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Title REDUCTION OF THE NUMBER OF ELEMENTS IN THE

SYNTHESIS OF RC AND RC-NIC NETWORKS

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This paper is a study of the procedures used in achieving element reduction in networks realized from a given network function. Reduction in the number of elements in RC-realization is possible in the synthesis of a transfer function and in the simultaneous realization of driving-point function and a transfer function, through proper partitioning of the function and through realization of all possible network configurations respectively.

Proper partioning of the specified polynomial in RC-NIC realization indicates the optimum expression for the selection of the auxiliary polynomials. The procedures developed for each form of RC-NIC synthesis will lead to the reduction of the number of elements in the realized networks.

REDUCTION OF THE NUMBER OF ELEMENTS IN THE
SYNTHESIS OF RC AND RC-NIC NETWORKS

by

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I. INTRODUCTION

1.1 Background Remarks

Unrestricted realization of performance characteristics as well as the elimination of inductors from low-frequency networks has always been the ambition of the circuit-designer. However, it was not until recently, with the development of the field of active networks, that this condition was fulfilled.

The current trend of active-network synthesis was first established by Linvill [33] with the realization of any transfer function. Since then, various techniques have been introduced for the realization of any network-function with the most extensively used methods being:

1. Yanagisawa's technique of transfer function realization.

2. Sipress's technique for realization of one or more characteristic functions.

Without exception the realization of a given function by any of the existing techniques requires the arbitrary selection of either additive or common divisor negative-real-root polynomials. The main criterion in the selection of these polynomials is to make

possible the realization of the given function; other criteria, set forth by many investigators, on this selection are: Minimization of pole or zero sensitivity [5, 6, 7, 22, 24] ; and unique realization of active-RC networks [2, 35, 36, 42]. Realization of RC networks with a minimum number of elements has received surprisingly little attention in the literature [8, 9]. In the presentation that follows, this criterion is fully treated.

1.2 Findings of This Thesis

This presentation assumes a knowledge of both passive-network and active-network synthesis. The different synthesis techniques, such as transfer function synthesis, active-RC realization with parallel connected networks, and others are reviewed briefly when they are first encountered but are not developed rigorously.

The discussion in this work is broken into three chapters. In Chapter II the well-known polynomial decomposition techniques are looked upon as means of element reduction in the synthesis of RC networks. Chapter III gives a complete treatment of element reduction in passive-RC network synthesis. Chapter IV presents a unification of the field of RC-NIC network synthesis and outlines procedures for the selection of the negative-real-root polynomials such that the number of elements in the resulting network realization is minimized.

The major contributions of this work are:

1. Introduction of a new method of passive-RC network synthesis from an open-circuit voltage-ratio transfer function which greatly reduces the number of elements in the resulting network and broadens the class of rational functions that can be realized by passive-RC networks.

2. Computer synthesis of RC-networks from simultaneous realization of a driving-point function and a transfer function.

3. Unification of the field of RC-NIC network synthesis.

4. Introduction of a new network configuration for RC-NIC network synthesis.

5. Outline of procedures for the selection of negative-real-root polynomials such that the number of elements in the resulting RC-NIC networks is minimized.

II. POLYNOMIAL DECOMPOSITION

2.1 Introduction

Polynomial decomposition has been widely used for the minimization of pole or zero sensitivity of the realized active RC-networks. The nature of the decomposition also lends itself to the reduction of the number of elements in these networks.

The advantage of this method is derived from the fact that if a given polynomial, $P(s)$, is Hurwitz, then it can be decomposed into two other polynomials, $A(s)$ and $B(s)$, in a manner that proves useful to network synthesis.

2.2 The Horowitz Decomposition

Horowitz [24, p. 299], has shown that if a polynomial, $P(s)$, is Hurwitz and with no roots on the negative real axis, it can be decomposed as,

$$P(s) = \prod_{i=1}^{m}(s+a_i) - \prod_{i=1}^{m-1}(s+b_i) = a^2(s) - sb^2(s) \qquad (2.1)$$

such that

$$b_1 < a_1 < a_2 < b_2 < b_3 \cdot \cdot \cdot \cdot < b_{m-2} < b_{m-1} < a_{m-1} < a_m$$

and that $a(s)/sb(s)$ and $b(s)/a(s)$ are both passive RC impedances. We will refer to the decompositon of equation (2.1) as the optimum

Horowitz decomposition where it is tacitly understood that $P(s)$ has

no zeros on the negative real axis.

If $P(s)$ has roots on the negative real axis from single algebraic

considerations, it can be concluded that $P(s)$ can be decomposed as

follows:

$$P(s) = \prod_{i=1}^{n}(s+h_i)\left[\prod_{i=1}^{m}(s+a_i) - \prod_{i=1}^{m-1}(s+b_i)\right]$$

$$= \prod_{i=1}^{n}(s+h_i)[a^2(s) - sb^2(s)] \qquad (2.2)$$

where the $\prod_{i=1}^{n}(s+h_i)$ contains all the roots on the negative real axis

and where the roots a_i and b_i and the polynomials $a(s)$ and $b(s)$ have

the same properties stated for the decomposition(2.1).

It is noted here that the optimum Horowitz decomposition

$P(s) = a^2(s) - sb^2(s)$, is unique and that it minimizes the root sensi-

tivity of every simple zero of $P(s)$ with respect to the conversion

ratio of the requisite NIC.

2.3 The Calahan Decomposition

Calahan has shown that a polynomial, $P(s)$, being Hurwitz and

having no roots on the negative real axis, can be also decomposed

in the following form

$$P(s) = \prod_{i=1}^{m}(s+c_i) + \prod_{i=1}^{m}(s+d_i) = c^2(s) + d^2(s) \qquad (2.3)$$

such that,

$$d_1 < d_2 < c_1 < c_2 < d_3 \quad \cdots \quad < d_{m-1} < d_m < c_{m-1} < c_m$$

and that

$c(s)/d(s)$ is an RC impedance.

As in Horowitz decomposition we will refer to (2. 3) as the optimum Calahan decomposition.

If $P(s)$ has roots on the negative real axis Calahan's decomposition will be written as,

$$P(s) = \prod_{i=1}^{n}(s+r_i) \left[\prod_{i=1}^{m}(s+c_i) + \prod_{i=1}^{m}(s+d_i) \right]$$

$$= \prod_{i=1}^{n}(s+r_i) [c^2(s) + d^2(s)] \tag{2.4}$$

where $\prod_{i=1}^{n}(s+r_i)$ contains all the negative real roots and where the roots c_i and d_i and the polynomials $c(s)$ and $d(s)$ have the same properties stated for decomposition (2. 3).

