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Linear programming is a recent development in the field of Mathematics, having its origin in the past seventy-five years. The purpose of this study was to identify several methods for solving linear programming problems. The algorithm for each of the methods is described in detail along with an analysis of its effectiveness in solving real-life problems.

Also included as part of the paper is a computer program written in Basic. The program solves linear programming problems in two dimensions using the Graphical Method.

A Survey of Linear Programming Algorithms

by

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TABLE OF CONTENTS

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LIST OF FIGURES

LIST OF TABLES

A SURVEY OF LINEAR PROGRAMMING ALGORITHMS

INTRODUCTION

Beginning with George Dantzig's work in the late 1940's to the present day work of Narendra Karmarkar, the study of linear programming and its applications to real life problems have been an important part of Applied Mathematics. Linear programming is a standard tool which has saved millions of dollars annually for businesses throughout the industrialized world. It has also been a major reason for the great advancements in scientific calculations on computers.

Linear programming is used in solving the problem of allocating a limited amount of resources among several competing activities in the "best" possible way. One example of this idea is the need for a company to allocate its limited resources to the many different products it manufactures in a way which maximizes the profits for the company. All linear programming methods use a mathematical model to describe the problem which must be solved. The "linear" part of linear programming means that all of the functions which are used in the problem are linear functions. "Programming", though, does not mean that a computer program needs to be used to solve the problem. It means that a systematic set of operations are taken to solve the problem in the most efficient way.

The general linear programming problem has the following standard form :

Select the values for $x_1, x_2, \ldots, x_n$ so as to

maximize $\quad Z = c_1x_1 + c_2x_2 + \ldots + c_n$

subject to $\quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$

and $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, m \geq 2, n \geq 2.$

The function being maximized, $c_1x_1 + \ldots + c_nx_n$, is called the objective function. The restrictions on the x_i's are referred to as the constraint equations. The first m of these constraint equations are generally referred to as the functional constraints. Similarly, the restrictions of the form $x_i \geq 0$ are called nonnegativity constraints, and the x_i variables are referred to as the decision variables, i.e. the variables that must be determined for the particular problem. All of the constants, i.e. a_{ij}, b_i, and c_j are called the parameters of the problem. There are other forms of the linear programming problem which are slight variations of the general problem stated above. The four most common differences are the following :

1. Minimizing rather than maximizing the objective function, i.e. minimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$.

2. One or more of the functional constraints are greater than or equal to inequalities, i.e. $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$ for some value of i.

3. One or more of the functional constraints are in equation form, i.e. $a_{i1}x_1 + ... + a_{in}x_n = b_i$ for some i.

4. The nonnegativity constraint can be removed for some decision variables, i.e., x_j unrestricted in sign for some values of j. All of the algorithms described in this paper can be adjusted to solve problems in these "nonstandard" forms.

In linear programming the terminology for a solution to a problem takes on several forms. Any specification of values for the decision variables, $(x_1, x_2, ..., x_n)$, is called a solution, even if it is not desireable or even allowable. A feasible solution is a specification of values for the decision variables for which all of the constraints are satisfied. Finally, a feasible solution that has the most favorable value of the objective function is called the optimal solution.

Since there are n decision variables that must be determined, then the solution to the problem is a point in $\mathbb{R}^n$. Thus, the set of feasible solutions to the problem forms a polygon in $\mathbb{R}^n$. The purpose of linear programming is to determine the optimal solution from the polygon of feasible solutions.

The polygon of feasible solutions in $\mathbb{R}^n$ has up to C(m,n) vertices, the number of possible combinations of n constraints equations from the m total constraint equations. A vertex of the feasible set is the intersection point of n of the constraint equations that satisfies all of the constraint equations. The Fundamental Theorem of Linear Programming shows that the optimal solution for any linear programming problem occurs at a vertex point of the feasible region.

There is a wide variety of techniques that are used to solve linear programmming problems. If there are only two decision variables in the problem, then the feasible region can be graphed and the optimal solution can be determined using the Graphical Method. The most commonly used method for solving linear programming problems is the Simplex Method developed by George Dantzig in the late 1940's. It has been found to be a very efficient method for solving the problem in all but a few specially formulated problems.

The transportation problem and the assignment problem, two special problems, can be solved using very specific linear programming techniques. The Transportation Method and the Hungarian Method are very specific methods which solve these problems. Since 1975, there have been several new techniques for solving the general linear programming problem. These new methods are attempts to find the solution to the problem in a shorter amount of time than the Simplex Method. Two of these methods, the Ellipsoidal Algorithm and Karmarkar's Algorithm have been effective in solving linear programming problems. As of the date of this paper, though, the savings in time when using these methods are minimal.

This paper discusses several of the techniques used in solving linear programming problems. The first chapter describes the linear programming problem in two dimensions and how the Graphical Method can be used to solve the problem. Chapter 2 describes the use of the Simplex Method in solving linear programming problems. The transportation problem is solved using the Transportation Method in Chapter 3. Also, a special case of the transportation problem, the assignment problem, is solved using the Hungarian Method.

Chapters 5 and 6 discuss two recent algorithms used in solving linear programming problems having thousands of constraints. The Ellipsoidal Method is discussed in Chapter 5. Finally, in Chapter 6, Karmarkar's Algorithm for solving linear programming problems is described. Along with a discussion of each of the algorithms, the effectiveness of each of the methods is evaluated.

Also included as part of this paper is a program written in Apple Basic which solves the linear programming problem in two dimensions. The program has three linear programming example problems. One problem has a finite feasible region, while the other two show examples with infinite feasible regions, one with no maximum value of the objective function and one with no minimum value.

Also, the user may input his own problem and let the program find the optimal solution. The program has the user input the coefficients of each of the constraint equations and for the objective function. The constraint equations are then transformed into standard form and graphed one by one. For each equation, the region which is NOT feasible is shaded. After all of the equations are graphed, the area not shaded is the feasible region. Finally, the objective function is evaluated for each of the vertex points and the optimal value is determined.

CHAPTER 1: THE GRAPHICAL METHOD

The most elementary case of a linear programming problem is one in which only two variables are involved in the problem. In this case, the easiest method for solving the problem is the Graphical Method. When the Graphical Method is used to solve a linear programming problem the following set of steps are followed to find the optimal solution.

Step 1 : The problem must be translated into mathematical language.

First, all the data must be organized for the problem. Also, all of the unknown quantities in the problem must be identified and a corresponding variable name must be defined. All of the restrictions on the variables in the problem are then translated into linear inequalities. Finally, the objective function for the problem must be determined.

Step 2 : Graph the feasible set.

To graph each of the inequalities found in step 1, it is best to put each equation in standard form, i.e., either $y \leq mx + b$, $y \geq mx + b$, or $y = mx + b$. Now, the straight line corresponding to each inequality can be graphed and the region of the graph which satisfies the inequality can be determined. The region of the graph which does NOT satisfy the equation is then crossed out. After all of the inequalities have been graphed and the corresponding regions have been crossed out, the feasible region is the region which has not been crossed out. The boundary of the feasible region is formed by the set of equality equations corresponding to the inequality constraints of the problem.

Step 3 : Determine the vertices of the feasible set.

By the Fundamental Theorem of Linear Programming, the optimal point is one of the vertices of the feasible set. Each vertex point of the feasible region can be found using two of the inequality constraints.

Equate the two inequalities that intersect at the vertex and solve for the x and y values of the vertex point.

Step 4 : Determine the optimum point.

To find the optimum point, i.e. the solution to the linear programming problem, the objective function is evaluated at each vertex point of the feasible region. The point with the optimal value of the objective function is the solution to the linear programming problem.

There is one problem which may occur with the Graphical Method. If the feasible region is not bounded, then the objective function can have either a maximum or a minimum value in the feasible region, but not both. Thus, in cases where the feasible region is unbounded, an optimum value may be impossible to obtain.

The following example will illustrate the Graphical Method where the feasible region is unbounded.

Example : Suppose that in a developing nation the government wants to encourage everyone to make rice and soybeans part of his staple diet. The object is to design a "lowest cost" diet which provides certain mimimum levels of protein, calories and vitamin B_2 (riboflavin). Suppose that one cup of uncooked rice costs 21 cents and contains 15 grams of protein, 810 calories and 1/9 milligram of riboflavin. On the other hand, one cup of uncooked soybeans costs 14 cents and contains 22.5 grams of protein, 270 calories and 1/3 milligram of riboflavin. Suppose that the minimum daily requirements for each person are 90 grams of protein, 1620 calories and 1 milligram of riboflavin. Design the "lowest cost" diet meeting these specifications.

Step 1 : Translate the problem into mathematical language.

Organize the data.

	Rice	Soybeans	Required level / day
Protein(grams/cup)	15	22.5	90
Calories (per cup)	810	270	1620
Riboflavin (mg/cup)	1/9	1/3	1
Cost (cents/cup)	21	14	

Table 1.1. Data for the diet problem.

Identify and define variables

x = number of cups of rice per day

y = number of cups of soybeans per day

Variable restrictions

$15x + 22.5y \geq 90$

$810x + 270y \geq 1620$ Satisfy daily requirements

$(1/9)x + (1/3)y \geq 1$

$x \geq 0, y \geq 0$

Objective function

$[\text{cost}] = 21x + 14y$

Step 2 : Graph the feasible set

Put the restrictions (inequalities) into standard form.

Inequality	Standard form
$15x + 22.5y \geq 90$	$y \geq (-2/3)x + 4$
$810x + 270y \geq 1620$	$y \geq -3x + 6$
$(1/9)x + (1/3)y \geq 1$	$y \geq (-1/3)x + 3$
$x \geq 0$	$x \geq 0$
$y \geq 0$	$y \geq 0$

Determine the area in the feasible set for each line.

Line	Feasible set
$y = (-2/3)x + 4$	above
$y = -3x + 6$	above
$y = (-1/3)x + 3$	above
$x \geq 0$	right
$y \geq 0$	above

Picture with feasible set unshaded.

