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Title: SIMPLIFICATION OF LARGE LINEAR SYSTEMS USING TWO-STEP ITERATIVE METHOD

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The order of the practical system (e.g. nuclear power plants, electrical power network and chemical plant) is quite large. However, there are many limitations in computing facilities for the large system. Because of these limitations, it is often necessary to reduce the order of the large system using an approximation.

Here the simple iterative technique which is free of certain shortcomings of the previous method is proposed for the approximation of large linear systems by a low-order model. A measure of the goodness of the model is the value of the integral-square error between the step responses of the exact and the simplified system.

The proposed technique consists of a two-step iterative scheme. In the first step, the optimum residues are

obtained by the minimization of the objective function, while the poles (or eigenvalues) are kept constant. In the second step, the poles (or eigenvalues) are optimized while the residues remain fixed. This procedure is continued cyclically until the objective function is satisfactorily minimized.

The necessary and sufficient conditions for existence of an optimum are satisfied in each step. The residues, poles (or eigenvalues) and objective functions always converge monotomically. The resulting reduced-order model obtained by this method is stable if the original system is stable. The method can be applied not only to single-variable systems, but also to systems with repeated poles (or eigenvalues) and to multivariable systems. The results are superior to those obtained previously in the steady-state and transient responses, and the value of the integral-square error.

Illustrative examples are presented.

Simplification of Large Linear Systems Using Two-Step Iterative Method

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CHAPTER I. INTRODUCTION

For many processes (e.g. nuclear power plants, electrical power networks and chemical plants), the order of the system is quite large. Thus for the large system, there are many limitations in computing facilities. Implementation on a digital computer is prohibitive in terms of time used, cost of computation and available space. For reasons such as these limitations, it is often desirable and sometimes necessary to reduce the order of the large system by means of an approximation.

A number of papers have been presented about the reduction of the system. These methods have their own advantages and disadvantages. However, the point to which one approximate model is superior to another can be decided in the light of the measure of performance of the same task.

The following criterion is used to define a measure of quality of an approximation. The integral-square error between the step responses of the exact and simplified models is used.

1.1 Statement of Problem

In many cases the dynamic behavior of the physical system is represented by the state-variable equations or transfer function matrices (or transfer functions).

Given:

(a) The original mathematical system is given by an nth-order transfer function matrix.

$$[H_n(S)] = [A_{2m+1}\, S^m + A_{2m}\, S^{m-1} + \ldots + A_{22}S + A_{21}] \times [A_{1n+1}\, S^n + A_{2n}\, S^{n-1} + \ldots + A_{12}\, S + A_{11}]^{-1} \quad (1\text{-}1)$$

where A_{ij} are constant matrices and $m \leq n$.

For a single-variable system of course matrix coefficients and matrix inversion are replaced by a constant coefficients and division respectively.

(b) An exact (v-input, w-output) nth-order linear time-invariant system is described by the state-variable equation.

$$\dot{\underline{x}} = A\,\underline{x} + D\,u$$

$$\underline{y} = H'\underline{x} \quad (1\text{-}2)$$

where $\underline{x}$ is an n dimensional vector

$\underline{y}$ is a w dimensional vector

A is an n x n matrix

D is an n x v matrix

H' is a W x n matrix

A prime denotes the transpose.

Assume that

$$\begin{aligned} D\,u &= 0 \qquad t < 0 \\ &= D \qquad t \geq 0 \end{aligned} \tag{1-3}$$

It is assumed in Eq. (1-2) that the eigenvalues of A have negative-real parts.

Problem:

(c) It is desired to find an ℓth-order ($\ell << n$) simplified transfer function matrix, given by

$$[\hat{H}_\ell(s)] = [\hat{A}_{2k+1}s^k + \hat{A}_{2k}s^{k-1} + \ldots + \hat{A}_{22}s + \hat{A}_{21}] \times [\hat{A}_{1\ell+1}s^\ell + \hat{A}_{1\ell}s^{\ell-1} + \ldots + \hat{A}_{12}s + \hat{A}_{11}]^{-1} \tag{1-4}$$

where A_{ij} are constant matrices and $k \leq \ell$.

(d) To find a simplified model of a reduced-order ($\ell << n$) in the form of

$$\begin{aligned} \dot{\underline{\hat{x}}} &= \hat{A}_0\,\underline{\hat{x}} + \hat{B}_0\,u \\ \underline{\hat{Y}}_\ell &= \hat{H}_0'\,\underline{\hat{x}} \end{aligned} \tag{1-5}$$

where $\hat{A}_0$ is an ℓ x ℓ matrix

$\hat{B}_0$ is an ℓ x v matrix

$\hat{H}_0'$ is a w x ℓ matrix.

1.2 A Survey of Reduced-Order Modeling Techniques

There is generally an implicit constraint on the simplification process. One is that the simplified-model is computationally simpler than an original system. The other is that the simplified-model must retain those features of the original system which are considered important. Various techniques for the reduction of the linear system, developed during the past decade or so, can be classified into two main categories, namely:

(i) Simplification in frequency-domain.

(ii) Simplification in time-domain.

In this section these methods are summarized and results of previous works are discussed.

1.2.1 Simplification in Frequency-Domain

At present three methods are available for the frequency-domain model reduction techniques. They are:

(i) Continued-fraction-expansion method

(ii) Moment-matching method

(iii) Hsia's method.

The first method is due to Chen and Shieh in which system transfer function is reduced by truncating coefficients of its continued-fraction-expansion [9]. The idea behind this

is as follows. Consider the typical feedback system shown in Figure 1 with the overall transfer function given by

$$H_n(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad (1\text{-}6)$$

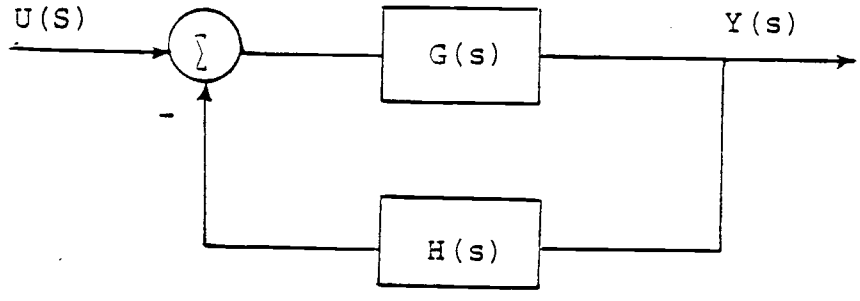

Figure 1. Block diagram of a feedback system.

Dividing the numerator and denominator in eq. (1-6) by G(s) we have:

$$H_n(s) = \frac{Y(s)}{U(s)} = \frac{1}{G(s) + \frac{1}{H(s)}} \qquad (1\text{-}7)$$

Eq. (1-7) can be considered as the simplest continued-fraction-expansion. It is easily seen that when G(s) is high, the overall gain can be approximated by 1/H(s). In other words, H(s) dominates the behavior of the system. In general, the continued-fraction will be obtained as follows (second Cauer form is shown as an example).

$$H_n(s) = \frac{Y(s)}{U(s)} = \cfrac{1}{h_1 + \cfrac{1}{\frac{h_2}{s} + \cfrac{1}{h_3 + \cfrac{1}{\frac{h_4}{s} + \cfrac{1}{\ddots}}}}} \qquad (1\text{-}8)$$

where h_j, $j = 1,2, \ldots$ are the coefficients in the continued-fraction-expansion.

It should be noted that the most dominant term is h_1 and the second influence term is h_2/s. As the coefficients in the continued-fraction descend lower and lower in position, they are less and less important as far as their influence on the performance of the system is concerned. This observation is the general base for the simplification techniques which will be developed in the continued-fraction-expansion.

Several continued-fraction forms can be considered, however, there are three major forms. Consider a rational transfer function of a single-variable system;

$$H_n(s) = \frac{Y(s)}{U(s)} = \frac{a_{2m+1}s^m + a_{2m}s^{m-1} + \ldots + a_{22}s + a_{21}}{a_{1n+1}s^n + a_{1n}s^{n-1} + \ldots + a_{12}s + a_{11}} \qquad (1\text{-}9)$$

where a_{ij} are constants, $m \leq n$.

The first Cauer form (Weiberg, 1962 [21]) is;

$$H_n(s) = \cfrac{1}{h_1's + \cfrac{1}{h_2' + \cfrac{1}{h_3's + \cfrac{1}{h_4' + \cfrac{1}{\ddots}}}}} \qquad (1\text{-}10)$$

The second Cauer form (Weiberg, 1962 [21]) is;

$$H_n(s) = \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{s} + \cfrac{1}{h_3 + \cfrac{1}{\cfrac{h_4}{s} + \cfrac{1}{\ddots}}}}} \tag{1-11}$$

The mixed Cauer form (Shieh and Goldman, 1974 [21]) is:

$$H_n(s) = \cfrac{1}{k_1+k_1's + \cfrac{1}{\cfrac{k_2}{s}+k_2' + \cfrac{1}{k_3+k_3's + \cfrac{1}{\cfrac{k_4}{s}+k_4' + \cfrac{1}{\ddots}}}}} \tag{1-12}$$

In the same fashion, the multivariable system can be expressed by a transfer function matrix:

$$[H_n(s)] = [A_{2m+1}s^m + A_{2m}s^{m-1} + \ldots + A_{22}s + A_{21}] \times [A_{1n+1}s^n + A_{1n}s^{n-1} + \ldots + A_{12}s + A_{11}]^{-1} \tag{1-13}$$

where A_{ij} are constant matrices, $m \leq n$.

In this case, the coefficients in the three Cauer forms are replaced by matrix coefficients and the division in the continued-fraction process is replaced by a matrix inversion.

The first-matrix Cauer form is:

$$[H_4(s)] = [H_1's + [H_2' + [H_3's + H_4' + [\ \ ]^{-1}]^{-1}]^{-1}]^{-1}]^{-1} \tag{1-14}$$

The second-matrix Cauer form is:

$$[H_4(s)] = [H_1 + [H_2 \frac{1}{s} + [H_3 + [H_4 \frac{1}{s} + [\ ]^{-1}]^{-1}]^{-1}]^{-1}]^{-1} \quad (1\text{-}15)$$

The mixed-matrix Cauer form is;

$$[H_4(s)] = [k_1 + k_1's + [k_2 \frac{1}{s} + k_2' + [k_3 + k_3' \frac{1}{s} + k_4 \frac{1}{s} + k_4' + [\ldots]^{-1}]^{-1}]^{-1}]^{-1} \quad (1\text{-}16)$$

where h_i', h_i and k_i are matrix coefficients of constant.

To evaluate the matrix coefficients of the mixed-matrix Cauer form, k_i and k_i', the matrix Routh algorithm can be used (Shieh and Goldman [21]).

$$\begin{array}{lll} & A_{11}A_{12}\cdots\cdots\cdots A_{1n+1} & \\ k_1 = A_{11}A_{21}^{-1} & & k_1' = A_{n+1}A_{2n}^{-1} \\ & A_{21}A_{22}\cdots\cdots\cdots A_{2n} & \\ k_2 = A_{21}A_{31}^{-1} & & k_2' = A_{2n}A_{3n-1}^{-1} \\ & A_{31}A_{32}\cdots A_{3n-1} & \\ & \vdots \qquad\qquad \vdots & \\ & A_{n-11}A_n \quad A_{n-13} & \\ k_{n-1} = A_{n-11}A_{n1}^{-1} & & k_{n-1} = A_{n-13}A_{n2}^{-1} \\ & A_{n1}A_{n2} & \\ k_n = A_n A_{n+11}^{-1} & & k_n' = A_{n2}A_{n+11} \\ & A_{n+11} & \end{array} \quad (1\text{-}17)$$

Thus k_i and k_i' in Eq. (1-16) are evaluated by the first and last column of the Routh table. $A_{k\ell}$ in the Routh table (1-17) are obtained using the following relationship.

$$A_{j\ell} = A_{j-2\ \ell+1} - k_{j-2}A_{j-1\ \ell+1} - k'_{j-2}A_{j-1\ell}$$

$$j = 3,4 \ldots n+1,\ \ell=1,2,\ \ldots.$$

and (1-18)

$$k'_p = A_{p1}(A_{p+1\ 1})^{-1} \quad : \quad k'_p = A_{p(n+2-p)}(A_{p+1\ n+1-p})^{-1}$$

$$\det A_{p+11} \neq 0 \quad : \quad \det A_{p+1\ n+1-p} \neq 0$$

$$p = 1,2, \ldots$$

The matrix coefficients of the second-matrix Cauer form is simply obtained by letting k_i' equal zero and relacing k_i by H_i in eq. (1-17) and (1-18).

Whereas the first-matrix Cauer form is similar to the second-matrix Cauer form, the matrix coefficients of eq. (1-13) should be modified so that the matrix Routh algorithm of eq. (1-17) and (1-18). By defining a new matrix coefficients B_{ij}, eq. (1-13) can be rewritten as follows:

$$[H_n(s)] = [B_{21}s^m + B_{22}s^{m-1} + \ldots. + B_{2m-1}s + B_{2m}] \quad (1\text{-}19)$$

$$\times\ [B_{11}s^n + B_{12}s^{n-1} + \ldots. + B_{1n}s + B_{1n+1}]^{-1}$$

where

$$B_{1i} = A_{1(n+2-i)} \quad i = 1,2, \ldots n+1$$

$$B_{2j} = A_{2(m+1-j)} \quad j = 1,2, \ldots m$$

The matrix coefficients can be evaluated by the following matrix Routh algorithm.

$$B_{j\ell} = B_{j-2\ \ell+1} - H'_{j-2}B_{j-1\ \ell+1} \qquad \begin{matrix} j=3,4,\ldots 2n+1 \\ \ell=1,2,\ldots n \end{matrix}$$

and (1-20)

$$H'_p = B_{1p}(B_{p+11})^{-1} \qquad p=1,2,\ldots.$$

$$\det B_{p+1\ 1} \neq 0$$

The state-space equation for the second-matrix Cauer form has been formulated by Chen and Shieh [9] as follows;

$$\dot{\underline{x}} = A\ \underline{x} + B\ u$$

$$\underline{y} = C'\underline{x} \qquad (1\text{-}21)$$

where

$$A = -\begin{bmatrix} H_1H_2 & (H_1)H_4 & (H_1)H_6 & \cdots & (H_1)H_{2n} \\ H_1H_2 & (H_1+H_3)H_4 & (H_1+H_3)H_6 & \cdots & (H_1+H_3)H_{2n} \\ H_1H_2 & (H_1+H_3)H_4 & (H_1+H_3+H_5)H_6 & \cdots & (H_1+H_3+H_6)H_{2n} \\ \vdots & & & & \\ H_1H_2 & (H_1+H_3)H_4 & (H_1+H_3+H_5)H_6 & \cdots & (H_1+H_3+\ldots+H_{2n-1})H_{2n} \end{bmatrix}$$

$$B' = [I,\ I,\ \ldots,\ I]$$

$$C' = [H_2,\ H_4,\ \ldots,\ H_{2n}]$$

$$\underline{x}' = [x_{11}, x_{12},\ \ldots,\ x_{1n}]$$

A prime denotes the transpose.

The n is the order of the matrix transfer function of eq. (1-19) and [I] and [H_i] are the identity matrix and matrix coefficients of dimension m x m respectively. This method can be applied to the transfer function which has repeated poles (Chen and Shieh [9]).

Hutton and Friedland proposed the Routh approximation based on the first Cauer form [31]. Shamash shows that the continued-fraction-expansion is also applied to the reduction of a discrete-time system. The discrete transfer function is given by the Z-transform [19]. The second technique is a moment-matching one which consists of the power-series expansion of the given system transfer function [23].

$$H_n(s) = \frac{A_{21} + A_{22}s + A_{23}s^2 + \ldots + A_{2\ m+1}s^m}{1 + A_{12}s + A_{13}s^2 + \ldots + A_{1\ n+1}s^n} \quad (m \leq n) \qquad (1\text{-}22)$$

where $A_{k\ell}$ are constant coefficients and $m \leq n$.

$H_n(s)$ may be expanded in a series of positive powers of s as follows

$$H_n(s) = \sum_{i=0}^{\infty} C_i s^i \qquad (1\text{-}23)$$

where the constants C_i are related to the moments M_i by the relation

$$C_i = (-1)^i \frac{1}{i!} M_i \qquad (1\text{-}24)$$

By direct division, one can get from (1-22)

$$H_n(s) = A_{21} - A_{31}s + A_{41}s^2 - A_{51}s^3 + \ldots \qquad (1\text{-}25)$$

The coefficients $A_{k\ell}$ are given by the relation

$$A_{k\ell} = A_{k-1}A_{1\ \ell+1} - A_{k-1\ \ell+1} \qquad (1\text{-}26)$$

The above algorithm gives a set of moments M_i (or C_i) for the original system. Assuming the unknown simplified-model as:

$$H_\ell(s) = \frac{\hat{A}_{21} + \hat{A}_{22}s + \hat{A}_{23}s^2 + \ldots + \hat{A}_{2,k+1}s^k}{1 + \hat{A}_{12}s + \hat{A}_{13}s^2 + \ldots + \hat{A}_{1,\ \ell+1}s^\ell} \qquad (1\text{-}27)$$

Using eq. (1-23), (1-25) and (1-26) the following relation is obtained.

$$\begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_k \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & & \\ -C_0 & 0 & \cdots & 0 & & \\ -C_1 & -C_0 & \cdots & \vdots & & 0 \\ \vdots & \vdots & & & & \\ -C_{k-1} & -C_{k-2} & \cdots -C_0 & \cdots\ 0 & & \end{bmatrix} \begin{bmatrix} \hat{A}_{12} \\ \hat{A}_{13} \\ \vdots \\ \hat{A}_{1\ell+1} \end{bmatrix} + \begin{bmatrix} \hat{A}_{21} \\ \hat{A}_{22} \\ \vdots \\ \hat{A}_{2k+1} \end{bmatrix}$$

$$\begin{bmatrix} C_{k+1} \\ C_{k+2} \\ \vdots \\ C_{k+\ell} \end{bmatrix} = \begin{bmatrix} -C_k & -C_{k-1} & \cdots -C_1 & -C_0 & & \cdots\cdots\ 0 \\ -C_{k+1} & -C_k & \cdots & -C_1 & -C_1 & \cdots\cdots\ 0 \\ \vdots & \vdots & & & \vdots & \\ -C_{k+\ell-1} & -C_{k+\ell-2} & \cdots & & \cdots -C_0\ 0 & 0 \\ & & & & \cdots -C_1\ -C_0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(ℓth column; $(k+1)$th row)

(1-28)

Solving eq. (1-28) the unknown coefficients $\hat{A}_{k\ell}$ in the reduced model are obtained.

M. Lal and R. Mitra establish the equivalence of the moment-matching and simplification by continued-fraction-expansion and truncation [25]. They give the coefficients relationship between these methods.

The other method is due to Hsia, which is based on the requirement that the magnitude ratio of the frequency responses of the simplified-model and the original system deviate the least at various frequencies [13]. Let the magnitude of the frequency function

$$\lambda(w) = \left|\frac{H(jw)}{L(jw)}\right|^2 = \frac{M(jw)M(-jw)}{\Delta(jw)\Delta(-jw)}$$

$$= \frac{M_0 + M_2w^2 + M_4w^4 + \ldots.}{\Delta_0 + \Delta_2w^2 + \Delta_4w^4 + \ldots.} \qquad (1\text{-}29)$$

where $H(jw)$ is an original system, and

$L(jw)$ is a simplified model

$$M_{2\ell} = \sum_{k=0}^{2\ell} (-1)^{k+\ell} \frac{M^{(k)}(s)\ M^{(2\ell-k)}(s)}{k!\ (2\ell-k)!}\bigg|_{s=0}$$

$$M^{(k)}(s) = \frac{d^k}{ds^k} M(s)$$

$\Delta_{2\ell}$ is defined similarily $M_{2\ell}$, replacing M by Δ.

