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PATRICIA CARRIE PIERCE (Name) for the M.S. (Degree) in Mathematics (Major)

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Abstract approved ______________________ (Major professor)

In this thesis, we study conditions not involving density which guarantee that a given positive integer is contained in a sum of sets of nonnegative integers. We survey the literature, give more detailed proofs of some known theorems, develop some new theorems, and make some conjectures.

SOME CONDITIONS FOR A GIVEN POSITIVE INTEGER TO BE CONTAINED IN A SUM OF SETS OF NONNEGATIVE INTEGERS

by

PATRICIA CARRIE PIERCE

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Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

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CHAPTER I

INTRODUCTION

Let k be any integer where $k \geq 2$, and let A_i $(i = 1, \cdots, k)$ denote subsets of the set of nonnegative integers.

Definition 1.1. $\sum_{i=1}^{k} A_i = \{ \sum_{i=1}^{k} a_i \mid a_i \in A_i, \ i = 1, \cdots, k \}$.

Let A be any subset of the set of nonnegative integers.

Definition 1.2. For n a positive integer, $A(n)$ denotes the number of positive integers not greater than n.

In this thesis we study conditions on $\sum_{i=1}^{k} A_i(n)$, not involving density, which guarantee that a given positive integer n is in $\sum_{i=1}^{k} A_i$, or equivalently, restrictions on $\sum_{i=1}^{k} A_i(n)$ by assuming that n is missing from $\sum_{i=1}^{k} A_i$.

Definition 1.3. A positive integer not occurring in A is a <u>gap</u> in A.

Definition 1.4. The <u>complement</u> of A, denoted by $\overline{A}$, is the set of all gaps in A.

As a special case of a result of Schur (10, p. 275), Rohrbach proved, in 1939, the following theorem (9, p. 206).

Theorem 1.1. If $0 \epsilon A_1$, $0 \epsilon A_2$, and n is a gap in $A_1 + A_2$, then

$$A_1(n) + A_2(n) \leq n-1 .$$

In 1952, Khinchin gave a proof of Theorem 1.1 independent of Schur's result (6, pp. 24-25).

In 1958, Erdős and Scherk stated the following generalization (4, p. 45).

Theorem 1.2. If $0 \epsilon A_i$ $(i = 1, 2, \cdots, k)$, and n is a gap in $\sum_{i=1}^{k} A_i$, then

$$\sum_{i=1}^{k} A_i(n) \leq \frac{k}{2}(n-1).$$

We prove Theorems 1.1 and 1.2 in Chapter V. We also show there that Theorem 1.2 is the "best possible" inequality by using an example of Erdős and Scherk (4, p. 45).

If n is further restricted to be the smallest gap in $\sum_{i=1}^{k} A_i$, then stronger results can be obtained. In 1955, Lin proved the following theorem (7, pp. 31-40).

Theorem 1.3. If $0 \in A_i$ $(i = 1, 2, 3)$, $0 < n < 15$, and n is the smallest gap in $A_1 + A_2 + A_3$, then

$$A_1(n) + A_2(n) + A_3(n) \leq n-1 \ .$$

Lin gives a counterexample which shows that the smallest gap can be made no larger in this theorem (7, p. 41). In Chapters II, III, and IV, we give a proof of Theorem 1.3 that is a more readable version of Lin's proof with more details and some changes. In Chapters II and III, we give background material to make the proof self contained. In Chapter III, we describe Mann's set transformation and establish its properties which are needed in the proof. Examples, to provide insight into the mechanics of the proof are supplied by the author.

Definition 1.5. If n is a gap in $\sum_{i=1}^{k} A_i$, then

$$f_k(n) = \max \sum_{i=1}^{k} A_i(n) \ ,$$

over all collections $\{A_i \mid 1 \leq i \leq k\}$.

If n is a gap in $\sum_{i=1}^{k} A_i$, then

$$\sum_{i=1}^{k} A_i(n) \le f_k(n),$$

for all collections $\{A_i \mid 1 \le i \le k\}$, while

$$\sum_{i=1}^{k} A_i(n) = f_k(n),$$

for at least one collection $\{A_i \mid 1 \le i \le k\}$. Thus the central problem of this thesis is to find information about $f_k(n)$, for example, upper bounds or asymptotic formulas.

In 1958, Erdös and Scherk proved the following theorem (4, pp. 46-55).

Theorem 1.4. If $0 \in A_i$ $(i = 1, 2, \cdots, k)$, $k \ge 3$, and n is the smallest gap in $\sum_{i=1}^{k} A_i$, then

$$\frac{1}{2}kn - a_k n^{\frac{k-1}{k}} < f_k(n) < \frac{1}{2}kn - \gamma_k n^{\frac{k-1}{k}},$$

where a_k and γ_k are positive constants.

Erdös and Scherk showed furthermore that

$$a_k = (k+1)2^{2k-3},$$

and

$$\gamma_k = \frac{1}{2^{\frac{k}{2}+4}(k-1)!}.$$

They also conjectured the following stronger asymptotic formula (4, p. 46).

Conjecture 1.1. If $0 \in A_i\,(i = 1, 2, \cdots, k)$, $k \geq 3$, and n is the smallest gap in $\sum_{i=1}^{k} A_i$, then

$$f_k(n) = \frac{1}{2}kn - (\beta_k + o(1))n^{\frac{k-1}{k}}$$

holds, where n tends to infinity, for some constant β_k which is positive. 1/

In 1964, Kemperman proved a theorem which yields Conjecture 1.1 (5, pp. 376-385). He gave the following definition and proved the following theorem.

Definition 1.6. If n is a gap in $\sum_{i=1}^{k} A_i$, then

$$\phi_k(n) = \min \{\frac{k}{2}(n-1) - \sum_{i=1}^{k} A_i(n)\}$$

over all collections $\{A_i \mid 1 \leq i \leq k\}$.

1/ In the original paper this formula is given with a plus sign in place of the minus sign, clearly an error.

Theorem 1.5 If $0 \epsilon A_i$ $(i = 1, 2, \cdots, k)$, $k \geq 3$, and n is the smallest gap in $\sum_{i=1}^{k} A_i$, then

$$\lim_{n \to \infty} \phi_k(n) n^{-\frac{k-1}{k}} = k2^{-\frac{k-1}{k}} . \tag{1.1}$$

We show that Conjecture 1.1 follows readily from Theorem 1.5. Since

$$\phi_k(n) = \frac{k}{2}(n-1) - f_k(n),$$

then equation (1.1) may be written

$$\lim_{n \to \infty} [\frac{k}{2}(n-1) - f_k(n)] n^{-\frac{k-1}{k}} = k2^{-\frac{k-1}{k}} ,$$

and so

$$\lim_{n \to \infty} [\frac{k}{2}n - f_k(n)] n^{-\frac{k-1}{k}} = k2^{-\frac{k-1}{k}} .$$

Hence, this equation may be written

$$[\frac{k}{2}n - f_k(n)] n^{-\frac{k-1}{k}} = o(1) + k2^{-\frac{k-1}{k}} ,$$

where n tends to infinity. Finally, this equation yields

$$\frac{k}{2}n - f_k(n) = (o(1) + k2^{-\frac{k-1}{2}}) n^{\frac{k-1}{k}} ,$$

or equivalently,

$$f_k(n) = \frac{k}{2}n - (o(1) + k2^{-\frac{k-1}{2}}) n^{\frac{k-1}{k}} .$$

This is the equation of Conjecture 1. 1 with $\beta_k = k2^{-\frac{k-1}{2}}$. Since $k2^{-\frac{k-1}{k}} > 0$ and since the hypotheses of Conjecture 1. 1 are the same as those of Theorem 1. 5, Conjecture 1. 1 is proved.

Kemperman also showed that the condition $\{i \mid 1 \leq i \leq n\} \subseteq \sum_{i=1}^{k} A_i$ of Theorem 1. 5 can be generalized (5, pp. 381 - 387).

In Chapter V, we include proofs of several further miscellaneous smaller theorems. Also some conjectures are made.

CHAPTER II

EVOLUTION OF A CONJECTURE

Our purpose in this chapter is to investigate in detail the manner in which a particular conjecture, Conjecture 2.7, is formulated. From Conjecture 2.7 evolves Theorem 4.1, proved in Chapter IV, which is a powerful tool in the proof of Theorem 1.3.

To formulate Conjecture 2.7, we start with Conjecture 2.1. While Conjecture 2.1 is a very natural extension of Mann's fundamental inequality (8, pp. 523-524), it is seen to be invalid. From Conjecture 2.1, we formulate Conjecture 2.2, a special case of Conjecture 2.1 closely related to Theorem 1.3, which is shown here also to be invalid. We then introduce several definitions, and Conjectures 2.3 and 2.4 are reformulations of Conjecture 2.2 using these definitions. Following Conjecture 2.4, we define a new set, the inversion set, and prove several properties of this set. We introduce the inversion set into our conjecture by making a set substitution when formulating Conjecture 2.5. We thereby not only reduce our consideration to two, instead of three, sets but we have all the properties of the inversion set available to us. Making use of these properties and previous definitions, we formulate Conjectures 2.6 and 2.7. Our evolution terminates with Conjecture 2.7.

The purpose of this sequence of conjectures, Conjecture 2.3 through Conjecture 2.7, is to put Conjecture 2.2 in a form more easily handled in the proof of Theorem 4.1, which is Conjecture 2.7 under a certain restriction. Each conjecture in our sequence is implied by the next one and in some cases the conjectures are equivalent. This implication or equivalence is shown in each case. We include a two page summary of this evolution at the end of this chapter.

The formulation of Conjecture 2.3 through Conjecture 2.6, as well as the proofs of the equivalences and implications of these conjectures and the summary page, is the work of the author. Also, for easy referral, we formalize definitions into statements which are displayed and numbered, which in most instances Lin does not do.

First we introduce Conjecture 2.1 (7, p. 21). In 1942, Mann gave the first proof of the $\alpha\beta$ Theorem for Schnirelmann density (8, pp. 523-527). The $\alpha\beta$ Theorem is an immediate consequence of the following more fundamental theorem which he proves in the same paper. In this proof, Mann introduces his method of set construction which we introduce in Chapter III and use for our purposes in this thesis.

Theorem 2.1. If $0 \in A_1$ and $0 \in A_2$, then for any positive integer n, either

$$\frac{(A_1+A_2)(n)}{n} = 1 ,$$

or

$$\frac{(A_1+A_2)(n)}{n} \geq \min_{\substack{1 \leq m \leq n \\ m \notin A_1+A_2}} \frac{A_1(m)+A_2(m)}{m} .$$

Our Conjecture 2.1 is the generalization of Theorem 2.1 to k sets where $k \geq 2$.

Conjecture 2.1. If $0 \in A_i$ $(i = 1, 2, \cdots, k)$, then for any positive integer n, either

$$\frac{\left(\sum_{i=1}^{k} A_i\right)(n)}{n} = 1 ,$$

or

$$\frac{\left(\sum_{i=1}^{k} A_i\right)(n)}{n} \geq \min_{\substack{1 \leq m \leq n \\ m \notin \sum_{i=1}^{k} A_i}} \frac{\sum_{i=1}^{k} A_i(m)}{m} .$$

This conjecture is not true according to an example following Conjecture 2.2.

Next we formulate Conjecture 2.2 (7, p. 21). In Conjecture 2.1 we set $k = 3$ and restrict n to be the smallest gap in $\sum_{i=1}^{3} A_i$. Since

$$\frac{(\sum_{i=1}^{3} A_i)(n)}{n} = \frac{n-1}{n} < 1 ,$$

we have

$$\frac{(\sum_{i=1}^{3} A_i)(n)}{n} \geq \min_{\substack{1 \leq m \leq n \\ m \notin \sum_{i=1}^{3} A_i}} \frac{\sum_{i=1}^{3} A_i(m)}{m} = \frac{\sum_{i=1}^{3} A_i(n)}{n} ,$$

and so

$$(\sum_{i=1}^{3} A_i)(n) \geq \sum_{i=1}^{3} A_i(n) .$$

Hence the following conjecture follows from Conjecture 2.1.

Conjecture 2.2. If $0 \in A_i$ $(i = 1, 2, 3)$, and if n is the smallest gap in $\sum_{i=1}^{3} A_i$, then

$$\sum_{i=1}^{3} A_i(n) \leq \left(\sum_{i=1}^{3} A_i\right)(n) .$$

Clearly Conjecture 2.2 does not imply Conjecture 2.1 because Conjecture 2.2 is obtained from Conjecture 2.1 by restricting n and k.

Since in Conjecture 2.2 we have $\left(\sum_{i=1}^{3} A_i\right)(n) = n-1$, it follows that Conjecture 2.2 differs from Theorem 1.3 only by not having the hypothesis $0 < n < 15$ of Theorem 1.3.

Since Conjecture 2.2 is a special case of Conjecture 2.1, the following example shows that both are invalid. Let

$$A_1 = \{0, 1, 8, 10, 12, 14\},$$

$$A_2 = \{0, 2, 8, 9, 12, 13\},$$

$$A_3 = \{0, 4, 8, 9, 10, 11\} .$$

Then

$$A_1+A_2+A_3 = \{i \mid 1 \leq i \leq 14\} \cup \{i \mid 16 \leq i \leq 38\},$$

and so $n = 15$ is the smallest gap in $\sum_{i=1}^{3} A_i$. Thus the hypotheses of Conjecture 2.2 are satisfied, but

$$A_1(15)+A_2(15)+A_3(15) = 15 > n-1 .$$

We cannot proceed to Conjecture 2.3 without some preliminary groundwork.. In the first place, we wish to avoid the confusion which arises in Lin's work with the use of the same set notation for the possibly infinite sets A_i and the finite sets $A_i \cap \{0, 1, 2, \cdots, n\}$ $(i = 1, 2, 3)$. To this end, we make the following definition.

Definition 2.1. The set N is defined by $N = \{j \mid 0 \leq j \leq n\}$. In addition $A = A_1 \cap N$, $B = A_2 \cap N$, $C = A_3 \cap N$ and $A+B+C = (\sum_{i=1}^{3} A_i) \cap N$, with similar definitions for all possible combinations of the A_i's and for the complements of these sets, e.g., $A+B = (A_1+A_2) \cap N$ and $\bar{A} = \bar{A}_1 \cap N$.

We also adopt the convention that any new, possibly infinite, sets introduced in the discussion to follow will be regarded as their intersections with the set N. Thus, for example, if we introduce the two sets X and Y, both subsets of the set of nonnegative integers, and state that X is a subset of Y, we mean $X \cap N \subseteq Y \cap N$.

In the second place, to make our notation more concise, we introduce the following definitions.

Definition 2.4. The set I_n is defined by $I_n = \{0, 1, 2, \cdots, n-1\}$.

Definition 2.3. The expression $A(n-1) + B(n-1) - (A+B)(n)$ will be referred to as the excess of $A(n-1) + B(n-1)$ over $(A+B)(n)$ and will be denoted by $\epsilon\,(A, B, n)$. This definition can, of course, be extended to three or more sets, e.g., $\epsilon\,(A, B, C, n) = A(n-1) + B(n-1) + C(n-1) - (A+B+C)(n)$.

