Numerical Considerations on Fibonacci Numbers of Order r

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Abstract

Recently generalized Fibonacci numbers have received increasing attention. Some properties that are well known for traditional Fibonacci numbers do not generalize easily, some others do not generalize at all. In this paper we report some properties that we have generalized. Section 1 introduces the notation and a theorem due to Miller ([14]). Section 2 shows how generalized Fibonacci numbers can be expressed as rounded power of the dominant root of the characteristic equation. This generalizes a known result on Fibonacci numbers. Section 3 lists some properties of the roots of the characteristic equation. Some of these properties are, to our knowledge, new. Section 4 introduces the Zeckendorf representation of integers and lists some of its most relevant properties. Finally, in Section 5 the asymptotic proportion of ones in the Zeckendorf representation is computed and an easy to compute closed formula is given.

1 Introduction.

We define the *Fibonacci numbers of order r* , in the sequel called *R-nacci numbers* , as follows:

$$F_0^{(r)} = 0,\ F_1^{(r)} = 1\ ,...,\ F_j^{(r)} = 2^{j-2}\ , j = 2, 3, 4, ..., r-1;$$
$$F_j^{(r)} = F_{j-1}^{(r)} + F_{j-2}^{(r)} + F_{j-3}^{(r)} + \cdots + F_{j-r+1}^{(r)} + F_{j-r}^{(r)}, j \geq r. \tag{1}$$

The well known *Fibonacci* , *Tribonacci* [8], and *Quadranacci* [9] numbers arise as a special case of (1) by setting $r = 2, r = 3, r = 4$, respectively.

In studying R-nacci numbers it is necessary to consider the *characteristic equation* $f(x) = x^r - x^{r-1} - \cdots - x - 1 = 0$ of the above difference equation. Miles [13] , by reducing the characteristic equation to a form where Rouché's Theorem can be applied , has shown that the roots of the equation are distinct . Moreover , the dominant root , denoted by $\Phi_1^{(r)}$, is real and lies between 1 and 2, and the remaining r-1 roots, denoted by $\Phi_i^{(r)}$, $i \in \{2, r\}$, lie within the unit circle of the complex plane.

Miller has shown [14] that these properties can also be derived using only elementary theory of equations.

The following theorem gives a complete characterization of the expansion of R-nacci numbers in terms of the roots of the characteristic equation.

Theorem 1. Let $F_n^{(r)}$ be the n-th R-nacci number.. Then

$$F_n^{(r)} = \sum_{i=1}^{r} b_i^{(r)} (\Phi_i^{(r)})^n, \tag{2}$$

where

$$b_i^{(r)} = \frac{(\Phi_i^{(r)})^{-1}(\Phi_i^{(r)} - 1)}{(r+1)\Phi_i^{(r)} - 2r}. \tag{3}$$

Proof. Miles [13] has shown that the solution of recurrence (1) is (2) with

$$b_i^{(r)} = \frac{(\Phi_i^{(r)})^{r-2}}{(\Phi_i^{(r)} - \Phi_r^{(r)})(\Phi_i^{(r)} - \Phi_{r-1}^{(r)})\dots(\Phi_i^{(r)} - \Phi_{i+1}^{(r)})(\Phi_i^{(r)} - \Phi_{i-1}^{(r)})\dots(\Phi_i^{(r)} - \Phi_1^{(r)})}.$$

We observe that $f(x) = x^r - x^{r-1} - \dots - x - 1 = (x - \Phi_1^{(r)})(x - \Phi_2^{(r)})\dots(x - \Phi_r^{(r)})$, and set

$$\begin{aligned} F(x) &= (x^r - x^{r-1} - \dots - x - 1)(x-1) = (x-1)(x - \Phi_1^{(r)})(x - \Phi_2^{(r)})\dots(x - \Phi_r^{(r)}) = \\ & x^{r+1} - 2x^r + 1. \end{aligned}$$

It turns out that $F(\Phi_i^{(r)}) = (\Phi_i^{(r)})^{r+1} - 2(\Phi_i^{(r)})^r + 1 = 0$ and $(\Phi_i^{(r)})^r = 1/(2 - (\Phi_i^{(r)}))$.