It is noted here that the optimum Calahan decomposition $P(s) = c^2(s) + d^2(s)$ is unique and that it minimizes the root sensitivity of every simple zero of $P(s)$ with respect to the inversion ratios of requisite impedance inverter.

2. 4 Preselection of Zeros in Polynomial Decomposition

In the optimum decomposition of both Horowitz and Calahan no preselection of zeros is possible. However, it has been shown, by

both investigators that if a polynomial P(s) of degree $\underline{n}$ has no roots

on the negative real axis, a preselection of at least $\underline{n}$ roots can be

made in both the Horowitz and the Calahan decompositions. If such

a preselection is made, P(s) will have the following form:

$$P(s) = A(s) - B(s) \text{ in Horowitz decomposition.} \tag{2.5}$$

and

$$P(s) = C(s) + D(s) \text{ in Calahan decomposition.} \tag{2.6}$$

Two important theorems stating relationships between Horowitz

decompositions have been derived by Thomas [42, p. 271], and are

stated here without proof.

Theorem 2.1

Let $P_1(s)$ be a polynomial of degree $\underline{n_1}$ with Horowitz decompo-

sition $\pm [a_1^2(s) - sb_1^2(s)]$. Let $P_2(s)$ be another polynomial or degree

$\underline{n_2}$ with $n_2 \geq n_1$ and with no zeros on the positive real axis. Then

$P_2(s)$ can always be decomposed as

$$P_2(s) = \pm [a_1(s)a_2(s) - sb_1(s)b_2(s)]$$

such that $a_1(s)/sb_1(s)$, $b_2(s)/a_1(s)$, $a_2(s)/sb_1(s)$ and $b_1(s)/a_2(s)$ are

all passive RC impedances.

Theorem 2.2

Let P(s) be a polynomial of degree $\underline{n}$ with Horowitz

decomposition $\pm [a_1^2(s) - sb_1^2(s)]$. Let $P_2(s)$ be any polynomial of

degree not greater than $\underline{n}$. Then $P_2(s)$ can always be decomposed as,

$$P_2(s) = a_1(s)b_2(s) - a_2(s)b_1(s)$$

such that $a_1(s)/sb_2(s)$, $b_2(s)/a_1(s)$, $a_2(s)/sb_1(s)$ and $b_1(s)/a_2(s)$ are

all passive RC-impedances.

The last two theorems represent a vast number of applications

to network synthesis.

2.5 Decomposition

From theorems (1) and (2) it can be concluded that any rational

function $F(s) = p(s)/q(s)$ with the degree of the numerator equal to or

one different from the degree of the denominator can be decomposed

into the following form:

$$F(s) = \frac{p(s)}{q(s)} = \frac{a_1(s)a_2(s) - sb_1(s)b_2(s)}{b_2(s)a_1(s) - a_2(s)b_1(s)}$$

$$\text{or} \quad \frac{b_2(s)a_1(s) - a_2(s)b_1(s)}{a_1(s)a_2(s) - sb_1(s)b_2(s)} \tag{2.7}$$

where the ratios $a_i(s)/b_j(s)$ and $sb_i(s)/a_j(s)$ ($i = 1, 2$; $j = 1, 2$),

satisfy the conditions for RC driving-point functions. Note that if

the root at zero of $sb_1(s)b_2(s)$ can be shifted to any location $- \alpha$

where $|\alpha| \leq |$ first root of $a_1(s)a_2(s)|$ the ratios will still satisfy

the conditions for RC driving-point functions.

The procedure of obtaining the decomposition of Eq. (2.7) is as follows [36, p. 190],

1. Given $F(s) = p(s)/q(s)$, obtain the polynomial $Q(s) = q(s^2) - sp(s^2)$,

2. Find the zeros of polynomial $Q(s)$. Form the polynomial $a_1(s)^2 + b_1(s)^2$ from the left-half plane zeros of $Q(s)$ including the zeros at the origin, if any, and from the right-half plane zeros of $Q(s)$ form the polynomial $a_2(s^2) - b_2(s^2)$.

3. From $a_1(s^2) + b_1(s^2)$ and $a_2(s^2) - b_2(s^2)$ obtain the even and odd parts and by s^2 to s transformation obtain $a_1(s)$, $a_2(s)$, $b_1(s)$, and $b_2(s)$.

Decomposing a polynomial by the above method, especially step 2, often is a rather long and cumbersome task. If the decomposition is done using the basic step of polynomial decomposition, i.e., partial-fraction expansion, the set of decomposition polynomials are less dependent on the degree of the original polynomial and the numerical computations for the decomposition become considerably easier. The difference introduced via the use of this decomposition is in the form of equation (2.7) which changes from $a_1(s)a_2(s) - sb_1(\varepsilon)b_2(s)/b_2(s)a_1(s) - a_2(s)b_1(s)$ to $a_1(s)a_2(s) - sb_1(s)b_2(s)/b_3(s)a_1(s) - a_3(s)b_1(s)$. Since the relative location of the roots in both forms of decomposition is the same, the ratio of $a_i(s)/b_j(s)$, $(s + a)b_i(s)/a_j(s)$ still satisfies the conditions for an RC

driving-point function.

The following numerical examples have been decomposed by this method.

III. PASSIVE RC NETWORKS

3.1 Introduction

One or two-port passive RC-networks without ideal trans-
formers are obtained from synthesizing:

1. A driving-point function

2. A transfer function

3. Part of a characteristic function

4. A driving-point and a transfer function.

This chapter develops the constraints for the class of functions
by which RC networks are realized with a minimum number of ele-
ments. These constraints are then used in the derivation of syn-
thesis techniques used to obtain a reduction in the number of ele-
ments of RC networks.

3.2 Synthesis of a Driving-Point Function

A rational function, $p(s)/q(s)$, describes a driving-point func-
tion, if

1. $p(s)/q(s)$ belongs to the class of positive real functions.