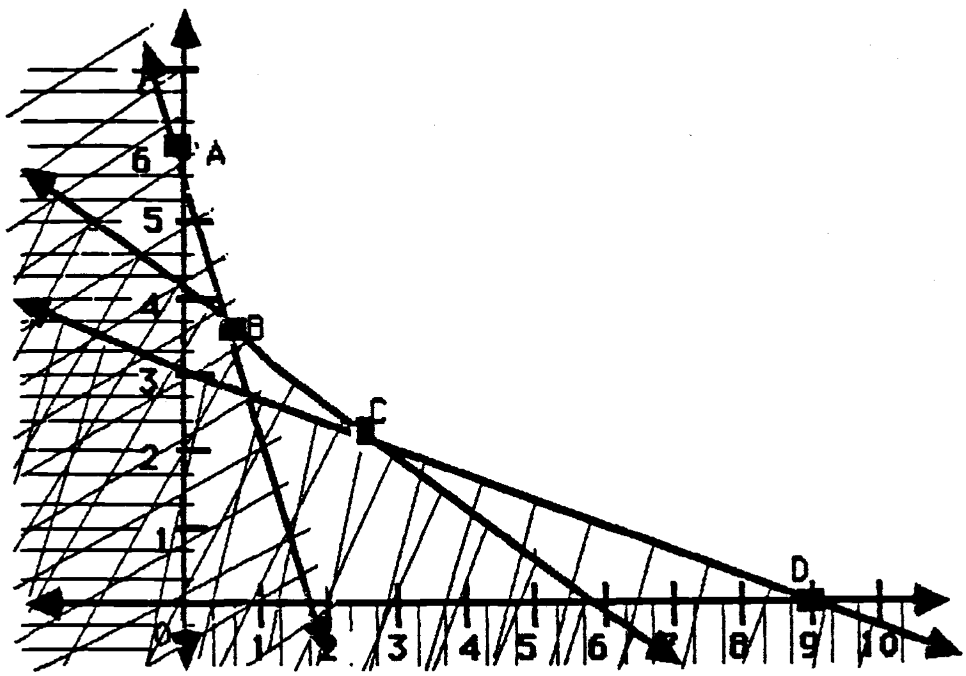

Figure 1.1. Feasible set for the diet problem.

Step 3 : Determine the vertices of the feasible set.

Point	Equations	Solution (vertex point)
A	$x = 0$	(0,6)
	$y = -3x + 6$	
B	$y = -3x + 6$	(6/7,24/7)
	$y = -2/3x + 4$	
C	$y = -2/3x + 4$	(3,2)
	$y = -1/3x + 3$	
D	$y = -1/3x + 3$	(9,0)
	$y = 0$	

Step 4 : Determine the optimal point.

Vertex	[cost] = 21x + 14y
(0,6)	84
(6/7,24/7)	66
(3,2)	91
(9,0)	189

SOLUTION : Minimum cost = 66, Optimal point (6/7, 24/7). Thus, each person should eat (6/7) of a cup of rice and 3 (3/7) cups of soybeans, which will cost 66 cents per day.

The following example will illustrate the Graphical Method where the feasible region is bounded.

Example : The Bluejay Lacrosse Stick Company makes 2 kinds of lacrosse sticks. Type A sticks require 2 man-hours for cutting, 1 man-hour for stringing, and 2 man-hours for finishing and is sold for a profit of $8. Type B sticks require 1 man-hour for cutting, 3 man-hours for stringing, and 2 man-hours for finishing and is sold for a profit of $10. Each day the company has available 24 man-hours for cutting, 30 man hours for stringing and 28 hours for finishing. How many lacrosse sticks of each kind should be manufactured each day in order to maximize profits?

Step 1 : Translate the problem into mathematical language.

Organize the data.

	Type A	Type B	Max(man-hrs/day)
Cutting (man-hours)	2	1	24
Stringing (man-hours)	1	3	30
Finishing (man-hours)	2	2	28
Profit (dollars)	8	10	

Table 1.2. Data for the lacrosse problem.

Identify and define variables

x = number of Type A lacrosse sticks produced each day
y = number of Type B lacrosse sticks produced each day

Variable restrictions

$2x + y \leq 24$
$x + 3y \leq 30$
$2x + 2y \leq 28$
$x \geq 0, \quad y \geq 0$

Objective function

[profit] = 8x + 10y

Step 2 : Graph the feasible set.

Put the restrictions (inequalities) into standard form.

Inequality	Standard form
$2x + y \leq 24$	$y \leq -2x + 24$
$x + 3y \leq 30$	$y \leq -(1/3)x + 10$
$2x + 2y \leq 28$	$y \leq -x + 14$
$x \geq 0$	$x \geq 0$
$y \geq 0$	$y \geq 0$

Determine the area in the feasible set for each line.

Line	Feasible set
$y = -2x + 24$	below
$y = (-1/3)x + 10$	below
$y = -x + 14$	below
$x = 0$	right
$y = 0$	above

Picture with feasible set, the unshaded region.

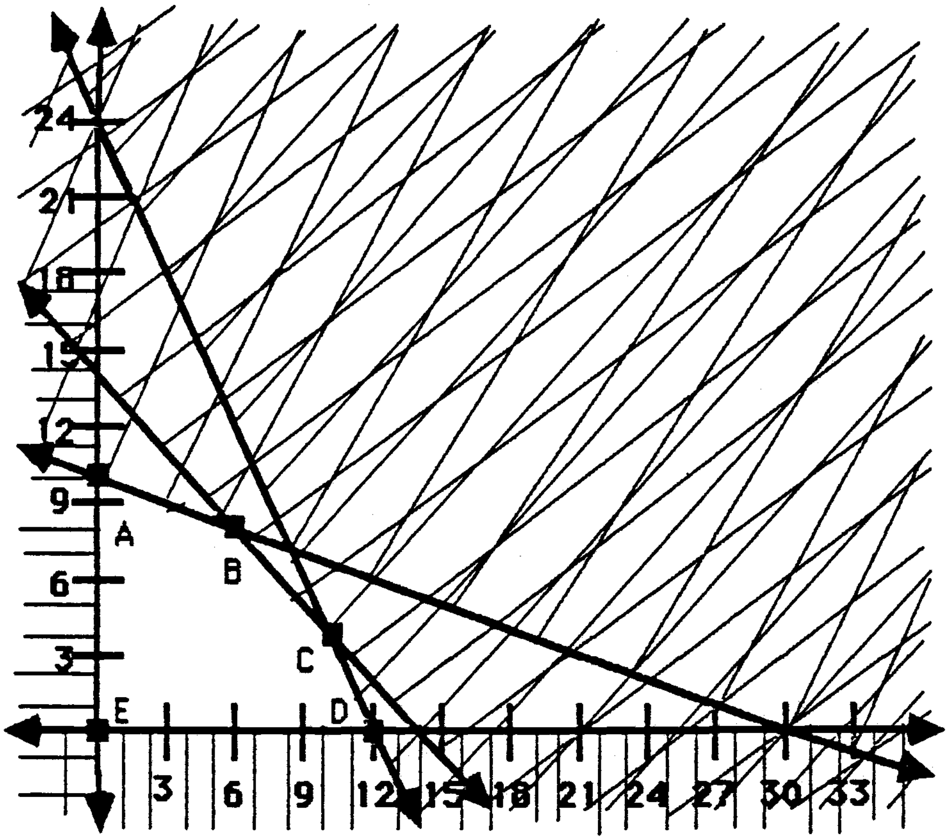

Figure 1.2. Feasible set for the lacrosse problem.

Step 3 : Determine the vertices of the feasible set.

Point	Equations	Solution (vertex point)
A	$x = 0$ $y = -(1/3)x + 10$	(0,10)
B	$y = -(1/3)x + 10$ $y = -x + 14$	(6,8)
C	$y = -x + 14$ $y = -2x + 24$	(10,4)
D	$y = -2x + 24$ $y = 0$	(12,0)
E	$x = 0$ $y = 0$	(0,0)

Step 4 : Determine the optimal point.

Vertex	[profit] = 8x + 10y
(0,10)	100
(6,8)	128
(10,4)	120
(12,0)	96
(0,0)	0

SOLUTION : Maximum profit = $128, Optimal point (6,8). Thus, the company should produce 6 Type A rackets and 8 Type B rackets each day for a net profit of $128 per day.

As these two examples show, the Graphical Method is a very efficient method for solving linear programming problems having two decision variables. The method finds all of the vertices of the feasible region, checks the objective function value for each vertex and determines the optimal solution. From this method, other, more complex algorithms for solving linear programming problems have been constructed.

CHAPTER 2: THE SIMPLEX METHOD

Most linear programming problems have more than two variables which must be considered. It is quite common for a problem to have thousands of variables and thousands of constraint equations. The Graphical Method cannot be used in these instances, since the graph would be n-dimensional in nature. However, the idea of finding the corner points of the n-dimensional figure will still be used in solving the problem. All of the corners will not be found, though, since the number of corners grows exponentially with the number of constraint equations.

In 1949, George Dantzig developed a method to solve the general linear programming problem. The method starts with an initial feasible solution (corner feasible point) and then works to find the optimal solution by improving the value of the objective function. This is done by moving from one vertex to another in a way which improves the objective function at each step and avoids checking vertices which are known to be suboptimal. The method is called the Simplex Method and is the most widely used method for solving linear programming problems.

The Simplex Method makes use of the following four assumptions of linear programming.

1. Proportionality: This assumption deals with the decision variables considered independently of the others. The assumption is that the measure of the effectiveness of a variable, x_i, equals $c_i x_i$ and the usage of each resource i equals $a_{ik} x_k$. Thus, both of the quantities are directly proportional to the level that the variable is used. So, there is no startup cost with the beginning of an activity and the proportionality holds throughout the entire range level of the activity.

2. Additivity : This particular assumption guarantees that the objective function and constraint equations are linear, i.e. there are no interactions between activities that could change the total measure of effectiveness or the total usage of some resource. So, given any activity levels, i.e. assigned values for x_1, x_2, ... x_n, the total usage of each resource and the resulting total effectiveness of the assignment is the sum of the corresponding quantities generated by each of the activities performed individually.

3. Divisibility : Most linear programming problems need decision variables with integral values, but this does not occur often in linear programming. So, this assumption says that the activity units can be divided into fractional levels and these fractional levels are permissable in the solution of the problem.

4. Certainty : The assumption here is that all of the parameters of the model a_{ij}, b_i and c_j are known constants. This assumption is usually not satisfied, but the best approximations are made for these values and are used as if they were the exact known constants.

These assumptions are not used explicitly in the solution of the problem, but they are the reason that the method holds true for such a diversified set of linear programming problems. The 5 basic steps of the Simplex Method can now be outlined as the following :

1. Define the problem variables.

This step includes the identification and assignment of names to each of the variables in the problem. Also at this step, the numerical values for the various parameters a_{ij}, b_i, and c_j are found.

2. State the problem mathematically.

State the linear objective function and the linear constraint equations.

3. Modify the problem to allow for a simplex solution.

This step includes changing all of the constraint inequalities into equality form by the use of slack variables. Also, it is often necessary to introduce artificial variables to satisfy certain requirements of the method.

4. Construct the initial simplex tableau.

The information of the problem is now summarized in tabular form. Also, the tableau represents the initial feasible solution to the problem.

5. Solve the problem.

This step can be separated into the following two parts

a. Find a basic feasible solution which is superior to the existing basic feasible solution.

b. Test the new solution for optimality. If the current solution is optimal, then the problem is solved. If the solution is not optimal, return to part (a) and repeat the procedure.

The procedure, as well as the terms used in the preceding set of steps, can be best described with a simple example.

