how to find it, but not how to find it by like the power rule or something.

100 Ernest: Oh.

101 MZ: Like the first thing you learned, this is a derivative.

102 Ernest: I suppose you could find it by slopes--
103 MZ: OK.
104 Ernest: -- from the graph. You find the tangent line, the slope of like 1, find the tangent line, and on another graph put that answer. And you keep doing that.
105 MZ: Say I had a graph, but I don't know the equation for it. And I want to know what the derivative is at some arbitrary point, like right here. Say this is $x = 5$. [sketches a smooth curve on a pair of axes; marks $x = 5$ on the x-axis with a dotted line up to the curve] And say this is a real nice graph so that you can figure out what's going on. Pretend it has graph paper behind it or something. How can you figure out what the derivative is at that point?
106 Ernest: You mean with just using the geometrical shapes--
107 MZ: Yeah--
108 Ernest: --or using your calculator.
109 MZ: --and using that idea of slope that you had.
110 Ernest: OK, well, I suppose-- Did you say we can draw a tangent line?
111 MZ: Uhm, you can draw one. How would you know what the slope of it was?
112 Ernest: Well if you draw it, then you-- It would be pretty easy to find the slope.
113 MZ: Oh, from like the graph paper, right?
114 Ernest: Yeah.
115 MZ: Now what if you weren't sure if that is exactly the tangent line though?
116 Ernest: Should be, wait. Still not being able to use mathematical means?
117 MZ: Well, I guess you could use mathematical means. What would you do?
118 Ernest: What would I do?
119 MZ: Say you had the equation of that graph.
120 Ernest: If I had the equation of that graph, I'd just take the derivative.
121 MZ: Just take the derivative.
122 Ernest: If not, let's see--
123 MZ: So any equation I give you, you could take the derivative of it?
124 Ernest: Yes. Yeah, unless there's some kind of equation that I haven't seen.

Table A.13 summarizes Ernest's first interview transcript. Ernest recalls that the derivative is related to slope and velocity and acceleration. Later in the interview, when discussing velocity and acceleration in more detail, Ernest also remembers that derivative is rate of change. It is only within the graphical interpretation that Ernest discusses the instantaneous nature of the derivative, a limiting process and that the derivative is a function. At no point does Ernest recall a symbolic difference quotient.

Ernest recognizes some connections between speed, acceleration, slope and rate of change. When he is asked if derivative is related to speed or velocity, he immediately mentions slope, making slope and speed sound functionally equivalent [ln 34]. When

Table A.13. Ernest: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎			○	misstatement
What can derivatives be useful for?			o o		acceleration tool
How can you tell if a function is differentiable?	◎				
Explain what a derivative is to someone who's an AB student or precalc student who hasn't studied it yet.	●				
Is derivative related to speed or velocity?	o	o	o o		acceleration
Is derivative related to change or rate of change?	o	o	o		in/decreasing concave up concave down
Is derivative related to measurement?			o		
Is derivative related to prediction or approximation?	○				concave down max/min
Is the derivative a function?	○				
Did you learn a formal definition of derivative?	◎			↦	
Summary	◎	o	o	○	

asked for clarification he mentions rate of change for a curve whose height is the speed [ln 36]. His graphical interpretation is that the rate of change increases when the curve is concave up. He finishes the discussion by deciding that slope and rate of change are the same but that velocity is not the same as the other two in this setting.

Ernest's only misstatement is his initial description of derivative as "the slope of the tangent lines leading up to the point, leading up to the limit" [ln 2]. He correctly indicates a limiting process involving slopes. However, the lines should be called secant lines not tangent lines, and the derivative should be not the collection of those slope values, but the limit of those values. His use of the word limit here is interesting, since he does not use the word again in the interview except, when asked specifically, to say that he can not explain how derivatives and limits are related.

QOTD #6

Find the derivatives of the following four functions:

$f(x) = (x-1)^2(x^2-4)$

$g(x) = \dfrac{x-1}{\sqrt{5-x^3}}$

$h(x) = \sin x$

$j(x) = \ln x$

Date: September 20, 1993. This question occurs prior to the class' learning about short-cut rules for taking derivatives of various forms.

Response:

$f'(x) = 2(x-1)(x^2-4) + (x-1)(2x)$

$g'(x) = \dfrac{(5-x^3)^{\frac{1}{2}} - (x-1)(\frac{1}{2})(5-x^3)^{-\frac{1}{2}}}{(5-x^3)}$

$h'(x) = \cos x$

$j'(x) = \mathrm{e}$

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be using your knowledge of derivative.

$y = \log(x^3)$ $\quad y = \dfrac{x}{|x|}$ $\quad y = x^e$ $\quad y = e$

Date: September 21, 1993. This question also occurs before the class studies short-cut rules for taking derivatives but after they have studied the limit definition of derivative. Response: "$y = \dfrac{x}{|x|}$ because $\ln x$ has a derivative at $x = 0$ and this example doesn't."

Comment: This question is presented to the students since no student correctly stated the derivative of $y = \ln x$ in the previous Question of the Day. Notice that Ernest is correct about $y = \dfrac{x}{|x|}$, but not about $y = \ln x$. It is not clear how he is making either determination.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$? Why not?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: "a) No, Because from $(0, \frac{\pi}{2})$ the slope of the tangent lines are positive, whereas $\cot x$ is negative $(0, \frac{\pi}{2})$. b) $y = \tan x$" Ernest also includes two drawings. The first is a sketch of the graph of the function $y = \tan x$ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$. The second sketch is of $-\cos x + 1$ from $-\pi$ to π.

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. Ernest states the definition of derivative incorrectly as $\lim\limits_{x_0 \to x} \dfrac{f(x - x_0) + f(x)}{x - x_0}$. He substitutes into the quotient correctly for the given function, but never completes the problem to the point of taking the limit due to algebraic difficulties. Given the incorrect definition of derivative above, it is interesting that on the very next problem Ernest uses the expression $\frac{f(x+h)-f(x)}{h}$ to correctly estimate the derivative of a function at a point with a given h value of 0.001. Perhaps the direct mention of h in the problem statement triggers this memory for Ernest. However, it does not cause him to go back to correct the previous problem.

Ernest shows that he can estimate the derivative at a particular point by zooming in to find two nearby points and calculating a difference quotient for those two points. Given the graph of a position function for a car he correctly answers questions about the speed and direction of the car. Given the graph of a function, he is able to sketch a correct graph for the derivative function.

Ernest correctly solves one complicated chain rule derivative, $f(x)=\tan^{\frac{3}{2}}(2x+1)$, but he seems to omit a step in finding the derivative of $f(x)=\cos(3x^2+4)^5$. His solution is $f'(x)=-\sin(3x^2+4)^4\cdot 5(6x)$.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: Ernest is absent on this day.

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "a) $f'(x) > 0 \quad \forall \in (a, b) \rightarrow$ increasing

$f'(x) < 0 \quad \forall \in (a, b) \rightarrow$ decreasing

b) I learned that critical points are the only candidates for extrema."

Comment: Since the researcher had not been present the day prior, this question is presented both as a means for the researcher to see the material covered and to ascertain the students' understanding of it. Notice that Ernest uses mathematical symbols in a very curious way. At first glance, it might be natural to assume that he was looking at his notes from class. However, he uses the notation incorrectly. It should be $\forall x \in (a, b)$...

Test 3

This test includes material on Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Ernest correctly calculates a third degree Taylor polynomial, and he is able to use the first and second derivatives of the position function to find the speed and acceleration of an object at a given time. He is able to use the graph of a derivative function to estimate when the original function is increasing or decreasing, concave up, or concave down and where it has inflection points. In addition he correctly describes three of the four extrema, omitting one of them even though he lists the location as a critical point.

QOTD #11

Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: Ernest is absent on this day.

Test 4

Two weeks later the class has a test on the applications of derivatives. Ernest correctly uses derivatives to solve three traditional max/min problems. He also correctly calculates the derivative of an implicitly defined function. He correctly solves two traditional related rate problems, but on a third problem he takes the derivative with respect to time of $V = \frac{1}{3}\pi r^2 h$ and, failing to apply the product rule, gets $\frac{dV}{dt} = \frac{2}{3}\pi r \frac{dr}{dt}\frac{dh}{dt}$.

Interview 2

The second interview occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by Table A.14, which provides a summary, and a discussion.

1 MZ: What is a derivative?

2 Ernest: It's the slope of the tangent line at a point.

3 MZ: Does anything else come to mind in addition to that?

4 Ernest: [pause] Well, it also-- If you take the derivative of a function, like a position function, you can find it's velocity and acceleration. And you can use it to find what would maximize your profits in business.

5 MZ: What are derivatives useful for? I guess you started giving me a list.

6 Ernest: Yeah, some problems in the book where it said pretend like you are working for Platypus company, and you need to make a fence around a rectangular yard. What would be the dimensions? You can use the derivative for that. That was pretty much common sense for-- You can still use the derivative to show how you did it.

7 MZ: Well, how does-- I guess I have two questions. One is, you're saying it's pretty much common sense, like you probably figure it out without using a derivative?

8 Ernest: The problems we did in the book, yeah. Like 600 feet of total fence and you have to find the maximum area. Well, that's just 150 for the width, and then 300 for the length. Add it up for a total of 600, and that gives you the maximum.

9 MZ: So how do you know that's the maximum? I mean, by common sense.

10 Ernest: Just-- Well, and also if you wanted to figure it out you could do this.

11 Ernest: You could do 149 by 302 and you get an answer. Or you could do 151 by 298. You get another answer. You do 150 by 300, and this answer is-- y, z, x. x is greater than y which is equal to z.

[has written:

$$149 \times 302 = y \qquad x > (y = z)$$
$$151 \times 298 = z$$
$$150 \times 300 = x$$

]

12 MZ: Oh. So you can kind of tell that it's symmetric?

13 Ernest: Yeah, without using the graph.

14 MZ: When you do it with the derivative, how does the derivative fit into this?

15 Ernest: I'm at a loss. I really couldn't give an answer.

16 MZ: OK. We'll continue. How would you explain a derivative to someone who hasn't had much math?

17 Ernest: Well, I'd tell them to learn limits first. Then-- [pause] I'd explain it as what you use when-- [pause]

18 MZ: Can you think of some way to explain it to someone who is so small in math background that limits is just not going to get them very far?

19 Ernest: Well, I'd tell them to wait until they had a proper math background to know what I'm talking about.

20 MZ: So there's just no way without having more math background?

21 Ernest: Not really, not for them to understand it. You couldn't just tell someone in third grade how to do it.

22 MZ: OK, then what if somebody had just finished doing limits, but hadn't studied derivatives yet. They were saying, "We're about to learn derivatives. What are derivatives anyway?"

23 Ernest: This.

24 MZ: You would say this, the slope of the tangent line? So what does the slope of the tangent line have to do with limits?

25 Ernest: What does it have to do with limits? I was never good at textbook definitions and stuff.

26 MZ: Somehow it does have to do with limits in your mind though?

27 Ernest: Yeah, kind of. Again, I can't give textbook definitions.

28 MZ: OK. Is there anything that occurs to you that's associated with limit and with derivative of why they're somehow related?

29 Ernest: I guess today wasn't a very good day for me.

30 MZ: How can you tell if a function's differentiable?

31 Ernest: It's continuous.

32 Ernest: And-- Well, if it's continuous, then-- Well, if it's differentiable, it's continuous. If it's continuous it's not necessarily differentiable, isn't it?

33 MZ: OK. So, what would be an example that shows that that's true, what you just said?

34 Ernest: I believe, the inverse of x cubed. Is that it?

35 Ernest: 'Cause I think x^3 goes similar to this. [sketches $y = x^3$ on a pair of axes]

36 Ernest: And the inverse is-- It's like what we did yesterday in the problems. [sketches the inverse function of $y = x^3$, $y = x^{\frac{1}{3}}$, on the same axes]

37 MZ: OK, so--

38 Ernest: 'Cause right here you get a vertical line, and a vertical line doesn't. So it wouldn't be differentiable at $x = 0$, but it's still continuous. 'Cause there's a value there.

39 MZ: So it is continuous, but it's not differentiable and that's because the slope is undefined.

40 Ernest: Mm hmm. [agrees]

41 MZ: Can you give a real world situation that involves the concept of derivative?
42 Ernest: Yeah. Like a factory, costs so much a day to run a factory, and to produce something costs a certain amount. What would be the maximum profit you could make? To find out how much of a product you would make to get the maximum profit.
43 MZ: OK, and how does the derivative fit into that situation?
44 Ernest: Well, you set up an equation. You set up two equations, and then the second equation multiply-- take the derivative of that equation. You plug in the value you got in the first equation to that derivative. And set the derivative equal to zero.
45 MZ: Why is it we get a maximum when the derivative is equal to zero?
46 Ernest: Because that's when-- If the derivative is equal to 0, you've either got a maximum--
47 Ernest: [sketches a local max with a horizontal tangent line at the maximum point and a local min with a horizontal tangent line at the minimum point] You set it equal to zero because the slope of both of these lines is zero. You're trying to-- The derivative is the tangent line at a point. So, the tangent line has a slope of 0. So you set the equation equal to 0.
48 MZ: OK. That makes sense. These are these words. Say if it has to do with derivative. Like, slope you already just explained what that has to do with derivative. Velocity?
49 Ernest: That's the derivative of a position equation.
50 MZ: OK. Change or rate of change? Does that have to do with derivative?
51 Ernest: Yeah. [short pause] That's like the slope at a point of a tangent line and like a car velocity is changing. Again, I can't give a good definition of this--
52 MZ: You don't have to mark it out. [Ernest had started sketching a curve with a steep positive slope becoming gradually less steep but staying positive, but then stopped and marked it out.]
53 Ernest: --but it does have something to do with it. I guess what I was kind of saying is that up here the rate of change is less than down here.
54 MZ: So whatever it is that that's a graph of you can talk about the rate of change being different.
55 Ernest: Whether it's distance or speed.
56 MZ: OK. And how does the derivative fit into that, what you just said? [pause; Ernest makes a face and hand motion.] Next one? Line or linear? [pause; Ernest makes same or similar expression.] Next one?
57 MZ: Measurement?
58 Ernest: Measurement. What do you mean?
59 MZ: Well, is measurement associated with derivative in any way?
60 Ernest: Yeah. Measurement of like a yard. Measurement in feet. Measurement of velocity, of acceleration.
61 MZ: Yeah. Prediction or approximation?
62 Ernest: Actually that's more of a limit.
63 MZ: Which one, prediction or approximation?
64 Ernest: Both.
65 MZ: Limit?
66 Ernest: [pause]
67 MZ: You know it has to do with it, but you weren't sure how. Do you know what an antiderivative is?
68 Ernest: Yeah, that's where if you've got $6x^2 + x$ is A one, and you've got to find what A is. [writes: $A' = 6x^2 + x$]

69 MZ: OK. So--

70 Ernest: So you take $2x^3 + \frac{1}{2}x^2 + C$. [writes: $A = 2x^3 + \frac{1}{2}x^2 + C$] And then I forgot exactly how you find what this is.

71 MZ: So then this is the antiderivative of A'?

72 Ernest: Yeah.

73 MZ: OK. A formal definition of derivative?

74 Ernest: [pause]

75 MZ: What comes to your mind? [Ernest underlines first thing written on the page.] Slope of a tangent line? That's about as formal as it gets. Let's see. Have you ever heard of any of these? Newton's Method?

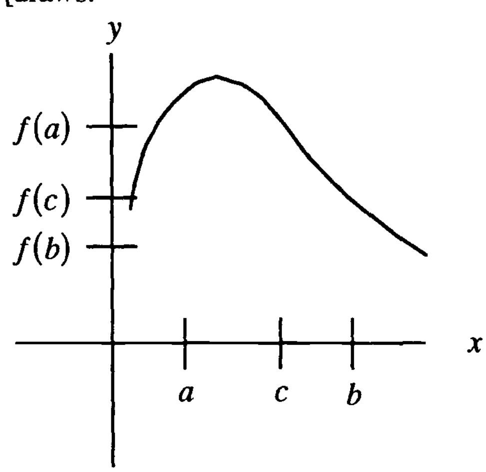
76 Ernest: I've heard of it.

77 MZ: Do you remember anything about it?

78 Ernest: I don't know.

79 MZ: Intermediate Value Theorem?

80 Ernest: Here's $f(a)$ and $f(b)$. [sketches a pair of axes and labels $f(a)$ and $f(b)$ on the vertical axis] Basically the-- That's the y and this is x. [labels the vertical axis y and the horizontal axis x] And this would be like a and b. [labels a and b on the x axis] If there's a number c in between $f(a)$ and $f(b)$-- [writes $f(c)$ between $f(a)$ and $f(b)$] If there's a value $f(c)$ between $f(a)$ and $f(b)$, there's got to be a number c in between a and b. [marks a spot c on the x-axis between a and b]. [draws:



81 MZ: And then if these two y values are on that function, then there's this in between point somewhere.

82 Ernest: But c doesn't necessarily have to yield a value, $f(c)$ in between $f(a)$ and $f(b)$. Because you can have like-- Well, it could go up here. The graph could go up here. [sketches in a graph that is above the interval $f(b)$ to $f(a)$].

83 MZ: Right. So if I just pick any c, it doesn't have to be in there.

84 Ernest: But if you pick any $f(c)$ it has to.

85 MZ: That makes sense. Does this have to do with derivative at all?

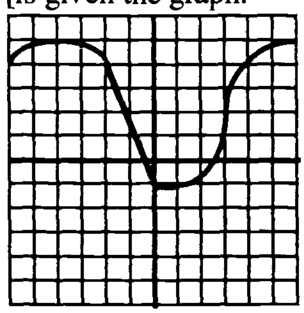
86 Ernest: Sure it does. I can't think of how though.

87 MZ: OK. How about Mean Value Theorem?

88 Ernest: Skip this one.

89 MZ: OK. Say I give you a function. And I said this is a derivative. Would you be able to tell me about the function that it came from?

90 Ernest: Yeah.
91 MZ: What kinds of things do you think you'd be able to tell me?
92 Ernest: You can tell inflection points, whether it's concave up or concave down, and like a maximum.
93 MZ: Would it be easier for you to find those things if you had the equation or if you had the graph?
94 Ernest: I think if we had the equation and knew how to do the antiderivative-- We don't even need to come up with the antiderivative. If you just give an equation it'd be easier to find because you can just set your derivative equal to zero and get your increasing, decreasing and uhm-- What do you call those, critical numbers?
95 MZ: Oh, right.
96 Ernest: And then you take the derivative of that and you can get your concavity. So it'd be easier if you give an equation.
97 MZ: OK. Well, I'm going to give you a graph.
98 MZ: The idea is, pretend this is the derivative of some function and see if you can sketch the original function.
[is given the graph:



99 Ernest: [pause]
100 MZ: Well, what were some of the things you told me you would be able to find about the original function?
101 Ernest: This is just a guess but-- See this is concave up.
102 MZ: Yeah.
103 Ernest: But wouldn't that mean the inflection's concave down.
104 MZ: Uhm, why?
105 Ernest: Well, like-- Well, I guess it wouldn't necessarily.
106 MZ: [pause] Well, let's see. You told me that you could probably figure out where the max and min points were.
107 Ernest: Here. [Ernest points to the graph]
108 MZ: At one of those two--
109 Ernest: Where the derivative's zero because the slope is zero of the original graph. And since this was-- I got a plus slope and I went to a minus slope so that would be a max. This went from a minus slope to a plus slope so that would be a min.
110 MZ: OK. So that should get us started.
111 Ernest: And concavity would be these points.
112 MZ: So the max and min of the derivative are going to tell us what about the original?
113 Ernest: Concavity.
114 MZ: OK. So is it going to be concave up, down or neither at say this first maximum point?
115 Ernest: [pause] I think it's concave up [because?] this is concave down.
116 MZ: OK. Let's see. What do we have so far?

117 Ernest: We have max, min, points of inflection at these three points. I know it's concave up. This is concave down.
118 MZ: Let's not try to do it too perfect, but let's see if we can make a max curve thing and a little min curve thing and see if we can play connect the dots.
119 Ernest: [sketches a curve] Something like that.
120 MZ: OK. So this max is around this, whatever, negative a half or something.
121 Ernest: And this min--
122 MZ: --is over there around 2 or so. And then where are the inflection points on this thing you just drew?
123 Ernest: It would be here which would be about here. And then maybe like that or something which would be-- See this is here which is about here. And then this is about here. [marks dots on his sketch where the inflection points are]
124 MZ: About at that second max type place. Here's a question. We kind of sketched this on here. Is it possible to know how high or low this should be?
125 Ernest: Yeah, you can tell by the slope how big-- how high the slope is.
126 MZ: So from this derivative graph you can tell how--
127 Ernest: In other words if the slope up here is negative four--
128 MZ: Oh, you mean the actual slope of the derivative graph?
129 Ernest: The actually slope here. Yeah, you just like put a straight edge down, and then if that's four, whatever, then it's at four. If it's a negative four then it's at negative four.
130 MZ: OK. So then that would give me the value on this one?
131 Ernest: Yes.

As the table suggests, Ernest mentions derivative most often as slope or in terms of velocity or acceleration. He only mentions rate of change when asked specifically. He only mentions the instantaneous nature of the derivative in the context of slope and in discussing that the derivative involves limits. Ernest does not describe the details of the ratio, limit, or function processes in any context. He can not remember how limits are related to derivatives, and does not state a symbolic formal definition for derivative.

Ernest does not make any direct connections between different interpretations of the derivative. When asked, "What is a derivative?" he mentions slope of a tangent line at a point and velocity and acceleration in succession, but does not directly relate the two. Also, when asked whether derivative is related to rate of change, Ernest mentions slope of a tangent line at a point, velocity and acceleration, and rate of change, but does not explicitly link them.

Table A.14. Ernest: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎		o o		acceleration maximum
What can derivatives be used for?	o			↦	maximum misstatement (d=tl)
Explain what a derivative is to someone who's an AB student or precalc student who hasn't studied it yet.	●			○	
Is derivative related to speed or velocity?			o		
Is derivative related to change or rate of change?	◎	o	o o		
Is derivative related to measurement?			o o		acceleration
Is derivative related to prediction or approximation?				○	
Is derivative related to integral?				↦	
Did you learn a formal definition of derivative?	●				
Given the derivative, what can you tell me about the original function?	o			↦	inflection pts concave up concave down maximum in/decreasing
Summary	◎	o	o	○	

While Ernest does not demonstrate a very complex understanding of derivative, he also does not make many misstatements. Prior to his difficulty in remembering the various theorems asked near the end of the interview, Ernest's only misstatement involves whether the derivative is the tangent line. Ernest is explaining why one sets the derivative equal to 0 to find a maximum or minimum location. He sketches a curve and draws a horizontal tangent line at the maximum and minimum points on a curve. He explains, "The slope of both of these lines is zero. The derivative is the tangent line at a point. So, the tangent line has a slope of 0. So you set the equation equal to 0" [ln 47].

Ernest equates the slope of the tangent line and the derivative equation for his calculation. He also states in other parts of the interview that the derivative is the slope of the tangent line. Perhaps in this situation he abbreviated the phrase "slope of the tangent line" to "the tangent line" for simplicity of speech without considering that he was changing the meaning of his statement. Using part of a phrase to represent the whole is an example of individual metonymy.

For the last part of the second interview Ernest is asked to graph an original function when given the graph of the derivative function. He sketches a reasonable graph using his knowledge that derivative is slope to help him determine where the function is increasing or decreasing and where its extrema lie. After having sketched a graph of the original function he correctly identifies the location of its inflection points. However, he initially makes a couple of misstatements about concavity. He says that the extrema of the derivative tell the concavity. When pushed to describe the concavity, he says that he thinks that the original function would be concave up at the maximum of the derivative.

Ernest's other confusion is that when asked if it's possible to know how high or low the curve of the original function should be, he says that it is possible. His explanation interchanges the role of the function and its derivative. He says to find the slope of the derivative graph at a point and that will be the value of the function.

Test 3 given two weeks prior to the interview includes series of problems asking for information about the original function given the graph of the derivative. On the test Ernest does not make any errors caused by confusing the two functions or misunderstanding concavity. During the beginning of his discussion of the graphing problem in the interview Ernest says, "On a normal day, I'd be able to give you the graph. Today, however --" Whether this is an excuse or an explanation is unclear, but perhaps Ernest's errors during the interview are more a lack of focus than an indication of a decrease in understanding from the test to the interview.

In general Ernest's second interview describes a similar but somewhat less complete understanding of derivative than his first interview. In both interviews Ernest focuses on slope and velocity and acceleration, with the word "limit" and the idea of calculating derivatives by the derivative rules as his only symbolic statements. The first interview statements are more complete because there, unlike in the second interview, Ernest mentions the details of the ratio, limit, and function processes for the graphical interpretation and states connections between the words slope, rate, and velocity. As with his work on the derivative graph problem, the change between the two interviews may be caused more by a lack of effort than a decrease in understanding. Ernest's absence from one third of the class periods between the first and second interviews may contribute to his lack of focus and his failure to show an increase in understanding.

QOTD #12

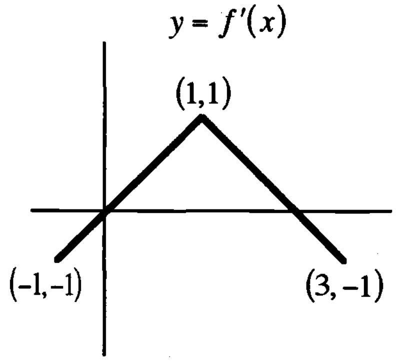
What is the most important idea that we have studied so far in this class?

Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

Response: "The 1st and 2nd Fundamental Law of Calculus"

Interview 3

The third interview occurs during the three days after the test on differential equations and antiderivatives by substitution and by parts. The first part of the interview is a summary of Ernest's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike the second interview, the point $(1,0)$ on the original function is given so that only one solution is possible.



1 MZ: This first question is just sort of a problem. I'm going to give you the graph of the derivative. [starts drawing the axes]

2 MZ: Yeah. This is the point $(1,1)$. This is the point $(-1,-1)$. This is the line connecting them. This is $(3,-1)$. [marks each of those points with dots; connects $(1,1)$ to each of the other two by line segments] And I don't care what happens to the function out side of -1 and 3. So this is $y = f'(x)$, the derivative. [writes: $y = f'(x)$] And you're going to graph the antiderivative or the original function you might say, f. And I'll also tell you that $f(1) = 0$ so you know. [writes: $f(1) = 0$]

3 MZ: So what were you thinking?

4 Ernest: I was just thinking that this is telling what the slope is.

5 MZ: Mm hmm.

6 Ernest: The original function. Since this is one, the slope would be one. So the slope of one is a 45 degree angle.

7 MZ: Right.

8 Ernest: [quietly, some parts inaudible] slope would be 0 ... How 'bout that?..

9 MZ: OK.

10 Ernest: And then this is a positive slope 'til it gets to here to a negative slope. So it's kind of like that. [has sketched in correctly shaped graph from -1 to 3]

11 MZ: OK.

12 Ernest: Kind of looks likes like that cosine curve.

13 MZ: Yeah. Uhm, how confident do you feel about how high or low this corner is for example? I mean, could it have gone down to like -3?

14 Ernest: Well, not really because there's-- the slope only gets to -1 and 1 so in the space allotted it wouldn't be able to get down to -3.

15 MZ: Does this original function that you just sketched have an inflection point?

16 Ernest: Yes. Right here.

17 MZ: OK. At $(1,0)$?

18 Ernest: At $(1,0)$. And well here and here if that would continue. [Put dots at $(-1,0)$ and $(3,0)$]

19 MZ: Oh.

20 Ernest: Possibly. It could possibly have it 'cause it could possibly go like this. [sketches another "period" of the "cosine like" curve to the left of the first one]

21 MZ: Right.

22 Ernest: If this kept going like that. And this kept going. [sketch of curve starting to continue in the same pattern to the right of the original] If the f prime kept going in the same pattern, then it would keep going like that.

23 MZ: OK, yeah. So it would be repeating itself.

24 Ernest: Yeah. Like the cosine curve.

25 MZ: Right.

26 Ernest: Oh, negative cosine, I'm sorry.

27 MZ: Yeah, you're right. It would be negative. Uhm, how do you know that's the inflection point, the $(1,0)$?

28 Ernest: Because that's where the slope goes from negative to positive or positive to negative in this case. And-- [short pause]

29 MZ: The slope meaning-- I mean, not the slope of the original function but the--

30 Ernest: That's where derivative goes from a positive slope to a negative slope is where you have the point of inflection. You take the second derivative of it. And-- [short pause]

31 MZ: Well, what would be the value of the second derivative at that inflection point?

32 Ernest: It's undefined isn't it?

33 MZ: Yeah. Why do *you* say it's undefined?

34 Ernest: Because when you make a sharp turn like that or a cusp even, it doesn't have a-- you can't take the derivative at that point.

Ernest concentrates on his knowledge that the derivative function tells the slope of the original function to sketch a reasonable graph of the original function. He also knows that the point of inflection of the original function occurs where the derivative function changes from a positive slope to a negative slope. Ernest's work here is consistent with his ability to make a similar sketch as part of the second interview. One thing to note is that Ernest does not mention concavity during the third interview discussion, and it was in discussing concavity that Ernest made several misstatements in the second interview.

Ernest's only misstatement during the third interview discussion of this problem is to mark the y values of the extrema as ± 1 instead of the correct answer of $\pm \frac{1}{2}$. He uses slope to reasonably approximate these, but he does not think of using areas to find exact values.

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

35 MZ: Right. What is a definite integral?
36 Ernest: [short pause] Definite integral. You mean, what do you use definite integrals for?
37 MZ: Uhm, yeah that would be fine.
38 Ernest: To find the area under a curve.
39 MZ: OK.
40 Ernest: And-- [short pause]
41 MZ: Yeah, anything else come to mind?
42 Ernest: Not particularly.
43 MZ: OK What about the indefinite integral?
44 Ernest: That one I really have problems with.
45 MZ: OK. [short pause] How's it different from the definite integral?
46 Ernest: [short pause] Definite integral. [Another short pause]
47 MZ: Well, OK. More just straight forward question like-- How do they look different on the page? I mean, if you had an integral there, how would you know if it was a definite or an indefinite?
48 Ernest: That's the-- [writes: $\int_a^b \rightarrow$ definite] and that's-- [writes: $\int \rightarrow$ indefinite]
49 MZ: OK. So just the difference in the-- whatever the a to b.
50 Ernest: the [inaudible]
51 MZ: So we know the definite one has to do with area. If you do an indefinite integral calculation, what kind of thing do you get? [Ernest shrugs] Well, if you did a definite integral calculation you would get some area value I guess. [short pause] But if you do the indefinite integral-- [10 second pause]
52 Ernest: I don't know.
53 MZ: What's the relationship between derivatives and integrals? I mean, do you see derivatives as having anything to do with either one of these types of integrals.
54 Ernest: Well, yeah it's a pretty integral part of it. [MZ laughs at the pun.] Like when we're doing parts and substitution--
55 MZ: Uh huh.
56 Ernest: -- you have to take derivatives and integrals all the time.
57 MZ: So it's really tied into the finding those. Right? [Ernest nods] OK. Do you happen to remember what either one of the fundamental theorems say-- at least in general?
58 Ernest: That's one of those things where you know it. You know it, but you just can't put it into words.
59 MZ: OK. Does anything come to your mind if I say fundamental theorem? Like any sort of picture or symbol or--
60 Ernest: $F(b) - F(a)$ equals something.
61 MZ: Mm hmm. [short pause] OK. Well, let me ask another sort of problem type thing. What if I gave you-- OK, this is just going to be $y = x^2$, a

parabola. [starts sketching the curve and labels as $y = x^2$] And I wanted to know the area from 0 to 1 underneath this curve, between this curve and the x -axis. So just that area right there. [shades in the appropriate area] How would you calculate that?

62 Ernest: [25 second pause]

63 MZ: No idea?

64 Ernest: Yeah.

65 MZ: Let me ask you a different question. Could you estimate what that area is?

66 Ernest: Yeah. You want a rough estimate?

67 MZ: Yeah, go for a rough estimate first.

68 Ernest: [short pause] Well that right there is a half. This one is about 1 so [inaudible] The midpoint theorem. [draws a vertical line from the x -axis to the curve at about $x = .5$ and a horizontal line segment at that height, $y(.5)$, from $x = 0$ to $x = 1$.]

69 MZ: I just wasn't sure how you were using your rectangles. Whether you were--

70 Ernest: Well, actually what you're suppose to do is find the midpoint from 0 to 1 which is .5 and then plug it into the equation which should be $\frac{1}{4}$.

71 MZ: OK, you mean the height--

72 Ernest: The height is $\frac{1}{4}$.

73 MZ: Uh huh.

74 Ernest: So you take 1 times $\frac{1}{4}$ so it's $\frac{1}{4}$. That would be the midpoint theorem. And these areas should cross out each other [marks area under the curve but not covered by the rectangle and the area in the rectangle but not underneath the curve].

75 MZ: OK. So you said 1 was the length of this whole bottom part and you multiplied that by the midpoint.

76 Ernest: Yeah.

77 MZ: OK. Do you know any other way to calculate that area? You don't have to do the whole calculation.

78 Ernest: Yeah, you could use Reimann sum [pronounced with a long i sound] which is midpoints with trapezoids and that which gives you better idea what it is. But if we've learned how to find exactly what the area is under the curve, I forgot.

79 MZ: Do you remember when we did that thing called the area function?

80 Ernest: Mm hmm.

81 MZ: It was like $A(x)$. Do you remember anything about that?

82 Ernest: [writes: $A(x) = \int_a^b f(x)\,dx$]

83 MZ: OK. So $A(x)$, from a to b –

84 Ernest: Which is the same thing as-- [in front of $A(x)$ writes: $F(x) =$] Which is-- [underneath the equality writes: $F'(x) = f(x)$]

85 MZ: So the derivative of F –

86 Ernest: Yeah, so $f(x)$ -- The antiderivative of $f(x)$ is $F(x)$.

87 MZ: OK and that's the same thing as the area.

88 Ernest: Yeah.

89 MZ: Yeah. I mean that's getting pretty close to stating one of the fundamental theorems.

90 Ernest: I suppose you're right.
91 MZ: How about slope fields?
92 MZ: Right. Uhm, do you think of the slope fields as having more to do with derivatives or integrals?
93 Ernest: More to do with derivatives because it's dealing with slopes and derivatives usually deal with the slope of a function.
94 MZ: Mm hmm.
95 Ernest: 'Cause integrals deal with the area. But I don't know what indefinite integrals do.
96 MZ: [laughs] He's working on that one. Do you think the slope fields could have something to do with integrals?
97 Ernest: I suppose so. What it does I don't know.
98 MZ: OK. I think that's it.

Ernest associates definite integral with area, but about indefinite integral he says only, "That one I really have problems with" [ln 44]. Ernest is able to estimate the area under a curve by using rectangles, but he does not remember how to calculate an exact answer. He remembers that the Fundamental Theorem has something to do with $F(b) - F(a)$ and also with $A(x) = \int_a^b f(x)dx$ but he does not connect these two. His principle statement about the Fundamental Theorem is close to that of the First Fundamental Theorem. He writes: $F(x) = A(x) = \int_a^b f(x)\,dx$ and $F'(x) = f(x)$.

When asked about slope fields, Ernest says they are more related to derivatives than integrals because they involve slope [ln 93]. He can not think of how slope fields would relate to integrals because he associates integrals with area. Even though Ernest has stated a symbolic connection between antiderivatives and area when discussing the Fundamental Theorem, he does not make that connection in any other setting.

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: While it is recorded that Ernest answered this question incorrectly, his exact response is not recorded.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Ernest's exact answer is not available. It is known though that he answers the question incorrectly.

Test 9: Semester final

Between the third and fourth interviews the class studies applications of integration and sequences. Early in this time period the class takes a semester exam that covers all of material on functions, limits, derivatives, areas, and volumes. Five questions of the day also occur during this time period. Both the test questions and the QOTD are largely computational.

On the semester final, Ernest solves an optimization problem, a related rate problem, and problems involving domain, range, continuity, and differentiation. He misses two integration calculations and a problem that gives the velocity function equation and asks for the distance traveled. On a later test Ernest corrects this latter error. Ernest also cannot find the inverse of $f(x) = \ln\left(\frac{x}{x-1}\right)$.

One other error points out Ernest's continued confusion with the relationship between limits and derivatives. The test question asks, "If $\lim_{x \to a} f(x) = L$, where L is a real number, which of the following must be true?" Ernest chooses the only answer that mentions the derivative, "$f'(x)$ exists". This is reminiscent of his inability in the first two interviews to explain the relationship between limits and derivatives. He knows the two are related but cannot say how. He is the only student to make no attempt to state the limit of the difference quotient definition on either of the first two interviews. When asked to state and use the definition of derivative on a test between the two interviews, Ernest states it incorrectly as $\lim_{x_0 \to x} \frac{f(x - x_0) + f(x)}{x - x_0}$.

QOTD #15

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: "It is continuous at every point and differentiable at every point."

Interview 4

The discussion for the fourth interview is broken into four parts. The first section includes general questions about derivatives. The second part asks the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity, and acceleration, given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the fourth interview.

1	MZ:	What is a derivative?"
2	Ernest:	That is the slope of a function at a given point.
3	MZ:	OK. Anything else come to mind?
4	Ernest:	Nope.
5	MZ:	You're sure?
6	Ernest:	[acknowledgment]
7	MZ:	Have you ever heard that sometimes people say that the derivative is instantaneous rate of change?
8	Ernest:	Yeah.
9	MZ:	What do you figure they mean by that?
10	Ernest:	Deals with change in slope. Slope changes, has different values and so it's a rate.

As in previous interviews, Ernest mentions slope at a given point as his first answer to what a derivative is. Ernest does not voluntarily mention other ways of describing derivatives. His response to the question about rate of change is to discuss the slope changing instead of stating that the slope is a rate of change [ln 10]. This distinction is consistent with Ernest's discussion of rate of change in the first two interviews. In the first interview he relates rate of change to acceleration and concavity. When asked about rate of change in the second interview, he discusses the slope or velocity changing. Since the unadorned phrase "rate of change" traditionally refers to the rate of change of the function values and not of the slope or derivative values, this may cause Ernest confusion or difficulty when communicating with others.

The next part of the fourth interview is a summary of Ernest's solution to the first of three problems of the interview. Given a table of values with x varying by .1, Ernest is asked to estimate $f'(2)$, the derivative of the function at $x = 2$. Ernest initially suggests that 20 is a good estimate. When asked for details, he explains that 20 is approximately the result of the slope calculation for the points with x values 2 and 2.1. When asked if he can find a different or better estimate, Ernest says that if he had the equation of the function he could take the derivative of it and plug in 2 to get the slope, or he could use the graph of the function to estimate the steepness. Ernest's focus throughout the problem is on the notion of derivative as slope, but he does not suggest any other numeric calculations of the slope between two points that are given or two points closer to $x = 2$ that might give a better estimate.

The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car takes 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far does the car travel in the 2 hour period?

Ernest's first reaction is that the car has traveled 60 miles. He reasons that the car averaged 30 miles per hour for two hours. When asked if there is a way to use calculus to solve the problem, he says, "Yeah, you could take the integral of that part inside the area." Although this statement sounds confused, Ernest correctly sketches a graph of the velocity function and uses the geometry of triangles to find the area underneath the curve to be 60. When asked, Ernest explains that if the graph were curved instead of straight, he would have to add up the area of rectangles. He mentions left, right and midpoint sums and says, "You take the limit of the integral, don't you, as x approaches infinity." When asked if there is a short cut to adding up everything and taking the limit, Ernest has no answer, but he is able to correctly use a definite integral to find the area under the graph of $y = x^2$.

Ernest finds the correct answer using both an intuitive and a graphical approach. He never chooses to use a symbolic approach. He mentions an integral only in two misstatements, and these are tied more to the notion of area than to antiderivatives.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Ernest is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. He is asked if the top of the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Ernest guesses that the ladder would be sliding down at a different rate than it is being pulled out of the wall. When asked to show how the rates are related, he begins solving the problem by labeling the wall as y and the floor as x and completing the following sequence of calculations:

$$y^2 + x^2 = 14^2$$

$$y = \sqrt{14^2 - x^2}$$

$$\frac{dy}{dt} = \tfrac{1}{2}\left(14^2 - x^2\right)^{-\frac{1}{2}}(-2x)\frac{dx}{dt}$$

Upon completion he states, "And then we need to know some values." When reminded of the question and that the y and x values will be different at different times, Ernest responds, "Right, but in these kind of problems, they tell you what x is usually." Note that Ernest's calculations are correct but his efforts seem to be focused on some prototypical problem that he remembers. Rather than trying to find the relationship between the two rates over time, Ernest wants the interviewer to provide values for $\frac{dx}{dt}$, y and x so that he can calculate $\frac{dy}{dt}$. This would be typical of the related rate problems he has seen in class and in his homework assignments. When provided with numeric values for the quantities, Ernest is able to determine that $\frac{dy}{dt}$ is not a constant rate and that its rate seems to be increasing in magnitude as the ladder slides down.