2. The sets of $\{a_i\}$, $\{b_i\}$, which represent the roots of the
numerator and denominator polynomials respectively arranged in
order of increasing magnitude, are such that:

a.　$b_1 < a_2 \ldots < a_n < b_n$, for RC driving -point impedance.

b.　$a_1 < b_2 \ldots < b_n < a_n$, for RC driving-point admittance.

Both conditions being satisfied, realization of an RC-network can

be accomplished by either Cauer's or Foster's Method of Synthesis.

Regardless of the method used, the realized network will have

the minimum possible number of elements. These networks are

referred to as canonical networks.

3.3 Synthesis of a Transfer Function

Given a transfer function $Z_{12}(s)$, $Y_{12}(s)$, or $G_{12}(s)$ of the form

$\dfrac{p(s)}{q(s)}$, the realization can be obtained by a symmetrical lattice net-

work, which in many cases can be decomposed into a grounded ladder-

network. The constraints for the realizability of RC transfer func-

tions are listed.

Case I; $G_{12}(s)$ Realization

If $G_{12}(s)$ is realized with a symmetrical lattice network,

Figure 3.1a, it can be easily shown that, within a constant multiplier

$\underline{k}$, realization of $G_{12}(s)$ will give

$$\frac{Z_a(s)}{Z_b(s)} = \frac{1 - kG_{12}(s)}{1 + kG_{12}(s)} = \frac{q(s) - kp(s)}{q(s) + kp(s)} = \frac{P(S)}{Q(S)} \qquad (3.1)$$

The most direct choice in satisfying equation (3.1) is to let

$$Z_a = 1 - kG_{12}(s) \quad \text{and} \quad Z_b = 1 + kG_{12}(s)$$

or

$$Z_a = \frac{1}{1 + kG_{12}(s)} \quad \text{and} \quad Z_b = \frac{1}{1 - kG_{12}(s)}$$
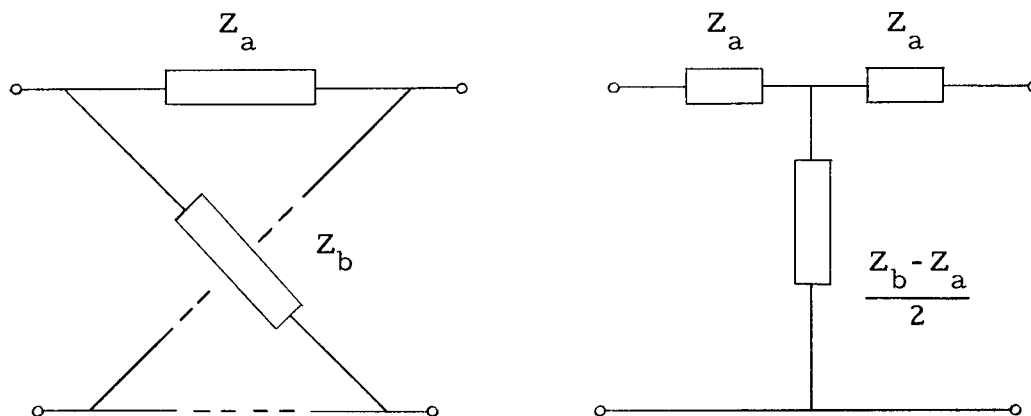
Figure 3. 1.   Form of lattice realization and its equivalent ladder decomposed network.

If a common divisor, r(s), is selected such that $\frac{p(s)}{r(s)}$ and $\frac{q(s)}{r(s)}$ are both RC driving-point impedance functions, a significant reduction in the number of elements is possible in the network realization. This approach will also broaden the class of $G_{12}(s)$ functions that can be realized with a passive RC-network.   The classes of functions that can be realized with passive RC-networks and a result of the introduction of the polynomial r(s) are discussed in the following pages.

Theorem I:

If p(s) and q(s) have only real roots, let the number of zeros of

p(s) and q(s) more positive than any $s = -\sigma_0$, $(s = \sigma + jw)$ be X_A and

X_B respectively; then, if

$$n \geq |X_A - X_B| \geq 0 \qquad -\infty \leq \sigma \leq 0$$

the function $G_{12}(s)$ is RC realizable. $\underline{n}$ is a positive integer.

Proof:

The proof will be accomplished by examining the real axis

from $s = 0$ to $s = -\infty$ and revealing the relative locations of zeros

of p(s) and q(s), subjected to the constraint that either

$$\frac{P(S)}{q(s)} \text{ and } \frac{Q(S)}{q(s)} \text{ are driving-point impedance functions} \qquad (3.2)$$

or,

$$\frac{P(S)}{r(s)} \text{ and } \frac{Q(S)}{r(s)} \text{ are driving-point impedance functions} \qquad (3.3)$$

where r(s) is an arbitrary-selected polynomial with roots on the

negative real axis.

To satisfy the constraint imposed by equation (3.2), if the

real axis is scanned beginning with $s = 0$, the first zero must belong

to q(s), Figure 3.2a, the second to P(S), and third to Q(S) or vice

versa, the fourth to q(s) and so on. Next, assuming a root-locus

viewpoint, the roots of P(S) and Q(S) are obtained from the following expressions:

$$1 - k\frac{p(s)}{q(s)} = 0 \qquad\qquad (3.4)$$

$$1 + k\frac{p(s)}{q(s)} = 0 \qquad\qquad (3.5)$$

The second zero, therefore, must belong to p(s) to satisfy constraints (3.5) and (3.2), as $\underline{k}$ varies from zero to infinity, Figure 3.2b. It cannot depart from the real axis to the interval -$\underline{a}$, -$\underline{b}$; for, to do so would require Q(S) to have complex coefficients.

The third zero has to be a zero of q(s) to satisfy constraints (3.4) and (3.3), Figure 3.2c, and the same argument is applied as for the second zero; thus a cycle is completed in that the next zeros can be assigned to either p(s) or q(s), using identical arguments. In any case, the resemblance of all other cycles to the first demands that the zeros of q(s) and p(s) alternate starting with a zero of q(s), and the theorem holds under constraint (3.2). Note that constraint (3.2) requires P(S)/q(s) to be of the form of RC driving-point impedance.