It is required that $\lambda(w) = 1$ and $\lambda(w)$ are expanded in Taylor series. From the condition $\lambda(w) = 1$, we obtain

$$M_{2\ell} = \Delta_{2\ell} \quad \ell = 1,2, \ldots \qquad (1\text{-}30)$$

Solving eq. (1-30), the stable set of solutions are chosen.

1.2.2 Simplification in Time-Domain

Model reduction can also be accomplished in the time domain. Two main ideas have appeared in the time-domain technique, namely, Davison's method and Aoki's method.

The principle of Davison's method is neglect eigenvalues of the original system which are the farthest from origin and retain only the dominant eigenvalues [1]. By the similarity transformation

$$\underline{x} = P\ \underline{z} \tag{1-31}$$

Eq. (1-2) can be rewritten by

$$\begin{bmatrix} \underline{z}_\ell \\ \dot{\underline{z}}_n \end{bmatrix} = \begin{bmatrix} {}^{\ell}J & 0 \\ 0 & {}^{n}J \end{bmatrix} \begin{bmatrix} \underline{z}_\ell \\ \underline{z}_n \end{bmatrix} + P^{-1}B\ u \tag{1-32}$$

where ${}^{\ell}J$ is an ℓ x ℓ matrix and chosen eigenvalues are the same as the predominant eigenvalues of A.

${}^{n}J$ is an $(n-\ell)$ x $(n-\ell)$ matrix

It is assumed that ${}^{n}J$ whose eigenvalues have negligible effect on the system response. From eq. (1-32), the reduced-order model is given by

$$\dot{\hat{\underline{z}}}_\ell = {}^{\ell}J\ \hat{\underline{z}}_\ell + \text{first } \ell \text{ rows of } P^{-1}B\ u \tag{1-33}$$

This is the same as the solution of eq. (1-2).

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_\ell \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \frac{-1+e^{\lambda_1 t}}{\lambda_1} \begin{bmatrix} x_1^1 \\ x_2^1 \\ \cdot \\ \cdot \\ x_\ell^1 \\ \cdot \\ \cdot \\ x_n^1 \end{bmatrix} (\phi_1^1 b_1 + \phi_1^2 b_2 + \ldots + \phi_1^\ell b_\ell + \ldots + \phi_1^n b_n) + \ldots$$

$$+ \frac{-1+e^{\lambda_\ell t}}{\lambda_\ell} \begin{bmatrix} x_1^\ell \\ x_2^\ell \\ \cdot \\ \cdot \\ x_\ell^\ell \\ \cdot \\ \cdot \\ x_n^\ell \end{bmatrix} (\phi_\ell^1 b_1 + \phi_\ell^2 b_2 + \ldots + \phi_\ell^\ell b_\ell + \ldots + \phi_\ell^n b_n) + \ldots$$

$$+ \frac{-1+e^{\lambda_n t}}{\lambda_n} \begin{bmatrix} x_1^n \\ x_2^n \\ \cdot \\ \cdot \\ x_\ell^n \\ \cdot \\ \cdot \\ x_n^n \end{bmatrix} (\phi_n^1 b_1 + \phi_n^2 b_2 + \ldots + \phi_n^\ell b_\ell + \ldots + \phi_n^n b_n) \qquad (1\text{-}34)$$

Then, neglecting terms of the type

$$\frac{-1+e^{\lambda_k t}}{\lambda_k} \begin{bmatrix} x_1^k \\ x_2^k \\ \cdot \\ \cdot \\ x_n^k \end{bmatrix} (\phi_k^1 b_1 + \phi_k^2 b_2 + \ldots + \phi_k^n b_k) \qquad (1\text{-}35)$$

where $k = \ell + 1, \ldots, n$

Chidambara suggested that term (1-35) cannot be completely neglected because (1-35) has, in general, significant contributions to the response [2]. The steady-state response of the ℓth-order reduced model will, in general, be significantly different from that of the exact system. To overcome this problem he introduced the R matrix as

$$\dot{\hat{z}} = {}^{\ell}\hat{J}\ z_{\ell} + G_{\ell} u$$

$$\hat{Y}_{\ell} = [k_{\ell} + R]\ \hat{z}_{\ell} \qquad (1\text{-}36)$$

where R is a constant matrix to be determined such that it minimizes the integral-square error between the step response of the exact $z_{ex}(t)$ and of the reduced model, $\hat{z}_{\ell}(t)$ and $(z_{ex} - \hat{z}_{\ell}) \to 0$ as $t \to \infty$.

In the methods based on the aggregation principle, a reduced-order model

$$\dot{\hat{z}}_{\ell} = F\ \hat{z}_{\ell} + B\ u \qquad (1\text{-}37)$$

is obtained for the system described by eq. (1-2), where the ℓ-dimensional $\hat{z}_{\ell}$ vector ($\ell << n$) is related to the n-dimensional x vector through the aggregation matrix C, as (Aoki [8]).

$$\hat{z}_{\ell}(t) = C\ x\ (t) \qquad (1\text{-}38)$$

where C is an ℓ x n constant matrix, $\ell << n$, and rank-C = ℓ is assumed.

The dynamic exactness is achieved if and only if

$$FC = CA$$
$$G = CB \qquad (1\text{-}40)$$

The matrix F will be referred to as the aggregated matrix or aggregation of A. Eq. (1-38) and (1-40) imply that $\hat{\underline{z}}(t)$ is a linear combination of certain modes of $\underline{x}(t)$. In this case, the eigenvalues of F are the eigenvalues of A corresponding to those modes of $\underline{x}(t)$ which are retained in $\underline{z}(t)$.

Thus the notion of aggregation for a linear system is a generalization of 'Davison's method' for simplifying the dynamics of linear systems by retaining the dominant modes.

1.3 Consideration of Reduced-Order Modeling Techniques

As shown in a previous section, a number of methods have appeared in the literature for developing a lower-order model. These methods have their own advantages and disadvantages. To investigate the superiority among these methods, the step responses of the simplified-models are computed to the same original system. An objective function is shown as an integral-square error between these responses.

From the consideration of the computational results, the following can be stated:

(i) On the measure of goodness of the approximation -

By computing the unit-step responses of the various methods, it is concluded that the second Cauer form in the continued-fraction-expansion and Hsia's method yields a small value of the objective function compared with all the other methods.

(ii) On the time responses of the time-domain techniques -

In a transient-state, Davison's model, neglecting the non-dominant eigenvalues, responds close to the original system for the unit-step input.

While the modified Davison's model due to Chidambara is close to the original system in a steady-state.

The previous time-domain techniques are interpretable in the frequency-domain so that the denominator polynomial of the transfer function is prespecified and only the numerator polynomial is subject to the approximation. Therefore, the time-domain techniques need to be changed so that the numerator polynomial as well as the denominator polynomial can be approximated.

(iii) On the stability of the continued-fraction-expansion method

The continued-fraction-expansion method has a drawback in that the reduced-order model may be unstable even though

the original system is stable. This is because the approximation does not necessarily give a stable model.

To overcome this instability, Shieh and Wei introduced the dominant eigenvalue concept to the Routh algorithm and obtained the stable reduced-order model [32], which is called 'Mixed method'. Routh approximation due to Hutton and Friedland also gives a stable reduced-order model [31].

However, these methods do not give satisfactory small values of the objective function, in the sense of error criterion. The stability condition on the reduced-order model obtained by the continued-fraction-expansion method must be considered.

(iv) On a multivariable system -

The present time-domain techniques are very hard to apply to a multivariable system. However, the frequency-domain techniques are applicable. A more successful model is desired so as to minimize the integral-square error between the step responses of the reduced-order model and of the exact system.

(v) On a system with repeated eigenvalues (or poles) -

Using the similarity transformation to the original system, the original system transforms into Jordan canonical form. The system can be considered as two separate blocks; one is a block which has non-repeated eigenvalues, and the other is the block with repeated eigenvalues. There are dominant eigenvalues for the block with non-repeated

eigenvalues. However, for the block with repeated eigenvalues, the procedure breaks down. Therefore, the present time-domain techniques, neglecting the non-dominant eigenvalues, can not be applied for this system.

The continued-fraction-expansion method can be used. The model obtained by this method, however, does not give satisfactory results in the sense of the integral-square error.

CHAPTER II. MODEL REDUCTION BY MEANS OF TWO-STEP ITERATIVE METHOD

As mentioned above, there are many drawbacks of the previous methods. In this chapter, the basic philosophy of a new approach for the model reduction of a large linear system is shown. The new technique is simple and improves on the various disadvantages of the existing reduced-order modeling techniques. In Chapter III, illustrative examples are presented.

2.1 Philosophy of Approach

The proposed technique is based on the philosophy that a good reduced-order model must be close to the original system of both the steady-state and transient responses.

To achieve this goal, the proposed method consists of a two-step iterative scheme. In the first step, the steady-state response is improved. This is obtained by optimizing the residues so as to minimize the objective function while the poles (or eigenvalues) are kept constant. The objective function is the value of the integral-square error between the step responses of the exact and simplified system.

In the second step, the transient response is developed. This result is obtained by optimizing the poles (or eigenvalues) while the residues remain fixed.

These two steps are continued cyclically until the objective function is satisfactorily minimized. The necessary and sufficient conditions for existence of an optimum are satisfied in each step. The residues, poles (or eigenvalues) and objective functions always converge monotonically.

This method gives a stable reduced-order model if the original system is stable. The method can be applied not only to single-variable systems but also to systems with repeated poles (or eigenvalues) and to multivariable systems. The results are superior to those obtained previously in the steady-state and transient responses, and the integral-square error.

2.2 Two-Step Iterative Method

The proposed technique is applied to the reduction of a transfer function and state-space systems. The objective function to be minimized has the form

$$J_s = \int_0^\infty (y_n - \hat{y}_\ell^i)^2 dt \qquad (2\text{-}1)$$

for a single-variable case

where y_n is the step response of the original system

$\hat{y}_\ell^i$ is the step response of the reduced-order model

The superscript 'i' denotes the number of iterative steps (i=0,1,2,...).

The subscript 'ℓ' is the order of the reduced model.

For a multivariable system

$$J_M = \int_0^\infty \hat{\underline{e}}^{i\prime} \, Q \, \hat{\underline{e}}^{i} \, dt \qquad (2\text{-}2)$$

where $\hat{\underline{e}}^{i\prime} = [\hat{e}_1^i \; \hat{e}_2^i \; \ldots \; \hat{e}_w^i]$

$$\hat{e}_j^i = Y_{n,j} - \hat{Y}_{\ell,j}^i \qquad j = 1,2, \ldots 2$$

w is the number of the system outputs

A prime denotes the transpose

Q is a positive-definitive weighting matrix.

2.2.1 Simplification of Transfer Function

In this section, consider the simplification of the single-variable system (single-input, single-output) and the multivariable system (v-input,w-output). There are no essential differences for the simplification procedure among these systems.

For the case of the multivariable system, the system equation and the solution have forms of the matrix and vector respectively, otherwise all the procedure is the same as the single-variable system.

The system with repeated poles is also reduced here.

1. Single-Variable System

Consider a single-variable system with the transfer function

$$H_n(s) = \frac{Y(s)}{U(s)} = \frac{a_{2m+1}s^m + a_{2m}s^{m-1} + \ldots + a_{22}s + a_{21}}{a_{1n+1}s^n + a_{1n}s^{n-1} + \ldots + a_{12}s + a_{11}} \qquad (2\text{-}3)$$

where The poles are at $s = \lambda_1, \lambda_2, \ldots \lambda_\ell, \lambda_{\ell+1}, \ldots \lambda_n$ and $\lambda_1\ \lambda_2, \ldots \lambda_\ell$ are the dominant poles of the transfer function and the remaining $(n-\ell)$ poles are assumed to have negligible effect on the transient response.

All the coefficients in the transfer function are constants and $m \leq n$.

Here it is assumed in eq. (2-3) that all the poles are real, distinct and lie in the negative-real part in a complex plane.

The proposed method can be applied for the system with complex poles by optimizing both real-parts and imaginary-parts of residues and poles (or eigenvalues) at each step.

An ℓth-order ($\ell << n$) transfer function to be found is in the form of

$$\hat{H}^i_\ell(s) = \frac{\hat{a}^i_{2k+1}s^k + \hat{a}^i_{2k}s^{k-1} + \ldots + \hat{a}^i_{22}s + \hat{a}^i_{21}}{\hat{a}^i_{1\ell+1}s^\ell + \hat{a}^i_{1\ell}s^{\ell-1} + \ldots + \hat{a}^i_{12}s + \hat{a}^i_{11}} \qquad (2\text{-}4)$$

where The poles are at $s = \hat{\lambda}^i_1, \hat{\lambda}^i_2, \ldots \hat{\lambda}^i_\ell$ (real, distinct) and lie in the negative-real part in the complex plane.

All the coefficients in eq. (2-4) are constants and $k \leq \ell$.

'i' is an iteration number ($i = 0,1,2, \ldots$).

The time-domain solution of eq. (2-3) for the unit-step

input is represented by

$$y_n(t) = \alpha_0+\alpha_1 e^{\lambda_1 t}+\alpha_2 e^{\lambda_2 t}+\ldots+\alpha_\ell e^{\lambda_\ell t}+\ldots+\alpha_{11} e^{\lambda_n t} \qquad (2\text{-}5)$$

Assume that the unit-step response of the approximate-model in the ith-step has the form

$$\hat{y}^i_\ell(t) = \hat{\alpha}^i_0 + \hat{\alpha}^i_1 e^{\hat{\lambda}^i_1 t} + \hat{\alpha}^i_2 e^{\hat{\lambda}^i_2 t} + \ldots + \hat{\alpha}^i_\ell e^{\hat{\lambda}^i_\ell t} \qquad (2\text{-}6)$$

where i=0,1,2, ...

In each step the residues ($\hat{\alpha}^i_0$, $\hat{\alpha}^i_1$, ..., $\hat{\alpha}^i_\ell$) and poles ($\hat{\lambda}^i_1, \hat{\lambda}^i_2$, ..., $\hat{\lambda}^i_\ell$) are found through a minimization of the objective function, eq. (2-1).

A. Computational Procedure

The General computational procedure by the two-step iterative method is summarized as follows:

Step 1: Initial step (i=0)

The approximate-model is assumed to have the same dominant poles as the exact system.

$$\underline{\hat{\lambda}}^0 = \underline{\lambda} \qquad (2\text{-}7)$$

where $\underline{\hat{\lambda}}^{0\prime} = [\hat{\lambda}^0_1 \ \hat{\lambda}^0_2 \ \ldots \ \hat{\lambda}^0_\ell]$

$\underline{\lambda}' = [\lambda_1 \ \lambda_2 \ \ldots \ \lambda_\ell]$

In order that the objective function, eq. (2-1) be a finite, it is necessary that $(y_n - \hat{y}^i_\ell) \to 0$ as $t \to \infty$ in each

iterative step. This condition gives

$$\hat{\alpha}_0^i = \alpha_0 \qquad (2\text{-}8)$$

for all i (i = 0,1,2, ...)

where $\hat{\alpha}_0^i$ is the first residue of the approximate-model, shown in eq. (2-6).

α_0 is the first residue of the original system, represented by eq. (2-5).

The general recursive formulae for the optimum residues and poles are computed next.

Step 2: Residue optimization (i=N)

The unit-step response of the approximate-model at the Nth-step is, setting i=N in eq. (2-6),

$$\hat{y}_\ell^N = \hat{\alpha}_0^N + \hat{\alpha}_1^N e^{\hat{\lambda}_1^N t} + \hat{\alpha}_2^N e^{\hat{\lambda}_2^N t} + \ldots + \hat{\alpha}_\ell^N e^{\hat{\lambda}_\ell^N t} \qquad (2\text{-}9)$$

where the initial condition is prescribed by eq. (2-8)

$$\hat{\alpha}_0^N = \alpha_0 \qquad (2\text{-}10)$$

The pole vector, $\underline{\hat{\lambda}}^N$, is kept the same as $\underline{\hat{\lambda}}^{N-1}$ in the previous step.

The residue vector, $\underline{\hat{\alpha}}^N$, of the approximate system is determined by minimizing the objective function, eq. (2-1). This is derived by a differentiation of the objective function with respect to $\underline{\hat{\alpha}}^N$, so that

$$\frac{\partial J_s}{\partial (\hat{\underline{\alpha}}^N)} = \underline{0} \tag{2-11}$$

where $\underline{0}$ is the ℓ null vector

Then a set of ℓth-order simultaneous equations is obtained. In this step the following two equations are used.

$$\hat{\underline{\lambda}}^N = \hat{\underline{\lambda}}^{N-1} \tag{2-12}$$

where $\underline{\lambda}^N = [\hat{\lambda}_1^N \ \hat{\lambda}_2^N \ \dots \ \hat{\lambda}_\ell^N]$

$$\hat{\underline{\lambda}}^{N-1} = [\hat{\lambda}_1^{N-1} \ \hat{\lambda}_2^{N-1} \ \dots \ \hat{\lambda}_\ell^{N-1}]$$

$$\hat{\psi}^N \cdot \hat{\underline{\alpha}}^N = \hat{\Gamma}^N \tag{2-13}$$

where

$$\hat{\psi}^N = \begin{bmatrix} \frac{1}{\hat{\lambda}_1^{N-1}} & \frac{2}{\hat{\lambda}_1^{N-1}+\hat{\lambda}_2^{N-1}} & \cdots & \frac{2}{\hat{\lambda}_1^{N-1}+\hat{\lambda}_\ell^{N-1}} \\ \frac{2}{\hat{\lambda}_2^{N-1}+\hat{\lambda}_1^{N-1}} & \frac{1}{\hat{\lambda}_2^{N-1}} & \cdots & \frac{2}{\hat{\lambda}_2^{N-1}+\hat{\lambda}_\ell^{N-1}} \\ \vdots & & & \vdots \\ \frac{2}{\hat{\lambda}_\ell^{N-1}+\hat{\lambda}_1^{N-1}} & \frac{2}{\hat{\lambda}_\ell^{N-1}+\hat{\lambda}_2^{N-1}} & \cdots & \frac{1}{\hat{\lambda}_\ell^{N-1}} \end{bmatrix}$$

$$\hat{\underline{\alpha}}^{N'} = [\hat{\alpha}_1^N \ \hat{\alpha}_2^N \ \dots \ \hat{\alpha}_\ell^N]$$

$$\hat{\Gamma}^N = 2 \times \begin{bmatrix} \sum_{q=1}^{n} \frac{\alpha_q}{\hat{\lambda}_1^{N-1} + \lambda_q} \\ \sum_{q=1}^{n} \frac{\alpha_q}{\hat{\lambda}_2^{N-1} + \lambda_q} \\ \vdots \\ \sum_{q=1}^{n} \frac{\alpha_q}{\hat{\lambda}_\ell^{N-1} + \lambda_q} \end{bmatrix}$$

Solving eq. (2-13), the residue vector, $\underline{\hat{\alpha}}^N$ is obtained. Eq. (2-12) and (2-13) are used in the first iterative step (N=1) and thereafter.