Since

$$\begin{aligned} f'(\Phi_i^{(r)}) &= (\Phi_i^{(r)} - \Phi_r^{(r)})\dots(\Phi_i^{(r)} - \Phi_{i-1}^{(r)})(\Phi_i^{(r)} - \Phi_{i+1}^{(r)})\dots(\Phi_i^{(r)} - \Phi_1^{(r)}) = \\ & r\,(\Phi_i^{(r)})^{r-1} - \sum_{j=1}^{r-1} j\,(\Phi_i^{(r)})^{j-1} = \\ & \frac{(r+1)(\Phi_i^{(r)})^r - 2r(\Phi_i^{(r)})^{r-1}}{\Phi_i^{(r)} - 1}, \end{aligned}$$

we finally obtain

$$b_i^{(r)} = \frac{(\Phi_i^{(r)})^{r-2}}{r\,(\Phi_i^{(r)})^{r-1} - \sum_{j=1}^{r-1} j\,(\Phi_i^{(r)})^{j-1}} = \frac{(\Phi_i^{(r)})^{r-2}(\Phi_i^{(r)} - 1)}{(r+1)\,(\Phi_i^{(r)})^r - 2r\,(\Phi_i^{(r)})^{r-1}} = \frac{(\Phi_i^{(r)})^{-1}(\Phi_i^{(r)} - 1)}{(r+1)\,\Phi_i^{(r)} - 2r}. \square$$

As we shall see in the foregoing, $b_1^{(r)}$ plays an important role. Thus in particular we have:

$$b_1^{(r)} = \frac{(\Phi_1^{(r)})^{-1}(\Phi_1^{(r)} - 1)}{(r+1)\Phi_1^{(r)} - 2r}. \tag{4}$$

Using fact R2 in Section 4 we observe that as r becomes large , $b_1^{(r)}$ tends to $1/4$, and for $i > 1$, $b_1^{(r)}$ tends to 0.

Table 1 lists the values of the dominant root $\Phi_1^{(r)}$ and the constant $b_1^{(r)}$ for $r \in [2, 11]$.

r	$\Phi_1^{(r)}$	$b_1^{(r)}$	r	$\Phi_1^{(r)}$	$b_1^{(r)}$
2	1.61803	.44721	7	1.99196	.25726
3	1.83928	.33623	8	1.99603	.25404
4	1.92756	.29381	9	1.99802	.25223
5	1.96594	.27362	10	1.99901	.25123
6	1.98358	.26304	11	1.99951	.25067

Table 1.

2 Approximation of R-nacci numbers.

As r becomes large , $F_n^{(r)} \approx b_1^{(r)}\Phi_1^{(r)}$. Indeed, a stronger result holds: R-nacci numbers are rounded powers. The following theorem generalizes Binet's formula [12] for Fibonacci numbers and Spickerman's [15] for Tribonacci.

Theorem 2. The r^{th} degree R-nacci numbers can be computed from the formula

$$F_n^{(r)} = \left\lfloor b_i^{(r)} \left(\Phi_1^{(r)}\right)^n + .5\right\rfloor, n \geq 0; \tag{5}$$

where $\lfloor x \rfloor$ denotes the integral part of x.

Once the order (r) of the R-nacci number has been determined, Table 1 can then be used to get the related constant and dominant root needed in (5). It should be noticed that the approximation error in (5) becomes small for large n, hence using the appropriate rounding we are able to evaluate the exact R-nacci numbers.

In the foregoing we shall drop the superscript r, indicating the order of the R-nacci numbers, whenever no confusion arises.

The strategy we will use in proving the theorem consists in considering the deviations, d_n, defined by

$$d_n = F_n^{(r)} - b_1\Phi_1^n$$

and showing that for all $n \geq 0, \mid d_n \mid < \frac{1}{2}$.

The following 5 Lemmas will prove the theorem.

Lemma 1. Let L be the companion matrix of the difference equation, that is

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

then the column eigenvectors of L are $\mathbf{C}_i = ((\Phi_i)^{r-1}, (\Phi_i)^{r-2}, \ldots, 1)$ and the row eigenvectors of L are $\mathbf{R}_i = (1, \Phi_i - 1, (\Phi_i)^2 - \Phi_i - 1, \ldots, (\Phi_i)^{r-1} - (\Phi_i)^{r-2} - \cdots - 1)$ and $(\mathbf{R}_i \circ \mathbf{C}_i) = P'(\Phi_i)$ where $P(x)$ is the characteristic polynomial. If $i \neq j$, $(\mathbf{R}_i \circ \mathbf{C}_j) = 0$.