Using the principle of scanning of the real axis, it can be easily shown that constraint (3.3) is satisfied if the roots of P(S) and Q(S) alternate in any of the sequences shown in Figure 3.3. Then utilizing the root-locus expressions (3.4) and (3.5) and keeping in mind that r(s) is an arbitrary polynomial in $\underline{s}$, it can be concluded that the roots
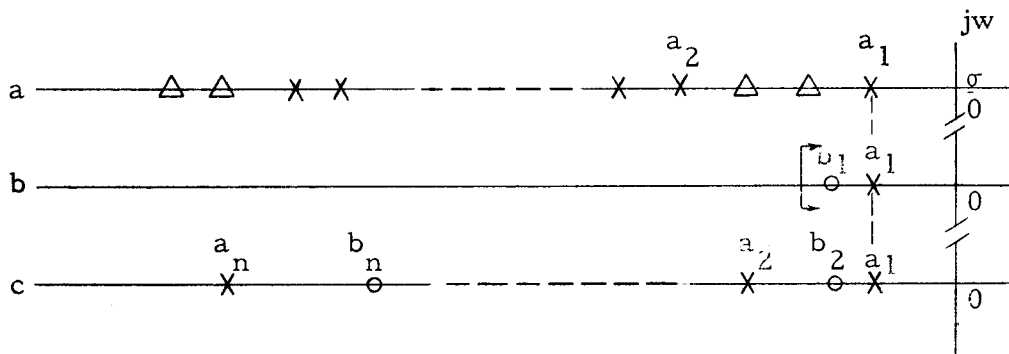
Figure 3. 2.   Zeros of p(s) and q(s).
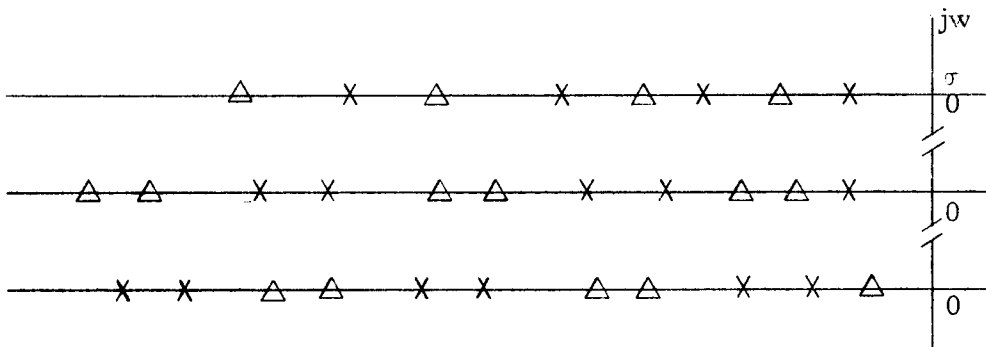
Figure 3. 3.   Possible alternations of zeros of P(S) and Q(S).
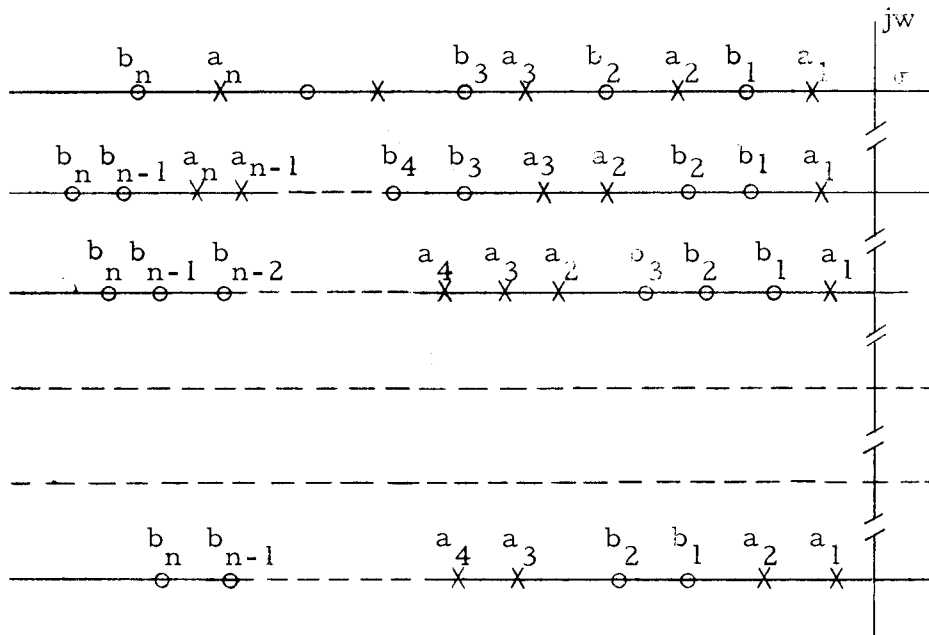
Figure 3. 4.   Examples in zero alternation of p(s) and q(s).

of p(s) and q(s) must alternate on the real axis in groups of $\underline{m}$, where

m is any positive integer, Figure 3. 4, and the theorem is proved.

Theorem I imposes constraints for the RC-network realizability

of $G_{12}(s)$, if both p(s) and q(s) have all their roots on the negative

real axis. Another class of functions, $G_{12}(s)$, having the polynomial

p(s) with complex roots can also be realized by passive RC-networks.

This can be concluded from the root locus shown in Figure 3. 5, which

indicates that if q(s) has roots on the negative real axis while p(s)

has roots on the complex plane, the roots of their sum and difference

for some $k = k_a$ can all be made negative real in all cases but the one

in which there is a critical frequency at zero.

Case II.   Realization of $Z_{12}(s)$

Realization of $Z_{12}(s)$ with a symmetrical lattice network,

Figure 3. 1, will give

$$kZ_{12}(s) = \frac{kP(s)}{q(s)} = \frac{k}{2}[Z_a(s) - Z_b(s)] = \frac{a_2(s)}{b_1(s)} - \frac{b_2(s)}{a_1(s)} \qquad (3.6)$$

where $kp(s) = a_1(s)a_2(s) - b_1(s)b_2(s)$, and $q(s) = a_1(s)b_1(s)$.

From equation (3. 6), it can be seen that a necessary condition

for $Z_{12}(s)$ to be RC realizable is that q(s) has negative real roots.