Step 3. Pole optimization (i = N+1)

The solution of the approximate-model is, letting i=N+1, in eq. (2-6),

$$\hat{y}_\ell^{N+1} = \hat{\alpha}_0^{N+1} + \hat{\alpha}_1^{N+1} e^{\hat{\lambda}_1^{N+1} t} + \hat{\alpha}_2^{N+1} e^{\hat{\lambda}_2^{N+1} t} + \ldots + \hat{\alpha}^{N+1} e^{\hat{\lambda}_\ell^{N+1} t} \tag{2-14}$$

The residue vector $\underline{\hat{\alpha}}^{N+1}$ is fixed. The pole vector $\underline{\hat{\lambda}}^{N+1}$ will be optimized. Let

$$\underline{\hat{\lambda}}^{N+1} = \underline{\hat{\lambda}}^N + \Delta\underline{\hat{\lambda}}^{N+1} \tag{2-15}$$

where $\Delta\underline{\hat{\lambda}}^{N+1}$ is assumed to be small, and it is obtained within the context of the necessary condition for the existence of an optimum, defined by the vanishing of all

the first derivatives of the objective function, eq. (2-1), so that

$$\frac{\partial J_s}{\partial(\Delta\underline{\hat{\lambda}}^{N+1})} = \underline{0} \tag{2-16}$$

where $\underline{0}$ is the ℓ null vector.

The recursive formulae in this step are as follows:

$$\underline{\hat{\alpha}}^{N+1} = \underline{\hat{\alpha}}^{N} \tag{2-17}$$

where $\underline{\hat{\alpha}}^{N+1\prime} = [\hat{\alpha}_1^{N+1} \ \hat{\alpha}_2^{N+1} \ \ldots \ \hat{\alpha}_\ell^{N+1}]$

$$\underline{\hat{\alpha}}^{N\prime} = [\hat{\alpha}_1^{N} \ \hat{\alpha}_2^{N} \ \ldots \ \hat{\alpha}^{N}]$$

$$\hat{\phi}^{N+1} \cdot \Delta\underline{\hat{\lambda}}^{N+1} = \hat{\pi}^{N+1} \tag{2-18}$$

$$\hat{\phi}^{N+1} = \begin{bmatrix} \frac{\hat{\alpha}_1^N}{2(\hat{\lambda}_1^N)^3} & \frac{4\hat{\alpha}_2^N}{(\hat{\lambda}_1^N+\hat{\lambda}_2^N)^3} & \cdots & \frac{4\hat{\alpha}_\ell^N}{(\hat{\lambda}_1^N+\hat{\lambda}_\ell^N)^3} \\ \frac{4\hat{\alpha}_1^N}{(\hat{\lambda}_2^N+\hat{\lambda}_1^N)^3} & \frac{\hat{\alpha}_2^N}{2(\hat{\lambda}_2^N)^3} & \cdots & \frac{4\hat{\alpha}_\ell^N}{(\hat{\lambda}_2^N+\hat{\lambda}_\ell^N)^3} \\ \vdots & \vdots & & \\ \frac{4\hat{\alpha}_1^N}{(\hat{\lambda}_\ell^N+\hat{\lambda}_1^N)^3} & \frac{4\hat{\alpha}_2^N}{(\hat{\lambda}_\ell^N+\hat{\lambda}_2^N)^3} & \cdots & \frac{\hat{\alpha}_\ell^N}{2(\hat{\lambda}_\ell^N)^3} \end{bmatrix}$$

$$\Delta\underline{\hat{\lambda}}^{N+1\prime} = [\Delta\hat{\lambda}_1^{N+1} \ \Delta\hat{\lambda}_2^{N+1} \ \ldots \ \Delta\hat{\lambda}_\ell^{N+1}]$$

$$\hat{\Gamma}^{N+1} = -2 \times \begin{bmatrix} \sum_{q=1}^{n} \frac{\alpha_q}{(\hat{\lambda}_1^N+\lambda_q)^2} - \sum_{r=1}^{\ell} \frac{\hat{\alpha}_r^N}{(\hat{\lambda}_1^N+\hat{\lambda}_r^N)^2} \\ \sum_{q=1}^{n} \frac{\alpha_q}{(\hat{\lambda}_2^N+\lambda_q)^2} - \sum_{r=1}^{\ell} \frac{\hat{\alpha}_r^N}{(\hat{\lambda}_2^N+\hat{\lambda}_r^N)^2} \\ \vdots \\ \sum_{q=1}^{n} \frac{\alpha_q}{(\hat{\lambda}^N+\lambda_q)^2} - \sum_{r=1}^{\ell} \frac{\hat{\alpha}_r^N}{(\hat{\lambda}^N+\hat{\lambda}_r^N)^2} \end{bmatrix}$$

To obtain eq. (2-18), the approximation

$$e^{\Delta\hat{\lambda}_r^{N+1}t} \simeq 1 + \Delta\hat{\lambda}_r^{N+1}\, t \qquad (2\text{-}19)$$

where $r = 1,2, \ldots \ell$

is used.

The pole vector, $\Delta\hat{\underline{\lambda}}^{N+1}$ is given by a solution of eq. (2-18). These equations, eq. (2-17) and (2-18), are applicable with N=2 and so on.

Step 4: i=(N+2)th-step

For the (N+2)th step, set N=N+2 and go to step 2. This procedure is repeated through a succession of stages until the objective function becomes sufficiently small.

The sufficient condition for an extreme value to exist is that the second derivatives of the objective function does not change sign. The second derivative of the objective function is evaluated for a confirmation of this condition.

With respect to residue vector, the second derivative of the objective function becomes

$$\frac{\partial^2 J_s}{\partial (\underline{\hat{\alpha}}^N)^2} = \begin{bmatrix} -\frac{1}{\hat{\lambda}_1^{N-1}} & 0 & \cdots & 0 \\ 0 & -\frac{1}{\hat{\lambda}_2^{N-1}} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & -\frac{1}{\hat{\lambda}_\ell^{N-1}} \end{bmatrix} \qquad (2\text{-}20)$$

Since the original system is stable, all the poles have negative-real parts. The increments of the poles are very small, hence the poles at the Nth-step have also negative-real parts. Therefore, the right side of eq. (2-20) is always positive-definite matrix.

$$\frac{\partial^2 J_s}{\partial (\underline{\alpha}^N)^2} > \underline{0} \qquad (2\text{-}21)$$

where $\underline{0}$ is the ℓ null vector.

Eq. (2-21) shows that a local minimum will exist at the optimum point.

With respect to the pole vector, the second derivative of the objective function becomes

$$\frac{\partial^2 J_s}{\partial(\Delta\underline{\hat{\lambda}}^{N+1})^2} = \begin{bmatrix} -\frac{(\hat{\alpha}_1^N)^2}{(\hat{\lambda}_1^N)^3} & 0 & \cdots & 0 \\ 0 & -\frac{(\hat{\alpha}_2^N)^2}{(\hat{\lambda}_2^N)^3} & & \vdots \\ \vdots & & & \vdots \\ 0 & & & -\frac{(\hat{\alpha}_\ell^N)^2}{(\hat{\lambda}_\ell)^3} \end{bmatrix}$$

Since all the poles lie in the negative-real part in a complex s-plane, the right side of Eq. (2-22) is always positive-definite matrix.

$$\frac{\partial^2 J_s}{\partial(\Delta\underline{\hat{\lambda}}^{N+1})^2} > \underline{0} \qquad (2\text{-}23)$$

where $\underline{0}$ is the ℓ null vector.

From eq. (2-23) the sufficient condition for a minimum to exist at the optimum point is satisfied.

It is found from eq. (2-21) and (2-23) that in these steps there exist minimum values of the objective function at the optimum points.

2. Multivariable System

The original system is represented by an nth-order transfer function matrix instead of a scalar transfer function.

$$[H_n(s)] = [A_{2m+1}s^m + A_{2m}s^{m-1} + \ldots + A_{22}s + A_{21}] \times [A_{1n+1}s^n + A_{1n}s^{n-1} + \ldots + A_{12}s + A_{11}]^{-1} \quad (2\text{-}24)$$

where the coefficients of s are constant matrices and $m \leq n$.

It is necessary to find the ℓth-order ($\ell << n$) function matrix.

$$[\hat{H}^i_\ell(s)] = [\hat{A}^i_{2k+1}s^k + \hat{A}^i_{2k}s^{k-1} + \ldots + \hat{A}^i_{22}s + \hat{A}^i_{21}2 \times [\hat{A}^i_{1\ell+1}s^\ell + \hat{A}^i_{1\ell}s^{\ell-1} + \ldots + \hat{A}^i_{12}s + \hat{A}^i_{11}]^{-1} \quad (2\text{-}25)$$

where all the coefficients are constant matrices and $k \leq \ell$

The solution of the exact system, eq. (2-24) and the reduced-order model, eq. (2-25) are given in the vector form respectively.

$$\underline{Y}'_n = [Y_{n,1}\ Y_{n,2} \cdots Y_{n,w}] \quad (2\text{-}26)$$

$$\underline{\hat{Y}}^{i\,\prime}_\ell = [\hat{Y}^i_{\ell,1}\ \hat{Y}^i_{\ell,2} \cdots \hat{Y}^i_{\ell,w}] \quad (2\text{-}27)$$

where w is a number of system outputs

i is an iteration number (i=0,1,2, ...).

In this case the objective function to be minimized is used in eq. (2- 2) replacing eq. (2- 1) for a single-variable

system. Minimizing eq. (2-2) with respect to the residues and poles, the similar simultaneous equations are obtained in the case of a single-variable system. Solving these simultaneous equations, the optimum residues and poles are yielded.

An illustrative example is shown in example 2.

3. System with Repeated Poles

In the system with repeated poles, the denominator of the transfer function can be separated by the product of the non-repeated poles and the repeated poles. There are dominant poles for the part of non-repeated poles. However, the dominant poles cannot be selected for the part of the repeated poles because all the poles have the same values.

Since the proposed method is based on the selection of the dominant poles in the initial step, the method is not able to apply this system directly. But the method is also applicable for the system by the modification of the original system.

In this case the part of the repeated poles in the original system are made slightly different from each other. These poles are used as the dominant poles in the original system.

Let the poles in the original system have k-repeated poles and (n-k) non-repeated poles:

$$s = \lambda_1 = \lambda_2 = \ldots = \lambda_k \equiv \lambda$$
$$= \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_\ell, \ldots, \lambda_n \qquad (2\text{-}28)$$

There are no dominant poles for the k-repeated poles.

Now the ℓth-order reduced model ($\ell << n$, and $\ell \geq k$) is required. In the initial iterative step, the poles to be fixed are chosen as follows:

$$\hat{\lambda}_1^0 = \lambda + \varepsilon_1$$
$$\hat{\lambda}_2^0 = \lambda + \varepsilon_2 \qquad (2\text{-}29)$$
$$\vdots$$
$$\hat{\lambda}_k^0 = \lambda + \varepsilon_k$$

where ε_i (i=1,2, ..., k) are arbitrarily small numbers

A superscript '0' denotes the initial step.

For (ℓ-k), the dominant poles are chosen in the reduced-order model. With eq. (2-29), the reduced-order model is made to have no repeated poles. As long as the set of ε_i (i=1,2, ..., k) consist of sufficiently small elements, then the reduced-order model is nearly identical in terms of the objective function regardless of the value of ε_i (i=1,2,...,k). After setting up the initial condition as eq. (2-29), a similar iterative scheme is carried out, which is shown in example 3.

2.2.2 Simplification of Linear Time-Invariant System

The same iterative method is applicable even if the original linear time-invariant system is described by state-variable equations. The optimization in the control-vector is equivalent to the optimization in the output-vector because they both give the same effect to the output. Therefore, there are two ways of simplification: one way is by the optimization of the control-vector, and the other is by the optimization of the output-vector.

1. Single-Variable System

Assume that an nth-order linear time-invariant system can be written as:

$$\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + \underline{d}\,u \\ y_n &= \underline{c}'\underline{x} \end{aligned} \tag{2-30}$$

where A is an n x n matrix

$$\underline{c}' = [c_1\, c_2 \ldots c_\ell \ldots c_n]$$

$$\underline{d}' = [d_1\, d_2 \ldots d_\ell \ldots d_n]$$

and

$$\underline{d}u = 0, \quad t < 0$$

$$= \underline{d}, \quad t \geq 0$$

The eigenvalues of A have negative-real parts in the complex plane and are distinct. The proposed method can be

used for the system with real eigenvalues and complex eigenvalues.

Suppose it is desired to find a simplified-model of a reduced order ($\ell << n$) in the form of

$$\dot{\hat{\underline{x}}}^i = \hat{A}_0^i \, \hat{\underline{x}}^i + \hat{\underline{b}}_0^i \, u$$
$$\hat{y}_\ell^i = \hat{\underline{h}}_0^{i\prime} \, \hat{\underline{x}}^i \qquad (2\text{-}31)$$

where $\hat{A}_0^i$ is an $\ell \times \ell$ matrix

A subscript '0' denotes the simplified-model in the x-domain

A subscript 'ℓ' signifies the output of the ℓth-order reduced model.

A. Computational Procedure

The application of a similarity transformation $\underline{x} = P_n \underline{Z}$ to the original system, eq. (2-30) results in

$$\dot{\underline{Z}} = \Lambda \, \underline{Z} + \underline{b} \, u$$
$$y_n = \underline{h}' \, \underline{Z} \qquad (2\text{-}32)$$

where $\Lambda = P_n^{-1} \, A \, P_n = \begin{bmatrix} \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & & & & \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot \lambda_\ell & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & & \cdot \lambda_n \end{bmatrix}$

$$\underline{b}' = P_n^{-1} \, \underline{d}' = [b_1 \;\; b_2 \;\; \ldots \;\; b_\ell \;\; \ldots \;\; b_n]$$

$$\underline{h}' = \underline{c}' P_n = [h_1 \; h_2 \; \ldots \; h_\ell \; \ldots \; h_n]$$

The solution of eq. (2-32) is,

$$y_n = -\sum_{k=1}^{n} \frac{b_k}{\lambda_k} h_k + \sum_{k=1}^{n} \frac{b_k}{\lambda_k} h_k \, e^{\lambda_k t} \tag{2-33}$$

To facilitate their later use, let

$$\bar{b}_m = \frac{b_m}{\lambda_m} \quad (m = 1,2, \ldots n) \tag{2-34}$$

then the solution of eq. (2-32) is rearranged into the form

$$y_n = -\sum_{k=1}^{n} \bar{b}_k h_k + \sum_{k=1}^{n} \bar{b}_k h_k \, e^{\lambda_k t} \tag{2-35}$$

The ℓth-order reduced model in the ith-step is obtained by the elimination of the non-dominant eigenvalues and the use of eq. (2-34), then

$$\begin{aligned} \dot{\hat{\underline{z}}}^i &= \hat{\Lambda}^i \, \hat{\underline{z}}^i + \hat{\underline{b}}^i \, u \\ \hat{y}_\ell^i &= \hat{\underline{h}}^{i\prime} \, \hat{\underline{z}}^i \end{aligned} \tag{2-36}$$

where $\hat{\Lambda}^i = P_\ell^{-1} \hat{A}^i P_\ell = \begin{bmatrix} \hat{\lambda}_1^i & 0 & \ldots & 0 \\ 0 & \hat{\lambda}_2^i & & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & & \hat{\lambda}_\ell \end{bmatrix}$

$$\underline{\hat{b}}^{i\prime} = P_\ell^{-1} \, \underline{\hat{d}}^{i\prime} = [\hat{b}_1^i \;\; \hat{b}_2^i \; \ldots\ldots\ldots \; \hat{b}_\ell^i]$$

$$= [\hat{\lambda}_1^i \bar{b}_1^i \;\; \hat{\lambda}_2^i \bar{b}_2^i \; \ldots\ldots \; \hat{\lambda}_\ell^i \bar{b}_\ell^i], \text{ from eq. (2-24)}$$

$$\underline{\hat{h}}^{i\prime} = \underline{\hat{c}}^{i\prime} \, P_\ell = [\hat{h}_1^i \; \hat{h}_2^i \; \ldots\ldots \; \hat{h}_\ell^i]$$

P_ℓ is an ℓ x ℓ matrix and is a truncation of the n x n modal matrix.

The solution of eq. (2-36) is

$$\hat{y}_\ell^i = -\sum_{m=1}^{\ell} \bar{b}_m^i \hat{h}_m^i + \sum_{m=1}^{\ell} \bar{b}_m^i \hat{h}_m^i e^{\hat{\lambda}_m^i t} \qquad (2\text{-}37)$$

i) Control-Vector Optimization

Step 1: Initial step (i=0)

Assume that the reduced-order model, eq. (2-36) has the same dominant eigenvalues as the original system, eq. (2-32).

$$\text{diag}[\hat{\lambda}_1^0 \ \hat{\lambda}_2^0 \ \ldots \ \hat{\lambda}_\ell^0] = \text{diag}]\lambda_1 \ \lambda_2 \ \ldots \ \lambda_\ell] \qquad (2\text{-}38)$$

The first residue and output-vector are fixed for all steps.

$$-\sum_{m=1}^{\ell} \bar{b}_m^i \hat{h}_m^i = \hat{\alpha}_0^i = \alpha_0 \quad \text{for all i} \qquad (2\text{-}39)$$

$$\underline{\hat{h}}^{i\prime} = \underline{\hat{h}}' \quad \text{for all i} \qquad (2\text{-}40)$$

Step 2: Residue optimization (i=N)

The state-space equations are, setting i=N in eq. (2-36), and using eq. (2-40),

$$\begin{aligned} \dot{\underline{\hat{z}}}^N &= \hat{\Lambda}^N \underline{\hat{z}}^N + \underline{\hat{b}}^N u \\ \hat{y}_\ell^N &= \underline{\hat{h}}' \ \underline{\hat{z}}^N \end{aligned} \qquad (2\text{-}41)$$

where

$$\hat{\Lambda}^N = \begin{bmatrix} \hat{\lambda}_1^N & 0 & \cdots & 0 \\ 0 & \hat{\lambda}_2^N & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \hat{\lambda}_\ell \end{bmatrix}$$

$$\underline{\hat{b}}^{N\prime} = [\hat{b}_1^N \quad \hat{b}_2^N \quad \cdots \quad \hat{b}_\ell^N]$$

$$= [\hat{\lambda}_1^N \bar{b}_1^N \quad \hat{\lambda}_2^N \bar{b}_2^N \quad \cdots \quad \hat{\lambda}_\ell^N \bar{b}_\ell^N]$$

$$\underline{\hat{h}}' = [\hat{h}_1 \quad \hat{h}_2 \quad \cdots \quad \hat{h}_\ell]$$

The solution of eq. (2-41) is, using eq. (2-40)

$$\hat{y}_\ell^N = -\sum_{m=1}^{\ell} \bar{b}_m^N \hat{h}_m + \sum_{m=1}^{\ell} \bar{b}_m^N \hat{h}_m e^{\hat{\lambda}_m^N t} \qquad (2\text{-}42)$$

The eigenvalues of the diagonalized model are considered fixed.

The control-vector, $\underline{\bar{b}}^N$ is optimized through eq. (2-1).

$$\frac{\partial J_s}{\partial (\underline{\bar{b}}^N)} = \underline{0} \qquad (2\text{-}43)$$

where $\underline{0}$ is the ℓ null vector.