PROBLEM : A furniture manufacturer makes two types of furniture, chairs and sofas. The production process for each can be divided into the operations of carpentry, finishing, and upholstery. The manufacture of a sofa requires 3 hours of carpentry, 1 hour of finishing, and 6 hours of upholstery. The manufacture of a chair requires 6 hours of carpentry, 1 hour of finishing, and 2 hours of upholstery. The company is limited in skilled labor as well as in equipment. Thus, the factory has 96 man-hours available for carpentry, 18 man-hours of finishing, and 72 man-hours for upholstery each day. The profit for for each sofa is $70 and the profit for the company per chair is $80.

QUESTION : How many chairs and how many sofas should be produced each day in order to maximize the profits for the day?

1. Define the problem variables

Let x_1 = number of sofas made per day

x_2 = number of chairs made per day

a_{11} = number of hours of carpentry needed per sofa

a_{12} = number of hours of carpentry needed per chair

a_{21} = number of hours of finishing needed per sofa

a_{22} = number of hours of finishing needed per chair

a_{31} = number of hours of upholstery needed per sofa

a_{32} = number of hours of upholstery needed per chair

b_1 = total number of hours available for carpentry

b_2 = total number of hours available for finishing

b_3 = total number of hours available for upholstery

c_1 = profit made from each sofa

c_2 = profit made from each chair

For this problem the values are as follows :

$a_{11} = 3$	$a_{21} = 1$	$a_{31} = 6$
$a_{12} = 6$	$a_{22} = 1$	$a_{32} = 2$
$b_1 = 96$	$b_2 = 18$	$b_3 = 72$
$c_1 = 70$	$c_2 = 80$	

2. State the problem mathematically.

The general form is

Maximize $Z = \sum_{j=1,n} c_j x_j$

Subject to $\sum_{j=1,n} a_{ij} x_j \leq b_i$ (for $i = 1,2,\ldots,m$)

and $x_j \geq 0$ (for $j = 1,2,\ldots,n$)

For the current problem, the model is the following :

Maximize $Z = 70x_1 + 80x_2$

Subject to $3x_1 + 6x_2 \leq 96$

$x_1 + x_2 \leq 18$

$6x_1 + 2x_2 \leq 72$

$x_1 \geq 0, \quad x_2 \geq 0$

Thus, the problem is to maximize the profit $Z = 70x_1 + 80x_2$ subject to the three man-hour constraints for each of the three operations.

3. Modify the problem for a simplex solution.

The Simplex Method requires that there be equalities instead of inequalities in our problem. Thus, the three constraint equations must be modified. The first constraint is $3x_1 + 6x_2 \leq 96$ and to make this an equality, a non-negative slack variable x_3 must be added. The slack variable picks up the "slack" man-hours that are not used in the carpentry of the sofas and chairs, if there is any extra. This same idea is applied to the second and third constraints using slack variables x_4 and x_5. Now, the problem can be stated in terms of the equality conditions.

Maximize $Z = 70x_1 + 80x_2 + 0x_3 + 0x_4 + 0x_5$

Subject to $3x_1 + 6x_2 + x_3 = 96$

$x_1 + x_2 + x_4 = 18$

$6x_1 + 2x_2 + x_5 = 72$

and $(x_1, x_2, x_3, x_4, x_5) \geq 0$

In the profit function, Z, the zero values for the contributions of the slack variables x_3, x_4 and x_5 shows that the idle process times do not affect the value of the objective function. These values can now be disregarded.

4. Construct the initial Simplex tableau.

Notice that the current system has two more variables than equations. This means that any two variables can be chosen to be set equal to any arbitrary value in order to solve the three equations in terms of the other three variables. The arbitrary value used in the Simplex Method is 0. The variables that are set equal to zero are called the non-basic variables, and the others are called the basic variables. In this problem, the Simplex Method sets the slack variables as the basic variables and the variables x_1 and x_2 as the non-basic variables. Thus, the initial solution $x_1 = 0$, $x_2 = 0$, $x_3 = 96$, $x_4 = 18$ and $x_5 = 72$ is called a basic solution. This solution is also called a basic feasible solution since all of the variables have non-negative values. The initial value of the objective function, Z, is zero.

Also, before putting the information in tabular form, the objective function is rewritten by bringing all of the variables to the left hand side of the equation. In this case the objective function becomes $Z - 70x_1 - 80x_2 = 0$. The initial Simplex tableau can be constructed using the four equations, one objective equation, and three constraint equations. The initial tableau for this problem is shown below. Since Z is the objective function variable, it is always a basic variable, and the other current basic variables are x_3, x_4 and x_5. The coefficients for the equations are in their respective positions.

Basic Variable	Eq. No.	Z	x1	x2	x3	x4	x5	Right Side
Z	0	1	-70	-80	0	0	0	0
X3	1	0	3	6	1	0	0	96
X4	2	0	1	1	0	1	0	18
X5	3	0	6	2	0	0	1	72

Table 2.1. Initial Simplex tableau for the furniture problem.

Note, since each equation contains only one basic variable, with a coefficient of +1, then each basic variable equals the constant on the right hand side of its equation.

5. Solve the problem.

Check to see if the solution is optimal. The solution is optimal if every coefficient in row 0 is non-negative. The coefficients in row 0 are the costs of not having each variable in the current solution. Thus, it is costing the company $70 per unit for not having x_1 in the solution and $80 for not having x_2 in the solution. Therefore, to find a better solution, replace one of the basic variables by a non-basic variable.

Finding a new basic feasible solution which is superior to the present one takes three steps.

a. Determine the entering basic variable by choosing the non-basic variable which will increase Z at the fastest rate, i.e. the variable with the largest negative value in row 0. Put a box around the column containing this coefficient-this column is the pivot column. In this problem the entering basic variable is x_2.

b. Determine the leaving basic variable by taking each of the positive coefficients in the pivot column and dividing them into the right hand side of the same row. Identify the equation which has the smallest ratio. The basic variable in this equation is the variable which will reach zero first as the entering basic variable increases. Thus, this variable is the leaving basic variable. Put a box around this row which is called the pivot row. The number that is in both of the boxes is the pivot number. The ratios for this problem are 96/6 = 16 for equation 1, 18/1 = 18 for equation 2 and 72/2 = 36 for equation 3. Thus row 1 is the pivot row, x_3 is the leaving basic variable and the coefficient 6 in row 1 is the pivot number. Currently the tableau is as follows.

Basic Variable	Eq. No.	Coefficient of Z	x1	x2	x3	x4	x5	Right Side
Z	0	1	-70	-80	0	0	0	0
X3	1	0	3	6	1	0	0	96
X4	2	0	1	1	0	1	0	18
X5	3	0	6	2	0	0	1	72

Table 2.2 Finding the pivot number.

Determine the new basic feasible solution by creating a new simplex tableau. The first column of the tableau should be the same, except x_2 replaces x_3 as a basic variable. The second and third columns are unchanged. The boxed coefficient of the new basic variable should be changed to +1 by dividing the entire pivot row, including the right, by the pivot number. The new tableau should now have the pivot row as follows.

Basic Variable	Eq. No.	Z	x1	x2	x3	x4	x5	Right Side
Z	0	1						
X2	1	0	1/2	1	1/6	0	0	16
X4	2	0						
X5	3	0						

Table 2.3. The new pivot row.

Also, it is necessary to eliminate the basic variable from each of the other equations, so that the value of the other basic variables can be determined immediately from the tableau. To do this, take each row in the tableau, including row 0, and subtract from it a multiple of the new pivot row. The multiple we need for each row is the number that is in the pivot column for each row. For row 0 the value is - 80. So the following operation is performed.

	[-70	-80	0	0	0	0]
-(-80)	[1/2	1	1/6	0	0	16]
new row	[-30	0	40/3	0	0	1280]

After performing this operation on rows 2 and 3 we have the following tableau:

Basic Variable	Eq. No.	Z	x1	x2	x3	x4	x5	Right Side
Z	0	1	-30	0	40/3	0	0	1280
X2	1	0	1/2	1	1/6	0	0	16
X4	2	0	1/2	0	-1/6	1	0	2
X5	3	0	5	0	-1/3	0	1	40

Table 2.4. The new Simplex tableau.

Thus, the new basic feasible solution is $x_1 = 0$, $x_2 = 16$, $x_3 = 0$, $x_4 = 2$, $x_5 = 40$ and $Z = 1280$.

Now, return to check if the solution is optimal. Since row 0 has a value of -30 for x_1, the solution is not optimal. Thus the column for x_1 is the new pivot column, and x_1 is the new entering basic variable. The ratios for the rows are 16/(1/2) = 32, 2/(1/2) = 4 and 40/5. So, row 2 is the pivot row and x_4 is the new lcaving basic variable.

The new tableau, after removing x_4 and bringing x_1 into the solution is as follows.

Basic Variable	Eq. No.	Z	x1	x2	x3	x4	x5	Right Side
Z	0	1	0	0	10/3	60	0	1400
X2	1	0	0	1	1/3	-1	0	14
X1	2	0	1	0	-1/3	2	0	4
X5	3	0	0	0	4/3	-10	1	30

Table 2.5. The final Simplex tableau.

Therefore, the new basic feasible solution is $x_1 = 4$, $x_2 = 14$, $x_3 = 0$, $x_4 = 0$, $x_5 = 30$ and $Z = 1400$. Since all the values in row 1 are non-negative, this is an optimal solution . The manufacturer should produce 4 sofas and 14 chairs to achieve a profit of $1400 per day. Since $x_5 = 30$, then 30 man-hours of available upholstery time are not used in the optimal solution, but all the carpentry and finishing time are used.

While computing the entering and leaving basic variables, there is a chance that two variables will tie according to the rule in the algorithm described. If there is a tie for the entering basic variable, then the selection of one or the other is arbitrary. The optimum solution will be found eventually, no matter which variable is chosen.

When determining the leaving basic variable, it can be very important if there is a tie. If a tie occurs, then both variables will become zero at the same time as the entering basic variable increases. Thus, the new feasible solution will be degenerate, i.e. there will be one more non-basic variable, variable with value zero, than there was in the previous solution. This could cause a loop in the process for finding the optimum solution. In practical problems, an infinite loop rarely occurs, so the variable used as the leaving variable is usually chosen arbitrarily.

Another problem which may occur when determining the leaving basic variable is that all the values in the pivot column may be negative or zero. If this occurs, then the entering basic entering variable can be increased indefinitely. This allows the objective function Z to increase without bound. In other words, the objective function is unbounded.

It is also possible that there is more than one optimal solution for a linear programming problem. If this is the case, then at least one of the non-basic variables will have a coefficient of zero in row 0 of the final tableau. This means that increasing the variable does not change the value of Z. Thus, it can be used as an entering basic variable, and the new optimal solution has a different set of basic variables. Other optimal solutions can be found by choosing a non-basic variable with a zero coefficient as the entering basic variable and calculating the corresponding solution.

Linear programming models other than the standard model can also be solved using the Simplex Method. Each model can be transformed into the standard form by a set of linear programming techniques. Several problem changes will be introduced here and the corresponding techniques for transforming the problem into standard form will be shown.