Proof. Direct computation and use of $\Phi_i^r = \Phi_i^{r-1} + \Phi_i^{r-2} + \cdots + 1$ shows that $L \circ \mathbf{C}_i = \Phi_i \mathbf{C}_i$ and $\mathbf{R}_i \circ L = \Phi_i \mathbf{R}_i$. Consider $\mathbf{R}_i \circ L \circ \mathbf{C}_j$, associating to the right gives $\Phi_j(\mathbf{R}_i \circ \mathbf{C}_j)$ while associating to the left gives $\Phi_i(\mathbf{R}_i \circ \mathbf{C}_j)$. Hence if $\Phi_i \neq \Phi_j$, $(\mathbf{R}_i \circ \mathbf{C}_j) = 0$. Direct computation shows that $(\mathbf{R}_i \circ \mathbf{C}_i) = P'(\Phi_i)$.□

Lemma 2. At most $r - 1$ consecutive deviations have the same sign.

Proof. From the definitions of deviations, the deviations have a zero coefficient in the direction of the eigenvector associated with Φ_1. If r consecutive deviations have the same sign and at least one of these deviations is nonzero then the deviations would have a nonzero coefficient in the direction of the Φ_1 eigenvector because $\mathbf{R}_1$ from Lemma 1 has all positive components and hence the inner product of $\mathbf{R}_1$ with these deviations would be nonzero.□

Lemma 3. The deviations satisfy the recurrence relation

$$d_{n+1} = 2d_n - d_{n-r}.$$

Proof. Since $F_n^{(r)}$ and Φ_1 satisfy the generalized Fibonacci recurrence relation of degree r and the deviations are a linear combination of $F_n^{(r)}$ and Φ_1, the deviations satisfy

$$d_{n+1} = d_n + d_{n-1} + \cdots + d_{n-r+1} = d_n + (d_{n-1} + \cdots + d_{n-r+1} + d_{n-r}) - d_{n-r} = 2d_n - d_{n-r}.\square$$

Lemma 4. If $| d_n | \geq \frac{1}{2}$, then $| d_{n-i} | > \frac{1}{2}$ for some i such that $r \geq i \geq 2$.

Proof. If d_n and d_{n+1} have different signs then since

$$d_{n+1} = 2d_n - d_{n-r}, \quad d_{n+1}^2 = 2d_n d_{n+1} - d_{n-r} d_{n+1} > 1$$

and since d_{n+1} and d_{n-r} have different signs

$$-2 \mid d_n \mid\mid d_{n+1} \mid + \mid d_{n-r} \mid\mid d_{n+1} \mid > 1. \quad \text{So } \mid d_{n-r} \mid > 2 \mid d_n \mid \geq 1 > \frac{1}{2}.$$

If d_n and d_{n+1} have the same sign then either $\mid d_{n-r} \mid > \frac{1}{2}$, or $\mid d_{n-r} \mid \leq \frac{1}{2}$ and $d_{n-r} = 2d_n - d_{n+1} = sgn(d_n)(2 \mid d_n \mid - \mid d_{n+1} \mid)$ and $-\frac{1}{2} \leq 2 \mid d_n \mid - \mid d_{n+1} \mid \leq \frac{1}{2}$ so $\mid d_{n+1} \mid \geq 2 \mid d_n \mid -\frac{1}{2} \geq \frac{1}{2}$.

If $\mid d_{n-r} \mid \leq \frac{1}{2}$ we can repeat our argument with d_{n+1} and d_{n+2}, and so forth, but we will not have to proceed beyond d_{n+r-2} because at most $r-1$ consecutive deviations can have the same sign. Hence at least one among $d_{n-r}, d_{n-r+1}, d_{n-r+2}, \ldots, d_{n-2}$, will have absolute value greater than $\frac{1}{2}$.□

To complete the proof we need to show that r consecutive deviations each have absolute value less than $\frac{1}{2}$.

We chose to show that $d_1, d_0, \ldots, d_{-r+2}$ all have absolute value less than $\frac{1}{2}$, and then conclude from Lemma 7 that $\mid d_n \mid < \frac{1}{2}$ for all $n \geq 0$, in fact this is true for $n \geq -r+2$.

Lemma 5. $\mid d_i \mid < \frac{1}{2}$ for i such that $1 \geq i \geq -r+2$.