Definition 2.4. Let $\rho\,(n, \overline{A}, \overline{B})$ denote the number of representations of n as a sum $n = \overline{a} + \overline{b}$, where $\overline{a} \epsilon \overline{A}$ and $\overline{b} \epsilon \overline{B}$.

With the necessary groundwork laid, we may now proceed to a new formulation of our conjecture.

Conjecture 2.3. If $A+B+C = I_n$, then

$$A(n) + B(n) + C(n) - (A+B+C)(n) \leq 0 .$$

That the hypothesis $A+B+C=I_n$ is equivalent to the hypotheses of Conjecture 2.2 as previously stated, namely, $0 \epsilon A_i\ (i = 1, 2, 3)$, and n is the smallest gap in $\sum_{i=1}^{3} A_i$, is made clear by close examination of Definitions 1.1, 2.1 and 2.2. That is, $A+B+C = I_n$ implies that n is the smallest gap in $A+B+C$ by the definition of I_n (Definition 2.2). But by Definition 2.1,

$A+B+C = (\sum_{i=1}^{3} A_i) \cap N$. Hence $I_n = (\sum_{i=1}^{3} A_i) \cap N$, and so n is the smallest gap in $\sum_{i=1}^{3} A_i$. Conversely, if n is the smallest gap in $\sum_{i=1}^{3} A_i$, then n is the smallest gap in $(\sum_{i=1}^{3} A_i) \cap N$. But $(\sum_{i=1}^{3} A_i) \cap N = A+B+C$, and so n is the smallest gap in $A+B+C$. Thus equivalence is established for one of our two hypotheses.

Since $0 \in I_n$ and $A+B+C = I_n$, it follows by the definition of the set sum (Definition 1.1) that 0 is in $A+B+C = (\sum_{i=1}^{3} A_i) \cap N$, and so $0 \in A_i$ $(i = 1, 2, 3)$. Conversely, if $0 \in A_i$ $(i = 1, 2, 3)$, then $0 \in A_i \cap N$. Hence 0 is in each of the sets A, B and C by Definition 2.1. Thus, the equivalence of hypotheses is substantiated.

Let us note here that the step from Conjecture 2.2 to Conjecture 2.3 leads us from consideration of possibly infinite sets to that of finite sets.

Conjecture 2.4 is a reformulation of Conjecture 2.3. Thus the hypotheses are the same and imply that n is not in A, B, or C. For example, suppose $n \in A$. Then $0 \in B$ and $0 \in C$ imply that $n+0+0 = n$ is in $A+B$, contradicting our hypothesis $A+B+C = I_n$.

Hence n is not in A, B, or C. Therefore, we can say that $A(n) = A(n-1)$, $B(n) = B(n-1)$, and $C(n) = C(n-1)$. This shows that the conclusions are the same by Definition 2.3.

Conjecture 2.4. If $A+B+C = I_n$, then

$$\epsilon(A, B, C, n) \leq 0 .$$

Proceeding to Conjecture 2.5 involves the set substitution previously mentioned. Therefore before stating this conjecture, we define two new sets and prove several properties of one of these sets needed for future reference. To this end, let $0 < n_1 < n_2 < \cdots < n_r = n$ be the gaps in the sum of some fixed two of our three sets, say $A+B$.

Definition 2.5. The set D is defined by $D = \{ d_i \mid d_i = n-n_i, \ i = 1, \cdots, r \}$.

Definition 2.6. Given a set X, for any positive integer n, the inversion of X is defined to be the set

$$\tilde{X} = \{ n-\bar{x} \mid \bar{x} \in \bar{X} \},$$

also denoted $X^{\sim}$. The set $\tilde{X}$ is empty if $X = N$. (Recall our convention regarding the introduction of new sets.)

Two observations should now be made: (1) comparison of

Definitions 2.5 and 2.6 shows us that $D = (A+B)^{\sim}$, and (2) comparison of Definitions 1.4 and 2.6 shows us that $n \notin \tilde{X}$ since $\bar{x} > 0$ for all $\bar{x} \in \bar{X}$.

Lemma 2.1. For any set X, the following properties hold:

a) $\tilde{\tilde{X}} = X$.

b) $n \notin X$ implies $0 \in \tilde{X}$.

c) $n \notin X + \tilde{X}$. (If $X = \Phi$, then $X + \tilde{X} = \Phi$.)

d) $0 \in (X + \tilde{X})^{\sim}$.

e) $n \notin X + Y$ if, and only if, $Y \subseteq \tilde{X}$.

f) $\tilde{X}(n-1) = \bar{X}(n-1)$.

g) $\epsilon(X, \tilde{X}, n) = X(n-1) + \tilde{X}(n-1) - (X + \tilde{X})(n) = n-1-(X+\tilde{X})(n) \geq 0$,

equality holding if $X + \tilde{X} = I_n$.

Proof. a) If $a \in \tilde{\tilde{X}}$, then $a = n-b$ where $b \notin \tilde{X}$ and $0 < b \leq n$. Thus $b = n-a$, and so $a \in X$. Hence $\tilde{\tilde{X}} \subseteq X$. If $a \in X$, then $n-a \notin \tilde{X}$. Consequently, $n-(n-a) = a \in \tilde{\tilde{X}}$. Hence $X \subseteq \tilde{\tilde{X}}$. This proves $\tilde{\tilde{X}} = X$.

b) If $n \notin X$, then $n \in \bar{X}$. Therefore $n-n = 0 \in \tilde{X}$.

c) Suppose $n \in X + \widetilde{X}$. Then $n = x + (n - \overline{x})$ where $x \in X$ and $\overline{x} \in \overline{X}$. Hence $x = \overline{x}$ contrary to the definition of $\overline{X}$. Therefore, $n \notin X + \widetilde{X}$.

d) By (c), $n \notin X + \widetilde{X}$ By (b), $n \notin X + \widetilde{X}$ implies $0 \in (X + \widetilde{X})^{\sim}$.

e) If $Y \subseteq \widetilde{X}$, then $X + Y \subseteq X + \widetilde{X}$. By (c), $n \notin X + \widetilde{X}$. Thus, $n \notin X + \widetilde{X}$ and $X + Y \subseteq X + \widetilde{X}$ implies that $n \notin X + Y$. Therefore, if $Y \subseteq \widetilde{X}$, then $n \notin X + Y$.

Conversely, let $y \in Y$ and assume $y \notin \widetilde{X}$ Then $y = n - x$, $x \in X$, or $n = x + y \in X + Y$, contrary to our hypothesis, $n \notin X + Y$. Therefore, our assumption $y \notin \widetilde{X}$ is false. Thus, $Y \subseteq \widetilde{X}$. This proves (e).

f) For all $\overline{x} \in \overline{X}$ such that $1 \leq \overline{x} \leq n-1$, we have $1 \leq n - \overline{x} \leq n-1$. Therefore, $\widetilde{X}(n-1) \geq \overline{X}(n-1)$. For all $n - \overline{x} \in \widetilde{X}$ such that $1 \leq n - \overline{x} < n-1$, we have $1 \leq \overline{x} \leq n-1$. Therefore, $\overline{X}(n-1) \geq \widetilde{X}(n-1)$. Hence $\widetilde{X}(n-1) = \overline{X}(n-1)$.

g) By (f) ,

$$\epsilon(X, \widetilde{X}, n) = X(n-1) + \widetilde{X}(n-1) - (X + \widetilde{X})(n)$$

$$= X(n-1) + \overline{X}(n-1) - (X + \widetilde{X})(n)$$

$$= n - 1 - (X + \widetilde{X})(n),$$

since $X(n-1)+\bar{X}(n-1) = n-1$. By (c), $n \notin X+\tilde{X}$. Hence

$$(X+\tilde{X})(n) = (X+\tilde{X})(n-1) \leq n-1.$$

Therefore,

$$\epsilon\,(X,\tilde{X},n) \geq n-1-(n-1) = 0.$$

However, we have also to show that if $X+\tilde{X} = I_n$, then $\epsilon\,(X,\tilde{X},n) = 0$. But if $X+\tilde{X} = I_n$, then $(X+\tilde{X})(n) = n-1$, and so $\epsilon\,(X,\tilde{X},n) = 0$.

This completes the proof of Lemma 2.1. Observe here that (a) implies that $\tilde{D} = A+B$. The proof of Lemma 2.1, and also that of Corollary 2.1 which follows, is not done by Lin (7, pp. 22-23). The next result is an important one in that it justifies the set substitution that we wish to make in Conjecture 2.5.

Corollary 2.1. If $X+Y = I_n$, then $Y \subseteq \tilde{X}$ and $X+\tilde{X} = I_n$.

Proof. If $X+Y = I_n$, then $n \notin X+Y$. By Lemma 2.1 (e), if $n \notin X+Y$, then $Y \subseteq \tilde{X}$. Therefore, $X+Y \subseteq X+\tilde{X}$. By Lemma 2.1 (c), $n \notin X+\tilde{X}$. Hence, $X+Y = X+\tilde{X} = I_n$.

This completes the proof of Corollary 2.1 and gives us the material we need to state Conjecture 2.5, but before we terminate our discussion of the inversion set, let us prove another property which is needed in Chapter IV (7, p. 23).

Theorem 2.2. A necessary and sufficient condition for X to satisfy $X+\tilde{X} = I_n$ is that, for every h where $0 < h \leq n$, we have

$$X + \{0,h\} \neq X.$$

Proof. If $X+\tilde{X} = I_n$, then

$$X+\tilde{X} + \{0,h\} = I_n + \{0,h\},$$

for any h. By hypothesis, $0 < h \leq n$, so that $0 \leq n-h < n$, which implies that $n-h \in I_n$. Consequently, $n = (n-h)+h \in I_n + \{0,h\}$ and $I_n + \{0,h\} \neq I_n$. Now suppose $X + \{0,h\} = X$. Then

$$X+\tilde{X} + \{0,h\} = I_n + \{0,h\} = X+\tilde{X} = I_n,$$

contradicting our previous result $I_n + \{0,h\} \neq I_n$. Hence $X+\{0,h\} \neq X$.

Conversely, let $X+\{0,h\} \neq X$ for all h where $0 < h \leq n$. Then, for all $0 \leq i < n$, we have $0 < n-i \leq n$. Consequently, our hypothesis tells us there exists an $x \in X$ such that $x+(n-i) \leq n$ and $x+(n-i) \notin X$. That is $x+(n-i) = \bar{x} \in \bar{X}$. Hence

$$i = x+n-\bar{x} \in X+\tilde{X} .$$

Thus $x+n-\bar{x}$ also satisfies $0 \leq x+n-\bar{x} < n$. Consequently, $X+\tilde{X} = I_n$, for by Lemma 2.1 (d), $n \notin X+\tilde{X}$.

We now formulate the next conjecture in our evolution.

Conjecture 2.5. If $A+B+(A+B)^{\sim} = I_n$, then

$$\epsilon(A, B, (A+B)^{\sim}, n) \leq 0 . \qquad (2.1)$$

We wish to show now that Conjecture 2.4 is implied by Conjecture 2.5. First observe that the hypothesis of Conjecture 2.4, $A+B+C = I_n$, satisfies the hypothesis of Corollary 2.1, $X+Y = I_n$, where we are letting $X = A+B$ and $Y = C$. Consequently, we have that $C \subseteq (A+B)^{\sim}$ and $A+B+(A+B)^{\sim} = I_n$. Hence, we have the hypothesis of Conjecture 2.5. Consider now the conclusions of Conjectures 2.4 and 2.5. We have in Conjecture 2.4

$$\epsilon(A, B, C, n) = A(n-1)+B(n-1)+C(n-1)-(A+B+C)(n) \leq 0 ,$$

and in Conjecture 2.5, we have

$$\epsilon(A, B, (A+B)^{\sim}, n) = A(n-1)+B(n-1)+(A+B)^{\sim}(n-1)-(A+B+(A+B)^{\sim})(n) \leq 0.$$

However, since $A+B+C = A+B+(A+B)^{\sim} = I_n$, we have

$$(A+B+C)(n) = (A+B+(A+B)^{\sim})(n) = n-1 .$$

Also, since $C \subseteq (A+B)^{\sim}$, and $n \notin C$ and $n \notin (A+B)^{\sim}$, we have $C(n-1) \leq (A+B)^{\sim}(n-1)$. Hence,

$$\epsilon(A, B, C, n) \leq \epsilon(A, B, (A+B)^{\sim}, n)$$

and the implication is established.

Let us progress another step in our evolution by exploring (2. 1) more thoroughly. Recalling our previous observations, $D = (A+B)^{\sim}$ and $\tilde{D} = A+B$, we may write

$$(2.2)\quad \epsilon(A, B, (A+B)^{\sim}, n) = A(n-1)+B(n-1)+(A+B)^{\sim}(n-1)-(A+B+(A+B)^{\sim})(n)$$

$$= A(n-1)+B(n-1)-(A+B)(n)+D(n-1)+\tilde{D}(n-1)-(D+\tilde{D})(n)$$

$$= \epsilon(A, B, n) + \epsilon(D, \tilde{D}, n).$$

Since $(A+B)(n) = \tilde{D}(n)$ and $\tilde{D}(n) = \tilde{D}(n-1)$, we have merely added and subtracted $\tilde{D}(n)$ on the right side of (2. 2) in order to obtain the desired result. Observe here that $D + \tilde{D} = I_n$ and so, by Lemma 2. 1 (g), $\epsilon(D, \tilde{D}, n) = 0$. Thus (2. 2) becomes

$$(2.3)\qquad \epsilon(A, B, (A+B)^{\sim}, n) = \epsilon(A, B, n).$$

To formulate Conjecture 2. 6, we substitute (2. 3) in Conjecture 2. 5.

Conjecture 2. 6. If $A+B+(A+B)^{\sim} = I_n$, then

$$\epsilon(A, B, n) \leq 0 .$$

Clearly Conjectures 2.5 and 2.6 are equivalent.

Let us concentrate now on the last step of our evolution which involves the use of Definition 2.4. How Definition 2.4 is applicable here is made clear by considering three subsets of the set $\{1, 2, \cdots, n-1\}$. To be more precise, we introduce the following definition.

Definition 2.7. The sets A', B', and L are defined by:

1) $A' = \{n-a \mid 1 \leq a \leq n-1, \quad a \epsilon A\}$,

2) $B' = \{b \mid 1 \leq b \leq n-1, \quad b \epsilon B\}$,

3) $L = \{\bar{b} \mid \bar{b} = n-\bar{a}, \quad \bar{a} \epsilon A, \quad \bar{b} \epsilon \bar{B}\}$.

The proofs of the following two lemmas concerning these sets are not given by Lin.

Lemma 2.2. The sets A', B', and L are mutually disjoint.