1. Equality constraint :

If one of the constraint equations is of the following form, $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$, then the method of artificial variables is used. If in the previous problem, equation number 1 was $3x_1 + 6x_2 = 96$, then an artificial variable, $\bar{x}_3$, is added as if it were a slack variable, and the equation becomes $3x_1 + 6x_2 + \bar{x}_3 = 96$. Adding an artificial variable to an equation enlarges the feasible region for the problem. If the problem is now solved using the Simplex Method, there is a chance that the solution found will not be feasible, since $\bar{x}_3$ may be a basic variable which would not satisfy the equality constraint. To eliminate the chance that $\bar{x}_3$ is a basic variable in the solution using the Simplex Method, the objective function is changed. A method known as the Big M Method is used to assign an overwhelming penalty for having $\bar{x}_3$ nonzero in the solution. The original function $Z = 70x_1 + 80x_2$ is changed to

$Z = 70x_1 + 80x_2 - M\bar{x}_3$, where M is a large number. This makes it so that the maximum value for Z occurs when $\bar{x}_3 = 0$.

Equation 0 is now $Z - 70x_1 - 80x_2 + M\bar{x}_3 = 0$, and in tabular form [-70 -80 M 0 0 0]. This equation cannot be used as equation 0 in the initial tableau, though. The reason is that in the initial tableau every basic variable has a coefficient of zero in row 0, and $\bar{x}_3$ is a basic variable. So, proceeding as if the column for the artificial variable, $\bar{x}_3$, were the pivot column and its equality constraint were the pivot row, eliminate the M from equation 0 as follows :

	Row 0	[-70	-80	M	0	0	0]
-M •	Row 1	[3	6	1	0	0	96]
		[-70-3M	-80-6M	0	0	0	-96M]

Now, the Simplex Method can be performed as previously illustrated. Quantities involving M are only in Row 0, so care must be taken when determining the entering basic variable. If there is more than one equality constraint in the problem, then this process is repeated for each equation.

2. Minimize the objective function.

In this case the objective function is minimize $Z = \Sigma_{j=1,n} c_j x_j$. The change here is very straight forward, just multiply the function Z by

-1 and the objective function becomes maximize $-Z = \Sigma_{j=1,n}(-c_j x_j)$. These two problems are the same since the smallest value of Z in the feasible region must also give the largest value of -Z in the region. The Simplex Method is now used on the modified problem.

3. Greater-than-or-equal-to inequality constraints.

Suppose the constraint $x_1 + x_2 \leq 18$ is changed to $x_1 + x_2 \geq 18$ in the previous problem. If the equation is multiplied by -1, then it becomes $-x_1 - x_2 \leq -18$. Now a slack variable, x_3, can be added to get $-x_1 - x_2 + x_3 = -18$. In the setup of the standard form of the Simplex Method, the right hand side must be positive. The reason for this is that a negative value on the right hand side makes the slack variable x_3 negative in the initial solution. All of the variables in the initial solution, though, are supposed to be non-negative. Multiplying the equation by -1 will make the right hand side positive : $x_1 + x_2 - x_3 = 18$. This equation, though, can be transformed into the standard form through the use of the artificial variable technique. So, add the artificial variable, $\bar{x}_4$, to make the equation $x_1 + x_2 - x_3 + \bar{x}_4 = 18$ and the basic variable for the equation is $\bar{x}_4$ and x_3 is a non-basic variable. In the equation, the variable x_3 is called a surplus variable, since it subtracts the "surplus" of the left hand side over to the right side, while maintaining an equivalent equation. Also, the Big M Method must be applied to row 0 to make it consistent with the standard form.

4. Variables allowed to be negative.

If a decision variable, x_j, can take on a negative value, but has a lower bound such that $x_j \geq L_j$, where L_j is some negative constant, then the problem can be converted in the following way. Let $x_j' = x_j - L_j$, then

$x_j' \geq 0$. Now, the term $(x_j' + L_j)$ can be substituted in for x_j throughout the problem and the problem fits the standard form.

On the other hand, if a decision variable, x_j, can take on any negative value, then it can be replaced by the difference of two variables, i.e. $x_j = x_j' - x_j''$ where $x_j' \geq 0$ and $x_j'' \geq 0$. The difference between the two variables can be any value, positive or negative. In the solution, though, either x_j' or x_j'' will be zero.

Other changes in linear programming problems can also be converted into a form suitable for solving using the Simplex Method. The techniques that are used to convert the problem into a workable form are similar to those described in this paper. Thus, the Simplex Method can be used to solve many problems that occur in real life.

In some cases of linear programming problems, the Simplex method works, but at a slow rate. When this occurs, it is sometimes easier to solve another problem called the Dual problem. If the original problem, which is called the Primal problem, is : Find $x_1, x_2, \ldots, x_n$ so as to

$$\text{maximize} \quad Z = \sum_{j=1,n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1,n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m,$$

then the Dual problem is : Find $y_1, y_2, \ldots, y_m$ so as to

$$\text{minimize} \quad y_0 = \sum_{i=1,m} b_i y_i$$

$$\text{subject to} \quad \sum_{i=1,m} a_{ij} y_i \geq c_j \quad \text{for } j = 1, 2, \ldots, n.$$

Thus, the parameters for a constraint in the Primal problem are the coefficients of a variable in the Dual problem and vice versa. Also, the coefficients for the objective function of the Primal problem are the right sides for the Dual problem and vice versa. If there are m basic and n non-basic variables in the Primal problem, then there are n basic and m non-basic variables in the Dual problem.

It has been determined that the solution for the Dual problem can be found by checking the final tableau for the Primal problem. The optimal solution for the Dual problem is the set of values in row 0 of the tableau in the columns corresponding to the initial basic variables. Thus, for the previous problem, the values for x_3, x_4 and x_5 in the final tableau, i.e. $x_3 = 10/3$, $x_4 = 60$, and $x_5 = 0$, are y_1, y_2 and y_3 for the Dual problem and $y_0 = 1400$. This is the exact optimal value for the Primal problem. This did not occur by accident, it is a characteristic held by all linear programming problems, i.e. $\sum_{j=1,n} c_j x_j^* = \sum_{i=1,m} b_i y_i^*$ if $(x_1^*, x_2^*, \ldots x_n^*)$ and $(y_1^*, y_2^*, \ldots y_m^*)$ are the optimal solutions for the Primal and Dual problems respectively. Therefore, if there are fewer constraint equations in the Dual problem, then it is sometimes advantageous to solve the Dual problem, and determine the solution to the Primal problem from the final tableau for the Dual problem. This can be done since the dual of the Dual problem is the Primal problem.

The Simplex Method is the most commonly used method for solving linear programming problems. In the worst case, the method has a running time which is exponential, in nature, but it is quite rare that it takes this amount of time. In most cases, this method is as efficient as any other method used to solve the same problem.

CHAPTER 3: TRANSPORTATION AND HUNGARIAN METHODS

The Simplex Method is an algorithm through which many linear programming problem are solved. For certain types of problems, though, special procedures have been developed which simplify the problem-solving process. One of the most commonly used of these is the Transportation Method.

The Transportation Method was formulated, as its name suggests, as a special technique for determining the minimal cost of transporting a product from several manufacturing points to a number of different cities. In the framework of this problem, there are m plants or sources of the product and n cities or destinations for the product. Each source has a certain capacity and each destination has a certain requirement. Also, each source-destination combination has an associated cost for transporting one unit of the product. The problem that must be solved is how to minimize the total cost of transporting the product while satisfying the capacities for the sources and the requirements for the destinations.

The general transportation problem can be expressed using the following equations.

$$\text{Minimize } Z = \sum_{j=1,n} \sum_{i=1,m} c_{ij} x_{ij}$$

$$\text{subject to } \sum_{j=1,n} x_{ij} = a_i \quad \text{for } i = 1,2,\ldots,m$$

$$\sum_{i=1,m} x_{ij} = b_j \quad \text{for } j = 1,2,\ldots,n$$

$$\sum_{i=1,m} a_i = \sum_{j=1,n} b_j \quad \text{and } x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

The variables in these equations are defined as follows :

x_{ij} is the number of units to be shipped from the i^{th} source to the j^{th} destination.

c_{ij} is the cost of shipping one unit from the i^{th} source to the j^{th} destination.

a_i is the capacity of the i^{th} source.

b_j is the requirement of the j^{th} destination.

When the Transportation Method is used to solve a problem, the variables x_{ij}, a_i and b_j are usually restricted to non-negative integer values.

For the third constraint equation, $\sum_{i=1,m} a_i = \sum_{j=1,n} b_j$, to be satisfied, the sum of the source capacities must equal the sum of the destination requirements. This is rarely the case in most applications. To meet all of the requirements for the method, a slack variable must be added to account for the excess capacity or the excess requirements. When the Transportation Method is used to solve a linear programming problem, the following set of steps are followed to find the optimal solution.

Step 1 : Formalize the problem requirements

In this step, all of the capacity and requirement constraints must be determined. Also, the slack capacity or the slack requirement, if any exist, must be calculated at this time. Finally, the values of the variables c_{ij}, a_{ij} and b_j are found. These pieces are all put together to form a mathematical model for the problem.

Step 2 : Determine an initial solution

Now, the initial transportation tableau can be constructed. The transportation tableau contains all of the capacity and requirement specifications and all of the relevant cost data. An initial assignment of the capacities to the requirements which satisfy the restrictions of the problem can be determined. This information will also be part of the tableau.

Step 3 : Solve the problem

The procedure for finding the optimal solution is an iterative one. First, check the initial solution calculated in step 2 for optimality. If the solution is optimal, then the problem is solved. If the solution is not optimal, then the solution is modified to improve the value of the objective functive. The process is then repeated until an optimal solution is found. The techniques which are used to determine the inital solution, modify the present solution, and check for optimality are best described using an example.

EXAMPLE : One of the main products of the P & T Company is canned peas. The peas are prepared at three canneries located in Washington, Oregon, and Minnesota and are shipped by truck to four distributing warehouses in California, Utah, South Dakota, and New Mexico. For the upcoming season, an estimate has been made for the capacity from each cannery and for the requirement at each warehouse. The estimates are as follows :

Cannery	Capacity	Warehouse	Requirement
1	90	1	80
2	130	2	65
3	115	3	70
		4	85

Table 3.1. Capacity-Requirement data for the Transportation problem.

The cost of transporting one unit (truckload) of peas from each cannery to each warehouse is as follows :

		Warehouse			
		1	2	3	4
	1	450	500	650	850
Cannery	2	350	400	700	800
	3	1000	700	400	700

Table 3.2. Cost data for the Transportation problem.

Determine a plan for assigning the shipments to the various cannery-warehouse combinations which minimizes the total shipping costs.