Polynomial p(s) can have complex roots provided that it has a decom-

position such that $a_2(s)/b_1(s)$ and $b_2(s)/a_1(s)$ are RC driving-point

impedance functions. Since any rational function p(s) can be
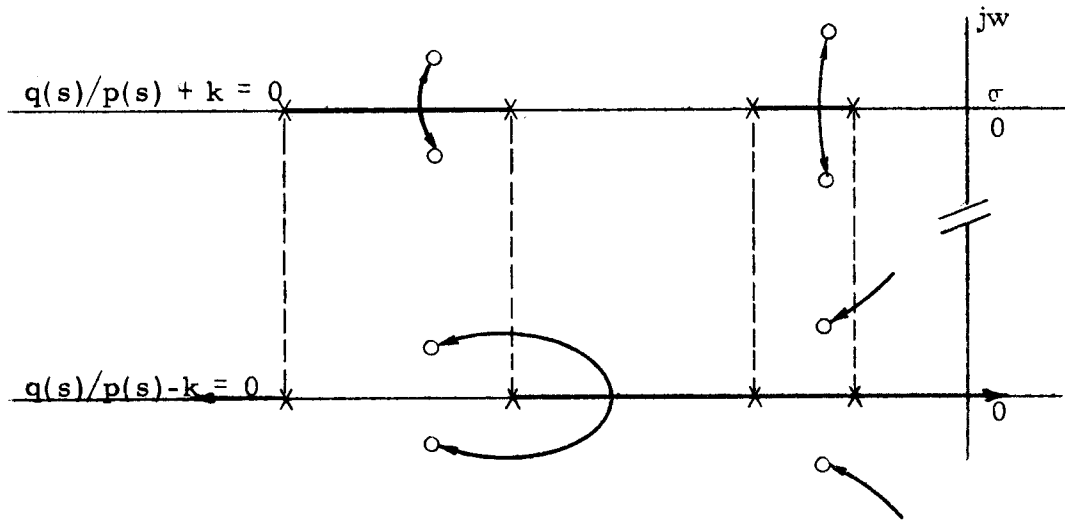
Figure 3. 5. Zeros of p(s) and q(s).
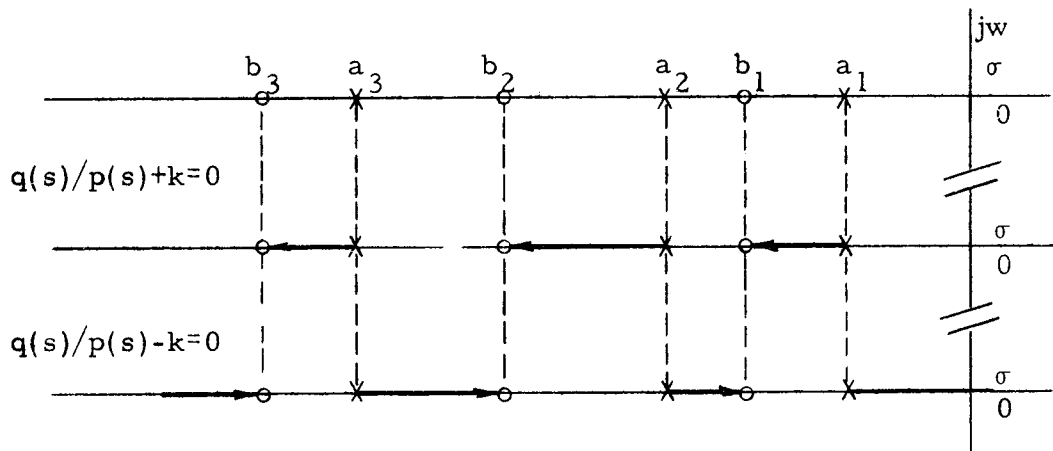
Figure 3. 6. Roots of p(s) and q(s) with relation
of those of kp(s) - q(s) and kp(s) + q(s).

decomposed into the form $p(s) = a_1(s)a_2(s) - b_1(s)b_2(s)$, so that

$a_1(s)/b_1(s)$ and $b_2(s)/a_1(s)$ are RC driving-point impedance functions,

RC-realization of $Z_{12}(s)$, under the above restriction is always pos-

sible.

In summary, transfer function RC realizability requires that

the function, $p(s)/q(s)$, satisfy the following restrictions:

I.   for $G_{12}(s)$:

1. $p(s)/q(s)$ is of the form of RC driving-point

impedance function.

2. $p(s)/q(s)$ has poles and zeros on the real axis and

they alternate in groups of m.

3.   If $p(s)$ has complex roots, they must lie on the

left plane while the roots of $q(s)$ must always be

negative real.

II.   for $Z_{12}(s)$:

The roots of $q(s)$ must lie on the negative real axis.

3.4   Reduction of the Number of Elements in RC Networks
Realized from a Transfer Function

The root-locus plot of Figure 3.6 indicates that if $[q(s) - p(s)]$ /

$q(s)$ and $[q(s) + kp(s)]$ /$q(s)$ are to be RC driving-point impedances,

then $p(s)/q(s)$ is necessarily an RC-impedance function. This class

of functions is, therefore, the only class of functions that can be

realized with RC passive-networks if $Z_a(s)$ and $Z_b(s)$ are as expressed

on page 13. The number of elements in the realized network is de-

duced from the expressions of $Z_a(s)$ and $Z_b(s)$ for a canonical Foster

form of RC-network realization, i. e.,

$$\left.\begin{array}{c} Z_a \\ Z_b \end{array}\right\} = \frac{k_o}{s} + \sum_{i=1}^{m} \frac{k_i}{s+\sigma_i} + k_\infty \qquad (3.7)$$

where m equals the number of roots, σ_i, of q(s) and the k's the

residues of $Z_a(s)$ and $Z_b(s)$ at all critical frequencies.

From equation (3.7) it can be concluded that the number of

elements in the realized network will be equal to 8(m + 1), assuming

that both k_o and k_∞ are present. In comparison, the introduction of

the polynomial, r(s), if it achieves, for instance, t_1 and t_2 cancella-

tions in $Z_a(s)$ and $Z_b(s)$ respectively, will realize the network with a

reduction in the number of elements equal to $4(t_1 + t_2)$, i. e., the

network will have $8(m + 1) - 4(t_1 + t_2)$ elements.

In general, for functions which satisfy constraints (1), (2), and

(3) of $G_{12}(s)$, reduction in the number of elements is made if r(s) is

selected so that maximum cancellation occurs between r(s) and the

numerator polynomials of $Z_a(s)$ and/or $Z_b(s)$, while at the same time

it satisfies constraint (3.3).