Since eq. (2-9) is similar to eq. (2-42), wherein $\hat{\underline{\alpha}}^N$ in eq. (2-11) is replaced by $\underline{\bar{b}}^N \underline{\hat{h}}'$ in eq. (2-43). The Nth-step optimum solution is obtained as follows.

$$\text{diag}[\hat{\lambda}_1^N \quad \hat{\lambda}_2^N \quad \cdots \quad \hat{\lambda}_\ell^N] = \text{diag}[\hat{\lambda}_1^{N-1} \quad \hat{\lambda}_2^{N-1} \quad \cdots \quad \hat{\lambda}_\ell^{N-1}] \qquad (2\text{-}44)$$

Then, eq. (2-13), replacing $\hat{\underline{\alpha}}^N$ by $\bar{\underline{b}}^N\ \hat{\underline{h}}'$, that is

$$\hat{\psi}^N(\bar{\underline{b}}^N\ \hat{\underline{h}}') = \hat{\Gamma}^N \tag{2-45}$$

where $\hat{\psi}^N$ and $\hat{\Gamma}^N$ are given by eq. (2-13).

Eq. (2-44) and (2-45) are effective in the first iterative step (N=1) and thereafter.

Step 3: Eigenvalue optimization (i=N+1)

The state-space equations in the (N+1)th-step are expressed as:

$$\dot{\hat{\underline{Z}}}^{N+1} = \hat{\Lambda}^{N+1}\ \hat{\underline{Z}}^{N+1} + \hat{\underline{b}}^{N+1}\ u \tag{2-46}$$

$$\hat{y}_\ell^{N+1} = \hat{\underline{h}}'\ \hat{\underline{Z}}^{N+1}$$

where

$$\hat{\Lambda}^{N+1} = \begin{bmatrix} \hat{\lambda}_1^{N+1} & 0 & \cdots & 0 \\ 0 & \hat{\lambda}_2^{N+1} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \hat{\lambda}_\ell \end{bmatrix}$$

$$\hat{\underline{b}}^{N+1\prime} = [\hat{b}_1^{N+1}\ \ \hat{b}_2^{N+1}\ \ \cdots\ \ \hat{b}_\ell^{N+1}]$$

$$= [\hat{\lambda}_1^{N+1}\ \ \bar{b}_1^{N+1}\ \ \hat{\lambda}_2^{N+1}\ \ \bar{b}_2^{N+1}\ \ \cdots\ \ \hat{\lambda}_\ell^{N+1}\ \ b_\ell^{N+1}]$$

$$\hat{\underline{h}}' = [\hat{h}_1\ \ \hat{h}_2\ \ \cdots\ \ \hat{h}_\ell]$$

The solution of eq. (2-46) is, using eq. (2.40),

$$\hat{y}_{\ell}^{N+1} = -\sum_{m=1}^{\ell} \bar{b}_m^{N+1}\hat{h}_m + \sum_{m=1}^{\ell} \bar{b}_m^{N+1}\hat{h}_m\, e^{\hat{\lambda}_m^{N+1}t} \qquad (2\text{-}47)$$

The control-vector $\underline{\bar{b}}^N$ is kept constant.

The eigenvalues in the $\hat{\Lambda}^{N+1}$ are assumed to be given by

$$\begin{aligned} &\mathrm{diag}[\hat{\lambda}_1^{N+1}\ \hat{\lambda}_2^{N+1}\ \ldots\ \hat{\lambda}_{\ell}^{N+1}] \\ &\quad = \mathrm{diag}[\hat{\lambda}_1^{N}+\Delta\hat{\lambda}_1^{N+1}\ \ \hat{\lambda}_2^{N}+\Delta\hat{\lambda}_2^{N+1}\ \ldots\ \ \hat{\lambda}_{\ell}^{N}+\Delta\hat{\lambda}_{\ell}^{N+1}] \end{aligned} \qquad (2\text{-}48)$$

To obtain $\Delta\underline{\hat{\lambda}}^{N+1}$, the following differentiation is evaluated.

$$\frac{\partial J_s}{\partial(\Delta\underline{\hat{\lambda}}^{N+1})} = \underline{0} \qquad (2\text{-}49)$$

where $\underline{0}$ is the ℓ null vector.

The resulting equation is the same as eq. (2-18), replacing $\underline{\hat{\alpha}}^N$ by $\underline{\bar{b}}^N\ \underline{h}'$. Solving eq. (2-49), the optimum $\Delta\underline{\hat{\lambda}}^{N+1}$ are obtained.

The recursive formulae in the (N+1)th-step are,

$$\underline{\bar{b}}^{N+1} = \underline{\bar{b}}^N \qquad (2\text{-}50)$$

where $\underline{\bar{b}}^{N+1\prime} = [b_1^{N+1}\ b_2^{N+1}\ \ldots\ \bar{b}_{\ell}^{N+1}]$

$$\underline{\bar{b}}^{N\prime} = [b_1^{N}\ b_2^{N}\ \ldots\ \bar{b}_{\ell}^{N}]$$

$$\hat{\varphi}^{N+1}\cdot\Delta\underline{\hat{\lambda}}^{N+1} = \hat{\pi}^{N+1} \qquad (2\text{-}51)$$

where $\hat{\phi}^{N+1}$ and $\hat{\pi}^{N+1}$ are given by eq. (2-18), replacing $\underline{\hat{\alpha}}^N$, by $\underline{\bar{b}}^N\ \underline{\hat{h}}'$

Eq. (2-50) and (2-51) are used for N+2 and so on.

Step 4: i=(N+2)th-step

Set N=N+2 and go to step 2.

This procedure is repeated until the objective function is satisfactorily minimized.

The original x-domain reduced-order model shown by eq. (2-31) is obtained through the following transformation to these steps.

$$\hat{A}_0^i = P_\ell \hat{\Lambda}^i P_\ell^{-1}$$

$$\underline{\hat{b}}_0^{i'} = P_\ell \underline{\hat{b}}^{i'} = P_\ell (\underline{\hat{\lambda}^i \bar{b}^i})' \qquad (2\text{-}52)$$

$$\underline{\hat{h}}_0^{i'} = \underline{\hat{h}}' P_\ell^{-1}$$

where i=0,1,2, ..., N, N+1, ...

The necessary and sufficient conditions for existence of an optimum are satisfied in each step which is proved the same as for reducing the transfer function.

ii) Output-Vector Optimization

The output-vector and eigenvalues are optimized through the objective function, and the control-vector is kept constant for all steps.

$$\underline{\hat{h}}^{i'} = [\hat{h}_1^i \ \hat{h}_2^i \ \ldots \ \hat{h}_\ell^i] \qquad (2\text{-}53)$$

where i=0,1,2, ...

$$\hat{\underline{b}}^i = \hat{\underline{b}} \quad \text{for all } i \tag{2-54}$$

$$\hat{\underline{b}}^{i\prime} = [\hat{b}_1^i \quad \hat{b}_2^i \quad \cdots \quad \hat{b}_\ell^i]$$

$$\hat{\underline{b}}' = [\hat{b}_1 \quad \hat{b}_2 \quad \cdots \quad \hat{b}_\ell]$$

Interchanging the roles of the control-vector and output-vector in the control-vector optimization, similar equations are obtained to those in case (i).

2. Multivariable System

The original nth-order linear time-invariant system (v-input, w-output) can be written as:

$$\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + D\,u \\ \underline{y}_n &= H'\underline{x} \end{aligned} \tag{2-55}$$

where A is an n x n matrix with real and distinct eigenvalues

D is an n x v matrix

H' is a w x n matrix

It is desired to find an ℓth-order simplified model.

The solutions of the exact system $\underline{y}_n$ and the simplified-model, $\hat{\underline{y}}_\ell^i$, are now in a vector form.

$$\begin{aligned} \underline{y}_n' &= [y_{n,1} \; y_{n,2} \; \cdots \; y_{n,w}] \\ \hat{\underline{y}}_\ell^{i\prime} &= [\hat{y}_{\ell,1}^i \; \hat{y}_{\ell,2}^i \; \cdots \; \hat{y}_{\ell,w}^i] \end{aligned} \tag{2-56}$$

The objective function to be minimized is used in eq. (2- 2). Exactly the same procedure is carried out with the multi-variable transfer matrix.

3. System with Repeated Eigenvalues

Suppose an nth-order system has k repeated eigenvalues. It is necessary to reduce the system to ℓth-order ($\ell << n$, $\ell \geq k$). The original system is transformed into a Jordan canonical form by $\underline{x} = P\ \underline{Z}$, and truncating the eigenvalues of the order greater than ℓ.

Step 1: Initial step (i=0)

The k-repeated eigenvalues are made slightly different from each other. Hence the reduced-order model no longer has repeated eigenvalues. Thus the ℓth-order model for the control-vector optimization is:

$$\begin{aligned} \dot{\underline{\hat{Z}}}^0 &= \hat{\Lambda}_r^0\ \underline{\hat{Z}}^0 + \underline{\hat{b}}_r^0\ u \\ \hat{y}_\ell^0 &= \underline{\hat{h}}^{0\prime}\ \underline{Z} \end{aligned} \tag{2-57}$$

where

$$\hat{\Lambda}_r^0 = \begin{bmatrix} \hat{\lambda}_1^0+\varepsilon_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \hat{\lambda}_2^0+\varepsilon_2 & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \hat{\lambda}_k^0+\varepsilon_k & & \cdot \\ \cdot & & & & \hat{\lambda}_{k+1}^0 & \\ 0 & & \cdot & \cdot & & \hat{\lambda}_\ell^0 \end{bmatrix}$$

$$\underline{\hat{b}}_r^{0\prime} = [\hat{b}_{r,1}^0 \ \hat{b}_{r,2}^0 \cdots \hat{b}_{r,k}^0 \ \hat{b}_{r,k+1}^0 \cdots \hat{b}_{r,\ell}^0]$$

$$= [(\hat{\lambda}_1^0+\varepsilon_1)\bar{b}_{r,1}^0 \ (\hat{\lambda}_2^0+\varepsilon_2)\bar{b}_{r,2}^0 \cdots (\hat{\lambda}_k^0+\varepsilon_k)\bar{b}_{r,k}^0$$

$$\hat{\lambda}_{k+1}^0\bar{b}_{r,k+1}^0 \cdots \hat{\lambda}_\ell^0\bar{b}_{r,\ell}^0]$$

$$\hat{\lambda}_1^0 = \hat{\lambda}_2^0 = \quad = \hat{\lambda}_k^0 \equiv \lambda \ \text{(k-repeated eigenvalues)}$$

$$\text{diag}[\hat{\lambda}_{k+1}^0 \ \hat{\lambda}_{k+2}^0 \cdots \hat{\lambda}_\ell^0] = \text{diag}[\lambda_{k+1} \ \lambda_{k+2} \cdots \lambda_\ell]$$

$$\underline{\hat{h}}_r^{0\prime} = \hat{h}_r^{i\prime} = \underline{\hat{h}}' \quad \text{for all } i$$

$$\underline{\hat{h}}' = [\hat{h}_1 \ \hat{h}_2 \cdots \hat{h}_k \cdots, \hat{h}_\ell]$$

Assume ε_i $(i=0,1,2,\ldots,k)$ are arbitrarily small constants. Thereafter the same iterative step for the case of single-variable system is computed. In the second step the eigenvalues are fixed, and the control-vector $\underline{\hat{b}}^i$ are optimized. In the third step the control vector $\underline{\hat{b}}^i$ remains fixed, the eigenvalues are optimized.

CHAPTER III. EXAMPLES OF SIMPLIFICATION OF SYSTEMS

In this chapter the computational procedure by the two-step iterative method is illustrated by examples and compared with other investigations.

Example 1

Consider the problems of reducing a fourth-order transfer function into a first-order and second-order models.

$$H_4 = \frac{s^3 + 7\,s^2 + 24\,s + 24}{s^4 + 10\,s^3 + 35\,s^2 + 50\,s + 24} \tag{3-1}$$

The poles are -1, -2, -3 and -4. The unit-step response of the original system is;

$$y_4 = 1 - e^{-t} - e^{-2t} + 2e^{-3t} - e^{-4t} \tag{3-2}$$

therefore

$$\alpha_0 = 1,\ \alpha_1 = -1,\ \alpha_2 = -1,\ \alpha_3 = 2,\ \alpha_4 = -1$$
$$\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3,\ \lambda_4 = -4 \tag{3-3}$$

(i) First-Order Reduced Model

Suppose that the original system is to be approximated by a first-order model with

$$\hat{y}_1^i = \hat{\alpha}_0^i + \hat{\alpha}_1^i e^{\hat{\lambda}_1^i t} \tag{3-4}$$

Then the transfer function of the reduced-order model is given by the form

$$\hat{H}_1^i(s) = \frac{(\hat{\alpha}_0^i + \hat{\alpha}_1^i)s - \hat{\alpha}_0^i\hat{\lambda}_1^i}{s - \hat{\lambda}_1^i} \tag{3-5}$$

For given residues and poles in eq. (2-60), the optimum $\hat{\alpha}_0^i$, $\hat{\alpha}_1^i$ and $\hat{\lambda}_1^i$ are to be determined through a minimization of the objective function.

$$J_s = \int_0^\infty (y_4 - \hat{y}_1^i)^2 \, dt \tag{3-6}$$

A. Computational Procedure

Step 1: Initial step (i=0)

Assuming the $\hat{\lambda}_1^0$ has the same value as the original system,

$$\hat{\lambda}_1^0 = \lambda_1 \tag{3-7}$$

Let $(y_4 - \hat{y}_1^i) \to 0$ as $t \to \infty$, then

$$\hat{\alpha}_0^i = \alpha_0 \quad \text{for all } i \tag{3-8}$$

The general recursive formulae of the preceeding section 2.2 are applied.

Step 2: Residue optimization (i=N)

The approximating function is, putting i=N in eq. (3-4)

$$\hat{y}_1^N = \hat{\alpha}_0^N + \hat{\alpha}_1^N e^{\hat{\lambda}_1^N t} \tag{3-9}$$

The $\hat{\lambda}_1^N$ is kept constant, as the previous step $\hat{\lambda}_1^{N-1}$. The $\hat{\alpha}_1^N$ is optimized using eq. (2-13). Let n=4 and ℓ=1 in the general recursive formulae.

$$\hat{\lambda}_1^N = \hat{\lambda}_1^{N-1} \tag{3-10}$$

$$\hat{\psi}^N \cdot \hat{\alpha}_1^N = \hat{\Gamma}^N \tag{3-11}$$

where $\hat{\psi}^N = \dfrac{1}{\hat{\lambda}_1^{N-1}}$

$$\hat{\Gamma}^N = 2 \sum_{q=1}^{4} \frac{\alpha_q}{\hat{\lambda}^{N-1} + \lambda_q}$$

Thus,

$$\hat{\alpha}_1^N = 2\hat{\lambda}_1^{N-1} \left(\frac{\alpha_1}{\hat{\lambda}_1^{N-1}+\lambda_1} + \frac{\alpha_2}{\hat{\lambda}_1^{N-1}+\lambda_2} + \frac{\alpha_3}{\hat{\lambda}_1^{N-1}+\lambda_3} + \frac{\alpha_q}{\hat{\lambda}_1^{N-1}+\lambda_4} \right) \tag{3-12}$$

Eq. (3-10) and (3-12) are used N=1 and so on.

Step 3: Pole optimization (i=N+1)

The solution of the reduced-order model has the form, setting i=N+1 in eq. (3-4),

$$\hat{y}_1^{N+1} = \hat{\alpha}_0^{N+1} + \hat{\alpha}_1^{N+1} e^{\hat{\lambda}_1^{N+1} t} \tag{3-13}$$

The $\hat{\alpha}_1^{N+1}$ is fixed as the previous step $\hat{\alpha}_1^N$. The $\hat{\lambda}_1^{N+1}$ is determined by using eq. (2-18). The general recursive formulae are as follows:

$$\hat{\alpha}_1^{N+1} = \hat{\alpha}_1^N \tag{3-14}$$

$$\hat{\phi}^{N+1} \cdot \Delta\hat{\lambda}_1^{N+1} = \hat{\pi}^{N+1} \tag{3-15}$$

where $$\hat{\phi}^{N+1} = \frac{\hat{\alpha}_1^N}{2(\hat{\lambda}_1^N)^3}$$

$$\hat{\pi}^{N+1} = -2 \times \left(\sum_{q=1}^{4} \frac{\alpha_q}{(\hat{\lambda}_1^N + \lambda_q)^2} - \frac{\hat{\alpha}_1^N}{(\hat{\lambda}_1^N + \hat{\lambda}_1^N)^2} \right)$$

so that

$$\Delta\hat{\lambda}_1^{N+1} = -\frac{4(\hat{\lambda}_1^N)^3}{\hat{\alpha}_1^N} \left(\frac{\alpha_1}{\hat{\lambda}_1^N + \lambda_1} + \frac{\alpha_2}{\hat{\lambda}_1^N + \lambda_2} + \frac{\alpha_3}{\hat{\lambda}_1^N + \lambda_3} + \frac{\alpha_4}{\hat{\lambda}_1^N + \lambda_4} + \frac{\hat{\alpha}_1^N}{4(\hat{\lambda}_1^N)^3} \right) \tag{3-16}$$

$$\hat{\lambda}_1^{N+1} = \hat{\lambda}_1^N + \Delta\hat{\lambda}_1^{N+1} \tag{3-17}$$

Step 4: i=(N+2)th-step

Set N=N+2, and go to step 2.

To ascertain the existence of an optimum, the sign of the second derivative of the objective function (i.e., the sufficient condition for the existence of an extreme) is checked.

$$\frac{\partial^2 J_s}{\partial (\hat{\alpha}_1^N)^2} = -\frac{1}{\hat{\lambda}_1^N} \tag{3-18}$$

$$\frac{\partial^2 J_s}{\partial (\Delta\hat{\lambda}_1^{N+1})^2} = -\frac{(\hat{\alpha}_1^N)^2}{2\hat{\lambda}_1^N} \tag{3-19}$$

Since $\hat{\lambda}_1^N$ is negative, both of these derivatives are positive-definite when the residue and pole are optimum. Thus a minimum will then exist.

The computational results are listed in Table 1 and depicted in Figure 2. The effects of the optimization procedures for i=1,2,3 are shown in Figure 3. Comparisons with other methods are shown in Table 2.