Step 1 : Formalize the problem requirements

It should first be noted that the sum of the capacities exceeds the sum of the requirements by 35 units. So, it is necessary that a "slack" warehouse, warehouse 5, with an assigned requirement of 35 units be introduced. Using the general form of the Transportation problem, this problem can be stated as follows :

Let x_{ij} = the number of truckloads shipped from cannery i to warehouse j.

$$\text{Minimize} \quad Z = 450x_{11} + 500x_{12} + 650x_{13} + 850x_{14} + 0x_{15} + 350x_{21} + 400x_{22} + 700x_{23} + 800x_{24} + 0x_{25} + 1000x_{31} + 700x_{32} + 400x_{33} + 700x_{34} + 0x_{35}$$

Subject to $x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 90$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 130$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 115$$

$$x_{11} + x_{21} + x_{31} = 80$$

$$x_{12} + x_{22} + x_{32} = 65$$

$$x_{13} + x_{23} + x_{33} = 70$$

$$x_{14} + x_{24} + x_{34} = 85$$

$$x_{15} + x_{25} + x_{35} = 35$$

and also $\sum_{i=1,m} a_i = \sum_{j=1,n} b_j = 335$

and $x_{ij} \geq 0$ for all values of i and j.

Step 2 : Determine an initial solution

Looking at the first eight constraint equations, it should be noted that if any seven of them are true, then the eighth is automatically satisfied. Thus, one of the constraint equations is redundant and can be removed from the problem. If this problem was being solved using the Simplex Method, each of the remaining seven equations would be assigned an artificial and a slack variable and the initial simplex tableau would be constructed. The tableau would have seven rows and twenty-two columns. Thus, it can be seen that a basic feasible Simplex solution to this problem would have exactly seven positive valued x_{ij} variables in the solution. So, in general, the transportation problem with m capacity constraints and n requirement constraints has exactly m+n-1 variables having positive values in any solution.

An initial solution to this problem is as follows :

Ci \ Wj	W1	W2	W3	W4	W5	Cap
C1	450 / 80	500 / 10	650	850	0	90
C2	350	400 / 55	700 / 70	800 / 5	0	130
C3	1000	700	400	700 / 80	0 / 35	115
Req	80	65	70	85	35	335

Table 3.3. Initial solution for the Transportation problem.

Each cell in the tableau is divided into two parts. The upper left hand space is the unit cost associated with that particular assignment. For example, the cell (C_1,W_1) specifies the cost of shipping one unit from cannery 1 to warehouse 1 is \$450. The zeroes in column W_5 means there is no shipping costs associated with assignments to this column.

The lower right half of each cell specifies the number of units assigned to that cell. For example, cell (C_2,W_3) has an assignment of 70 units. This means that 70 units should be shipped from cannery 2 to warehouse 3. Cells with no entries have a value of zero for the variable in the initial solution. Notice that exactly seven of the x_{ij} variables have non-zero values and that all of the constraints have been satisfied. So the initial solution is $x_{11} = 80$, $x_{12} = 10$, $x_{22} = 55$, $x_{23} = 70$, $x_{24} = 5$, $x_{34} = 80$ and $x_{35} = 35$. The value of the objective function, Z, for this solution is \$172,000.

The method used to determine the initial solution is called the northwest corner rule. The northwest corner solution is a random solution. So, it may not be a good solution, but it is a simple method for finding an initial solution. When using the northwest corner rule, start in the upper left hand, or northwest, corner (C_1,W_1). A value sufficient to satisfy either the requirement W_1 or the capacity C_1, whichever is less, is assigned to cell (C_1,W_1). In this example, the value 80 is assigned to the cell, since it satisfies the requirement W_1. The capacity constraint C_1 is so far only partially met. So, move to cell (C_1,W_2) where an assignment of 10 units is made, which completely satisfies the capacity C_1. Now, move to cell (C_2,W_2), where 55 units is allocated, which completes the requirement for warehouse W_2. This process continues until all of the allocations and requirements are fulfilled. The last allocation will be in the southeast corner. The initial solution is now completely determined.

Step 3 : Solve the problem

To solve the problem, given the initial solution, the following set of steps are repeated until an optimal solution is obtained.

a) The existing solution is tested to determine whether it is optimal.

b) If the existing solution is optimal, the problem has been solved. If not, the pattern of assignments is altered giving a new solution, and the two step process is repeated.

To test a particular solution for optimality, an examination of the effect upon the objective function of shifting an assignment of cells that are presently being used to those presently unused must be performed. In other words, each unused cell must be checked to see if it would be advantageous to make it part of the solution. Consider the cell (C_2,W_1). If one unit is added to cell (C_2,W_1), then one unit must be removed from cell (C_1,W_1), one must be added to cell (C_1,W_2), and one must be removed from cell (C_2,W_2) to maintain the capacity and requirement constraints. Now, the net cost of this reassignment can be evaluated. One unit is added to cell (C_2,W_1) at a cost of \$350. Cell (C_1,W_1) is reduced by one unit which subtracts \$400 from the cost. The cost of adding one unit to cell (C_1,W_2) is \$500 and \$400 is subtracted from the cost since one unit is removed from the cell (C_2,W_2). Thus the net effect is

$\Delta C_{21} = C_{21} - C_{11} + C_{12} - C_2 = 350 - 450 + 500 - 400 = 0$ Thus, the value of the objective function can't be reduced by bringing x_{21} into the solution at this time. It is required that ΔC_{ij} is computed for each of the remaining unused cells. The values for ΔC_{ij} are recorded in the tableau which follows. The ΔC_{ij}'s are circled in the tableau.

Ci \ Wj	W1	W2	W3	W4	W5	Cap
C1	450 / 80	500 / 10	650 / (450)	850 / (-50)	0 / (-200)	90
C2	350 / (0)	400 / 55	700 / 70	800 / 5	0 / (-100)	130
C3	1000 / (750)	700 / (400)	400 / (-200)	700 / 80	0 / 35	115
Req	80	65	70	85	35	335

Table 3.4. Solution including the initial effects.

From the tableau, it can be seen that the introduction of x_{14}, x_{15}, x_{25} or x_{33} would be advantageous, since they would all decrease the objective function value. To determine which variable should enter the solution, choose the variable that indicates the greatest unit change. In this example x_{33} and x_{15} both have a value of - \$200, so the choice can be arbitrary, say x_{33}.

The value that x_{33} has in the new solution should be as large as possible. To determine the value, the loop through which the unused cell

(C_3,W_3) was evaluated must be examined. The loop for (C_3,W_3) includes (C_3,W_3), (C_2,W_3), (C_2,W_4), and (C_3,W_4). Whatever value is added to cell (C_3,W_3) must be removed from cells (C_2,W_3) and (C_3,W_4) and added to cell (C_2,W_4). Since the cell (C_2,W_3) can only afford to lose 70 units and cell (C_3,W_4) can lose 80 units, only 70 units can be relocated.

The new solution is shown in the following tableau :

Ci \ Wj	W1	W2	W3	W4	W5	Cap
C1	450 / 80	500 / 10	650 / (50)	850 / (-100)	0 / (-200)	90
C2	350 / (0)	400 / 55	700 / (200)	800 / 75	0 / (-100)	130
C3	1000 / (750)	700 / (400)	400 / 70	700 / 10	0 / 35	115
Req	80	65	70	85	35	335

Table 3.5. Second solution.

A check of this new solution, shows that there are still exactly seven cells allocated units, and the requirements are still fulfilled. Also included in the tableau are the new indicators for the unused cells. Since some of the indicators are negative, this solution is not optimal. These new indicators show that x_{15} should be brought into the solution, and x_{12} should be removed.

The resulting assignments are the following :

Ci \ Wj	W1	W2	W3	W4	W5	Cap
C1	450 / 80	500 / (250)	650 / (150)	850 / (150)	0 / 10	90
C2	350 / (-200)	400 / 65	700 / (200)	800 / 65	0 / (-100)	130
C3	1000 / (550)	700 / (400)	400 / 70	700 / 20	0 / 25	115
Req	80	65	70	85	35	335

Table 3.6. Third solution.

The new indicators show that the solution is still not optimal. So, x_{21} should be introduced and x_{35} should be removed to form the new tableau:

Ci \ Wj	W1	W2	W3	W4	W5	Cap
C1	450 / 55	500 / (0)	650 / (100)	850 / (-50)	0 / 35	90
C2	350 / 25	400 / 65	700 / (200)	800 / 40	0 / (100)	130
C3	1000 / (750)	700 / (400)	400 / 70	700 / 45	0 / (200)	115
Req	80	65	70	85	35	335

Table 3.7. Fourth solution.

Now, there is only one indicator which is negative. So, introduce the variable x_{14} into the solution and remove the variable x_{24}. This gives the following final tableau.

Ci \ Wj	W1	W2	W3	W4	W5	Cap
C1	450 / 15	500 / (0)	650 / (100)	850 / 40	0 / 35	90
C2	350 / 65	400 / 65	700 / (250)	800 / (50)	0 / (100)	130
C3	1000 / (900)	700 / (550)	400 / 70	700 / 45	0 / (150)	115
Req	80	65	70	85	35	335

Table 3.8. Final solution for the Transportation problem.

Since all of the indicators are zero or positive, the optimal solution has been found. The optimal solution is Z = \$149,000 when $x_{11} = 15$, $x_{14} = 40$, $x_{15} = 35$, $x_{21} = 65$, $x_{22} = 65$, $x_{33} = 70$ and $x_{34} = 45$. To solve the problem, the initial solution is altered by introducing a non-used variable and removing a used variable until all of the indicator variables are non-negative. At this point, the optimal solution can be read from the tableau and the value of the objective function can be determined.

A special case of the transportation problem is the assignment problem. In the assignment problem, each $a_i = b_j = 1$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. The assignment problem can be solved using the Transportation Method or the general Simplex Method. In this paper, the "Hungarian Method" will be used to make the needed assignments. The procedure for solving the assignment problem using this method entails the following 4 steps.

Step 1 : For each row of the assignment matrix, subtract the smallest value in the row from each element in the row.

Step 2 : For each column in the matrix, subtract the smallest value in the column from each element in the column. These two steps form an equivalent reduced matrix for the problem.

Step 3 : Determine whether an optimal solution can be formulated, i.e. check to see if there are m "independent zeroes" in the reduced matrix. A set of zeroes are called "independent" if there are m of them and no more than one of them lies on each horizontal or vertical line. If the solution is not optimal, then go to step 4.

Step 4 : Change the arrangement or number of zero elements in the matrix, and return to step 3. The procedure for rearranging the zeroes and for checking for "independence" are best described using the following simple problem.

PROBLEM : A rent-a-car company has 4 cars which it must send to four separate cities. The time it will take for each car to arrive at a city varies with the city. The company has estimated the amount of time it would take each car to reach each city. The estimates are listed in the following matrix. Determine the minimum total time for the cars to arrive at the cities.

		City 1	2	3	4
	1	6	1	5	9
	2	8	7	2	4
Car	3	7	3	6	9
	4	2	4	5	7

Table 3.9. Data for the Assignment problem.

Step 1 : For each row, subtract the smallest value in the row from each element in the row, to get the following matrix.