Proof. We have three statements to prove: a) $A' \cap B' = \Phi$, b) $A' \cap L = \Phi$, and c) $B' \cap L = \Phi$.

a) Assume $A' \cap B' \neq \Phi$. Then there exists an element $b \epsilon B'$ and an element $n-a \epsilon A'$ such that $n-a = b$, or $n = a+b$.

This contradicts $n \notin A+B$. Hence $A' \cap B' = \Phi$.

b) Assume $A' \cap L \neq \Phi$. Then there exists an element $n-a \in A'$ and an element $n-\bar{a} \in L$ such that $n-a = n-\bar{a}$. But this implies $a = \bar{a}$ contrary to the definition of $\bar{A}$. Hence $A' \cap L = \Phi$.

c) Assume $B' \cap L \neq \Phi$. Then there exists an element $b \in B'$ and an element $\bar{b} \in \bar{B}$ such that $b = \bar{b}$, contradicting the definition of $\bar{B}$. Consequently, $B' \cap L = \Phi$.

Lemma 2.3. If n is a gap in $A+B$, then

$$A(n-1) + B(n-1) + \rho(n, \bar{A}, \bar{B}) = n-1.$$

Proof. We show first that $A' \cup B' \cup L = \{1, 2, \cdots, n-1\}$. By Definition 2.7, A', B', and L are all subsets of the set $\{1, 2, \cdots, n-1\}$, and so it follows that $A' \cup B' \cup L \subseteq \{1, 2, \cdots, n-1\}$. Suppose $x \in \{1, 2, \cdots, n-1\}$. Then either $x = b$ where $b \in B$ and $1 \leq b \leq n-1$, and so $x \in B'$, or $x = \bar{b}$ where $\bar{b} \in \bar{B}$ and $1 \leq \bar{b} \leq n-1$. In the latter case, there are only two possibilities. Either $\bar{b} = n-\bar{a}$ and so $\bar{b} \in L$, or $\bar{b} = n-a$, and so $\bar{b} \in A'$. Consequently, if $x \in \{1, 2, \cdots, n-1\}$, then $x \in A' \cup B' \cup L$, and so $\{1, 2, \cdots, n-1\} \subseteq A' \cup B' \cup L$. This proves $A' \cup B' \cup L = \{1, 2, \cdots, n-1\}$. This result, in addition to Lemma 2.2, shows that

$$(2.4) \qquad A'(n-1)+B'(n-1)+L(n-1) = n-1.$$

Definitions 2.4 and 2.7 yield

$$A'(n-1) = A(n-1),$$

$$B'(n-1) = B(n-1),$$

$$L(n-1) = \rho(n, \overline{A}, \overline{B}).$$

Making these substitutions in (2.4), we obtain

$$A(n-1)+B(n-1)+\rho(n, \overline{A}, \overline{B}) = n-1,$$

and the proof is complete.

Using Lemma 2.3, we have

$$\begin{aligned} \epsilon(A, B, n) &= A(n-1) + B(n-1) - (A+B)(n) \\ (2.5) \qquad &= n-1-\rho(n, \overline{A}, \overline{B}) - (A+B)(n). \end{aligned}$$

However, since $n \notin A+B$, we have

$$(A+B)(n) + (\overline{A+B})(n-1) = n-1,$$

or

$$(2.6) \qquad (A+B)(n) = n-1-(\overline{A+B})(n-1).$$

Substituting (2.6) in (2.5) gives

$$\epsilon(A, B, n) = n-1-\rho(n, \overline{A}, \overline{B}) - [n-1-(\overline{A+B})(n-1)]$$

$$(2.7) \qquad = (\overline{A+B})(n-1) - \rho(n, \overline{A}, \overline{B}).$$

Now recall that we let $0 < n_1 < n_2 < \cdots < n_r = n$ be the gaps in $A+B$, and so $A+B$ has r gaps. Thus, $(\overline{A+B})(n) = r$, or, since $n \epsilon \overline{A+B}$,

$$(2.8) \qquad (\overline{A+B})(n-1) = r-1 .$$

Substituting (2.8) in (2.7) gives

$$(2.9) \qquad \epsilon(A, B, n) = r-1-\rho(n, \overline{A}, \overline{B}).$$

Hence, by substituting (2.9) in Conjecture 2.6, we complete our evolution (7, p. 26).

Conjecture 2.7. If $A+B+(A+B)^{\sim} = I_n$, and if $A+B$ has r gaps, then

$$\rho(n, \overline{A}, \overline{B}) \geq r-1 .$$

Since substitution of equivalent quantities was our only modification, clearly Conjecture 2.6 is equivalent to Conjecture 2.7.

The theorem proved in Chapter IV, Theorem 4.1, is Conjecture 2.7 under the restriction $r \leq 5$. Hence, since we have shown that Conjecture 2.2 is implied by Conjecture 2.7, Theorem 4.1 shows that Conjecture 2.2 is valid under the added restriction that the sum of some fixed two of our three sets, A, B, and C, has less than six gaps.

In the following chapter is a brief discussion of Mann's set transformation. We bring into this discussion only those properties of this transformation which are needed in the proof of Theorem 4.1, so that this work is self contained.

SUMMARY OF THE EVOLUTION

Conjecture	Statement	Arguments
2. 1	If $0 \in A_i (i = 1, \cdots, k)$, then for any positive integer n, either $\dfrac{(\sum_{i=1}^{k} A_i)(n)}{n} = 1$, or $\dfrac{(\sum_{i=1}^{k} A_i)(n)}{n} \geq \min_{\substack{1 \leq m \leq n \\ m \notin \sum_{i=1}^{k} A_i}} \dfrac{\sum_{i=1}^{k} A_i(m)}{m}$.	Generalization of Theorem 2. 1 to k sets where $k \geq 2$.
2. 2	If $0 \in A_i (i = 1, 2, 3)$, and if n is the smallest gap in $\sum_{i=1}^{3} A_i$, then $\sum_{i=1}^{3} A_i(n) \leq (\sum_{i=1}^{3} A_i)(n)$.	Conjecture 2. 1 for $k = 3$ and n the smallest gap in $\sum_{i=1}^{3} A_i$.
2. 3	If $A+B+C = I_n$, then $A(n)+B(n)+C(n)-(A+B+C)(n) \leq 0$.	Definitions 2. 1, 2. 2

2.4	If $A+B+C = I_n$, then $\epsilon(A, B, C, n) \le 0$.	Definition 2.3
2.5	If $A+B+(A+B)^{\sim} = I_n$, then $\epsilon(A, B, (A+B)^{\sim}, n) \le 0$.	Definition 2.6 Corollary 2.1
2.6	If $A+B+(A+B)^{\sim} = I_n$, then $\epsilon(A, B, n) \le 0$.	Lemma 2.1 (g) Definition 2.5
2.7	If $A+B+(A+B)^{\sim} = I_n$, and if $A+B$ has r gaps, then $\rho(n, \overline{A}, \overline{B}) \ge r-1$.	Definition 2.4 Lemma 2.3 Equation (2.9)

CHAPTER III

SYNOPSIS OF MANN'S TRANSFORMATION

It is not our intent here to scrutinize minutely Mann's transformation, but only to dwell on it briefly so that it may be employed in Chapter IV without loss of continuity in this work. We introduce only those properties which are needed for our purposes and of these properties, only those whose proofs are not immediately available in the literature (2, pp. 10-18; 5, pp. 4-6; 6, pp. 524-526) will be discussed in detail. The sets A, B, and D were defined in Chapter II. We emphasize here that in the following definition we are assuming that at least one step in the transformation is possible.

Definition 3. 1. (Mann's Transformation)

a) Let $B = B_0 = B_0^*$, $T = T_0 = T_0^* = \Phi$ and $e_{j+1} (j \geq 0)$ be any integer in $B_j = B_0 \cup B_1^* \cup \cdots \cup B_j^*$.

b) The subscript $i \in T_{j+1}^*$ if there exist $a \in A$, $d_i \in D$ $(1 \leq i \leq r-1)$, and a subscript $k \notin T_j = T_0^* \cup T_1^* \cup \cdots \cup T_j^*$ such that

$$n_k - a = e_{j+1} + d_i, \quad i \notin T_j .$$

c) Let $B^*_{j+1} = \{ e_{j+1} + d_i \mid i \in T^*_{j+1} \}$.

This defines Mann's transformation up to uniqueness. Therefore, to arrive at a uniquely determined construction, choose e_{j+1} as the least element $e \in B_j$ such that T^*_{j+1} is not empty. This process terminates at B_J, when T^*_{J+1} is empty. Thus for $j = 0$, we choose the least element $e_1 \in B$ such that the equation

$$e_1 + d_t = n_v - a \tag{3.1}$$

is solvable for some $d_t \in D$, $n_v \in \overline{A+B}$, and $a \in A$. Notice that equation (3.1) may be rewritten

$$e_1 + d_v = n_t - a,$$

so that v as well as t is in T^*_1. We then continue the construction, if possible, with the sets A, D, and $B_1 = B \cup B^*_1$, where $e_2 \in B_1$.

The following properties are readily verified.

Lemma 3.1. a) B^*_{j+1} has the same number of elements as T^*_{j+1};

b) $B_j \cap B^*_{j+1} = \Phi$;

c) $T_j \cap T^*_{j+1} = \Phi$;

d) No element in B^*_{j+1} is of the form $n-a$, where $a \epsilon A$;

e) $n \notin A+B_{j+1}$;

f) $n_t \epsilon A+B_{j+1}$ if, and only if, $t \epsilon T_{j+1}$.

Lemma 3.2. Let s be the least subscript, if any, not in T_J. If s exists and if $n_t > d_s$, then $n_t \epsilon A+B_J$ (7, p. 11).

Proof. If $n_t > d_s$, then $n_s > d_t$. Thus

$$0 < n_s - d_t < n.$$

Consequently, either $n_s - d_t \epsilon A+B \subseteq A+B_J$, or $n_s - d_t$ is a gap in $A+B$, say $n_s - d_t = n_u$, $u < s$. Since s is the least subscript not in T_J, we must have $u \epsilon T_J$. Hence, by Lemma 3.1 (f), $n_s - d_t \epsilon A+B_J$. Therefore, in either case,

$$n_s - d_t = a + b^*, \quad a \epsilon A, \quad b^* \epsilon B_J,$$

and so $t \epsilon T_J$, for if t were not in T_J our construction could be carried at least one step further. Thus $n_t \epsilon A+B_J$.

Corollary 3.1. If $0 \epsilon A$ and $0 \epsilon B$, then $n_i \neq d_s$ for all i where $i < r$.

Proof. Suppose $n_i = d_s$ for some $i < r$. Then, as the first step in Mann's construction, choose $e_1 = 0 \in B$ with

$$e_1 + d_s = n_1 - a, \quad a = 0.$$

We are immediately led to the false conclusion that $s \in T_1$. Hence $n_i \neq d_s$ for all i where $i < r$.

The proof of the following lemma is not given by Lin.

Lemma 3.3. We have $B_J \cap \overline{B} = B_1^* \cup B_2^* \cup \cdots \cup B_J^*$ and all elements in $B_J \cap \overline{B}$ are distinct values of $\overline{b}$ satisfying the equation $n = \overline{a} + \overline{b}$, where $\overline{a} \in \overline{A}$.

Proof. If $\overline{b} \in B_J \cap \overline{B}$, then

$$\overline{b} = e_{j+1} + d_i, \quad i \in T_{j+1}^*$$

for some $j \geq 0$, by the definition of B_J. Hence $\overline{b} \in B_1^* \cup \cdots \cup B_J^*$ and so $B_J \cap \overline{B} \subseteq B_1^* \cup \cdots \cup B_J^*$.

If $b^* \in B_1^* \cup \cdots \cup B_J^*$, then

$$b^* = e_{j+1} + d_i, \quad i \in T_{j+1}^*$$

for some $j \geq 0$, by the definition of B_{j+1}^*. Suppose $b^* = b$ where $b \in B$. Then

$$b = e_{j+1} + d_i = n_k - a,$$

by the definition of T^*_{j+1}, and so

$$a+b = n_k$$

This contradicts n_k being a gap in A+B. Hence $b^* = \bar{b}$ where $\bar{b} \in \bar{B}$. Consequently, $b^* \in B_J \cap \bar{B}$ and so $B^*_1 \cup \cdots \cup B^*_J \subseteq B_J \quad \bar{B}$. This proves $B_J \cap \bar{B} = B^*_1 \cup \cdots \cup B^*_J$.

By Lemma 3.1(d), no element in $B^*_1 \cup \cdots \cup B^*_J$ is of the form n-a. Hence, all elements in $B_J \cap \bar{B}$ are of the form $\bar{b} = n - \bar{a}$ where $\bar{a} \in \bar{A}$. That these elements are distinct is immediate, for in the first place the B^*_j's are disjoint by Lemma 3.1(b), and in the second place, if

$$e_{j+1} + d_i = e_{j+1} + d_k,$$

where $i \in T^*_{j+1}$ and $k \in T^*_{j+1}$, then $i = k$. Hence all elements in $B_J \cap \bar{B}$ are distinct values of $\bar{b}$ satisfying $n = \bar{a} + \bar{b}$, where $\bar{a} \in \bar{A}$.

Definition 3.2. The phrase "e corresponds to k" means "e is the least element in B_j such that $k \in T^*_{j+1}$."

This concludes our discussion of Mann's transformation.

CHAPTER IV

TWO THEOREMS

In this chapter, we prove two theorems, Theorem 4.1 which is Conjecture 2.7 under the restriction $r \leq 5$, and Theorem 1.3 which we restate as Theorem 4.2. The proofs of Theorems 4.1 and 4.2 are reorganized and clarified, and in the case of Theorem 4.1, examples are introduced. However, the underlying theory in most instances is still that of Lin (7, pp. 27-40).

We begin by turning our attention back to Conjecture 2.7. Our purpose here is to lay the background for the proof of Theorem 4.1 by proving three lemmas pertaining to Conjecture 2.7 now that Mann's set transformation is available to us. We again state

Conjecture 2.7. If $A+B+(A+B)^{\sim} = I_n$, and if $A+B$ has r gaps, then

$$\rho(n, \bar{A}, \bar{B}) \geq r-1 .$$

Notice that this conjecture is trivially true for $r = 1$, since $\rho(n, \bar{A}, \bar{B}) \geq 0$ is always satisfied. Therefore, we assume that $r > 1$ in the argument to follow.

The necessity of Mann's transformation lies in Lemma 3.3, for we construct $r-1$ elements $\bar{b} \in \bar{B}$ that are distinct solutions of the equation $n = \bar{a}+\bar{b}$ where $\bar{a} \in \bar{A}$, some of which are in $B_J \cap \bar{B}$.

Lemma 4.1. The set T_J is not empty; in particular, $r-1 \in T_J$.

Note that the hypothesis of Conjecture 2.7 is assumed in Lemmas 4.1, 4.2 and 4.3.