Interview 5

Ernest's fifth interview occurs almost two weeks after he takes the BC version of the AP exam. During that week the class discusses the written questions from the exam. Between the fourth and fifth interviews, the class studies series and integration techniques, and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Ernest's response to a set of questions on a particular topic and provides an analysis of those responses.

1 MZ: What is a derivative?
2 Ernest: It's the slope of a function at a certain point.
3 MZ: And what else is a derivative?
4 Ernest: [long pause]
5 MZ: Can you think of any other ways to describe what a derivative is?
6 Ernest: Nope.
7 MZ: OK. Do you happen to remember that formal definition for derivative?
8 Ernest: [writes: $\frac{f(x+h)+f(x)}{(x+h)}$] I'm not really sure.
9 MZ: Is there any way to relate this idea of slope of a function at a point to some kind of symbolic thing, perhaps even this?
10 Ernest: Well, if you use integrals you can find area under the slope-- under a curve by using the derivative.
11 MZ: How do derivatives come into play there?
12 Ernest: Well, antiderivatives and integrals are really the same thing, except the opposite.
13 MZ: OK. Well, how about this. What are derivatives useful for?

14 Ernest: To find critical numbers, to find when a function is increasing or decreasing, the concavity's concave up or down.

15 MZ: OK. What else?

16 Ernest: You know that if a function's differentiable at a certain point then it's continuous.

17 MZ: OK. What would be a real world situation that involves derivatives?

18 Ernest: A business wanting to know in a certain market how many cars to produce. For instance, if they produce a thousand then make the biggest profit, or if they produce two thousand then they won't be making as big a profit. You can use derivatives to figure that out.

19 MZ: How do derivatives come into play there?

20 Ernest: Well, to find if your profits were increasing or decreasing, you can graph it and take the derivative and that would tell you the function.

21 MZ: OK, what aspect of that graph would you be looking for?

22 Ernest: Critical numbers.

23 MZ: OK, and on the graph what does the critical number tell you?

24 Ernest: [pause] Well, a critical number has a derivative of zero.

25 MZ: OK.

26 Ernest: And it says when the graph is changing from increasing to decreasing or if you take the second derivative, concave up or concave down, a point of inflection.

27 MZ: OK. Explain what is meant by a rate of change.

28 Ernest: That's at what rate the slope is changing.

29 MZ: Does that have anything to do with derivatives?

30 Ernest: Yes.

31 MZ: [pause] You said, at what rate the slope is changing. Could it be at what rate something else is changing or is it mostly focused on the slope?

32 Ernest: Yeah, it could be. Like-- [short pause] I don't know.

33 MZ: OK, let's go on. Does the derivative involve a limiting process?

34 Ernest: What do you mean?

35 MZ: Well, what do you think of if I say a limiting process?

36 Ernest: Not much.

37 MZ: What about, do derivatives involve limits in any way?

38 Ernest: [pause] I don't know.

39 MZ: Is the derivative of a function a function?

40 Ernest: Yes, usually. Well yeah, it's a function. If you have x^2 and you take the derivative of that, it's $2x$. So that's a function.

41 MZ: And would it always be a function, no matter what you started with? I mean, as long as you started with a function.

42 Ernest: No, I don't think so. If the function is x , then the derivative of that is 1 Is 1 a function? It's a line, but it's not a function. A function has to have an x in it.

43 MZ: Well, $y = 1$ is a function.

44 Ernest: What about $f(x) = 1$?

45 MZ: Yeah, that would be the same thing.

46 Ernest: OK, then yes, it's always a function. Because then you get 1. You take the derivative of that and that's 0. Well, that's a function too. So it's always a function.

47 MZ: OK. What if I didn't have a polynomial type thing?

48 Ernest: What do you mean?

49 MZ: Like a trig function or a log, exponents or something else.

50 Ernest: Well, certain trig functions don't have a derivative-- No, they do have a derivative, but-- I don't know what I'm saying.
51 MZ: That's OK. I just want to kind of see how you're thinking about it. Explain what is meant by a differentiable function.
52 Ernest: It's a function that you can take the derivative of. It's also continuous because differentiable implies continuous.
53 MZ: OK.
54 Ernest: And then you get the point there's a value for the function because it has to be in order to be continuous.
55 MZ: OK. Can you give an example of a differentiable and a nondifferentiable function?
56 Ernest: Differentiable x^2. Non differentiable $\frac{1}{x}$.
57 MZ: And why isn't $\frac{1}{x}$ differentiable?
58 Ernest: Because at $x = 0$ it's undefined.
59 MZ: OK. Can you think of one that is defined and continuous, but it's just no differentiable?
60 Ernest: Uhm. I think it's $\frac{1}{x^3}$-- x^{-3}? What ever the function is, it's-- [draws a pair of axes with a curve resembling $y = \arctan x$ or $y = x^{\frac{1}{3}}$]. It's not differentiable here [at the origin].
61 MZ: OK. And why isn't it differentiable there?
62 Ernest: Because you can't have-- You can't have a slope.
63 MZ: So you're saying the slope there would be infinite?
64 Ernest: Mm hmm.
65 MZ: And that's why it can't be differentiable at 0?

As in previous interviews, Ernest uses a graphical interpretation, "the slope of a function at a certain point" [ln 2], as his first response to "what is a derivative?" (see Table A.15). Even though the portion of the fifth interview focusing on general questions about the derivative has fewer questions than similar sections of the first and second interviews, Ernest gives almost as complete of answers here as in the second interview. The two major omissions are that Ernest discusses velocity and the derivative being a limit in the second interview, and does not in the fifth interview. The additions for the fifth interview include the acknowledgment that there is a symbolic ratio for the formal definition, even though Ernest misstates it, and acknowledgment that the derivative is a function. Although Ernest makes connections between several of the interpretations of

Table A.15. Ernest: Interview 5 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎				
Did you learn a formal definition of derivative?				●	misstatement
How does the formal definition of derivative relate to slope or rate of change?	○				
What are derivatives useful for?					increasing/ decreasing concave up/down pt of inflection
What is meant by rate of change?	○	○			Possible misstatement (d=what rate the slope is changing)
Is the derivative of a function a function?				↦ ◯	
What is meant by a differentiable function?	○				
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.		○	○ ○	↦	acceleration decreasing maximum concavity incorrect calc
Summary	◎	○	○	⊙	

the derivative in the first interview, he does not make any direct connections in the second, fourth, or fifth interviews.

Ernest makes one misstatement in the fifth interview. He states that the derivative is change instead of rate of change. Ernest says that rate of change means, "at what rate the slope is changing" [ln 28]. While this is not the rate of change of the function itself, but is accurate in the sense that it refers to the rate of change of the rate of change function. In the fourth interview Ernest makes a similar statement explaining that rate of change "deals with slope. Slope changes, has different values and so it's a rate." In the fourth interview, the misstatement is more obvious. Slope is not a rate because it changes, but because it describes the covariation of two other changes.

For the second part of the fifth interview Ernest is asked if he remembers the Mean Value Theorem. His first statement is incorrect but it contains both symbolic and graphic underpinnings. He says, "In an interval $[f(a), f(b)]$ there has to be some point $f(c)$ that's the average of $f(a)$ and $f(b)$. There's got to be a point -- I remember the basic concept, and if I saw it on multiple choice, I'd probably get it." When asked if he has a picture in mind, he sketches a pair of axes with no curve. Contrary to usual notation but fitting his comment about the interval $[f(a), f(b)]$, he marks $f(a)$ and $f(b)$ on the horizontal axis, and a and b on the vertical axis. He places a dot in the center of his marks and says, "There's got to be some point here."

Ernest is then shown the symbolic statement of the Mean Value Theorem including the expression $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. He cannot interpret the statement except to say that c should be somewhere between a and b on the y-axis. He is not able to explain what $f'(c)$ or the ratio represent.

In the context of discussing the Mean Value Theorem, Ernest does not even apply the knowledge that the derivative is slope, something that he discusses correctly in the first interview through the current interview. Even more unexpected is his labeling of $f(a)$ and $f(b)$ on the horizontal axis and a and b on the vertical axis. He has shown knowledge of the $f(a)$ notation throughout the course including a graph he draws during the second interview in which he tries to explain the Intermediate Value Theorem with a sketch including correctly labeling $f(a)$ and $f(b)$ on the vertical axis and a and b on the horizontal axis.

The next question in the fifth interview involves a problem from the AB version of the AP exam which Ernest has taken. He reports that he remembers working on the question. The question is as follows:

> Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.
> (a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.
> (b) On what intervals is F increasing?
> (c) If the average rate of change of F on the closed interval $[1,3]$ is k, find $\int_1^3 \sin(t^2)\,dt$ in terms of k.

The interviewer asks Ernest to discuss his methods for solving parts (a) and (b), but does not require him to complete the solution of either part. For part (a) Ernest correctly explains how to apply the trapezoid rule to find the definite integral. For part (b) Ernest reports, "I think I took the derivative twice, no once, and did the sign graph, and whatever the critical numbers are--" Ernest explains that when he took the exam he took the derivative of $\sin(t^2)$ to find the critical numbers. Now he thinks that he should have used $\sin(t^2)$ to find the critical numbers since F is the function being asked about.

For part (c) Ernest says, "Either I didn't do it or I don't know how." When asked, he responds that he does not know what is meant by the phrase, the average rate of change of F and "that's why I can't answer the question."

Ernest does not use the phrase "average rate of change" in any of his interviews. He mentions the average value of a function once in stating that if the speed of a car increases at a constant rate from 0 to 60 then its average rate on the interval is 30. He also mentions the word average when first asked if he remembers the Mean Value Theorem in the fifth interview. He says, "In an interval $[f(a), f(b)]$ there has to be some point $f(c)$ that's the average of $f(a)$ and $f(b)$." He cannot get further without being given the statement of the theorem, and even then does not mention average again, or explain what the ratio means.

Throughout the school year Ernest is unable to state the formal definition of derivative correctly. In four of the five interviews he does not make any attempt to state the formal definition. On Test 2 Ernest writes that the formal definition is $\lim_{x_0 \to x} \frac{f(x - x_0) + f(x)}{x - x_0}$. In the fifth interview he makes the incorrect guess that the definition involves $\frac{f(x+h) + f(x)}{x+h}$. The only other time in all the interviews that Ernest tries to mention the details of the ratio in any context is in the first interview, where he describes a vertical line as having an undefined derivative because "1 over 0 would be the slope."

The next section of the interview concerns Taylor polynomials. Ernest is not able to answers any of the interviewer's questions on this topic. The records do not show whether Ernest attended the class discussion or Taylor polynomials and series five weeks previous to the fifth interview, but he receives 0 points on in-class test question asking him to calculate and use a fourth degree Taylor polynomial. He also did not need to know Taylor polynomials for the AP exam since he decided to take the lower level AB version.

The final section of the fifth interview concerns a function, f, that at any time x, given in hours, tells the outside temperature in degrees Fahrenheit. Ernest is shown a series of symbolic expressions and asked what information each one provided about the outside temperature.

For $f'(3) = 4$ Ernest first states that the temperature at 3 is 4 degrees but quickly changes his answer to a "change in temperature." He continues, "At time 3 it's risen 4 degrees in the last hour." For $f''(3) = -2$ he explains that the rate of change is changing, at 2 the rate would have been 6 to be a rate of 4 at 2. He continues, "Like acceleration. The acceleration decreased 2 but it still could have a positive velocity."

When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Ernest knows that the temperature is rising 4 degrees every hour. For $f''(x) = -2$ for $3 \le x \le 6$ Ernest says, "It's still getting warmer, but it's getting less warm. So instead of raising 4 degrees per hour, it's only raising like 3 degrees per hour."

When asked, Ernest says that he thinks the maximum would occur at 6 since "the temperature goes up." When asked to sketch a graph, he draws two straight line segments. The first has a slope of 4 connecting $(0,0)$ to $(3,12)$. The second connects $(3,12)$ to $(6,18)$. When asked specifically what $f''(x) = -2$ for $3 \le x \le 6$ tells about the graph, Ernest recognizes that it means the graph is concave down and that his graph is not. However, he does not correct his graph and answers a numerical question about the temperature value at 5 as if the rate of change was constant for that interval.

Ernest is next asked to find an equation for the temperature. Once he is given a hint that he needs to work backward from the second derivative, Ernest correctly uses antiderivatives and an initial condition to find $f'(x) = -2x + 10$ and $f(x) = -x^2 + 10x + C$. He states that he does not have enough information to find C. Given the hint to use $f(3) = 62$, he finds that $f(x) = -x^2 + 10x + 41$. He sketches this incorrectly as a concave up parabola with a vertex at $x = 0$. At this juncture time runs out for the interview.

Even though Ernest knows that the a negative second derivative indicates a concave down curve, he does not use this fact to find more accurate numeric estimates or to sketch a more appropriate graph. He recognizes that the second derivative may be thought of in terms of acceleration and that a function may have negative acceleration and positive rate of change but he has trouble stating exactly what is happening. His comment is that the temperature is "still getting warmer, but it's getting less warm. So instead of raising 4 degrees per hour, it's only raising like 3 degrees per hour." Even this verbal statement indicates a different but *constant* rate of change.

Case Study 6 — Frances

Academic record

*National Merit Scholar.

*Other AP courses: US. History (junior year), Spanish, Chemistry.

*Writing tutor at the high school writing center.

*Plans to major in chemistry in college and go to medical school.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: " A function is an equation to show how the dependent variable changes when the independent variable changes."

Comment: Notice that her answer emphasizes covariation of variables as well as a symbolic representation.

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different?

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: "a) $f(x) = 3x + 2$

$f(x) = x^{\frac{-2}{3}}$

They are very different because the first one is a linear function and the second one is a rational power function. The first one has a domain of **R** while the second one is restricted to the open interval $(0, \infty)$.

b) Something that is not a function but is almost a function would be an equation that had two different y values for the same x value."

QOTD #3

Give an example of a function <u>without</u> using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Date: August 26, 1993. This question occurs while the class is doing a quick review of functions.

Response: "1) An example of a function could be a cost function describing the cost of producing x amount of an object.. 2) this is a function" Frances also includes a sketch of the graph $y = x$.

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) non integers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: "a) yes--- $f(x) = \begin{cases} x^2 & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$

b) yes--- $f(x) = \begin{cases} x - \dfrac{1}{x} & \text{for all integers} \\ x\left(\dfrac{1}{x}\right) & \text{for all non integers} \end{cases}$ "

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to class discussion on limits.

Response: "The limit of a function at a certain point is what value $f(x)$ approaches as it gets closer and closer to x from the left and from the right."

Test 1

On a test on limits a week later, Frances is able to correctly find limits by reading values from a graph, by substituting into a piecewise function and by using algebra to simplify a limit calculation. She is able to work with the formal definition of limit to the extent of finding a δ for a given ε in both a symbolic and a graphical setting.

Interview 1

This interview occurs after the test on limits but prior the class discussing derivatives. Therefore, Frances' answers are presumed to be based on what she remembers from her junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table A.16, which codes these responses. A summary discussion follows.

1 MZ: What is a derivative?
2 Frances: I don't know. Well, I know it's the slope of the tangent line, you know to the -- Like if you're taking the derivative of x^2 so it would be $2x$.
3 MZ: OK.
4 Frances: I mean, I know how to take the derivative. The slope of the tangent line at a certain point.
5 MZ: What are derivatives useful for?
6 Frances: Uhm-- Well, like to find how like the graph behaves. Cause if you take the derivative you can find the critical numbers -- then use the critical numbers to find the max or min and like use the second derivative test to find like when it changes between -- like concavity changes and stuff. That makes it easier to graph. Also like the derivative of the position is the velocity and the derivative of the velocity is the acceleration.
7 MZ: Yeah. Can you think of anything else that it would be useful for?
8 Frances: Nope. [laughs]

9 MZ: How would you explain what a derivative to someone who's in AB and hasn't gotten to it yet or in precalculus?

10 Frances: [pause] If you had a graph and you wanted to find out the slope at any
11 particular point if you take the derivative it'll give you like an expression with x in it or something. And if you plug in like two -- if you want to at 2 what the slope of the tangent line is then you plug in two into the expression and that would give you the slope.

12 MZ: Now the second part of the question is , what if you had to explain it to someone who has no math clue?

13 Frances: I don't know how to explain it with out using like terms that you'd already have to know some math to understand.

14 MZ: Say I give you a function. How can you tell if it's a differentiable function?

15 Frances: Differentiable. Well, if the slope-- The limit of the slope from the left and the right has to be equal. You have to be able to like -- cause at like a cusp you can't take the derivative because the slope goes from like a negative to a positive and like the positive uhm-- Like at a cusp the slope -- there's not a limit. The limit of the slope is not defined.

16 Frances: All polynomials are differentiable.

17 MZ: OK. And how do you know that?

18 Frances: [pause, both laugh] You just know it. I don't know.

19 MZ: Why --

20 Frances: Because they're continuous everywhere.

21 MZ: So is that true in general that if a function's continuous then it's differentiable?

22 Frances: No, because like at a cusp it's continuous but not differentiable.

23 MZ: Oh, OK.

24 Frances: If it's continuous and the limit of the slope exists --

25 MZ: If it's continuous and the limit of the slope exists it should work.

26 Frances: Yeah.

27 MZ: OK. Do you know of an equation of a function that has a cusp?

28 Frances: No.

29 MZ: But you would just know it if you saw it?

30 Frances: Yeah, well -- I don't know. I mean, if I saw the graph I would know it was a cusp.

31 MZ: Right.

32 Frances: Or if it had a sharp turn.

33 MZ: Right. OK. Here's a list of words. I'll just show them to you one at a time. I was going have you tell me if they have to do with derivative or not.

34 MZ: We can go on. Speed or velocity?

35 Frances: The derivative of the position equation would be the velocity.

36 MZ: Change or rate of change?

37 Frances: Well like the slope at a certain point -- the average slope is like -- Well, say you had a curve. If you took the slope between two points on that curve that would be like the average -- you know, the average rate of change. But if you took the derivative, that would be the instantaneous rate of change there.

38 MZ: So to make derivative related to rate of change you have to be careful about how you say it. You have to say instantaneous as opposed to average. Line or linear?

39 Frances: Well, the derivative of a line is a constant.

40 MZ: True.

41 Frances: But I don't know what else.

42 MZ: What about linear?

43 Frances: Well, if it's a linear function, then the slope is a constant. So the limit of the slope would be a constant.

44 MZ: Measurement?

45 Frances: Oh, is it like-- If there's an error in the measurement you can use derivatives to see how it would change the results? Using, uhm --

46 Frances: This is differentiation. Like if you -- Let's say you have to make a box
47 of a certain volume, but your pieces have a percent -- like a plus or minus .01 cm error in it -- like how that would change the volume of the cube when you're done. We used differentiation to do that.

48 MZ: Prediction or approximation?

49 Frances: [pause only]

50 MZ: Should we skip this one?

51 Frances: Yes. We better.

52 MZ: Optimization?

53 Frances: Well, if you want at the maximum value, you could take a derivative and find the critical numbers.

54 MZ: How does that work?

55 Frances: Oh, uhm. Set the derivative equal to zero and those are your critical numbers and those are where you would have the maximum or minimum points. You do the first derivative test, and you can see.

56 MZ: So why is it true that when the derivative is equal to zero that's when you have a max or a min?

57 Frances: The tangent line would be horizontal and a horizontal line has a slope of zero. So the derivative would be zero where you had like a lump [laughs]-- when the tangent is zero.

58 MZ: Continuity?

59 Frances: Uhm-- Well, you have to have a continuous function to be differentiable.

60 MZ: You had kinda already talked about it. What about if you have a differentiable function does it have to be continuous?

61 Frances: A differentiable function has to be continuous but a continuous function doesn't have to be differentiable.

62 MZ: Limit?

63 Frances: Well, for it to be differentiable the limit of the slope has to exist.

64 Frances: That's what I was saying before.

65 MZ: You guys didn't do integrals did you?

66 Frances: I know it's like the antiderivative.

67 Frances: But then in the next chapter they call it integral. But I don't know why --

68 MZ: Function?

69 Frances: [pause]

70 MZ: That's kind of a weird question. Is the derivative a function?

71 Frances: [long pause] Uh, yeah. I don't know. I guess so.

72 MZ: So it seems like it should be. Why does it seem like it should be?

73 Frances: Because when you take the derivative you're just like decreasing the exponent for it.

74 MZ: Uh huh.

75 Frances: And I don't really see how that would -- I mean, I just can't figure out how that could not be a function.

76 MZ: Yeah.

77 Frances: I just don't know how to explain it right.

78 MZ: That's OK. Do you remember a formal definition of derivative? Not epsilon delta that formal but just sort of --

79 Frances: Uhm-- I didn't know there was one. Well, like-- uhm-- [pause]

80 MZ: What kind of thing do you think it is?
81 Frances: It's like the limit of the s -- They say it is like the limit of the slope.
82 MZ: The limit of the slope.
83 Frances: It's like so they do -- Uhm-- The limit as x approaches h of -- of-- uhm-- of--[pause] I don't know. It's like $x+h$ somewhere. I don't know. I just don't remember how-- Wait, the slope of the change in-- I don't know.

Table A.16, summarizes Frances' first interview transcript. Frances recalls that the derivative is related to slope and velocity and acceleration. When asked specifically, she also remembers that derivative is related to rate of change. She states the instantaneous nature of derivative when discussing slope and rate of change, but not while discussing velocity or acceleration. She does not describe a limiting process or the details of a ratio.

Regarding the formal symbolic definition of derivative, she can only remember that it is a limit and that it contains an $x+h$. When asked whether the derivative is a function, she guesses that it is based on the symbolic expressions one obtains when calculating derivatives by the power rule.

On two occasions Frances makes a connection between different models for derivative. When asked about rate of change, Frances discusses that the average slope, the slope between two points, is the average rate of change. When attempting to state the formal definition she remembers that the symbols are related to the limit of the slope. She is never directly asked to make any connections [ln 70-75].

QOTD #6

Find the derivatives of the following four functions:

$f(x)=(x-1)^2(x^2-4)$

$g(x)=\dfrac{x-1}{\sqrt{5-x^3}}$

$h(x)=\sin x$

$j(x)=\ln x$

Table A.16. Frances: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎			↦	
What can derivatives be useful for?			∘ ∘		max/min concavity acceleration
Explain what a derivative is to someone who's an AB student or precalc student who hasn't studied it yet.	◎			↦	
How can you tell if a function is differentiable?	∘ ○				
Is derivative related to speed or velocity?			∘		
Is derivative related to change or rate of change?	◎	◎			
Is derivative related to line or linear?	○ ∘				
Is derivative related to measurement?				↦	
Is derivative related to optimization?	∘				max/min
Is derivative related to limit?	○ ∘				
Is the derivative a function?				○ ↦	
Did you learn a formal definition of derivative?	∘			◎	
Summary	◎	◎	∘	⦾	

Date: September 20, 1993. This question occurs prior to the class learning about short-cut rules for taking derivatives of various forms.

Response:

$f'(x) = (x-1)^2(2x) + (x^2-4)(2)(x-1)(1)$

$g'(x) = \dfrac{(5-x^3)^{\frac{1}{2}}(1) - (x-1)(\frac{1}{2})(5-x^3)^{-\frac{1}{2}}(-3x^2)}{5-x^3}$

$h'(x) = -\cos x$

$j'(x) =$ I DONT KNOW

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be using your knowledge of derivative.

$y = \log(x^3) \qquad y = \dfrac{x}{|x|} \qquad y = x^e \qquad y = e$

Date: September 21, 1993. This question also occurs before the class studies short-cut rules for taking derivatives but after they have studied the limit definition of derivative.

Response: Frances is absent on this day.

Comment: This question is presented to the students since no student correctly stated the derivative of $y = \ln x$ in the previous Question of the Day.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$? Why <u>not</u>?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: "a) No, because $\tan x = \dfrac{\sin x}{\cos x}$ so when you take the derivative you don't get $\cot x$."

For part b, Frances correctly applies the quotient rule and simplifies to get $y' = \sec^2 x$.

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. Frances shows that she can correctly state the limit of the difference quotient definition of derivative and use it both symbolically and numerically to calculate a derivative value. She also knows to estimate the derivative at a particular point by finding two nearby points and calculating a difference quotient for those two points. Given the graph of a position function for a car she correctly answers questions about the time and value of the maximum speed and the time interval that the car is stopped. However, she thinks that the car is moving backward when the position is negative instead of when the slope or velocity is negative.

Given the graph of a function, Frances is able to sketch a correct graph for the derivative function. Her paper emphasizes the connection between the extrema of the function and the zeros of the derivative. Frances also correctly works through two complex chain rule derivatives with only one minus sign error for the derivative of the tangent function.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: "I really think I understood derivatives pretty well last year, but I did understand them better after relearning the formal definition of the derivative instead of just the chain rule and stuff.."

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "a) We talked about critical points, extrema, and when a function was increasing or decreasing.

b) It was pretty much all review from last year except I understood this time how on an open interval (a, b) the points $(a, f(a))$ and $(b, f(b))$ can't be max or min points because they are never reached."

Comment: Since the researcher had not been present the day prior, this question is presented both as a means for the researcher to see the material covered and to ascertain the students' understanding of it.

Test 3

About two weeks later the class is tested on Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Frances correctly calculates a third degree Taylor polynomial. She correctly uses the first and second derivatives of the position function to find the speed and acceleration of an object at a given time. She is able to use the graph of a derivative function to estimate when the original function is increasing or decreasing, concave up or concave down, and where it has extrema. She correctly notes that the zeros of the derivative are the critical numbers of the original function and only misses that the value where the derivative is undefined must be considered a critical point as well.

QOTD #11

Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: "If you have a swimming pool where water is going out the drain while you are filling it you can use derivatives to find the rate of change of the volume of the water in the pool at a certain time."

Test 4

Two weeks later the class has a test on the applications of derivatives. Frances correctly uses derivatives to solve three traditional max/min problems and three traditional related rate problems. She also correctly calculates the derivative of an implicitly defined function.

Interview 2

The second occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by a summary table (Table A.17) and a discussion.

1	MZ:	What is a derivative?
2	Frances:	The slope of the tangent line to a point.
3	MZ:	OK. Does anything else come to mind in terms of describing what a derivative is?
4	Frances:	The rate of change.
5	MZ:	OK.
6	Frances:	Instantaneous, I mean you know, instead of the average, it's the instantaneous rate of change.
7	MZ:	What can derivatives be useful for?
8	Frances:	[short pause] Knowing the behavior of the function. Also those word problems, the related rate problems, where something's like changing. Like if the volume's changing or the surface area with time, then you can see how the other things change, like the radius.
9	MZ:	OK, and what aspects of that problem is the derivative?
10	Frances:	It's the rate of the change.
11	MZ:	OK, any other examples?
12	Frances:	No, next question.
13	MZ:	How would you explain what a derivative is to someone without very much math background?
14	Frances:	Like how fast it's changing. How fast like a curve is changing at a certain point.
15	MZ:	OK. [short pause] Can you give example of a real world situation that has to do with derivative?

16 Frances: Like those problems we did. Like if you have a cost function and you want to like maximize, uh minimize cost.

17 MZ: How do those problems work generally?

18 Frances: What do you mean?

19 MZ: What usually happens in a problem like that? How does the derivative fit into that problem? What does it have to do with it?

20 Frances: Well, if you have like the cost function and you have to make it so that you have only one variable in the equation, in the function.

21 Frances: And then if you take the derivative of that, you try to maximize. The variable in the function-- you find what value it maximizes or minimizes the cost.

22 MZ: Did you say how the derivative fit into that?

23 Frances: You take the derivative to find what value of the variable gives you the maximum.

24 MZ: OK, so I take the derivative and then what happens?

25 Frances: You set it to zero. 'Cause those are the critical numbers.

26 MZ: What is it about setting it equal to zero that makes it the maximum?

27 Frances: See if you have a curve and when the slope is zero you're going to have a max or min point.

28 MZ: OK, she drew this-- [sketches smooth local max and a horizontal tangent line]

29 Frances: Yeah, and the slope there will be zero so you want to set the derivative equal to zero.

30 MZ: How can you tell if a function's differentiable?

31 Frances: If it's continuous over the interval and uhm-- I don't remember. If it's continuous and there's another--

32 Frances: It doesn't have any sharp turns.

33 MZ: OK, no sharp turns. What's bad about having a sharp turn?

34 Frances: The limit of the-- The limit of the slope is like-- I think the slope is changing too fast because first it'll be negative and then it'll be positive right away. I don't know.

35 MZ: OK, let's do these. Do these words have to do with derivative? Slope?

36 Frances: Mm hmm. [yes]

37 MZ: Velocity?

38 Frances: The first derivative of the position function is the velocity.

39 MZ: Line or linear?

40 Frances: Uhm, the derivative of a linear function is a constant.

41 MZ: OK. Measurement?

42 Frances: You could do those problems with uhm-- If they tell you that you make something, but there's a four percent error in the length of it, how would it change the volume of it.

43 MZ: OK, so where does the derivative fit into that?

44 Frances: Because you'd see how-- I don't know anymore. You're seeing how that change would change the volume.

45 MZ: Prediction or approximation?

46 Frances: Uhm, well we did those problems where you can-- instead of finding the exact slope of something at a point, like the instantaneous one-- instead of finding the exact derivative, you could approximate it with that by using a really small interval and seeing the average rate of change for that time.

47 MZ: So that would be like an approximation of the actual derivative?

48 Frances: Yeah.

49 MZ: Continuity? Did you say what continuity has to do with derivatives?

50 Frances: I said it had to be continuous to be differentiable.

51 MZ: OK. And would every continuous function be differentiable?

52 Frances: No, if we had a sharp turn.

53 MZ: Limit? Does limit have to do with derivatives?

54 Frances: Yeah. The derivative is a limit-- The formal definition of derivative is the limit as--

55 Frances: [writes: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$]

56 MZ: So this is the formal definition?

57 Frances: Mm hmm. [yes]

58 MZ: How does the formal definition relate to the idea of slope or rate of change that you were talking about earlier?

59 Frances: 'Cause if you had like a function and you wanted to know like where-- You make the interval, the Δx, like the change in a really small interval, then it's not going to change the-- Cause, let's see-- So if Δx was approaching 0 it would be changing-- The instantaneous slope-- The average slope gets closer and closer to the instantaneous slope when Δx goes to 0. [has written: $m = \frac{\Delta y}{\Delta x}$]

60 MZ: So this, what you wrote-- [referring to $m = \frac{\Delta y}{\Delta x}$]

61 Frances: Yeah, that's the average change-- rate of change.

62 MZ: OK, so that's the average rate of change, and that relates to this, I guess, how?

63 Frances: Because-- The function doesn't have-- The closer you are to the point you want, the smaller that the-- There's not a lot of room for the function to change in between there. [has sketched a smooth curve on a pair of axes and marked one point and put a pair of parentheses around it]

64 MZ: OK. So is this exactly the same? I mean Δx here is the same as the Δx here?

65 Frances: This is the change in y.

66 MZ: That's the change in y on top?

67 Frances: Yeah.

68 MZ: Where would these different things be on this picture?

69 Frances: [pause] Uhm-- I don't know. I have no clue.

70 MZ: No idea. Somehow it's related to that picture?

71 Frances: Yeah.

72 MZ: How about, is the derivative a function?

73 Frances: Derivative a function? [short pause] OK. [laughs; pause] Sure. I don't know. I can't think of a--

74 MZ: Why does it seem like it should be? I mean, you said sure.

75 Frances: I don't know. Cause if you do something to a function, I guess it stays one. I don't know.

76 MZ: Newton's Method? Did you study that before?

77 Frances: Yeah.

78 MZ: Do you think it has anything to do with derivative? Do you happen to remember?

79 Frances: I remember it's the way to find the roots of something. Is that right?

80 MZ: Yeah.

81 Frances: But I don't remember what to do.

82 MZ: OK, and so you don't have a feel for if derivative is involved in that process or not?

83 Frances: No. I don't know.

84 MZ: Intermediate Value Theorem?

85 Frances: It has to do with something-- I don't know. It's one of those ones that if over the closed interval something-- [laughs]

86 MZ: Well, there's two of them here, the Mean Value--?

87 Frances: Yeah, those two are really close. One of them is more-- I don't know. This one says that there's a-- Isn't this one that says there's a c on the interval where the derivative is 0.

88 MZ: She's talking about the mean value theorem.

89 Frances: If you have over this interval, and a and b are the-- $f(a)$ equals $f(b)$. [sketch of a curve with a single max on a pair of axes; a and b are marked on the x -axis so that their y values on the curve are equal]

90 MZ: OK, $f(a)$ equals $f(b)$.

91 Frances: Then there's a c in there where the derivative at c is equal to 0.

92 MZ: OK.

93 Frances: Because you would have a maximum somewhere.

94 MZ: Somewhere in there. Yeah, that's good. What if $f(a)$ and $f(b)$ were different? They weren't equal.

95 Frances: Then it would probably be the Intermediate Value Theorem. [laughs]

96 MZ: Well, not exactly. Say I give you a function and I say, this is a derivative. What can you tell me about the original function?

97 Frances: Uhm, you know where it's increasing and decreasing.

98 MZ: OK.

99 Frances: 'Cause the derivative is the slope so if the slope is positive then you know that the line is going like that.

100 MZ: Yeah, up towards the right.

101 Frances: And if the slope is negative, then you know it's decreasing.

102 MZ: OK. Would it be easier or harder if I gave you a graph and said, this is the derivative?

103 Frances: Harder.

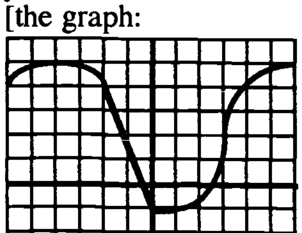
104 MZ: Harder? To do the calculation?

105 Frances: Well, I mean like I know that wherever it was above the y-axis then the slope is positive, but I would have to think about it more.

106 MZ: So if I just gave you an equation you would just-- I guess, you would just see where it was positive.

107 Frances: If that was the equation of the derivative, I would do the first derivative test. I would find the critical numbers and do the first derivative test.

108 MZ: I'm going to give you a sketch and say this is the derivative and see if you can tell me the function.
[the graph:



]

109 Frances: This is the derivative. [Noting x -intercepts of the derivative graph]

110	MZ:	Yeah, you can approximate those points. Don't make the problem any harder than necessary.
111	Frances:	[pause, working] OK. [working] Well, I don't know where it is? I mean, cause [inaudible]
112	MZ:	You said you didn't know where it was?
113	Frances:	I know that it goes from being increasing to decreasing at point 5 so there's a max there, but I don't know where.
114	MZ:	How high it is or whatever?
115	Frances:	Yeah.
116	MZ:	OK.
117	Frances:	And then it goes to 2. [working] Wait. What am I doing? [pause] At 2 there'd be a--. I don't know where anything is.
118	MZ:	OK. Now are you thinking that you don't know where anything is because it's not possible to know or you just don't happen to know.
119	Frances:	I just don't happen to know.
120	MZ:	But you're thinking there might be a way.
121	Frances:	I guess, yeah.
122	MZ:	You don't have to know. Where would be an inflection point of the original function?
123	Frances:	The max and min points of the derivative. Here, here, and over there.
124	MZ:	OK. How come it is that the max of the derivative gives you an inflection point?
125	Frances:	Because if you took the first derivative of the first derivative-- You know like [heading it?] this way. Because the second derivative is how you find an inflection point, when the second derivative is equal to zero. So if you took the derivative of this, it would be the second derivative, and the slope is zero at the max and min for this.
126	MZ:	OK. Thank you.

Frances mentions a graphical interpretation of derivative first and more often than any other interpretation, but rate of change is mentioned almost as frequently. Frances is able to explain the details of the ratio and the limiting process in terms of slope, rate of change, and the formal symbolic definition. She only mentions the example of velocity when asked about it specifically. In Table A.17, Frances is given credit for knowing that derivative is a function when she states the formal definition as $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. This notation is correct for defining the derivative function, but Frances may think of x as a specific point. When asked specifically whether the derivative is a function, she guesses that it is by saying, "if you do something to a function, I guess it stays one" [ln 75].

Table A.17. Frances: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎	◎			
What can derivatives be useful for?		○ ◎			related rate
Give an example of a real world situation that involves the concept of derivative.	●				max/min
How can you tell if a function is differentiable?	○ ○				
Is derivative related to slope?	○				
Is derivative related to speed or velocity?			○		
Is derivative related to change or rate of change?		●			
Is derivative related to line or linear?				↦	
Is derivative related to prediction or approximation?	◎	◎			
Is derivative related to limit?				◉	
How does the formal definition of derivative relate to slope or rate of change?	◉	◉		◉	
Is the derivative a function?				○ ↦	
Given the derivative, what can you tell me about the original function?	○				in/decreasing max/min inflection pt
Summary	◉	◉	○	◉	

Frances mentions a graphical interpretation of derivative first and more often than any other interpretation, but rate of change is mentioned almost as frequently. Frances is able to explain the details of the ratio and the limiting process in terms of slope, rate of change, and the formal symbolic definition. She only mentions the example of velocity when asked about it specifically. In Table A.17, Frances is given credit for knowing that derivative is a function when she states the formal definition as $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. This notation is correct for defining the derivative function, but Frances may think of x as a specific point. When asked specifically whether the derivative is a function, she guesses that it is by saying, "if you do something to a function, I guess it stays one" [ln 75].

Frances makes a connection between her statement of the formal definition, $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, and slope. She notes that the difference quotient represents $m = \frac{\Delta y}{\Delta x}$ and explains that Δx approaching 0 means the "average slope gets closer and closer to the instantaneous slope" [ln 59]. Without prompting she also states that $\frac{\Delta y}{\Delta x}$ is the average rate of change and begins describing a limiting process in that context before stopping herself.

For the last part of the second interview Frances is asked to graph an original function when given the graph of the derivative function. She sketches a reasonable graph using her knowledge that derivative is slope to help her determine where the function is increasing or decreasing and where its extrema lie. Her only confusion is that she does not know at what height the curve should be located. She thinks that maybe there is a way to tell how high it should be that she is not remembering when actually a curve of the right shape at any height would be correct.

In comparing Frances' responses during the first two interviews it is clear that she shows a more complete understanding during the second interview . Her replies in both interviews are similar in that a graphical interpretation is the dominant representation. However, in the second interview, rate of change also plays a frequent role in her comments. Frances' first interview contains no description of the details of the ratio or limit processes. In the second interview she states these in terms of slope, rate of change, and a symbolic notation as well as relating the three to each other. Unlike in the first interview, Frances' can now state the correct limit of the difference quotient definition.

QOTD #12

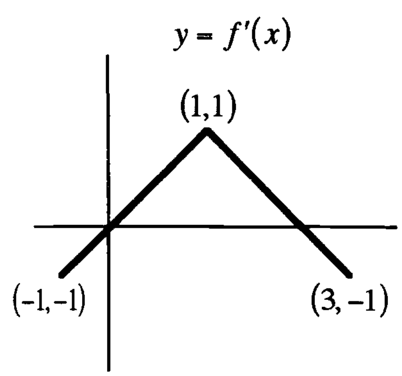
What is the most important idea that we have studied so far in this class?

Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

Response: "How the integral and the derivative are related by the Fundamental Theorem of Calculus"

Interview 3

The third interview occurs during the three days after the test on differential equations, and antiderivatives by substitution, and by parts. The first part of the interview is a summary of Frances's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike the second interview, the point (1,0) on the original function is given so that only one solution is possible.



1 MZ: OK. The first question-- is actually a problem. Here's going to be the graph of the derivative [sketches axes].

2 Frances: Mm hmm.

3 MZ: And this is the point $(1,1)$ and this is the point $(-1,-1)$ and this is the line connecting them. And this is $(3,-1)$ and this is the line-- [sketches the three points indicated and the line segments connecting $(1,1)$ to the other two] So you have this function. OK. So you have this upside-down v function. This is $y = f'(x)$ [writes: $y = f'(x)$]. And so I want you to sketch the graph of f, the original function. And don't worry about outside of that domain, just from -1 to 3. [short pause; Frances draws an axes] I should probably give you an initial condition--

4 Frances: OK.

5 MZ: So you know $f(1) = 0$ [writes: $f(1) = 0$].

6 Frances: Uhm--

7 MZ: OK.

8 Frances: [short pause] I don't know. I mean I know how to do a slope field, but I don't--

9 MZ: [15 second pause] What are you thinking?

10 Frances: That the slope is positive here--

11 MZ: OK.

12 Frances: So I'm-- So the function is increasing before 1 and then the-- [15 second pause] I don't remember. OK. Now I'm thinking about integrals. I can't do this.