	1	2	3	4
1	5	0	4	8
2	6	5	0	2
3	4	0	3	6
4	0	2	3	5

Table 3.10. After subtracting smallest value for each row.

Step 2 : For each column, subtract the smallest value in the column from each element in the column, to get the following matrix.

	1	2	3	4
1	5	0	4	6
2	6	5	0	0
3	4	0	3	4
4	0	2	3	3

Table 3.11. After subtracting smallest value for each column.

Step 3 : Determine whether an optimal solution can be formulated.

To determine whether there are m = 4 "independent" zeroes, draw lines (vertically and horizontally) through the matrix in such a manner as to minimize the number of lines. If the minimum number of lines is four, then the set of zeroes is "independent" and form the optimal solution. To find the minimum number of lines, use the following procedure.

a. Begin with row 1 and find the first row with exactly one zero. Now, put a box around the zero and draw a line through the column containing the zero. Repeat this procedure for the rest of the rows while neglecting all elements with a line through them.

After completing this process, the matrix should be as follows.

	1	2	3	4
1	5	0	4	6
2	6	5	0	0
3	4	0	3	4
4	0	2	3	3

Table 3.12 After eliminating columns.

b. After all of the rows have been examined, the same procedure is applied to the columns. Find each column, starting with column 1, containing exactly one zero that has not been crossed out. Place a box around the zero, and draw a line through the row.

After completing this process the matrix should look like the following.

	1	2	3	4
1	5	[0]	4	6
2	---6---	---5---	[0]	---0---
3	4	0	3	4
4	[0]	2	3	3

Table 3.13. After eliminating rows.

Now, all of the zeroes have a box around them or are crossed out. Since only three lines have been used, this is not an optimal solution. If all of the zeroes had not been crossed out yet, steps (a) and (b) should be repeated.

Step 4 : Change the arrangement or number of zero elements in the matrix.

Let the set A be all of the elements in the matrix not covered by a line in the reduced matrix. Define the set B as the set of elements covered by one line and the set C as the elements covered by two lines. Construct a new reduced matrix by 1) Subtracting from each element of A the smallest element of the set A. 2) Add to each element of C the smallest element of the set A, and 3) leaving the elements of B unchanged.

The smallest element of A is 3, so after making these changes, the matrix is as follows.

	1	2	3	4
1	5	0	1	3
2	9	8	0	0
3	4	0	0	1
4	0	2	0	0

Table 3.14. After arranging elements in Step 4.

Now repeat step 3, to check if there are 4 "independent" zeroes in the matrix. After completing this step the new matrix is :

	1	2	3	4
1	5	[0]	1	3
2	9	8	0	[0]
3	4	0	[0]	1
4	[0]	2	0	0

Table 3.15. Final solution for the Assignment problem.

It has taken four lines to cover all of the zeroes in the matrix. Thus there are 4 "independent" zeroes. The solution to the problem is to send car 1 to city 2, car 2 to city 4, car 3 to city 3 and car 4 to city 1, which takes a total time of 13 hours. The 13 hours is determined from the original matrix for the amount of time for each car to reach each city.

Thus, it can be seen that for certain problems, there are several alternatives to the Simplex Method for finding the optimal solution. The transportation problem and the assignment problem can be solved using specific algorithms which are highly efficient. These methods, though, solve only a narrow class of problems, while the Simplex Method solves most general linear programming problems.

CHAPTER 4: THE ELLIPSOIDAL ALGORITHM

The Simplex Method and its variations can be used to solve linear programming problems of any kind. The only difficulty that can occur with the method is that the length of time necessary to solve the problem can be excessively long. In the worst case, the number of iterations necessary to find a sufficient solution is an exponential function of the number of constraint equations in the standard model. Thus, ever since the Simplex Method was developed for solving linear programming problems, mathematicians have been searching for an algorithm which has a polynomial function for the number of iterations to solve the problem in the worst case. In other words, the algorithm can solve the problem in polynomial time.

In 1979, the Russian mathematician, L.G. Khachiyan, published a proof that there exists a polynomial-bound algorithm to solve linear programming problems. Khachiyan looked at the following three problems.

1. The Decision problem : Decide whether the set $P = \{x \mid Ax \le b\}$ is empty or not.

2. The Feasibility problem : Given $P = \{x \mid Ax \le b\}$, find an $x \in P$, or show $P = \emptyset$.

3. The Optimization problem : The usual linear programming problem, i.e. $\max \{ cx \mid x \in P\}$.

In these problems, the mxn matrix A is the matrix of coefficients for the constraint equations, the m-dimensional vector b is the vector of constraint values and the n-dimensional vector c is the objective function vector. It can be shown that if a polynomial algorithm can be found for one of the problems, a polynomial algorithm can be found for the other two problems. Khachiyan looked at the Feasibility problem and created an algorithm to find a point x satisfying $Ax \le b$ or proved there was no such x.

Khachiyan considered a system of $m \ge 2$ linear inequalities for $n \ge 2$ real variables $x_1, x_2, \ldots, x_n$ of the following form.

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i \quad \text{for } i = 1,2,\ldots,m \tag{1}$$

with a_{ij}, b_i having integer values.

Let $L = [\sum_{i=1,m} \sum_{j=1,n} \log_2(|a_{ij}| + 1) + \sum_{i=1,m} \log_2(|b_i| + 1) + \log_2(mn)] + 1$ which is the length of the input of the system. In other words, this is the number of 0's and 1's that are necessary to write the m inequalities in the binary number system.

The algorithm created to determine the consistency or inconsistency of the system (1) in %^{n} is polynomial in L. The memory size required is $O(nm + n^2)$ numbers, each with $O(nL)$ places in the binary system of notation with fixed point. On these numbers, $O(n^3(n^2 + m)L)$ operations of the form +, -, *, %, $\sqrt{\ }$, and max are performed, with a required accuracy of $O(nL)$.

Khachiyan used the following two lemmas as a basis for his algorithm.

Lemma 1 : If the system of linear inequalities with input L is consistent, then there exists a solution x^0 in the Euclidean ball, $S = \{x \mid ||x|| \le 2^L\}$. If we let $\Theta(x) = \max \{a_{i1}x_1 + \ldots + a_{in}x_n - b_i\}$ for $i = 1,2,\ldots,m$ then, $\Theta(x)$ is the residual of the system at the point $x \in \mathbb{R}^n$. Thus, if x^0 is a solution of the system of linear inequalities, then $\Theta(x) \le 0$.

Lemma 2 : If the system of linear inequalities with input L is inconsistent, then for any $x \in \mathbb{R}^n$, the residual $\Theta(x) \ge 2 \cdot 2^{-L}$.

In order to check if the system of linear inequalities is consistent or not, it is necessary and sufficient to find a point $x \in \mathbb{R}^n$ such that

$\Theta(x) \leq \Theta_s + 2^{-L}$ where Θ_s is the minimum residual on the ball S. Thus, $\Theta(x) \leq 2^{-L}$, in which case the system is consistent, or $\Theta(x) \geq 2 \cdot 2^{-L}$ and the system is inconsistent.

The method used to solve the system of linear inequalities is to enclose the Euclidean ball with an ellipsoid in $\mathbb{R}^n$ and to construct new ellipsoids of decreasing volumes until a point is found solving $Ax \leq b$ or it can be shown that no such point exists. Thus, the following information about ellipsoids will be important in the algorithm to solve the feasibility problem.

Consider an ellipsoid E in $\mathbb{R}^n$ given by the pair (x,Q), where $x \in \mathbb{R}^n$ is the center of the ellipsoid and $Q = ||q_{ij}||$ is an nxn matrix. The ellipsoid is the image of the Euclidean sphere $||z|| < 1$ under the transformation Q, shifted to the point x. In other words, $E = \{y \mid y = x + Qz, \text{ where } ||z|| \leq 1\}$. Thus if $Det(Q) \neq 0$, then E is nondegenerate. Define $||Q|| = \sqrt{\Sigma_{i=1,m}\Sigma_{j=1,n}q_{ij}^2}$ and say that the ellipsoid $E' \sim (x',Q')$ approximates the ellipsoid $E \sim (x,Q)$ with accuracy δ if $||x-x'|| + ||Q-Q'|| \leq \delta$.

Now, let a nondegenerate ellipsoid $E \sim (x,Q)$ and a non-zero n dimensional vector R be given. Let (1/2) E_R be the half-ellipsoid obtained by intersecting E with the half space $R^t(y-x) \geq 0$. The definition of E' approximating E and the construction of (1/2) E_R are now used in the following lemma.

Lemma 3 : Let $E \sim (x,Q)$ be a nondegenerate ellipsoid and (1/2) E_R, $(R \neq 0)$, be a half-ellipsoid of E. Consider the ellipsoid $E^R \sim (x^R,Q^R)$, where $x^R = x + QQ^tR/[(n+1)||Q^tR||]$ and $Q^R = 2^{(1/8)n \wedge 2} \cdot Q \cdot Ort(Q^tR) \cdot \Lambda_n$ where $Ort(Q^tR)$ is an orthogonal nxn matrix whose first column is the vector $Q^tR/||Q^tR||$ and Λn is the nxn diagonal matrix $\Lambda_n = \text{diag}(n/(n+1), \sqrt{1/(1-1/n^2)}, \ldots, \sqrt{1/(1-1/n^2)})$. Now if the ellipsoid $E' = (x', Q')$ is a δ approximation for E^R, where $\delta \leq n^{-4n} \cdot |\det Q| / ||Q||^{n-1}$, then E' will completely contain the half-ellipsoid (1/2) E^R, and $||x'|| \leq ||x|| + ||Q||/n$, $||Q'|| \leq ||Q|| \cdot 2^{(1/n)^2}$ and $2^{-1/n} \leq \det Q'/\det Q \leq 2^{-1/(4n)}$.

This lemma describes how to circumscribe a new ellipsoid E' around the half-ellipsoid (1/2) E^R in a way so that

1.) only approximate calculations are performed,
2.) the norms $||x'||$ and $||Q'||$ do not exceed $||x||$ and $||Q||$ by too much and,
3.) the ratio of the volumes of the ellipsoids,

meas E'/meas E = $\det Q'/\det Q$, is within the bounds $2^{-1/n}$ and $2^{-1/(4n)}$.

Using the preceding lemmas and definitions, the algorithm for solving the feasibility problem is as follows.

Write the original system of linear inequalities in the form $A_i^t x \leq b_i$ for $i = 1,2,\ldots,m$ where the A_i are the rows of the system. Without loss of generality, it can be assumed that for all i, $A_i \neq 0$. The algorithm consists of $N = 16n^2L$ iterations with indices $K = 0,1,\ldots,N$. At the k^{th} iteration, the ellipsoid $E_k \sim (x_k, Q_k)$ and the scaler Θ_k are obtained. For the initial iteration, $K = 0$, let $x_0 = 0$, $Q_0 = \text{diag}(2^L, 2^L, \ldots, 2^L)$, $\Theta_0 = \max\{-b_i\}$. Thus, the ellipsoid E_0 coincides with the ball S, and Θ_0 is the magnitude of the residual at the center of the sphere.