Proof. By hypothesis, $A+B+(A+B)^{\sim} = I_n$. Thus, since $(A+B)^{\sim} = D$, every gap n_i $(1 \leq i \leq r-1)$ in $A+B$ can be expressed as a sum

$$(4.1) \qquad a+b+d_t = n_i ,$$

where $a \in A$, $b \in B$, $d_t \in D$, $d_t \neq 0$, or

$$n_i - a = b+d_t .$$

Therefore, at least one step in Mann's construction is possible. Let us emphasize here that we can say no more than this. That is, since we choose $b \in B$ to be the smallest b such that

$$n_i - a = b+d_t ,$$

and since the T_j^*'s are disjoint by Lemma 3.1(c), it is entirely possible that the construction can be carried no further. However, we have shown that at least T_1^* is not empty, and hence, T_J is not empty. This concludes the first part of the proof.

If no subscript i where $1 \leq i \leq r-1$ is missing from T_J, then, of course, $r-1 \in T_J$. If, on the other hand, there exist subscripts missing from T_J, let s be the least such subscript. By (4.1), there exists a subscript v $(1 \leq v \leq r-1)$ such that

$$a+b+d_v = n_1 ,$$

and so

$$d_v \leq n_1 \leq n_s .$$

By the definition of the set D, we have

$$d_{r-1} \leq d_i \quad (1 \leq i \leq r-1) .$$

Hence,

$$d_{r-1} \leq d_v \leq n_1 \leq n_s ,$$

and so

$$n_{r-1} > d_s ,$$

since equality cannot hold by Corollary 3.1. Consequently, by Lemma 3.2, $n_{r-1} \in A+B_J$, and therefore, $r-1 \in T_J$ by Lemma 3.1 (f).

Lemma 4.2. Conjecture 2.7 is true if at most one subscript

i $(1 \leq i \leq r-1)$ is not in T_J.

Proof. If no subscript i $(1 \leq i \leq r-1)$ is missing from T_J, then the equation $n = \bar{a}+\bar{b}$ has $r-1$ distinct solutions where $\bar{b} \epsilon B_J \cap \bar{B}$ by Lemma 3.3, and Conjecture 2.7 follows.

Assume there exists one subscript $s \notin T_J$. Again by Lemma 3.3, we then have $r-2$ distinct solutions of $n = \bar{a}+\bar{b}$ where $\bar{b} \epsilon B_J \cap \bar{B}$. We want to show that we also have at least one more solution $\bar{b} \epsilon \bar{B}$, but $\bar{b} \notin B_J \cap \bar{B}$. By the hypothesis of Conjecture 2.7, we know that there exists a subscript t $(1 \leq t \leq r-1)$ such that

$$a+b+d_t = n_s .$$

Then $b+d_t \notin B_J$, for suppose $b+d_t \epsilon B_J$. Then, since we also have $a+b+d_s = n_t$, we would have $b+d_s \epsilon B_J$, and so $s \epsilon T_J$, contrary to our assumption. Consequently, we assert that $\bar{b} = b+d_t$ is another distinct solution of $n = \bar{a}+\bar{b}$ where $\bar{a} = a+d_s$. To show this, suppose that $b' = b+d_t$ where $b' \epsilon B$. Then also $b' = n_s - a$, and so $a+b' = n_s$, contradicting n_s being a gap in $A+B$. Suppose further that $b+d_t = n-a'$, where $a' \epsilon A$. Then $a'+b = n_t$ and we reach the same contradiction. Hence $\bar{b} = b+d_t$ is a solution of $n = \bar{a}+\bar{b}$ with $\bar{a} = a+d_s$. Distinctness follows from the fact that $b+d_t \notin B_J$, and hence, $b+d_t \notin B_J \cap \bar{B}$, and so $\rho(n, \bar{A}, \bar{B}) \geq r-1$.

Lemma 4.3. Suppose there are subscripts not in T_J. Let s be the least subscript, and t any subscript, not in T_J. Let

$$a + b + d_k = n_t ,$$

where $a \epsilon A$, $b \epsilon B$, $d_k \epsilon D$, $d_k \neq 0$. Then

a) $k \epsilon T_J$ and $k > s$;

b) if e corresponds to k (Definition 3.2), then $b > e$;

c) if $t = s$, then all subscripts $k, k+1, \cdots, r-1$ are in T_J.

Proof. a) Suppose $k \notin T_J$. Then, since $t \notin T_J$ and

$$a + b + d_k = n_t ,$$

T_{J+1} is not empty. This contradicts the definition of T_J. Hence $k \epsilon T_J$. Since $t \notin T_J$, then $n_t \notin A+B_J$ by Lemma 3.1(f). Since $n_t \notin A+B_J$, then $n_t < d_s$ by Lemma 3.2 and Corollary 3.1, and so

$$d_k \leq n_t < d_s .$$

Hence $k > s$ by the definition of the set D.

b) Since $k \in T_J$ by (a), we let e correspond to k. Then $e+d_k \in B_J$. Suppose $b = e$. Then

$$a + e + d_k = n_t ,$$

and so $t \in T_J$, contradicting our hypothesis $t \notin T_J$. Hence $b \neq e$. Suppose $b < e$. This contradicts our hypothesis that e corresponds to k by Definition 3.2. Hence $b \nless e$, and so $b > e$ follows.

c) Let the subscript i satisfy $k \leq i \leq r-1$. Then, by the definition of the set D, $d_i \leq d_k$. Suppose $t = s$. Then, by hypothesis,

$$a + b + d_k = n_s ,$$

and so $d_k < n_s$, strict inequality holding by Corollary 3.1. Hence,

$$d_i \leq d_k < n_s ,$$

and so

$$n_i > d_s .$$

Thus, $n_i \in A+B_J$ by Lemma 3.2, and therefore, $i \in T_J$ by Lemma 3.1(f). This proves (c), thus completing the proof of Lemma 4.3.

We now state Conjecture 2.7 under the restriction $r \leq 5$, and proceed with its proof.

Theorem 4.1. If $A+B+(A+B)^{\sim} = I_n$, and if $A+B$ has $r \leq 5$ gaps, then

$$\rho(n, \overline{A}, \overline{B}) \geq r-1.$$

Proof. We divide the proof into four parts corresponding to $r = 2, 3, 4, 5$ respectively.

i) Suppose $r=2$. Then $1 = r-1 \in T_J$ by Lemma 4.1. Hence no subscripts are missing from T_J and Theorem 4.1 follows by Lemma 4.2.

ii) Suppose $r = 3$. Then $2 = r-1 \in T_J$ by Lemma 4.1. Hence there is at most one subscript missing from T_J and Theorem 4.1 again is valid by Lemma 4.2.

iii) Suppose $r = 4$. Then, again by Lemma 4.1, $3 = r-1 \in T_J$. If at most one subscript is missing from T_J, then Theorem 4.1 again is valid by Lemma 4.2. Hence, assume the remaining two subscripts 1 and 2 are not in T_J. Thus, if s is the least subscript not in T_J, then $s = 1$. Since $r = 4$, then $n_1 \in I_n$, and so by hypothesis, there exists a subscript k such that

$$a+b+d_k = n_1 .$$

By Lemma 4.3(a), we have $k \in T_J$ and $k > s$. By Lemma 4.3(c), all the subscripts greater than or equal to k are in T_J. Hence, since $2 \notin T_J$ and $k > s$, we have $k = 3$. Then

$$a+b+d_3 = n_1,$$

or equivalently,

$$a+b+d_1 = n_3 ,$$

where $b+d_3$ and $b+d_1$ are values of $\bar{b} \in \bar{B}$ satisfying $n = \bar{a}+\bar{b}$, as shown in the proof of Lemma 4.2. However, $b+d_3$ and $b+d_1$ are not in $B_J \cap \bar{B}$ since the subscript $s = 1 \notin T_J$. Let e correspond to 3. Then $e+d_3 \in B_J \cap \bar{B}$, and $e+d_3$ is a value of $\bar{b}$ satisfying $n = \bar{a}+\bar{b}$ by Lemma 3.3. Also, we have that $b > e$ by Lemma 4.3(b). Hence

$$e + d_3 < b+d_3 < b+d_1,$$

and so we have at least three distinct values of $\bar{b} \in \bar{B}$ satisfying $n = \bar{a} + \bar{b}$, namely $e+d_3$, $b+d_3$, and $b+d_1$. This establishes our theorem for $r = 4$.

iv) Let $r = 5$. Then $4 = r-1 \in T_J$ again by Lemma 4.1. Hence there are at most three subscripts missing from T_J. If not more than one of these three subscripts is missing from T_J, Theorem 4.1 follows from Lemma 4.2 again. Hence we assume there are at least two subscripts missing from T_J. Let s be the least, and t be any subscript not in T_J. We have two cases to consider, $s = 1$ and $s = 2$.

Suppose $s = 2$. Let $t \neq s$. Then $t = 3$ and so $1, 4 \in T_J$. Furthermore, since $n_2 \in I_n$ and $n_3 \in I_n$, we have by hypothesis that there exist subscripts i and k such that

$$a+b+d_i = n_2,$$

$$a'+b'+d_k = n_3 .$$

Consequently, by Lemma 4.3(a), $i \in T_J$, $k \in T_J$, and $i > s$, $k > s$. Hence $i = k = 4$, the only subscript larger than s in T_J, and so

$$a+b+d_4 = n_2,$$

$$a'+b'+d_4 = n_3 .$$

Again we have that

$$b+d_4,\quad b+d_2,\quad b'+d_4,\quad b'+d_3$$

are not in $B_J \cap \overline{B}$ since the subscripts 2 and 3 are not in T_J. However, as before these four elements are values of $\overline{b} \in \overline{B}$

satisfying $n = \bar{a}+\bar{b}$, but are not necessarily distinct. Let e and e' correspond to the subscripts 1 and 4 respectively. Then

$$e+d_1, \quad e'+d_4$$

are in $B_J \cap \bar{B}$, and therefore, are two distinct values of $\bar{b}$ satisfying $n = \bar{a}+\bar{b}$ by Lemma 3.3. Furthermore, by Lemma 4.3(b), we have $b' > e'$ and $b > e'$, and so

$$b+d_4 > e'+d_4 ,$$

$$b'+d_4 > e'+d_4 .$$

Consequently, if $b \neq b'$, then our four distinct solutions are

$$e+d_1, \ e'+d_4, \ b+d_4, \ b'+d_4 ,$$

since $e+d_1 \neq b+d_4$ and $e+d_1 \neq b'+d_4$. To show this, we note that $e+d_1 = b+d_4$ or $e+d_1 = b'+d_4$ implies that

$$a+e+d_1 = n_2$$

or

$$a'+e+d_1 = n_3,$$

and so either the subscript $2 \epsilon T_J$ or the subscript $3 \epsilon T_J$, or both. Now suppose $b = b'$. Then, we have

$$e'+d_4 < b+d_4 < b'+d_3 < b+d_2,$$

and so our four values of $\bar{b} \epsilon \bar{B}$ satisfying $n = \bar{a}+\bar{b}$ are

$$e'+d_4, \quad b+d_4, \quad b'+d_3, \quad b+d_2 .$$

Observe here that the solution $\bar{b} = e+d_1$ is not available to us, since we do not know that $e+d_1 \neq b+d_2$ or that $e+d_1 \neq b'+d_3$. This completes the proof for $r = 5$, $s = 2$.

For $s = 1$, we argue as follows. By hypothesis, there exist subscripts i and k such that

$$a+b+d_i = n_1 ,$$

$$a'+b'+d_k = n_t, \quad t \epsilon \{2, 3\},$$

and so, by Lemma 4.3(a), $i \epsilon T_J$, $k \epsilon T_J$ and $i > s$, $k > s$. Suppose $i = 2$. Then, by Lemma 4.3(c), all the subscripts greater than 2 are in T_J, contradicting our assumption that at least two subscripts are missing from T_J. Hence, either $i = 3$ or $i = 4$.

Suppose $i = 3$. Then $3, 4 \epsilon T_J$ and so $t = 2$. Hence, $k = 3$ or $k = 4$. Let e and e' correspond to the subscripts 3 and 4 respectively. Then

$$e+d_3, \quad e'+d_4$$

are in $B_J \cap \overline{B}$, and therefore, we have two distinct solutions of $n = \overline{a}+\overline{b}$ by Lemma 3.3.

Let $k = 3$. Then we have

$$a+b+d_3 = n_1 ,$$

$$a'+b'+d_3 = n_2 ,$$

where, as before,

$$b+d_3, \quad b+d_1, \quad b'+d_3, \quad b'+d_2$$

are values of $\overline{b} \in \overline{B}$, but not in $B_J \cap \overline{B}$, satisfying $n = \overline{a}+\overline{b}$. Also, by Lemma 4.3(b), we have $b' > e$, $b > e$, and so

$$b'+d_3 > e+d_3 ,$$

$$b+d_3 > e+d_3 .$$

Hence, if $b \neq b'$, our four distinct values of $\overline{b} \in \overline{B}$ satisfying $n = \overline{a}+\overline{b}$ are

$$e+d_3, \quad e'+d_4, \quad b+d_3, \quad b'+d_3,$$

for again $e'+d_4 \neq b+d_3$ and $e'+d_4 \neq b'+d_3$. On the other hand, if $b = b'$, then

$$e+d_3 < b+d_3 < b'+d_2 < b+d_1,$$

and these four values of $\overline{b}$ are our solutions. Again we observe that $\overline{b} = e'+d_4$ is not usable since we do not know that $e'+d_4 \neq b+d_1$ or that $e'+d_4 \neq b'+d_2$.

Now let $k = 4$. Then

$$a+b+d_3 = n_1,$$

$$a'+b'+d_4 = n_2,$$

with

$$b+d_3 > e+d_3,$$

$$b'+d_4 > e'+d_4.$$

Here, we have that

$$b+d_3, \ b+d_1, \ b'+d_4, \ b'+d_2$$

are not in $B_J \cap \overline{B}$. Hence, if $e+d_3 < e'+d_4$, (recall that $e+d_3 \neq e'+d_4$), then

$$e+d_3 < e'+d_4 < b'+d_4 < b'+d_2,$$

or if $e'+d_4 < e+d_3$, then

$$e'+d_4 < e+d_3 < b+d_3 < b+d_1,$$

and so, in either case, we have at least four distinct solutions of $n = \bar{a}+\bar{b}$. This completes the proof for $s = 1$, $i = 3$.

Suppose $i = 4$. Then we have

$$a+b+d_4 = n_1$$

$$a'+b'+d_k = n_t, \quad t \in \{2,3\}.$$

Let e and e' correspond to 4 and k respectively. Then

$$e+d_4, \quad e'+d_k$$

are in $B_J \cap \bar{B}$ and satisfy $n = \bar{a}+\bar{b}$ again by Lemma 3.3. Also, again by Lemma 4.3(b), $b > e$ and $b' > e'$. We have three cases to consider, $k = 2,3,4$.