Proof. Using the difference equation backwards, we find that $F_0^{(r)} = F_{-1}^{(r)} = F_{-2}^{(r)} = F_{-r+2}^{(r)} = 0$, and that $d_0 = -b_1$, $d_{-1} = -\frac{b_1}{\Phi_1}, \ldots, d_{-r+2} = -\frac{b_1}{\Phi_1^{r-2}}$. Since $b_1 > 0$ and $\Phi_1 > 1$, we have $\mid d_0 \mid > \mid d_{-1} \mid > \ldots > \mid d_{-r+2} \mid$. So we have only to show that $\frac{1}{2} > \mid d_0 \mid$.
This is equivalent to

$$\frac{1}{2} > \frac{\Phi_1 - 1}{\Phi_1[(r+1)\Phi_1 - 2r]} \text{ or } (r+1)\Phi_1(\Phi_1 - 2) + 2 > 0.$$

For this, using Lemma 1, we find that $\frac{r+1}{2^{r-1}}\Phi_1 < 2$ is sufficient and hence that $\frac{r+1}{2^{r-1}} \leq 1$ is sufficient.
This sufficient condition holds for all $r \geq 3$. Although this sufficient condition does not hold for $r = 2$, it is easy to check that $\frac{1}{2} > \mid d_0 \mid$ in this case by substituting $\Phi_1 = \frac{1+\sqrt{5}}{2}$.
Since $d_0, d_{-1}, \ldots, d_{-r+2}$ are negative, d_1 is necessarily positive, so we only have to show that $\frac{1}{2} > 1 - \frac{\Phi_1 - 1}{(r+1)\Phi_1 - 2r}$, but this is equivalent to $\Phi_1 < 2$ which is true.□

3 Properties of the roots.

Theorem 2 allows us to compute R-nacci numbers by using only the dominant root and neglecting the contribution of all the other roots. It is, however, of mathematical interest to know the properties of the roots of the characteristic equations. In this paragraph we list, without proof, some of them. Some of the properties listed here are known from the literature, some others are, to our knowledge, new.

We recall that the characteristic equation associated to R-nacci numbers is: $f(x) = x^r - x^{r-1} - \cdots - x - 1$ whose roots are called $\Phi_i^{(r)}$. Let $z = a + ib$ be a complex number, we denote $a = Re(z)$, $\mid z \mid = \sqrt{a^2 + b^2}$, and $arg(z)$ the angle of z in its polar coordinates representation.

R1. The characteristic equation has r distinct roots one of which, called *dominant*, is positive and lies between 1 and 2.

R2. The dominant roots of the characteristic equation for different r's form a strictly increasing succession:

$$1.618 < \Phi_1^{(2)} < \Phi_1^{(3)} < \ldots < \Phi_1^{(r)} < \ldots < 2$$

R3. For every $r \geq 2$:

$$2 - \frac{2}{2^r} < \Phi_1^{(r)} < 2 - \frac{1}{2^r}.$$

Note that this is weaker than R2 but gives an immediate range for the value of any dominant root given r.

R4. If r is even, the characteristic equation has 1 negative root which lies in $(-1, \frac{1-\sqrt{5}}{2}]$ and tends to -1 as r increases. The remaining $r-2$ roots are complex and form conjugate pairs. These complex roots have absolute values less than 1 and as r increases these absolute values approach 1.

R5. If r is odd, the characteristic equation has $r-1$ complex roots that form conjugate pairs. These complex roots have absolute values less than 1 and as r increases these absolute values approach 1.

For **R6** , **R7** , **R8** , and **R9** we shall call $\Phi_i^{(r)}$ with $1 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$ the distinct and non conjugate roots of the characteristic equation.

R6. This rule gives a bound on the modulus of the non dominant roots. For every $2 \leq i \leq r$:

$$\frac{1}{\sqrt[r]{3}} <\mid \Phi_i^{(r)} \mid< 1$$

R7. This rule gives a bound on the argument of the complex roots. For each $1 \leq i \leq \frac{r-1}{2}$, the argument of complex roots $\Phi_i^{(r)}$, satisfies

$$\frac{2i\pi}{r} < arg\left[\Phi_i^{(r)}\right] \leq \frac{2i\pi}{r-1}.$$

R8. This rule computes the sum of the real part of the complex roots.

$$-1 < \sum_{j=1}^{\frac{r-1}{2}} Re(\Phi_j^{(r)}) = \frac{1-\Phi_1^{(r)}}{2} < 0$$

R9. This rule computes the product of the moduli of the complex roots:

$$\Pi_{j=1}^{\frac{r-1}{2}} \mid \Phi_j^{(r)} \mid^2 = (\Phi_1^{(r)})^{-1}$$

4 Zeckendorf representation of integers: Properties and Extraspace.

Recently the Zeckendorf representation of integers has been shown to be a valid alternative to the binary one. In this section, after the initial definitions, we shall list some of the properties of Zeckendorf representation and prove that it requires more space than the binary representation. The space being measured in number of bits needed to encode a given set of characters. However, we shall also prove that for a large enough order ($r = 6$) the two representations "practically" require the same amount of space.