13 MZ: OK. Just you mean in general you're thinking about integrals?

14 Frances: Yeah. And like I don't remember if this is--

15 Frances: [40 second pause] I can't think about this anymore.

16 MZ: So this doesn't seem to have anything to do with integrals which is kind of the stuff that's in your mind now.

17 Frances: No, I mean I know it does but-- OK, let me think. [30 second pause] Well, except if you think about it like-- as like integral, the area under the curve here at 0.

18 MZ: OK, so you plotted this point. Is this like sort of negative a half, zero or something like that? [the point is actually marked as $(0,-.5)$]

19 Frances: Yeah.

20 MZ: And how did you get that point?

21 Frances: I don't know. It's not math hour anymore. I can't think about math. [both laugh]

22 MZ: Well, you said something about area under the curve. Were you using that to get that $(\frac{1}{2},0)$ point?

23 Frances: Yeah. Well just look at 1. You've got negative one half and positive one half so they'll cancel out.

24 MZ: OK.

25 Frances: I can't do anymore.

26 MZ: So does that-- what you just explained to me about these canceling out, does that relate to one of these points that you just graphed on the other function?

27 Frances: Huh?

28 MZ: Did you use that, that you just explained to me about the area canceling out, to plot one of these two points or was that a different thing you were talking about?

29 Frances: Well, you already gave me that point.
30 MZ: Yeah, I already gave you that, $(1,0)$.
31 Frances: [25 second pause] It doesn't seem right though.
32 MZ: What doesn't seem right?
33 Frances: It's not.
34 MZ: The points you plotted, you mean?
35 Frances: Yeah.
36 MZ: Why do you think they're wrong?
37 Frances: I don't know. [10 second pause] Oh, wait. I'm thinking right. The slope is negative here so it's going to be decreasing there and since that's-- [short pause] So it's decreasing until there. I don't know about that point though. [sketched decreasing graph from about $x = -1$ to $x = 0$ going through $(0, .5)$]
38 MZ: OK.
39 Frances: And then it's increasing until 2 and then it's decreasing. [sketches a function increasing through $(1,0)$ and about $(2, .5)$ and then decreasing from there to about $(3,0)$]
40 MZ: OK. But you said you weren't that confident about that point, about it going down to $-.5$. I mean, do you feel like it could go down to like -3 or it just doesn't go down that far?
41 Frances: I don't know.
42 MZ: Does this function, the original function, have an inflection point?
43 Frances: Yeah.
44 MZ: Where?
45 Frances: Here.
46 MZ: Uhm, at $(1,1)$ of the derivative function.
47 Frances: Yeah at here.
48 MZ: So which is going to be $(1,0)$ of the original function. So what's the value of the second derivative at that point, at $x = 1$, at the inflection point?
49 Frances: Zero.
50 MZ: And how do you know that it's zero?
51 Frances: Because if you took the derivative of this--
52 MZ: OK, you took the derivative of the derivative.
53 Frances: The slope there is zero.
54 MZ: The slope there is zero at the-- corner, whatever. Is it always true that the inflection point occurs where the second derivative is zero or could the second derivative be something else not zero?
55 Frances: Well, if it's like undefined?
56 MZ: So the inflection point could occur where--?
57 Frances: Because if you had an asymptote and over here it was going like that and over here it's going like, I don't know, like that [sketches graph of a function similar to $y = \frac{1}{x}$; marks the y -axis with a dotted vertical asymptote line]
58 MZ: OK.
59 Frances: Then your infection point's at the asymptote, so--
60 MZ: Yeah, the function's not even defined so the second derivative's not defined either. True.

As in the second interview, Frances is able to use her knowledge that the derivative is slope to describe where the original function is increasing and decreasing and to discuss the location of its extrema. In addition, she is able to use her newly acquired knowledge of the relationship between antiderivatives and area under a curve to use area calculations to help her plot specific points on the original function. This combination allows her to sketch a curve that not only has the correct general shape but the correct height as well. Her only misstatement comes in discussing the location of the inflection point of the original function. She knows that it occurs where the derivative has its maximum, but says that the derivative function has a slope of zero at this corner point [ln 53].

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

61	MZ	Let's keep going. What is a definite integral?
62	Frances:	It's the area under a curve bounded by the x axis in between two definite points.
63	MZ:	OK. What's the indefinite integral?
64	Frances:	[short pause] Well, it's like if you don't have an initial condition, you know, because there's a constant. Like if you took the derivative of something any constant won't show up in the derivative. So if you go backwards there's going to be a constant added that's not there.
65	MZ:	OK. Explain the part about why the constant doesn't show up when you take the derivative.
66	Frances:	Because the derivative of a constant is zero.
67	MZ:	So it just goes away in the calculation. So then it gets put back in-- When you go the other way, it gets put back in?
68	Frances:	Yeah.
69	MZ:	So the indefinite integrals are the ones that have the extra constant?
70	Frances:	Yeah. Well, yeah because like on a definite integral like you take the-- you take the f -- the-- You know you take the b minus the a.
71	MZ:	We're talking about the definite integral?
72	Frances:	Yeah. So then the C's would be-- the constants would be canceled out.
73	MZ:	Oh, yeah, right. So do a really easy example to show how that works, like uhm I don't know, like x^2 or something.
74	Frances:	So it would be x^2 plus C and then you're going from-- And then you do that. [writes: $\Big[_a^b x^2 + C$]
75	MZ:	OK. Like from a to b.

76 Frances: So you would have $b^2 + C - a^2 - C$. [writes: $b^2 + C - a^2 - C$] [marks a line through both C's]

77 MZ: Yeah, the C's go away.

78 Frances: So you just have $b^2 - a^2$. [writes: $b^2 - a^2$]

79 MZ: So how are the definite and the indefinite integrals related then?

80 Frances: [short pause] The definite integral is the indefinite integral at b minus the indefinite integral at a.

81 MZ: OK. So you kind of use the indefinite to get the definite?

82 Frances: Yeah.

83 MZ: How is the derivative related to the integral?

84 Frances: The derivative of the integral is-- You take the-- [short pause] Like if $f(x)$ is the original function--

85 MZ: OK.

86 Frances: and $f'(x)$ is the derivative--

87 MZ: OK.

88 Frances: then if you take the integral of $f'(x)$ you get $f(x)$.

89 MZ: OK. Do you remember what the fundamental theorem says?

90 Frances: [short pause] Mmm. Well, the second one says what I said before.

91 MZ: OK. About the b minus a part?

92 Frances: And the first part says that-- [short pause] the integral of the derivative is the original function.

93 MZ: OK. Do you remember when we talked about area functions?

94 Frances: The area under the curve?

95 MZ: Like that $A(x)$ function?

96 Frances: Yeah.

97 MZ: Do you remember how that function was set up?

98 Frances: What do you mean?

99 MZ: Uhm. What is an area function? What does it show or-- [short pause] I guess, there's two things I'm thinking of. One of them is, what is $A(x)$ equal to as like an equation and the other one thing I was thinking of is what does it represent graphically.

100 Frances: [short pause; next said quietly] The area under the curve is equal to the indefinite integral. [laughs] I don't remember.

101 MZ: OK. Say you have this $y = x^2$ and you want to know the area from 0 to 1. So I want to know what this area is right here. [sketches the parabola, shades the appropriate region and labels the graph as $y = x^2$] How would you find that area?

102 Frances: Well, you could either-- like you can approximate it using those stupid things--

103 MZ: [laughs]

104 Frances: Or you can take the definite integral from 0 to 1.

105 MZ: OK. So actually find that area.

106 Frances: [short pause] Wait. I have to think about this a second. Uhm, I'm confusing myself. So wait is this the-- What is this?

107 MZ: $y = x^2$ is the equation for this curve right here that I just drew.

108 Frances: Yeah, I know but-- [inaudible] [30 second pause while writing:
$\int_0^1 x^2\, dx$
$\frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$]
109 MZ: So then this area is one third? Is that what you're thinking?
110 Frances: Yeah. [laughs]
111 MZ: Which part seems confusing?
112 Frances: I don't know. I just don't remember what I'm doing. [laughs]
113 MZ: OK.
114 Frances: I mean I--
115 MZ: That's the calculation I would have done too. OK. Do you think slope fields have more to do with derivatives or integrals?
116 Frances: Integrals. Well, you need to know the derivative to find the slope field, but it's like showing you like the shape of your integral.
117 MZ: OK. So you would be given not the original function then. You would be given the derivative?
118 Frances: Yeah.
119 MZ: And then you would use that--
120 Frances: Because if you had a slope field and you knew that it-- I don't know like [inaudible]-- [sketches three slope field lines on the x -axis each one of a steeper slope than the one before] you know and it got higher you'd know that the-- like original function had a shape that went something like-- [sketches an increasing function across those x values]
121 MZ: Right, that it increased. OK. That's it.

Frances associates definite integral with area. She associates indefinite integral with antiderivative and emphasizes that there is a constant involved because the derivative of a constant is 0. She relates definite and indefinite integrals by stating how the Second Fundamental Theorem uses "indefinite integrals" to calculate definite integrals [ln 80-82]. Later she correctly calculates an area by applying this method to a definite integral. She thinks of derivatives and integrals as opposite operations, and she states the First Fundamental Theorem as saying, "the integral of the derivative is the original function" [ln 92]. Here she is using derivative in the sense of the derivative operator, not the derivative at a point or the derivative function.

Frances does not know what is meant by an area function except in the sense that the integral represents the area under the curve. When asked whether slope fields have more to do with derivatives or integrals, Frances responds, "Integrals. Well, you need to know the derivative to find the slope field, but it's like showing you the shape of your integral" [ln 116]. She already connects antiderivatives and area in terms of the Second Fundamental Theorem and describes the integral and derivative as opposite operations, but she does not try to answer why slopes and areas should be related in this way.

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: While it is recorded that Frances answered this question incorrectly, her exact response is not recorded.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Frances' exact answer is not available. It is known though that Frances answers the question correctly.

Test 9: Semester final

On the semester final, Frances correctly solves an optimization problem as well as questions of domain, inverse functions, and continuity. She also computes limits, derivatives, and integrals correctly. She only makes two errors. One is regarding the domain of a function, and the other involves finding the inverse of a function. On the latter she uses the derivative of the function f to find the derivative of the inverse, instead of the inverse itself.

QOTD #15

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: "The function is continuous everywhere because the $\lim_{x \to a} x^{\frac{2}{3}}$ will always equal $f(a)$, however it is not differentiable everywhere because at $x = 0$ there is a sharp turn."

Interview 4

The discussion for the fourth interview is broken into four parts. The first section includes general questions about derivatives. The second part asks the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity, and acceleration, given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the fourth interview.

1	MZ:	What is a derivative?
2	Frances:	It's the slope of the tangent line to the graph at a certain point. Like the instantaneous slope.
3	MZ:	OK. Does anything else come to mind when you think of what a derivative is?
4	Frances:	No.
5	MZ:	Are you sure?
6	Frances:	Uhm, derivatives-- [short pause] When were we using derivatives? I don't remember anymore. Uhm-- [pause]
7	MZ:	Well, have you ever heard when sometimes people say derivative is the instantaneous rate of change?
8	Frances:	Mm hmm.
9	MZ:	What do you think they mean by that?
10	Frances:	Like how fast it's changing at a certain spot.
11	MZ:	OK. So the instantaneous part is that at a certain spot part?
12	Frances:	Yeah, like-- You could say at any-- You could find it at any point. It's not just-- Like 'cause when we did slope, like a long time ago, we were always doing like between two points. Like here's at a certain spot how fast it's changing.
13	MZ:	OK. Then the rate of change part is the how fast it's changing part.
14	Frances:	Yeah.

As in previous interviews, Frances mentions slope at a point as her first answer to what a derivative is. However, when asked if anything else comes to mind, Frances says, "No" [ln 4]. To the same question in the second interview she replies "rate of change".

Frances's response to the fourth interview question about the meaning of rate of change is, "how fast it's changing at a certain spot" [ln 10]. Note that Frances' description uses the metaphor of speed and describes the gestalt layer of the object and not the details of its structure.

The next part of the fourth interview is a summary of Frances' solution to the first of the three problems in this interview. Given a table of values with x varying by .1, Frances is asked to estimate $f'(2)$, the derivative of the function at $x = 2$. Frances first suggests that she can find an estimate by finding the slope between two points. She does so and calculates that $f'(2)$ is approximately 20.4. When asked how she would find a different or better estimation, she only says that if she had the equation for the function, then she could find the derivative. Frances uses the slope model of derivative to calculate a ratio for her estimate. She does not discuss a limiting process for finding a more accurate estimate.

The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward, increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car takes 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far does the car travel in the 2 hour period?

Frances' first reaction is that the car has traveled 60 miles. She reasons that the car averages 30 miles per hour for two hours. She also recognizes that this estimate is accurate only if the car is increasing and decreasing at a constant rate. When asked to use calculus to solve the problem, she remarks that position is the antiderivative of velocity and that the acceleration in this problem is constant. She writes the following sequence of equations: $A(t) = c$, $v = ct$, and $s(t) = \frac{ct^2}{2}$. The sequence is correct for this problem since the initial position and initial velocity are both 0. However, she does not acknowledge the need to discuss initial values.

When asked for the value of the constant acceleration, Frances guesses 30 since the average velocity is 30 miles an hour. She is asked about the units on the acceleration and says miles per hour squared, but she does not explain further the connection between average velocity and acceleration. Perhaps she connects average velocity with average rate of change of velocity which would be an approximation for acceleration.

Next Frances is prompted to draw a graph of the velocity curve. She initially thinks it would be a concave down parabola until she is reminded of her velocity equation which she describes as linear. With a bit more prompting she draws a correct graph of the acceleration, both positive and negative constant accelerations, and draws a correct "inverted V" graph for the velocity. She explains that the velocity graph has constant positive and then constant negative slope to match the acceleration values. Without prompting she realizes that the area under this curve will be the distance traveled. She is not asked to sketch a graph of the position function.

Although it is with some prompting, Frances successfully coordinates a physical situation involving position, velocity, and acceleration with symbolic and graphical representations. Her first reaction is to stay within the physical setting and find the average speed. When asked to use calculus, she chooses a symbolic representation, but she is also able to use a graphical representation including information on slope and area.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Frances is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. She is asked if the top of the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

As the interviewer begins describing the problem, Frances indicates that she is familiar with the problem type, "I guess the ladder's slipping don't you think?" When the interviewer has finished the description, she begins solving the problem by labeling the wall as y and the floor as x and writing $\frac{dy}{dt} = c$. Without comment she completes the following sequence of calculations:

$$14^2 = x^2 + y^2$$
$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$-y\frac{dy}{dt} = x\frac{dx}{dt}$$
$$\frac{-yc}{x} = \frac{dx}{dt}$$

Upon completion she asks, "So, what am I trying to tell you?" Note that her calculations are all accurate and relevant, but they seem more directed at solving some prototypical ladder problem, rather than answering the questions asked by the interviewer.

When reminded of the questions, Frances says that she thinks the ladder is sliding down at a constant rate, but that the rate is not the same as the rate that the ladder is being pulled away from the wall. Frances' explanation seems to recall the set-up of similar problems found in her textbook: "When the y's at a certain point, then you could figure out where the x is at that point and then you could figure out the rate at that instant that the height of the ladder is changing." She does not recognize that the interviewer's question is not about the rate at a particular instant but about how the rate changes over time. When asked specifically if the rate is the same for two different y values, Frances calculates the rate for two arbitrary y values and determines that the rates are not identical and that the magnitude of the rate seems to be increasing as the ladder slides down.

Interview 5

Frances' fifth interview occurs approximately one week after she takes the BC version of the AP exam. During that week the class discusses the written questions from the BC version. Between the fourth and fifth interviews the class studies series and integration techniques and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Frances' response to a set of questions on a particular topic and provide an analysis of those responses.

1 MZ: What is a derivative?

2 Frances: The instantaneous slope at a point. There's the formal definition,--

3 Frances: [writes: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$]

4 MZ: OK. And what else?

5 Frances: It's how fast the function is changing.

6 MZ: Got another one?

7 Frances: Uhm. If the derivative is positive the function's increasing. If it's negative, it's decreasing.

8 MZ: OK.

9 Frances: The second derivative tells you about inflections.

10 MZ: OK. This is what I'm kind of more interested in, how this statement, instantaneous slope at a point, relates to this formal definition.

11 Frances: Because if you say, as h gets really small, then it's kind of like you're taking the slope of the secant line, but the slope of the secant line-- The end points of the secant line are getting closer and closer together. So it's almost like you're taking the slope at that exact point.

12 MZ: OK. So how do the different little symbols here fit into that thing you just described?

13 Frances: This is like $f(b)$-- If this was x , then that would be $x+h$, if that was a really small distance [sketches a pair of axes and marks x and $x+h$ on the horizontal axis].

14 MZ: OK.

15 Frances: So then this would be $f(b) - f(a)$ over $b-a$. But $x+h-x$, you just get h [writes: $x+h-x$ and then crosses out the x 's].

16 MZ: OK. And then that describes the slope right?

17 Frances: Yeah.

18 MZ: Then the limit does what you were saying about--

19 Frances: Because you're making it a tiny distance between x and $x+h$.

20 MZ: OK. What does this formal definition have to do with this statement that you just gave me, how fast the function is changing?

21 Frances: [short pause] I don't know. 'Cause if the derivative is 0, then-- If you have a line like-- If you have a function like that [draws a flat curve], the slope of that is small because it's not changing very fast. But if you have a function like that [draws a steep curve], then the derivative is big because it's changing a lot.

22 MZ: OK. Does the derivative involve a limiting process?

23 Frances: Like that?

24 MZ: Yeah, this would be the limit, and I was thinking more of the process, but that's kind of what you were explaining about the h's.

25 Frances: Mm hmm.

26 MZ: You know that velocity is related to derivative?

27 Frances: Mm hmm.

28 MZ: Could you describe the limiting process in terms of velocity or in terms of the position function who's derivative is velocity?

29 Frances: [short pause] Uhm, I don't know what you're asking for.

30 MZ: [short pause] Forget that question. Is the derivative of a function a function?

31 Frances: [long pause] I guess so. I don't know why though.

32 MZ: What makes you lean toward guessing that way as opposed to guessing not?

33 Frances: Wait. I'm not sure. I don't know. I'm thinking. [pause] A derivative of a polynomial is definitely a function, but I'm thinking about the other ones.

34 MZ: OK. [short pause] So you think there might be a case where it doesn't turn out to be a function? Is that what you're saying?

35 Frances: Yeah. If you have-- [short pause] I don't know.

36 MZ: Well, what were you thinking?

37 Frances: I was thinking like if you had $x^{\frac{1}{2}}$. So if you took the derivative, you would have $\frac{1}{2}x^{-\frac{1}{2}}$. But I couldn't figure out what I was thinking after that. [writes:

$$x^{\frac{1}{2}}$$
$$\frac{1}{2}x^{-\frac{1}{2}}$$
]

38 MZ: Trying to think if maybe there'd be an example like that that wouldn't quite work or something?

39 Frances: Yeah.

40 MZ: What is meant by a differentiable function?

41 Frances: To be differentiable, it has to be continuous. Like over-- If it's differentiable over an interval, it has to be continuous on that interval.

42 MZ: OK.

43 Frances: And it can't have a cusp or something because-- Like if you had something like-- [sketches a cusp pointing upward]. It's not differentiable there because the slope is-- Like this limit from the left and the right isn't the same at this point.

44 MZ: OK. So this would be an example one that's not differentiable.

45 Frances: Right.

46 MZ: What would be an example of one that is differentiable?

47 Frances: [Sketches a smooth curve on a pair of axes.]

48 MZ: Right. Do you happen to know the equation of a function that's not differentiable?

49 Frances: [long pause] I mean, I could just do something that's not continuous. It wouldn't be differentiable at the point where it's not continuous.

50 MZ: True.

51 MZ: OK. What are derivatives useful for?

52 Frances: Like finding velocity or acceleration if you have position. Finding rates of change. You know, when you have a leaky bucket or something like that.

Table A.18. Frances: Interview 5 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎			◉	
What else?		○			
How does the formal definition of derivative relate to slope or rate of change?	◉ ○	○		◉	
Does the derivative involve a limiting process?				○	
Is the derivative of a function a function?				○ ↦	
What is meant by a differentiable function?	○ ●				
What are derivatives useful for?		○	○ ○		acceleration
Asked to interpret the Mean Value Theorem.	●		◉	● ●	
Asked to find the average rate of change of a function defined as an integral.	●	●		●	
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.		◎		↦	misstatement (d=change) in/decreasing concavity
Summary	◉	◉	◉	◉	

Even though the portion of the fifth interview focusing on general questions about the derivative has fewer questions than similar sections of the first and second interviews, Frances gives an almost equally complete a description of the derivative concept here as she does in her very detailed account from the second interview. The only difference is that she places a bit more emphasis on the formal limit definition in the fifth interview versus more emphasis on the derivative as the instantaneous rate of change in the second interview. As in previous interviews, Frances mentions a graphical interpretation of derivative first. In the second interview she mentions instantaneous rate of change as a second interpretation. In the fourth interview she does not give an additional interpretation without a specific query. In the fifth interview she mentions the limit definition second and "how fast the function is changing" [ln 5] as her third interpretation.

In the fifth interview, Frances discusses the details of the ratio and the limiting process in the graphical and symbolic interpretations and makes the connections between the parallel process in the two interpretations. In the second interview, Frances not only compares those two interpretations but includes the details of the limiting process and the rate of change ratio in her comparisons as well. It is likely that Frances would have been able to make the connections to rate of change in the fifth interview if she had been asked to do so.

For the second part of the fifth interview Frances is asked about the Mean Value Theorem. She is given a formal statement of the theorem, including the equation $f'(c) = \frac{f(b) - f(a)}{b - a}$, and is asked to tell what it means. Without being given a graph or drawing her own graph, Frances describes the right side of the equation as the slope of the secant line between two endpoints and that there will be a point between a and b where the slope is equal to the slope of the secant line. When asked specifically to explain the equation for a function that describes the distance a person has walked and the derivative as the velocity, Frances says that the right side is the average speed for the trip and that at "some place in the trip they had to be going the average velocity".

Frances' first interpretation is in terms of slope but she is aware of the ratio layer in terms of velocity also and connects both to the symbolic representation. She is clear on the instantaneous nature of the derivative in these contexts. Earlier in the interview she describes the slope-symbolic connections, but this is her first chance in the interview to show her knowledge of the relationship between the velocity and symbolic representations of derivative.

The next question in the fifth interview involves a problem from the AB version of the AP exam, which Frances has not taken. She reports that she has not seen the question before. The question is as follows.

Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.

(a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.

(b) On what intervals is F increasing?

(c) If the average rate of change of F on the closed interval $[1,3]$ is k , find $\int_1^3 \sin(t^2)\,dt$ in terms of k .

The interviewer asks Frances to discuss her methods of solution for parts (a) and (b), but does not require her to complete the solution of either part. For part (a) Frances correctly describes how to apply the trapezoid rule to find area under the curve.

For part (b) Frances states that $\sin(t^2)$ is the derivative of F. She says that to solve the problem she would set $\sin(t^2)$ equal to zero and see where it is positive or negative. Frances is not asked to elaborate since she seems confident and has shown in the past that she knows that the function is positive when the derivative is increasing.

For part (c) Frances notes that the average rate of change is the slope of the secant line. She writes, $\frac{F(3) - F(1)}{3 - 1} = k$, but then pauses, unsure of how to proceed. Her next comment is that the definite integral, $\int_1^3 \sin(t^2)\,dt$, should be equal to $F(3) - F(1)$ and that this F is the same as in her ratio. Therefore the definite integral equals $2k$.

Note that in part (b), Frances suggests an algorithm that she is familiar with. In part (c), for a more unusual problem, Frances must reason from her knowledge of the meaning of the phrase "average rate of change". It is interesting that she translates to the graphical interpretation, "slope of the secant line", before giving the symbolic formulation she needs to solve the problem. This gives further evidence to the centrality of the graphic interpretation of derivative in Frances' thinking.

The next section of the interview concerns Taylor polynomials. Frances is asked what a function and its second degree Taylor polynomial have in common and how they differ. Frances responds that at one point they have the same function value, first derivative and second derivative. When asked how the graphs would compare, Frances says that they would have a common point where the slopes would be the same. When trying to say graphically what having the same second derivatives means, she says that if the function has an inflection point there, then the polynomial does also. This last statement is her only misstatement in that second degree polynomials do not have inflection points.

The final section of the fifth interview concerns a function, f, that at any time, x given in hours, tells the outside temperature in degrees Fahrenheit. Frances is shown a series of symbolic expressions and asked what information each one provides about the outside temperature.

For $f'(3) = 4$ Frances replies that the temperature is increasing since the derivative is positive. She knows that the information is given "at time 3" and that the 4 is "change in degrees over change in time". For $f''(3) = -2$ Frances explains that the graph of temperature versus time would be concave down. When asked what this has to do with temperature, she says that the temperature is increasing (because of the first derivative information) but not as fast as it was. She continues, "The rate at which it's increasing is decreasing."

When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Frances replies that "from 0 to 3 it was linear." When asked specifically about the temperature she says, "It was increasing at a constant rate." When asked to interpret $f''(x) = -2$ for $3 \le x \le 6$, Frances says, "The instantaneous change in the temperature was increasing, but the amount that it was going up was going down at a constant rate." Frances says the "change" in temperature and the "amount" instead of speaking in terms of a ratio or rate of change.

When asked for the temperature at time 3, Frances uses the antiderivative of -2 and $f'(3) = 4$ to find $f'(x) = -2x + 10$. From this she antidifferentiates to note that $f(x) = \frac{-2x^2}{2} + 10x + C$, but she does not know how to continue without an initial condition on f. Her next idea is to calculate $\int_0^3 4dx = 12$ which she then realizes is "the total number of degrees it went up" in the first 3 hours. When asked about the second three hours, Frances notes that $f'(x) = -2x + 10 = 0$ when $x = 5$. She recognizes that the temperature is increasing before 5 and decreasing after 5, but that the temperature would not have decreased much by 6 because "you hadn't been decreasing as long as you were increasing at that rate."

Overall Frances' initial reactions to the symbolic expressions are graphical. She says that f is concave down for $f''(3) = -2$ or linear for $f'(x) = 4$ for $0 \le x \le 3$. However when asked she is able to give a correct interpretations of the derivatives in terms of the physical situation. When asked for specific details of the situation she begins to use symbolic calculations that she is eventually able to use correctly to interpret how the temperature changes. Her one misstatement of using change instead of rate of change does not seem to affect her otherwise correct analysis.

Case Study 7 — Grace

Academic record

*National Merit Scholar.

*Other AP courses: US. History (junior year), English, Spanish, Chemistry.

*Award from National Council of Teachers of English.

*Writing tutor at the high school writing center.

*Plans to major in English in college.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: "Profound thoughts by ... A function is an equation with x or $f(x)$ in it."

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different?

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: "a) $9x+3=f(x)$

$f(x)=8x^3-2$

One is a linear function and the other is a polynomial function."

Grace includes a sketch of a graph of the function $y^2=(x-3)$.

"b) It's not a function because there can only be 1 $f(x)$ value for every x ."

QOTD #3

Give an example of a function without using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Date: August 26, 1993. This question occurs while the class is doing a quick review of functions.

Response: "An example of a function w/o an equation or whatever is beyond me."

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) non integers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: "a) yes, yes a polynomial. b) yes."

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to class discussion on limits.

Response: "A limit of a function f at $x = a$ is only applicable when a is defined. It is the point $f(a)$ (approached from both sides) that is the same.

Ex: $\underset{f \to a^-}{f(x)} = \underset{f \to a^+}{f(x)}$ If this is true, there is a limit."

Test 1

On a test on limits a week later, Grace is able to correctly find limits by reading values from a graph, by substituting into a piecewise function, and by using algebra to simplify a limit calculation. She is able to work with the formal definition of limit to the extent of finding a δ for a given ε in both a symbolic and graphical setting. Only one error seems to contradict her understanding of a dynamic conception of limit. On one of the test questions the students are asked to find $f(x)$ values for a sequence for x values that approach 0 from both negative and positive sides. Grace does this correctly. The next question asks Grace to find the $\lim_{x \to 0} f(x)$. Grace answers incorrectly. Her answer is equal to $f(1)$ which is the first value she had calculated in the previous problem.

Interview 1

This interview occurs after the test on limits but prior to the class' discussing derivatives. Therefore Grace's answers are presumed to be based on what she remembers from her junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table A.19, which codes these responses. A summary discussion follows.

1 MZ: What do you remember, what's a derivative?
2 Grace: It's the tangent to the slope at a given point.
3 MZ: What does that mean really, the tangent to the slope?
4 Grace: That means, I don't know, at a certain point -- at a given point, that's where your line is going to be.
5 MZ: OK, so if I had a graph of something [sketches arbitrary graph] and a certain point then I would draw a -- What would I draw?
6 Grace: I don't know. Wouldn't you just -- [motions to paper]
7 MZ: Draw something like that? [sketches in a tangent line]
8 Grace: I don't know, it's instantaneous rate of change. At which it's changing as it's-- I can't explain it. [laughs]
9 MZ: That's no problem. Instantaneous rate of change -- So is instantaneous rate of change the same thing as saying this up here [sketch of tangent line to curve]?
10 Grace: Uhm, [pause] I guess not. Well, I'm not sure because instantaneous rate of change is just exactly at that moment. I guess.
11 MZ: OK, so what would it be saying about this moment?
12 Grace: At that moment?
13 MZ: Yeah.

14 Grace: That your derivative is going to be -- exactly opposite to your slope, I guess.
15 MZ: What do you mean exactly opposite to your slope?
16 Grace: It's going to be the tangent of your slope.
17 MZ: Yeah. OK, my other questions was what are derivatives useful for?
18 Grace: Well, we did a lot of those application problem things last semester so I suppose you could use them for those. Application problems like, well I don't know, like a tank is filling up at a certain rate and the rate of change is so and so and all that stuff. As for what they're really useful for I'm not sure. We just do them because they tell us to.
19 MZ: What other kinds of applications did you guys do?
20 Grace: [long pause] I don't remember. [laughs] I don't remember anything from that class.
21 MZ: Say you had a function. How would you know if it was differentiable or not?
22 Grace: Well, it's differentiable everywhere if it doesn't have any like weird things. Like if it doesn't have a cusp or if it doesn't have -- I don't remember. Something else.
23 MZ: Well, what would be an example of a function that's not differentiable?
24 Grace: Something like that. [Sketches a graph of a function with a cusp]
25 MZ: Do you know what kind of formula would cause a function to have a weird kind of sharp turn like that?
26 Grace: Mnn, no.
27 MZ: How is the derivative related to slope?
28 Grace: It's the tangent of slope.
29 MZ: How about speed or velocity?
30 Grace: Isn't the derivative of velocity acceleration?
31 MZ: True. How about change or rate of change?
32 Grace: It measures the rate of change at the given instant.
33 MZ: It, you mean the derivative?
34 Grace: Yeah.
35 MZ: Limit? What does derivative have to do with limit, if anything?
36 Grace: Good question. And I don't really know the answer so I'll say they have nothing to do with each other.
37 MZ: So does the derivative have anything to do with optimization? Or maybe you could tell me what you think I mean by optimization?
38 Grace: Doesn't that mean like estimating your maximum or minimum stuff?
39 MZ: Yeah.
40 Grace: It [derivative] helps you find it.
41 MZ: Do you happen to remember how that works that it's able to help you find it?
42 Grace: No. [laughs]
43 MZ: How about prediction?
44 Grace: I suppose if you have a function and you can find it's derivative then you can kind of see where it's going. Maybe if you find like it over a certain area.
45 MZ: So if you find the derivative or the function over a certain area?
46 Grace: Both I guess. You can kind of predict where it's going to go next. If it's like--
47 MZ: OK. Did you guys learn an official definition of derivative?
48 Grace: Probably. You mean like mathematical?
49 MZ: Yeah, mathematical. Not like epsilon delta that stuff, but like a sort of mathematical definition.

50 Grace: Like the $f(x+h)$ thing?

51 MZ: Yeah.

52 Grace: How about $\frac{f(x+h)-f(x)}{h}$?

53 MZ: What does this definition have to do with some of your earlier definitions like tangent to the slope or rate of change?

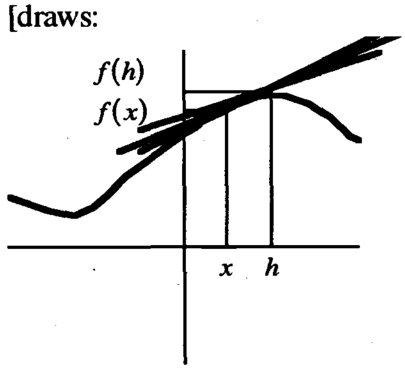
54 Grace: Do you want me to like prove it?

55 MZ: Uhm, if you can sure. Or just describe how they are related or why they are not related or --

56 Grace: I think it works like this. If you have a graph and you pick two points, and you want to find the -- what do you call it -- the change between the two of them. Right? I guess this is more accurate.

57 MZ: OK.

58 Grace: The closer these two points move together the more accurate it is. So what they're doing here I bet is they're moving their points together. So I guess this is suppose to be x and it probably isn't but I'm going to say that is h. [pause] I don't really know what I'm doing here so-- [pause] OK.
[draws:



]

59 MZ: Just take a stab at it.

60 Grace: All I'm going to say is that you have two points and the closer they move together the more accurate your slope is going to measure and the rate of change.

61 MZ: OK. Let me go back. I understood what you said about this picture. How does this picture relate to this quotient? [the difference quotient she gave]

62 Grace: Good question. Hmm.

63 MZ: Like I mean is this the same h as is here and this is the same $f(x)$ as there?

64 Grace: Maybe. [laughs]

65 MZ: It's possible.

66 Grace: Yeah. If I want it to be. Maybe.

67 MZ: Would you know where this $f(x+h)$ would be on this picture?

68 Grace: [pause] No. Maybe in the middle.

69 MZ: In the middle of here-- I mean up here I guess with your outputs? [points to location between $f(x)$ and $f(h)$]

70 MZ: OK. If you had to explain to somebody what a derivative is that doesn't know about it yet, like maybe in the AB class. They are about to learn it, but they haven't learned it yet. What do you think would be the easiest to explain it to them?

71 Grace: I'd just explain it as, I guess, the slope. How it changes as it approaches a certain point and when you're like catching it like a certain instant. That's the derivative.

Table A.19 summarizes Grace's first interview transcript. Grace recalls that the derivative is related to slope and rate of change. When asked specifically, she also remembers that derivative is related to velocity and a symbolic difference quotient. She states the instantaneous nature of derivative as slope or rate of change, but not while discussing velocity or acceleration. Without using the word limit, she recalls a limiting process when trying to relate the symbolic difference quotient to slope. However, she fails to remember the limit when writing the symbolic definition of derivative. Further, when asked specifically whether derivative is related to limit she says they are not related.

When asked early in the interview, Grace is unable to clearly state a connection between slope and rate of change other than that they are both related to derivatives. Later she makes a weak attempt at relating the symbolic difference quotient to slope or rate of change. The connection she identifies is that the symbolic difference quotient has to do with two points getting closer together [ln 58]. She incorrectly states that "the change" between the two points is the quantity of interest, although she corrects this to say that the process creates more accurate slope or rate of change estimates [ln 60]. She cannot relate the structure of the difference quotient to a structured ratio for slope or rate of change.

Grace's other significant misstatements are that the derivative is "the tangent to the slope at a given point" [ln 2], "exactly opposite to your slope" [ln 14], "the tangent of your slope" [ln 16] or "the tangent of slope" [ln 28]. When first asked to clarify her statement she refers to a line at a point [ln 3-4]. Later she equates "exactly opposite to your slope" and "tangent of your slope" [ln 14-16]. I propose that in these phrases when she says "tangent" she means tangent line and when she says "slope" she means the

Table A.19. Grace: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎ ○	◎			misstatement (d=tl) misstatement (your derivative is opposite to your slope)
What are derivatives useful for?		○			
Is derivative related to slope?	○				misstatement (d=tl)
Is derivative related to speed or velocity?			○ ○		acceleration
Is derivative related to change or rate of change?		◎			
Do you know an official definition of derivative?				◉	
How is the definition related to slope or rate of change?	○ ●	○			misstatements
Explain what a derivative is to someone who's an AB student or precalculus student who hasn't studied it yet.	◉				
Summary	◉	◎	○	◉	

curve, the slope or shape of the curve, and when she says "opposite" she means next to or adjacent. Thus her statement can be rephrased as the derivative is the tangent line to the curve at a given point. Each of these -- using tangent as an abbreviation for tangent line, using slope for the sloping of the curve, and referring to the derivative concept by its most visible aspect, the tangent line -- is an example of individual metonymy. The reader may not agree with this interpretation based on the data presented thus far. Grace's statements in her second interview, discussed below, give this interpretation further weight.

QOTD #6

Find the derivatives of the following four functions:

$f(x) = (x-1)^2(x^2-4)$

$g(x) = \dfrac{x-1}{\sqrt{5-x^3}}$

$h(x) = \sin x$

$j(x) = \ln x$

Date: September 20, 1993. This question occurs prior to the class' learni

cut rules for taking derivatives of various forms.

Response:

$f'(x) = (2)(x-1)(1)(x^2-4) + (2x)(x-1)^2$

$g'(x) = (1)(5-x^3)^{-\frac{1}{2}} + \frac{-1}{2}(5-x^3)^{-\frac{3}{2}}(-3x^2)(x-1)$

$h'(x) = \cos x$

$j'(x) =$ logx^3

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one

why it could not be using your knowledge of derivative.

$y = \log(x^3)$ $\quad y = \dfrac{x}{|x|}$ $\quad y = x^e$ $\quad y = e$

Date: September 21, 1993. This question also occurs before the class st

rules for taking derivatives but after they have studied the limit definitio

Response: Grace includes a sketch of the graph of the function $y = e^x$. S

"$y = \ln x \quad y = \log_e x \quad e^y = x$ I dunno."

Comment: This question is presented to the students since no student co

derivative of $y = \ln x$ in the previous Question of the Day.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$?

Why not?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: "a) NO. Because $\sin x$ is not the cofunction of $\cos x$. (If that makes any sense). (which it doesn't)

b) deriv of $\tan x$ $y' = \cot^2 x$ or not"

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. Grace shows that she can correctly state the limit of the difference quotient definition of derivative and use it both symbolically and numerically to calculate a derivative value. She also knows how to estimate the derivative at a particular point by finding two nearby points and calculating a difference quotient for those two points. Given the graph of a position function for a car she correctly answers questions about the speed of the car. However, given the graph of another function, she is unwilling even to attempt drawing the graph of its derivative. She also has problems computing some complicated chain and quotient rule derivatives. Grace does write up corrections later for the problems she missed.

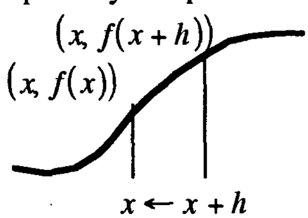
Grace's inability to draw a graph of the derivative given the graph of the function is curious considering her other correct answers. She correctly calculates the greatest speed from a piecewise linear position graph. Although it is never specifically stated, she likely knows that the derivative at a point is the slope of the original function graph at that point. Potential difficulties arise in estimating slope values for a nonlinear curve and in coordinating individual derivative values into a derivative function by plotting appropriate input-output pairs. She probably does not yet have a clear recognition of the distinction and relationship between the word derivative as referring to a value and the word derivative referring to a function.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: "Truth: I understand what a derivative is a little better than last year. Especially this picture:



helped me to understand how the definition is produced $\frac{f(x+h)-f(x)}{h}$."

Comment: It is interesting to note that Grace omits the limit from the symbolic representation, but represents a limiting processes with the arrow in the diagram. She includes the limit in the symbolic definition on the exam. Also note that there are no tangent or secant lines on her sketch.

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "We studied how derivatives predict function behavior.