For $k = 2$, we have $2,4 \in T_J$ and so $t = 3$. Thus, we have

$$a+b+d_4 = n_1,$$

$$a'+b'+d_2 = n_3,$$

and as before,

$$b+d_4, \quad b+d_1, \ b'+d_2, \ b'+d_3$$

are in $\bar{B}$, and of the form $n-\bar{a}$, but are not in $B_J \cap \bar{B}$. Hence, we have

a) $e'+d_2 < e+d_4 < b+d_4 < b+d_1$, if $e'+d_2 < e+d_4$;

b) $e+d_4$, $e'+d_2$, $b+d_4$, $b'+d_2$, if $e+d_4 < e'+d_2$ and $b+d_4 \neq b'+d_2$;

c) $e+d_4 < e'+d_2 < b'+d_2 < b+d_1$, if $e+d_4 < e'+d_2$ and $b+d_4 = b'+d_2$, since if $b+d_4 = b'+d_2$, then $b-b' = d_2-d_4 > 0$, or $b > b'$. Hence, in all three cases, we have four distinct solutions of $n = \bar{a}+\bar{b}$.

For $k = 3$, we have $3, 4 \epsilon T_J$ and so $t = 2$. We need only interchange the subscripts 2 and 3 in the preceding argument for $k = 2$, and the result follows.

For $k = 4$, we have $t = 2$ or $t = 3$, $e = e'$ and both $b > e$ and $b' > e$. Thus, we have

$$a+b+d_4 = n_1,$$

$$a'+b'+d_4 = n_t,$$

and

$$b+d_4, \quad b+d_1, \quad b'+d_4, \quad b'+d_t$$

are not in $B_J \cap \bar{B}$. In addition, we have that $e+d_4 \epsilon B_J \cap \bar{B}$. From these five values of $\bar{b}$ satisfying $n = \bar{a}+\bar{b}$, we find four distinct ones as follows:

a) $e+d_4 < b+d_4 < b'+d_t < b+d_1$, if $b = b'$;

b) $e+d_4 < b+d_4 < b'+d_4 < b'+d_t$, if $b < b'$;

c) $e+d_4 < b'+d_4 < b+d_4 < b+d_1$, if $b > b'$.

This concludes the proof for $r = 5$, and therefore, Theorem 4.1 is established.

The example following Conjecture 2.2 in Chapter II, also serves to show that Theorem 4.1 cannot be improved. In this example, $n = 15$, $A+B+(A+B)^{\sim} = I_n$ and $r = 6$, since $A+B = (A_1+A_2) \cap N = \{0, 1, 2, 3, 8, 9, 10, 12, 13, 14\}$. However, $A(n)+B(n)+(A+B)^{\sim}(n) > n-1$.

At this point, let us emphasize the significance of Theorem 4.1 relative to the proof of Theorem 1.3. Theorem 4.1 implies the following statement.

If $A+B+C = I_n$ and if the sum of some fixed two of these three sets has less than six gaps, then

$$A(n)+B(n)+C(n) \leq n-1 .$$

Therefore, we may make the assumption in the proof of Theorem 1.3 that none of the sets $A+B$, $A+C$, and $B+C$ have less than six gaps. We now restate Theorem 1.3 as Theorem 4.2, and proceed with its proof.

Theorem 4.2. If $A+B+C = I_n$ and $0 < n < 15$, then

$$A(n)+B(n)+C(n) \leq n-1 .$$

Proof. By Corollary 2.1, we may assume that

$$A = (B+C)^{\sim}, \quad B = (A+C)^{\sim}, \quad C = (A+B)^{\sim} ,$$

thereby maximizing $A(n)+B(n)+C(n)$. By Theorem 4.1, we may assume that the sets $A+B$, $A+C$, and $B+C$ all have more than five gaps. These two assumptions imply that

$$A(n) \geq 5, \quad B(n) \geq 5, \quad C(n) \geq 5.$$

The proof of Theorem 4.2 consists in showing that $A(n) \geq 5$, $B(n) \geq 5$, $C(n) \geq 5$ and our hypotheses, $A+B+C = I_n$ and $0 < n < 15$, are contradictory.

By Lemma 2.1(g), we have

$$(A+B)(n) + (A+B)^{\sim}(n) = n-1.$$

Combining this result with our assumptions, $A(n) \geq 5$, $B(n) \geq 5$, and $C(n) \geq 5$, we conclude that

$$5 \leq B(n) \leq (A+B)(n) = n-1-C(n) \leq n-6. \tag{4.2}$$

From (4.2) and the hypothesis $0 < n < 15$, we have two results:

$$(4.3) \qquad (A+B)(n) \leq 8,$$

with similar inequalities for $A+C$ and $B+C$, and

$$n \geq 11.$$

This last result establishes Theorem 4.2 for $n < 11$.

By hypothesis, $1 \epsilon A+B+C$ and must, therefore, be in one of our three sets. Without loss of generality, we assume that $1 \epsilon A$. Let

$$A = \{0, 1, a_2, \cdots, a_h\}, \quad h \geq 5,$$

$$B = \{0, b_1, \cdots, b_k\}, \quad k \geq 5.$$

Then $A+B$ contains at least the following numbers:

$$(4.4) \qquad 1 \leq b_1 < 1+b_1 \leq b_2 < 1+b_2 \leq \cdots \leq b_k < 1+b_k,$$

where $1+b_k \leq n-1$, or $b_k \leq n-2$. Hence, since $1+b_k \notin B$, we can sharpen (4.2) to

$$6 \leq B(n) + 1 \leq (A+B)(n) \leq n-6,$$

and so

$$n \geq 12.$$

This completes the proof for $n < 12$.

Suppose that equality holds for all the signs "$\leq$" of (4.4).

Then $b_m = m \ (1 \leq m \leq k)$, and so

$$B = \{0, 1, \cdots, k\},$$

$$A + B \supseteq \{0, 1, \cdots, k+1\}.$$

Furthermore, we assert that $a_5+k < n$, and therefore, that the elements $a_5+m \ (1 \leq m \leq k)$ are also in $A+B$. To show this, suppose that $a_5+k \geq n$. Then $k \geq n-a_5 > 0$ and so $n-a_5 \in B$. Thus, $n = a_5+(n-a_5) \in A+B$ contrary to our hypothesis. Hence $a_5+k < n$. But $a_5+k < n$ and $a_5 \geq 5$ imply that $A+B$ contains not only the elements $0, 1, \cdots, k+1$, but also the elements $a_5+k-3 \ (a_5+k-3 \geq k+2)$, a_5+k-2, a_5+k-1, and $a_5+k \ (a_5+k < n)$. Hence,

$$(A+B)(n) \geq k+5 \geq 10,$$

contradicting (4.3), $(A+B)(n) \leq 8$, and so equality cannot hold for all the signs "$\leq$" in (4.4). We conclude that at least one of the inequalities "$\leq$" in (4.4) must be strict inequality. Hence, we can again sharpen (4.2) to

(4.5) $$7 \leq B(n)+2 \leq (A+B)(n) = n-1-C(n) \leq n-6,$$

and so

$$n \geq 13.$$

Consequently, Theorem 4.2 is valid for $0 < n < 13$.

Furthermore from (4. 5) and our assumptions $B(n) \geq 5$ and $C(n) \geq 5$, we establish the following inequalities:

$$(4.6) \qquad \begin{cases} 7 \leq (A+B)(n) \leq 8 , \\ 10 \leq B(n)+C(n) \leq 11 . \end{cases}$$

We have two cases to consider, $(A+B)(n) = 7$ and $(A+B)(n) = 8$. These two cases incorporate $n = 13$ and $n = 14$. We observe here that (4. 6) and our assumptions, $B(n) \geq 5$ and $C(n) \geq 5$, give that either $B(n) = 5$ or $C(n) = 5$. Hence, without loss of generality, we assume that $B(n) = 5$, or equivalently $k = 5$, in the argument to follow. However, before we proceed to this argument, let us display here all the assumptions upon which this argument is based. They are:

1) $A+B+C = I_n$;

2) $12 < n < 15$;

3) $A = (B+C)^{\sim}$, $B = (A+C)^{\sim}$, $C = (A+B)^{\sim}$;

4) $A(n) \geq 5$, $B(n) = 5$, $5 \leq C(n) \leq 6$;

5) $1 \in A$; and

6) at least one of the inequalities "$\leq$" in (4. 4) must be strict inequality.

Case I. Let $(A+B)(n) = 7$ and $13 \leq n \leq 14$. Since $(A+B)(n) = 7$, we have that strict inequality holds for one of the signs "$\leq$" in (4.4). Thus, there exists a subscript s such that

$$(4.7) \qquad 0 \leq b_s < 1+b_s < b_{s+1} \leq b_k = b_5 ,$$

where $0 \leq s < 5$ and $b_0 = 0$. Hence, since $(A+B)(n) = 7$, there are exactly two elements in $A+B$ which are not in B, namely, $1+b_s$ and $1+b_5$. We also have

$$b_m = m \ (m = 1, \cdots, s),$$

$$b_{s+j} = b_{s+1} + j-1 \ (j = 1, \cdots, 5-s),$$

and so

$$B = \{0, 1, \cdots, s, b_{s+1}, b_{s+1}+1, \cdots, b_{s+1}+4-s\},$$

$$A+B = \{0, 1, \cdots, s+1, b_{s+1}, \cdots, b_{s+1}+5-s\},$$

since $b_{s+1}+5-s = 1+b_5$ is the largest element in $A+B$. Let

$$t = b_{s+1} - b_s .$$

Then, since $b_{s+1} > b_s+1$, we have $t > 1$. We also have

$$b_{s+1} = b_s + t = s+t > s+1,$$

$$b_{s+2} = b_{s+1} + 1 = s+t+1,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$b_{s+j=5} = b_{s+j} + j - 1 = s+t+(5-s)-1 = 4+t .$$

Consequently, B and A+B can be written in the form

$$B = \{0, 1, \cdots, s, s+t, \cdots, 4+t\},$$

$$A+B = \{0, 1, \cdots, s+1, s+t, \cdots, 5+t\}.$$

Let us now examine the elements in A+B a little more closely. For instance, we know that for $i = 2, \cdots, 5$ and $m = 1, \cdots, 5$,

$$a_i + b_m \leq 1 + b_5 \quad \text{if} \quad a_i + b_m \in A+B;$$

$$a_i + b_m > n \quad \text{if} \quad a_i + b_m \notin A+B;$$

$$a_i + b_0 = a_i \in A+B;$$

$$a_i + b_5 > n,$$

for $a_i + b_5 \geq 2 + b_5 \notin A+B$. Therefore, there exists a subscript v, where $0 \leq v < 5$, such that

$$a_i+b_v \le 1+b_5 < a_i+b_{v+1} .$$

Thus,

$$(4.8) \qquad \begin{cases} a_i+b_v \le 1+b_5 \le n-1 \\ a_i+b_{v+1} > n, \end{cases}$$

and so

$$a_i+b_v+1 \le n < a_i+b_{v+1}$$

or

$$b_v+1 < b_{v+1} .$$

Hence $v=s$ by (4.7), and so by (4.8), we have

$$(4.9) \qquad \begin{cases} a_i+s = a_i+b_s \le 1+b_5 = 5+t, \\ a_i+s+t = a_i+b_{s+1} > n , \end{cases}$$

and so

$$(4.10) \qquad n-(s+t) < a_i \le 5+t-s \quad (i = 2, 3, 4, 5).$$

We note here that (4.9) gives

$$a_2+s+3 \le a_5+s \le 5+t \le n-1,$$

$$a_2+s+t > n, \quad \text{or} \quad a_2+s+3 > n-t+3,$$

and so

(4.11) $$n-t+3 < a_2+s+3 \leq 5+t \leq n-1 .$$

Hence, since $13 \leq n \leq 14$, we have

$$16-t < 5+t \leq 13,$$

and so

(4.12) $$6 \leq t \leq 8.$$

Furthermore, we assert that $a_i \geq s+t$ $(i = 2, 3, 4, 5)$, for suppose $a_i < s+t$. Then, since $a_i \in A+B$, we must have

$$2 \leq a_i \leq s+1,$$

and so $s \geq 1$. Then it follows that

$$1 \leq s+2-a_i \leq s,$$

and so, by the construction of the set B, we have $s+2-a_i \in B$. Hence

$$s+2 = a_i + (s+2-a_i) \in A+B,$$

and, by (4.12), $s+2 < s+t$, contradicting our assumption that no number in $A+B$ lies between $s+1$ and $s+t$. Consequently, $a_i \geq s+t$ $(i = 2, 3, 4, 5)$.

Hence, this result and (4.10) give

(4.13) $$s+t \le a_i \le 5+t-s \quad (i = 2, 3, 4, 5).$$

We now have the desired contradiction for the case $(A+B)(n) = 7$, for suppose $b \in B$ and $b \le s$. Then $a_2+b \le 5+t$ by (4.9), and therefore, $a_2+b \in A+B$. Consequently,

$$s+t \le a_2+b < a_5+s \le 5+t,$$

and so

$$s+t \le a_2+b \le 4+t.$$

However, by assumption, B contains all numbers y such that $s+t \le y \le 4+t$. Hence, $a_2+b \in B$, and so

$$\{0, a_2\} + B = B .$$

Therefore, $B + \widetilde{B} \ne I_n$ by Theorem 2.2, contradicting our hypothesis, since by assumption $B+\widetilde{B} = B+A+C = I_n$. This completes the proof of Theorem 4.2 for $n = 13$ or $n = 14$, and $(A+B)(n) = 7$.

Before we proceed to the case $(A+B)(n) = 8$, let us make some further observations about the preceding proof, and look at some examples. First, we observe that (4.13) gives

$$s+t \le a_2 \le 2+t-s,$$

since $a_2 \le a_5-3$, and so we have the added restriction that

$$0 \le s \le 1 .$$

We could, from this point, proceed with a proof by considering all the possible cases reamining under the restrictions $0 \leq s \leq 1$ and $6 \leq t \leq 8$ (4.12). However, this is not our purpose. We wish merely to get some insight into the mechanics of the preceding proof. Therefore, suppose $s = 1$ and $t = 6$. Then, from $n = 13$, we have

$$B = \{0, 1, 7, 8, 9, 10\},$$

$$B + \{0, 8\} = B,$$

and so $B + \tilde{B} \neq I_n$. For $n = 14$, let $s = 0$, $t = 8$. Then

$$B = \{0, 8, 9, 10, 11, 12\},$$

$$B + \{0, 8\} = B,$$

and again we conclude that $B + \tilde{B} \neq I_n$ by Theorem 2.2. We observe here that we could compute the set $\tilde{B}$ directly to show that $B + \tilde{B} \neq I_n$, and thereby bypass Theorem 2.2.

Let us now consider the last case in the proof of Theorem 4.2.

Case II. Let $(A+B)(n) = 8$. Then we have

$$5 \leq C(n) = (n-1) - (A+B)(n) = (n-1) - 8 = n - 9,$$

and so $n = 14$. With $(A+B)(n) = 8$, we have two possibilities arising from (4.4). If only one of the signs "$\leq$" in (4.4) is strict inequality, then, as in the case $(A+B)(n) = 7$, we have

$$B = \{0, 1, \cdots, s, s+t, \cdots, 4+t\},$$

and so either

$$A+B = \{0, 1, \cdots, s+1, x, s+t, \cdots, 5+t\},$$

or

$$A+B = \{0, 1, \cdots, s+1, s+t, \cdots, 5+t, x\},$$

where the extra element x arises from our assumption $(A+B)(n) = 8$.