Each nonnegative integer N has the following unique (*canonical*) Zeckendorf representation in terms of Fibonacci numbers of degree r [9].

$$N = \alpha_2 F_2^{(r)} + \alpha_3 F_3^{(r)} + \alpha_4 F_4^{(r)} + ... + \alpha_j F_j^{(r)}$$

where $\alpha_i \in \{0,1\}$ and $\alpha_i\alpha_{i-1}\alpha_{i-2}\alpha_{i-3}\ldots\alpha_{i-r+1} = 0$ (no r consecutive α's are 1).

Like the binary representation of integers, Zeckendorf representation can be written as a string of 0's and 1's; i.e. $\alpha_j\alpha_{j-1}\alpha_{j-2}.....\alpha_3\alpha_2$ (where leading zeros are ineffective) . We notice that it supplies the set of binary sequences that do not contain r consecutive ones. We call such sequences *Fibonacci sequences of order r* or *R-nacci sequences.*

Capocelli [2] gives an efficient and elegant algorithm to derive the Zeckendorf representation of integers. We recall that the Zeckendorf representation is a lexicographic ordering [10].

We list , without proof , the following properties of Zeckendorf representations.

Z1. The number of R-nacci sequences of length n-1 is $F_{n+1}^{(r)}$.

Z2. The number of R-nacci sequences of length n-1 that begin with the symbol 1 is $F_{n+1}^{(r)} - F_n^{(r)}$.

Z3. The Zeckendorf representation of $F_{n+1}^{(r)}$ is $1\ \underbrace{00\ldots0}_{n-1}$.

Z4. The Zeckendorf representation of $\sum_{i=1}^{h} F_{n+i}^{(r)}$ is $\underbrace{11\ldots1}_{h}\underbrace{00\ldots0}_{n-1}$; $h < r$.

Z5. Let t and j be integers, $t > 0$ and $F_{n-t}^{(r)} \leq j < F_{n-t+1}^{(r)}$, and let α be the Zeckendorf representation of j. Then, the Zeckendorf representation of $j + \sum_{i=1}^{h} F_{n+i}^{(r)}$, $h < r$, is $\underbrace{11\ldots1}_{h}\underbrace{00\ldots0}_{t}\ \alpha$.

Z6. The Zeckendorf representation provides a *standard representation* of integers in the sense that if x is less than y , the representation for x is no longer than the representation for y .

The following proposition gives an estimate of the extra space required if we use Zeckendorf representation instead of the binary one. Table 2 lists the extraspace required by the Zeckendorf representation for a few values of r. We notice that the two representation become practically equivalent from $r = 6$ onward.

r	%	r	%
2	44.0%	3	13.0%
4	5.0%	5	2.0%
6	1.0%	7	0.5%

Table 2. % indicates the approximate extra space in percentage.

Proposition 1. For large enough n, the percentage of extraspace required by the Zeckendorf representation (wrt the binary one) in encoding a set of F_{n+1}^{r} characters is $(\log(\Phi_1^{(r)}))^{-1}$. For large values of r the two representations use almost the same space.

Proof. Suppose we want to represent $F_{n+1}^{(r)}$ characters. Using the binary representation we need $\lceil \log(F_{n+1}^{(r)}) \rceil$ bits. Instead, using Zeckendorf's, from fact Z1 we deduce that $n-1$ bits are needed. To compute the extraspace it will then suffice to estimate the ratio:

$$s_n^{(r)} = \frac{\lceil \log(F_{n+1}^{(r)}) \rceil}{n-1}.$$

Since we know that for large values of n approximation (5) holds , we have:

$$s^{(r)} = \log(\Phi_1^{(r)}) + O(\frac{1}{n})$$

From R2 in section 4 we know that $\lim_{r\to\infty} \Phi_1^{(r)} = 2$ hence $\lim_{r,n\to\infty} s_n^{(r)} = 1$ and the proposition follows. □

5 Asymptotic Proportion of ones.

Table 3 shows a possible Zeckendorf representation for the first $F_6^{(3)}$ numbers.

0	00000	6	00110	12	01101	18	10101
1	00001	7	01000	13	10000	19	10110
2	00010	8	01001	14	10001	20	11000
3	00011	9	01010	15	10010	21	11001
4	00100	10	01011	16	10011	22	11010
5	00101	11	01100	17	10100	23	11011

Table 3.