$f(x) = 4x^3 - 3x$

$f'(x) = 12x^2 - 3$

Ex: $12x^2 = 3$ 1st derivative to find critical numbers.

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

b) ∅ "

Comment: Since the researcher had not been present the day prior, this question is presented both as a means for the researcher to see the material covered and to ascertain the students' understanding of it.

Test 3

About two weeks later the class is tested on Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Grace correctly calculates the third degree Taylor polynomial for $\tan(x)$ at $x = 0$. However, when asked to find the error in using this polynomial to approximate $\tan(\pi/4)$, she lists two points with x coordinates .7 and .8 and y coordinates found by substituting .7 and .8 into the Taylor polynomial, and then calculates the difference quotient for those two points. Grace seems to know that this calculation is an approximation for the derivative and she connects that to being asked for the error in an approximation.

Grace correctly uses the first and second derivatives of the position function to find the speed and acceleration of an object at a given time. She has more difficulty using the graph of a derivative function to estimate information about the original function. She correctly notes that the zeros of the derivative are the critical numbers of the original function and that the maximum and the minimum of the derivative are the locations of the inflection points on the original function. However, when asked where the original function is increasing and decreasing, Grace lists the intervals in which the derivative is increasing and decreasing. When asked where the original function is concave up, she lists values for which the derivative function is positive. Similarly, for concave down she lists values for which the derivative function is negative.

Just as in the previous test, Grace shows difficulty with what Monk (1990) labels as across-time questions. Questions about a single value of a function or derivative are handled correctly, but questions that ask her to make statements about an interval of values are missed. Grace does not coordinate her knowledge of the meaning of derivative at a point into the covariation of input and output values involved in a function.

QOTD #11

Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: Grace does not respond to this question.

Test 4

Two weeks later the class has a test on the applications of derivatives. Grace correctly uses derivatives to solve three traditional max/min problems and three traditional related rate problems. She also correctly calculates the derivative of an implicitly defined function.

Interview 2

The second interview occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by a summary table (Table A.20), and a discussion.

1 MZ: What is a derivative?
2 Grace: It's a slope. Tangent of the slope at a point, at a certain time.
3 MZ: What does that mean?

4 Grace: That means the line that's tangent of the slope [laughs].

5 MZ: It means the line that's tangent, OK. And the derivative is what? How does the derivative relate to this line that's tangent?

6 Grace: I know that if you have a curve-- We'll make it more curvy [draws a smooth curve].

7 Grace: You have your point here [marks a point on the curve]. And if you have your tangent line [draws a short tangent line at that point].

8 MZ: Yes.

9 Grace: What it gives you I guess is like-- Well, what you're seeing, I guess, is every time you [draws a box around the tangent line point and then a smaller box inside that one], that's a zoom box-- every time you move in--

10 MZ: OK

11 Grace: --the tangent line is going to become that thing.

12 MZ: The curve?

13 Grace: I mean it's not going to become, but it's going to get really really close. The derivative is what tells you that I guess. I'm not good at explaining things.

14 MZ: OK. What are derivatives useful for?

15 Grace: Doing all that stuff that we're doing now like optimization.

16 MZ: Mm hmm. [writes: optimization]

17 Grace: Optimization like using them in the real business world. Like for those related rates problems, stuff like that.

18 MZ: What's the set up for an optimization problem?

19 Grace: Like if you're redecorating your house or something, and you want to use the least amount of-- You want to have the least amount of paint, but you want to get the maximum thing done, you know?

20 MZ: How would you explain what a derivative is to someone without very much math background?

21 Grace: How to explain a derivative. I guess, the rate of change. It's just like slope. Because if you have two points and you can find the slope between them, but that's like an average slope. It's like average velocity when we studied that in physics last year.

22 MZ: OK.

23 Grace: There's a difference between average velocity and instantaneous velocity because when you have an average velocity that just means it could be really high and really low at one point, but if it's like kind of in the middle for the rest of the time, then it's just like middle.

24 Grace: But if you have instantaneous velocity, that tells you what it is at exactly an instant and it doesn't have time or room to move or anything. It doesn't have time to change.

25 MZ: We were talking about rate of change and slope and then you also mentioned velocity. Are those three things the same or almost the same? How are they related to each other?

26 Grace: Couldn't tell you. I mean, I could.

27 MZ: Well, are they related to each other?

28 Grace: Well yeah, they're related. They are kind of the same. Almost the same, but not the same.

29 MZ: OK, and the difference is? [both laugh]

30 Grace: It doesn't go any more specific than that, sorry.

31 MZ: OK. How about a real world situation involving the concept of derivative?

32 Grace: Involving the concept of derivatives. Like related rates? Can I use those?

33 MZ: Sure.

34 Grace: OK. How about we pick-- Let's say you have a hot air balloon, and you're filling a hot air balloon and it's like going in, but there's a hole in it so it's leaking out like at a lower rate of growth. And you want to know how long it's going to take you to fill up the balloon or something.

35 MZ: So what aspect of that problem is the derivative? How does the derivative fit into that problem?

36 Grace: Do you want me to tell you how to do it?

37 MZ: Uh, not necessarily, but, I mean-- You just described a nice thing, and I understood what you described, but I didn't see which part of that description I should tag the label derivative to.

38 Grace: Well, the derivative comes in when you do like maximum or minimum stuff.

39 MZ: OK.

40 Grace: Because you know like in derivatives that's going to be at zero. So when you have your thing, your equation I guess, you find the derivative of that and then you set it equal to zero because you know that's going to give you your maximum or minimum.

41 MZ: So how come where the derivative is equal to zero is the maximum or minimum?

42 Grace: I would know if I had more time to think about it, but I don't know.

43 MZ: How can you tell if a function is differentiable?

44 Grace: Like if it doesn't have any really weird things, like a cusp or stuff like that.

45 MZ: What is it about a cusp that makes it not have a derivative?

46 Grace: Because you're never going to be able to-- Like what I was talking about here [first sketch with tangent lines and zoom boxes]

47 MZ: Yeah.

48 Grace: Like eventually when you zoom in, that curve right there is going to get closer and closer to the tangent line. And when you're at a corner here [sketches a cusp], no matter what you do there's no way that that is ever going to get closer to each other. It's just a point.

49 MZ: So that corner that you drew doesn't become a line.

50 Grace: Yeah and when you have something like this. You know how your slope gets more and more positive and then less and less positive until it's just-- Is that zero or is that undefined? No that's zero. Until it becomes zero. It gets more and more negative and less and less negative. Well here it just gets more and more positive all the way up to here and then it doesn't do anything there. It doesn't like become something like this. It all of a sudden just goes from more and more positive to like more and more negative like that. [during this has sketched various lines touching at the cusp point]

51 MZ: How did you say velocity was related to derivative?

52 Grace: It's the derivative of the position equation.

53 MZ: OK. You said rate of change was related to derivative.

54 Grace: Yeah because your derivative is your rate of change only it's like an instantaneous rate of change. It's not like an average rate of change.

55 MZ: Line or linear. Is that related to derivative?

56 Grace: Well, I suppose so. I mean your derivative is like a line. [inaudible, laughing] It's a line. I guess that's kind of the context, isn't it, of derivative. When you have a point and you're finding tangent lines getting closer to that point, approaching that point.

57 MZ: Does measurement have to do with the derivatives?

58 Grace: Well yeah because derivative gives you a really accurate measurement. Like I was saying before. When you have just like an average measurement-- OK, like if you had an 80% average for a class.

59 MZ: OK.

60 Grace: Like you could have had a 70, a 90 and an 80, but you'd still come out with an 80. But if you have your derivative, they'd be telling you-- Like if you have an 80, it can't be anything higher or lower.

61 Grace: Yeah because it tells you what's going on at one time.

62 MZ: Is derivative related to either prediction or approximation?

63 Grace: Yeah. Just leave it at that. [laughs]

64 MZ: Can you tell me why?

65 Grace: I think I know. Because you know how like when two points move closer together?

66 Grace: And you have that equation, $f(x+h)-f(x)-$

67 Grace: [writes: $\frac{f(x+h)-f(x)}{h}$] I think that's it.

68 Grace: So when you have like this equation, and like when these two points are moving closer together. [sketches a curve (no axes), marks two points of the curve, draws vertical line from each point down, draws an arrow pointing from right line to left line]

69 Grace: Like you get closer and closer. I mean, you're approximating but you get closer and closer to an instantaneous value, I guess. I don't know if you can ever get exactly to an instantaneous value because there are too many points for that.

70 MZ: So, this equation relates to this picture, right?

71 Grace: Uh huh. [yes]

72 MZ: Could you tell me how the parts of this equation relate to this picture?

73 Grace: Yeah because this is-- That point is $(x, f(x))$. [labels the left point on the curve]

74 Grace: And this point is $x+h$. [labels abscissa of right point] Let's say that's h. [writes h between the two vertical lines]

75 Grace: And that's $f(x+h)$. [labels ordinate of the right point] And you have the slope because you have $f(x+h)-f(x)$ over $x+h-x$ which is just h. And that's the slope. And as h grows smaller, as it moves toward this first point, this line in between gets more accurate.

76 MZ: OK. So this equation right here is the slope of what?

77 Grace: Is the slope of these two point?

78 MZ: The slope of those two points.

79 Grace: Right, except it's changing. You know, I mean, it could be like-- just at this graph. The way it looks. That could just be it. As you're moving h-- Because h is also change in x [writes to the side: Δx] which means it's changing. It just keeps changing.

80 MZ: Continuity?

81 Grace: If it's not continuous, it's not differentiable. Is that right? No. Mr. Forrest would shoot me. [laughs, pause] If it's continuous, it doesn't have to be -- No wait. If it's differentiable, it has to be continuous.

82 MZ: Differentiable, it has to be continuous?

83 Grace: Yeah, I think so. If it's continuous it doesn't have to be differentiable. Yes, that's right. [points to the previously drawn cusp] Because you think about it this way, and that's continuous, but it doesn't have to be differentiable. But if it's differentiable it has to be continuous.

84 MZ: Does limit have to do with derivative?
85 Grace: Yes, I forgot. [writes in front of difference quotient: $\lim_{h \to 0}$] That's in here.
86 MZ: OK, limit as h goes to zero.
87 Grace: Yeah and that's because the thing of h-- As h gets smaller and smaller and smaller, it gets closer to 0.
88 MZ: Is a derivative a function?
89 Grace: Always?
90 MZ: Yeah, always.
91 Grace: [pause] Mm, probably. No. No, it's not.
92 MZ: [both laugh] Well--
93 Grace: I don't think I know. I'll say yes.
94 MZ: Based on anything?
95 Grace: Not really.
96 MZ: Well, what were you thinking when you were thinking, "well no. I don't think it is," for just a second. What caused you to think that?
97 Grace: You don't want to know.
98 MZ: I don't want to know. It's not very mathematical?
99 Grace: [both laugh] I'm not going to tell you.

From Table A.20, it is evident that Grace mentions a graphical interpretation of derivative first and more often than any other interpretation. However, she does mention rate of change, velocity, and a symbolic formulation of derivative, each without being specifically asked to do so. Grace recognizes the instantaneous nature of the derivative in each of these forms. She repeatedly describes a limiting process in the graphical interpretation and eventually with respect to the symbolic interpretation, but not in the other two representations. The role of derivative as a function is only mentioned in the symbolic formulation and even there not with great certainty.

Grace mentions rate of change, slope and velocity in rapid succession [ln 21], but cannot specify the relationship between the three [ln 25-30]. Without prompting she attempts to relate the difference quotient to the limiting process of two points coming closer together. It is interesting that her first connecting statement relates a graphical limiting process, with no mention of slope, to a symbolic difference quotient that has no mention of limit [ln 65-69]. When prompted for details she correctly describes how the difference quotient is the slope between two points [ln 72-79]. A few minutes later when

Table A.20. Grace: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎				misstatement (d=tl)
How does derivative relate to this line that's tangent?	●				misstatement
What are derivatives useful for?		○			optimization related rate
Explain what a derivative is to someone without much math background.	○	○	○ ◎		
Real world situation involving the concept of derivative.		○		↦	related rates max/min
How can you tell if a function is differentiable?	● ○ ●			↦	
Is derivative related to slope?	○				
Is derivative related to speed or velocity?			○		
Is derivative related to change or rate of change?		◎ ○			
Is derivative related to line or linear?	●				misstatement (derivative is a line)
Is derivative related to prediction or approximation?	●			◉	misstatement
Could you tell me how this equation [difference quotient] relates to this picture?	◉			•	
Does limit have to do with derivative?				◉	
Is the derivative a function?				◯	uncertain
Summary	◉	◎	◎	◉	

asked specifically whether limit is related to derivative she adds limit notation in front of her difference quotient [ln 85].

As in the first interview, Grace makes misstatements which refer to the derivative as the tangent line. She says the derivative is "tangent of the slope" and when questioned clarifies by saying "the line that's tangent of the slope" [ln 2-4]. She also says that the derivative is a line when asked to relate derivative to line or linear [ln 56].

Grace mentions two types of limiting processes in association with the derivative as a line. She describes zooming in on a curve in association with the tangent line [ln 9-11]. Later in explaining differentiability she also describes zooming in on the curve so that it gets closer to the tangent line [ln 48]. The second limiting process is a more traditional description of secant lines becoming "more accurate" [ln 75]. In each case she makes the misstatement that the limiting process never actually reaches its limit, in the latter case "because there are too many points for that" [ln 69].

During the last part of the interview Grace is asked what she can tell about the original function when she is given the derivative. Her initial response is, "Probably everything except for how to draw it. I was never very good at those." She correctly explains that the maximum and minimum of the original function occur where the derivative is equal to zero. Additionally, she says that the slope of the original is positive where the derivative is above the x -axis and negative where the derivative is below the x -axis.

When given the graph of derivative function, she is further able to note which zero of the derivative is a maximum and which is a minimum by analyzing the change in the slope of the original function. She also knows that the inflection points of the original function would occur at the maxima or minima of the derivative, but she is not able to say why this is so. Despite all these correct responses, when asked to sketch a graph of the original function, she draws a graph with maximum and minimum occurring almost exactly at the same x-values as on the original function. When asked for clarification she becomes more confused and realizes that the maximum and minimum of the original should line up with the zeros of the derivative.

In comparing Grace's responses during the first and second interviews it is clear that she exhibits a more complete understanding during the second interview . She mentions a limiting process more frequently and remembers, albeit with some prodding, to write limit notation in front of her symbolic difference quotient. Her replies in both interviews are similar in that a graphical interpretation is the dominant representation. Her most persistent misstatement in both interviews is to say that the derivative is the tangent line itself although she does also recognize the derivative as the slope of that line.

The last part of the second interview is Grace's attempt to graph an original function when given the graph of the derivative function. Her ability to give verbal information about the original function is an improvement from her performance on Test 3. Her inability to graph the original function and her repeated statements that she is not good at graphing in this context is reminiscent of her work on Test 2. In that instance she correctly answers questions about the derivative function given the graph of the original, but refuses to attempt a graph of a derivative function given the graph of the original.

Grace's aversion to drawing graphs in these contexts is seemingly contradictory to her frequent use of a graphical description for derivative and her tendency to provide sketches of some sort when answering interview questions or questions of the day. Perhaps her inability to graph the original function given the derivative function, her misstatements that the derivative is the tangent line, and her confusion about whether the derivative is a function [ln 88-99], are all indications of the same phenomenon. Grace focuses on the derivative at a point, and her graphical representation of derivative as a tangent line occur at a point. She does not think about the derivative in terms of input-output pairs that join to form a function. There is no sense of covariation of these input-output pairs. Using Monk's (1990) language, Grace has across-time difficulties with questions.

During the early QOTD's on functions, Grace primarily discusses functions as formulas in terms of x . Perhaps her further understanding that for a function, "...there can only be 1 $f(x)$ value for every x ," has not been evoked in the similar context of a derivative function.

QOTD #12

What is the most important idea that we have studied so far in this class?

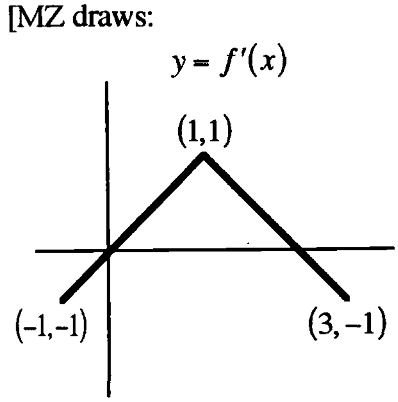
Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

Response: "I know nothing."

Interview 3

The third interview occurs during the three days after the test on differential equations, and antiderivatives by substitution and by parts. The first part of the interview is a summary of Grace's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike the second interview, the point $(1,0)$ on theoriginal function is given so that only one solution is possible.

[MZ draws:



]

1 MZ: I'm going to give you the graph of the derivative. And here it is. So this is the point $(1,1)$, $(-1,-1)$ [labeling these points on the graph] This goes through there. [connects points with straight line] This is $(3,-1)$. And this goes through there. [connects this$(3,-1)$ to $(1,1)$ with straight line] OK, so this is $y = f'(x)$, the derivative. [writes $y = f'(x)$] And don't worry about what happens outside of, you know, negative one and three.

2 Grace: OK.

3 MZ: And you're going to draw the graph of --

4 Grace: -- the original function. [simultaneously]

5 Grace: Hey, I can't do this. I really can't do this.

6 MZ: I bet you can. And I'm even going to give you another piece of information. f of 1 is 0. [writes: $f(1) = 0$] So you have one point on the original function.

7 Grace: f of 1 is 0.

8 MZ: Yeah. Here you go. [Slides her the notebook.]

9 Grace: [25 second pause; then writes as described below]

10 MZ: She just put dots at $(1,0)$, $(3,0)$ and $(-1,0)$. And how did you figure out -- I mean, I guess $(1,0)$ is that point you already know--

11 Grace: How did I figure out those? Are they right?

12 MZ: Uhm -- possibly. [both laugh] How did *you* know?

13 Grace: Because -- oh stupid, wait, that's not right.

14 MZ: Why not? What were you thinking?

15 Grace: Mm wait.

16 Grace: OK, I know where the derivative is equal to zero you get a max or a min. [writes: $f'(x)$ max/min where $f'(x) = 0$]

17 MZ: OK.

18 Grace: [short pause] I can't do this. I can't seem to relate the two of them. [short pause]

19 MZ: OK, so putting this at $(3,0)$ for example is somehow related to this idea that --

20 Grace: I know this idea works [pointing to what she has written about $f'(x) = 0$]. I'm pretty sure that idea works.

21 MZ: Right.

22 Grace: Except I can't seem to take this and -- [short pause] relate it to the picture. [pause, makes a hasty sketch of a curve on an axes with no hash marks, labels as $f(x)$]

23 MZ: So this sketch you just made is sort of just a general idea for f? Is that what you were thinking?

24 Grace: No, no. It doesn't --

25 MZ: It doesn't relate, OK.

26 Grace: Well, I don't know. [Marks out sketch.]

27 MZ: Don't mark it out. [both laugh] I want to see it later. OK, let's go back to this idea that you already started with. So when the derivative is equal to 0 you're suppose to have a max or a min of the original function? Is that -- this is a max or min of the original function, where f prime is 0, or is this a max or min of the derivative, where f prime is 0?

28 Grace: [laughs] I don't know.

29 MZ: Seems like it must be a max or a min of the original function.
30 Grace: Yeah that probably is true.
31 MZ: So there's the derivative.
32 Grace: This is -- I know it's decreasing from 1 to 3. Is it? Yeah it is.
33 MZ: Well, what is decreasing?
34 Grace: The original function -- because the slope is negative.
35 MZ: OK, so the slope of -- You mean the slope of the derivative is negative.
36 Grace: Right.
37 MZ: OK.
38 Grace: Doesn't that mean that in the original function that part of the graph is going to be negative?
39 MZ: Uhm -- [5 second pause] Hmm. [10 more second pause] Hmm -- I'm going to go back to this idea again [indicates what Grace wrote about $f'(x) = 0$]. The derivative is equal to 0. We have the graph of the derivative. Where on the graph of the derivative is it equal to 0?
40 Grace: [short pause] Right there. [points to $(0,0)$]
41 MZ: Right. Like at $(0,0)$ for example. So at x equals 0 it should be a max or min of the original function.
42 Grace: Right.
43 MZ: OK. Is there any way we can figure out which one it is?
44 Grace: Yeah, but I couldn't tell you how. [laughs] To figure out whether it's a max or a min?
45 MZ: Yeah.
46 Grace: Uhm -- Actually it's neither isn't it?
47 MZ: How come?
48 Grace: I would see where it was if it were here.
49 MZ: At the point $(1,1)$?
50 Grace: Right, because it's increasing up to this point and then it's decreasing. If you think about that on the original function that means it has to be a max, or a local max.
51 MZ: If it's increasing and decreasing?
52 Grace: Right.
53 MZ: OK.
54 Grace: But since it's all increasing I don't know.
55 MZ: Oh. Yeah. [short pause] I think -- you're getting confused the derivative increasing versus the original function increasing.

As in the second interview, Grace adamantly states her inability to do this type of problem, and her only sketch is of a curve that mimics the ups and downs of the derivative function. Grace does remember that maximum and minimum of the original function occur where the derivative is zero, but she is unable to apply this information to her sketch. Unlike the second interview, Grace is also unable to state whether a particular zero of the derivative function is a maximum or minimum. At one point she states that the original function is decreasing where the derivative is decreasing. A few lines later she suggests that perhaps the original function is negative when the derivative is increasing. Of course, both are incorrect.

Grace's performance on this task is worse than her performance on the task from the second interview, despite her claim after the *Mathematica* lab about gaining a better understanding of the relationship between the two graphs. Perhaps this is because she was not asked leading questions about what she could tell from the derivative graph before being asked to sketch the curve as she was during the second interview. Instead of looking at input-output pairs and seeing the outputs as slopes of the original function, she mimics the derivative graph, thus treating it as an object without internal structure. Perhaps her poorer performance is due to the intervening time spent on other topics. However, Grace could have attempted newer strategies such as slope fields and areas that might be useful in solving this type of problem.

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

56	MZ:	What's a definite integral?
57	Grace:	The area under a curve?
58	MZ:	Yeah. Anything else come to mind?
59	Grace:	Uhm no, I mean, how much more specific do you want me to get?
60	MZ:	Well, that's OK. What's an indefinite integral? How is that different?
61	Grace:	Indefinite integral -- It's an antiderivative. It doesn't have those numbers.
62	MZ:	You mean the --
63	Grace:	Like, when it says going from 0 to x or whatever. [she writes: $\int_0^x \quad \int$]
64	MZ:	OK.
65	Grace:	That's just a little sign.
66	MZ:	OK. What's the difference graphically? I mean, you said the first one was the area under the curve. Does this indefinite integral have any graphical association?
67	Grace:	I don't know. I didn't even know you could graph it.
68	MZ:	OK. Uhm, what do either one of these integrals have to do with derivatives?
69	Grace:	Well, that's the opposite of the derivative. [points to $\int$]
70	MZ:	The antiderivative -- I mean, the indefinite integral --
71	Grace:	Yeah, it's the antiderivative so its -- [mumbling together]

72 MZ: -- antiderivative so its the opposite of the derivative. [mumbling together] OK.

73 Grace: I suppose it has something to do with it. Uhm, that -- the second fundamental theorem.

74 MZ: Oh, the second fundamental theorem?

75 Grace: Yeah and the first one. I mean, I know that's how they're related. I just can't figure it out. I know this deals with antiderivatives. Like you take the antiderivative of some number minus the antiderivative of something else. [writes: $F'(x) - F'(0)$] And you get --

76 MZ: OK, so the capital F prime is the antiderivative?

77 Grace: Yeah.

78 MZ: OK. So this is related to one of the fundamental theorems? Is that what you're saying?

79 Grace: It is the fundamental theorem, the second one, only I can't think of it.

80 MZ: But you don't know what it's equal to, is what you're saying? You know that this is there but --

81 Grace: Yeah. No, that's equal to area. [after $F'(x) - F'(0)$ writes: $= A(x)$]

82 MZ: Here's $y = x^2$ and we want to know the area under this curve from 0 to 1. Like that area right there. [draws the graph of $y = x^2$ and labels it as such, labels 0, 1 on the x axis and shades in the appropriate area]

83 Grace: And you want me to solve it using this?

84 MZ: Uh, sure.

85 Grace: [takes about 30 seconds to write the following: $F'(x) = \frac{1}{3}x^3$

$$\tfrac{1}{3}(1)^3 - \tfrac{1}{3}(0)^3 = A(x)$$

$$\tfrac{1}{3}$$]

No that doesn't sound right at all.

86 MZ: What?

87 Grace: $\frac{1}{3}$.

88 MZ: $\frac{1}{3}$? How come it doesn't sound right?

89 Grace: I don't know. It seems like it should be bigger. Is it right?

90 MZ: I *think* it's OK. Why does it seem like it should be bigger?

91 Grace: I don't know. Because it looks bigger.

92 MZ: It looks bigger.

93 Grace: We don't get very mathematical here. It doesn't look right.

94 MZ: OK. Well, I don't know. I mean, this is a distance of 1 [pointing to length under area on the x axis from 0 to 1]

95 Grace: That's a distance of 1 [pointing to distance from x axis to curve at $x = 1$].

96 MZ: Yeah right, the height at that point is 1. So, I mean, it's definitely smaller than 1, the answer. Right? -- because this rectangle would be 1.

97 Grace: OK. Well, then I'm happy with that.

98 MZ: Do you know another method of finding that area besides this second fundamental theorem stuff?

99 Grace: Ah, those -- We learned a million ways to do it. Do you mean like Riemann sums and stuff?

100 MZ: Yeah.

101 Grace: Like trapezoidal stuff.

102 MZ: Mm hmm. [short pause] You're not going to remember the first fundamental theorem ? Do you happen to remember anything about it?

103 Grace: Oh, I remember that I didn't understand it, but then I learned it, but now I've forgotten it. It has to do with the definite integral. I know that.
104 MZ: OK.
105 Grace: Not this one. Not the second one.
106 MZ: The second one has to do with the indefinite integral?
107 Grace: Yeah.
108 MZ: So-- But -- you didn't use the indefinite integral, did you, in this? [short pause] Oh, because this is the antiderivative, the F'. That's --
109 Grace: Yeah.
110 MZ: -- where it was, the indefinite integral. OK, OK. But the first one has to do with the *definite* integral?
111 Grace: Yeah. Now that I'm pretty sure of.
112 MZ: OK.
113 Grace: Only -- [laughs] Uhm, maybe.
114 MZ: [short pause] Do you remember when we did that area function? When we had that capital A of x equals to something?
115 Grace: Mm hmm.
116 MZ: Do you happen to remember what it was equal to?
117 Grace: On a graph?
118 MZ: Either way. Yeah. I was going to ask you both things -- what it means on the graph and how you wrote it as a symbolic --
119 Grace: Well, do you mean something like this? [writes: $A(x) = \int_0^x x\,dx$]
120 MZ: Yeah, something like that.
121 Grace: Uhm, you have a function. And here's x. [starts drawing a graph of a positive valued curve that has both positive and negative slopes and concavities; i.e. not $y = x$, $y = x^2$ or anything I recognize off hand. The area under the curve is shaded from 0 to x.] And then here's the area going from x to 0. [draws an arrow from x pointing toward 0]
122 MZ: All right, now --
123 Grace: Or was it 0 to x ?
124 MZ: Uhm --
125 Grace: Did I have it right the first time?
126 MZ: Well, what would be the difference in this area going from 0 to x instead of the area going from x to 0?
127 Grace: One would be negative.
128 MZ: Oh. I see what you mean.
129 Grace: And the way I wrote it. I think it's 0 going to x, isn't it?
130 MZ: Yeah, I guess I would say this [referring to $A(x)$ integral] is the area going from 0 to x.
131 Grace: [changes arrow on graph to point from 0 to x]
132 MZ: OK. So that's what this function is. Now is this curve that you drew specific to this integral that you wrote down, the integral of $x\ dx$?
133 Grace: No.
134 MZ: No. This is just sort of more general?
135 Grace: Uhm. It's kind of an illustration. I mean, I remember learning something in class. Except I don't -- It's one of those rule things. Except I don't think it's the fundamental theorem.
136 MZ: What were you thinking?

137 Grace: [writes: $\int_a^b x\,dx + \int_b^c x\,dx = \int_a^c x\,dx$]
138 MZ: OK, right. Yeah, you're right. That's one of those -- I don't know what you want to call it -- rule things. [both laugh]
139 Grace: Like this one. [writes: $\int_a^a x\,dx$]
140 MZ: Yeah. Well, what does this mean graphically though, the a to a?
141 Grace: Well, if you're going from a to a, you're not moving, so that means you can't have any area under it.
142 MZ: Yeah you're just right there. That's good. Yeah, this one has to do with the first fundamental theorem, this $A(x)$ function.
143 Grace: Mm hmm.
144 MZ: But I didn't know if you remembered that. Uhm, how bout -- Oh, is this area function related to derivatives at all?
145 Grace: [short pause] Uhm, probably. I don't know that.
146 MZ: OK. Uhm, [bell for change of class] Do you know anything about slope fields?
147 Grace: Do I know anything about them?
148 MZ: Yeah. What's a slope field?
149 Grace: It's a little graph showing your slopes at different points.
150 MZ: OK. Do slope fields have more to do with derivatives or integrals?
151 Grace: [short pause] Mm, derivatives?
152 MZ: Why would you say that?
153 Grace: Because derivatives are slopes.
154 MZ: True. [both laugh] Do they have *anything* to do with integrals?
155 Grace: Probably.
156 MZ: Probably. I mean do you -- Last question. Do you know what they were useful for, why we bothered to do them?
157 Grace: No. To draw graphs.
158 MZ: -- of --?
159 Grace: Of functions. Like when you didn't know the constant. You have a function and you have like something like that [sketches a slope field]. Like that's what a slope field looks like.
160 MZ: OK.
161 Grace: And if you didn't know where your constant is -- you can like put it anywhere -- but it just shows you what the slope is like at a certain point on the graph.
162 MZ: OK. Thanks.

Grace associates definite integral with area and indefinite integral with antiderivative. She knows that derivatives and antiderivatives, hence derivatives and indefinite integrals, are opposite operations. Here she is using derivative in the sense of the derivative operator, not the derivative at a point or the derivative function. She remembers how antiderivatives are used in the Second Fundamental Theorem and that the difference of the two antiderivative values gives an area. She also remembers studying ways of approximating area with sums. Grace cannot recall the First Fundamental Theorem of Calculus.

When prodded about area functions, Grace gives a symbolic example and sketches a graph that correctly demonstrates the area idea except the graph is for a different function than the one she has given symbolically. When questioned she acknowledges that it is just "an illustration" [ln 135]. Perhaps Grace is using prototypical examples. For her, the function x is a paradigmatic symbolic function whereas a wavy curve is a paradigmatic graphical function.

When asked, Grace replies that derivatives "probably" have to do with area functions, but she does not know what that relationship is [ln 145]. She knows slope fields have to do with slopes, hence derivatives, and when asked, thinks they "probably" have something to do with integrals [ln 155]. She also knows slope fields are about showing the shape of a curve that might reside at a variety of locations. She mentions not knowing "the constant" which probably means that she associates slope fields with antiderivatives. She already connects antiderivatives and area in terms of the Second Fundamental Theorem, but she does not try to answer why slopes and areas should be related in this way.

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: While it is recorded that Grace answered this question incorrectly, her exact response is not recorded.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Grace's exact answer is not available. It is known though that Grace answers the question correctly.

Test 9: Semester final

This test, which is a cumulative semester exam, covers all of material on functions, limits, derivatives, areas, and volumes. The test questions are largely computational. Grace has trouble setting up integrals for complicated areas and volumes, but she has no trouble solving arc length, polar integral, or sequence questions. Grace forgets twice to apply the chain rule when differentiating. The second instance involves implicit differentiation. When asked to find $\frac{dy}{dx}$ for $x + \sin(xy)$, Grace answers, $1 + \cos(xy)$. Curiously, on a different problem when differentiating in order to find the minimum of $SA = 2\pi r^2 + \frac{2V}{r}$, Grace includes an unnecessary term. She writes $SA' = 4\pi r dr + \frac{-2V}{r^2} dr$ and nothing further. Even though Grace correctly completes optimization problems early in the term and even on a different problem on this exam, she does not set this expression equal to zero or try to solve for r. Perhaps her inclusion of the dr term reminds her of a related rate problem and thus she is confused as to how to proceed. Perhaps the fact that in this problem no value is given for volume, V, makes the problem confusing. The other optimization problems that she has been tested on had no constants of this type. However, this constant does not limit Grace's ability to combine her first surface area equation with a constant equation in order to find the surface area equation listed above.

Two other small errors involve limits and continuity. Grace marks $y = x^{-2}$ as a continuous function, and she marks that if $\lim_{x \to a} f(x) = L$ then $f(a) = L$.

QOTD #15

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: " $f(x) = x^{\frac{2}{3}} = \sqrt[3]{x^2}$ continuous for all x not differentiable at $x = 0$."

Interview 4

The discussion of the fourth interview is broken up into four parts. The first section includes general questions about derivatives. The second part asks the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity, and acceleration given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the fourth interview.

1 MZ: What is a derivative?
2 Grace: Uhm. I guess, the slope of the tangent line.
3 MZ: OK.
4 Grace: A function.
5 MZ: Anything else?
6 Grace: No.
7 MZ: Well, sometimes people say the derivative is the instantaneous rate of change.
8 Grace: Mm hmm.
9 MZ: What do people usually mean if they say that?
10 Grace: Like, uhm, if you're measuring something that's always changing, but you catch it like at an exact moment.
11 MZ: OK. So that's the instantaneous part, is the fact that it's at an exact moment?
12 Grace: Mm hmm.
13 MZ: How about what the rate of change part means, more specifically?
14 Grace: Rate of change?
15 MZ: Yeah, what does rate of change mean?
16 Grace: Average rate of change?
17 MZ: Uhm-- In general rate of change.
18 Grace: The rate that something is changing. [laughs] Like how fast the thing is changing, and the derivative is just when you catch it at one point.

As in previous interviews, Grace mentions slope as her first answer to what a derivative is. However, here for the first time, she says that derivative is the slope of the tangent line instead of saying that it is the tangent to the slope. She also mentions, without prompting, that the derivative is a function. Unfortunately, no follow-up question probes this statement. Grace does not voluntarily mention any other ways of describing derivatives. Her first response to the question about rate of change is to ask, "Average rate of change?" [ln 16] Perhaps if the interviewer had said yes, Grace would have given a description of the ratio involved. When the interviewer asks for a general description, Grace talks at the level of a gestalt, "the rate that something is changing" [ln 18]. When pressed Grace uses the metaphor of speed, "how fast the thing is changing" [ln 18].

The next part of the fourth interview is a summary of Grace's solution to the first of three problems involved in the interview. Given a table of values with x varying by .1, Grace is asked to estimate $f'(2)$, the derivative of the function at $x = 2$. Grace's first reaction is to graph the function given by the values in the table. From the graph she states that the derivative at 2 is positive and when pressed she calculates that its value is approximately 20.4. At first she thinks this value is too large, but decides that it's OK. When asked how she calculated 20.4, she says, "I picked two points and found the slope between them." Grace says she picked these two points because "they're close, better than any one." She is unable to state how she would find a different estimation.

Grace seems to think of derivative as slope from the beginning. She uses a graphical approach and seems to note the steepness of the graph both when stating that it is positive and then when wondering if her estimate is too large. However, she does not communicate that she is thinking of slope until she is asked to explain her calculation. Grace demonstrates both a process and object concept of slope. She does not discuss a limiting process for finding a more accurate estimate.

The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car takes 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far does the car travel in the 2 hour period?

Grace's first reaction is to sketch a graph of miles per hour versus time. Her graph is correct. She decides to find the distance traveled by finding the "average of how fast it was going". She finds the average for several intervals correctly, seemingly by taking the average of the highest and lowest speeds on the interval. She multiplies these average speeds by the time to get the distance traveled for several intervals of time. When asked to sketch a graph of the distance function, she sketches an increasing concave up curve with no points labeled. She incorrectly states that the curve is concave down and remains confused about what shapes are concave up or concave down. When asked, she plots a few points on the curve. She becomes momentarily confused as to whether her curve should be everywhere increasing. "He wouldn't really go up to 60 because he's decreasing." Then she decides that the curve is OK.

Grace is not asked, nor does she attempt, to use any formulas. She is correct in calculating the distance using averages, but this method does not allow her to show how to use calculus to relate distance and rate. The only error in her graph of the distance function is that it should be concave down over the second half of the interval.

In both the previous problem and this problem, Grace's first reaction is to draw a graph. This is interesting considering her expressed aversion to solving problems that give her the graph of one function and ask her to draw the graph of another. However, in the two problems of this interview Grace has data or explicit physical information about the curve she wants to draw. The difficulty in the other type problem arises because Grace does not recognize the need to extract explicit information from one graph before trying to draw the other. She tries to draw the second graph based on a holistic view of the first graph. In the current problem regarding the velocity and position of the car, her earlier struggles recur momentarily. She sees that the rate curve is decreasing from 1 to 2 and wants to apply that to the distance function. The points that she has already calculated and plotted seem to help to keep her from making that error again.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Grace is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. She is asked if the top of the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Grace's first reply is, "Are you asking me how to figure it out first?" She begins solving the problem by labeling the wall as dy, the floor as dx and the ladder as dz and writing $(dx)^2 + (dy)^2 = (dz)^2$. She wants to take the derivative next and knows that the derivative of a constant is special, but she's uncertain whether the answer is 1 or 0. She convinces herself it is 0 during the following exchange.

Grace: Oh. Because if you were going to take the derivative of a constant number, like on a graph, then the slope is 0, so it's 0.
MZ: Did you have a picture in your mind of a constant graph when you were talking about that?
Grace: No, just like a point on a graph.

Grace does not further explain her last statement, but decides to change her variables to x, y, and z. She writes $x^2 + y^2 = z^2$ followed by $2xdx + 2ydy = 0$. At this point, when asked how the rate the ladder is sliding out compares to the rate it is sliding down, she says that the rate it is sliding down, dy, is constant. She performs algebra to find $dx = \frac{-y}{x}dy$ and claims dy is constant because the other three terms in the last equation are constant. When the interviewer insists that x and y are changing and not fixed, Grace agrees that dy is not constant. She makes no attempt to answer whether it is increasing or decreasing.

Grace's problem solving attempt initially does not focus on answering the interviewer's question about the relationship of the two rates, but seems to focus on a familiar procedure for solving problems with scenarios of this type. Related rate problems with the sides of a right triangle as the principle construction were likely common in Grace's junior year study of calculus and have been assigned and tested in her senior year study of calculus. In these traditional problems the student is usually asked to find a numerical value for one rate given numerical values for the other quantities involved. Grace seems to be operating under the assumption that all quantities in this problem will have a single numerical value.

Grace also is confused how best to label her variables. Initially, dx is used as a length, instead of x, and later, dx is used as a rate of change, instead of $\frac{dx}{dt}$. The former choice might occur because she is labeling a length that is changing, while the latter might occur because she is concentrating on the length changing and the notion that this is with respect to time is implicit, and hence hidden from view. Grace's test on related rate problems, which occurred five months prior to this interview, did not exhibit either of these errors.

Interview 5

Grace's fifth interview occurs exactly one week after she takes the BC version of the AP exam. During that week the class discusses the written questions from the BC version. Between the fourth and fifth interviews the class studies series and integration techniques and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Grace's response to a set of questions on a particular topic and provides an analysis of those responses.

1 MZ: What is a derivative?

2 Grace: Instantaneous change in rate.

3 MZ: Well, what does that mean, instantaneous change in rate?

4 Grace: Like, as opposed to the average. Like if you have a curve and you take the end points and then you could say the average of this curve is whatever, but at a certain point the instantaneous change, like right at that second that it's changing, that's the derivative.

5 MZ: OK. What other kinds of descriptions of what a derivative is do you know?

6 Grace: The slope of the line tangent to a curve at a certain point.

7 Grace: Well, it's change in y over change in x which is the change in rate pretty much.

8 MZ: OK. How do you write that?

9 Grace: Like $\frac{\Delta y}{\Delta x}$.

10 MZ: [writes: $\frac{\Delta y}{\Delta x}$]

11 MZ: Do you know that formal definition [of derivative]?

12 Grace: [writes: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$]

13 MZ: Yeah. I was going to ask, what does the formal definition have to do with some of these previous ones?

14 Grace: I know this. I'm just not sure how to explain it. If you have a point and the point is $(x, f(x))$ --

15 MZ: OK.

16 Grace: [sketches a smooth curve on a pair of axes] Like here's x [labels a spot on the x -axis]. And this point right here is $(x, f(x))$ [labels the point and draws a vertical line to the x mark on the x -axis].