On the other hand, if two of the "$\leq$" signs of (4.4) are strict inequality, then we have that there exists a subscript u, where $0 \leq u < 5$ and $u \neq s$, such that

$$0 \leq b_u < 1+b_u < b_{u+1} \leq b_k .$$

Hence, instead of two, we have three elements in $A+B$ which are not in B, namely, $1+b_s$, $1+b_u$, and $1+b_5$, and so $(A+B)(n) = B(n) + 3 = 8$, satisfying our assumption. Clearly we cannot have more than two of the signs "$\leq$" in (4.4) being strict inequality, for then $(A+B)(n) > 8$.

Therefore, we have three cases to consider:

i) one of the signs "$\leq$" in (4.4) is strict inequality and $1+b_s < x < b_{s+1}$;

ii) one of the signs "$\leq$" in (4.4) is strict inequality and $1+b_5 < x < n$; and

iii) two of the signs "$\leq$" are strict inequalities and so $x = 1+b_u$ $(0 \leq u < 5,\ u \neq s)$. We treat these three cases separately.

i) Suppose one of the signs "$\leq$" in (4.4) is strict inequality, and that the extra element arising from our assumption $(A+B)(n) = 8$ satisfies $1+b_s < x < b_{s+1}$. Then we have

$$B = \{0, 1, \cdots, s, s+t, \cdots, 4+t\},$$

$$A+B = \{0, 1, \cdots, s+1, x, s+t, \cdots, 5+t\}.$$

Let $x = a+b$ where $a \in A$ and $b \in B$. Then since $s+1 < x < s+t$, we have $b < s+t$ and so $b \leq s$. Consequently, $a > s+1-b \geq s+1-s > 1$, and so $a \geq a_2$.

Since the set B is the same as in the case $(A+B)(n) = 7$ and since $1+b_5$ is still the largest element in $A+B$, we have as before that there exists a subscript v, where $0 \leq v < 5$, such that

$$a_i+b_v \leq 1+b_5 < a_i+b_{v+1} \quad (i \geq 2).$$

We conclude in the same manner that $v = s$, and so

$$(4.14)\qquad \begin{cases} a_i + s = a_i + b_s \le 1 + b_5 = 5+t\,, \\ a_i + s + t = a_i + b_{s+1} > n, \\ n-(s+t) < a_i \le 5+t-s \quad (i \ge 2). \end{cases}$$

Letting $i = 5$ in the first, and $i = 2$ in the second, inequality (4.14), and noting that $a_5 \ge a_2 + 3$, we have

$$n-t+3 \le a_2+s+3 \le a_5+s \le 5+t \le n-1,$$

and so

$$(4.15)\qquad 7 \le t \le 8\,.$$

Then, using (4.14) and (4.15), we have

$$(4.16)\qquad a_i \ge 3 \ (i \ge 2).$$

Since $a_2 \le a \le x < b_{s+1}$, it follows from (4.14) that

$$2b_{s+1} > a+b_{s+1} > n,$$

and so $b_{s+1} \ge 8$. This result, together with (4.15), eliminates the case $t = 7$, $b_s = 0$. Consequently, if $b_s = 0$, then $t = 8$,

$$B = \{0, 8, 9, 10, 11, 12\},$$

and so

$$B + \{0, 8\} = B.$$

Hence, $B+A+C = B + \widetilde{B} \neq I_n$ by Theorem 2.2, contradicting our hypothesis. Therefore, $b_s \geq 1$ and $1 \in B$.

Furthermore, we assert that $a_i \geq b_{s+1}-1$, where $i \geq 2$, for suppose first that $1+b_s < a_i \leq b_{s+1}-2$. Then not only a_i but a_i+1 would lie between $1+b_s$ and b_{s+1} and both be in $A+B$, contrary to our assumption, for since $1 \in B$, we have $1+b_s < a_i < a_i+1 < b_{s+1}$. Suppose now that $a_i \leq 1+b_s$. Then, since $a_i \geq 3$ by (4.16), we have

$$-(1+b_s) \leq -a_i \leq -3,$$

and so adding b_s+2, we obtain

$$1 \leq b_s+2-a_i \leq b_s-1 ,$$

$$2 \leq b_s+3-a_i \leq b_s .$$

Hence, both b_s+2-a_i and b_s+3-a_i are in B, and so both

$$b_s+2 = a_i+(b_s+2-a_i)$$

and

$$b_s+3 = a_i+(b_s+3-a_i)$$

are in $A+B$, a contradiction since by assumption x is the only member of $A+B$ between b_s+1 and b_{s+1}. Consequently, $a_i \geq b_{s+1}-1$.

Hence we have the following results.

$$a \geq a_2 ,$$

$$a_i \geq b_{s+1} - 1 \quad (i \geq 2),$$

$$x = a+b < b_{s+1} ,$$

$$x > b_s + 1,$$

and so we conclude that

$$(4.17) \qquad \begin{cases} x = a_2 = b_{s+1} - 1 , \\ a_i \geq b_{s+1} \quad (i \geq 3). \end{cases}$$

Combining (4. 14) with (4. 17) we obtain

$$s+t+2 \leq a_5 \leq 5+t-s ,$$

since $a_5 \geq a_3 + 2$, and so $s \leq 1$. Consequently, $s = 1$ and $t = 7, 8$ respectively are the only remaining cases to consider. Therefore let $s = 1$. Then

$$B = \{0, 1, 8, 9, 10, 11\} \quad (t = 7),$$

and so

$$B + \{0, 8\} = B,$$

and

$$B = \{0, 1, 9, 10, 11, 12\} \ (t = 8),$$

and so

$$B + \{0, 9\} = B.$$

In either case, we reach the same contradiction $B+A+C = B+\widetilde{B} \neq I_n$, which concludes the proof for (i).

ii) Suppose of the signs "$\leq$" in (4.4) one is strict inequality and $1+b_5 < x < n$. Then again we have

$$B = \{0, 1, \cdots, s, s+t, \cdots, 4+t\},$$

and so

$$A+B = \{0, 1, \cdots, s+1, s+t, \cdots, 5+t, x\}.$$

Furthermore, if $x = a+b$ where $a \in A$, $b \in B$, then $1+b \notin B$, for suppose $1+b \in B$. Then $a+(1+b) = 1+x > n$, since x is the largest number in $A+B$. But since $x \in A+B$, we have $x \leq n-1$, and so $x+1 \leq n$. Hence, since both inequalities cannot be satisfied at the same time, we conclude that $1+b \notin B$. Thus it follows, by the construction of B, that $b = b_s$ or $b = b_5$. We treat these two cases separately.

Suppose $b = b_s$. Suppose also that $x = 2+b_5$. Then $a+b_s = 2+b_5$, and so $a+b_{s+1} > n$. Consequently, it follows that

$$a+s = 6+t,$$

$$a+s+t > n,$$

and so

$$6+2t > 14,$$

or $t \geq 5$. Furthermore, since $b_5+2 \leq n-1$, we have $4+t = b_5 \leq n-3$, and so $t \geq 7$. Hence,

(4.18) $$5 \leq t \leq 7.$$

We know that $a_i \leq a$, where $i \geq 2$, for suppose first that $a_2 > a$. Then $a \leq a_1 = 1$ and $x = a+b \in \{b, 1+b\}$, a contradiction. Next suppose that $a_i > a$ where $i \geq 3$. Then, since by assumption $a+s = 2+b_5$, it follows that $a_i+s > n$, where $a_i \leq n-1$. However s satisfies $0 \leq s \leq 4$. Hence,

$$14 < a_i+s \leq a_i+4 ,$$

and so

$$11 \leq a_i \leq 13 \quad (i \geq 3).$$

But these two results, $a_i+s > 14$ and $11 \leq a_i \leq 13$, imply that $n \in A+B$, since all the numbers less than or equal to s are in $A+B$, a contradiction. For example, suppose $a_i = 11$ for some $i \geq 3$. Then, since $a_i+s > n$, we have $s = 4$, and so $3 \in B$. Hence $11+3 = n \in A+B$. We conclude that $a_i \leq a$ where $i \geq 2$.

In addition, we assert that $a_i \geq s+t$ where $i \geq 2$. We proved this before in a different situation so let us be brief. Suppose $a_i < s+t$. Then

$$2 \leq a_i \leq s+1 ,$$

and so $s \geq 1$. It follows that

$$1 \leq s+2-a_i \leq s,$$

and so $s+2-a_i \in B$. Thus

$$s+2 = a_i + (a+2-a_i) \in A+B ,$$

but

$$s+2 < s+t,$$

since $t \geq 5$. Hence $s+2 \in A+B$, and we have a contradiction. Therefore $a_i \geq s+t$ $(i \geq 2)$.

Hence, from these two results, $a_i \leq a$ and $a_i \geq s+t$, we obtain

$$s+t \leq a_i \leq a \quad (i \geq 2). \tag{4.19}$$

Consequently,

$$2s+t \leq a_2+s \leq a_5+s-3 \leq a+s-3 < b_5 = 4+t,$$

since by assumption $a+s = 2+b_5$, and so

(4.20) $$0 \leq s \leq 1.$$

Consequently, we have $s = 0$, or $s = 1$.

We assert further that $a_2+s+t > n$, for suppose $a_2+s+t \leq n$. Then since $a_2+s+t \in A+B$, we have

$$a_2+s+t \leq x = a+s = 6+t.$$

Lin claims strict inequality here, but we are unable to show that this follows from his method. Hence, we use an alternate method for obtaining a contradiction. By (4.19), we have

$$s+t \leq a_2 \leq 6+t-(s+t) = 6-s ,$$

and so

$$6-2s \geq t \geq 5.$$

Therefore, $s=0$ and so using this result and (4.18), we have $5 \leq t \leq 6$. Hence, we have

$$5 \leq s+t \leq a_2 \leq 6-s = 6.$$

Our problem is reduced to considering the cases, $s = 0$, $t = 5$, $5 \leq a_2 \leq 6$ and $s = 0$, $t=6$, $a_2 = 6$. Therefore, we have

$$B = \{0,5,6,7,8,9\} \ (t = 5) ,$$

and

$$B = \{0, 6, 7, 8, 9, 10\} \ (t = 6).$$

Thus, in either case, $a_2 = 5$ or $a_2 = 6$, we have $n \in A+B$, a contradiction. Consequently

$$(4.21) \qquad a_2+s+t > n .$$

We now use (4.21) to sharpen (4.18). We have

$$n < a_2+s+t < a_5-3+s+t \leq a-3+s+t = 3+2t,$$

and so

$$6 \leq t \leq 7.$$

This result, together with (4.20), shows that we need only consider the four cases $s = 0,1$ and $t = 6,7$ respectively to get our final contradiction. Let us then look at these four cases. For $t = 6$, we have

$$B = \{0, 6, 7, 8, 9, 10\} \ (s = 0),$$
$$B = \{0, 1, 7, 8, 9, 10\} \ (s = 1),$$

and so

$$B + \{0, 9\} = B.$$

For $t = 7$, we have

$$B = \{0, 7, 8, 9, 10, 11\} \ (s = 0),$$
$$B = \{0, 1, 8, 9, 10, 11\} \ (s = 1),$$

and so

$$B + \{0, 8\} = B.$$

Hence, in all cases, we have $B+\tilde{B} \neq I_n$. Thus we must reject the assumption $b = b_s$ and $x = 2+b_5$.

Now suppose $b = b_s$ and also suppose $x > 1+b_5$. Then $a+b_s \geq 3+b_5$. Let us again determine the subscript v, where $0 \leq v < 5$, such that

$$a_i+b_v \leq 1+b_5 < a_i+b_{v+1} \quad (i \geq 2).$$

In this situation, we have

$$a_i+b_{v+1} \geq x,$$

and so

$$x-b_{v+1} \leq a_i \leq 1+b_5-b_v,$$

or

$$b_{v+1}-b_v \geq x-(1+b_5).$$

By our assumption $x > 1+b_5$, we have

$$x-(1+b_5) > 1,$$

and so

$$b_{v+1}-b_v > 1.$$

Hence $v = s$, and so

$$a_i+b_s \leq 1+b_5,$$

contradicting $a+b_s \geq 3+b_5$. Therefore, $b=b_s$ and $x > 2+b_5$ are impossible.

Let $b = b_5$ and $x = 2+b_5$. Then $a = a_2 = 2$. Consequently, $s+2 \in A+B$, and since no numbers in $A+B$ lie between $s+1$ and $s+t$, we conclude that $s+2 = s+t$, and so $t = 2$. Hence, we have $b_5 = 4+t = 6$ and $x = 2+b_5 = 8$. Since x is the largest number in $A+B$, it follows that $3 \leq a_i \leq 8$, where $i \geq 3$. In particular, we have that a_3 satisfies

$$3 \leq a_3 \leq 6,$$

and so

$$9 \leq a_3+b_5 \leq 12.$$

Hence, $a_3+b_5 \in A+B$ and $a_3+b_5 > x$, contradicting our choice of x. This shows that we cannot have $b = b_5$ and $x = 2+b_5$.

Suppose $b = b_5$ and $x > 2+b_5$. Then $a \geq 3$. As in the case $b = b_s$ and $x > 2+b_5$, we have

$$b_{v+1}-b_v \geq x-(1+b_5) > 1,$$

and so $v = s$. Hence,

$$a_i+b_s \leq 1+b_5 ,$$

$$a_i+b_{s+1} \geq x \quad (i \geq 2).$$

Consequently,

$$x = a + b_5 \geq a_2+b_{s+1} \geq x ,$$

since $a \geq a_2$ and $b_5 \geq b_{s+1}$, and so

$$a+b_5 = a_2+b_{s+1} .$$

Hence, $a = a_2$ and $b_5 = b_{s+1}$, and so $s = 4$. It follows that $a_i > a$ and $a_i+b_5 > n$, where $i \geq 3$. Therefore, since $a_2+b_s > 1+b_s$ and since there are no numbers in $A+B$ between b_s+1 and b_{s+1}, we conclude that $a_2+b_s \geq b_{s+1} = b_5$. But this is a contradiction for then

$$b_5 \leq a_2+b_s \leq a_5-3+b_s \leq 1+b_5-3 = b_5-2.$$

This completes the proof for $b = b_5$ and $x > 1+b_5$.

iii) Suppose two of the signs "$\leq$" of (4.4) are strict inequalities and so $x = 1+b_u$, where $u \neq s$, $u \neq 5$ and $0 \leq s < u \leq 4$. Then

$$B = \{0, 1, \cdots, b_s, b_{s+1}, \cdots, b_u, b_{u+1}, \cdots, b_5\},$$

$$A+B = \{0, 1, \cdots, b_s+1, b_{s+1}, \cdots, b_u, b_u+1, b_{u+1}, \cdots, 1+b_5\}.$$

Let

$$t = b_{s+1}-b_s > 1, \quad w = b_{u+1}-b_u > 1.$$

If we consider the numbers

$$a_i+b_0, \quad a_i+b_1, \quad \cdots, \quad a_i+b_5,$$

where $i \geq 2$, then as earlier we can find a subscript v, where $0 \leq v < 5$, such that

$$a_i+b_v \leq 1+b_5 ,$$

$$a_i+b_{v+1} > n ,$$

and conclude in the same manner that $b_{v+1}-b_v > 1$. Consequently, either $v = s$ or $v = u$.