The k^{th} iteration starts by evaluating the residual $\Theta(x_k)$ at the center of the current ellipsoid E_k. Since $||A_i||$ and $||b_i|| \leq 2^L$ and vectors A_i are integral, then the magnitude of the $\Theta(x_k)$ can be determined exactly and the index i_k of the row in which the maximum $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n - b_i$ is attained at the point $x = x_k$. Now, let $\Theta_{k+1} = \min(\Theta_k, \Theta(x_k))$. The quantity Θ_{k+1}, when evaluated exactly, is the smallest minimal value of the residual over the approximations, $x_0, x_1, \dots, x_k$ obtained so far.

The next step is to find a new ellipsoid $E_{k+1} \sim (x_{k+1}, Q_{k+1})$ that fully contains the half-ellipsoid (1/2) E_k obtained from E_k by cutting off the region $A(x-x_k) > 0$, in which the residual $\Theta(x)$ always exceeds $\Theta(x_k)$. To obtain the new ellipsoid, first find the vector $F_k = - Q_k{}^t A_{ik}$, which can be found exactly since A_{ik} is integral. Since $A_{ik} \neq 0$ and $\det Q_k \neq 0$, then $F_k \neq 0$. Finally $E_{k+1} \sim (x_{k+1}, Q_{k+1})$ is found by letting

$x_{k+1} \approx x_k + Q_k F_k / ((n+1) \,||F_k||)$ and

$Q_{k+1} \approx 2^{1/(8n\,\hat{}2)} \cdot Q_k \cdot \mathrm{Ort}(F_k) \cdot \Lambda_n$. These calculations are performed with an accuracy of $\delta = 2^{-37nL}$ which takes $O(n^3)$ operations of the form +, -, *, %, $\sqrt{\ }$, and max with an accuracy of $O(nL)$ places.

Using this algorithm, the following inequalities hold at each step k

$$||x_k|| \le (k/n) \cdot 2^{18L},\ ||Q_k|| \le 2^{2L+k/(n)\hat{}2},\ |\Theta_k| \le 2^{23L}$$
$$\text{and} \quad 2^{nL-k/n} \le \det Q_k \le 2^{nL-k/(4n)}.$$

After $k = N = 16n^2L$ iterations , the smallest minimal residual Θ_{N+1} is output from the algorithm. Following the $k = N = 16n^2L$ iterations, the following theorem holds.

Theorem : The smallest minimal residual of the residual Θ_{N+1} satisfies the inequality $\Theta_{k+1} \leq \Theta_s + 2^{-L}$, where Θ_s is the minimum residual on the ball S.

From this theorem and lemmas 1 and 2 either $\Theta_{N+1} \leq 2^{-L}$ and the system of linear inequalities is consistent or $\Theta_{N+1} > 2 \cdot 2^{-L}$ and the system of inequalities is inconsistent. Thus, the Feasibility problem can be solved in polynomial time.

Using this algorithm, the general linear programming problem can be solved in polynomial time. The linear programming problem is to maximize cx subject to the constraint $Ax \leq b$, and is denoted by LP. Let $P = \{x \mid Ax \leq b\}$. Use the previous algorithm to check if P is empty. If P is empty, the problem has no solution. If P is nonempty, check to see whether cx is bounded. This can be done, since using linear programming duality theory, the linear programming problem LP is bounded if and only if the Dual LP given by $\{\min yb \mid A^t y = c, y \geq 0\}$ is feasible, in other words, if and only if $\{y \in \mathbb{R}^{n+} \mid A^t y = c\} \neq \emptyset$. This can be checked by the ellipsoidal method. If LP is bounded, then let it be bounded by $\psi^+ = cx^*$ where ψ^+ is a rational number of the form t/s, t and s are relatively prime integers, and $|t| \leq 2^{L1}$, $|s| \leq 2^{L1}$ where

$L1 = L + \sum_{j=1,n} \log_2(|c_j| + 1)$, the length of the input for LP. The value of ψ^+ can be approximated to any desired accuracy, ε, using a binary search. Begin with the interval $[2^{-L1}, 2^{L1}]$ and represent ψ^+ as $[\psi^+] + t'/s'$ with $0 \leq t' \leq s'$, t' and s' are relatively prime and [x] is defined to be the greatest integer less than x.

Now, do the following binary search:

For $k = 0$ to $\lceil \log_2(2^{L1+2}/\varepsilon) \rceil$ solve the decision problem P_k. in other words, check if P_k is empty, for

$$P_k : Ax \le b$$
$$cx \le \psi_k$$

where ψ_k is the midpoint of $[\alpha k, \beta k]$ with $\alpha_0 = 2^{-L1}$ and $\beta_0 = 2^{L1}$. The decision problem P_k is also solved using the ellipsoid algorithm. If P_k is empty, then let $\alpha_{k+1} = \alpha_k$ and $\beta_{k+1} = \psi_k$, and if P_k is nonempty, let $\alpha_{k+1} = \psi_k$ and $\beta_{k+1} = \beta_k$. After completing the search, an approximation $\tilde{\psi}$ to ψ^+ satisfying $|\tilde{\psi} - \psi^+| \le \varepsilon$ is found.

After determining $\psi^+ = \tilde{\psi}$, any solution to $P' : Ax \le b$

$$cx \ge \psi$$

will be an optimal solution to LP. A solution to P' is found using the Ellipsoidal Method. Thus to solve the general linear programming problem, the Ellipsoidal Method, which is a polynomial time algorithm, is used a polynomial number of times in solving different decision problems and then is used a final time to find the optimum value for the problem.

Several variations of the Ellipsoidal Method have been developed by mathematicians such as Gacs, Lovasz, Judin, Shor and Nemirovskii. Each version, though, finds a solution to the general linear programming problem in polynomial time. It has been found, by solving the identical problem using both the Simplex Method and the Ellipsoidal Method, that none of the ellipsoidal methods solve the standard linear programming problem any faster, on the average, than the Simplex Method. Thus the Ellipsoidal Method is not an obvious choice over the Simplex Method when solving linear programming problems.

CHAPTER 5: KARMARKAR'S ALGORITHM

In 1984, Narendra Karmarkar, a scientist at AT&T Bell Laboratories, discovered a new method for solving linear programming problems with thousands of variables. This new method works from the interior of the polygon of feasible solutions using projective geometry to transform the structure so that a selected interior point becomes the center. A new point is then located in the direction that will further minimize or maximize the objective function and then the process is repeated until the optimum solution is reached.

As is the case with the Ellipsoidal Method, Karmarkar's Algorithm solves the problem in polynomial time. The worst case running time for this algorithm is $O(n^{3.5}L^2)$, where n is the dimension of the problem and L is the number of bits in the input. On the other hand, the running time of the Ellipsoidal Method is $O(n^6L^2)$ and for the Simplex Method the worst case running time is $O(L^n)$, but in practice, all three methods are comparable in speed.

Karmarkar's Algorithm uses several of the ideas of the Ellipsoidal Method, but extends the ideas onto the sphere in the following way. Consider the linear programming problem:

$$\text{Minimize } c^t x \quad c, x \in \mathbb{R}^n$$
$$\text{subject to } Ax = b, \quad x \geq 0,$$

where the n-dimensional vector c is the objective function vector, the mxn matrix A is the matrix of coefficients for the constraint equations, and the m-dimensional vector b contains the constraint values.

Let Λ be the affine space $\{x \mid Ax = b\}$, and P_+ denote the positive orthant, the region of $\mathbb{R}^n$ with $x \geq 0$, i.e. each $x_j \geq 0$ for $j = 1,2,\ldots,n$. Now, if the constraint $x \in P_+$ is replaced by the constraint $x \in E$, where E is an ellipsoid, the problem is simplified and can be solved in the following way.

First, apply a linear transformation T that transforms the ellipsoid E to the sphere S. This transforms the affine space Λ into the affine space Λ' and transforms the objective function vector c to c'. The new transformed problem is :

$$\text{Minimize } c'^t x \quad \text{where } c', x \in \mathbb{R}^n$$

$$\text{subject to } x \in S \cap \Lambda'.$$

The intersection of a sphere and an affine space is a lower dimensional sphere inside the affine space. So the problem is to find x so that $c''^t x$ is a minimum subject to $x \in S'$, where c" is the orthogonal projection of c' onto Λ'. This problem, though, can be solved by starting at the center of the sphere and moving in the direction of - c" a distance equal to the radius of the sphere.

Looking at this idea closer, it becomes very important to find out how the ellipsoid should be chosen. Letting P be the n-dimensional polytope defined by $Ax = b$, $x \geq 0$, i.e. $\Lambda \cap P_+$, and $a_0 \in P$, a strictly interior point. With a_0 as the center, draw an ellipsoid E contained in P and solve the optimization problem over E instead of P. To see how good the solution over E is compared to the true solution over P, construct an ellipsoid E' by magnifying E by a factor of ν large enough for E' to contain P. Thus, $E \subset P \subset E'$, $E' = \nu E$. Now, let f_E, f_P and f_E' be the minimum values of the objective function $f(x) = c^t x$, on E, P and E', respectively. Since $f_E' \leq f_P \leq f_E$, the following holds true.

$$f(a_0) - f_E \leq f(a_0) - f_P \leq f(a_0) - f_E' = \nu(f(a_0) - f_E$$

The equality portion above follows from the linearity of f. So, $1 \leq \nu(f(a_0) - f_E)/(f(a_0) - f_P)$ and $(f(a_0) - f_E)/(f(a_0) - f_P) \geq 1/\nu$ and, after some algebra, $(f_E - f_P)/\ (f(a_0) - f_P) \leq 1 - 1/\nu$. Thus, by going from a_0 to some point, say $a' = f_E$, the value of the objective function becomes closer to the minimum value by a factor of $(1 - 1/\nu)$. The process can then be repeated using a' instead of a_0 and the rate of convergence to the optimal solution f_p depends on ν. The closer ν is to 1, the more rapid the convergence. It can be shown that for any linear programming problems with n variables, there exists a projective transformation such that $\nu = n$.

From the above set of steps, a sequence of points $x^{(0)}, x^{(1)}, \ldots, x^{(k)} \ldots$ having decreasing values for the objective function are created. In the k^{th} step, the point $x^{(k)}$ is brought into the center by a projective transformation. The objective function is then optimized over the intersection of the inscribed sphere and the affine subspace to find the next point $x^{(k+1)}$. This allows a reduction in the objective function by a factor of $(1 - 1/n)$, at least.

There is one other point which must be looked at when using the previous ideas. Linear functions, such as the objective function, are not invariant under a projective transformation, so they need not be linear after being transformed. On the other hand, ratios of linear functions are transformed into ratios of linear functions. So, with every linear function f(x), associate a "potential function" g(x) expressed in terms of ratios of linear functions. Let $g(x) = \sum_{j=1,n} \ln(f(x)/x_j) + k$, where k is some constant. The function g(x) has the following properties:

1. Any desired reduction in the function f(x) can be obtained by a sufficient reduction in the value of g(x).

2. The function g(x) is invariant under any projective transformation. Thus, it is transformed into a function of the same form.