(ii) Second-Order Reduced Model

Consider the same original system for a reduction to a second-order model. The unit-step response of the original system and the residues and poles are given by eq. (3-2) and eq. (3-3) respectively. The unit-step response of the reduced-order model is:

$$\hat{y}_2^i = \hat{\alpha}_0^i + \hat{\alpha}_1^i e^{\hat{\lambda}_1^i t} + \hat{\alpha}_2^i e^{\hat{\lambda}_2^i t} \tag{3-20}$$

The transfer function is of the form:

$$\hat{H}_2^i(s) = \frac{(\hat{\alpha}_0^i+\hat{\alpha}_1^i+\hat{\alpha}_2^i)s^2-(\hat{\alpha}_0^i\hat{\lambda}_1^i+\hat{\alpha}_0^i\hat{\lambda}_2^i+\hat{\alpha}_1^i\hat{\lambda}_2^i+\hat{\alpha}_2^i\hat{\lambda}_1^i)s+\hat{\alpha}_0^i\hat{\lambda}_1^i\hat{\lambda}_2^i}{(s-\hat{\lambda}_1^i)(s-\hat{\lambda}_2^i)} \tag{3-21}$$

The objective function to be minimized is:

Table 1. Optimum residues, optimum poles and integral-square errors versus iterations for first-order reduced models.

$$\hat{H}_1^i(s) = \frac{(\hat{\alpha}_0^i + \hat{\alpha}_1^i)s - \hat{\alpha}_0^i\hat{\lambda}_1^i}{(s - \hat{\lambda}_1^i)}$$

$$\hat{y}_1^i = \hat{\alpha}_0^i + \hat{\alpha}_1^i e^{\hat{\lambda}_1^i t}, \quad \hat{\alpha}_0^i = \alpha_0$$

i	$\hat{\alpha}_1^i$	$\hat{\lambda}_1^i$	$\Delta\hat{\lambda}_1^i$	$\int_0^\infty \left(\text{Step Response Error}\right)^2 dt$
0	-1.0000	-1.0000	0.0000	0.00357107
1	-1.0667	-1.0000	0.0000	0.00134720
2	-1.0667	-0.9646	0.0354	0.00099558
3	-1.0481	-0.9646	0.0354	0.00081727
4	-1.0481	-0.9489	0.0157	0.00074352
5	-1.0396	-0.9489	0.0157	0.00070593
6	-1.0396	-0.9417	0.0072	0.00069003
7	-1.0357	-0.9417	0.0072	0.00068194
8	-1.0357	-0.9383	0.0033	0.00067847
9	-1.0339	-0.9383	0.0033	0.00067672
10	-1.0339	-0.9368	0.0016	0.00067597
11	-1.0330	-0.9368	0.0016	0.00067559
12	-1.0330	-0.9360	0.0007	0.00067543
13	-0.0326	-0.9360	0.0007	0.00067535
14	-1.0326	-0.9357	0.0003	0.00067531
15	-1.0324	-0.9357	0.0003	0.00067530

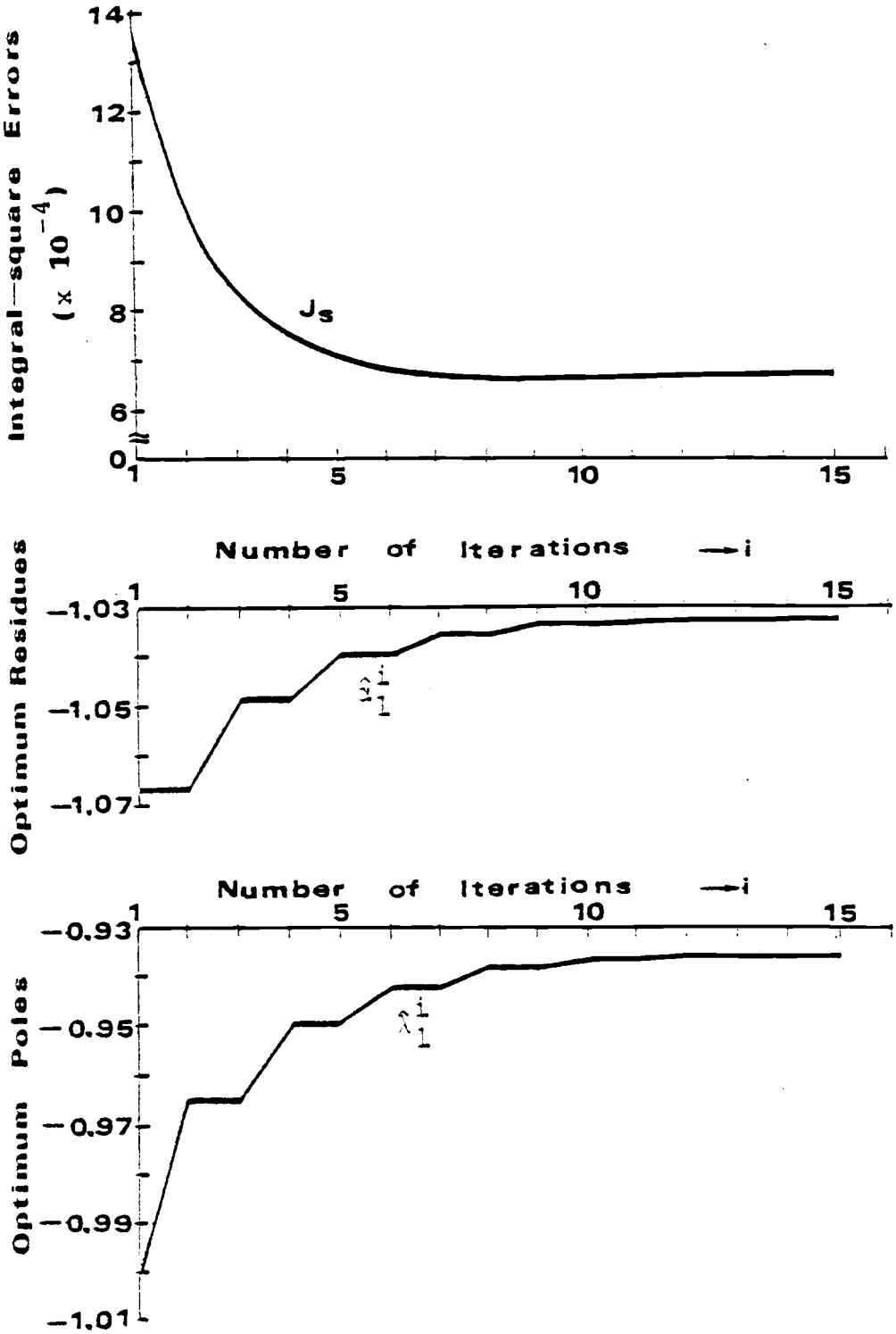

Figure 2. Integral-square errors, optimum residues and optimum poles versus iterations for first-order reduced models of a fourth-order system.

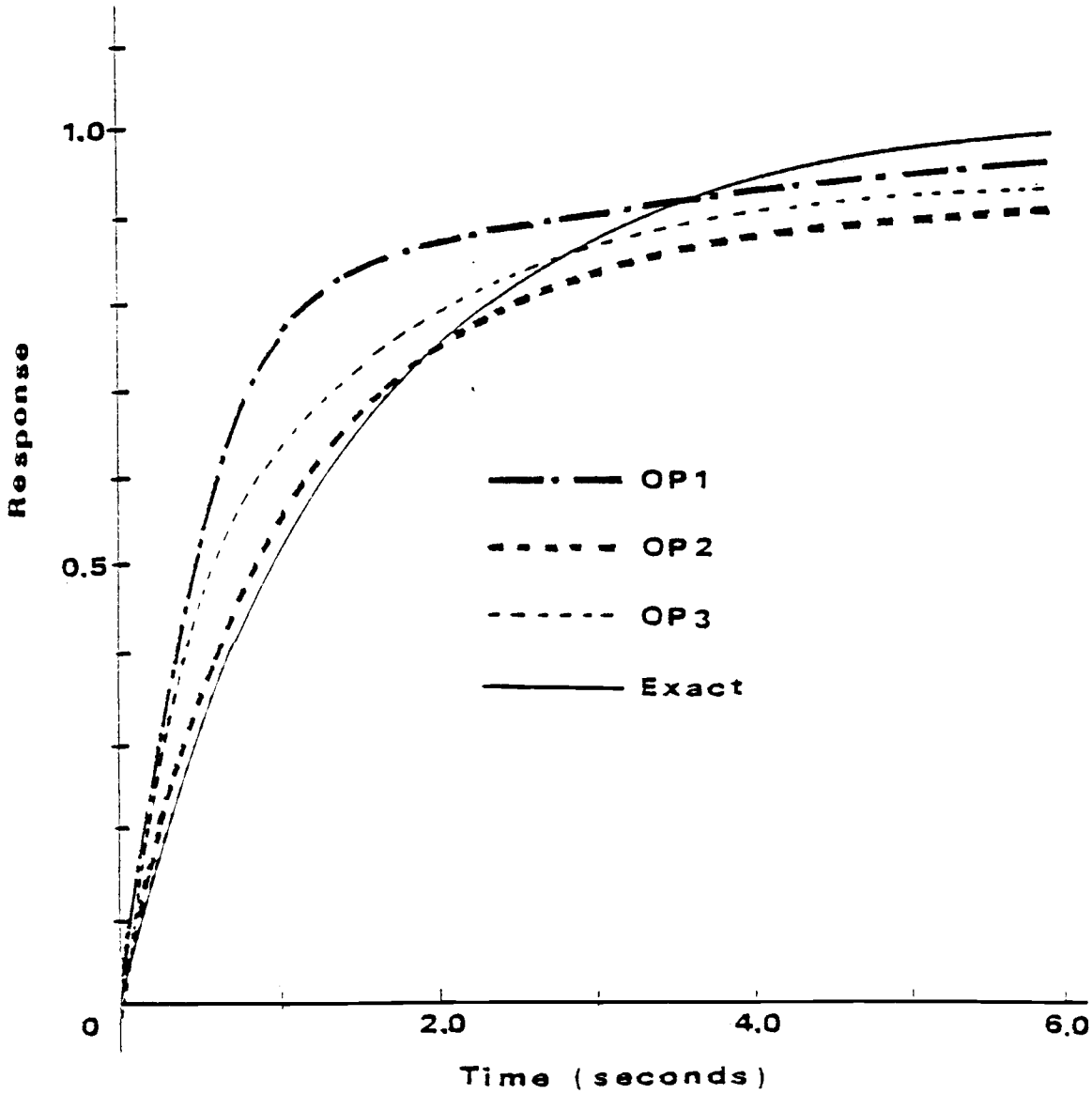

Figure 3. Unit-step response characteristic of the iterative optimization procedure from a fourth-order system into first-order models.

Table 2. Comparison of the iterative method with other techniques: reduction from a fourth-order system to first-order models.

Original system $H_4(s) = \dfrac{s^3 + 7s + 24s + 24}{s^4 + 10s + 35s^2 + 50s + 24}$

Unit Step Response $y_4 = 1 - e^{-t} - e^{-2t} + 2e^{-3t} - e^{-4t}$

Reduced Model	Unit-Step Response	Transfer Function	$\int_0^\infty \left(\text{Step Response Error}\right)^2 dt$
i=5 step OP5	$1-1.0396e^{-0.9489t}$	$\dfrac{-0.0396s+0.9489}{s+0.9489}$	0.0007059 3
i=4 step OP4	$1-1.0481e^{-0.9489t}$	$\dfrac{-0.0481s+0.9489}{s+0.9489}$	0.00074352
i=3 step OP3	$1-1.0481e^{-0.9646t}$	$\dfrac{-0.0481s+0.9646}{s+0.9646}$	0.00081727
i=2 step OP2	$1-1.0667e^{-0.9646t}$	$\dfrac{-0.0667s+0.9646}{s+0.9646}$	0.00099558
N_4(IIsia) Second Cauer form (Weiberg: Shieh)	$1-3^{-0.9231t}$	$\dfrac{0.9231}{s+0.9231}$	0.00105241
i=1 step OP1	$1-1.0667e^{-t}$	$\dfrac{-0.0667s+1}{s+1}$	0.00134720
i=0 step OP0	$1-e^{-t}$	$\dfrac{1}{s+1}$	0.00357106
Routh Approximation (Futton & Fiedland) Third Cauer form (Shieh & Goldman)	$1-e^{-048t}$	$\dfrac{0.48}{s+0.48}$	0.15710238

$$J_s = \int_0^\infty (y_4 - \hat{y}_2^i)^2 \, dt \tag{3-22}$$

A. Computational Procedure

To obtain the residues and poles in the reduced-order model, set n=4 and ℓ=2 in eq. (2-7)-(2-18). The results are summarized as follows:

Step 1: Initial step (i=0)

$$\underline{\hat{\lambda}}^0 = \underline{\lambda}$$

where $\underline{\hat{\lambda}}^{0\prime} = [\hat{\lambda}_1^0 \ \hat{\lambda}_2^0]$

$$\underline{\lambda}' = [\lambda_1 \ \lambda_2]$$

$$\hat{\alpha}_0^i = \alpha_0 \quad \text{for all i} \tag{3-24}$$

Step 2: Residue optimization (i=N)

$$\underline{\hat{\lambda}}^N = \underline{\hat{\lambda}}^{N-1} \tag{3-25}$$

where $\underline{\hat{\lambda}}^{N\prime} = [\hat{\lambda}_1^N \ \hat{\lambda}_2^N]$

$$\underline{\hat{\lambda}}^{N-1\prime} = [\hat{\lambda}_1^{N-1} \ \hat{\lambda}_2^{N-1}]$$

$$\hat{\psi}^N \cdot \underline{\hat{\alpha}}^N = \hat{r}^N \tag{3-26}$$

where

$$\hat{\psi}^N = \begin{bmatrix} \frac{1}{\hat{\lambda}_1^{N-1}} & \frac{2}{\hat{\lambda}_1^{N-1}+\hat{\lambda}_2^{N-1}} \\ \frac{2}{\hat{\lambda}_2^{N-1}+\hat{\lambda}_1^{N-1}} & \frac{1}{\hat{\lambda}_2^{N-1}} \end{bmatrix}$$

$$\underline{\hat{\alpha}}^{N\prime} = [\hat{\alpha}_1^N \quad \hat{\alpha}_1^N]$$

$$\hat{\pi}^N = 2 \times \begin{bmatrix} \sum_{q=1}^{4} \frac{\alpha_q}{\hat{\lambda}_1^{N-1}+\lambda_q} \\ \sum_{q=1}^{4} \frac{\alpha_q}{\hat{\lambda}_2^{N-1}+\lambda_q} \end{bmatrix}$$

Solving eq. (3-26), $\underline{\hat{\alpha}}^N$ is obtained.

Step 3: Pole optimization (i=N+1)

$$\underline{\hat{\alpha}}^{N+1} = \underline{\hat{\alpha}}^N \tag{3-27}$$

where $\underline{\hat{\alpha}}^{N+1\prime} = [\hat{\alpha}_1^{N+1} \quad \hat{\alpha}_2^{N+1}]$

$$\underline{\hat{\alpha}}^{N\prime} = [\hat{\alpha}_1^N \quad \hat{\alpha}_2^N]$$

$$\hat{\phi}^{N+1} \cdot \Delta\underline{\hat{\lambda}}^{N+1} = \hat{\pi}^{N+1} \tag{3-28}$$

where

$$\hat{\phi}^{N+1} = \begin{bmatrix} \frac{\hat{\alpha}_1^N}{2(\hat{\lambda}_1^N)^3} & \frac{4\hat{\alpha}_2^N}{(\hat{\lambda}_1^N+\hat{\lambda}_2^N)^3} \\ \frac{4\hat{\alpha}_1^N}{(\hat{\lambda}_2^N+\hat{\lambda}_1^N)^3} & \frac{\hat{\alpha}_2^N}{2(\hat{\lambda}_2^N)^3} \end{bmatrix}$$

$$\Delta\underline{\hat{\lambda}}^{N+1\prime} = [\Delta\hat{\lambda}_1^{N+1} \quad \Delta\hat{\lambda}_2^{N+1}]$$

$$\hat{\pi}^{N+1} = -2 \times \begin{bmatrix} \sum_{q=1}^{4} \frac{\alpha_q}{\hat{\lambda}_1^N+\lambda_q} - \sum_{r=1}^{2} \frac{\hat{\alpha}_r^N}{(\hat{\lambda}_1^N+\hat{\lambda}_r^N)^2} \\ \sum_{q=1}^{4} \frac{\alpha_q}{\hat{\lambda}_2^N+\lambda_q} - \sum_{r=1}^{2} \frac{\hat{\alpha}_r^N}{(\hat{\lambda}_2^N+\hat{\lambda}_r^N)^2} \end{bmatrix}$$

Solving eq. (3-28), $\Delta\underline{\hat{\lambda}}^{N+1}$ is obtained.

Thus,

$$\underline{\hat{\lambda}}^{N+1} = \underline{\hat{\lambda}}^{N} + \Delta\underline{\hat{\lambda}}^{N+1} \tag{3-29}$$

Step 4: i=(N+2)th-step

Set N=N+2 and go to step 2.

The sufficient condition for the existence of a minimum value of the objective function at the optimum point is investigated.

From ℓ=2 in eq. (2-20) and (2-22)

$$\frac{\partial^2 J_s}{\partial(\underline{\hat{\alpha}}^N)^2} = \begin{bmatrix} -\frac{1}{\hat{\lambda}_1^{N-1}} & 0 \\ 0 & -\frac{1}{\hat{\lambda}_2^{N-1}} \end{bmatrix} \tag{3-30}$$

$$\frac{\partial^2 J_s}{\partial(\Delta\underline{\hat{\lambda}}^{N+1})^2} = \begin{bmatrix} -\frac{(\hat{\alpha}_1^N)^2}{(\hat{\lambda}_1^N)^3} & 0 \\ 0 & -\frac{(\hat{\alpha}_2^N)^2}{(\hat{\lambda}_2^N)^3} \end{bmatrix} \tag{3-31}$$

Since eq. (3-30) and (3-31) are positive-definite matrices, the sufficient condition is satisfied.

The computational results by the iterative method are shown in Table 3 and Figure 4.

As shown in Table 4, Figure 5 and Table 5, the iterative method is superior to other methods of model reduction of the unit-step response in the transient and steady-state regions, and the value of the objective function.