If $v = s$, then

$$(4.22) \qquad \begin{cases} a_i+b_s \leq 1+b_5 , \\ a_i+b_{s+1} > n, \end{cases}$$

and if $v = u$, then

$$(4.23) \qquad \begin{cases} a_i + b_u \le 1 + b_5, \\ a_i + b_{u+1} > n \quad (i \ge 2). \end{cases}$$

Therefore, either

$$(4.24) \qquad n - b_{s+1} < a_i \le 1 + b_5 - b_s,$$

or

$$(4.25) \qquad n - b_{u+1} < a_i \le 1 + b_5 - b_u \quad (i \ge 2).$$

Observe here that in either case, $v = s$ or $v = u$, we have

$$(4.26) \qquad \begin{cases} a_i + b_s \le 1 + b_5, \\ a_i + b_{u+1} > n \quad (i \ge 2), \end{cases}$$

due to our assumption $0 \le s < u \le 4$.

Due to the construction of the set B, we may write

$$b_s = s, b_{s+1} = s+t, \quad b_u = t+u-1.$$

Then

$$b_{u+1} = t+u-1+w,$$

$$b_5 = 5+(t-1)+(w-1) = t+w+3 \le n-2 = 12,$$

and so

$$t+w \leq 9.$$

Using these identities, (4.24) and (4.25) become

$$14-(s+t) < a_i \leq t+w+4-s,$$

$$14-(t+u+w-1) < a_i \leq t+w+4-(t+u-1),$$

and so either

$$2t+w \geq 11,$$

or

$$t+2w \geq 11.$$

By the definition of t and w, we have $t \geq 2$ and $w \geq 2$. Now suppose $t+w \leq 6$. Then since $2t+w \geq 11$, we have $t \geq 5$, and so $w \leq 6-t \leq 1$, a contradiction. Similarly, since $t+2w \geq 11$, we have $w \geq 5$, and so $t \leq 6-w \leq 1$, again a contradiction. Hence $t+w \geq 7$, and so

$$7 \leq t+w \leq 9. \tag{4.27}$$

We have that

$$a_i \geq \max(t, w, b_{s+1}), \tag{4.28}$$

where $i \geq 2$. We may suppose that $i = 2$. Further suppose that

$a_2 < t$. Then

$$2+b_s \leq a_2+b_s \leq b_{s+1} ,$$

contradicting our assumption that no number in A+B lies between $1+b_s$ and b_{s+1}. Hence $a_2 \geq t$. We have a similar contradiction for $a_2 < w$, for then

$$2+b_u \leq a_2+b_u < b_{u+1} ,$$

and again there is no number in A+B between $1+b_u$ and b_{u+1}. Hence $a_2 \geq w$. The last possibility, $a_2 < b_{s+1}$ implies that $a_2 \leq b_s+1 = s+1$. Hence, $2 \leq a_2 \leq s+1$, or $-(s+1) \leq a_2 \leq -2$. Thus, adding $s+2$, we obtain

$$1 \leq s+2-a_2 \leq s,$$

and so

$$a_2+(s+2-a_2) = s+2$$

is in A+B. Hence, $s+2 = s+t$, and so $t = 2$. By (4.27), we have $w \geq 5$, and by what we have just shown $a_2 \geq w$. Hence,

$$5 \leq w \leq a_2 \leq s+1 \leq 4,$$

since $0 \leq s \leq 3$, and we have a contradiction. This completes the proof of (4.28).

From (4.28), it follows that

(4.29) $$a_i \geq 4,$$

where $i \geq 2$, since by (4.27) either $t \geq 4$ or $w \geq 4$.

If we can show that

(4.30) $$\{0, a_i\} + B = B,$$

for some fixed $i \geq 2$, then, by Theorem 2.2, $B + \widetilde{B} \neq I_n$. This is the contradiction that we wish to obtain. Consequently, we wish to show that for some fixed i, where $i \geq 2$, there does not exist an $e_i \epsilon B$ such that

$$a_i + e_i < n$$

and

$$a_i + e_i \notin B,$$

thereby showing that

$$\{0, a_i\} + B = B$$

for that fixed value of i. Hence, we assume the contrary. That is, we assume that for each i where $i \geq 2$, there exists an $e_i \epsilon B$ such that

$$(4.31)\qquad \begin{cases} a_i+e_i < n, \\ \\ a_i+e_i \notin B . \end{cases}$$

Then, since $a_i+e_i \notin B$ and $a_i+e_i \in A+B$, for each i at least one of the following three equations holds:

a) $a_i+e_i = 1+b_s$;

b) $a_i+e_i = 1+b_5$;

c) $a_i+e_i = 1+b_u$.

We obtain a contradiction to our assumption by producing a value of i for which these three equations all fail to hold.

a) This equation fails to hold for each $i \geq 2$, for by (4.28), $a_i \geq b_{s+1}$. Hence,

$$a_i+e_i \geq b_{s+1} > 1+b_s .$$

b) Suppose $a_i+e_i = 1+b_5$. Then

$$e_i = 1+b_5-a_i < b_5 .$$

Hence $e_i = b_s$ or $e_i = b_u$, for if this were not the case, then $1+e_i \in B$ and so $2+b_5 = a_i+(1+e_i) \in A+B$ since $2+b_5 \leq n$. We consider these two subcases separately.

Suppose $a_i+e_i = 1+b_5$ and $e_i = b_s$. Then since by (4.26),

$$a_j+b_s \le 1+b_5,$$

where $j \ge 2$, and by assumption

$$a_i+b_s = 1+b_5,$$

we conclude that a_i is the largest element in A+B. Hence, since $A(n) \ge 5$, at least for $j = 2, 3, 4$, we have

$$a_j+b_s < 1+b_5, \tag{4.32}$$

a contradiction to the supposition $a_i+e_i = 1+b_5$ if $i = 2, 3, 4$. Therefore, we have shown that if $e_i = b_s$, then the equation $a_i+e_i = 1+b_5$ fails to hold for $i = 2, 3, 4$.

Next suppose that $a_i+e_i = 1+b_5$ and $e_i = b_u$, for i fixed and $2 \le i \le 3$. Then

$$a_i+e_i = a_i+b_u = 1+b_5,$$

and so

$$a_4+b_u > 1+b_5.$$

Hence, by (4.23), $v \ne u$, and so $v = s$. Thus, by (4.22), we have

$$a_4+b_{s+1} > n,$$

and

$$a_4+b_s < 1+b_5,$$

strict inequality holding by (4.32). Using this result and (4.28), we have

$$b_{s+1} \leq a_4 \leq a_4+b_j \leq b_5,$$

where $0 \leq j \leq s$. By (4.31), $a_4+e_4 < n$ and $a_4+e_4 \notin B$. Hence $e_4 \leq b_s$, for suppose $e_4 > b_s$. Then $e_4 \geq b_{s+1}$, and so $a_4+e_4 \geq a_4+b_{s+1} > n$, contradicting $a_4+e_4 < n$. Thus, it follows that

$$b_{s+1} \leq a_4+e_4 \leq b_5,$$

and so $a_4+e_4 = 1+b_u$, since $1+b_u$ is the only member of $A+B$ satisfying the above inequalities and which is not in B. But if $a_4+e_4 = 1+b_u$, then for $2 \leq i \leq 3$ we have by (4.28) that

$$n \geq 2+b_5 = a_i+(1+b_u) = a_i+a_4+e_4 \geq b_{s+1}+a_4+e_4 > n,$$

a contradiction. Hence, if $e_i = b_u$, then the equation $a_i+e_i = 1+b_5$ fails to hold for $i = 2, 3$. Thus for both subcases of case (b) we see that the equation $a_i+e_i = 1+b_5$ fails to hold for $i = 2, 3$.

c) Suppose $a_i+e_i = 1+b_u$, where $i = 2, 3$. Then

$$a_2+e_2 = a_3+e_3,$$

and so since $a_3 > a_2$, we have $e_2 > e_3$. By (4.29), $a_i \geq 4$, so we also have

$$e_3 < e_2 < b_u.$$

We wish to show that $e_2 < b_{s+1}$. Therefore, suppose first that $e_2 > b_{s+1}$. Then, since $a_2+e_2 = 1+b_u$, we have

$$a_2-1 = b_u-e_2 < b_u-b_{s+1} .$$

But, by the definitions of b_u and b_{s+1}, we have

$$b_u-b_{s+1} = t+u-1-(s+t) = u-1-s,$$

and so

$$a_2-1 < u-1-s.$$

Thus

$$4 \leq a_2 < u-s \leq 4,$$

which is a contradiction. Hence, $e_2 \leq b_{s+1}$.

Next suppose $e_2 = b_{s+1}$. Then repeating the same argument with equality, we conclude that

$$4 \leq a_2 = u-s \leq 4,$$

and so

$$a_2 = 4, \quad s = 0, \quad u = 4 .$$

Consequently, since $e_2 = b_{s+1}$ and $s = 0$, we have $e_2 = b_1 = t$, and so $1 < t \leq 4$ since $4 = a_2 \epsilon A+B$ and again no numbers in A+B are between $s+1$ and $s+t$. Also since all the numbers between b_{s+1} and b_u are in B, we have

$$2 \leq b_1 < b_2 = b_1+1 < b_3 = b_1+2 < b_4 = b_u = b_1+3 \leq 7.$$

Hence, since $a_2 = 4$, we have

$$6 \leq a_2+b_1 = 1+b_4 < a_2+b_2 = 2+b_4 < a_2+b_3 = 3+b_4 < a_2+b_4 = 4+b_4 \leq 11 < n,$$

and therefore,

$$1+b_u, \quad 2+b_u, \quad 3+b_u, \quad 4+b_u$$

are all greater than or equal to $1+b_u$ and are in A+B, contradicting $u = 4$. We conclude that $e_2 \neq b_{s+1}$. Hence $e_2 < b_{s+1}$.

We now have

$$0 \leq e_3 < e_2 \leq s,$$

and so

$$s \geq 1, \quad u \geq 2,$$

$$1 \leq 1+e_3 < 1+e_2 \leq s+1 .$$

Consequently, $1+e_3 \epsilon B$. Hence, since $1+e_3 \leq s$, we have by (4.26) that

$$2+b_u = a_3 + (1+e_3) \leq a_3+b_s \leq 1+b_5.$$

Thus $2+b_u \epsilon A+B$. Since there exist in $A+B$ no numbers between b_u+1 and b_{u+1}, we must have

$$2+b_u = b_{u+1},$$

and so

$$w = b_{u+1}-b_u = 2.$$

Thus, by (4.27) and (4.28), we have

$$5 \leq t \leq 7,$$

$$a_2 \geq s+t > 5,$$

since $s \geq 1$. Recalling that

$$b_5 = t+w+3,$$

$$b_u = t+u-1,$$

we have

$$b_5-b_u = t+w+3-(t+u-1) = w-u+4 \leq w+2 = 4.$$

since $u \geq 2$. Thus,

$$b_u \geq b_5 - 4,$$

and so

$$5 + b_u \geq 1 + b_5 .$$

Hence, since $a_2 > 5$, we have

$$a_2 + b_u > 5 + b_u \geq 1 + b_5,$$

and so, by (4.23), $v = s$. Therefore, by (4.22), we have

$$a_2 + b_{s+1} > n,$$

$$a_2 + s < a_5 + s \leq 1 + b_5 .$$

The proof can now be completed with the help of the following three inequalities, validity of which has already been established (see 4.22 and 4.28).

(4.33)
$$\begin{cases} a_2 \geq b_{s+1} , \\ a_2 + (1 + b_s) \leq b_5, \\ a_2 + b_{s+1} > n. \end{cases}$$

We show that

$$\{0, a_2\} + A + B = A + B .$$

Recall that $2+b_u = b_{u+1}$. Therefore, the set

$$A+B = \{0, 1, \cdots, b_s+1, b_{s+1}, \cdots, 1+b_5\}.$$

Consequently, if z satisfies $0 \leq z \leq b_s+1$, then $z \in A+B$. Thus, by (4.33),

$$b_{s+1} \leq a_2+z \leq a_2+(1+b_s) \leq 1+b_5.$$

Hence not only z but a_2+z is in $A+B$. Now suppose z satisfies $b_{s+1} \leq z \leq 1+b_5$. Then again $z \in A+B$ and, by (4.33), we have

$$a_2+z \geq a_2+b_{s+1} > n.$$

Hence $a_2+z \notin A+B$. Thus, we have shown that

$$\{0, a_2\} + A+B = A+B,$$

and so $A+B+(A+B)^{\sim} \neq I_n$, a contradiction.

We conclude that we must reject the assumption that $a_i+e_i = 1+b_u$, where $i = 2, 3$. This means that this equation fails to hold for $i = 2$, or $i = 3$, or both of these values for i. Hence in all three cases (a), (b), and (c) our equation fails to hold for $i = 2$, or $i = 3$, and we have obtained our contradiction to (4.30). This completes the proof of Theorem 4.2.

The example to show that Theorem 4.2 cannot be improved was given in Chapter II following Conjecture 2.2. Theorem 4.2 does not say, of course, that it is not possible to find three sets, for $n \geq 15$, such that $A+B+C = I_n$ and

$$A(n) + B(n) + C(n) \leq n-1,$$

as we will see in the next chapter.

CHAPTER V

RESULTS FOR $k \geq 2$ SETS

In this final chapter, we turn our attention to the proofs of several miscellaneous theorems, some of which were mentioned in Chapter I. Also included here is a discussion of the possibility of generalizing Theorems 4.1 and 4.2 to k sets where $k \geq 3$. We close the chapter with a few unanswered questions.

Theorem 1.1 is a necessary tool in the proof of Erdős and Scherk's generalization, Theorem 1.2 (4, p. 45). To keep this work self contained, we restate Theorem 1.1 as Lemma 5.1 and proceed with its proof.

Lemma 5.1. If $0 \in A_1$, $0 \in A_2$ and n is a gap in $A_1 + A_2$, then

$$A_1(n) + A_2(n) \leq n-1.$$

Proof. Suppose $a \in A_1$ and $1 \leq a \leq n-1$. Then $1 \leq n-a \leq n-1$. Furthermore, $n-a \notin A_2$, for if $n-a \in A_2$, then $n = a+(n-a) \in A_1+A_2$, contrary to the hypothesis. Hence

$$A_1(n-1) \leq \bar{A}_2(n-1) = n-1-A_2(n-1)$$

and so

$$A_1(n-1) + A_2(n-1) \leq n-1.$$

However, if $0 \in A_1$, $0 \in A_2$ and $n \notin A_1+A_2$, then $n \notin A_1$ and $n \notin A_2$. Thus, $A_1(n) = A_1(n-1)$ and $A_2(n) = A_2(n-1)$, and therefore,

$$A_1(n)+A_2(n) \leq n-1,$$

which concludes the proof.