17 MZ: OK.

18 Grace: And then you have a point h. [draws in a second point to the right of $(x, f(x))$] No, you don't have a point h. This distance is h [between the x values of the two points].

19 MZ: OK.

20 Grace: And this is $x+h$ [labels on x -axis]. And this point is $(x+h, f(x+h))$ [labels new point and draws a vertical line to the $x+h$ mark].

21 MZ: OK.

22 Grace: And as h approaches 0 means when that comes that way [draws an arrow from the $x+h$ mark toward the x mark]. What you're doing is you have a line between these two points [secant line drawn connecting two labeled points; note, the curve is drawn flat enough such that the secant line almost looks like a tangent line]. That's your tangent line. And when your two points get closer together, the line becomes more exact. Like it becomes a more exact tangent to those points. And when h approaches 0 and these two points are getting closer together, that's

when-- It's like the delta y over delta x. You have f of $x+h$ minus f of x over $x+h$ minus x which is just h.

23 MZ: Is there a physical quantity in the picture that we can point to and say that's what this ratio is?

24 Grace: Well, this whole thing.

25 MZ: Mm, OK. Well, what's a specific example the derivative would give us?

26 Grace: That one on the highway, right?

27 MZ: OK.

28 Grace: If the guy's going an average of 50 miles per hour. Like he could be going from 30 miles an hour to 70 miles an hour, but the average could still be 50 miles per hour. But like at a certain point he could be going 90 miles per hour, his instantaneous rate. Cause like at that instant he's going 90 miles per hour, but his average is 50.

29 MZ: OK. So, how does this symbolic limit and this ratio [indicates the formal definition], fit in that situation that you just described?

30 Grace: Because it's like a-- It's like if he was going 90 miles an hour and then-- It's like that measure thing. How like a cars pass between two lines, that kind of thing.

31 MZ: OK, so a car passes between two lines--

32 Grace: Right, and like if he's going 90 miles at one point and like 10 feet from there, they want to know how fast he's going or something. But he could brake and slow down a lot. But as the distance between those two points gets smaller, that's h, as it comes from like 5 feet and 2 feet, there's like no-- It's more accurate because you can't brake in 2 feet. You know what I mean?

33 MZ: He can't change his speed as much in that short of time?

34 Grace: Yeah, right. I mean, like when you had it this way, you can have lines that go like that and lines that go opposite. [draws some sketchy lines almost perpendicular to the line connecting $(x, f(x))$ and $(x+h, f(x+h))$]

35 MZ: Oh, the other way?

36 Grace: Yeah. But as it comes closer together and the distance between it gets really really small, there's less of a chance that it's going to do a lot of really weird stuff.

37 MZ: Mm hmm. OK. Let's see. Does the derivative involve a limiting process?

38 Grace: What do you mean a limiting process?

39 MZ: I mean, there's obviously a limit here [indicates the limit definition], but I was thinking of the limit-- the limiting process involved in the picture or in the description you just gave.

40 Grace: I'm not sure I understand that.

41 MZ: OK. Is the derivative of a function a function?

42 Grace: I don't think.

43 MZ: Why not?

44 Grace: Well, because if you have something that looks-- Wait a minute. Wait. I know this is going to sound really silly-- [both laugh] Does a function have to be continuous to be a function? No, right? No.

45 MZ: No.

46 Grace: It doesn't. Right. [short pause] Yeah, it can be. It has to be.

47 MZ: It has to be?

48 Grace: I think so, yeah.

49 MZ: OK, why?

50 Grace: Just because. [laughs] There isn't really a reason for it. [short pause] Well, cause I know one of the things about when something can't be a function. It's like do the vertical line test and stuff.

51 MZ: OK.

52 Grace: And when you have a line and you take the derivative of it, you can't like have it coming the other way. You know what I mean? Like if you have a function, your derivative can't be something like that, you know? [draws an $x = y^2$ type curve on a pair of axes]

53 MZ: Why do you know that whenever you take the derivative of some function it never turns around and has this type of situation.

54 Grace: I guess just because-- I mean, if you pick any point, you know. It's like-- [sketches a smooth curve on a pair of axes]

55 Grace: And it's like a-- And you have a line there. [marks a point on the curve and draws a line touching the curve at that point]

56 MZ: OK. Is it a tangent line?

57 Grace: Yeah, it's a tangent line. It's like that doesn't correspond here. Like you can't have one point here that corresponds to two points on this graph. You know?

58 MZ: Yeah, I do know.

59 MZ: Explain what is meant by a differentiable function. Give an example of a differentiable function and an example of a nondifferentiable function.

60 Grace: One where you can take the derivative at all points.

61 MZ: OK. So what's an example of a differentiable or a nondifferentiable function?

62 Grace: Uhm, that's a differentiable function. [has drawn a smooth curve]

63 MZ: OK. And why is that a differentiable function?

64 Grace: Because there's no place where you can't take the derivative.

65 MZ: How do you know that?

66 Grace: Because there are-- [draws a function with a cusp pointing upward on a pair of axes] That's a nondifferentiable function because that has a cusp.

67 MZ: OK.

68 Grace: Well the reason why this one isn't is because like if you put tangents to every point. Like they're suppose to like just come together. I don't know how to explain it. Like this one. [indicates a previously drawn differentiable function] No matter how many changes you keep taking, they're never going to come close together. They are never going to become-- Do you know what I mean?

69 MZ: At this point. You're talking about specifically at that cusp point.

70 Grace: Right.

71 MZ: So like if I took a tangent along here and then came in closer and took another tangent, then something bad is going to happen at that point? Is that what you're saying?

72 Grace: Yeah, because the thing is-- If you have two points like that. [marks two points on the smooth curve and draws a tangent line] The point is when you look at it--

73 Grace: If this is your curve and it's touching right there, and then like if you come in close in, that, the tangent line, eventually-- If you get close enough, you can almost say they're the same line. [has drawn to the side a part of a curve, marked a point and drawn a tangent line; there are three circles of smaller and smaller radius drawn centered at the marked point]

74 MZ: OK.

75 Grace: For this one though, [referring to the graph with a cusp] you can't do that because there's a sharp turn. You can't say that they are ever going to

become a straight line because it's not. It's just a straight point-- a straight-- a sharp curve-- turn. [laughs]

76 MZ: OK. Do you know the equation of a nondifferentiable function?

77 Grace: [pause] I don't remember if this is one, but I'll try one over x. [writes: $\frac{1}{x}$]

78 MZ: And why is this one not differentiable or for what point?

79 Grace: Because there's an asymptote at 0.

80 MZ: Asymptote at 0. That's not continuous at 0 either, right?

81 Grace: Right.

82 MZ: How about one that is continuous but not differentiable. You know, it could be like the one you drew.

83 Grace: Of course I wouldn't know an equation for the one I drew. [short pause] I don't know. I could do a piece function, but--

84 MZ: Could you do a piece function that's continuous but not differentiable?

85 Grace: Possibly. Could it-- I mean, if you had--

86 Grace: If you had something like this. [sketches a curve that comes to a pointed up cusp at $(0,1)$]

87 MZ: OK.

88 Grace: That would be not differentiable.

89 MZ: OK.

90 Grace: And if you had--

91 Grace: How to do this-- So it would be-- Right? [has written:

$$f(x)=\begin{cases} e^{x} & \text{if } x\le 0 \\ & \end{cases}$$
]

92 MZ: OK.

93 Grace: And then uhm-- [pause] I can't get that one. What would that be? It's a parabola. I don't know how to figure it out.

94 MZ: Oh, this is like some parabola, on the right side?

95 Grace: Yeah.

96 MZ: OK. Well, you don't have to figure it out, but some parabola that goes--

97 Grace: And maybe something-- x is greater than 0 or it could be greater than or equal to.

98 MZ: Because it's going to hit that point?

99 Grace: Yeah.

100 MZ: That's fun. I thought you were going to say e^{-x} for the other side.

101 Grace: Oh, Oh. I suppose I could do that. [At some point Grace added to the curve on the right side so that it could not be simply $y=e^{-x}$]

102 MZ: That'd be another way.

103 MZ: Explain what a derivative is without using the symbolic definition or mentioning slope or rate of change.

104 Grace: [pause] Hmm. I don't know. [pause] Can I say line?

105 MZ: What about the line?

106 Grace: Just a line.

107 MZ: The line itself is the derivative?

108 Grace: No, the point. I don't know. I don't know how to explain that. Cause, it's like-- You don't really think about it except in terms of math. It's always math. Cause it's math.

109 MZ: Well-- How about, what are derivatives useful for?

110 Grace: Let's say you're a business, and you want to figure out maximum output or minimum cost or something. You can use the derivative there.

111 MZ: OK.
112 Grace: Cause like if you take the derivative of something and set it equal to 0, you can get your numbers and stuff. You can figure them out for businesses. Or if you have a field that you are trying to fence, and you want to find out what is the minimum amount you need to fence your yard, you can use the derivative for that.
113 MZ: OK. What about something besides a max-min problem? What would they be useful for?
114 Grace: Hmm. Like you could also use it for measuring the rate. If you're filling a swimming pool, how long it's going to take or you know, going at this rate or whatever. You can use derivatives there. In case you ever decide to haul a leaky bucket. You can use the derivative there to figure it out.

As in previous interviews, Grace mentions a graphical interpretation of derivative more frequently than the other interpretation (see Table A.21). Even though the portion of the fifth interview focusing on general questions about the derivative has fewer questions than similar sections of the first and second interviews, Grace gives more complete answers here . She relates the details of the ratio in three different interpretations, and she discusses a limiting process in three different notations. She also realizes the derivative is a function in both a graphical and symbolic sense.

Unlike previous interviews, Grace's first answer to, "What is a derivative?" refers to rate [ln 2]. The question about the meaning of rate of change during the previous interview may have influenced Grace's choice to mention rate first in this interview. Previously, her first answer referred to a graphical representation, which becomes her second response in this interview. With minimal prompting, Grace mentions slope [ln 6], rate [ln 2], and a symbolic ratio [ln 12] and relates them correctly [ln 14-22]. She also correctly states the complete formal definition and relates the limit of the ratio to both a graphical interpretation and an interpretation in terms of velocity [ln 30]. Grace, for the first time, uses the definition of a function and the idea of one tangent line per point to determine whether the derivative of a function is a function [ln 50-57].

In the first and second interviews Grace makes the statement that "derivative is tangent to the slope". During the fourth and fifth interviews Grace no longer makes that misstatement. However, she continues to emphasizes the tangent line, instead of

Table A.21. Grace: Interview 5 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?		◎			misstatement (change in rate)
What else?	◎ ○	•		•	
Did you learn a formal definition of derivative?				◉	
How does the formal definition of derivative relate to slope or rate of change?	◉		◉		misstatement
Is the derivative of a function a function?	◎				
What is meant by a differentiable function?	●			↦	
Explain what a derivative is without mentioning slope, velocity, rate of change or the formal definition.	○				misstatement: (d=tl)
What are derivatives useful for?		○			max/min rate
Asked to interpret the Mean Value Theorem.	◉	○ ○	◉	• •	misstatement: (d=tl)
Asked to find the average rate of change of a function defined as an integral.		○			misstatement (d=average roc)
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.	○			↦	misstatement concavity incorrect calc
Summary	◉	◉	●	◉	

mentioning the slope of the tangent line. While focusing on tangent line instead of slope, at one point Grace misstates that the derivative is the tangent line. In addition, when relating the symbolic ratio to a graphical picture of secant lines approaching a tangent line, she calls the ratio "that whole thing" [ln 24].

Grace makes a misstatement not found in her previous interviews when she describes derivative as change in rate instead of rate of change [ln 2]. Her initial statement might be taken as a simple transposition of words but later she emphasizes this word choice with the description "change in y over change in x which is the change in rate" [ln 7]. The use of, "change in" reflects the general pattern of the previous interviews. Why rate should be equated with y or x is unknown. Even though Grace does not make this error in earlier interviews, her earlier interviews do not as clearly relate rate to other interpretations of derivative as she does here.

For the second part of the fifth interview Grace is asked about the Mean Value Theorem. First she is asked if she remembers what it says. Then she is further encouraged to tell what it means in a graphical context and in the context where the function represents distance traveled.

Grace recalls the main statement of the Mean Value theorem and writes it as $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. She first says, "It's change. I think" and continues by describing that the right side of the equation represents a slope like $\dfrac{y_2 - y_1}{x_2 - x_1}$ The left side of the equation is harder for Grace to describe. After sketching a curve with $(a, f(a))$ and $(b, f(b))$ marked, she notes that c is on the x -axis between a and b and continues, "f prime of c will be the line that's-- for those two points-- I can't think of the word." Grace sketches a line on her graph through $(a, f(a))$ and another point near both $(b, f(b))$ and the point on the curve above c. When asked what point or points the line passes through, Grace replies "a and b." When questioned as to whether it also passes through the c point, Grace replies that she thinks it goes through c. Grace says she keeps thinking of the Intermediate Value Theorem where one thinks about $f(c)$ instead of the derivative at c. Grace appears unable to further clarify her understanding of $f'(c)$ in this context.

When Grace is directed to think of f as a function for the distance, she recognizes that the derivative of position gives the speed or velocity, and that the ratio is distance over time, hence velocity. Knowing that both sides of the Mean Value Theorem equality represent velocity, she is unclear on what the difference between the two is in terms of the physical situation.

Grace's ability to recognize the ratio as slope and as velocity indicates her understanding of the ratio level of the derivative in these two contexts, and additionally her understanding of the relationship of these to the symbolic representation. Her inability to make a clear the distinction between the derivative at a point and the ratio in the context of the Mean Value Theorem is puzzling considering her comments emphasizing this distinction in other interviews. In the second interview and again in the fifth, Grace states, without prompting, that the distinction between average and instantaneous is important and that instantaneous refers to the value of the derivative "at a certain point" or "at that instant." Her prior comments about average rates or average slopes range from correct to confused. In the second interview Grace describes the difference between average and instantaneous as follows:

Grace: If you have two points and you can find the slope between them, but that's like an average slope. It's like average velocity when we studied that in physics last year.
MZ: OK.
Grace: There's a difference between average velocity and instantaneous velocity because when you have an average velocity that just means it could be really high and really low at one point, but if it's like kind of in the middle for the rest of the time, then it's just like middle. Know what I mean?

Her description of average slope is correct and may imply a difference quotient calculation for slope. Her description of average velocity conveys the idea that an average is in between a range of possible values that may occur, but it does not seem to imply a ratio of distance over time. She may be thinking of an arithmetic average as seems the case in a later excerpt from the second interview.

MZ: Does measurement have to do with the derivatives?
Grace: Yeah because derivative gives you a really accurate measurement. When you have an average measurement-- OK, like if you had an 80% average for a class.
MZ: OK.
Grace: You could have had a 70, a 90 and an 80, but you'd still come out with an 80. But if you have your derivative, that'd be telling you-- Like if you have an 80, it can't be anything higher or lower.

In the fifth interview, Grace's comments about average versus instantaneous are vague and do not specify a difference quotient.

3 MZ: Well, what does that mean, instantaneous change in rate?
4 Grace: Like, as opposed to the average. Like if you have a curve and you take the end points and then you could say the average of this curve is whatever, but at a certain point the instantaneous change, like right at that second that it's changing, that's the derivative.

It is unclear what Grace means by "average of this curve." Since she mentions the end points she may mean the slope of the line through those two points. However, average of the curve sounds more like an average value of the function or of the slope values for all points, a type of arithmetic average. Later in the fifth interview Grace speaks about average and instantaneous again when asked to relate the limit of the difference quotient to instantaneous rate of change.

28 Grace: If the guy's going an average of 50 miles per hour. Like he could be going from 30 miles an hour to 70 miles an hour, but the average could still be 50 miles per hour. But like at a certain point he could be going 90 miles per hour, his instantaneous rate. Cause like at that instant he's going 90 miles per hour, but his average is 50.
29 MZ: OK. So, how does this symbolic limit and this ratio [indicates the formal definition], fit in that situation that you just described?
30 Grace: Because it's like a-- It's like if he was going 90 miles an hour and then-- It's like that measure thing. How like a cars pass between two lines, that kind of thing.

31 MZ: OK, so a car passes between two lines--

32 Grace: Right, and like if he's going 90 miles at one point and like 10 feet from there, they want to know how fast he's going or something. But he could brake and slow down a lot. But as the distance between those two points gets smaller, that's h, as it comes from like 5 feet and 2 feet, there's like no-- It's more accurate because you can't brake in 2 feet. You know what I mean?

33 MZ: He can't change his speed as much in that short of time?

36 Grace: Yeah, as it comes closer together and the distance between it gets really really small, there's less of a chance that it's going to do a lot of really weird stuff.

Grace is referring to h from the formal definition as a distance when it should be an amount of time. She has the general idea of the limiting process correct, but she does not seem to realize that the ratio is one of distance over time. Again her discussion of average rates seems to emphasize the nature of average as a value in a range of values instead of the result of a difference quotient.

Grace struggles with the notion of average slope or average velocity throughout the interviews. When discussing the Mean Value theorem she recognizes the ratio as a slope or velocity, but does not mention the word average or discuss the distinction between instantaneous slope or velocity. With the graphical interpretation, Grace has further problems because she never states that the derivative is the slope of the tangent line. She recalls the idea of a line, but does not get any farther. Her persistent misstatement that the derivative is the tangent line may also contribute to her inability to make the distinction between average and instantaneous slopes in the context of the Mean Value Theorem.

The next question on the fifth interview involves a problem from the AB version of the AP exam which Grace has not taken. She reports that she has not seen the question before. The question is as follows:

Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.

(a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.

(b) On what intervals is F increasing?

(c) If the average rate of change of F on the closed interval $[1,3]$ is k , find $\int_1^3 \sin(t^2)\,dt$ in terms of k .

The interviewer first asks Grace to discuss her methods for solving parts (a) and (b), but does not require her to complete the solution of either part. For part (a) Grace recognizes that she must find the definite integral $\int_0^1 \sin(t^2)\,dt$. She attempts to do this by finding the antiderivative of $\sin(t^2)$ ignoring the instructions to use the trapezoid rule. After the interviewer helps Grace see why this antiderivative can not be found using the substitution method, Grace attempts to remember the trapezoid rule. Grace is unsure what the curve $y = \sin(t^2)$ looks like so she says she will "pretend" and draws a curve similar to a sine wave. Grace sketches four trapezoids under this curve and correctly finds the area of the first one before the interviewer suggests she move on to part (b).

For part (b) Grace initially takes two derivatives of F. She finds $F'(x) = \sin(t^2)$ and then must have the interviewer help to correctly use the chain rule to find the second derivative $F''(t) = 2t\cos(t^2)$. At first she decides to set the second derivative equal to 0 to find the critical points. While describing the process for charting where the second derivative is negative or positive, she realizes that it is the first derivative not the second derivative that she should be looking at to determine where F is increasing. She recognizes that the first derivative is positive where the original function is increasing, and the interviewer asks her to continue to part (c).

For part (c) Grace does not know how to interpret the question. When asked to talk about what is meant by the phrase, the average rate of change of F is k, Grace responds, "It's just like your derivative except average. It's not like at a certain point." When asked how to find the average rate of change, she suggests taking the derivative of F and solving from 0 to 1. Grace is unsure if this would work and the interviewer does not pursue the question further.

In this section Grace struggles to calculate a symbolic antiderivative and a symbolic derivative. After some initial confusion she remembers how to find area using trapezoids and that a function is increasing when its derivative is positive. Even though F is defined as an integral, Grace has no problem finding $F(1)$ and $F'(x)$. In part c, however, Grace does not know how to interpret the average rate of change of F. This may be a difficulty with F being defined as an integral or it may be another example of Grace's lack of specificity when thinking of average rate of change, as discussed after the Mean Value Theorem problem above.

The next section of the interview concerns Taylor polynomials. Grace responds that she remembers nothing about Taylor polynomials and answers none of the interviewers questions on this topic. Grace suggests that the class did not study Taylor polynomials this semester. However, records indicate that the class studied and was in fact tested on Taylor polynomials, Taylor series and power series approximately five weeks before the interview. That test accounts for Grace's lowest test score of the course.

The final section of the interview concerns a function, f, that at any time, x, given in hours tells the outside temperature in degrees Fahrenheit. Grace is shown a series of symbolic expressions and is then asked what information each one provides about the outside temperature.

Grace is unable to give any answer for $f'(3) = 4$ and $f''(3) = -2$. When asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Grace replies that the temperature is rising and that the slope of the function is positive. When asked to further clarify how the temperature is rising or if the temperature would be rising differently if the derivative were equal to 8 instead of 4, Grace pauses for a long moment and then asks, "Could it be that it's not changing? It's just one temperature?" For clarification the interviewer asks Grace what the temperature would be at 1 o'clock if the temperature at time equals 0, noon, was 50 degrees. Grace replies that the temperature would still be 50 degrees. She explains that the derivative is just the graph of a flat line. When asked about the equation for the function of temperature, Grace recognizes that it would be an increasing line and is able to determine the equation, $f(x) = 4x + C$ or $f(x) = 4x + 50$ using the initial condition. Given a correct graph and equation she can now answer that the temperature is 54 degrees at 1 o'clock.

Next Grace is asked to interpret $f''(x) = -2$ for $3 \le x \le 6$. Grace says that she thinks "you'd still be going up," and, "there would be concavity." When asked what the graph would look like after $x = 3$, she attaches an increasing concave down curve for $x \ge 3$ to the straight line she has already drawn for $x \le 3$. When asked what the temperature would be at 4 o'clock, Grace guesses 70 and then 65. When the interviewer presses her to use the information that $f''(x) = -2$, she calculates $f'(x) = -2x$ adding $+C$ at the interviewers request. From this information Grace states that the temperature would be going down. When the interviewer points out that what she has just written is only the first derivative, Grace calculates the original function as $f(x) = -x^2$.

Since the time allotted for the interview is running short the interviewer does not ask Grace further questions about her calculations, but asks whether Grace thinks the temperature at 6 o'clock would be greater than any other time since noon or back down to 50 degrees or somewhere in the middle. Grace says that she thinks it would be greatest at 6 o'clock.

Grace knows that a positive first derivative means that the function is increasing and has a positive slope and that a negative first derivative means that the function is concave down.

Without being asked, Grace quickly mentions graphical interpretations such as slope and concavity. The graphical answers come more easily than verbal descriptions of the change in temperature. Grace's graphical emphasis may be due to her preference for graphical interpretations, as shown in her circle diagrams, and her tendency to sketch a graph when solving problems, as noted in the analysis of the fourth interview above. However, Grace's graphical emphasis on this problem is probably also caused by her lack of experience with verbal interpretation of physical situations. Graphical interpretations are frequently discussed and tested in this class; verbal interpretations are not.

Grace is only able to state a verbal or graphical interpretation when the information is given for the first and second derivative over an interval of time, instead of only at a single point. Interpreting a constant first or second derivative over an interval, especially graphically, is to treat the function or graph as an object without considering the underlying process. To state an interpretation for the first and second derivative of a function at a point, Grace would have had to explain the ratio involved. This difficulty is also involved in Grace's failure to realize that a second derivative of -2 would eventually cause the temperature function to decrease. Her sketch and numeric guesses are of increasing functions. She never sees the ratio involved, or more specifically that the rate the temperature is increasing is itself decreasing by 2 degrees per hour every hour.

Grace's failure to examine the details of the ratio involved is reminiscent of her failure to consider the details of what is meant by an average rate in AB#6 part (c) and as discussed after the Mean Value Theorem problem. Grace has shown that she knows how to interpret the ratio in terms of slope and is also able to interpret the ratio in terms of velocity when asked about it directly on the Mean Value Theorem problem. However, she fails to evoke this information in other situations where it would be appropriate and helpful to do so.

Case Study 8 — Helen

Academic record

*National Merit Scholar.

*Other AP courses: US. History (junior year), English, Spanish, Chemistry.

*Writing tutor at the high school writing center.

*Math team participant.

*Plans to major in environmental engineering in college.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: "A function is an equation that has only one y for every x when graphed."

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different?

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: "a) A constant function is very different from a polynomial function. A constant function has no " x " value. When graphed it is always a straight line unlike a polynomial function which can be either odd or even and is never a straight line.
b) One example of an equation that is almost a function is the equation of a parabola that opens to the left or right."

QOTD #3

Give an example of a function <u>without</u> using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Date: August 26, 1993. This question occurs while the class is doing a quick review of functions.

Response: Helen provides a sketch of the graph of the function $y = (x-1)^2$.

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) non integers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: "a) possibly b) possibly"

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to class discussion on limits.

Response: "A limit is the number y approaches as the x-value approaches a certain point."

Test 1

On a test on limits a week later, Helen is able to correctly find limits by reading values from a graph, by substituting into a piecewise function and by using algebra to simplify a limit calculation. She is able to work with the formal definition of limit to the extent of finding a δ for a given ε in a symbolic setting, but she is unable to do this in a graphical setting.

Interview 1

This interview occurs after the test on limits but prior the class's discussing derivatives. Therefore, Helen's answers are presumed to be based on what she remembers from her junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table A.22, which codes these responses. A summary discussion follows.

1 MZ: What is a derivative, if you remember?
2 Helen: It's -- I can tell you the equation.
3 MZ: OK, we'll go with that.
4 Helen [writes: $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$]
5 MZ: What else comes to your mind when you think of derivative?
6 Helen: [pause] Lots of messy equations.
7 MZ: What can derivatives be useful for?
8 Helen: To help you graph things. You can use them for that. Find the max or min or --
9 Helen: You can use it like to figure out how you can like find the -- to find the most you can get from something. Like if you're figuring out like a word problem.
10 Helen: Like if they say the cost of making this is you know, such and such and you want to figure out how much money you can get -- like how you can get the most money.
11 MZ: Do you remember how those work?
12 Helen: You find the derivative, and then you set it equal to zero.
13 MZ: Do you remember why it works that you set it equal to zero? I mean like why not set it equal to 5 or something?
14 Helen: Uhm, it's kind of like when you graph it, when you do the first derivative test --
15 Helen: -- and that tells you the max/min. I'm not sure why exactly you set it equal to zero. Oh, yeah I am. Because won't that give you the roots of the derivative?
16 MZ: Yeah --
17 Helen: And those are your critical points. On the graph those would be your critical points and those would be your max and min.
18 MZ: OK. Uhm. How come it turns out that the roots of the derivative give you the max or min?
19 Helen: I don't know.
20 Helen: Yeah.
21 MZ: How can you tell, if I give you a function, how can you tell if it's differentiable?
22 Helen: If it's -- it has to be continuous. [pause] I don't know. I don't remember.
23 MZ: We'll put it has to be continuous. [writes continuous] Let's see. So I have a list of words here. So for each one I want you to tell me if you think it has to do with derivative and if so what does it have to do -- So the first one is slope.
24 Helen: Yeah It's the slope of -- Like the slope of the tangent is the derivative or the -- aah.
25 Helen: Yeah, it gives you the slope of the tangent line-- if you plug in -- into the derivative of an equation --

26 MZ: OK, so if I have the derivative of an equation and I plug in the x value -- then I'll get the slope.
27 Helen: then you'll get the slope. [said almost simultaneously]
28 MZ: Speed or velocity?
29 Helen: Well, if you have like the position graph and you take the derivative of that, then it'll give you the velocity graph.
30 MZ: OK. Change or rate of change?
31 Helen: Like the acceleration?
32 MZ: Uhm, let's see. Acceleration is like rate of change isn't it? From velocity, yeah.
33 Helen: Oh. [inaudible] Yeah.
34 MZ: How about generic rate of change, not regarding a particular thing?
35 Helen: Yeah, I mean -- I remember going over this last year, but I don't remember --
36 MZ: Since you brought it up, what does acceleration have to do with the derivative?
37 Helen: If you find the derivative of the velocity graph then that'll give you the acceleration graph.
38 MZ: Line or linear?
39 Helen: Uhm. Well, if you find the derivative of a line that'll give you like the line $x = 0$.
40 Helen: Oh, no. Not necessarily. If it's a horizontal line --
41 MZ: OK. So if I had a horizontal line and I took the derivative of that -- [sketches axes with horizontal line at $y = a$ positive constant]
42 Helen: You would get the line $x = 0$.
43 Helen: Oh, no I'm sorry. $y = 0$.
44 MZ: But if you had a diagonal line, you wouldn't necessarily get that.
45 Helen: That would give you a horizontal line.
46 MZ: OK, so if I took the derivative --
47 Helen: Because then the slope of that would be the y value.
48 MZ: OK, so if I took the derivative of this line, say this is the line $y = x$, then the line I get is--
49 Helen: $y = 1$.
50 MZ: $y = 1$ and you just said how it related to this line. [pause] It was the slope, right?
51 Helen: Yeah.
52 MZ: Measurement?
53 Helen: Uhm, you could figure out -- like error, maximum error.
54 MZ: Maximum error --?
55 Helen: Like if you have -- if you're trying to find the volume of a cube and you measure one side and you say maybe it's 10 cm plus or minus .5 cm, something like that. You can figure out the total error in the cube or the total possible error.
56 MZ: So, where does the derivative come into that then?
57 Helen: You have to use like d of y. It's like-- [pause] I don't remember. It's --
58 Helen: It's like if you ask me how to do a question, I could probably do it, but then if you ask me to explain how I did it, I don't know.
59 MZ: OK. Prediction or approximation?
60 Helen: [pause] I don't know.
61 MZ: Optimization? [pause] Do you know what I mean by optimization?
62 Helen: Is that like finding the greatest value?

63 MZ: Yeah, that's like the cost example that you were already talking about. Continuity, what does continuity have to do with --?
64 Helen: Well, if it's not continuous at the point then you can't find the derivative.
65 MZ: True. What about if it is continuous can you tell?
66 Helen: You can't always find the derivative if it's continuous.
67 MZ: Well, what's an example of a function that's not differentiable at some point?
68 Helen: Like if there's a cusp.
69 Helen: Then it would still be continuous there.
70 MZ: It would still be continuous, right. Do you know how to write the equation of a function that has a cusp? [pause, Helen shakes head no] OK. Limit. What does limit have to do with derivative?
71 Helen: Oh, it is a limit. It's a limit of the equation -- [pointing to the difference quotient she gave earlier]
72 MZ: Oh OK, so the derivative was not just this equation --
73 Helen: No, it's the limit of that.
74 MZ: The limit of this. The limit of this as what?
75 Helen: x approaches change in x .
76 MZ: Integral? You guys didn't do integrals yet?
77 Helen: No.
78 MZ: I was going to ask you for a formal definition of derivative, but this is kind of what I was thinking of for the formal definition. [pointing to limit of difference quotient]
79 MZ: How does this formal definition relate to sort of a more informal definition?
80 Helen: I'm not sure what you mean.
81 MZ: Hmm. Maybe I should ask this in a different way. Forget that question. I'm going to start somewhere else and maybe get back. If you were going to explain to somebody in the AB class who hasn't quite gotten to derivatives yet what a derivative is, what would you tell them?
82 Helen: I would probably tell them that it was like -- use that equation.
83 MZ: Use this equation? [indicating the limit of the difference quotient]
84 Helen: Yeah. I don't know how I would say what it does, but -- I don't know. [pause] I wouldn't want to say like you take the exponent and you multiply it times --
85 Helen: Because they wouldn't understand what they're doing.
86 MZ: Yeah, so if they have this it's more -- What if you were going to explain to somebody who -- somebody who doesn't really know much about math so they're never going to understand this equation, but maybe you could give them some sort of general idea what a derivative is. [long pause] Kinda hard I guess. [pause] OK. Well, I guess that's it. Thanks.

Table A.22 summarizes Helen's first interview transcript. Helen's focus is on the symbolic difference quotient definition of derivative. The difference quotient without the limit is her only answer to "What is a derivative?" After being asked to relate limits and derivatives she remembers that the difference quotient is preceded by a limit but states it as $\lim_{x \to \Delta x}$. This incorrect limit of the difference quotient is her response to how she would

Table A.22. Helen: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?				◎	
What can derivatives be useful for?				↦	max/min
Is derivative related to slope?	○				
Is derivative related to speed or velocity?			○		
Is derivative related to change or rate of change?			○		acceleration
Is derivative related to line or linear?	○				
Is derivative related to optimization?					max/min
Is derivative related to limit?				◎	misstated the limit
Explain what a derivative is to someone who's an AB student or precalc student who hasn't studied it yet.				◎ ↦	misstated the limit
Summary	○		○	◎	

explain derivative to another student. Helen relates derivative to slope and velocity only when asked directly. When asked if derivative is related to rate of change, she mentions acceleration but does not discuss rate of change more generally. Except when stating the limit of the difference quotient definition, Helen does not describe the details of the ratio or mention the limit or function layers in either pseudostructural or operational form.

When asked if derivative is related to rate of change, Helen thinks first of acceleration [ln 30-31]. This is the only occasion where Helen make a connection between different models for derivative. When asked specifically to relate the formal symbolic definition to slope, she has no answer.

QOTD #6

Find the derivatives of the following four functions:

$f(x) = (x-1)^2(x^2-4)$ $\qquad$ $h(x) = \sin x$

$g(x) = \dfrac{x-1}{\sqrt{5-x^3}}$ $\qquad$ $j(x) = \ln x$

Date: September 20, 1993. This question occurs prior to the class learning about short-cut rules for taking derivatives of various forms.

Response:

$f'(x) = (x-1)^2(2x) + (x^2-4)(2)(x-1)$

$g'(x) = \dfrac{(5-x^3)^{\frac{1}{2}}(1) - (x-1)(\frac{1}{2})(-3x^2)(5-x^3)^{-\frac{1}{2}}}{5-x^3}$

$h'(x) = \cos x$

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be using your knowledge of derivative.

$y = \log(x^3)$ $\qquad$ $y = \dfrac{x}{|x|}$ $\qquad$ $y = x^e$ $\qquad$ $y = e$

Date: September 21, 1993. This question also occurs before the class studies short-cut rules for taking derivatives but after they have studied the limit definition of derivative.

Response: "No clue."

Comment: This question is presented to the students since no student correctly stated the derivative of $y = \ln x$ in the previous Question of the Day.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$? Why <u>not</u>?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: "a) No because if y=tanx than $y = \frac{\sin x}{\cos x}$ and then the quotient rules apply

b) $\frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \sec^2 x$."

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. Helen shows that she can correctly state the limit of the difference quotient definition of derivative and use it symbolically to calculate a derivative value. She also knows to estimate the derivative at a particular point by finding two nearby points and calculating a difference quotient for these two points. Given the graph of a position function for a car she correctly answers questions about the speed and direction of the car.

Given the graph of a function, Helen is able to sketch a correct graph for the derivative function. Her paper emphasizes the connection between the extrema of the function and the zeros of the derivative as well as the connection between the extrema of the function and the inflection points of the derivative. Helen also works through two complex chain rule derivatives with one minus sign error for the derivative of the tangent function and, as part of a chain rule, writes that the derivative of $2x + 1$ is $2x$.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: "I knew that the derivative was a limit but I didn't really think about it. Now, when I think of the derivative I think of the limit right away."

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "a) We reviewed our homework and then talked about critical pts and monotonic equations. b) I learned what a monotonic equation is."

Comment: Since the researcher had not been present the day prior, this question is presented both as a means for the researcher to see the material covered and to ascertain the students' understanding of it.

Test 3

About two weeks later the class is tested on Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Helen correctly calculates a third degree Taylor polynomial. She correctly uses the first and second derivatives of a position function to find the speed and acceleration of an object at a given time. She is able to use the graph of a derivative function to estimate when the original function is increasing or decreasing, concave up or concave down, and where it has extrema. Even though she correctly lists the location of all extrema on a previous problem, she omits two of these x values from her list of critical points. When asked for the location of the inflection points of the original function, she notes the location of the local maximum of the derivative and where the derivative is undefined. She omits the location of the local minima of the derivative.

QOTD #11

Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: "If you have x amount of fencing and you want to enclose a square section on your field, find the greatest possible area."

Test 4

Two weeks later the class has a test on the applications of derivatives. Helen correctly uses derivatives to solve three traditional max/min problems and three traditional related rate problems. She also correctly calculates the derivative of an implicitly defined function. The only error she makes is the consistent omission of a division sign in her derivative expressions. In other words, she writes $\begin{matrix} dy \\ dx \end{matrix}$ instead of $\frac{dy}{dx}$, $\begin{matrix} dr \\ dt \end{matrix}$ instead of $\frac{dr}{dt}$ and so forth. No other students in the class make this error.

Interview 2

The second interview occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by Table A.23 and a discussion.

1	MZ:	What is a derivative?
2	Helen:	The derivative is the limit of--
3	MZ:	Actually, why don't you write that for me. It'll be easier.
4	Helen:	OK. [writes: $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$]
5	MZ:	Does anything else come to your mind for what a derivative is in addition to that?
6	Helen:	Not really. I just picture the limit approaching the equation. That's about it.
7	MZ:	Oh. How do you picture the limit approaching the equation?
8	Helen:	Nothing specific. I just think of a limit, just like any other limit.
9	MZ:	Just the idea of getting closer to.
10	Helen:	Right.
11	MZ:	OK. What are derivatives useful for?
12	Helen:	Optimization, related rates, graphing things.
13	MZ:	How would you explain what a derivative is to someone who doesn't have very much math background?
14	Helen:	I don't know. I still don't know how to answer that. I would probably tell them that [points at difference quotient].

15 MZ: What if it was somebody who was clueless? They're not even going to know what a function notation is.
16 Helen: I don't know. I really don't know.
17 MZ: OK. What's an example of a real world situation that involves the concept of derivative?
18 Helen: Like if you have a cost function. Like if you're trying to find out the most money you can make and you know how much certain things are going to cost, you can decide how much you need to make to make the most profit.
19 MZ: OK. So you're trying to find the most profit. So how does the derivative fit into that scenario?
20 Helen: Like how would you figure it out?
21 MZ: Yeah or-- 'Cause like the description you gave, I know exactly what you're talking about, but you didn't use the word derivative anywhere in there. So I didn't see how derivative fit into the scenario. [short pause] If it's easier you could just say how you figure it out, if that makes more sense.
22 Helen: I don't know. Once you have your equations-- I don't know.
23 MZ: I don't know if there was a cost function on the test. There wasn't, was there? I guess there was in some of the homework.
24 Helen: Like if you have-- If you have to fence in an area and you want to get the biggest area and you have this much fencing and you're doing it with rectangles. And you want to figure out-- Like you would figure out an equation for x and y , like for the perimeter. And then you would say the area equals x times y . And then you'd get it so you just have x in the equation. Then you take the derivative of it and you'd solve for 0.
25 Helen: And then you just figure out what x is.
26 MZ: So when I take the derivative and solve for 0, are you thinking of setting it equal to 0 and then solving for x ?
27 Helen: Yeah.
28 MZ: Then how come it turns out that where the derivative is equal to 0 gives you the point you're looking for?
29 Helen: It's like when you're graphing, when you're doing the first derivative test, where ever it's 0 is going to be a maximum or minimum.
30 MZ: How come it turns that where the derivative is equal to 0 there's a maximum or a minimum?
31 Helen: I don't know.
32 MZ: OK. Say I give you a function, how do you know if that function is differentiable?
33 Helen: If it's continuous at the point that you want it to be differentiable.
34 MZ: OK, so we're doing it at a point.
35 Helen: I don't think it has to be defined there, taking the limit as it approaches that point.
36 MZ: So not necessarily defined, but it just has to be continuous?
37 Helen: Right.
38 MZ: OK. Not necessarily defined. Anything else? So as long as it's continuous, it'll be differentiable and you don't need--
39 Helen: [agreement]
40 MZ: Seems reasonable. Can you think of a function that wouldn't be differentiable, I mean, at a point?
41 Helen: [short pause] If it has like a short turn or a sharp point like that.
42 Helen: [Helen sketches a curve that is increasing and concave up on the left of the cusp and decreasing and concave up on the right of the cusp.]

43 MZ: OK. So, what is it about this sharp corner that makes it not differentiable there?

44 Helen: Because the slope is going up and up and up and then all of a sudden it's going down right away and you can't really say what it is. It's like approaching that, so there's two different slopes on either side. You can't find the limit there.

45 MZ: OK. I think that makes sense. Here's this list of words that we're going to see if they have to do with derivative. The first one is, does slope have to do with derivative?

46 Helen: Yeah, because the derivative is-- If you're graphing something, the slope of your original graph is the derivative.