As we know , using the binary representation the proportion of 0's and 1's in all the strings of length n is a constant equal to $\frac{1}{2}$. If we use Zeckendorf representation , instead , the proportion is not constant, for Table 3 the proportion is .416 (=50/120).

The following theorem gives a complete characterization.

<u>Theorem 3.</u> The proportion of 1's in the Zeckendorf representation of integers is:

$$A(1)_n^{(r)} = (\Phi_1^{(r)})^{-1} - r\, b_1^{(r)}\,(2 - \Phi_1^{(r)}) + O(\frac{1}{n}) \tag{6}$$

and tends to $\frac{1}{2}$ as r increases.

<u>*Proof.*</u> If we denote by $N(1)_n^{(r)}$ and by $N_n^{(r)}$ the number of ones and the total number of sequences in R-nacci sequences of length $n+1$, respectively, we have that the asymptotic proportion of ones is given by:

$$A(1)_n^{(r)} = \frac{N(1)_n^{(r)}}{N_n^{(r)}}.$$

By induction on n it is possible to prove that the total number of ones, $N(1)_n^{(r)}$, in Fibonacci sequences of length $n-1$ satisfies the following equation [2]:

$$
\begin{aligned}
N(1)_n^{(r)} &= N(1)_{n-1}^{(r)} + N(1)_{n-2}^{(r)} + F_{n-1}^{(r)} + N(1)_{n-3}^{(r)} + 2\,F_{n-2}^{(r)} + \ldots \\
&\quad + N(1)_{n-r+1}^{(r)} + (r-2)\,F_{n-r+2}^{(r)} + N(1)_{n-r}^{(r)} + (r-1)\,F_{n-r+1}^{(r)}
\end{aligned}
$$

which, applied recursively, gives:

$$
N(1)_n^{(r)} = \sum_{j=0}^{n-2} (F_{j+1}^{(r)}) \sum_{i=1}^{r-1} i\,(F_{n-i-j}^{(r)})
$$

From fact Z1 in section 5 we know that the total number of symbols in R-nacci sequences of length $n-1$ is $(n-1)F_{n+1}^{(r)}$. Thus we have:

$$
A(1)_n^{(r)} = \frac{\sum_{j=0}^{n-2} (F_{j+1}^{(r)}) \sum_{i=1}^{r-1} i\,(F_{n-i-j}^{(r)})}{(n-1)F_{n+1}^{(r)}} \tag{7}
$$

Using approximation (5) and after some algebra, (7) becomes:

$$
A(1)_n^{(r)} = b_1^{(r)} \sum_{i=1}^{r-1} i\,(\Phi_i^{(r)})^{-i} + O(\frac{1}{n}) \tag{8}
$$

It turns out that

$$
\sum_{i=1}^{r-1} i\,(\Phi_i^{(r)})^{-i} = (b_1^{(r)}\,\Phi_1^{(r)})^{-1} - r\,(\Phi_1^{(r)})^{-r}
$$

Thus (8) becomes:

$$
A(1)_n^{(r)} = (\Phi_1^{(r)})^{-1} - r\,b_1^{(r)}\,(\Phi_1^{(r)})^{-r} + O(\frac{1}{n}) \tag{9}
$$

and using fact R1 in section 4 we get : $\lim_{r,n\to\infty} A(1)_n^{(r)} = \frac{1}{2}$. In the proof of theorem 1 we showed that: $(\Phi_i^{(r)})^r = 1/(2 - (\Phi_i^{(r)}))$ which substituted into (9) proves (6). □

The asymptotic proportion of ones in R-nacci sequences has been recently calculated by Chang [5] using a somewhat different method. He found a slightly different, although equivalent algebraic expression. Our formula is considerably simpler. Moreover, it seems to give more accurate numerical values. Indeed, numerical values obtained from Chang's formula do not converge increasingly to 1/2 as is to be expected. This is essentially due to the fact that to calculate the constant $b_1^{(r)}$, Chang uses the identity $(2-\Phi_2^{(r)})\ldots(2-\Phi_r^{(r)}) = 1/(2-\Phi_1^{(r)})$; whereas we use the identity $(\Phi_1^{(r)})^r = 1/(2-\Phi_1^{(r)})$. The identity used by Chang is more sensitive to rouding errors. To estimate the asymptotic proportion of ones $R_1^{(r)}$ in R-nacci sequences , Table 1 can be used.

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