No detailed discussion as to the selection of the roots of r(s)

is made here, since their selection will follow directly from root

locus plots similar to those of Figures 3. 3, 3. 4, and 3. 5.

The aforementioned method for the realization of $Z_{12}(s)$ results in a network with a fixed number of elements. This can be easily concluded from the fact that the number of elements in the realized network depends wholly upon the number of poles of $Z_{12}(s)$. Under any partitioning of $Z_{12}(s)$ into $Z_a(s)$ and $Z_b(s)$, the total number of poles of $Z_a(s)$ and $Z_b(s)$ will always be equal to that of $Z_{12}(s)$. In general, if

$$Z_{12}(s) = \frac{p(s)}{q(s)} = \frac{a_1(s)a_3(s) - b_1(s)b_3(s)}{a_1(s)b_1(s)} \qquad (3.8)$$

then $Z_a(s)$ and $Z_b(s)$ will be

$$Z_a(s) = \frac{a_3(s)}{b_1(s)} \text{ and } Z_b(s) = \frac{b_3(s)}{a_1(s)} \qquad (3.9)$$

Examples:

3. 1 Consider the realization of the open-circuit voltage ratio transfer function having only negative real zeros

$$G_{12}(s) = \frac{s^3 + 9s^2 + 23s + 15}{s^3 + 6s^2 + 8s} = \frac{(s+1)(s+3)(s+5)}{s(s+2)(s+4)} \qquad (3.10)$$

equation (3. 10) is of the form of an RC driving-point impedance function and therefore can be realized either by the direct method assigning the expressions of page 13 to $Z_a(s)$ and $Z_b(s)$ or by the method discussed in this work.

a. Direct Method

Let

$$Z_a(s) = 1 - k\frac{s^3+9s^2+23s+15}{s^3+6s^2+8s}$$

(3.11)

$$= \frac{(1-k)s^3+(9-6k)s^2+(23-8k)s+15}{s^3+6s^2+8s}$$

and

$$Z_b(s) = 1 + k\frac{s^3+9s^2+23s+15}{s^3+6s^2+8s}$$

(3.12)

$$= \frac{(1+k)s^3+(9+6k)s^2+(23+8k)s+15}{s^3+6s^2+8s}$$

from the root locus plots, Figure 3.7, of equations (3.11) and (3.12) it can be seen that the values of $\underline{k}$ for RC realizability depends totally on those values which will make $Z_a(s)$ an RC impedance function. The RC impedance character of $Z_b(s)$ is independent of the value of $\underline{k}$.
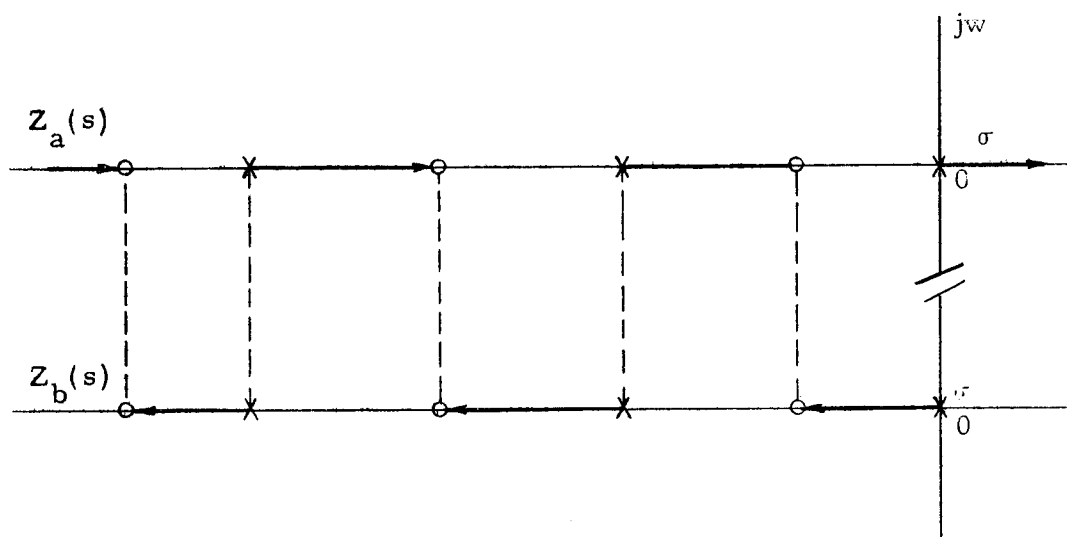
Figure 3.7. Root locus plots of equations (3.11) and (3.12).

A desirable value for $\underline{k}$ can be found, without plotting any of the above locii, through the following procedure: select a root to the left of the most negative zero, for instance $\underline{-6}$; consider it as a zero of $Z_a(s)$, factor it out of the numerator and obtain $\underline{k}$ by setting the remainder equal to zero.

$$\frac{(1-k)s^3+(9-6k)s^2+(23-8k)s+15}{\text{Remainder} \qquad\qquad 15-30+48k} \Bigg| \frac{(1-k)s^2+3s+(5-8k)}{s+6}$$

therefore $k = \dfrac{15}{48}$ and substituting this value in equations (3.11) and (3.12) the expressions for $Z_a(s)$ and $Z_b(s)$ are as follows:

$$Z_a = \frac{.69s^3+7.12s^2+20.5s+15}{s^3+6s^2+8s}$$

$$\text{(3.13)}$$

$$= .69\frac{(s+1.13)(s+3.2)(s+6)}{s(s+2)(s+4)}$$

$$Z_b = \frac{1.31s^3+10.88s^2+25.5s+15}{s^3+6s^2+8s}$$

$$\text{(3.14)}$$

$$= 1.31\frac{(s+.88)(s+2.26)(s+5.12)}{s(s+2)(s+4)}$$

The resulting network, realizing $Z_a(s)$ and $Z_b(s)$ by Foster's method of synthesis is shown in Figure 3.8 and has a total of 20 elements.
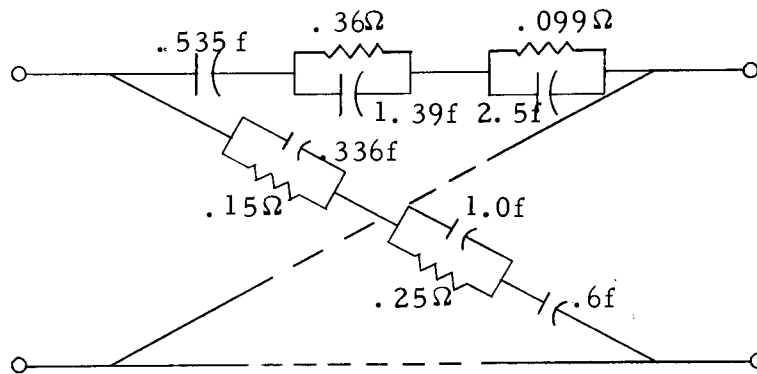
Figure 3.8. Realized network from $G_{12}(s)$ by the direct method.