47 MZ: OK. How about speed or velocity?

48 Helen: If you know the position of something at a certain time, you could find out it's speed. Like if you can differentiate the equation.

49 MZ: Find the derivative of the equation for the position?

50 Helen: Right.

51 MZ: OK. Change or rate of change?

52 Helen: I mean, you could find it. [laughs] I don't know.

53 MZ: I mean, so I can find it somehow related to derivative then.

54 Helen: Right.

55 MZ: OK. Line or linear?

56 Helen: If you have a graph of like a parabola or something where the highest power is x^2, then your derivative will be a line. If you have a line for your original graph then your derivative will be $y = 0$.

57 MZ: OK, so if I have a line, I take the derivative, I get $y = 0$.

58 Helen: Right.

59 MZ: OK. Measurement?

60 Helen: You can find the error in the measurement using the derivative.

61 MZ: Can you describe how the derivative helps you find the error or how it fits in to finding the error?

62 Helen: I mean, if you had a problem, I could probably solve it, but I don't know.

63 MZ: Yeah. Prediction or approximation?

64 Helen: [short pause] I don't know.

65 MZ: OK. Limit?

66 Helen: The derivative is a limit.

67 MZ: Do you know what is meant by the antiderivative?

68 Helen: Yeah, that's where you have the derivative and you can find the original equation.

69 MZ: So the original equation then, is that the antiderivative part?

70 Helen: Right.

71 MZ: OK. Is the derivative a function?

72 Helen: Yeah, I think so.

73 MZ: Why does it seem like it must be?

74 Helen: Because when you take the derivative usually you're taking the derivative of a function so I guess it should be a function.

75 MZ: How does this formal definition relate to slope?

76 Helen: Well, if you have the equation of like a graph, then you would just plug that into here and then that would give you your slope. And then if you wanted to find the equation of your tangent line there, you would just plug in two points, like the x value at that point where you wanted the tangent line.

77 MZ: OK, so first you would use this to get the slope and then you would use like the x value or whatever. OK. Can you relate this to rate of change?

78 Helen: Well, when I take the equation that's what I'm using. I'm just using a simple method instead of like writing that out every time.

79 MZ: Have you ever seen this and if so do you think it has anything to do with derivative? Newton's method?

80 Helen: I remember something about x' or the derivative at x, but I don't-- I remember that it helps you to find x -intercepts or like guess what they are, but I don't remember how.

81 MZ: So you're thinking it has something to do with derivatives because you are remembering the x' part?

82 Helen: Right.

83 MZ: Intermediate Value Theorem?

84 Helen: That's familiar. I know we've used it. I just don't remember which one it is.

85 MZ: OK. How about this one, Mean Value Theorem? Heard of that one?

86 Helen: Yeah.

87 MZ: So both of these it's something you've heard of but--

88 Helen: Right. I know we use it, but I'm not sure exactly what it is.

89 MZ: Say I gave you the derivative of some function, what kinds of things can you tell me about the original function?

90 Helen: I can tell you where it has a maximum and a minimum, and I can tell you where the slope is increasing or decreasing on the original function.

91 MZ: Do you think it would be easier for you to find the max and min and stuff like that if I gave you the graph of the derivative or if I give you the equation for the derivative?

92 Helen: Probably the equation because on the graph it's a lot less precise. You have to like get out your ruler and measure.

93 MZ: Yeah, you have to be real careful. I'm going to give you a graph, but don't stress about doing the preciseness, just the general idea stuff. Here's the graph of the derivative. See what you can tell me about the original function and maybe even be able to make a little sketch of it.

94 Helen: OK. At negative one half there's a min. Wait. No, there's a max because the equation is positive here and then negative so like if you did the first derivative, it'd be going up and then down. So there's a max there, and over here there's a min because it's negative and then positive. And right in here it would be going down, and then these two points it'd be going up.

95 MZ: On either-- Outside of that? Yeah. OK. Can you tell me anything about the concavity or inflection points of the original function?

96 Helen: [short pause] Those would be at the max's and min's because those would be where the second derivative is equal to 0.

97 MZ: OK.

98 Helen: I know there's going to be a change in concavity here and here.

99 MZ: Yeah, at that first max and at the min. OK.

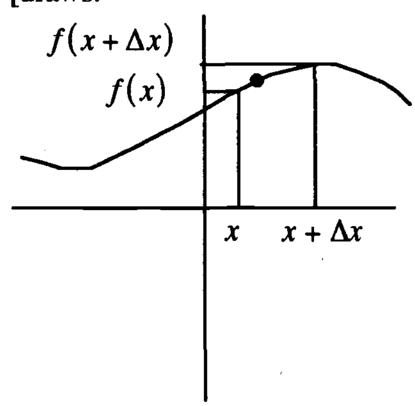
100 Helen: I could figure out which way it's going if I drew a graph of the derivative, but that'd be kind of ugly.

101 MZ: Oh, if you drew a graph of the derivative of this, you mean.

102 Helen: Yeah.

103 MZ: Well, see if you can piece together the information that you do have and see if you can draw a graph of the original function. Just a sketch, not anything perfect.

104 Helen: [sketches for a while, laughs]
105 MZ: Right, right. There we have it. We have the max at one half and a min about 2 or whatever that is.
106 Helen: And it could be anywhere up or down.
107 MZ: OK. So you could have drawn a similar graph higher or lower and have the same derivative.
108 Helen: Right. And I just remembered, this would be like concave up because the slope is increasing there.
109 MZ: Oh, the first little bit there.
110 Helen: Right, and this would be concave down because it's a negative slope.
111 MZ: Oh, OK. Yeah, that's kind of what you drew here without even trying to draw it. OK. Last question. Do you happen to remember that little picture that [Mr. Forrest] likes to draw when he's talking about this limit of the difference quotient thing?
112 Helen: I could draw one kind of like it.
113 MZ: OK.
114 Helen: [starts drawing] There's a point in here somewhere. [Helen marks a point on the smooth curve between $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$] [draws:



]

115 MZ: OK.
116 Helen: Actually, I think I'm just doing the limit in general. Maybe not. We'll see.
117 MZ: OK. Keep going. [short pause] So we have x and $x + \Delta x$ and we have $f(x)$ and $f(x + \Delta x)$. So I guess this fits in all these little parts.
118 Helen: This is at a point. Get rid of that [crosses out extra point marked earlier].
119 MZ: So we're suppose to be focusing in on the $(x, f(x))$ point. Let's see. What else? What is this quotient suppose to be related to in this picture?
120 Helen: It's kind of the slope, like from here to here. If you just took y minus-- You know, $y_2 - y_1$ over $x_2 - x_1$.
121 MZ: OK, and that's what this is supposed to be, the slope between those two.
122 Helen: Right.
123 MZ: And then where's the Δx on the picture?
124 Helen: It's the difference between these two.
125 MZ: OK.
126 Helen: So when Δx is closer to 0 the slope is more accurate.
127 MZ: You mean it's more accurate as to the slope at that point?
128 Helen: Right.
129 MZ: OK, thanks. That was good.

Table A.23. Helen: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?				◎	
What can derivatives be useful for?		○			optimization related rates
Explain what a derivative is to someone who doesn't know anything about math.				◎	
Give an example of a real world situation that involves the concept of derivative.					max/min
How can you tell if a function is differentiable?	○ ●				
Is derivative related to slope?	○				
Is derivative related to speed or velocity?			↦ ○		
Is derivative related to change or rate of change?		•			
Is derivative related to line or linear?				↦	
Is derivative related to limit?				◯	
Is derivative related to integral?				↦	
Is the derivative a function?				↦ ◯	
How does the formal definition of derivative relate to slope?	○			◎	
Given the derivative, what can you tell me about the original function?	○				max/min in/decreasing inflection pts concave up concave down
Do you remember the picture that Mr. Forrest draws when discussing the limit of the difference quotient definition of derivative?	◎			◎	
Summary	◎	○	○	◎	

From Table A.23, it is evident that Helen mentions a symbolic interpretation of derivative first and more often than any other interpretation. Helen mentions a graphical interpretation frequently as well. She only mentions velocity and rate of change when asked specifically. The one other mention of rate is that related rate problems are a use of derivatives. Helen is able to explain the details of the ratio and the limiting process in terms of slope and the formal symbolic definition, but she does not even mention the instantaneous nature of the derivative in terms of velocity or rate of change.

In Table A.23, Helen is given credit for knowing that derivative is a function when she states the formal definition as $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. This notation is correct for defining the derivative function, but Helen may think of x as a specific point. When asked specifically whether the derivative is a function, she guesses that it is by saying, "when you take the derivative usually you're taking the derivative of a function so I guess it should be a function" [ln 74].

When asked to explain how the formal definition of derivative is related to slope or rate of change, Helen states that she could use the formal definition to calculate the derivative and then plug in the x value of interest to find the slope or rate of change [ln 75-76]. At the end of the interview Helen is asked an unscripted question to determine whether Helen has a more detailed understanding of this relationship than she has stated. She is asked whether she remembers the picture that Mr. Forrest likes to draw when explaining the limit of the difference quotient. Helen draws a curve with two points labeled $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$. She explains that the difference quotient represents the slope between those points and that as Δx approaches 0 the slope of the line will become closer to the slope of the line at $(x, f(x))$.

In the last part of the second interview Helen is asked to graph an original function when given the graph of the derivative function. She sketches a reasonable graph. Although earlier in the interview she cannot explain why maxima and minima occur where the derivative is equal to zero, here she can at least determine which zeros of the function are maxima and which are minima. She can tell the concavity based on whether the slope of the derivative is increasing or decreasing. She knows that the curve she has drawn could be located at any height.

In comparing Helen's responses during the first two interviews it is clear that she states a more complete understanding during the second interview . Her replies in both interviews are similar in that a symbolic interpretation is the dominant representation. However, in the second interview, slope is also mentioned frequently. Helen's first interview contains no descriptions of the details of the ratio or limit processes except in the statement of the formal symbolic definition. In the second interview she states these in terms of slope as well as the symbolic definition and is able to relate the two. Unlike in the first interview, Helen can now state the correct limit in the limit of the difference quotient definition.

QOTD #12

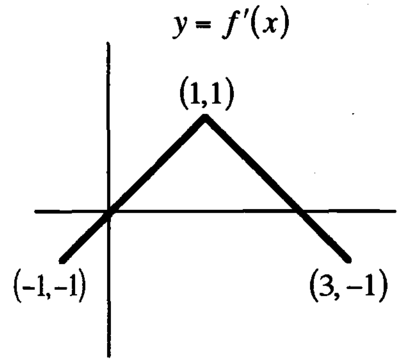
What is the most important idea that we have studied so far in this class?

Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

Response: "Area & Antiderivatives"

Interview 3

The third interview occurs during the three days after the test on differential equations and antiderivatives by substitution and by parts. The first part of the interview is a summary of Helen's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike the second interview, the point $(1,0)$ on the original function is given so that only one solution is



1 MZ: The first questions is -- It's actually more of a problem instead of an abstract question. This is going to be the graph of a derivative [sketches x and y -axes with hash marks]. So this is $(1,1)$ and this is $(-1,1)$ and this is the line connecting them [marks those points with a dot and sketches in line connecting them]. And this is $(3,-1)$ and this is the line connecting them [marks the point with a dot and connects it to the point $(1,1)$ with a straight line]. So you have this function -- and don't worry about what happens outside -1 and 3. So this is $y = f'(x)$ [writes $y = f'(x)$ next to graph]. And you also know that for the original function f, $f(1) = 0$ [writes $f(1) = 0$]. And then so the idea is: draw the graph of the original.

2 Helen: [17 second pause] Oh that would be --

3 MZ: What were you thinking?

4 Helen: I was accidentally taking the derivative of that [$y = f'(x)$] instead of -- so that would be 1 -- [inaudible] -- -1.

5 MZ: Oh, yeah. Right, right.

6 Helen: [15 second pause] I don't know. [quietly] [20 second pause] I don't know. I don't know what to do.

7 MZ: OK. What were you thinking of? Did you have any ideas of what might work or --

8 Helen: Not really.

9 MZ: Uhm, do you know how to figure out, if this is the derivative, where a max or a min on the original function might be?

10 Helen: Well, it would be at these two points. [circles two x -intercepts of f']

11 MZ: OK, so where the derivative crosses the x -axis?

12 Helen: Yeah.

13 MZ: And uhm, which one would be the max or the min?

14 Helen: This would be the max. [points to intercept at $x = 2$]

15 MZ: The right side one with the $x = 2$?

16 Helen: Yeah.

17 MZ: OK. What else can we tell? You have this point so that helps too. [points to statement $f(1) = 0$]

18 Helen: Mm hmm.

19 MZ: Do you know where there would be an inflection point of the original?

20 Helen: Well -- [5 second pause] I mean, I don't know.
21 MZ: Well, why don't you just sketch what we know so far. [Helen starts drawing an axis and marks $(1,0)$.] Throw that on there.
22 Helen: I mean, I don't know where to put it though.
23 MZ: Oh, you mean how high for the max and min?
24 Helen: Yeah.
25 MZ: That's true. Uhm, is there any way to figure out how high?
26 Helen: I don't know. I mean --
27 MZ: [10 second pause] OK. Let me ask you one other question I was asking some people. This is the derivative right. On the second derivative, what's the value of the second derivative at that point, $x = 1$?
28 Helen: It would be undefined.

Helen initially has no reply to a request to sketch the graph of a function given the graph of its derivative. She states that she does not know what to do. This is surprising since she sketches a reasonable graph in answer to a similar problem in the second interview. Her reluctance, however, belies some knowledge. When asked specifically about the locations of extrema, Helen immediately answers correctly, stating the locations of the extrema and whether each is a maximum or a minimum. Her reluctance to sketch the graph is apparently due to not knowing "where to put it" [ln 22], how high to locate the maximum and minimum. Note that Helen does not mention techniques such as slope fields and areas, which have been covered in class since the second interview and which could be helpful in solving this problem.

Another student might make a rough sketch based on the extrema locations and the given point $(1,0)$. Even Helen was willing to do this in the second interview when no initial value was given, and she knew that the graph might be located at any height. However, throughout the interviews, Helen seems reluctant to discuss her ideas unless she is sure that they are correct. She rarely makes a misstatement, but she sometimes fails to demonstrate the full extent of her knowledge.

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

29 MZ: Right. What's a definite integral or what comes to your mind?

30 Helen: Area. The area between x and the curve and between two points.

31 MZ: OK. What about an indefinite integral? How's that different?

32 Helen: Uhm, you don't really -- it's the antiderivative and you don't have -- it's not between two specific points.

33 MZ: OK. How is the derivative related to the integral?

34 Helen: [15 second pause] I don't know.

35 MZ: I mean, I guess -- I mean you already said the antiderivative. How about -- do you remember what the fundamental theorem says at least sort of? [Helen makes a face; MZ laughs]

36 Helen: Which one?

37 MZ: Which one is easier?

38 Helen: I don't know. I know the second one.

39 MZ: OK. What does the second one say?

40 Helen: Uhm, wait a minute. Maybe I don't. [short pause then writes $\int_a^b f(x)dx = F(b) - F(a)$]

41 MZ: [reading what Helen has written] And then, what's the distinction between the f and the F?

42 Helen: Uhm, this is the antiderivative of this equation. [pointing to F]

43 MZ: So the F is the antiderivative.

44 Helen: Right.

45 MZ: Yeah, that's the second one. Any remembrance about the first one?

46 Helen: Not really.

47 MZ: OK. Do you remember when we did that thing called the area function?

48 Helen: Mm hmm.

49 MZ: With the $A(x)$ thing?

50 Helen: Mm hmm.

51 MZ: Do you remember what that was supposed to be?

52 Helen: Uhm, I did. It's the area under a curve. It's -- I think it's the antiderivative also. I don't know.

53 MZ: Is it the same area under the curve as this? [pointing to left side of second Fundamental Theorem equation] Area under the curve from a to b?

54 Helen: I think so.

55 MZ: How about slope fields? Do you remember doing those?

56 Helen: Yeah.

57 MZ: Does it seem like those have more to do with derivatives or integrals?

58 Helen: Uhm, I don't know. [short pause] Derivatives maybe.

59 MZ: Why does it strike you --

60 Helen: Well, I just mostly think about the antiderivative, not really the derivative.

61 MZ: When you think of slope fields?

62 Helen: Yeah.

63 MZ: What are slope fields used for?

64 Helen: Uhm, it gives you like a general picture of what the graph's going to be like. And then you just have to figure out where the [inaudible], like what y coordinates.

65 MZ: Mm hmm. Here's another problem for you. Say you have $y = x^2$. And you want to know the area under $y = x^2$ from 0 to 1. [talks while

sketching x and y axes with graph of $y = x^2$, labeling as such and shading the appropriate area]

66 Helen: Mm hmm.

67 MZ: How would you do that?

68 Helen: [short pause] Uhm, I would probably do like Simon's or Simpson's rule or whatever his name is.

69 MZ: Oh yeah, Simpson's rule?

70 Helen: Yeah.

71 MZ: Can you remember how that works at all?

72 Helen: Yeah you take -- you find like from the left boundary -- you find the area and then you find it from the right boundary. Then you add those together and you divide by two and then you take the midpoint one and -- you multiply that times two and you divide it all by three, everything together by three.

[writes: $\dfrac{\frac{L+R}{2}+2M}{3}$]

73 MZ: Right. Well, that's really good. Uhm, do you know of any other way to do it, that's more-- where you don't have to add up the little rectangles?

74 Helen: You could use the second fundamental theorem. If you took the antiderivative of x^2 I think -- and then you plug in 1 and 0.

75 MZ: OK, why don't you just do that calculation real quick.

76 Helen: [writes: $\frac{1}{3}x^3 = F(x)$

$\frac{1}{3} - 0 = \frac{1}{3}$]

77 MZ: OK, I think that's it. You're done.

Helen associates definite integral with area and indefinite integral with antiderivative. She knows that derivatives and antiderivatives, hence derivatives and indefinite integrals, are opposite operations. She remembers how antiderivatives are used in the Second Fundamental Theorem and that the difference of the two antiderivative values gives an area. She also remembers studying ways of approximating area with sums, including how to combine left, right, and midpoint sums to calculate Simpson's rule, but she cannot recall the First Fundamental Theorem of Calculus.

When prodded about area functions she says, "It's the area under a curve. I think it's the antiderivative also" [ln 52]. When asked whether slope fields have more to do with derivatives or integrals, Helen responds, "Derivatives maybe. ... I just think more about the antiderivative, not really the derivative" [ln 58-60]. She already connects antiderivatives and area in terms of the Second Fundamental Theorem, but she does not try to explain why slopes and areas should be related in this way.

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: While it is recorded that Helen answers this question incorrectly, her exact response is not recorded.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Helen's exact answer is not available. It is known, though, that Helen answers the question correctly.

Test 9: Semester final

This test, which is a cumulative semester exam, covers all of the material on functions, limits, derivatives, areas, and volumes. The test questions are largely computational. On the semester final, Helen correctly solves an optimization problem as well as questions of domain, range, inverse functions, and continuity. She also computes limits, derivatives, and integrals correctly. She makes two computational errors that involve a forgotten chain rule and an incorrect substitution in solving a definite integral.

QOTD #15

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: Helen includes a sketch of the graph of the function $f(x) = x^{2/3}$. She also writes, "continuous from $(0, \infty)$ differentiable from $(0, \infty)$"

Interview 4

The fourth interview discussion is broken into four parts. The first section includes general questions about derivatives. The second part asks the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity, and acceleration given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the interview.

1	MZ:	What is a derivative?
2	Helen:	It's just a specific limit.
3	MZ:	Do you want to write it down, what you were thinking of, or do you want to just tell it, explain it?
4	Helen:	I don't remember, but I'll see if I can write it.-- [writes: $\lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$]
5	MZ:	OK. What else do you think of when you think of what a derivative is?
6	Helen:	Like a tangent line usually. Like if I picture a graph, I just picture a tangent line going to some kind of curve.
7	MZ:	OK. Anything else?
8	Helen:	Not really.
9	MZ:	Have you ever heard when sometimes people say the derivative is instantaneous rate of change?
10	Helen:	Mm hmm. [yes]
11	MZ:	What do you think they mean by that?
12	Helen:	Just-- I mean, it's a way of finding the speed at one exact moment instead of like the speed from one time to one time, you know. You know what I mean?
13	MZ:	OK. So the instantaneous part is referring to the--
14	Helen:	So you can pick any particular second or minute or whatever.
15	MZ:	And then the rate of change part is referring to something like speed? That's what you were thinking of?
16	Helen:	Yeah.

As in previous interviews, Helen states the limit of the difference quotient definition as her first answer to what a derivative is. When asked if anything else comes to mind, she mentions a tangent line, explaining further, "If I picture a graph, I just picture a tangent line going to some kind of curve" [ln 6]. Helen probably does not think that the derivative is the tangent line itself. She does not make this misstatement in other interviews. In fact, in the second interview she gives a clear explanation of how the difference quotient describes a slope. However, it is interesting that her fourth interview description does not include the word slope. The tangent line to a curve seems to be a particularly salient image whereas the slope of that line is a more subtle commodity.

When asked to explain what is meant by instantaneous rate of change, Helen responds, "It's the way of finding the speed at one exact moment." This is similar to her first interview response when she is asked if derivative is related to rate and she mentions acceleration without discussing rate more generally. Even in the second interview she only acknowledges that taking the derivative finds the rate of change and does not discuss it further during any part of the interview.

The next part of the fourth interview is a summary of Helen's solution to the first of three problems involved in this interview. Given a table of values with x varying by .1, Helen is asked to estimate $f'(2)$, the derivative of the function at $x = 2$. Helen states that she can calculate an estimate by finding the slope between the two points in the table with x values of 1.9 and 2.1. She picks these values, the closest points listed on either side of 2, because using them gives "the closest [value] to the instantaneous." When asked if there is a way to get a better estimate, she recognizes that points with x values closer to 2 would give a better estimate. Her statement hints at, but does not explicitly state, a limiting process for finding a more accurate estimate.

That Helen uses the slope model of derivative to calculate the ratio for her estimate of $f'(2)$ is clear not just from her statement, but also from the structure of the ratio she chooses. Helen regularly states the definition of derivative as

$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ using this as her first answer to "What is a derivative?" However, she does not use this difference quotient to calculate her derivative estimate to this problem. If she had done so, she would have used the points with x values 2 and 2.1. It is interesting that Helen's consistent first response to the question "What is a derivative?" is not always her first response to the need for a model of derivative in a problem solving context.

The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward, increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car takes 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far does the car travel in the 2 hour period?

Helen's first reaction is, "I don't remember how to do that." She says that she first thinks the car travels 120 miles, but realizes that the car does not travel 60 miles per hour for the whole 2 hours. Otherwise, even when she is asked how calculus relates to velocity, she says she has no idea how to proceed . When asked explicitly for the relationship between velocity, acceleration and position, she says, "velocity's the derivative of acceleration, and position's the derivative of velocity ... It's either like that or backwards." Helen's confusion about position, velocity and acceleration is surprising since she does not make this misstatement in any of the previous interviews.

When asked to draw the graph of velocity for this situation, Helen draws a correct graph and notes that the lines are straight because the car has a constant rate of increase. When asked to draw the graph of the acceleration for this situation, Helen draws a correct piece-wise, constant curve. With these two graphs on the page, Helen now correctly states that the acceleration is the derivative of velocity and the velocity is the derivative of position.

When asked to draw the graph of the position function for this situation, Helen states that she could use the values from the velocity curves as slopes for each x value on the position curve. She also remembers that she can just find the area under the curve to determine that the distance traveled is 60 miles. Helen's ability to use slopes and areas to gain information about a function from the graph of its derivative is an improvement from her efforts on a similar problem at the beginning of the third interview. During the third interview, she does not think to use slopes or areas to solve the problem and must be prompted to use information about the derivative to determine the extrema of the function.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Helen is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. She is asked if the top of the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Helen's first guess is that the rate the ladder is sliding down is the same rate as it is being pulled out. When asked to use calculus to confirm her guess, Helen begins by labeling the wall as a, the floor as c, and the ladder as b. She writes $a^2 + c^2 = b^2$ and notes that she wants to take the derivative to have the velocity. She writes $2ada + 2cdc = 0$ and indicates that dc is the rate that we know. Note that she does not use a ratio, e.g. $\frac{dc}{dt}$, for rate. Helen's previous work on related rate problems does not show this notational error. However, since the related rate test five months previous, the class has covered integrals which include the dx notation as a common element.

Helen explains her notation by describing how she could find the rate the ladder is sliding down at any given time. Helen's explanation seems to recall the set-up of similar problems found in her textbook. She does not acknowledge that the interviewer's question is not about the rate at a particular instant but about how the rate changes over time. When asked specifically if the rate the ladder is sliding down would be the same at two different points, she guesses that it would be. Then, when given a specific rate of 3 feet per second for dc, Helen calculates that $da = \frac{-3c}{a}$ and is able to use this with given values for c to determine that the rate is not the same for two different values of c. Helen has difficulty thinking about the scenario without being given specific values to work with.

Interview 5

Helen's fifth interview occurs almost two weeks after she takes the BC version of the AP exam. During those weeks, the class discusses the written questions from the BC version. Between the fourth and fifth interviews the class studies series and integration techniques and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Helen's response to a set of questions on a particular topic and provides an analysis of those responses.

1 MZ: What are the things that you know about what a derivative is?
2 Helen: It's the slope of an equation or it can give you the slope of an equation at a certain point. I don't know. There's all kinds of stuff.
3 MZ: Well let's see, let's just list a couple different things that you can think of for what a derivative is.
4 Helen: Well, I'll just right it down. [writes: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$]
5 MZ: What else could we put on a list of what it is?
6 Helen: I don't know.
7 Helen: You can use it to relate speed and acceleration and position. On a graph, that's the same as the-- The slope of one graph being the points on another graph.
8 Helen: You can use it to find out when a square has its maximum area or something like that. Max/min problems.
9 MZ: Well, how about on the what it is part, how do these two things relate to each other?
10 Helen: This is another-- This is the slope on a graph.
11 MZ: The whole--

12 Helen: Yeah, it's the slope, and you're taking the limit of it. I think. Hmm, draw the graph. And say-- I don't remember what I'm trying to say. I don't know. It's just the slope.

13 MZ: When you say, it's the slope, are you referring to the whole thing with the limit?

14 Helen: No, without the limit.

15 MZ: Without the limit? This thing is the slope?

16 Helen: Right.

17 MZ: OK. And then how is it different when the limit it added?

18 Helen: It's just making it more-- It's the slope at one specific point.

19 MZ: And without the limit it's-- It's not the slope at one specific point?

20 Helen: I guess.

21 MZ: Would you be able to show on a graph where Δx would be and $f(x)$ and $f(x+\Delta x)$?

22 Helen: It's just like two different points. It's like Δx and $f(\Delta x)$ and then x and $f(x)$, and you're just taking the slope between those two points.

23 MZ: OK, so one of the points is $(\Delta x, f(\Delta x))$, what did you just say?

24 Helen: Oh wait. It's $x+\Delta x$ and $f(x+\Delta x)$.

25 MZ: OK. So that's one of the points.

26 Helen: Right. [sketching] This is x . $f(x)$. Then this is $x+\Delta x$. This little thing here is Δx. And then it's the slope, but this is going to be really really really small. [Helen has sketched a curve on a pair of axes with x and $x+\Delta x$ marked on the horizontal axis and $f(x)$ and $f(x+\Delta x)$ marked on the vertical axis. There are vertical and horizontal lines connecting these marked spots to the curve. The distance between x and $x+\Delta x$ is labeled as Δx .]

27 MZ: So without the limit, it's the slope of what line? Can you draw the line that it's the slope of, without the limit?

28 Helen: Whatever this line is between these two points.

29 MZ: Then when you have the limit, where is the line that you have the slope of?

30 Helen: Right at x , $f(x)$. The tangent at $f(x)$. It looks pretty much the same on this graph.

31 MZ: Say that we had a position function and we knew that derivative is the velocity or speed. How would all this limit, f, stuff relate to that situation?

32 Helen: What do you mean?

33 MZ: Well, say this f function is the position function.

34 Helen: Mm hmm.

35 MZ: Well, then what's this ratio equal to in that situation?

36 Helen: I don't know. Distance over time? Something like that.

37 MZ: Yeah. And what would be the difference between the ratio without the limit and the ratio with the limit in that situation?

38 Helen: It would be just like when we talked about this before. The slope between these two points. It would be like the average velocity between those two points, and then with the limit it's the velocity at one point.

39 MZ: Does the derivative involve a limiting process? Explain.

40 Helen: All I know is it's a specific limit, but I don't know what you mean by limiting process.

41 MZ: Explain what is meant by rate of change or rate of change of a function.
42 Helen: It's just the velocity or how fast it moves from one point to the next.
43 MZ: Can you think of an example of rate of change other than velocity?
44 Helen: [short pause] No, not really.
45 MZ: Is the derivative of a function a function?
46 Helen: It should be.
47 MZ: OK, why?
48 Helen: I don't know. It's just a guess. I mean, it seems like it should be because if your function is a polynomial, then you're just going to get another polynomial.
49 MZ: So, do you think it would happen for other cases also?
50 Helen: It seems like it should. I don't know. [short pause] I don't know.
51 MZ: Explain what is meant by a differentiable function.
52 Helen: A differentiable function is something that you can take the derivative of.
53 MZ: OK.
54 Helen: I can't think of any examples where you can't take the derivative. I mean, I know there are some. I know you can't take it at certain points sometimes. Like if there's a sharp turn or something like that, you can't find the derivative there.
55 MZ: OK. So what would be an example of a function that has a sharp turn, some kind of nondifferentiable point.
56 Helen: I don't know. I'm not good with that.
57 MZ: Can you sketch one?
58 Helen: Yeah, I can sketch one.
59 Helen: [Helen sketches a curve with a cusp pointing down.] Like right here [circles the cusp] it wouldn't be differentiable.
60 MZ: OK, and what is it about that situation that makes it not differentiable there?
61 Helen: Because right here it has a positive slope and a little bit over it has a negative slope. There's not much-- I don't know. It just changes real quick.
62 MZ: OK. It changes too quickly. We don't know an equation for that.
63 Helen: [Agrees.]
64 MZ: Explain what a derivative is without mentioning the symbolic definition, slope, rate of change or velocity and acceleration.
65 Helen: No. Those are the only things I can-- Those are the only ways I use it. I don't know where else it's used.
66 MZ: That's when I asked you about if there was another way to think of rate of change without velocity.
67 Helen: I mean there might be, but it's not something that I know.

Even though the portion of fifth interview focusing on general questions about the derivative has fewer questions than similar sections of first and second interviews, Helen gives almost as detailed a set of answers here as in the second interview. The principal weakness of her fifth interview is her lack of discussion of a limiting process. She writes the limit in terms of the formal limit definition, and she mentions the instantaneous nature of the derivative in terms of slope, velocity and a symbolic form, but she does not discuss

Table A.24. Helen: Interview 5 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎			⦿	
"What a derivative does" i.e. What are derivatives useful for?	∘		∘		max/min
How does the formal definition of derivative relate to slope?	◉			◉	
How does the formal definition of derivative relate to velocity when f is a position function?			◉	◉	
Does the derivative involve a limiting process?				○	
What is meant by instantaneous rate of change?		∘	∘		
Does the derivative involve a limiting process?					
Is the derivative of a function a function?				◯	
What is meant by a differentiable function?	∘				
Asked to interpret the Mean Value Theorem.	◉			•	
Asked to find the average rate of change of a function defined as an integral.		∘			
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.	∘	◎	∘		increasing max/min incorrect calc
Summary	◉	◎	◉	⦿	

a limiting process. When asked directly, Helen says she knows that the derivative is a limit (as seen in the formal definition) and that the limit means that the slope is "at one specific point", but she does not know what is meant by the phrase "a limiting process".

In fifth interview, Helen continues her emphasis on the symbolic representation. However, for the first time, Helen mentions something other than the formal limit definition as her first answer to, "What is a derivative?". Her first answer in the fifth interview is that the derivative is "the slope of an equation at a certain point" [ln 2]. Helen also continues her tendency to emphasize the specific example of speed or velocity over the more general statement that the derivative is the instantaneous rate of change. When asked specifically to do so Helen correctly connects the details of the ratio and the instantaneous nature of the derivative in the graphical and symbolic forms and in terms of velocity and the symbolic form.

For the second part of the fifth interview Helen is asked if she remembers the Mean Value Theorem. Helen's first thinks that it has to do with the slope between two points and "a derivative somewhere in between those two points that equals the slope." When asked to write something she draws a correct graph showing a curve with a secant line connecting $(a, f(a))$ and $(b, f(b))$ and a parallel tangent line at a point with x-coordinate marked as c. When asked specifically to write out her ideas symbolically she writes, "Between a and b there is some c such that $\frac{f(b)-f(a)}{b-a} = f'(c)$."

Note that Helen remembers the Mean Value Theorem through its graphical representation. This representation may help Helen condense her knowledge of the theorem in a way that is easier to remember than memorizing the words and symbols of the statement of the theorem. Helen demonstrates her knowledge of the details of the ratio and the instantaneous nature of the derivative in the graphical representation and knows the relationship of these layers to the symbolic representation. This connection is one that Helen explains in terms of the formal definition earlier in the interview. Here she uses the connection to recreate the verbal and symbolic statement of the theorem from her memory of the graphical representation.

The next question on the fifth interview involves a problem from the AB version of the AP exam which Helen has not taken. She reports that she has not seen the question before. The question is as follows:

Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.
(a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.
(b) On what intervals is F increasing?
(c) If the average rate of change of F on the closed interval $[1,3]$ is k , find $\int_1^3 \sin(t^2)\,dt$ in terms of k .

The interviewer asks Helen to discuss her methods of solution for parts (a) and (b), but does not require her to complete the solution of either part. For part (a) Helen correctly describes how to apply the trapezoid rule to find area under the curve.

For part (b) Helen's first idea is that she needs to find the formula for F by finding the antiderivative of $\sin(t^2)$, but she quickly realizes that she does not need it. She says, "Usually you would take the antiderivative, but then to find where that's increasing you take the derivative and find where that's positive. So anywhere this graph [a sketch of $y = \sin(t^2)$] is positive capital F should be increasing." Note that Helen is generally following a series of steps for solving problems of this type, but that she has consolidated this procedure so that she is able to look ahead and judge that finding the antiderivative is unnecessary.

For part (c) Helen's first statement is, "I think when you're doing the average you would do--" She writes, $\dfrac{\int_1^3 \sin(t^2)\,dt}{3-1} = k$. When asked what this is the average of, she thinks this is perhaps not the average rate of change that the problem is asking for, but then decides that it is the average value (an integral calculation) of a rate of change function (the rate of change of F being $\sin(t^2)$). From there she concludes that the integral is equal to $2k$. It is interesting that Helen separates the phrase average rate of change. She knows that the derivative is rate of change, but she does not often use this phrase in her general discussions of derivative. More specifically, she does not discuss average rate of change per se in this or early interviews, even though she does discuss average velocity or the slope of a secant line.

The next section of the interview concerns Taylor polynomials. Helen is asked what a function and its second degree Taylor polynomial have in common and how they differ. Helen responds that at one point they have the same function value and first derivative value, and she thinks they have the same second derivative. She is able to correctly calculate the second degree Taylor polynomial for $f(x)=e^x$ at $x=0$. When asked, she says the purpose of Taylor polynomials is "to approximate an equation." When asked how the graphs of a function and its second degree Taylor polynomial would compare, Helen says that they would have "close" tangent lines.

The final section of the fifth interview concerns a function, f, that at any time, x given in hours tells the outside temperature in degrees Fahrenheit. Helen is shown a series of symbolic expressions and asked what information each one provided about the outside temperature.

For $f'(3)=4$, Helen says, that after 3 hours the temperature is changing at 4 degrees per hour. For $f''(3)=-2$ Helen explains, "The change in the change in the temperature is -2." In both cases Helen explains that something is changing but she does not say whether it is increasing or decreasing.

When asked about the expression $f'(x)=4$ for $0\le x\le 3$, Helen replies that, "it's going at 4 degrees per hour for those three hours." When asked to interpret $f''(x)=-2$ for $3\le x\le 6$, Helen says, "The change in the change in the velocity--" before changing her statement to, "The change in the change in the temperature is at -2 degrees per hour squared ... between 3 hours and 6 hours." Note that Helen says the change in the "change" in temperature instead of speaking in terms of the rate of change of temperature.

She also unintentionally mentions velocity and seems to hint at the notion of velocity with the phrase "it's going at."

When asked specifically about what is happening with the temperature during this time period, Helen says that it is increasing. "It increases at the same rate from $[0,3]$, but then it slows. It's still increasing from 3 to 6 but not as fast." Despite this accurate description of the interval from 0 to 3, she initially says that she does not know how to estimate the temperature at 1, given that the temperature at 0 is 50, and that she is not sure whether or not that segment is linear. When asked whether it would be easier for her to find an equation for f or to sketch a graph of f, she chooses to attempt a graph.

Helen's graph incorporates a line segment with slope 4 for the interval $[0,3]$, and a concave down increasing curve for the interval $[3,6]$. Her only error is to not realize that the curve will decrease after $x = 5$. Given this graph she is easily able to estimate temperature values for any time. She is also able to write a correct equation for the line segment but says she does not know how to find an equation for the curve on $[3,6]$.

Helen's initial reaction to the symbols is stilted but attempts to use the metaphor of velocity. When she is asked more directly about the temperature she replies correctly but cannot at first relate her statement to numeric, graphic or symbolic information. Eventually it is her use of the graphical that leads her to further insight about the numeric and symbolic results. Her inability to use the numeric or symbolic results seems to directly keep her from realizing that the graph will eventually decrease.

Case Study 9 — Ingrid

Academic record

*National Merit Scholar.

*Other AP courses: US. History (junior year), English, Spanish.

*Writing tutor at the high school writing center.

*Undecided about her college major.

QOTD #1

What is a function?

Date: August 24, 1993. The question occurs before the class has reviewed functions.

Response: "A function is a mathematical expression of a certain variable."

QOTD #2

a) Give an example of two functions that are very different from each other. In what way are they very different?

b) Give an example of something that is not a function, but is almost a function. Why isn't it a function?

Date: August 25, 1993. The question occurs before the class has reviewed functions.

Response: "a) $f(x) = 2x + 1$ $\quad f(x) = 4x^2 - 6$

The first function is linear and the second is quadratic. They are very different because the graph of the first is a line and the second is a parabola.

b) $x = y^2 - 1$

This is not a function because it is a parabola going sideways [small sketch of something like $x = y^2$] and a vertical line passing through it would hit it more than one place."

QOTD #3

Give an example of a function <u>without</u> using an equation or a mathematical expression. If you can think of more than one way to do this, give more than one example.

Date: August 26, 1993. This question occurs while the class is doing a quick review of functions.

Response: Ingrid provides two different sketches in answer to this question. In the first sketch, there are two ovals with three points in each. The first oval's points are labeled x_1, x_2, and x_3. Similarly, the other oval has points labeled y_1, y_2, and y_3. There are lines with arrows to indicate a one-to-one mapping between the points across the ovals. The second sketch is a graph of the function $y = x - .5$.

QOTD #4

a) Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

b) Does there exist a function whose values for (all) integers are not integers and whose values for (all) nonintegers are integers?

Date: August 27, 1993. This question occurs while the class is doing a quick review of functions.

Response: "a) No because for the equation $f(x) = x^2 \quad f(0) = 0$.

b) $f(x) = x^{-2} \quad f(2) = \frac{1}{4} \quad f(\frac{1}{2}) = -4$ "

QOTD #5

What is a limit?

What is a limit of a function f at a point $x = a$?

Date: August 30, 1993. This question occurs prior to any class discussion on limits.

Response: "A limit is the point approached by both sides of the equation, from the right and from the left."

Comment: It is unclear to what "equation" Ingrid refers. Perhaps she uses the word equation to mean the same thing as the word function.

Test 1

On a test on limits a week later, Ingrid is able to find limits correctly by reading values from a graph, by substituting into a piecewise function, and by using algebra to simplify a limit calculation. However, she is not able to work with the formal definition of limit to find a δ for a given ε in either a symbolic or a graphical setting.

Interview 1

This interview occurs after the test on limits but prior the class's discussing derivatives. Therefore, Ingrid's answers are presumed to be based on what she remembers from her junior year study of derivatives or any homework completed over the summer.

An edited version of the interview is followed by Table A.25,. which codes these responses. A summary discussion follows.