Example 2

Consider a multivariable gas-turbine system shown in eq. (3-32)

$$[H_4(s)] = \frac{\begin{bmatrix} 95164.960 \\ 124082.200 \end{bmatrix} s^2 + \begin{bmatrix} 11133616.130 \\ 1501230.668 \end{bmatrix} s + \begin{bmatrix} 1808490.200 \\ 2538178.498 \end{bmatrix}}{s^4 + 113225\, s^3 + 1357.275\, s^2 + 3499.750\, s + 2525} \tag{3-32}$$

The unit-step response for eq. (3-51) has the form

$$\underline{Y}_4 = \begin{bmatrix} Y_{4,1} \\ Y_{4,2} \end{bmatrix} = \begin{bmatrix} 716.242133 \\ 1005.219207 \end{bmatrix} + \begin{bmatrix} -749.755894 \\ -1223.504697 \end{bmatrix} e^{-1.3471t} + \begin{bmatrix} 23.75188 \\ 204.773557 \end{bmatrix} e^{-1.8735t} + \begin{bmatrix} 0.119204 \\ 0.962824 \end{bmatrix} e^{-10.0047t} + \begin{bmatrix} 9.643081 \\ 12.549109 \end{bmatrix} e^{-99.9997t} \tag{3-33}$$

Assume that the second-order reduced model can be expressed as:

$$\underline{y}_2^i = \begin{bmatrix} \hat{y}_{2,1}^i \\ \hat{y}_{2,2}^i \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{0,1}^i \\ \hat{\alpha}_{0,2}^i \end{bmatrix} + \begin{bmatrix} \hat{\alpha}_{1,1}^i \\ \hat{\alpha}_{1,2}^i \end{bmatrix} e^{\hat{\lambda}_1^i t} + \begin{bmatrix} \hat{\alpha}_{2,1}^i \\ \hat{\alpha}_{2,2}^i \end{bmatrix} e^{\hat{\lambda}_2^i t} \tag{3-34}$$

Table 3. Optimum residues, optimum poles and integral-square errors for successive iterative-steps for second-order reduced models of a fourth-order system.

$$\hat{y}_2^i = \hat{\alpha}_0^i + \hat{\alpha}_1^i e^{\hat{\lambda}_1^i t} + \hat{\alpha}_2^i e^{\hat{\lambda}_2^i t}, \quad \hat{\alpha}_0^i = \alpha_0$$

i	$\hat{\alpha}_1^i$	$\hat{\lambda}_1^i$	$\hat{\alpha}_2^i$	$\hat{\lambda}_2^i$	$\int_0^\infty \left(\text{Step Response Error}\right)^2 dt$
0	-1.0000	-1.0000	-1.0000	-2.0000	0.22024248
1	-1.2000	-1.0000	0.2000	-2.0000	0.00023915
2	-1.2000	-1.0334	0.2000	-2.5641	0.00014024
3	-1.2239	-1.0334	0.2442	-2.5641	0.00007157
4	-1.2239	-1.0393	0.2442	-2.6655	0.00006676
5	-1.2277	-1.0393	0.2516	-2.6655	0.00006502
6	-1.2277	-1.0411	0.2516	-2.6851	0.00006454
7	-1.2294	-1.0411	0.2543	-2.6851	0.00006433
8	-1.2294	-1.0419	0.2543	-2.6845	0.00006410
9	-1.2308	-1.0419	0.2558	-2.6845	0.00006394
10	-1.2308	-1.0424	0.2558	-2.6789	0.00006372
11	-1.2320	-1.0424	0.2571	-2.6789	0.00006356
12	-1.2320	-1.0429	0.2571	-2.6721	0.00006334
13	-1.2332	-1.0429	0.2583	-2.6721	0.00006318
14	-1.2332	-1.0433	0.2583	-2.6652	0.00006297
15	-1.2344	-1.0433	0.2594	-2.6652	0.00006281
16	-1.2344	-1.0437	0.2594	-2.6584	0.00006261
17	-1.2355	-1.0437	0.2605	-2.6584	0.00006245
18	-1.2355	-1.0441	0.2605	-2.6517	0.00006225
19	-1.2367	-1.0441	0.2616	-2.6517	0.00006211
20	-1.2367	-1.0446	0.2616	-2.6451	0.00006191

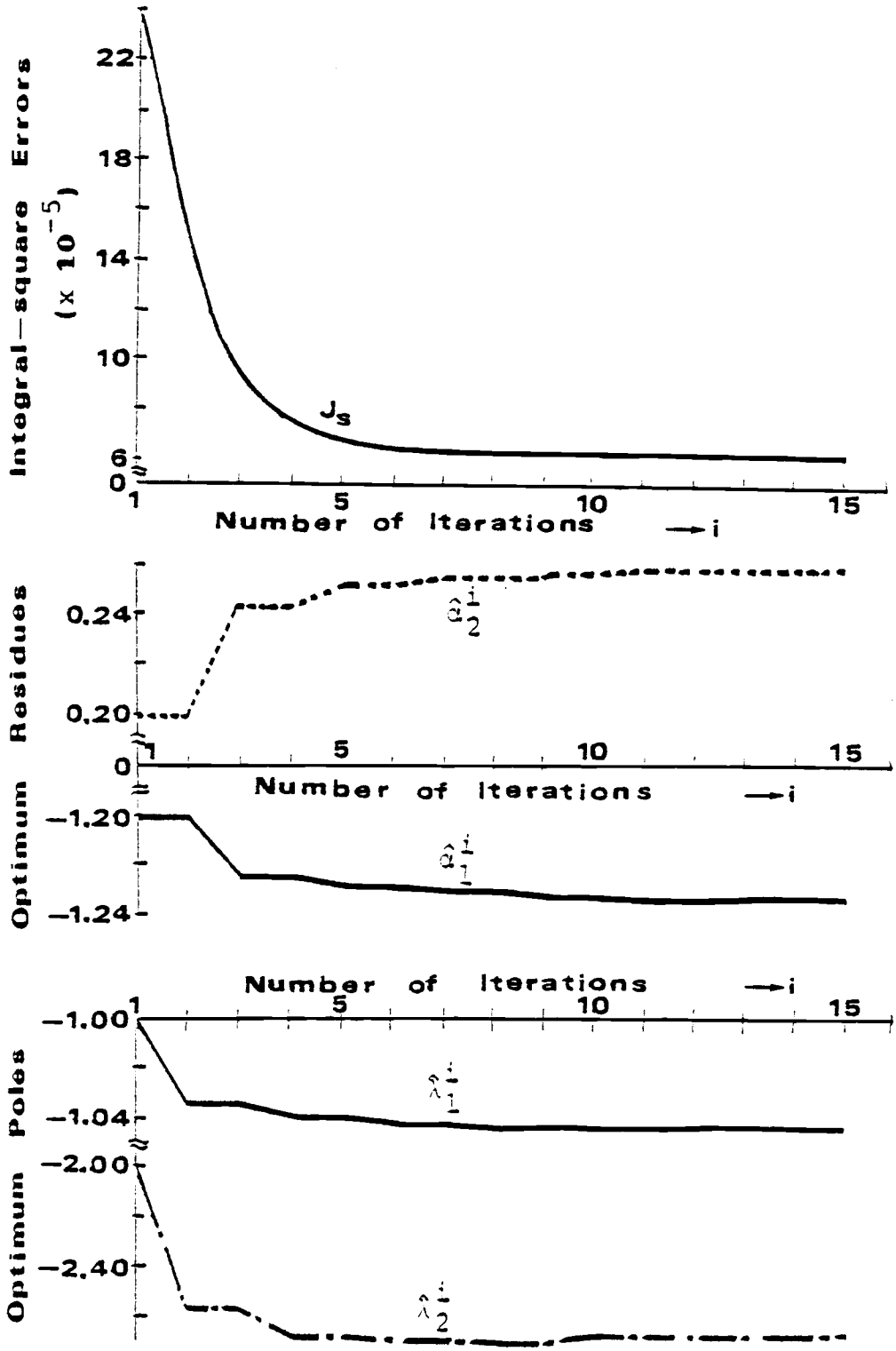

Figure 4. Integral-square errors, optimum residues and optimum poles versus iterations for second-order reduced models of a fourth-order system.

Table 4. Comparison of the iterative method with various second-order reduced models of a fourth-order system.

Original System: $H_4(s) = \dfrac{s^3 + 7s^2 + 24s + 24}{s^4 + 10s^3 + 35s^2 + 50s + 24}$

Unit-step Response: $y_4 = 1 - e^{-t} - e^{-2t} + 2e^{-3t} - e^{-4t}$

Reduced Model	Unit-Step Response	Transfer Function	$\int_0^\infty \left(\frac{\text{Step Response}}{\text{Error}}\right)^2 dt$
i=5 step OP5	$1-1.2277e^{-1.0393t}+0.2516e^{-26655t}$	$\dfrac{0.0239s^2+0.6939s+2.7703}{s^2+3.7048s+2.7703}$	0.00006502
i=4 step OP4	$1-1.2239e^{-1.0393t}+0.2442e^{-2.6655t}$	$\dfrac{0.0203s^2+0.6963s+2.7703}{s^2+3.7048s+2.7703}$	0.00006676
i=3 step OP3	$1-1.2239e^{-1.0334t}+0.2442e^{-2.5641t}$	$\dfrac{0.0203s^2+0.7117s+2.6497}{s^2+3.5975s+2.6497}$	0.00007157
Second Cauer form (Weiberg; Shieh)	$1-1.2309e^{-1.0435t}+0.2309e^{-2.3994t}$	$\dfrac{0.7305s+2.5038}{s^2+3.4429s+2.5037}$	0.00012404
N_3 (Hsia)	$1-1.2292e^{-1.0429t}+0.2292e^{-2.4032t}$	$\dfrac{0.7311s+2.5063}{s^2+3.4461s+2.5063}$	0.00012495
i=2 step OP2	$1-1.2e^{-1.0334t}+0.2e^{-2.5641t}$	$\dfrac{0.7273s+2.6497}{s^2+3.5975s+2.6497}$	0.00014024

Table 4 (continued)

Reduced Model	Unit-Step Response	Transfer Function	$\int_0^\infty \left(\text{Step Response Error}\right)^2 dt$
i=1 step OP1	$1-1.2e^{-t}+0.2e^{-2t}$	$\frac{0.8s+2}{s^2+3s+2}$	0.00023915
N_2 (Hsia) Mixed Method (Shieh & Wei)	$1-\frac{7}{6}e^{-t}+\frac{1}{6}e^{-2t}$	$\frac{0.8333s+2}{s^2+3s+2}$	0.00033127
N_1 (Hsia)	$1-1.0972e^{-t}+0.0278e^{-2t}$	$\frac{-0.0694s^2+0.8333s+2}{s^2+3s+2}$	0.00113126
Routh Approximation (Futton & Friedland)	$1-e^{-0.8279t}$ x{0.5041 cos (0.3307t) - 0.1 sin (0.3307t)}	$\frac{0.7947(s+1)}{s^2+1.6557s+0.7947}$	0.00114932
D_n (Davison)	$1-\frac{11}{12}e^{-t}-\frac{1}{12}e^{-2t}$	$\frac{1.0833s+2}{s^2+3s+2}$	0.00792561
C_2 (Chidambara)	$1=\frac{5}{6}e^{-t}-\frac{1}{6}e^{-2t}$	$\frac{0.4833s+2}{s^2+3s+2}$	0.01144288
Third Cauer form (Shieh & Goldman)	$1-0.8843e^{-0.4289t}-0.1157e^{-5.3657t}$	$\frac{s+2.3014}{s^2+5.7946s+2.3014}$	0.15485461
i=0 step OP0 C_1 (Chidambra) D_m (Davison)	$1-e^{-t}-e^{-2t}$	$\frac{-s^2+2}{s^2+3s+2}$	0.22024248

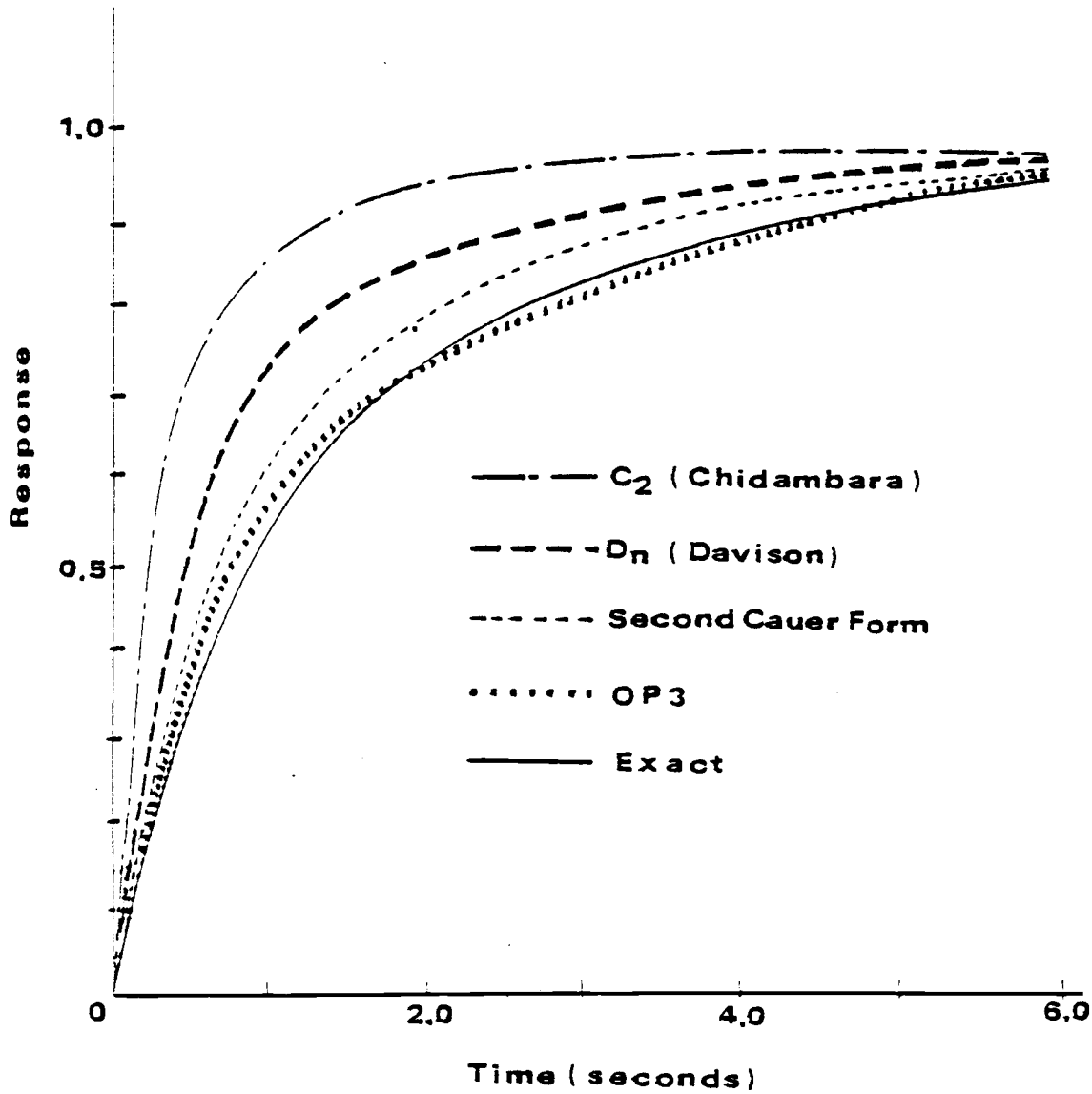

Figure 5. Comparison of unit-step responses according to various methods of second-order reduction of a fourth-order system.

Table 5. Numerical representations of unit-step responses according to various methods of reduction of a fourth-order system into second-order models.

Time (seconds)	Exact	i=1 step OP1	i=2 step OP2	i=3 step OP3
0.0	0.000000	0.000000	0.000000	0.020359
1.0	0.578044	0.585612	0.588448	0.583363
2.0	0.850971	0.841261	0.849280	0.846522
3.0	0.947975	0.940751	0.946045	0.944990
4.0	0.981361	0.978088	0.980778	0.980397
5.0	0.993273	0.991924	0.993159	0.993023
6.0	0.997515	0.997027	0.997566	0.997517

Time (seconds)	Second Cauer Form[1/]	D_n (Davison)	C_2 (Chidambra)	C_1 (Chidambra) D_m (Davison)
0.0	0.000000	0.000000	0.000000	-1.000000
1.0	0.587412	0.651491	0.670886	0.496785
2.0	0.849199	0.874412	0.884172	0.846349
3.0	0.946487	0.954154	0.958099	0.947734
4.0	0.981071	0.983182	0.984682	0.981349
5.0	0.993329	0.993820	0.994378	0.993217
6.0	0.997650	0.997727	0.997933	0.997515

[1/] Weiberg; Shieh

The objective function is;

$$J_M = \int_0^\infty \{(y_{4,1} - \hat{y}^i_{2,1})^2 + Q(y_{4,2} - \hat{y}^i_{2,2})^2\}dt \quad (3\text{-}35)$$

where Q is the weighting constant.

The results for Q=1 are shown together with the results of the second-matrix Cauer form expansion by Chen [18] in Table 6. The integral-square error is bigger than other examples, but error percentage (i.e., the error is normalized by the initial residues) is small.

Example 3

Next it is investigated how well a third-order system with double poles is approximated by a second-order model.

The original system is

$$H_3(s) = \frac{5}{(s+1)^2(s+3)} \quad (3\text{-}36)$$

The unit-step response of the original system is

$$y_3 = \frac{5}{3} - \frac{5}{4}e^{-t} - \frac{5}{2}te^{-t} - \frac{5}{12}e^{-3t} \quad (3\text{-}37)$$

$$\alpha_0 = \frac{5}{3}, \quad \alpha_1 = -\frac{5}{4}, \quad \alpha_2 = -\frac{5}{2}, \quad \alpha_3 = -\frac{5}{12}$$

$$\lambda_1 = \lambda_2 = -1, \quad \lambda_3 = -3 \quad (3\text{-}38)$$

In the initial step, the repeated poles in the original system are transformed into two single poles which are

Table 6. Simplification of a fourth-order gas-turbine system to second-order models.

Exact $\begin{bmatrix} 716.242123 \\ 1005.219207 \end{bmatrix} + \begin{bmatrix} -749.755894 \\ -1223.504697 \end{bmatrix} e^{-1.3471t} + \begin{bmatrix} 23.751488 \\ 204.773557 \end{bmatrix} e^{-1.8735t} + \begin{bmatrix} 0.119204 \\ 0.962824 \end{bmatrix} e^{-10.0047t} + \begin{bmatrix} 9.643081 \\ 12.549109 \end{bmatrix} e^{-99.9997t}$

Model	Unit-Step Response	$\int_0^\infty \left(\text{Step Response Error}\right)^2 dt$
OP1	$\begin{bmatrix} 716.242123 \\ 1005.219207 \end{bmatrix} + \begin{bmatrix} -751.385095 \\ -1226.427730 \end{bmatrix} e^{-1.3471t} + \begin{bmatrix} 26.039264 \\ 208.939647 \end{bmatrix} e^{-1.8735t}$	7.932244
OP2	$\begin{bmatrix} 716.242123 \\ 1005.219207 \end{bmatrix} + \begin{bmatrix} -751.284095 \\ -1226.427730 \end{bmatrix} e^{-1.3477t} + \begin{bmatrix} 26.039264 \\ 208.939647 \end{bmatrix} e^{-1.8806t}$	7.909576
OP3	$\begin{bmatrix} 716.242123 \\ 1005.219207 \end{bmatrix} + \begin{bmatrix} -752.351699 \\ -1226.410759 \end{bmatrix} e^{-1.3477t} + \begin{bmatrix} 27.04369 \\ 209.035801 \end{bmatrix} e^{-1.8806t}$	7.808392
Second-matrix Cauer form expansion	$\begin{bmatrix} 716.236153 \\ 1005.211092 \end{bmatrix} + \begin{bmatrix} -705.416131 \\ -1224.439730 \end{bmatrix} e^{-1.3118t} + \begin{bmatrix} -10.82002 \\ 239.228637 \end{bmatrix} e^{-1.6062t}$	70.549237

slightly different from each other, and they are chosen as the dominant poles. Therefore, the reduced-order model has no repeated poles and has a form of

$$\hat{H}_2^i(s) = \frac{(\hat{\alpha}_0^i + \hat{\alpha}_1^i + \hat{\alpha}_2^i)s^2 - (\hat{\alpha}_0^i\hat{\lambda}_1^i + \hat{\alpha}_0^i\hat{\lambda}_2^i + \hat{\alpha}_2^i\hat{\lambda}_1^i)s + \hat{\alpha}_0^i\hat{\lambda}_1^i\hat{\lambda}_2^i}{(s - \hat{\lambda}_1^i)(s - \hat{\lambda}_2^i)} \tag{3-39}$$

The unit-step response of the reduced-order model is

$$\hat{y}_2^i = \hat{\alpha}_0^i + \hat{\alpha}_1^i e^{\hat{\lambda}_1^i t} + \hat{\alpha}_2^i e^{\hat{\lambda}_2^i t} \tag{3-40}$$

The objective function is

$$J_r = \int_0^\infty (y_3 - \hat{y}_2^i)^2 \, dt \tag{3-41}$$

A. Computational Procedure

The poles and residues in the reduced-order model are obtained as follows.