b. Alternate Method Introduced in This Work

From equation (3.1) and for $k = \dfrac{15}{48}$ $Z_a(s)/Z_b(s)$ takes the following form

$$\frac{Z_a(s)}{Z_b(s)} = \frac{.69(s+1.13)(s+3.2)(s+6)}{1.13(s+.88)(s+2.26)(s+5.12)} \qquad (3.15)$$

Select $r(s) = (s+.88)(s+1.13)(s+5.12)$; division of both numerator and denominator polynomials of equation (3.15) by $r(s)$

$$\frac{Z_a(s)}{Z_b(s)} = \frac{\dfrac{.69(s+3.2)(s+6)}{(s+.88)(s+5.12)}}{\dfrac{1.13(s+2.26)}{s+1.13}} \qquad (3.16)$$

results in the following expressions for $Z_a(s)$ and $Z_b(s)$.

$$Z_a(s) = .69 \frac{(s+3.2)(s+6)}{(s+.88)(s+5.12)}$$

and

$$Z_b(s) = 1.13 \frac{s+2.26}{s+1.13} \qquad (3.17)$$

The resulting network, realizing $Z_a(s)$ and $Z_b(s)$ by Foster's method of synthesis is shown in Figure 3.9 and has a total of $\underline{16}$ elements.
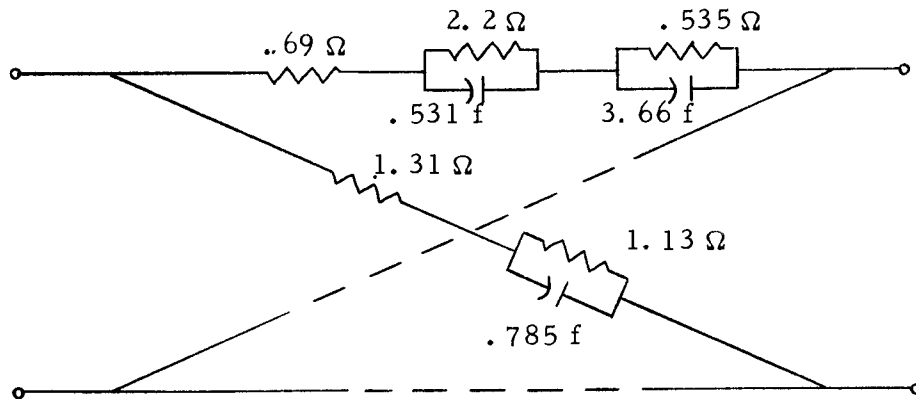
Figure 3.9. Realized network from $G_{12}(s)$ with
$$r(s) = (s+.88)(s+1.13)(s+5.12)$$

3.2 The transfer function of equation (3.18) is an example of a function in which the poles and zeros do not alternate.

$$G_{12}(s) = k \frac{s^2+8s+11.6}{s^3+18.32s^2+86.24s+53.24}$$

$$\qquad (3.18)$$

$$= k \frac{(s+1.8)(s+6.2)}{(s+.72)(s+6.82)(s+10.77)}$$

Since the function is not of the form of an RC driving-point impedance function it cannot be realized

by the direct method. Because it satisfied constraint (2), page 19,

it can be realized by the alternate method. From the root locus plots,

Figure 3. 10 , it can be seen that for $k = 1$ all the roots of $1 - kG_{12}(s)$

and $1 + kG_{12}(s)$ will be negative real and

$$\frac{Z_a(s)}{Z_b(s)} = \frac{(s^3+18.32s^2+86.24s+53.24) - (s^2+8s+11.6)}{(s^3+18.32s^2+86.24s+53.24) + (s^2+8s+11.6)}$$

$$= \frac{(s+.62)(s+7)(s+9.7)}{(s+.82)(s+11.8)(s+6.7)}$$

(3. 19)
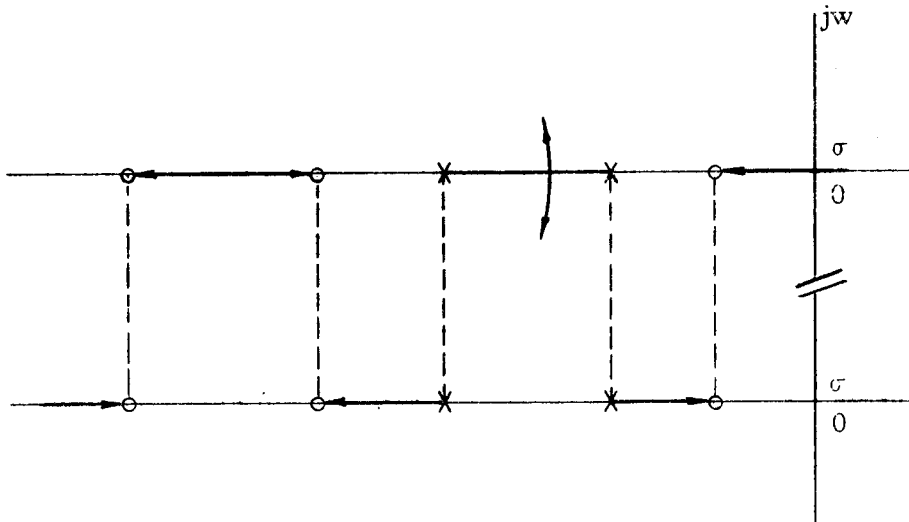
Figure 3. 10.   Root locus plots of

$$1 - k\frac{s^2+8s+11.6}{s^3+18.32s+86.24s+53.24}$$

The most desirable expression for $r(s)$ which will make $Z_a(s)$

and $Z_b(s)$ RC realizable with the minimum number of elements is

$r(s) = (s+.62)(s+.82)(s+9.7)$. $Z_a(s)$ and $Z_b(s)$ will take the following

form

$$Z_a(s) = \frac{s+7}{s+.82} \quad \text{and} \quad Z_b(s) = \frac{(s+6.7)(s+11.8)}{(s+.62)\ (s+9.7)}$$