1	MZ:	What do you remember, what is a derivative?
2	Ingrid:	A derivative is the slope of line tangent to a certain point on a location.
3	MZ:	What do you remember derivatives being useful for?
4	Ingrid:	A cone of water is filling up at the rate of this per this, and the rate is increasing at this per this. How much more is going to be in the pond or something.
5	MZ:	So which part of that does the derivative plug into?
6	Ingrid:	The rate of change.
7	MZ:	Do you remember any other types of uses?
8	Ingrid:	[pause] No.
9	MZ:	Can you think of any real world situations? I don't know if you count the cone filling up as a real world situation, but-- any real world situations that derivative applies to.
10	Ingrid:	[10 second pause] I guess problems like that I guess.
11	MZ:	OK. Say I give you a function, how can you tell if that function is differentiable?
12	Ingrid:	I could graph it and see if there's a problem, if there's like a sharp turn or something or a hole.
13	MZ:	So if there is a sharp turn or a hole, it's not differentiable? Is that what you were going to say?
14	Ingrid:	Yeah.
15	MZ:	Can you think of an equation of something that has a sharp turn, for example?
16	Ingrid:	[pause] I don't know.
17	MZ:	I believe you. If you had to explain to someone who is in precalc or AB what a derivative is, but they hadn't had it yet, what would you tell them?

18 Ingrid: I'd probably show them a graph and draw a tangent line. I'd say you can find the equation of any line, any tangent line, any line that's tangent tothe graph at any point, by taking the derivative. And instead of like showing them the short cut, the way you learn, you'd have to show it like-- x plus change in x or something minus x over x minus h or something like that.

19 MZ: Something like x plus change in x minus x over h? [writes: $\frac{x + \Delta x - x}{h}$]

20 Ingrid: That's probably not right. $f(x + \Delta x)$ minus $f(x)$ over change in x. Guess you could use h.

21 MZ: [Edits previous to: $\frac{f(x + \Delta x) - f(x)}{\Delta x}$] Something like that. So this is like the non-shortcut way.

22 Ingrid: Well, that's like the definition at a certain point of derivative, isn't it?

23 MZ: Yeah, something like that.

24 MZ: What if you had to explain to somebody who didn't know hardly anything about math, just sort of general interest?

25 Ingrid: I think I'd just do that with the graph and show them the line. Find the equation of that line.

26 MZ: How do you tell someone what a tangent line is?

27 Ingrid: It's a line that only touches the graph at one place.

28 MZ: [pause] Could it touch the graph at more than one place?

29 Ingrid: [pause] I don't know.

30 MZ: [sketches the graph of a tangent to a curve that also touches the curve further away again] Touches the graph at two places.

31 Ingrid: Mm. [inaudible] It could be the tangent is like where it's drawn.

32 MZ: So is this a tangent line? This one I just drew.

33 Ingrid: I don't know.

34 MZ: So. I'm going to show you a list of words and then you're going to tell me if a word has to do with derivative or not and if so what. So the first one is slope.

35 Ingrid: Well, derivative is the slope.

36 MZ: Yeah, you kind of already said that. Speed or velocity?

37 Ingrid: Well, you can find that if you have the position. If you have like $s(t) = x^2 + 1$. You can find the velocity by taking the derivative of that.

38 MZ: OK. Change or rate of change?

39 Ingrid: That's what derivative is too.

40 MZ: Line or linear?

41 Ingrid: You can use it to find the slope of a line.

42 MZ: True. What about linear?

43 Ingrid: Isn't there something like linear derivatives? I don't think I've heard it too much in conjunction with derivative.

44 MZ: Measurement?

45 Ingrid: [pause] [shrugs]

46 MZ: Prediction or approximation?

47 Ingrid: Well if you're doing those weird [inaudible] functions you can approximate how long it would take to fill up or something.

48 MZ: So you could use the derivative and do a calculation to approximate how long it would take. Is that what you're thinking?

49 Ingrid: Yeah.

50 MZ: How about optimization?
51 Ingrid: You use like the first derivative and second derivative test.
52 MZ: Do you remember how that works, how the derivative fits into those kind of problems?
53 Ingrid: I know how to do it. I think. Oh, it's like graphing, finding turns and stuff.
54 MZ: Oh, like where the graph turns somewhere?
55 Ingrid: Yeah. It's useful. But I don't know like why.
56 MZ: So, if uhm, if you were trying to optimize a certain situation, you said you knew how to do it. What's the general procedure that you know?
57 Ingrid: Oh. Well, you find the derivative and set it equal to zero and find it's roots. And then put it on a, what do you call it, sign graph. See where it's positive and where it's negative.
58 MZ: So by sign you mean like plus and minus.
59 Ingrid: Yeah, not a sine graph.
60 MZ: You say you find the roots. Is that root going to be where the extremum is?
61 Ingrid: If it goes from positive to negative on the sign graph at that place.
62 MZ: So if it goes from positive to negative on the sign graph it's an extremum.
63 Ingrid: Yeah.
64 MZ: And is it a max or is it a min?
65 Ingrid: I don't remember how to find out whether it's a max or a min.
66 MZ: Do you happen to remember why it turns out that the zeros of the derivative give an extrema?
67 Ingrid: Nah. I don't remember.
68 MZ: OK. How about continuity? Does it have anything to do with derivatives?
69 Ingrid: Well, it can be continuous and not differentiable, but it can't be not continuous and differentiable.
70 MZ: So what would be an example for something that's continuous and not differentiable?
71 Ingrid: Something that has a sharp turn.
72 MZ: How about limit?
73 Ingrid: You can define the derivative by using derivatives of the limit as x approaches change in x or something with this one [points to difference quotient previously dictated by Ingrid and written on the page by MZ].
74 MZ: OK. So as x approaches the change in x, take that limit, of this thing? [writes in front of diff quotient: $\lim_{x \to \Delta x}$]
75 Ingrid: And that'll be the slope of the derivative.
76 MZ: The slope of the derivative?
77 Ingrid: That's just-- The derivative is the slope so that's the slope of the tangent line.
78 MZ: So this thing is the derivative, and you used limits to do that so that connects those. I was going to ask you a more specific question, is the derivative a function?
79 Ingrid: Yeah.
80 MZ: How do you know or why would you think the derivative was a function?
81 Ingrid: It has to be a line. A line is going to be a function.
82 MZ: 'Cause the derivative's a line and then line's are functions. What if I asked you for a formal mathematical definition of derivative? Not like epsilon delta, that formal, but just like in the book it probably says, definition--

83 Ingrid: Is this like an equation or something? [pause] Are you talking like an equation or like words?

84 MZ: Well, actually I was thinking like equation. Like, I don't know, I mean you might answer this [points to limit of diff quotient] as the formal definition.

85 Ingrid: Is that right? I mean is that like an equation?

86 MZ: It's a limit.

87 Ingrid: But I mean-- Is that like an actual equation like that?

88 MZ: I think so. I mean, this is one way to say it. There's probably some other words in the sentence, but the derivative is, $f'(x)$ can be defined like that. What I was going to ask you about this is-- We have this formula and we know this is supposed to be the same thing as the slope of the tangent line. Why should this be the slope of a tangent line?

89 Ingrid: [long pause]

90 MZ: What are you thinking?

91 Ingrid: I'm trying to do an example. [pause]

92 MZ: What kind of example?

93 Ingrid: I'm thinking of like an equation when you-- I'm trying to think of the graph, like say x^2+1. So you find the equation of the derivative or something was $y=x$. I'm trying figure out how that fits in.

94 MZ: How it fits into this thing.

95 Ingrid: [long pause]

96 MZ: Is that true that if you have x^2+1, you have a tangent line $y=x$.?

97 Ingrid: I think so.

98 MZ: Let me draw it. [Sketches graph of those two curves on the same axes; the curves do not touch or intersect in this drawing.]

99 Ingrid: If the graph was a little better-- [Sketches the right part of the parabola a bit wider so that it almost touches the line.]

100 MZ: Well, it depends on the graph, right.

101 Ingrid: x^2+1. Oh, it's 2x. It's $2x$ not $y=x$. [Sketches a more accurate graph of $y=x^2+1$ and $y=2x$]

102 MZ: [long pause during sketching] So is that a tangent line?

103 Ingrid: Yeah. It would be.

104 MZ: A tangent line at that point. Do you know how to figure out what that point is that it's tangent?

105 Ingrid: Well, if you've got like $y=2x$ and you've got $y=x^2+1$, then you can get where they--

106 MZ: Thinking about figuring out where they intersect each other?

107 Ingrid: Yeah.

108 MZ: [pause] What if I had, my function was $y=x^2$ then what would an example of a tangent line to that graph be?

109 Ingrid: [Writes: $y=x^2$

$y'=2x$ Then sketches the graph of $y=x^2$ on the same axes as the previous two functions. $y=2x$ is clearly not tangent to the graph of $y=x^2$.]

110 MZ: [pause] Something's wrong with this picture because--

111 Ingrid: [pause] It could be like $y=x^2+2$, $y=x^2+3$ --

112 MZ: Right, right. Well, we don't have to figure this out right now. That was the end of my series of questions.

113 Ingrid: It couldn't be $x^2 + x$. It'd be $2x + 1$. [Writes: $x^2 + x \quad 2x + 1$] [bell rings]
114 MZ: Then you'd have a different one, yeah. OK. Well, thanks.

Table A.25 summarizes Ingrid's first interview transcript. Ingrid recalls that the derivative is related to slope, rate of change, and a symbolic difference quotient. However, slope is the model mentioned first and most frequently. When asked specifically, she also remembers that derivative is related to velocity. Ingrid states the instantaneous nature of derivative as slope, but not while discussing rate of change or velocity. She only remembers to add a limit to her difference quotient when asked if limit and derivative are related. Even then she states the limit incorrectly as $\lim_{x \to \Delta x}$. When asked whether the derivative is a function, she guesses that it is. She reasons, when pressed, that a derivative is a function because it is a line [ln 78-81].

The only connection Ingrid makes between different models of the derivative is an attempt to relate the graphical to the symbolic formal definition. She knows that the difference quotient has something to do with a tangent line, but cannot give any more detailed description of the connection.

Ingrid shows confusion about the relationship between the tangent line and the derivative function. A minor foreshadowing of this occurs when Ingrid says she would explain derivative to another math student by drawing a tangent line and explaining that "you can find the equation of any line that's tangent to the graph by taking the derivative" [ln 18]. This statement is perfectly accurate, but it does not mention the role of slope. Later, when Ingrid is asked whether a derivative is a function she says that the derivative is a function because it is a line [ln 81]. This statement is false since the derivative function is only a line in the case that the original function was quadratic or linear. Finally, Ingrid is asked to explain the relationship between the limit of the difference quotient and the slope of the tangent line. She chooses to focus on the example of $y = x^2 + 1$. She finds its derivative to be $y = x$, later corrected to $y = 2x$, and tries to

Table A.25. Ingrid: Interview 1 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎				
What can derivatives be useful for?		○			
How can you tell if a function is differentiable?				↦	
Explain what a derivative is to someone who's an AB student or precalc student who hasn't studied it yet.	◯			◉	
Is derivative related to slope?	○				
Is derivative related to speed or velocity?			○	↦	
Is derivative related to change or rate of change?		○			
Is derivative related to line or linear?	○				
Is derivative related to optimization?					extrema
Is derivative related to limit?	○			◉	misstated the limit
Is the derivative a function?				◯	misstatement (derivative is a line)
How does the formal definition of derivative relate to slope?				↦	possible misstatement (d=tl)
Summary	◎	○	○	◉	

"figure out how that fits in" [ln 88-114]. Although the interviewer's questions may be misleading her, Ingrid seems to want the derivative function to be a tangent line to the original curve. Certainly Ingrid's example does not consider the difference quotient or slope. Unfortunately there is not enough time in the interview to pursue her understanding further.

Taken as a whole, Ingrid's misstatements seem to indicate that she does not distinguish slope of the tangent line and the tangent line itself. On several occasions in this interview, Ingrid clearly states that the derivative is the slope of the tangent line. Yet when she tries to explain in more detail she discusses the derivative equation and the tangent line without considering slope. It seems contradictory for Ingrid to say the derivative is both the slope of the tangent line and the tangent line itself when these two things are not equivalent. However, there are many words in the English language that have multiple meanings that are not equivalent. In fact, the word derivative itself has two nonequivalent meanings; it can refer to a value or a function. Often in English the nonequivalent meanings are related to each other metonymically. A slope value, derivative at a point, and the function with these slope values as outputs (the derivative function) have a metonymic relationship. Slope of the tangent line and the tangent line itself have a different metonymic relationship. Perhaps Ingrid's error is not quite as surprising as it initially seems.

QOTD #6

Find the derivatives of the following four functions:

$$f(x) = (x-1)^2(x^2-4)$$

$$g(x) = \frac{x-1}{\sqrt{5-x^3}}$$

$$h(x) = \sin x$$

$$j(x) = \ln x$$

Date: September 20, 1993. This question occurs prior to the class learning about short-cut rules for taking derivatives of various forms.

Response:

$$f'(x) = (2)(x-1)(1)(x^2-4) + (x-1)^2(2x)$$

$$g'(x) = \frac{(1)\left(\sqrt{5-x^3}\right) - (x-1)\left(\frac{1}{2}\right)(5-x^3)^{-\frac{1}{2}}(-3x^2)}{5-x^3}$$

$$h'(x) = \cos x$$

$$j'(x) = x^e$$

QOTD #7

The following are not the derivative of $y = \ln x$. Pick at least one and explain why it could not be using your knowledge of derivative.

$y = \log(x^3)$ $\quad y = \frac{x}{|x|}$ $\quad y = x^e$ $\quad y = e$

Date: September 21, 1993. This question also occurs before the class studies short-cut rules for taking derivatives but after they have studied the limit definition of derivative.

Response: Ingrid was absent on this day.

Comment: This question was presented to the students since no student correctly stated the derivative of $y = \ln x$ in the previous Question of the Day.

QOTD #8

a) If derivative of $y = \sin x$ is $y' = \cos x$, could the derivative of $y = \tan x$ be $y' = \cot x$? Why not?

b) What is the derivative of $y = \tan x$?

Date: September 22, 1993. This question occurs prior to the class discussion on the derivation of the formula for the derivative of $y = \tan x$.

Response: "a) because they have asymptotes in different places

b) because derivative of $y = \tan x = \sec x \tan x$ "

Comment: Originally, she does the correct calculation using the quotient rule and then incorrectly simplifies it to $\csc^2 x$. Later, after we discuss it, she changes it to $\sec^2 x$.

Test 2

After spending a week reviewing the concept of derivative, but before doing derivative applications, the class has its first test on derivatives. On this test, Ingrid incorrectly states the definition of the derivative as $\lim_{x \to h} \frac{f(x+h) - f(x)}{h}$. She makes an algebraic error that allows her to simplify the expression without applying the limit. Here the quotient simplifies to 2 so that the limit is 2 and $x \to h$ may be ignored.

Ingrid is able to estimate the derivative at a particular point by finding two nearby points and calculating a difference quotient for those two points. Given the graph of a position function for a car she correctly answers questions about the speed of the car and its direction of movement. However, given the graph of another function, she draws an incorrect graph of its derivative. Her derivative graph has extrema at both the locations of extrema for the original graph and at the inflection points of the original graph.

Ingrid's inability to draw a graph of the derivative given the graph of the function is curious considering her other correct answers. On other test problems she correctly calculates the derivative at a point numerically, labeling it as the slope. She also correctly calculates the maximum speed from the position graph, although this calculation is written as $r = \frac{d}{t}$, instead of being labeled as a slope. To apply these sorts of calculations to graphing the derivative in this problem she must estimate slope values for a nonlinear curve and coordinate individual derivative values into a derivative function by plotting appropriate input-output pairs. Ingrid may not yet have a clear recognition of the distinction or the relationship between the word derivative referring to a value and the word derivative referring to a function. Her graphing of extrema of the derivative as inflection points of the function may indicate a correct point-wise connection between the two graphs. Her graphing of a similar shape for parts of the function and derivative graphs may indicate the use of a global rather than pointwise view of the function graph.

Ingrid also has problems computing some complicated chain rule derivatives. Her major error is to treat $\tan^{\frac{1}{2}}(2x+1)$ as the product of $\tan^{\frac{1}{2}}$ and $(2x+1)$. This error indicates both a lack of recognition of the composition of functions notation involved and the application of an incorrect analogy based on surface similarity of symbolic representations. Since an expression such as $x^{\frac{1}{2}}(2x+1)$ indicates a product, a student may generalize that $\tan^{\frac{1}{2}}(2x+1)$ does also.

QOTD #9

What do you understand about derivatives now that you didn't know at the end of last year?

Date: September 28, 1993. This question occurs before the class studies the chapter on alternative representations of the derivative.

Response: "Yes, we never looked at the graph of $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ so I never understood what the equation really stood for and now I can picture it in my head."

Comment: Note that during the first interview Ingrid states the difference quotient as how she would explain derivative to an AB calculus student, but she doesn't think to include the limit until later in the interview when she is asked if limit and derivative are related. Now she seems to include this in her understanding of derivative.

QOTD #10

a) Mathematical Highlights of yesterday's class.

b) Any insight you gained from the class.

Date: October 10, 1993.

Response: "We went over the homework from the day before and spent pretty much time on that. I didn't understand this one long problem that had to do with rate of change of the area of a cone made from a section of a circle. We also went over the first derivative test used to find extrema."

Comment: Since the researcher had not been present the day prior, this question is presented both as a means for the researcher to see the material covered and to ascertain the students' understanding of it.

Test 3

About two weeks later the class is tested on Taylor polynomials, a simple velocity application, and the use of the derivative to analyze function behavior. Ingrid calculates the Taylor polynomial for $\tan(x)$ at $x = 0$ to be $\frac{x+2}{3!}x^3$. Since she shows no other work it is unclear how she calculates her incorrect answer.

Ingrid correctly uses the first and second derivatives of the position function to find the speed and acceleration of an object at a given time. She is able to use the graph of a derivative function to estimate when the original function is increasing or decreasing and concave up or concave down. She also correctly notes that the inflection points of the original function occur at the extrema of the derivative. However, when asked to find the extrema of the original function, she lists the extrema for the derivative. Her only other error is to omit the point where the derivative is undefined as a critical number of the original function.

Ingrid's work on this problem is consistent with her work on the previous exam where she is given the graph of the original function and asked to graph the derivative. There she creates extrema not only at the locations of the inflection points of the original curve but also at the extrema of the original curve.

QOTD #11

> Give an example of a real world situation involving the concept of derivative but not involving velocity or acceleration.

Date: October 14, 1993. Chapter 5 covers various applications of derivative.

Response: "If you wanted to find the rate of change of how fast people grow from year to year from when they are a baby to an older person."

Test 4

Two weeks later the class has a test on the applications of derivatives. Ingrid uses derivatives to solve three traditional max/min problems, with only a single error of a factor of 2 in a surface area formula. She also correctly solves two traditional related rate problems. On a third such problem she mistakenly solves for $\frac{dr}{dt}$, the change in the radius with respect to time, instead of $\frac{dh}{dt}$, the change in the height with respect to time. She also correctly calculates the derivative of an implicitly defined function.

Interview 2

The second interview occurs during the next few days after the test on applications of the derivative. During that time period the class completes worksheets on parametric and polar functions and their derivatives. Highlights of that interview are followed by a summary in Table A.26 and a discussion.

1 MZ: What is a derivative?

2 Ingrid: The slope of a line, the slope of a tangent line, the slope of a line tangent to a function at a certain point.

3 MZ: Does anything else come to your mind for what a derivative is?

4 Ingrid: The limit-- That little picture that [Mr. Forrest] wants us to have in our mind of the graph. [pause] The limit as the change in x approaches 0-- You know, that one equation.

5 Ingrid: [writes: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$]

6 MZ: And do you happen to remember that picture that goes with it?

7 Ingrid: I'm thinking about it. I don't know. Maybe it'll come to me later.

8 MZ: Do you happen to remember how these parts fit in, the f of --

9 Ingrid: Well, you have a function and this is like--- Whatever's over here would be x plus change in x and then $f(x)$, subtract those and divide by-- I don't know.

10 MZ: OK. I'm probably going to ask you a little more later. What are derivatives useful for?

11 Ingrid: Related rates. Finding maximum and minimum values.
12 MZ: How does the derivative fit into a max/min problem?
13 Ingrid: If you set the first derivative equal to 0, then what you get for x could be a maximum or minimum.
14 MZ: OK. Why is it that when you set the derivative equal to 0 you get a max or a min?
15 Ingrid: Because the graph of the derivative-- If you look on the graph--
16 MZ: On the graph of the derivative?
17 Ingrid: On the graph of the derivative they'll be--
18 MZ: Oh, it'll cross the x -axis at that point.
19 Ingrid: Yeah. The zeros.
20 MZ: And that's-- On the original function--?
21 Ingrid: It's a max or min.
22 MZ: It's a max or min. How can you tell which it is?
23 Ingrid: I set up a little sign graph. [draws a horizontal line segment and a vertical dotted line] And then the equation is like $f(x)$ equals x minus 3 [writes: $f(x) = x - 3$].
24 MZ: And this is the original function?
25 Ingrid: Yeah. There's the original function. See my equation is-- 3 negative. 4 positive. Negative, positive, so it would be a min.
26 MZ: How do you remember that, that if it goes from negative to positive it's a min?
27 Ingrid: How? Well, I just draw it in my mind. If you're negative and it's going to positive, it's going to-- Why, is it wrong?
28 MZ: No, it's right. I was just trying to figure out how you remembered it.
29 Ingrid: I just drew the picture. [picture: a point at the bottom of a concave up, increasing curve]
30 MZ: So in this picture that you just sketched, what's negative and what's positive?
31 Ingrid: [short pause] I don't know. Because I think of like down. I'll think of like, levels. I'll think of it going like-- Starting at negative and then it's going up. If it was positive negative, I'd think it'd be starting at positive and then going down. So you draw that in, for max or min.
32 MZ: How do you remember that?
33 Ingrid: I don't understand how you don't --
34 MZ: I'll tell you what I see and you can tell me how what you see is different. I see, I'm up here right, I'm in the left upper corner and I move my pencil down to the lower right corner. To me this-- Maybe it's a max because you started up. Is that why?
35 Ingrid: Yeah.
36 MZ: Oh. Because then the other way you start in the lower left and then you go up. But you started down so it's a min.
37 Ingrid: Yeah, I'm not talking about slopes or anything.
38 MZ: Just the way--
39 Ingrid: Just the way it looks.
40 MZ: How would you explain what a derivative is to someone without very much math background?
41 Ingrid: I would say-- You like have a line first. You have a graph. You want to find the equation of that line. Then you tell them, the derivative is the slope of that line. Like OK, no. I would tell them what a tangent line is. The tangent line will touch the graph at only one place. I'd tell them that the derivative is the slope of that line. But they wouldn't know what slope was. Would they know what slope is?

42 MZ: Well, I guess it depends on who it is.

43 Ingrid: Steepness. The steepness of that line or something.

44 MZ: OK. What's a real world situation that involves derivatives?

45 Ingrid: If your life is interesting and you have like cost and you have to pay. It costs 6 cents for every foot of these and 5 cents for every foot of these, what's your maximum or minimum cost. Then you can choose--

46 MZ: OK. I guess.

47 Ingrid: Like there's water in a river and it's 15 feet deep. And it's raining and water's coming and the height is increasing at a rate of 2 inches per hour. Not increasing at 2 inches per hour. You're going to find out how fast it's increasing when the height is 17 feet or something.

48 MZ: So what part of that situation is the derivative?

49 Ingrid: Well you'd have-- I guess, the depth of the-- The volume-- You know that the river is 15 feet high. That's a bad example. OK, it's a square river.

50 MZ: OK.

51 Ingrid: It's a square river. Volume equals s^3. $3s^2dsdt$ [writes:

$$V = s^3$$

$$\frac{dV}{dt} = 3s^2\frac{ds}{dt}$$]

But that wouldn't really increase.

52 MZ: Which part wouldn't increase?

53 Ingrid: Well, the length of the river.

54 MZ: Oh, right.

55 Ingrid: The length of the river wouldn't increase.

56 MZ: The length generally wouldn't.

57 Ingrid: It's a rectangular river. I'll draw a picture. So you've got-- Depending on how it looks like. The sides are squares. So the height is the same as the width. [Ingrid has drawn a box with a rectangular base.]

58 MZ: OK, I think I understand.

59 Ingrid: So you've got x x y y and that's x too [labeling the widths as x, the lengths as y and the height as x]. [writes: $V = x^2y$] Then you'll always want to omit one so you-- And they give you an equation like the area of the river is like 50 meters. This won't make sense. If I'm talking about feet. 50 feet equals x y and you want y in terms of x. [writes:

$$50 = xy \qquad V = x^2y$$

$$y = \frac{50}{x} \quad \text{and then} \quad V = x^2\frac{50}{x}$$

$$V = 50x$$

$$\frac{dV}{dt} = 50\frac{dx}{dt}$$]

60 Ingrid: Yeah, so you want $dxdt$. You want to know how fast the height is changing. They would give you $dVdt$ The volume is increasing at the rate of 1 foot cubed per hour [writes: $\frac{dV}{dt} = 1\,\text{ft}^3/\text{hr}$ and

$$1 = 50\frac{dx}{dt} \qquad \frac{dx}{dt} = \frac{1}{50}$$].

61 MZ: So what part or parts of this would you say is the derivative? [Ingrid points.] The *dVdt* part?

62 Ingrid: Yeah.

63 MZ: What about the *dxdt* part?

64 Ingrid: Well, they're both [hesitates] derivatives related to time.

65 MZ: Yeah, I think so too. Are these derivatives related to the slope idea?

66 Ingrid: Probably not. I don't know. I don't think so. Are all the derivatives that, slope?

67 MZ: [short pause] Well, I guess I'm thinking that if you say like what is a derivative and then you say it's one thing, it seems like somehow that should fit in to every case that a derivative shows up--

68 Ingrid: Yeah.

69 MZ: --although it might not always be that obvious.

70 Ingrid: I just don't know how it fits in.

71 MZ: What is it that makes you know that dV dt is a derivative? Just the form of those symbols or were you thinking of something else?

72 Ingrid: Well, because I have this, and I know the way to do it is to take the derivative.

73 MZ: OK. Say I give you a function. How do you know if it's differentiable?

74 Ingrid: [pause] I don't know. I guess you could graph it and see like if there's-- If it's not continuous it can't be differentiable.

75 Ingrid: If there's like a sharp turn, it's not differentiable. It has to have a-- OK, I got it.

76 Ingrid: The limit-- I think there's like these two steps. There has to be a limit as x approaches [hesitates] h or something like that, of the function. Then it has to be continuous at $x = h$ or something. [has written:

1) $\lim_{x \to h}$

2) cont at]

There's this 3 step thing. The limit has to be equal to the-- [pause] I think I'm going to have to quit on that.

77 MZ: I'm going to go back to something you already said. You said if it had a sharp turn. So why is it that if it has a sharp turn, it's not differentiable?

78 Ingrid: Something about the limit from both sides doesn't equal. [short pause] I never really knew. I've heard why but I never really understood.

79 MZ: It seems like it just might as well be differentiable?

80 Ingrid: Yeah, because you think its limit would be-- But I don't know.

81 MZ: Well, OK. Say it's differentiable. Then what would be the derivative at that corner point?

82 Ingrid: Well, this line. [By this point Ingrid has drawn a graph with a cusp and a horizontal line that touches the curve at the cusp.]

83 MZ: The-- whatever, the horizontal line?

84 Ingrid: Yeah.

85 MZ: And how do you know that that's the derivative?

86 Ingrid: Because it touches it at only one point.

87 MZ: What if I drew say this line [draws a diagonal line that touches the curve at the cusp point]?

88 Ingrid: [short pause] Well, then the derivative could be anywhere. Usually you want-- Say you want the derivative at-- It should probably be a constant.

89 MZ: So this line I drew was not the right line to draw for that point to talk about the derivative.

90 Ingrid: Well, the derivative is the slope so it's going to be just a line anywhere on here with the same slope is going to be 0.

91 MZ: OK. So I could draw any horizontal line.

92 Ingrid: And that would be the derivative.

93 MZ: And that would be for that point. OK. Continuing. You kind of did this one. For each of these words I'm going to ask you if it has to do with derivative and if so how, but slope you've already pretty much explained. Velocity?

94 Ingrid: Velocity. The derivative of the position function is the velocity.

95 MZ: OK. Change or rate of change?

96 Ingrid: The derivative-- Like these are rate of change.

97 MZ: Like the *dVdt* stuff?

98 Ingrid: Yeah.

99 MZ: OK. Line or linear?

100 Ingrid: [pause] Well, the derivative could be a line if its of the second degree.

101 MZ: Oh, if the function was second degree, the derivative would be a line?

102 Ingrid: Yeah.

103 MZ: True. OK. Measurement?

104 Ingrid: Well, you could use the derivative to measure rates of change. I don't know about that.

105 MZ: Yeah, OK. Prediction or approximation?

106 Ingrid: I don't know. It comes to mind that when [Mr. Forrest] would draw something that he'd be like 'around here'. [Ingrid drew a parabola, marked a point and then on the words 'around here' drew parentheses on either side of the marked point.] It sort of reminds you of approximation, as you're getting closer. It's sort of a approximation because limit is a function as it approaches something. So it's never like exact. That's what I would think.

107 MZ: So when he draws these pictures where there's a small region around the point is kind of what you think of. OK. Continuity?

108 Ingrid: A function has to be continuous to have a derivative.

109 MZ: Limit? Did you explain how limit is related?

110 Ingrid: [Indicates the difference quotient she had written earlier.]

111 MZ: I guess, it's in the definition. Integral?

112 Ingrid: Well, I know what is meant by that. I know what the antiderivative is.

113 MZ: OK, what's an antiderivative?

114 Ingrid: If this is the derivative [writes: $2x^2$] and you multiply-- [pause, writes: $\frac{2}{3}x^3$]

115 MZ: OK, so--

116 Ingrid: This is the derivative of this. [she correctly identifies the relation after some further discussion]

117 MZ: Is the derivative a function?

118 Ingrid: No.

119 MZ: OK, why not?

120 Ingrid: It's just a slope. It's like-- [short pause] It's not like y equals something. If you just have a derivative, the derivative is-- I guess you could say-- I just keep thinking, cause you can't graph just a slope.

121 MZ: Just a slope?

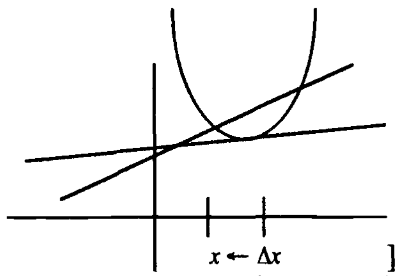
122 Ingrid: Or you can't graph a limit. But then if you say like what's the derivative. Like on a test, 'this is the graph of the derivative.' I guess it has to be. It could be a function.

123 MZ: OK. You gave the formal mathematical definition of derivative, and I guess I was hoping you were going to be able to tell me more about the picture that goes with that.

124 Ingrid: Except I'm drawing a total blank on that.

125 MZ: Yeah. OK, well maybe I'll ask you a little bit--

126 Ingrid: Wait. What if I just have a parabola and there's the point x and there's a point there. And I want to find-- And then I draw a line -- [draws:



]

127 MZ: OK. So the first line just goes through the point where you marked x .

128 Ingrid: Yeah. And this line is going through them both. And as this one is turning into the-- [inaudible] This one into this one.

129 MZ: So the one that goes through both is turning into the one that just goes through the x point? Is that what you were motioning?

130 Ingrid: Yeah. It's like becoming-- As change in x approaches x it's going to become more like the same line.

131 MZ: Is it possible now to fit in some of the other stuff from this equation into the picture?

132 Ingrid: As change in x approaches 0, so it's closer. This, f of this, x, plus change in x , is going to become more like just regular $f(x)$, and this is going to be really small.

133 MZ: The change in x ?

134 Ingrid: Right. So it's just going to be like the line.

135 MZ: So, so-- What part of this equation is the line?

136 Ingrid: $f(x)$? Oh, the line.

137 MZ: Yeah.

138 Ingrid: The result. This line. [refers to the tangent line]

139 Ingrid: The result of the equation.

140 MZ: Have you ever heard of Newton's method?

141 Ingrid: Yeah. Is that the one with-- Like there's a here and b there and if you have c –

142 MZ: Oh, that's different one. So you've heard of Newton's method?

143 Ingrid: We did it. At least I think.

144 MZ: OK. L'Hopital's rule? [Ingrid indicates no.] No. Intermediate Value Theorem?

145 Ingrid: That's the one I was thinking of.

146 MZ: Yeah, this one has some a's and b's. There's also this one, the Mean Value Theorem.

147 Ingrid: We did them both and I never remember them.

148 MZ: Do you remember which one of these that you've heard of having to do with derivative?

149 Ingrid: I don't know. Would it be Newton's?

150 MZ: Yeah, Newton's has to do with derivative.

151 Ingrid: I have no picture in my mind of what Newton's method is, but I didn't think it was the other two.

152 MZ: OK. If I give you the derivative of some function, what can you tell me about the original function?
153 Ingrid: If you give me a graph? Well, it doesn't really matter.
154 MZ: Well, I could give you either way. What would be easier?
155 Ingrid: Probably the graph. I could tell you-- Well, if you gave me an equation, then I could-- If you gave me the graph then I could tell you the zeros of the original equation and the max and min, where they would occur, but not what the x value is. I could tell you, but not exactly.
156 MZ: But not the height in other words.
157 Ingrid: Right.
158 MZ: Why is it that we can't tell the height?
159 Ingrid: Because the derivative can be the derivative of more than one equation. So you can have more than one answer.
160 MZ: More than one answer for the original you mean?
161 Ingrid: Yeah.
162 MZ: Here's a graph. Let me see what you can tell me. Say this is the graph of the derivative and maybe you could kind of sketch the graph of the original.
163 Ingrid: I got this wrong on the test. [pause]
164 MZ: So what are you thinking so far?
165 Ingrid: Points of inflection. [inaudible]
166 MZ: So you're marking the points of inflection of the original function and you marked-- One of them was at the first maximum of the derivative function? That's what you were counting, right?
167 Ingrid: Actually I'm not doing the derivative function. I'm doing--
168 MZ: So this is the derivative, right? You know that, right?
169 Ingrid: Yeah.
170 MZ: You're doing the original function.
171 Ingrid: I'm trying to think. Those aren't points of inflection. No. Forget points of inflection. Max and mins. [short pause] I guess it's going to be-- [pause]
172 MZ: OK, so now you've marked two more points.
173 Ingrid: Although now I'm marking-- I don't know. This is where I'm confused, but-- Like points of inflection on this graph are zeros here.
174 MZ: So points of inflection on the derivative graph you're putting as zeros on the original.
175 Ingrid: --zeros on the original [said simultaneously].
176 MZ: And why would that be the case?
177 Ingrid: I don't know. Because I'm trying to think if the-- OK, the max and the mins are-- The max and mins of the original graph are zeros on the derivative graph.
178 MZ: OK.
179 Ingrid: But I don't remember how to tell if they're maxes or mins.
180 MZ: OK, so for example this zero here at negative, let's say a half. We're thinking that this is a max or a min, but you weren't sure which one it was. [pause] Well, why is it that the zeros of the derivative give us a max or a min?
181 Ingrid: [inaudible] I don't know why. I just know that they do.
182 MZ: Well, you already told me that the derivative gives the slope so maybe we could use that information to tell us something about the original.
183 Ingrid: Well, the graph is increasing and then decreasing.
184 MZ: So the graph of the derivative is doing those things, right?
185 Ingrid: Yeah.

186 MZ: Can you tell me where the graph of the original function would be increasing?
187 Ingrid: Where the derivative is positive, like above the x -axis.
188 MZ: OK. So for all this first part here we would know that the original function was increasing.
189 Ingrid: So increasing until about-- So that would be a max, say here. And then it's decreasing until there. [sketches a graph of that]
190 MZ: Until a little after 2 or so. [pause] And then is it increasing or decreasing-
191 Ingrid: Yeah, it's increasing after that, but I don't know if it goes below.
192 MZ: Because we don't know how high it goes. Yeah, you told me before that we wouldn't be able to find out how high it goes, right?
193 Ingrid: Mm hmm.
194 MZ: So then that's OK. It's no big deal how high it goes. Let me ask you one last question.
195 Ingrid: OK. It looks the same as the--
196 MZ: It does kind of look the same. It's kind of shifted over. I mean the max here is at a different-- I just wanted to ask you one last question about the concavity of the original function. Here you kind of made it look like it changed from being concave up to concave down on your way to the first maximum. Were you basing that on something about the derivative?
197 Ingrid: I don't know. The slope here is positive. It's positive here. It'd be concave up-- Concave down this whole time
198 MZ: The derivative graph is concave down this whole time on this first little bit of it What information do we need to know to tell us if the original function is concave up or concave down?
199 Ingrid: Would that be the second derivative?
200 MZ: Yeah, that could help. What about the second derivative?
201 Ingrid: Where it equals 0 and then at the zeros do the sign graph thing.
202 MZ: Like you did up here where you draw and you mark.
203 Ingrid: Mmm hmm.
204 MZ: OK. Last question. From this derivative graph-- When you do the little sign graph thing you check and see--
205 MZ: Well, I'm going to let you go. I don't need to do this. I was going to ask you if you could tell from this derivative graph where the second derivative is positive for example?
206 Ingrid: [short pause] No. I don't know. I couldn't.
207 MZ: Not really.

Ingrid mentions a graphical interpretation of derivative and a symbolic interpretation most frequently. She mentions rate of change and velocity much less frequently and in less detail. For the graphical interpretation, Ingrid recognizes the instantaneous nature of the derivative and describes a graphical limiting process of two points getting closer together [ln 126-130]. She also uses a graphical argument to guess that a derivative is a function. She remembers test questions which give a graph and say

Table A.26. Ingrid: Interview 2 Circle Diagrams

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎			◉	
What can derivatives be useful for?		○			related rates max/min
Explain what a derivative is to someone who doesn't know anything about math?	●				
Give an example of a real world situation that involves the concept of derivative.			○	↦ ◎	max/min in/decreasing
How can you tell if a function is differentiable?	○			●	misstatement misstatement (d=tl) misstatement (confuse the derivative at a point with the derivative function)
Is derivative related to speed or velocity?			○		
Is derivative related to change or rate of change?		○		◎	
Is derivative related to measurement?		○			
Is derivative related to prediction or approximation?	●				
Is the derivative a function?	○ ◯			◯	misstatement (confuse the derivative at a point with the derivative function)
You mentioned a picture that goes with the formal definition of derivative. Can you remember anything more about it?	●			◉	misstatement (d=tl)
Summary	◎	○	○	◉	

this is the graph of the derivative. Since the graph is a function, the derivative is a function [117-122]. For the symbolic interpretation, Ingrid correctly states the formal definition as $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

Ingrid makes very few connections between the various interpretations for derivative. She notes that $\frac{dV}{dt}$ and $\frac{dx}{dt}$ denote rates of change [95-98]. She also attempts to relate her statement of the formal definition to a graphical interpretation. She recognizes that the limiting process means that two x values are getting closer together, but she has the x values listed as x and Δx. She believes that the result of the limit is a line [ln 134-136], as opposed to a slope, and that the resulting line is going through one point, but on her sketch this is not a tangent line. This confusion is surprising considering Ingrid's comment on a QOTD four weeks previous regarding what she has learned about derivative in Mr. Forrest's class that she had not known at the end of her precalculus study of derivatives. "We never looked at the graph of $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ so I never understood what the equation really stood for and now I can picture it in my head." It is unclear whether her understanding has deteriorated in the past four weeks or whether the picture in her head was never accurate.

Even though Ingrid mentions slope frequently during the interview, she consistently misses opportunities to relate slope to other aspects of her understanding of derivative. When asked how she determines whether a zero of the derivative is a maximum or minimum of the original function, she describes a mnemonic involving where the derivative is positive and negative [196-197]. When asked for clarification, she makes it clear that her mnemonic has nothing to do with slopes. In another instance when she is asked whether $\frac{dV}{dt}$ and $\frac{dx}{dt}$ are related to slope, she says, "Probably not. I don't think so. Are all derivatives that, slope?" [ln 66]

During a discussion of differentiability, Ingrid makes two statements that make it sound like she thinks that the derivative is the tangent line itself and not the slope of that line. She says that if the function is differentiable at a sharp corner, then the derivative would be a horizontal line because it touches the graph at one point. She even seems to use the knowledge that derivative is slope to argue that any horizontal line would be the derivative.