Now we restate Theorem 1.2 as Theorem 5.1 and proceed with its proof. Erdös and Scherk indicated the lines along which a proof of this theorem would proceed, but the details were supplied by the author.

Theorem 5.1. If $0 \in A_i$ $(i = 1, \cdots, k)$, and n is a gap in $\sum_{i=1}^{k} A_i$, then

$$\sum_{i=1}^{k} A_i(n) \leq n-1.$$

Proof. By Lemma 5.1, since $0 \in A_j$, $0 \in A_m$, and $n \notin A_j+A_m$, where $1 \leq j < m \leq k$, then

$$A_j(n) + A_m(n) \leq n-1.$$

Thus,

$$\sum_{j=1}^{k-1} \sum_{m=j+1}^{k} [A_j(n)+A_m(n)] \leq (n-1)[1+2+ \cdots + (k-1)],$$

and so

$$(k-1)\sum_{i=1}^{k} A_i(n) \leq \frac{k}{2}(k-1)(n-1) .$$

Finally,

$$\sum_{i=1}^{k} A_i(n) \leq \frac{k}{2}(n-1),$$

and the proof is complete.

That this is the "best possible" inequality is illustrated by the following example of Erdős and Scherk (4, p. 45). Let n be odd and let A_i $(i = 1,\cdots,k)$ contain 0 and the set of integers x such that $\frac{n-1}{2} < x \leq n-1$. The sets A_i thus defined satisfy the hypotheses of Theorem 5. 1, for each contains 0 and n is a gap in $\sum_{i=1}^{k} A_i$. Furthermore,

$$A_i(n) = n-1-\frac{n-1}{2} = \frac{n-1}{2} ,$$

and so

$$\sum_{i=1}^{k} A_i(n) = \frac{k}{2}(n-1).$$

The above example shows that Theorem 5. 1 cannot be improved without restricting n more severely than to be odd. Let us then discuss briefly some other possible restrictions which could be made on n. We question, for instance, if Theorem 5.1 could be improved if we restrict n to be even. Let us look at the example corresponding to the one above and let n be even. Then we let A_i $(i = 1, \cdots, k)$ contain 0 and the set of integers x such that $\frac{n}{2} < x \leq n-1$. Again the hypotheses of Theorem 5.1 are satisfied. Furthermore,

$$A_i(n) = n-1-\frac{n}{2} = \frac{n}{2} - 1 ,$$

and so

$$\sum_{i=1}^{k} A_i(n) = \frac{k}{2}(n-2) .$$

Although we don't know that $\sum_{i=1}^{k} A_i(n)$ is maximized, for n even, with this example, it does point out that Theorem 5.1 cannot be greatly improved under the added restriction that n is even.

We have already seen some of the results which have been obtained under the restriction that n is the smallest gap in $\sum_{i=1}^{k} A_i$, for instance, the results of Erdös and Scherk (4, p. 46), and Kemperman (5, p. 376) discussed in Chapter I, and the results of Lin (6, pp. 27, 31) proved previously in this work. However, there are still many questions unanswered relative to this restriction, and some of these are posed later in this chapter.

Another possibility for improving Theorem 5.1 would be to restrict n so that n is the s^{th} gap in $\sum_{i=1}^{k} A_i$, where $s \geq 1$. What could be done, for instance, if n were the second gap, or the third? We see that there are many questions yet to be answered.

The material in the remainder of this chapter is original with the author.

Before we terminate the discussion of Theorem 5.1, we observe here that with Theorem 5.1 available, a weaker version of Theorem 4.2 is immediate, namely, the following corollary.

Corollary 5.1. If $A+B+C = I_n$ and $0 < n < 11$, then

$$A(n)+B(n)+C(n) \leq n-1.$$

Proof. In order to maximize $A(n)+B(n)+C(n)$, we assume that

$$A = (B+C)^{\sim}, \quad B = (A+C)^{\sim}, \quad C = (A+B)^{\sim}.$$

By theorem 4.1, we may also assume that

$$A(n) \geq 5, \quad B(n) \geq 5, \quad C(n) \geq 5,$$

and so

(5.1) $$A(n)+B(n)+C(n) \geq 15.$$

We note here that these are the same assumptions made in the proof of Theorem 4.2.

Since the hypotheses of Theorem 5.1 are satisfied, we have

$$A(n)+B(n)+C(n) \leq \frac{3}{2}(n-1).$$

Combining this result with (5.1) yields

$$15 \leq A(n)+B(n)+C(n) \leq \frac{3}{2}(n-1),$$

and so

$$11 \leq n.$$

This contradicts $n < 11$ and the corollary is proved.

It seems that extending Corollary 5.1 to the stronger result of Theorem 4.2 is a very difficult process. In searching for some way to improve on the proof of Theorem 4.2, and in particular, searching for some way to bypass Theorem 4.1, and therefore Mann's transformation, the author noticed that the inequality $A(n)+B(n) > (A+B)(n)$ is implicitly contained in Theorem 4.2. For instance, using Theorem 4.1, we assumed that $A(n)+B(n) \geq 10$, but using other results also came to the conclusion that $(A+B)(n) \leq 8$, and eventually showed that this was impossible. In connection with these observations, we prove the following result.

Theorem 5.2. If $A+B+\widetilde{(A+B)} = I_n$, then

$A(n)+B(n) \leq (A+B)(n)$ if, and only if,

$$A(n)+B(n) + (A+B)^{\sim}(n) \leq n-1.$$

Proof. The proof is based on Lemma 2.1(g), which states: If $A+B+(A+B)^{\sim} = I_n$, then

$$(A+B)(n)+(A+B)^{\sim}(n) = n-1 .$$

Therefore, if $A(n)+B(n) \leq (A+B)(n)$, then

$$n-1 = (A+B)(n)+(A+B)^{\sim}(n) \geq A(n)+B(n)+(A+B)^{\sim}(n).$$

Conversely, if $A(n)+B(n)+(A+B)^{\sim} \leq n-1$, then

$$A(n)+B(n) \leq n-1-(A+B)^{\sim}(n),$$

and so

$$A(n)+B(n) \leq (A+B)(n) .$$

This completes the proof.

Let us illustrate this theorem by altering slightly the sets A_1 and A_2, without changing the set sum A_1+A_2, in the example used following Conjecture 2.2 in Chapter II. Let $n = 15$ and

$$A = \{0, 1, 8, 10, 14\},$$

$$B = \{0, 2, 9, 13\}.$$

Then

$$A+B = \{0, 1, 2, 3, 8, 9, 10, 12, 13, 14\},$$

$$(A+B)^{\sim} = \{0, 4, 8, 9, 10, 11\},$$

and so

$$A+B+(A+B)^{\sim} = I_n,$$

and

$$7 = A(n)+B(n) < (A+B)(n) = 9.$$

Hence,

$$A(n)+B(n)+(A+B)^{\sim}(n) \leq n-1,$$

by Theorem 5. 2, whereas, in the original example, $A_1(n)+A_2(n) > (A_1+A_2)(n)$, and consequently, $A_1(n)+A_2(n)+(A_1+A_2)^{\sim}(n) > n-1$. Also observe in both examples that $r = 6$. Hence, Theorem 4. 1 and Theorem 5. 2 are not related in the sense that $A+B+(A+B)^{\sim} = I_n$ and $A(n)+B(n) \leq (A+B)(n)$ imply that $r \leq 5$, for this is clearly not the case. However, the converse is true, which we state in the following corollary.

Corollary 5. 2. If $A+B+(A+B)^{\sim} = I_n$ and $A+B$ has $r \leq 5$ gaps, then

$$A(n)+B(n) \leq (A+B)(n) .$$

Proof. By Theorem 4. 1, under our hypotheses,

$$A(n)+B(n)+(A+B)^{\sim} \leq n-1 \;.$$

Consequently, by Theorem 5.3,

$$A(n)+B(n) \leq (A+B)(n),$$

and the proof is complete.

We note here that Theorem 5.2 can be generalized to k sets where $k \geq 3$. The proof is essentially the same as for the case $k = 3$. For instance, for $k = 4$, we have the statement:

If $A+B+C+(A+B+C)^{\sim} = I_n$, then $A(n)+B(n)+C(n) \leq (A+B+C)(n)$ if, and only if,

$$A(n)+B(n)+C(n)+(A+B+C)^{\sim}(n) \leq n-1 \;.$$

We wish now to investigate the possibility of generalizing Theorems 4.1 and 4.2 to k sets where $k \geq 3$. Therefore, suppose $k = 4$, then the following questions naturally arise:

1. If $A+B+C+(A+B+C)^{\sim} = I_n$ and if $A+B+C$ has r gaps, where $r \leq 5$, is

$$\epsilon\,(A, B, C, (A+B+C)^{\sim}, n) \leq 0 \;?$$

2. What is the largest positive integer p such that $A+B+C+E = I_n$ and $0 < n < p$ imply that

$$A(n)+B(n)+C(n)+E(n) \leq n-1 \quad ?$$

We answer "no" to the first question by exhibiting a counter example and can only make a conjecture concerning the second question. Observe here that the first question is not a generalization of Theorem 4. 1 to k sets where $k \geq 3$, but a generalization of Conjecture 2. 5 under the restriction $r \leq 5$ which, for $k = 3$, is implied by Theorem 4. 1. We were unable to produce a counter example for the generalization of Theorem 4. 1 to $k = 4$ sets.

Consider the following sets as a counter example for the first question, where $n = 16$. Let

$$A = \{0, 1, 9, 11, 13, 15\},$$

$$B = \{0, 2, 9, 10, 13, 14\},$$

$$C = \{0, 4, 9, 10, 11, 12\}.$$

Then

$$A+B+C = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15\},$$

$$(A+B+C)^{\sim} = E = \{0, 8\},$$

and

$$A+B+C+E = I_n .$$

Hence $r = 2$, but

$$\epsilon(A, B, C, E, n) = A(n-1)+B(n-1)+C(n-1)+E(n-1)-(A+B+C+E)(n)$$
$$= 5+5+5+1-15 > 0,$$

contradicting the conclusion of the first question. Also observe here that

$$15 = A(n)+B(n)+C(n) > (A+B+C)(n) = 14,$$

in keeping with the generalization of Theorem 5.2 to $k = 4$ sets.

We make the following conjecture concerning the second question and feel quite strongly that it is valid.

Conjecture 5.1. If $A+B+C+E = I_n$ and $0 < n < 16$, then

$$A(n)+B(n)+C(n)+E(n) \leq n-1,$$

where $A(n) \geq 1$, $B(n) \geq 1$, $C(n) \geq 1$, and $E(n) \geq 1$.

The preceding counter example used for the first question also shows that Conjecture 5.1 cannot be improved. Unfortunately, all attempts on the part of the author to prove this conjecture have met with failure, but on the other hand, all attempts to find a counter example for $n < 16$ have also met without success.

Another question which deserves consideration concerns the following generalization of Theorem 4.2 and Conjecture 5.1.

3. What is the largest positive integer $g(k)$, where $k \geq 3$, which perhaps depends upon whether k is even or odd, such that:

If $\left(\sum_{i=1}^{k} A_i\right) \cap N = I_n$ and $0 < n < g(k)$, then

$$\sum_{i=1}^{k} A_i(n) \leq n-1,$$

where $A_i(n) \geq 1$ $(i = 1, \cdots, k)$?

The best the author can do with this question is state that it appears that it is always possible to find a counter example for $n = 2^k$, where $k \geq 4$, by forming the sets in a manner which appears to maximize $\sum_{i=1}^{k} A_i(n)$. This method of set construction is as follows. We form the first $k-1$ sets by selecting as few numbers as possible in each set in the closed interval $[0, 2^{k-1}-1]$ such that the set sum includes all the numbers in the interval, and select as many numbers as possible in the interval $[2^{k-1}+1, n-1]$ so that the $\left(\sum_{i=1}^{k} A_i\right) \cap N = \{j \mid 0 \leq j \leq 2^{k-1}-1\} \cup \{j \mid 2^{k-1}+1 \leq j \leq n-1\}$. Then the k^{th} set is $\{0, 2^{k-1}\}$, and so $\left(\sum_{i=1}^{k} A_i\right) \cap N = I_n$. This procedure was used in constructing the counter example for the first question in this chapter. For instance, suppose $k = 5$. Then using this construction, with $n = 2^k$, we have

$$A = \{0, 1, 17, 19, 21, 23, 25, 27, 29, 31\},$$

$$B = \{0, 2, 17, 18, 21, 22, 25, 26, 29, 30\},$$

$$C = \{0, 4, 17, 18, 19, 20, 25, 26, 27, 28\},$$

$$E = \{0, 8, 17, 18, 19, 20, 21, 22, 23, 24\},$$

$$F = \{0, 16\}.$$

We also have

$$A+B+C+E+F = I_n,$$

but

$$A(n)+B(n)+C(n)+E(n)+F(n) > n-1 .$$

In our concluding remark, we wish to emphasize that the largest positive integer $g(k)$ we are seeking is not $g(k) = 2^k$, where $k \geq 4$, at least not for odd k, for we have a counter example for $k = 5$, $n = 30$. However, again we have not been able to construct a counter example for $n < 30$ and $k = 5$.

BIBLIOGRAPHY

1. Corput, J. G. van der. On sets of integers. (First communication.) Koninklijke Nederlandsche Akademie van Wetenschappen 50: 252-261. 1947.

2. Damewood, Leroy Mitchell. Proofs of the $\alpha + \beta$ theorem based on Mann's transformation. Master's thesis. Corvallis, Oregon State College, 1960. 58 numb. leaves.

3. Dyson, T. J. A theorem on the densities of sets of integers. Journal of the London Mathematical Society 20: 8-14. 1945.

4. Erdős, P. and P. Scherk. On a question of additive number theory. Acta Arithmatica 5: 45-55. 1958.

5. Kemperman, J.H.B. On sums of large sets of integers. Transactions of the American Mathematical Society 110: 375-392. 1964.

6. Khinchin, A.Y. Three pearls of number theory. Rochester, New York, Graylock Press, 1952. 64 p.

7. Lin, Chio-Shih. On sums of sets of integers. Ph.D. thesis. Columbus, Ohio State University, 1955. 54 numb. leaves. (Microfilm)

8. Mann, Henry B. A proof of the fundamental theorem on the density of sums of sets of positive integers. Annals of Mathematics 43: 523-527. 1942.

9. Rohrbach, H. Einige neuere Untersuchungen über die Dichte in der additiven Zahlentheorie. Jahresbericht der Deutschen Mathematiker Vereinigung. 48: 199-236. 1939.

10. Schur, I. Über den Begriff der Dichte in der additiven Zahlentheorie. Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse 21: 269-297. 1936.