Step 1: Initial step (i=0)

$$\hat{\lambda}_1^0 = \lambda_1 + \varepsilon_1$$
$$\hat{\lambda}_2^0 = \lambda_1 + \varepsilon_2 \tag{3-42}$$

Here, let

$$\varepsilon_1 = -\varepsilon_2 = \varepsilon \, , \tag{3-43}$$

where ε is an arbitrarily small number.

$$\hat{\alpha}_0^i = \alpha_0 \quad \text{for all } i \tag{3-44}$$

Step 2: Residue optimization (i=N)

The poles are kept constant at the previous step. Solving $\frac{\partial J_r}{\partial(\underline{\hat{\alpha}}^N)} = \underline{0}$, $\underline{\hat{\alpha}}^N$ is obtained.

$$\underline{\hat{\lambda}}^N = \underline{\hat{\lambda}}^{N-1} \tag{3-45}$$

where $\underline{\hat{\lambda}}^{N\prime} = [\hat{\lambda}_1^N \quad \hat{\lambda}_2^N]$

$$\hat{\lambda}^{N-1\prime} = [\hat{\lambda}_1^{N-1} \quad \hat{\lambda}_2^{N-1}]$$

$$\hat{\psi}^N \cdot \underline{\hat{\alpha}}^N = \hat{\Gamma}^N \tag{3-46}$$

where

$$\hat{\psi}^N = \begin{bmatrix} \frac{1}{\hat{\lambda}_1^{N-1}} & \frac{2}{\hat{\lambda}_1^{N-1}+\hat{\lambda}_2^{N-1}} \\ \frac{2}{\hat{\lambda}_2^{N-1}+\hat{\lambda}_1^{N-1}} & \frac{1}{\hat{\lambda}_2^{N-1}} \end{bmatrix}$$

$$\underline{\hat{\alpha}}^{N\prime} = [\hat{\alpha}_1^N \quad \hat{\alpha}_2^N]$$

$$\hat{\Gamma}^N = 2 \times \begin{bmatrix} \sum_{q=0}^{\frac{1}{}} \frac{\alpha_{2q+1}}{\hat{\lambda}_1^{N-1}+\lambda_{2q+1}} - \frac{\alpha_2}{(\hat{\lambda}_1^{N-1}+\lambda_1)^2} \\ \sum_{q=0}^{\frac{1}{}} \frac{\alpha_{2q+1}}{\hat{\lambda}_2^{N-1}+\lambda_{2q+1}} - \frac{\alpha_2}{(\hat{\lambda}_2^{N-1}+\lambda_2)^2} \end{bmatrix}$$

Step 3: Pole optimization (i=N+1)

The residues are kept constant.

Calculating $\frac{\partial J_r}{\partial(\Delta\hat{\underline{\lambda}}^{N+1})} = \underline{0}$, $\Delta\hat{\underline{\lambda}}^{N+1}$ is obtained.

$$\hat{\underline{\alpha}}^{N+1} = \hat{\underline{\alpha}}^{N} \tag{3-47}$$

where $\hat{\underline{\alpha}}^{N+1\prime} = [\hat{\alpha}_1^{N+1} \quad \hat{\alpha}_2^{N+1}]$

$$\hat{\underline{\alpha}}^{N\prime} = [\hat{\alpha}_1^{N} \quad \hat{\alpha}_2^{N}]$$

$$\hat{\Phi}^{N+1}.\Delta\hat{\lambda}^{N+1} = \hat{\pi}^{N+1} \tag{3-48}$$

where

$$\hat{\Phi}^{N+1} = \begin{bmatrix} \frac{\hat{\alpha}_1^N}{2(\hat{\lambda}_1^N)^3} & \frac{4\hat{\alpha}_2^N}{(\hat{\lambda}_1^N+\hat{\lambda}_2^N)^3} \\ \frac{4\hat{\alpha}_1^N}{(\hat{\lambda}_2^N+\hat{\lambda}_1^N)^3} & \frac{\hat{\alpha}_2^N}{2(\hat{\lambda}_2^N)^3} \end{bmatrix}$$

$$\Delta\hat{\underline{\lambda}}^{N+1\prime} = [\Delta\hat{\lambda}_1^{N+1} \quad \Delta\hat{\lambda}_2^{N+1}]$$

$$\hat{\pi}^{N+1} = -2 \times \begin{bmatrix} \sum_{q=0}^{1} \frac{\alpha_{2q+1}}{(\hat{\lambda}_1^N+\lambda_{2q+1})^2} - \sum_{r=1}^{2} \frac{\hat{\alpha}_r^N}{(\hat{\lambda}_1^N+\hat{\lambda}_r^N)^2} - \frac{2\alpha_2}{(\hat{\lambda}_1^N+\lambda_1)^3} \\ \sum_{q=0}^{1} \frac{\alpha_{2q+1}}{(\hat{\lambda}_2^N+\lambda_{2q+1})^2} - \sum_{r=1}^{2} \frac{\hat{\alpha}_r^N}{(\hat{\lambda}_2^N+\hat{\lambda}_r^N)^2} - \frac{2\alpha_2}{(\hat{\lambda}_2^N+\lambda_1)^3} \end{bmatrix}$$

$$\hat{\underline{\lambda}}^{N+1} = \hat{\underline{\lambda}}^{N} + \Delta\hat{\underline{\lambda}}^{N+1} \tag{3-49}$$

Step 4: i=(N+2)th-step

Set N=N+2 and go to step 2.

These steps satisfy the sufficient condition for the existence of a minimum in the objective function at the optimum point.

The numerical solutions for $\varepsilon = 0.01$ and $\varepsilon = 0.001$ are listed in Tables 7 and 8, and represented by Figures 6 and 7.

This procedure can be applied for the system with repeated poles of any order. The other time-domain techniques are difficult to apply to the system with repeated poles as mentioned in section 1.3(v) in Chapter I.

The continued-fraction-expansion method can be used. For a comparison, the second Cauer form expansion by Chen [18] is shown.

$$H_3(s) = \frac{5}{(s+1)^2(s+3)} = \frac{5}{3 + 7s + 5s^2 + s^3} \qquad (3\text{-}50)$$

$$= \cfrac{1}{\frac{3}{5} + \cfrac{1}{\frac{5}{7s} + \cfrac{1}{-\frac{49}{25} + \cfrac{1}{-\frac{125}{126s} + \cfrac{1}{\ddots}}}}} \qquad (3\text{-}51)$$

The truncation of the coefficients of orders greater than four leads to a second-order reduced model.

Table 7. Reduction of a third-order system with double poles into second-order models by the two-step iterative method, for $\varepsilon = 0.01$.

$$\hat{y}_2^i = \hat{\alpha}_0^i + \hat{\alpha}_1^i e^{\hat{\lambda}_1^i t} + \hat{\alpha}_2^i e^{\hat{\lambda}_2^i t} \qquad \hat{\lambda}_1^0 = \lambda_1 + \varepsilon \qquad \hat{\lambda}_2^0 = \lambda_1 - \varepsilon$$

$\varepsilon = 0.01$

i	$\hat{\alpha}_1^i$	$\hat{\lambda}_1^i$	$\hat{\alpha}_2^i$	$\hat{\lambda}_2^i$	$\int_0^\infty \left(\text{Step Response Error}\right)^2 dt$
0	-1.250000	-0.990000	-2.50000	-1.010000	1.201421961000
1	-115.348411	-0.990000	113.785815	-1.010000	0.001810667500
2	-115.348411	-0.960420	113.785815	-0.979100	0.000918579497
3	-113.778780	-0.960420	112.182931	-0.979100	0.000630503496
4	-113.778780	-0.951155	112.182931	-0.969428	0.000544522859
5	-113.247361	-0.951155	111.641029	-0.969428	0.000515576228
6	-113.247361	-0.948278	111.641029	-0.966425	0.000507270598
7	-113.078418	-0.948278	111.468825	-0.966425	0.000504396559
8	-113.078418	-0.947388	111.468825	-0.965496	0.000503578500
9	-113.025804	-0.947388	111.415201	-0.965496	0.000503282391
10	-113.025804	-0.947113	111.415201	-0.965209	0.000503196359
11	-113.009550	-0.947113	111.398636	-0.965209	0.000503161881
12	-113.009550	-0.947028	111.398636	-0.965121	0.000503151964
13	-113.004568	-0.947028	111.393557	-0.965121	0.000503146744
14	-113.004568	-0.947002	111.393557	-0.965093	0.000503144484
15	-113.003071	-0.947002	111.392031	-0.965093	0.000503143400
16	-113.003071	-0.946994	111.392031	-0.965085	0.000503142587
17	-113.002651	-0.946994	111.391602	-0.965085	0.000503142301
18	-113.002651	-0.946991	111.391602	-0.965082	0.000503142238
19	-113.002564	-0.946991	111.391511	-0.965082	0.000503142136
20	-113.002564	-0.946990	111.391511	-0.965081	0.000503142150

Table 8. Second-order reduced models for $\epsilon = 0.001$.

$\epsilon = 0.001$

i	$\hat{\alpha}_1^i$	$\hat{\lambda}_1^i$	$\hat{\alpha}_2^i$	$\hat{\lambda}_2^i$	$\int_0^\infty \left(\text{Step Response Error}\right)^2 dt$
0	-1.250000	-0.999000	-2.500000	-1.001000	1.200919034200
1	-1146.612888	-0.999000	1145.050387	-1.001000	0.001816373421
2	-1146.612888	-0.968764	1145.050387	-0.970631	0.000921336958
3	-1130.731887	-0.968764	1129.136055	-0.970631	0.000630703304
4	-1130.731887	-0.959294	1129.136055	-0.961121	0.000544075909
5	-1125.356617	-0.959294	1123.750277	-0.961121	0.000514997475
6	-1125.356617	-0.956354	1123.750277	-0.958169	0.000506807917
7	-1123.647512	-0.956354	1121.037803	-0.958169	0.000503930098
8	-1123.647412	-0.955445	1121.037803	-0.957255	0.000503003594
9	-1123.114658	-0.955445	1121.504036	-0.957255	0.000502705051
10	-1123.114658	-0.955164	1121.504036	-0.956973	0.000502572221
11	-1122.949630	-0.955164	1121.338696	-0.956973	0.000502537734
12	-1122.949630	-0.955077	1121.338696	-0.956886	0.000502519119
13	-1122.898618	-0.955077	1121.287588	-0.956886	0.000502513818
14	-1122.898618	-0.955050	1121.287588	-0.956859	0.000502529800
15	-1122.882870	-0.955050	1121.271809	-0.956859	0.000502528847
16	-1122.882870	-0.955042	1121.271809	-0.956850	0.000502673900
17	-1122.878019	-0.955042	1121.266950	-0.956850	0.000502674148
18	-1122.878019	-0.955039	1121.266950	-0.956848	0.000502537861
19	-1122.876526	-0.955039	1121.265453	-0.956848	0.000502537970
20	-1122.876526	-0.9550038	1121.265453	-0.956847	0.000502538919

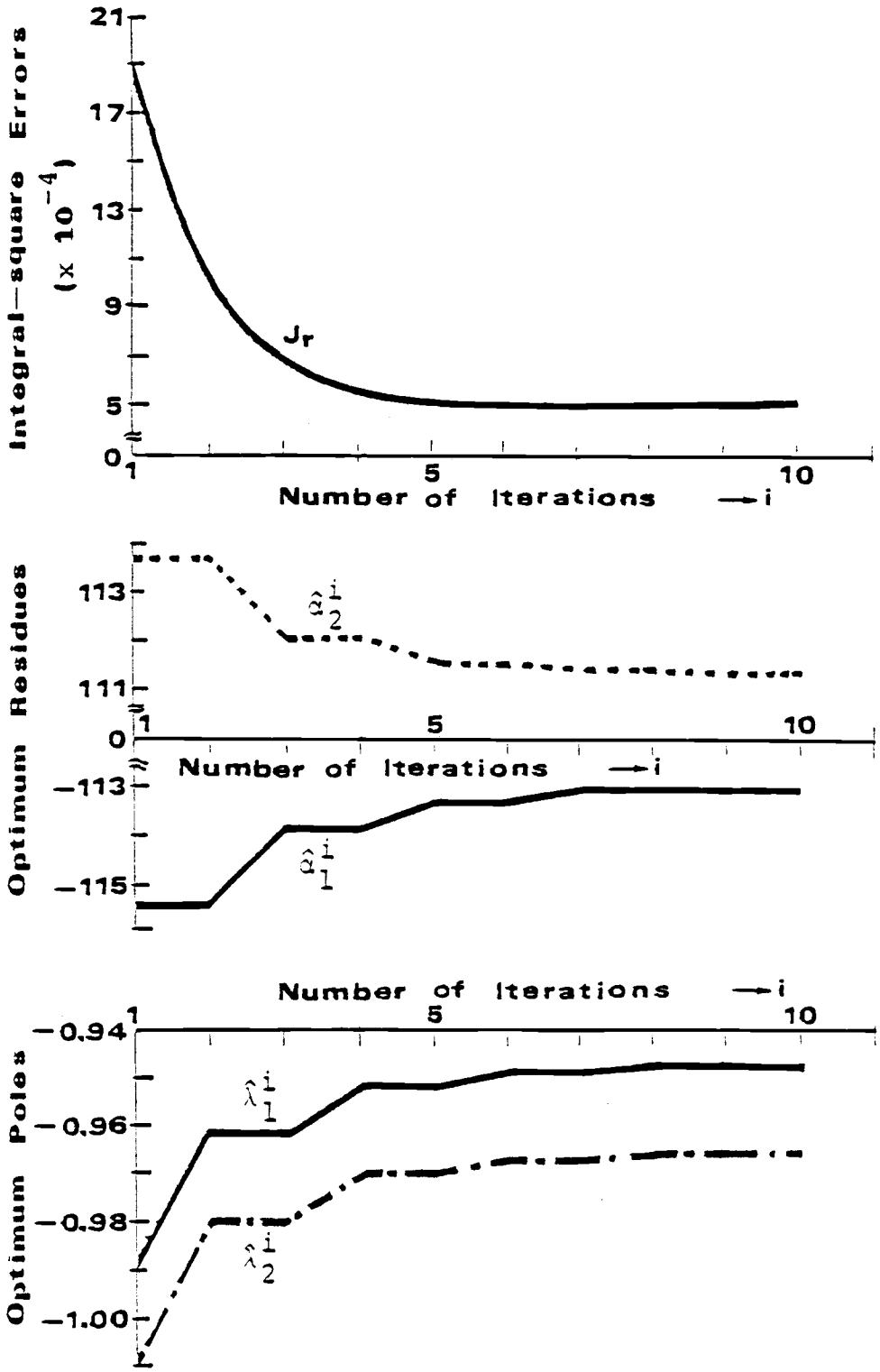

Figure 6. Integral-square errors, optimum residues and optimum poles versus iterations for second-order reduced models of a third-order system with double poles (for $\varepsilon = 0.01$).

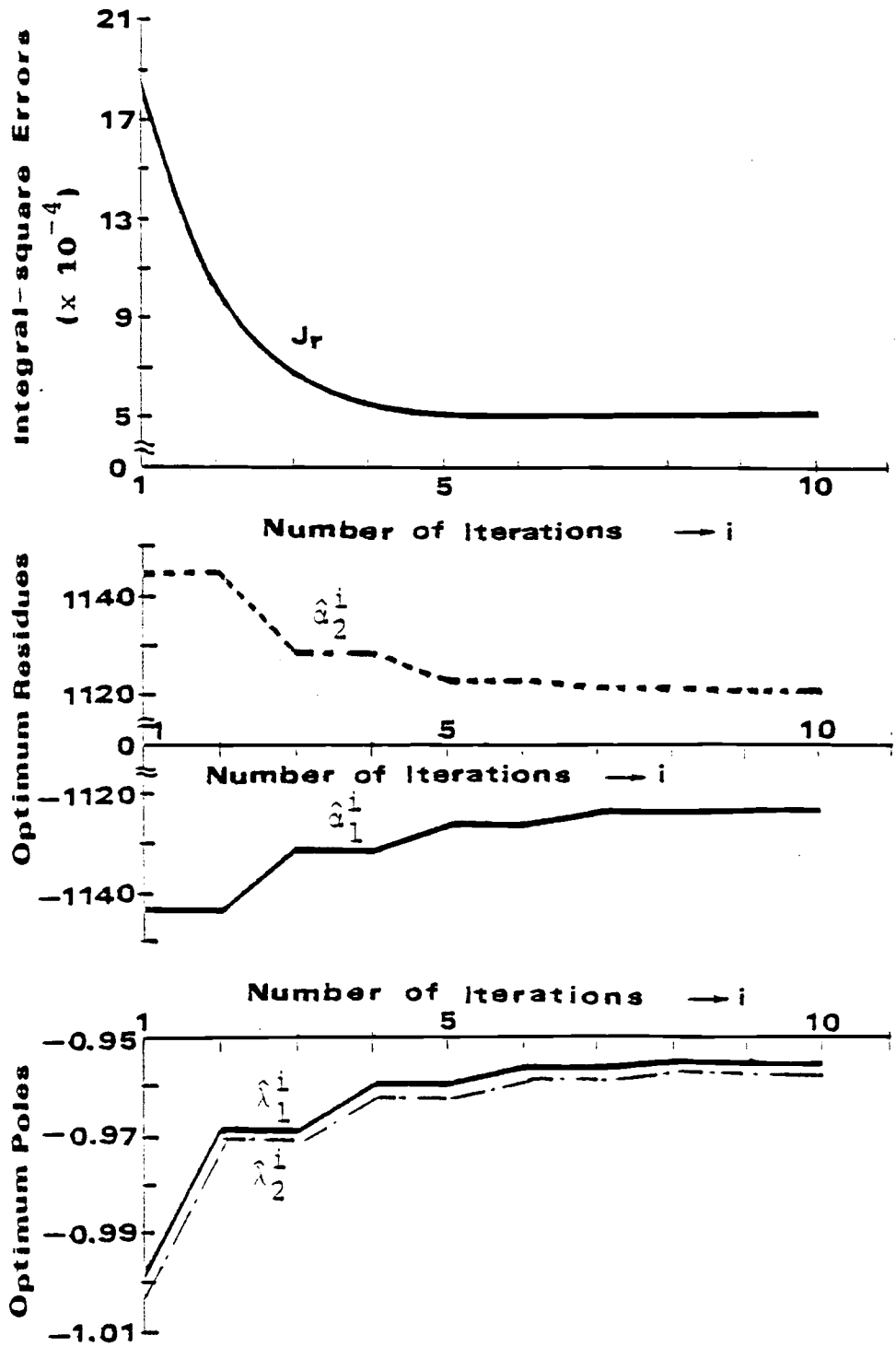

Figure 7. Integral-square errors, optimum residues and optimum poles versus iterations for second-order reduced models of a third-order system with double poles (for $\varepsilon = 0.001$).

$$\hat{H}_2(s) = \frac{-0.2778s + 1.3888}{s^2 + 1.7778s + 0.8333} \qquad (3\text{-}52)$$

The unit-step response is of the form

$$\hat{y}_2 = 1.6667 - 1.2000\, e^{-0.8889t}[0.2887 \cos (0.2079t) + 1.4660 \sin (0.2079t)] \qquad (3.53)$$

The integral-square error is

$$J_r = 2.71445781 \qquad (3\text{-}54)$$

As can be seen in Tables 7 and 8, the iterative method is superior.