90 Ingrid: Well, the derivative is the slope so it's going to be just a line anywhere on here with the same slope is going to be 0.
91 MZ: OK. So I could draw any horizontal line.
92 Ingrid: And that would be the derivative.

This misstatement also indicates a confusion between the derivative at a point, which is a single value like a slope, and the derivative function, which in some instances will be a line. This confusion also occurs in her first answer to whether the derivative is a function. She replies that it is not a function since "you can't graph a slope" and "you can't graph a limit" [ln 120]. Both a single limit value and a slope value would represent the derivative at a point, not a derivative function.

For the last part of the second interview Ingrid is asked to graph an original function when given the graph of the derivative function. After making a surprising misstatement that points of inflection on the derivative graph are zeros on the original function [ln 173], Ingrid goes on to make correct statements about the locations of the extrema of the original function and to draw a reasonable graph of the original function. Her inconsistency is similar to her work on Test 3. There she uses the graph of the derivative to correctly answer a number of questions about the original function, including finding the critical points and the intervals on which the function is increasing and decreasing. However, she answers the final question of the locations of the extrema for the function graph by listing the locations of the extrema for the derivative graph.

In comparing Ingrid's responses during the first two interviews it is clear that she states a more complete understanding during the second interview . Her replies in both interviews are similar in that a graphical interpretation is the dominant representation.

However, in the second interview, the symbolic representation has also gained prominence. Ingrid's first interview contains no descriptions of the details of the ratio or limit processes except in the statement of the formal symbolic definition. In the second interview she states a graphical limiting process in addition to the symbolic, and is able to state the limit part of the limit of the difference quotient correctly, which she did not do in her first interview.

Ingrid makes an attempt to connect the symbolic and graphical interpretations in both interviews. Both attempts are incomplete and include the misstatement that the derivative is the tangent line itself and not its slope. The main improvement between the two interviews is that in the second interview Ingrid recognizes that the symbolic limit represents two points getting closer together. In the first interview she makes no connections to the limit of the difference quotient. In both cases she fails to relate the difference quotient to slope.

QOTD #12

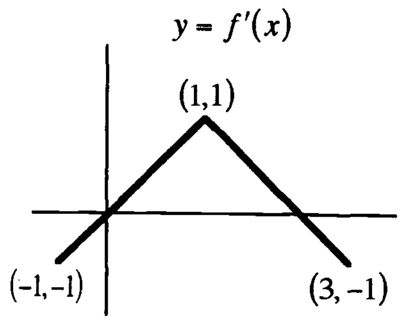
What is the most important idea that we have studied so far in this class?

Date: December 2, 1993. This question occurs immediately after the class has finished the chapter on integration, which includes a discussion of The Fundamental Theorem of Calculus.

Response: "The most important ideas are understanding the Fundamental Theorem of Calculus and knowing derivatives and antiderivatives."

Interview 3

The third interview occurs during the three days after the test on differential equations, and antiderivatives by substitution and by parts. The first part of the interview is a summary of Ingrid's attempts to graph a function given the graph of its derivative. In contrast to the same assignment at the end of the second interview, a piecewise linear function is used so that slope field or area calculations are easy if a student chooses either of those methods of solution. Also, unlike the second interview, the point $(1,0)$ on the original function is given so that only one solution is possible.



1 MZ: The first question is actually a problem to solve, and I'm about to draw the graph of the derivative on here. So this is the point $(1,1)$ and this is $(1,-1)$ and this is the line connecting them. [draws the points and line segment described on a labeled axes] This is the point $(3,-1)$ and this is that line. [draws in the point and connects it with a line segment to $(1,1)$] So this is some function, and you don't care what happens outside of this domain. Just worry about from -1 to 3. So the thing is-- this is supposed to be $y = f'(x)$ and we want to know what the graph of the original function would look like, f. [writes: $y = f'(x)$] And I better give you the initial condition, $f(1) = 0$. [writes: $f(1) = 0$]

2 Ingrid: [25 second pause] Well, I don't know. [short pause] Just in this range it's going to be?

3 MZ: Uhm-- Just in the domain from $x = -1$ to 3, but the y 's it's up to you how you do those. OK. So you put a dot at $(0,1)$ and $(2,-1)$?

4 Ingrid: Kind of thought this would be a max and min because uhm, these are zeros of the derivative and that's how you find max and min like if you use the [inaudible] we have.

5 MZ: OK. And how did you know the left one was the max and the right one was the min?

6 Ingrid: I wasn't sure. I just guessed because the slope is increasing here and decreasing there.

7 MZ: OK. So the slope is increasing-- you motioned from sort of -1 to 1. Is that right?

8 Ingrid: Yeah. It would be decreasing 1 to 3.

9 MZ: [short pause] When you said that about slope were you talking about the slope of the original function?

10 Ingrid: Well, the slope of the derivative.

11 MZ: It's the slope of the derivative, OK.

12 Ingrid: [25 second pause; Ingrid is sketching] I think the derivative-- Max and mins. I don't know. I don't think that's right though.

13 MZ: She put small dots on-- what is that? $(-1,0), (1,0)$ and $(3,0)$ [Ingrid has sketched not just these points but curves connecting these to the max and min she labeled earlier.]

14 Ingrid: I didn't mean to put a dot there.
15 MZ: Yeah, that's OK. [short pause] So tell me again why you put a dot at $(-1,0)$ for example, what you were thinking?
16 Ingrid: 'Cause it was a min of the derivative, I thought it might be a zero of the original function.
17 MZ: OK. Does this original function have an inflection point?
18 Ingrid: Yeah.
19 MZ: Where would it be? [Ingrid points.] At $(1,0)$?
20 Ingrid: Yeah.
21 MZ: How do you know that's an inflection point?
22 Ingrid: Because it's concave down and then concave up.
23 Ingrid: And it's--
24 Ingrid: -- a max on the derivative.
25 MZ: OK. Do you know what the value of the second derivative would be at that point?
26 Ingrid: At what point? This one?
27 MZ: At the inflection point, yeah.
28 Ingrid: Zero.
29 MZ: How do you know?
30 Ingrid: Because that's how you find inflection points.
31 MZ: Oh, because usually inflection points--
32 Ingrid: are the zeros of the--
33 MZ: are the zeros--
34 Ingrid: the derivative.
35 MZ: Is it true that the inflection points are always at a zero or could something else happen in the second derivative at an inflection point?
36 Ingrid: Zero or-- [short pause] --undefined. [short pause] Something like that.
37 MZ: OK. And so on this case, this inflection point you said was definitely a zero though as opposed to an undefined? [short pause; following overlaps Ingrid's next comment] For the second derivative.
38 Ingrid: Yeah, because it's a min-- a max. I guess.

Ingrid's work on this problem continues the trend of inconsistency found in her work on similar problems from the second interview and the Test 3. At the beginning of her effort to solve the problem she correctly notes that the zeros of the derivative are the extrema of the original function. She reverses this notion several minutes later when she explains why she has marked $(-1,0)$ as a point on the original function. "'Cause it was a min of the derivative, I thought it might be a zero of the original function" [ln 16]. Later in the interview she reverses the notion again, but this time in an incorrect setting. She says that at $x = 1$, the maximum of the derivative, the second derivative is equal to zero [ln 23-34]. The problem is that in this case the second derivative is undefined.

In the third interview Ingrid mislabels the maximum and minimum. Her argument is based not on using the derivative graph to determine information about the slope of the original function, but on considering the slope of the derivative function itself. She says that $x = 0$ is a maximum of the original function because the slope of the derivative is increasing there. Because of this confusion her graph of the original function is incorrect. Her performance on in the second interview is stronger than in the third interview in that she is able to correctly determine whether a zero of a function is a max or min based on whether the derivative is positive or negative to the left of the point in question. Her second interview method is a mnemonic that does not involve an understanding of derivative as slope, but at least it gives a correct answer.

Each of Ingrid's statements seems to be based on some memorized (perhaps incorrectly memorized) phrase or mnemonic and not on an attempt to think about what the derivative represents in a graphical setting. In addition, Ingrid does not mention techniques such as slope fields and areas, which have been covered in class since the second interview and which could be helpful in solving this problem.

The remainder of the third interview focuses on general questions about integrals, antiderivatives, slope fields, and the Fundamental Theorem of Calculus.

39 MZ: These are just more general questions. What is a definite integral?
40 Ingrid: It's the area under a curve between two specific like points.
41 MZ: OK. What about an indefinite integral?
42 Ingrid: It's the area under a curve.
43 MZ: So what would be the difference between the two?
44 Ingrid: Well, definite integral is like bounded by-- you know like x values.
45 MZ: OK.
46 Ingrid: [inaudible]
47 MZ: And then the indefinite's not bounded by x values so that would be the major difference? Uhm--
48 Ingrid: Indefinite integral would be the antiderivative.
49 MZ: OK.
50 Ingrid: I don't know about [inaudible]
51 MZ: What did you just say? You said the indefinite integral was the antiderivative, but the definite integral is not?
52 Ingrid: Well, I don't know.
53 MZ: Hmm.
54 Ingrid: I guess it is, you know.

55 MZ: What's the difference between the way the integrals are written, in symbols?
56 Ingrid: Well, it's the same except the definite has an a and b.
57 MZ: OK. What do these integrals have to do with derivatives?
58 Ingrid: They're both limits.
59 MZ: Oh, you mean derivatives and integrals are both limits.
60 Ingrid: Yeah.
61 MZ: OK. That's true. Are they related in any way besides having in common that they are both limits?
62 Ingrid: No. Well, probably but-- Well, the integral is the antiderivative.
63 MZ: True.
64 Ingrid: So if you take derivative of the integral, you can just get rid of the [inaudible].
65 MZ: Do you remember what the fundamental theorem says?
66 Ingrid: I'm thinking $A(x) = F'(x)$ and I never understood it.
67 MZ: Here write down what you just said.
68 Ingrid: [writes: $A(x) = F'(x)$]
69 MZ: So this F' what's that suppose to represent?
70 Ingrid: A derivative.
71 MZ: [short pause] OK. So, let's see.
72 Ingrid: The area. I guess this is like the area under the curve.
73 MZ: The $A(x)$ part? So the area under the curve is related somehow to this derivative of F?
74 Ingrid: Is that what you're asking or little? Well capital-- I guess it doesn't really matter. This should be little f. [changes F to lowercase f]
75 MZ: I mean what would be the difference in saying F or little f?
76 Ingrid: Well if I use-- I guess you should use like capital F for the original function or something.
77 MZ: Oh, OK. Hmm. Do you remember if this is supposed to be the first or second fundamental theorem?
78 Ingrid: The first.
79 MZ: And how does-- How do integrals fit into this?
80 Ingrid: I don't know.
81 MZ: How about the second fundamental theorem, do you remember what that one says?
82 Ingrid: $f(b) - f(a)$ is-- [writes: $f(b) - f(a) =$] I know you can use it to find definite integrals. Like if you had-- [writes: $\int_a^b 2x$]
83 MZ: OK and then this f in this case represents what function? The $2x$ function?
84 Ingrid: Yeah.
85 MZ: So if I wanted to solve this definite integral I can just plug in b into $2x$?
86 Ingrid: Mm hmm.
87 MZ: So I would like have $2b - 2a$ and that would be my answer?
88 Ingrid: Yeah.
89 MZ: OK, uhm. Let's see what questions have I not asked. [short pause] Have you ever heard of an area function?
90 Ingrid: I don't know.
91 MZ: I guess this $A(x)$ stuff could be called an area function.

92 Ingrid: Probably, but I don't know what it is.
93 MZ: Have you ever heard of slope field?
94 Ingrid: Yeah.
95 MZ: Do they have anything to do with derivatives?
96 Ingrid: Probably, but I don't know what. We've done slope fields, but --
97 MZ: Well, what's a slope field suppose to tell me?
98 Ingrid: Like, because if you find like an antiderivative, you don't know for sure like where it's at. It could be any point. Like it could be-- If you know $f(1) = 0$ then you know, but if you don't have it, you just have a bunch of slopes.
99 MZ: Do you see slope fields as having more to do with integrals or derivatives?
100 Ingrid: I don't really know what they're for. I don't remember.
101 MZ: They just seem mostly for looking at the different--
102 Ingrid: I guess maybe like derivatives because of the antiderivative.
103 MZ: OK. We're done.

Ingrid mentions area three times in this interview. She says that the definite integral is the "area under a curve between two points" [ln 40] and that the indefinite integral is "the area under a curve" [ln 42]. When asked about the First Fundamental Theorem of Calculus she says that she "never understood it", but that it has to do with the expression $A(x) = f'(x)$ where $A(x)$ is "the area under the curve" [ln 66-72]. She pronounces $f'(x)$ as "f one", and says that she does not know how this equation is related to integrals.

Ingrid also struggles in discussing the Second Fundamental Theorem [ln 81-92]. This is surprising since she had no trouble computing it on previous tests and writes about it in discussing what she learns in the *Mathematica* lab. Perhaps she is influenced by the interviewer's suggestion of the wrong answer, but this is after she has written $f(b) - f(a)$ instead of the traditional $F(b) - F(a)$.

Ingrid is unsure about the purpose of slope fields but knows they have to do with antiderivatives [ln 93-96]. Her emphasis is that slope fields are useful for discussing the fact that a function has more than one antiderivative. Ingrid says earlier in the interview that the "indefinite integral" or "the integral" is the antiderivative, and that the indefinite integral is the area under a curve, but she does not attempt to extend these connections to a relationship between slope fields and area under a curve.

QOTD #13

Find the derivative of $f(x) = \ln(x^2)$.

Date: January 5, 1994. This question occurs shortly after the students return from winter break.

Response: While it is recorded that Ingrid answered this question incorrectly, her exact response is not recorded.

QOTD #14

Find the derivative of $f(x) = \sec(x^2)$.

Date: January 6, 1994.

Response: Once again, Ingrid's exact answer is not available. It is known, though, that Ingrid answers the question incorrectly.

Test 9: Semester final

This test, which is a cumulative semester exam, covers all of the material on functions, limits, derivatives, areas, and volumes. The test questions are largely computational. On the semester final, Ingrid correctly solves one optimization problem and misses another. She states that a maximum occurs at the location of the minimum. Reversing the maximum and minimum locations is an error Ingrid also made during the third interview. Ingrid correctly answers two questions on continuity. She also computes several limits, derivatives, and integrals correctly, but misses one derivative because of a minus sign error, and another integral because of a substitution error. One other error occurs on a conceptual question about limits. She marks that if $\lim_{x \to a} f(x) = L$ then $f(x)$ is defined at a.

QOTD #15

Discuss the continuity and differentiability of $f(x) = x^{2/3}$.

Date: February 1, 1994. This question occurs after the semester final but before the class begins covering new material.

Response: "continuous everywhere not differentiable everywhere"

Interview 4

The transcript of the fourth interview is broken up into four parts. The first section includes general questions about derivatives. The second part asks the student to estimate the derivative from a table of values. The third part asks the student to relate information about distance, velocity, and acceleration given a verbal description of a situation. The fourth part is a standard related rate problem about which some nonstandard questions are asked. The following is a transcript of the first part of the fourth interview..

1 MZ: What is a derivative?
2 Ingrid: Derivative is the slope of a line tangent to the function at a certain point.
3 MZ: --slope of a line tangent to the function at a certain point. OK. Do you think of anything else when you think of what a derivative is?
4 Ingrid: It could be a rate of change.
5 MZ: OK. Uhm, what do you think of-- I mean, what do you mean by a rate of change?
6 Ingrid: The rate like at which like-- Like usually you think of something like going up or like leaking out. Like the rate at which like the speed increases or something. Something increases [inaudible].

As in previous interviews, Ingrid states the slope of the tangent line at a point as her first answer to what a derivative is. When asked if anything else comes to mind, Ingrid mentions rate of change. This follow-up is different from her answers in the first two interviews, in which she does not mention rate of change until specifically asked about it. However, in those interviews Ingrid does discuss related rate problems as examples of what the derivative is useful for. Her explanation of the meaning of the phrase "rate of change" seems to fit that focus on related rate problems as she mentions something "leaking out" [ln 6]. She also mentions acceleration, "the speed increasing" [ln 6].

The next part of the fourth interview is a summary of Ingrid's solution to the first of three problems of this interview. Given a table of values with x varying by .1, Ingrid is asked to estimate $f'(2)$, the derivative of the function at $x = 2$. Ingrid's first reaction is to try to find an equation for the function. She initially believes that the function could be a line. Without prompting she calculates the difference in a few $f(x)$ value pairs, each with x values that differ by .1. When asked, she states incorrectly that the differences do not have to be the same for a line. When told that the function is not a line, that the function decreases then increases, she says of estimating $f'(2)$, "I feel like I need an equation to find it." If she had an equation for the function, she would "take the derivative and plug in 2." She is unable to state how she would find an estimation without an equation. When prompted with the suggestion of sketching a graph of the points, she does so but does not think of calculating the slope at a point.

Even though Ingrid states earlier in this interview that the derivative is the slope of the tangent line at a point, she does not think of using that information in this problem solving situation. Neither does she try to use the notion of rate of change, also mentioned earlier this interview, nor the difference quotient of the formal definition which she states correctly in the second interview. These aspects of her understanding of derivative seem compartmentalized from the problem solving strategies that she evokes in this situation.

The next question concerns a scenario involving the movement of a car. A car is stopped. It then moves forward, increasing speed at a constant rate until it reaches 60 miles per hour. Then it continues moving forward, but it's speed decreases at a constant rate back down to 0 miles per hour. The car takes 1 hour to get up to 60 miles per hour and another hour to get back down to 0 miles per hour. How far does the car travel in the 2 hour period?

Ingrid's first reaction is to write $s(t) =$, $v(t) =$ and $a(t) =$ one above the other. However, she cannot proceed from there except to guess that $s(t) = t$. The interviewer suggests that Ingrid attempt to find the velocity or acceleration instead. Ingrid states that the acceleration is constant. When asked for the value of the constant, Ingrid says, "for every minute you increase another mile". The interviewer persuades Ingrid that every minute velocity increases another mile per hour. Ingrid says that the constant acceleration is 1/60 miles per minute squared and must be convinced that this translates to 60 miles per hour squared. Once this is established, Ingrid uses antiderivatives to find $v(t) = 60t$ and $s(t) = 30t^2$ and concludes that the car has traveled 120 miles, since $s(2) = 120$. Ingrid does not notice that the formulas she has calculated only hold when the acceleration is positive, i.e. for the first hour of the car's trip.

For this problem Ingrid concentrates on using a symbolic representation to interpret the physical situation. She does not attempt to draw a graph, nor is she asked to do so. Without using a second solution method or even an estimate, Ingrid does not see that 120 is twice the correct answer of 60.

The last question of the fourth interview involves a traditional scenario of a ladder sliding down a wall. Ingrid is told that a ladder is being pulled away from the wall, horizontally, at a constant rate. She is asked if the top of the ladder is sliding down the wall at a constant rate. If so, is it the same rate as it's being pulled out or different? If not, is it increasing in rate or decreasing in rate?

Ingrid begins solving the problem by labeling the wall as y and the floor as x and writing $\frac{dy}{dt}$ near the wall and $\frac{dx}{dt}$ near the floor. She comments that she is trying to find how $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are related, and completes the following sequence of calculations:

$$x^2 + y^2 = 196$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} = -y\frac{dy}{dt}$$

Upon completion, she states that the relationship depends on x and y. However, she seems to think of x and y as constant values when she answers that $\frac{dy}{dt}$ is constant "because there's nothing else in there"; i.e. there is nothing else in the equation except constants. Thinking of x and y as constant values would be typical of solving the problems given in her textbook. In these traditional problems, the student is usually asked to find a numerical value for one rate given numerical values for the other quantities involved.

Interview 5

Ingrid's fifth interview occurs exactly one week after she takes the BC version of the AP exam. During that week the class discusses the written questions from the BC version. Between the fourth and fifth interviews the class studies series and integration techniques and practices old AP exams.

The interview and analysis is divided into five sections. The first section includes a transcript of general questions about derivatives that parallel some of the questions from earlier interviews, a summary table with the circle diagrams, and a written analysis. The remaining four sections each summarize Ingrid's response to a set of questions on a particular topic and provides an analysis of those responses.

1 MZ: What's a derivative?
2 Ingrid: A derivative is the slope of a line tangent to a curve or a function or something at a certain point.
3 MZ: OK, anything else? What else is a derivative?
4 Ingrid: [writes: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$] Something like that.
5 MZ: What else is a derivative?
6 Ingrid: Well, it tells you like max and minimum points.
7 MZ: OK, go ahead.
8 Ingrid: Tells the max and minimum points of the original function, and it can tell you like concave up or concave down.
9 MZ: [writes: concave up or concave down] Uh, how does this first one that you gave me relate to the second one that you wrote down?

10	Ingrid:	It's like-- Oh, OK. It's like that drawing, isn't it? [sketches smooth curve on a pair of axes] Hmm. Oh, I guess-- You're trying to find like a certain point and you kind of like narrow it down. And you've got a distance of h like between your interval or something. [She has marked two points on the horizontal axis as a and b with a bracket connecting them labeled as h .]
11	Ingrid:	So each one would be-- Like the smaller h gets the closer, the more accurate you get.
12	MZ:	OK. How do some of the things in this limit definition relate to this picture? Like what is this $f(x+h)$ and--?
13	Ingrid:	I guess this would be like x [writes x over where she had a].
14	Ingrid:	And then like-- Any distance here is like h . Like $x+h$ here [marks $x+h$ over where she had b].
15	MZ:	OK.
16	Ingrid:	So as h approaches 0, $x+h$ approaches x . So you subtract $f(x)$ from that to get the difference and divide [inaudible].
17	MZ:	OK. How does the f come into play?
18	Ingrid:	Well, the function. You want the point on the function. You want the y -value.
19	MZ:	OK. So-- Let's see, what else. So this quantity, the $\frac{f(x+h)-f(x)}{h}$
20	Ingrid:	Some point-- [pause] Some point over here.
21	MZ:	Along the curve.
22	Ingrid:	A y -value between x and $x+h$ [marks a point on the curve with x-value between x and $x+h$]..
23	MZ:	OK. Let me ask you a different question. How does this picture relate to this, slope of the line tangent to the curve stuff?
24	Ingrid:	[long pause] Well, the more closer h gets to 0, the more accurate you slope is going to be.
25	MZ:	OK, so what line are we talking about then? The derivative is the slope of the tangent line to the curve. So what line would that be in this example?
26	Ingrid:	[Draws in a line touching the point she had marked earlier, parallel to the curve at that point.]
27	MZ:	So, that's going through this point?
28	Ingrid:	Mm hmm.
29	MZ:	OK. Now one thing you didn't happen to mention so far was--
30	Ingrid:	Oh.
31	MZ:	What were you thinking?
32	Ingrid:	Rate of change.
33	MZ:	I was wondering if you saw that as related to any of these other things. [short pause] I mean, are you saying that's an answer to "what is a derivative"?
34	Ingrid:	Hmm, yeah.
35	MZ:	So what does that mean, the rate of change?
36	Ingrid:	Rate of change. Like you've got a triangle and an x , y and z [draws a right triangle with the sides labeled x and y and the hypotenuse labeled z]. And you want to know-- As x changes-- Like as x grows bigger by like 2 feet a minute, what does y do?
37	MZ:	OK. Does rate of change have anything to do with the line tangent?

38 Ingrid: [pause] I guess the slope at a certain point is how the curve is changing at that point.

39 MZ: OK. Does the derivative involve a limiting process? Explain.

40 Ingrid: Well, it does [points at limit definition].

41 MZ: Yeah, so there's a limit in that definition. How would you describe that in terms of the slope of the tangent line? [pause] Is there a limiting process involved in finding the slope of the tangent line?

42 Ingrid: [pause] I don't know.

43 MZ: Or I could ask, is there a limiting process involved in finding the instantaneous rate of change.

44 Ingrid: [pause] No. Well, they're all limits. All the derivatives are like limits.

45 MZ: Mm hmm. [pause] OK. Is the derivative of a function a function? Explain why or why not.

46 Ingrid: It could be.

47 MZ: It could be, but it doesn't have to be?

48 Ingrid: [long pause]

49 MZ: What are you thinking of?

50 Ingrid: I'm trying to think of an example where it wouldn't be.

51 MZ: So you can think of an example where it would be?

52 Ingrid: Well, yeah. Like if you have $y = x^2$ and the derivative is $y = 2x$. They're both functions.

53 MZ: Right.

54 Ingrid: OK, I'll say yeah. Derivatives are functions.

55 MZ: Based on that you can't find a counterexample?

56 Ingrid: Yes.

57 MZ: OK. How were you checking to see, when you were trying different examples, to see if it was a function.

58 Ingrid: Vertical line test.

59 MZ: Oh, you were thinking of the vertical line test?

60 Ingrid: Yeah.

61 MZ: OK. Explain what is meant by a differentiable function. And give an example of a differentiable function and a nondifferentiable function.

62 Ingrid: A differentiable function is a function that can be differentiated.

63 MZ: OK.

64 Ingrid: A function where you can find the slope of a tangent line at any point. An example of a differentiable function would be $y = x^2$. A nondifferentiable would be like a function that wasn't continuous like $\frac{(x+1)^2}{x+1}$. There's a hole.

65 Ingrid: [writes: $\frac{(x+1)^2}{x+1}$]

66 MZ: See what I'm thinking is that if you simplify that you don't get a hole because there's an $x+1$ on top.

67 Ingrid: Could it be if it was like-- [writes: $\frac{(x+1)(x+3)}{x+1}$]

68 MZ: OK, so there's one with a hole. So it's not defined there. How about one that is continuous but is not differentiable? Is that possible?

69 Ingrid: Yeah, like if you have something that goes like that [sketches a curve on pair of axes with a cusp pointing upward; the cusp point and its x -value are marked].

70 MZ: Like a cusp.

71 Ingrid: Because the limit-- [pause] Something with limit. It like approaches to that or something like that.
72 MZ: So you're explaining why it's not differentiable at that point?
73 Ingrid: Yeah.
74 MZ: Because the limit of what approaches what?
75 Ingrid: The limit as x approaches this number [x -value of the cusp point].
76 MZ: OK.
77 Ingrid: [pause] Too close I guess.
78 MZ: Too close to each other, you mean?
79 Ingrid: Like it's approaching this number too fast, like the rate of change or something like that.
80 MZ: OK. Do you know an equation of a function that has a cusp or is not differentiable but is continuous?
81 Ingrid: [pause] Can't think of one.
82 MZ: Explain what a derivative is without mentioning the symbolic definition or slope or rate of change.
83 Ingrid: A number.
84 MZ: OK. Three is also a number.
85 Ingrid: Derivative could be three.
86 MZ: True. Can you give me any more information about this number that makes it a derivative?
87 Ingrid: [pause] The limit.
88 MZ: How about, what are derivatives useful for?
89 Ingrid: Like rate of change stuff. If you want to find out, how fast the water's draining out of the pool. [short pause] It's like for curve sketching. If you don't know what the function looks like, you take the derivative.

As in previous interviews, Ingrid mentions a graphical interpretation of derivative first (see Table A.27). Her follow-up answer, the limit of the difference quotient [ln 4], is the same as her follow-up during the second interview, but is different from the fourth interview, in which she mentions rate of change second. Even though the portion of the fifth interview focusing on general questions about the derivative has fewer questions than similar sections of the first two interviews, Ingrid answers nearly as completely here as in the second interview. The main difference is that Ingrid does not discuss the derivative as velocity or acceleration in the fifth interview. Also, when asked if the derivative is a function, Ingrid's second interview response is graphical and indicates that she does not realized that the derivative could be a function since it is just a limit or a slope value. In the fifth interview Ingrid answers in terms of a function's equation, and using derivative rules to take the derivative.

Table A.27 Ingrid: Interview 5 Circle Diagram

	Slope	Rate	Vel.	Sym.	
What is a derivative?	◎				
What else?				◉	
How does the formal definition of derivative relate to slope?	● ○	○		●	misstatement
What is meant by instantaneous rate of change?		○			related rate
Does the derivative involve a limiting process?				○	
Is the derivative of a function a function?				○	
What is meant by a differentiable function?	○	○			
What are derivatives useful for?		○			
Asked to interpret the Mean Value Theorem.				•	
Asked to find the average rate of change of a function defined as an integral.		○			misstatement (d=average roc)
Asked to interpret the derivative in the context of a function that gives the temperature for a given time.	○	○		↦	misstatement (d=change) decreasing maximum concavity incorrect calc
Summary	◎	○		◉	

As in previous interviews, Ingrid has trouble explaining the relationship between different representations of the derivative concept. She knows that the notion that "a derivative is the slope of a line tangent to a curve ... at a certain point" [ln 2] is related to the limit of the difference quotient by a drawing that she has seen before. When she tries to recreate it, she labels x and $x+h$ on the horizontal axis and denotes h as the distance between them. She indicates that "the smaller h gets, the more accurate you get" [ln 24], but she does not seem to know what gets more accurate. After some prompting by the interviewer for her to show what the ratio represents in her drawing or to indicate how the slope of the tangent line fits, she marks a point with x-value between x and $x+h$ and draws a tangent line through that point, but she does not explain further. Even though she does not connect slope to the symbolic form, she does spontaneously mention rate of change and state that the slope is "how the curve is changing at that point" [ln 38].

For the second part of the fifth interview, Ingrid is asked about the Mean Value Theorem. When asked if she remembers what it says, her first reaction is to write $\frac{f(b)-f(a)}{b-a} = f'(c)$. She also remembers that c is between a and b, but thinks that any c between a and b will satisfy the equation. The interviewer gives her the actual statement including the fact that not any c works and asks Ingrid what the theorem means "in terms of something else?" Ingrid's reaction is to draw a sketch with a curve and the points $(a, f(a))$ and $(b, f(b))$ marked. She also marks c between a and b on the horizontal axis, but she does not draw any secant or tangent lines. She is not able to explain what the ratio or $f'(c)$ represent in terms of her picture, "I don't know. I never understood those theorems."

Ingrid's inability to relate the symbolic statements to the notion of derivative as slope is consistent with her difficulties in the earlier part of the interview where she cannot give a graphical interpretation for the formal definition of derivative. In both cases she has memorized the symbols, and she knows there is some graphical association, but the symbols themselves have no specific meaning for her.

The next question on the fifth interview involves a problem from the AB version of the AP exam which Ingrid has not taken. The question is as follows:

> Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.
> (a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.
> (b) On what intervals is F increasing?
> (c) If the average rate of change of F on the closed interval $[1,3]$ is k , find $\int_1^3 \sin(t^2)\,dt$ in terms of k .

The interviewer asks Ingrid to discuss her methods of solution for parts (a) and (b), but does not require her to complete the solution of either part. For part (a) Ingrid correctly describes how to apply the trapezoid rule to find area under the curve.

For part (b) Ingrid initially takes two derivatives of F. She finds $F'(x) = \sin(t^2)$ and then $F''(t) = 2t\cos(t^2)$. At first she decides to set the second derivative equal to 0 and to find where it is positive or negative. While describing the process for charting where the second derivative is negative or positive, she realizes that it is the first derivative not the second derivative that she should be looking at to determine where F is increasing.

For part (c) Ingrid does not know how to interpret the question. She asks, "If they say that the average rate of change is k , does that mean the derivative is k ?" Ingrid does not decide whether or not this is true, but she says that if it is true, then she would take the derivative of F and set it equal to k . Ingrid then changes the subject and goes on to the next problem.

Ingrid's failure to make a distinction between average rate of change and instantaneous rate of change, the derivative, is consistent with her answers to the Mean Value Theorem question and other questions in previous interviews. To understand the distinction Ingrid needs to know that the derivative at one point may be approximated by a ratio, a difference quotient for two points. However, nowhere in any of the interviews does Ingrid describe the details of the ratio in any context except for symbolic. When asked to explain the symbolic ratio in any other context such as in the Mean Value Theorem question or in explaining the formal definition of derivative, she is unable to do so.

The next section of the interview concerns Taylor polynomials. Ingrid is asked what a function and its second degree Taylor polynomial have in common and how they differ. Ingrid responds that she does not know enough about Taylor polynomials to answer the question. When asked to calculate the second degree Taylor polynomial for $f(x) = e^x$ at $x = 0$, she says that she knows she has to calculate the first two derivatives but does not know what to do after that.

The final section of the fifth interview concerns a function, f, that at any time, x given in hours, tells the outside temperature in degrees Fahrenheit. Ingrid is shown a series of symbolic expressions and asked what information each one provides about the outside temperature.

Ingrid's initial reaction to $f'(3) = 4$ is to write $f'(x) = 4$ and $f(x) = 4x + C$. These two equations assume more than Ingrid is given, i.e. that $f'(x) = 4$ over some interval and not just at $x = 3$, and signal Ingrid's interest in a symbolic solution method. With some encouragement to focus on $f'(3) = 4$, Ingrid states that "after 3 hours the temperature changes by 4." When asked for the units, she gives them as degrees per hour. Later when asked about the expression $f'(x) = 4$ for $0 \le x \le 3$, Ingrid initially thinks that the temperature changes 4 degrees from noon to 3, but then changes her answer to a change of 3 degrees for each hour, giving correct numerical answers when asked the temperature at each hour.

For $f''(3) = -2$, Ingrid initially thinks that it means the temperature is decreasing, but she is not sure. For $f''(x) = -2$ for $3 \le x \le 6$. Ingrid wants to say that the temperature decreases by 2 each hour, but "because it's double prime, I guess it's going to be different. ... It's decreasing 2 degrees per degree per minute, or hour. The rate of change is changing, going down 2 for every hour." With this new insight she is able to give reasonable estimates for the temperature at 4 and 5.

When asked when the temperature is hottest and coldest for the six hour period, Ingrid says, "You can find a maximum when $f'(x)$ is 0, " but then she continues with a numeric argument comparing her estimates for the temperature at 4, 5, and 6. She figures that the maximum occurs at 4 since the change from 4 to 5 is 0. When asked to sketch a graph of the temperature function, she sketches 3 straight line segments. From 0 to 3 is the steepest positive slope. The slope is less steep but still positive from 3 to a maximum at 4 and then negative at about the same steepness from 4 to 6. She confirms that no part of the curve is meant to be concave up or down contradicting the information that $f''(x) = -2$ for the second half of the graph.

When asked for an equation on the interval $[3,6]$, Ingrid uses the points $(4, 64)$ and $(6, 62)$ that she has calculated earlier to find the equation of a line. The last question the interviewer asks is what does it say about the graph that $f''(x) = -2$. Ingrid responds, "Concave down?" She then makes her graph look concave down from 4 to 6 and marks out the equation of the line she found for that interval. The interview concludes with no further questions.

Appendix B — Topics Covered in Calculus BC (1993-1994)

A. Functions and Graphs

1. Properties of functions

   a. Domain and range

   b. Sum, product, quotient, and composition

   c. Odd and even functions

   e. Periodic functions

   f. Zeros of a function

   g. Vector functions

   h. Parametric equations

   i. Conversion between polar and rectangular coordinates

2. Properties of graphs

   a. Intercepts

   b. Symmetry

   c. Asymptotes

   d. Relationships between the graph of $y = f(x)$ and the graphs of $y = kf(x)$, $y = f(kx)$, $y - k = f(x - h)$, $y = |x|$, and $y = f(|x|)$.

   e. Parametrically defined curves

   f. Graphs in polar coordinates

B. Limits and Continuity

1. Finite limits

   a. Limit of a constant, sum, product, or quotient

   b. One-sided limits: $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$

   c. Limits at infinity: $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

   d. Rigorous definitions

(1) $\lim_{x \to a} f(x) = L$ (ε and δ)

(2) $\lim_{x \to a} f(x) = L$ (ε and N)

2. Nonexistent limits

a. Types of nonexistence, e.g., $\lim_{x \to 0} \frac{1}{x^2}$, $\lim_{x \to 0} \frac{|x|}{x}$, $\lim_{x \to 0} \sin \frac{1}{x}$ are each, for different reasons, nonexistent.

b. Infinite limits, e.g., $\lim_{x \to 0} \frac{1}{x^2} = \infty$ and $\lim_{x \to 0^+} \ln x = -\infty$

c. Rigorous definition of $\lim_{x \to a} f(x) = +\infty$ (N and δ)

3. Continuity

a. Definition: $\lim_{x \to a} f(x) = f(a)$

b. Graphical interpretation of continuity and discontinuity

c. Existence of absolute extrema of a continuous function on a closed interval $[a, b]$

d. Application of the Intermediate Value Theorem

C. **Differential Calculus**

1. The derivative

a. Definition: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

b. Derivative formulas

(1) Derivations of $\frac{d}{dx}(x^n) = nx^{n-1}$ for a positive integer n and $\frac{d}{dx} \sin x = \cos x$

(2) Derivatives of elementary functions

(3) Derivatives of sums, products, and quotients

(4) Derivatives of composite functions (chain rule)

(5) Derivatives of implicitly defined functions

(6) Derivatives of higher order

(7) Derivatives of inverse functions

(8) Logarithmic differentiation

(9) Derivatives of vector functions and parametrically defined functions

2. Statements and applications of theorems about derivatives

a. Relationship between differentiability and continuity

b. The Mean Value Theorem

c. L'Hôpital's Rule for the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$ and $0 \cdot \infty$

d. L'Hôpital's Rule for the indeterminate forms 0^0, 1^∞, ∞^0, and $\infty - \infty$

3. Applications of derivative

a. Geometric applications

(1) Slope of a curve; tangent and normal lines

(2) Increasing and decreasing functions

(3) Critical points

(4) Concavity

(5) Points of inflection

(6) Curve sketching

(7) Differentials and linear approximations

(8) Newton's method for approximating zeros of functions

(9) Tangent lines to parametrically defined curves

b. Optimization problems

(1) Relative and absolute maximum and minimum values

(2) Extreme value problems

c. Rate-of-change problems

(1) Average and instantaneous rates of change

(2) Velocity and acceleration in linear motion

(3) Related rates of change

(4) Velocity and acceleration vectors for motion on a plane curve

D. Integral Calculus

1. Antiderivatives (indefinite integrals)

   a. Techniques of integration

      (1) Basic integration formulas

      (2) Integration by substitution (change of variables)

      (3) Simple integration by parts, e.g., $\int x \ln x dx$, $\int x^2 \cos x dx$, $\int \text{Arctan } x dx$

      (4) Integration by trigonometric substitution

      (5) Other integration by parts, e.g., $\int e^x \cos x dx$

      (6) Integration by partial fraction (only linear factors in the denominator)

   b. Applications of antiderivatives

      (1) Distance and velocity from acceleration with initial conditions

      (2) Solutions of $f(x)dx = g(y)dy$ (separable differential equations)

      (3) Solutions of $y' = ky$ and applications to growth and decay

2. The definite integral

   a. Definition of the definite integral as a limit of sums

   b. Properties

      (1) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

      (2) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

      (3) $\int_a^a f(x)dx = 0$

      (4) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

      (5) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

      (6) If $f(x) \leq g(x)$ on $[a, b]$ then $\int_a^b f(x)dx \leq \int_b^a g(x)dx$

   c. Approximations of the definite integral

      (1) Rectangles (Riemann sums)

      (2) Trapezoids (Trapezoidal Rule)

      (3) Parabolas (Simpson's Rule)

   d. Fundamental theorems

      (1) $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

(2) $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$

(3) Composite functions defined by integrals, e.g., $f(x) = \int_0^{x^2} e^{-t^2}\,dt$

e. Applications of the definite integral

(1) Area under a curve; area between curves

(2) Average value of function on an interval

(3) Volumes of solids with known cross sections, including solids of revolution (disc and washer methods) about the x -axis, the y -axis or a line parallel to either axis

(4) Volumes of solids of revolution (shell method) about the x -axis, the y -axis or a line parallel to either axis

(5) Area bounded by polar curves

(6) Length of a path, including parametric curves

(7) Work as an integral, with either force or displacement as a variable, e.g., Hooke's Law (conversion of units not required)

f. Improper integrals (as limits of definite integrals)

E. Sequences and Series

1. Sequences of real numbers; convergence

2. Series of real numbers

a. Convergence

(1) Tests for convergence: comparison (including limit comparison), ratio, root, and integral tests

(2) Absolute and conditional convergence

b. Special series

(1) Geometric series

(2) Alternating series and error approximation

(3) p-series

3. Power series

a. Manipulation of series, e.g., addition of series, substitution, term-by-term differentiation and integration

b. Convergence

(1) Maclaurin series expansion for $\frac{1}{1-x}$, $\sin x$, and e^x

(2) Taylor series

(3) Taylor polynomials with remainder and Lagrange error approximation $\left[R_n(x) = \frac{f^{n+1}(c)(x-a)_{(n+1)}}{(n+1)!}\right]$