3. The optimization of g(x) in each step can be done approximately by optimizing some linear function, which changes with each step.

Using these ideas, Karmarkar formulated an algorithm to solve the following linear programming problem.

Minimize $c^t x$, $c, x \in \mathbb{R}^n$

Subject to $x \,[\, (\Lambda \cap S)$ where $\Lambda = \{x \mid Ax = 0\}$

and $S = \{x \mid x \geq 0, \sum_{j=1,n} x_j = 1\}$

This problem is a simplified version of all linear programming problems. It is used to bring out the main points of the algorithm by making several restrictions which can be removed by standard linear programming techniques.

For this problem, the following assumptions are made :

1. The feasible region is the intersection of an affine space with a simplex, rather than a positive orthant of $\mathbb{R}^n$. It usually takes one projective transformation to change a linear programming problem into this form.

2. The linear system of equations defining the affine space Λ is homogeneous, i.e. the right hand side equals 0. There is also an additional equation $\sum_{j=1,n} x_j = 1$, and by using this equation, any non-homogeneous system can be made homogeneous.

3. The minimum value of the objective function is assumed to be zero. If the minimum value, say c_m, is known and not zero, then modify the objective function to be $c^t x - c_m$, which is homogeneous. If the minimum value of the objective function is not known, then a variation on this algorithm using a "sliding objective function" can be used to solve the problem.

4. It is also assumed that the problem has a feasible solution, and that the center of the simplex, S, given by $a_0 = (1/n)e$, where e is the n-dimensional vector of all 1's, is a feasible point.

5. A termination parameter q for the problem is given. The purpose

of the algorithm is to find a feasible x such that $(c^t x)/(c^t a_0) \leq 2^{-q}$. As is the case of the ellipsoid method, if $q = O(L)$, the resulting approximate optimal solution can be converted to an exact optimal solution.

Now, Karmarkar's algorithm can be put into the following steps.

Step 1 : Initialize $x^{(0)} = a_0 = (1/n)e$, and $k = 0$.

Step 2 : While $c^t x^{(k)}$ is too large, i.e. $c^t x/c^t a_0 \geq 2^{-q}$, set $x^{(k+1)} = \Theta(x^{(k)})$, making a new approximation for the solution. To find $\Theta(x^{(k)})$ in step 2, let $D = \mathrm{diag}\{x_1, x_2, \ldots, x_n\}$ be the diagonal matrix whose j,j^{th} entry is x_j and perform the following operations.

a. Let $B = \begin{bmatrix} \underline{AD} \\ e^t \end{bmatrix}$. Augment the matrix AD with a row of 1's.

b. Set $c_p = [I - B^t(BB^t)^{-1}B]Dc$.

c. Set $\hat{c} = c_p/|c_p|$, find the unit vector $\hat{c}$ in the direction of c_p.

d. Set $x = a_0 - r\hat{c}$ where r is the radius of the largest inscribed sphere in S, $r = 1/\sqrt{n(n-1)}$, and $\alpha \in (0,1)$ which may be arbitrarily set equal to 1/4.

e. Set $x^{(k+1)} = (Dx)/(e^t Dx)$.

Now, using the "potential function" $f(x) = \sum_{j=1,n} \ln((c^t x)/x_j)$, the following two theorems arise.

Theorem : In $O(n(q+\log n))$ steps, the algorithm finds a feasible point x such that either (i) $c^t x = 0$ or (ii) $(c^t x)/(c^t a_0) \leq 2^{-q}$.

Theorem : Either (i) $c^t x^{(k+1)} = 0$ or (ii) $f(x^{(k+1)}) \leq f(x^{(k)}) - \delta$, where δ is a constant depending on α. If $\alpha = 1/4$, then set $\delta \geq 1/8$.

The steps in finding $\Theta(x^{(k)})$ can also be described in the following way. First, perform the projective transformation $T(x^{(k)},a_0)$ of the simplex, S, that maps the input point $x^{(k)}$ to the center a_0. Now, optimize approximately the transformed objective function over an inscribed sphere, of radius αr, to find a point x. Finally, apply the inverse of the transformation T to x to obtain the output $x^{(k+1)}$.

To perform the projective transformation $T(x^{(k)},a_0)$, let $D = \text{diag}\{x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\}$ be the diagonal matrix with diagonal entries $x_1^{(k)}, x_2^{(k)},\ldots,x_n^{(k)}$. For any vector $x \in \mathbb{R}^n$, the transformation $T(x,a_0)$ is given by $x' = (D^{-1}x)/(e^tD^{-1}x)$, where $e = (1,1,\ldots,1)$ and the inverse transformation is $x = (Dx')/(e^tDx')$. Let f' be the transformed potential function defined by $f(x) = f'(T(x))$ and thus,

$f'(y) = \sum_{j=1,n} \ln((c'^ty)/y_j) - \sum_{j=1,n} \ln(a_j)$, where $c' = Dc$. Also, let Λ' be the transformed affine space Λ. Then $Ax = 0 \Longleftrightarrow ADx' = 0$ by the definition of x'. Thus, Λ' is the null space of AD.

When the optimization of the transformed objective function over an inscribed sphere is taking place, the sphere, $B(a_0, \alpha r)$, having center a_0 and radius αr is used instead of $B(a_0,r)$ for two reasons. First, it allows the optimization of $f'(y)$ to be approximated very accurately by the optimization of a linear function. Also, if approximate arithmetic operations are performed, there is a margin to absorb errors without leaving the simplex.

Now, looking at the null space of AD, $\Lambda' = \{y \mid ADy = 0\}$, a problem can be formulated similar to the inital problem. If the constraint $\sum_{i=1,n} y_i = 1$ is added, then $\Lambda' \cap \{y \mid \sum_{i=1,n} y_i = 1\}$ is another affine space, Λ''.

The following theorem now becomes very important.

Theorem : There exists a point $y \in B(a_0, \alpha r) \cap \Lambda''$ such that either (i) $c'^t y = 0$ or (ii) $f'(y) \le f'(a_0) - \delta$, where δ is some constant greater than or equal to 1/8.

This theorem proves the existence of a point that achieves a constant reduction in the potential function. From this point, it can be shown that the minimization of the potential function $f'(x)$ can be approximated by the minimization of the linear function $c'^t x$, using the following theorem.

Theorem : Let y be the point that minimizes $c'^t x$ over $B(a_0, \alpha r) \cap \Lambda''$. Then either (i) $c'^t y = 0$ or (ii) $f'(y) \le f'(a_0) - \delta$, with δ a constant with a value greater than or equal to 1/8.

Finally, the function $c'^t x$ can be minimized as follows :

1. Project c' orthogonally onto the null space of B, i.e. set $c_p = [I - B^t(BB^t)^{-1}]c'$.
2. Normalize c_p, i.e. set $\hat{c} = c_p / |c_p|$.
3. Take a step of length αr in the direction $-\hat{c}$, i.e. $x = a_0 - \alpha r \hat{c}$.

The point x approximately optimizes the transformed objective function over an inscribed sphere.

Now to find $x^{(k+1)}$, just apply the inverse of the projective transformation T to the point x. Thus, $x^{(k+1)} = (Dx)/(e^t Dx)$ and the 5 parts of step 2 in the algorithm accomplish the desired result of finding a new estimate for the solution.

Looking at the steps of the algorithm, the major computational effort is in determining c_p. Since $B = \begin{bmatrix} AD \\ \hline e^t \end{bmatrix}$

then

$$BB^t = \begin{bmatrix} AD^2A^t & ADe \\ (ADe) & e^tDe \end{bmatrix} = \begin{bmatrix} AD^2A^t & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$(BB^t)^{-1} = \begin{bmatrix} (AD^2A^t)^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the only quantity that changes from step to step is the diagonal matrix D, since $D_{jj}^{(k)} = x_j^{(k)}$. To take advantage of the computations done in the previous steps, exploit the following facts about matrix inverses.

Let D and D' be nxn diagonal matrices. Then

1. if D and D' are "close" in some suitable norm, the inverse of AD'^2A^t can be used in place of the inverse of AD^2A^t and

2. if D and D' only differ in one entry, then the inverse of AD^2A^t can be computed in $O(n^2)$ arithmetic operations given the inverse of AD'^2A^t.

Define a diagonal matrix $D'^{(k)}$, a "working approximation" to $D^{(k)}$ at step k if $1/2 \leq [D'_{jj}{}^{(k)}/D_{jj}{}^{(k)}]^2 \leq 2$ for j = 1,2,...,n, and use $(AD'^{(k)\wedge 2}A^t)^{-1}$ in the place of $(AD^{(k)\wedge 2}A^t)^{-1}$. Update D' by $D'^{(k+1)} = \sigma^{(k)} \bullet D'^{(k)}$ where $\sigma^{(k)}$ is some scaling factor. For j = 1 to n, if $(D'_{jj}{}^{(k+1)}/D'_{jj}{}^{(k)}) \notin [1/2,2]$, then let $D'_{jj}{}^{(k+1)} = D_{jj}{}^{(k)}$ and update $(AD'^{(k+1)\wedge 2}A^t)^{-1}$. This adjustment reduces the average work per step to $O(n^{2.5})$ as compared to $O(n^3)$ for the simpler algorithm.

The values for the complexity of this algorithm, $O(n^{3.5}L^2)$, come from the following sources. The number of steps of the algorithm is O(nL), each step requires $O(n^{2.5})$ arithmetic operations in the adjusted algorithm. Finally, each arithmetic operation requires a precision of O(L) bits. Thus, the algorithm which Karmarkar developed is a polynomial time algorithm. It has been quite effective in finding the optimal solution to the most complicated linear programming problems known to man.

There are several ways of solving linear programming problems. The most elementary problems can be solved using the Graphical Method. If there are more than three decision variables, though, the Graphical Method does not work, because the graph can't be made in a higher dimension than three. All linear programming problems that can currently be solved can be solved by the Simplex Method, developed by George Dantzig. This method has several variations that make it the most widely used method for solving linear programming problems. The only drawback to the Simplex Method is that it is an exponential time algorithm.

Several algorithms have been developed to solve very specific linear programming problems. The Transportation Method solves the transportation problem quite rapidly. A special case of the transportation problem, the assignment problem, is readily solved using the Hungarian Method. In the past ten years, several algorithms having polynomial running time have been developed. In 1979, L. G. Khachiyan completed his formulation of the Ellipsoidal Method, and in late 1984, Narendra Karmarkar introduced his algorithm for solving linear programming problems. Both of these algorithms have polynomial computation times, but on the average, the time it takes these two algorithms to solve a linear programming problem is the same as the amount of time it takes to solve the problem using the Simplex Method. Thus, there are many algorithms that can be used to solve the standard linear programming problem, depending on the exact nature of the problem.

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