Example 4

Reduce a fourth-order linear time-invariant system to a second-order model. The original system in this example has the same output as in example 1(ii).

$$\dot{\underline{x}} = A\,\underline{x} + \underline{d}\,u$$
$$y_4 = \underline{c}'\underline{x} \qquad (3\text{-}55)$$

where

$$A = \begin{bmatrix} -1 & -1 & 6 & -2 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} \qquad \underline{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c' = [\,1 \quad 1 \quad 0 \quad 0\,]$$

By using the modal matrix P_4 ($\underline{x} = P_4\ \underline{Z}$),

$$P_4 = \begin{bmatrix} 1 & 1 & -4 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_4^{-1} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3-56}$$

the original system transforms into a purely diagonal form.

$$\dot{\underline{Z}} = \Lambda\ \underline{Z} + \underline{b}\ u$$
$$y_4 = \underline{h}'\underline{z} \tag{3-57}$$

where

$$\Lambda = P^{-1}\ A\ P_4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \underline{b} = P_4^{-1}\underline{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{h}' = \underline{c}'P_4 = [\ 1 \quad 2 \quad -6 \quad 4]$$

The unit-step response of eq. (3-57) is;

$$y_4 = 1 - e^{-t} - e^{-2t} + 2e^{-3t} - e^{-4t} \tag{3-58}$$

$$\alpha_0 = 1,\ \alpha_1 = -1,\ \alpha_2 = -1,\ \alpha_3 = 2,\ \alpha_4 = -1$$
$$\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3,\ \lambda_4 = -4 \tag{3-59}$$

The second-order reduced model in the ith-step is obtained by the elimination of the non-dominant eigenvalues, then using eq. (2-34)

$$\dot{\hat{\underline{z}}}^i = \hat{\Lambda}^i \hat{\underline{z}}^i + \hat{\underline{b}}^i u$$

$$\hat{y}_2^i = \hat{h}^{i'} \hat{\underline{z}}^i \qquad (3\text{-}60)$$

where $\hat{\Lambda}^i = P_2^{-1} \hat{A}^i P_2 = \begin{bmatrix} \hat{\lambda}_1^i & 0 \\ 0 & \hat{\lambda}_2^i \end{bmatrix}$

$$\hat{\underline{b}}^i = P_2^{-1} \hat{\underline{d}}^i = \begin{bmatrix} \hat{b}_1^i \\ \hat{b}_2^i \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_1^i \bar{b}_1^i \\ \hat{\lambda}_2^i \bar{b}_2^i \end{bmatrix}$$

$$\hat{\underline{h}}^{i'} = \hat{\underline{c}}^{i'} P_2 = [\hat{h}_1^i \quad \hat{h}_2^i]$$

P_2 is a 2x2 matrix and is truncated the modal matrix P_4, 4x4.

The solution of eq. (3-60) is

$$\hat{y}_2^i = \hat{\alpha}_0^i + \bar{b}_1^i \hat{h}_1^i e^{\hat{\lambda}_1^i t} + \bar{b}_2^i \hat{h}_2^i e^{\hat{\lambda}_2^i t} \qquad (3\text{-}61)$$

With the control-vector optimization method discussed in section 2.22(i), the control-vector $\bar{\underline{b}}^i$ and eigenvalues $\hat{\underline{\lambda}}^i$ are optimized so as to minimize the objective function.

$$J_s = \int_0^\infty (y_4 - \hat{y}_2^i)^2 \, dt \qquad (3\text{-}62)$$

A. Computational Procedure

Following section 2.22(i), the optimum control-vector and eigenvalues are obtained as follows.

Step 1: Initial step (i=0)

Assuming that the reduced-order model, eq. (3-60) consists of the dominant eigenvalues of the original system, eq. (3-55)

$$\begin{bmatrix} \hat{\lambda}_1^0 & 0 \\ 0 & \hat{\lambda}_2^0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \tag{3-63}$$

The first residue and output-vector are kept constant for all steps.

$$\hat{\alpha}_0^i = \alpha_0 = 1 \qquad \text{for all } i \tag{3-64}$$

$$\underline{\hat{h}}^{i'} = \underline{\hat{h}}' = [1 \quad 2] \qquad \text{for all } i \tag{3-65}$$

By substitution of eq. (3-63) and (3-64) into eq. (3-61) the output of reduced-order model is, using $\hat{b}_1 = -1$, and $\hat{b}_2 = -2$, the truncation of $\underline{b}$ vector, 4x1 into $\underline{\hat{b}}$, 2x1.

$$\hat{y}_2^i = 1 - e^{-t} - e^{-2t} \tag{3-66}$$

The state-space equation in the Z-domain is

$$\begin{aligned} \underline{\hat{z}}^0 &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underline{\hat{z}}^0 + \begin{bmatrix} (-1) & (-1) \\ (-2) & (0.5) \end{bmatrix} u \\ \hat{y}_2^0 &= [\,1 \quad 2\,]\,\underline{\hat{z}}^0 \end{aligned} \tag{3-67}$$

Step 2: Control-vector optimization (i=N)

The solution is, setting i=N in eq. (3-61) and substituting eq. (3-64) and (3-65) into eq. (3-61),

$$\hat{y}_2 = 1 + \bar{b}_1^N e^{\hat{\lambda}_1^N t} + \bar{b}_2^N e^{\hat{\lambda}_2^N t} \tag{3-68}$$

The eigenvalues are kept constant. To find $\underline{\bar{b}}^N$, solve $\frac{\partial J_s}{\partial(\underline{\bar{b}}^N)} = \underline{0}$. The resulting equation is the replacement of $\underline{\hat{\alpha}}^N$ by $\underline{\bar{b}}^N \underline{\hat{h}}'$ in eq. (2-13).

$$\hat{\psi}^N \cdot \underline{\hat{\alpha}}^N = \hat{\rho}^N \tag{3-69}$$

The solutions of eq. (3-69), $\hat{\alpha}_1^N$ and $\hat{\alpha}_2^N$ are shown in Table 3.

$$\begin{aligned} [\hat{\alpha}_1^N \quad \hat{\alpha}_2^N] &= [\bar{b}_1^N \hat{h}_1 \quad \bar{b}_2^N \hat{h}_2] \\ &= [\bar{b}_1^N \quad 2b_2^N] \end{aligned} \tag{3-70}$$

Thus, the recursive formulae in this step are

$$\begin{bmatrix} \hat{\lambda}_1^N & 0 \\ 0 & \hat{\lambda}_2^N \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_1^{N-1} & 0 \\ 0 & \hat{\lambda}_2^{N-1} \end{bmatrix} \tag{3-71}$$

From eq. (3-71)

$$\bar{b}_1^N = \hat{\alpha}_1^N , \quad \bar{b}_2^N = \frac{1}{2} \hat{\alpha}_2^N \tag{3-72}$$

The state-space equations are using eq. (3-71)

$$\dot{\hat{\underline{z}}}^N = \begin{bmatrix} \hat{\lambda}_1^{N-1} & 0 \\ 0 & \hat{\lambda}_2^{N-1} \end{bmatrix} \hat{\underline{z}}^N + \begin{bmatrix} \hat{\lambda}_1^{N-1}\bar{b}_1^N \\ \hat{\lambda}_2^{N-1}\bar{b}_2^N \end{bmatrix} u \tag{3-73}$$

$$\hat{y}_2^N = [1 \qquad 2]\,\hat{\underline{z}}^N$$

Step 3: Eigenvalue optimization (i=N+1)

The solution of eq. (2-41) is, set i=N+1 in eq. (3-61)

$$\hat{y}_2^{N+1} = 1 + \bar{b}_1^N e^{\hat{\lambda}_1^{N+1}t} + 2\bar{b}_2^N e^{\hat{\lambda}_2^{N+1}t} \tag{3-74}$$

and

$$\hat{\underline{\lambda}}^{N+1} = \hat{\underline{\lambda}}^N + \Delta\hat{\underline{\lambda}}^{N+1} \tag{3-75}$$

The control-vector is kept constant. $\Delta\hat{\underline{\lambda}}^{N+1}$ is obtained by solving $\frac{\partial J_s}{\partial(\Delta\hat{\underline{\lambda}}^{N+1})} = \underline{0}$. Replacing $\hat{\underline{\alpha}}^N$ in eq. (2-18) with $\bar{\underline{b}}^N\hat{\underline{h}}'$, the simultaneous equations are obtained.

The recursive formulae are as follows:

$$\bar{\underline{b}}^{N+1} = \bar{\underline{b}}^N \tag{3-76}$$

where $\bar{\underline{b}}^{N+1\prime} = [b_1^{N+1} \quad \bar{b}_2^{N+1}]$

$\bar{\underline{b}}^{N\prime} = [\bar{b}_1^N \quad \bar{b}_2^N]$

$$\hat{\phi}^{N+1}\cdot\Delta\hat{\underline{\lambda}}^{N+1} = \hat{\pi}^{N+1} \tag{3-77}$$

where $\hat{\phi}^{N+1}$ and $\hat{\pi}^{N+1}$ are given by eq. (2-18) and replace $\hat{\underline{\alpha}}^N$ by $\bar{\underline{b}}^N\hat{\underline{h}}'$. $\hat{\lambda}_1^{N+1}$ and $\hat{\lambda}_2^{N+1}$ are listed in Table 3.

The state-space equations are:

$$\dot{\underline{\hat{z}}}^{N+1} = \begin{bmatrix} \hat{\lambda}_1^{N+1} & 0 \\ 0 & \hat{\lambda}_2^{N+1} \end{bmatrix} \underline{\hat{z}}^{N+1} + \begin{bmatrix} \hat{\lambda}_1^{N+1} \bar{b}_1^N \\ \hat{\lambda}_2^{N+1} \bar{b}_2^N \end{bmatrix} u$$

$$\hat{y}_2^{N+1} = [1 \quad 2]\, \underline{\hat{z}}^{N+1} \tag{3-78}$$

where $\underline{\hat{\lambda}}^{N+1} = \underline{\hat{\lambda}}^N + \Delta\underline{\hat{\lambda}}^{N+1}$.

The original x-domain reduced-order model is obtained by the transformation, eq. (2-52) with $\ell=2$. As an example, the first two steps are summarized.

<u>i=1st-step</u>: Control-vector optimization

From the initial step (i=0)

$$\hat{\Lambda}^0 = \begin{bmatrix} \hat{\lambda}_1^0 & 0 \\ 0 & \hat{\lambda}_2^0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \tag{3-79}$$

Solving eq. (3-69), $\hat{\alpha}_1^1$ and $\hat{\alpha}_2^1$ are obtained.

$$\hat{\alpha}_1^1 = -1.2, \qquad \hat{\alpha}_2^1 = 0.2 \tag{3-80}$$

Using eq. (3-79), (3-80), and (3-72), the eigenvalues are fixed and the control-vector is optimized.

$$\hat{\Lambda}^1 = \begin{bmatrix} \hat{\lambda}_1^1 & 0 \\ 0 & \hat{\lambda}_2^1 \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_1^0 & 0 \\ 0 & \hat{\lambda}_2^0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \tag{3-81}$$

$$\bar{b}_1^1 = \hat{\alpha}_1^1 = -1.2, \qquad \bar{b}_2^1 = \tfrac{1}{2}\,\hat{\alpha}_2^1 = 0.1 \tag{3-82}$$

So that,

$$\dot{\hat{z}}^1 = \hat{\Lambda}^1 \hat{z}^1 + \hat{b}^1 u$$
$$\hat{y}_2^1 = \hat{h}^{1'} z \qquad (3\text{-}83)$$

where $\hat{\Lambda}^1 = P_2^{-1} A P_2 = \begin{bmatrix} \hat{\lambda}_1^0 & 0 \\ 0 & \hat{\lambda}_2^0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

$$\hat{b}^1 = P_2^{-1} d = \begin{bmatrix} \hat{\lambda}_1^0 b_1 \\ \hat{\lambda}_2^0 b_2 \end{bmatrix} = \begin{bmatrix} (-1)\cdot(-1.2) \\ (-2)\cdot(0.1) \end{bmatrix} = \begin{bmatrix} 1.2 \\ -0.2 \end{bmatrix}$$

$$\hat{h}^{1'} = c' P_2 = [1 \quad 2] \equiv \hat{h}' \quad \text{for all } i$$

P_2 is a truncation of the modal matrix P_4

$$P_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad P_2^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad (3\text{-}84)$$

The unit-step response of eq. (3-83) is

$$\hat{y}_2^1 = 1 - 1.2\, e^{-t} + 0.2\, e^{-2t} \qquad (3\text{-}85)$$

The original x-domain reduced-order model is obtained by inverse transformation.

$$\dot{\hat{x}}^1 = \hat{A}_0^1 \hat{x}^1 + \hat{b}_0^1 u$$
$$\hat{y}_2^1 = \hat{h}_0^{1'} \hat{x}^1 \qquad (3\text{-}86)$$

where $\hat{A}_0^1 = P_2 \hat{\Lambda}^1 P_2^{-1} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$

$$\underline{\hat{b}}_0^1 = P_2\,\underline{\hat{b}}^1 = P_2\begin{bmatrix}\hat{\lambda}_1^0\bar{b}_1^1\\ \hat{\lambda}_2^0\bar{b}_2^2\end{bmatrix} = \begin{bmatrix}1.0\\ -0.2\end{bmatrix}$$

$$\underline{\hat{h}}_0^{1\prime} = \hat{h}^{1\prime}P_2^{-1} = [1 \quad 1]$$

The output $\hat{y}_2^1$ is not affected by the transformation.

i=2nd-step: Eigenvalue optimization

The control-vector $\underline{\hat{b}}^2$ is fixed. The eigenvalues are optimized using eq. (2-18). Solving eq. (2-18), $\frac{\partial J_s}{\partial(\Delta\underline{\hat{\lambda}}^2)} = \underline{0}$, replacing $\underline{\hat{\alpha}}^2$ by $\underline{\hat{b}}^2\,\underline{\hat{h}}'$, $\Delta\hat{\lambda}_1^2$ and $\Delta\hat{\lambda}_2^2$ are obtained.

$$\Delta\hat{\lambda}_1^2 = -0.0334, \quad \Delta\hat{\lambda}_2^2 = -0.5641 \tag{3-86}$$

From the previous step

$$\hat{\lambda}_1^1 = \hat{\lambda}_1^0 = -1, \quad \hat{\lambda}_2^1 = \hat{\lambda}_2^0 = -2 \tag{3-88}$$

and the relationship,

$$\underline{\hat{\lambda}}^{N+1} = \underline{\hat{\lambda}}^N + \Delta\underline{\hat{\lambda}}^{N+1}$$

$$\begin{bmatrix}\hat{\lambda}_1^2 & 0\\ 0 & \hat{\lambda}_2^2\end{bmatrix} = \begin{bmatrix}\hat{\lambda}_1^1 + \Delta\hat{\lambda}_1^2 & 0\\ 0 & \hat{\lambda}_2^1 + \Delta\hat{\lambda}_2^2\end{bmatrix}$$

$$= \begin{bmatrix}-1.0334 & 0\\ 0 & -2.5641\end{bmatrix} \tag{3-89}$$

$$\underline{\bar{b}}^2 = \underline{\bar{b}}^1 \tag{3-90}$$

where $\underline{\bar{b}}^2 = \begin{bmatrix} \bar{b}_1^2 \\ \bar{b}_2^2 \end{bmatrix}$, $\underline{\bar{b}}^1 = \begin{bmatrix} \bar{b}_1^1 \\ \bar{b}_2^1 \end{bmatrix} = \begin{bmatrix} 1.2 \\ -0.1 \end{bmatrix}$

Thus

$$\begin{aligned} \dot{\hat{\underline{z}}}^2 &= \hat{\Lambda}^2 \, \hat{\underline{z}}^2 + \hat{\underline{b}}^2 \, u \\ \hat{y}_2^2 &= \hat{\underline{h}}^{2'} \hat{\underline{z}}^2 \end{aligned} \tag{3-91}$$

where $\hat{\Lambda}^2 = \begin{bmatrix} \hat{\lambda}_1^2 & 0 \\ 0 & \hat{\lambda}_2^2 \end{bmatrix} = \begin{bmatrix} -1.0334 & 0 \\ 0 & -2.5641 \end{bmatrix}$

$$\hat{\underline{b}}^2 = \begin{bmatrix} \hat{\lambda}_1^2 \bar{b}_1^2 \\ \hat{\lambda}_2^2 \bar{b}_2^2 \end{bmatrix} = \begin{bmatrix} (\hat{\lambda}_1^1 + \Delta\hat{\lambda}_1^2) \cdot \bar{b}_1^1 \\ (\hat{\lambda}_2^1 + \Delta\hat{\lambda}_2^2) \cdot \bar{b}_2^1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1 - 0.0334) \cdot (1.2) \\ (-2 - 0.5641) \cdot (0.1) \end{bmatrix} = \begin{bmatrix} 1.2400 \\ -0.2564 \end{bmatrix}$$

$$\hat{\underline{h}}^{2'} = \hat{\underline{h}}' = [1 \quad 2]$$

The unit-step response of eq. (3-91) is

$$\hat{y}_2^2 = 1 - 1.2 \, e^{-1.0334t} + 0.2 \, e^{-2.5651t} \tag{3-92}$$

The reduced-order model in the x-domain is:

$$\begin{aligned} \dot{\hat{\underline{x}}}^2 &= \hat{A}_0^2 \, \hat{\underline{x}}^2 + \hat{\underline{b}}_0^2 \, u \\ \hat{y}_2^2 &= \hat{\underline{h}}_0^{2'} \hat{\underline{x}}^2 \end{aligned} \tag{3-93}$$

where $\hat{A}_0^2 = P_2 \, \hat{\Lambda}^2 \, P_2^{-1} = \begin{bmatrix} -1.0334 & -1.5307 \\ 0 & -2.5641 \end{bmatrix}$

$$\underline{\hat{b}}_0^2 = P_2 \, \underline{\hat{b}}^2 = \begin{bmatrix} 0.9836 \\ -0.2564 \end{bmatrix}$$

$$\underline{\hat{h}}_0^{2'} = \hat{h}^{2'} P_2^{-1} = [1 \quad 1]$$

The state-space equations of the succeeding steps are obtained similarly.

CHAPTER IV. CONCLUSIONS

As discussed in the preceeding chapter, the proposed method is superior to other methods in the sense of the integral-square error as shown in Table 2.

Since the poles (or eigenvalues) are shifted slightly, the reduced-order model is stable if the original system is stable.

For the case of system with complex poles (or eigenvalues), both real parts and imaginary parts of residues and poles (or eigenvalues) are optimized respectively at each step.

The method can be easily extended to the reduction of other types of the large linear system, for example, a discrete-time system, an unknown original system and so on.

The other characters of the proposed method are as follows:

1) In each iterative step, there exists a minimum of the objective function. The necessary and sufficient conditions for existence of an optimum are satisfied in each step.
2) The iterative scheme improves the steady-state and transient responses cyclically as depicted in Figure 2, so that the method gives a good approximation for both responses.

3) A satisfactorily reduced-order model is attained after few iterative steps.

4) The proposed method is also effective to the multi-variable system. As shown in Table 6, the method is superior to the matrix continued-fraction-expansion method.

5) As shown in Tables 7 and 8, and Figures 6 and 7, the method can be applied for the system with repeated poles (or eigenvalues) by making the repeated poles (or eigenvalues) slightly different in the reduced-order system. This is found in comparison to the result obtained by the second Cauer form expansion method shown in (3-54) with the results of Tables 7 and 8.

6) The residues and poles (or eigenvalues) converge monotonically with the iterative steps.

7) As shown in Tables 1, 3, 7, and 8, the integral-square error decreases monotonically as the number of the iterative steps increase.

8) The integral-square error consists of the difference between the exact and approximate solutions. These solutions have in the form of exponential function. The difference of the exponential functions does not converge into a constant. Therefore the integral-square error converges into a very small value after a large number of iterations, thereafter it fluctuates around this value. The approximate-model at this step

might be the best approximation of the original system in the two-step iterative method. This is the limitation of the proposed method.

The fluctuation is found in Tables 7 and 8 for the reduction of a third-order system with double poles into a second-order model. As shown in Table 7, for $\varepsilon = 0.01$, the integral-square error decreases monotonically until the 19th iteration. However, after the 19th iterative step, the error fluctuates around

$$J_r = 0.0005031421 \tag{4-1}$$

This is found from Table 7:

Number of Iteration (i)	Integral-square Error (J_r)
18	0.000503142238
19	0.000503142136
20	0.000503142150

(4-2)

The approximate-model around the 19th-step will give the best approximation of the original system.

For $\varepsilon = 0.001$ in Table 8, the error fluctuates around

$$J_r = 0.0005025 \tag{4-3}$$

After the 15th iteration.

While the residues and poles still converge montonically.

The similar fluctuation occurs to the reduction of the other high-order systems.

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