and the resulting network realized with Foster's method of synthesis

is shown on Figure 3.11.
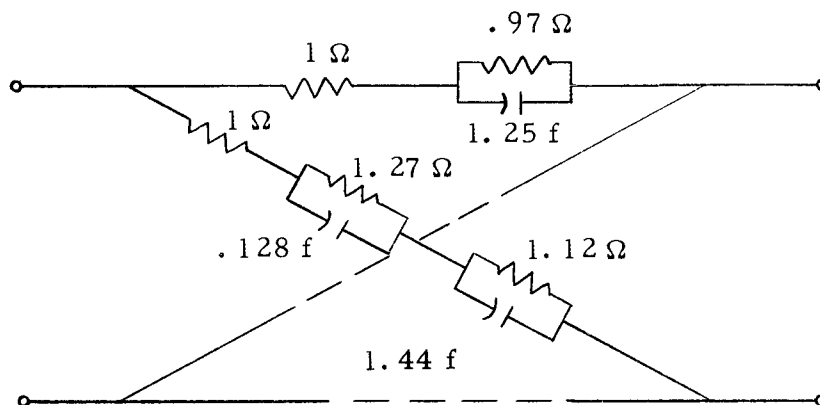
Figure 3.11 RC-realization of

$$G_{12}(s) = \frac{(s+1.8)(s+6.2)}{(s+.72)(s+6.82)(s+10.77)}$$

3.3 The following example is the realization of a transfer

function having only complex zeros and negative real poles.

$$G_{12}(s) = \frac{s^2+2s+2}{s^2+6s+8} \tag{3.20}$$

From the root locus plots, Figure 3.12, of expressions

$1 - kG_{12}(s)$ and $1 + kG_{12}(s)$ it can be seen that a $\underline{k}$ exists which will

make all their roots negative real and for k = .1 .

$$\frac{Z_a(s)}{Z_b(s)} = \frac{(s^2+6s+8) - .1(s^2+2s+2)}{(s^2+6s+8) + .77(s^2+2s+2)}$$

$$= \frac{.1(s+1.88)(s+4.68)}{(s+2.2)(s+3.44)}$$

(3. 21)
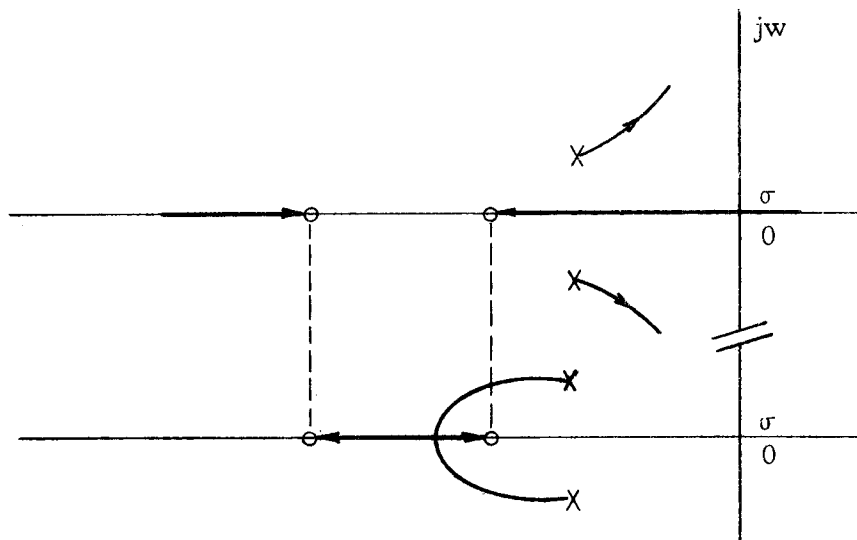
Figure 3. 12.   Root locus of $1 \pm k\dfrac{s^2+2s+2}{s^2+6s+8}$

For RC realizability, with the minimum number of elements, of $Z_a(s)$ and $Z_b(s)$ set $r(s) = (s+1.88)(s+2.2)$ and therefore

$$Z_a(s) = \frac{.1(s+4.68)}{s+2.2} \quad \text{and} \quad Z_b(s) = \frac{s+3.44}{s+1.88}$$

The resulting network realized with Foster's method of synthesis is shown on Figure 3. 13.
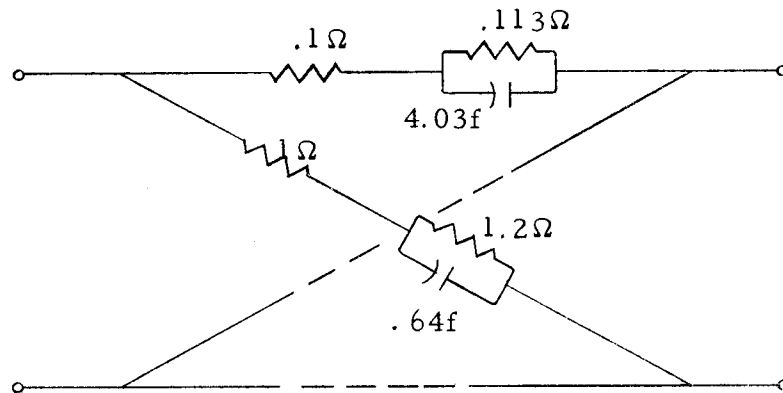
Figure 3. 13.   RC-realization of $G_{12}(s) = \dfrac{s^2+2s+2}{s^2+6s+8}$.

3.5 Synthesis of Parts of a Network Function

Parts of a network function, that is the real part, the imaginary

part or the angle function, are realized by an RC-network if their

corresponding driving-point function satisfies the conditions for RC

driving-point functions.   If a driving-point function, $F(s)$, is written

in the form

$$F(s) = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)}$$ (3. 22)

where m_1, m_2 and n_1, n_2 are the even and odd parts of its numerator

and denominator polynomials, then the even part Ev $F(s)$, odd part

Od $F(s)$ and angle function Arg $F(s)$ will take the following forms,

$$\text{Ev } F(s) = \frac{m_1(s)m_2(s) - n_1(s)n_2(s)}{m_2^2(s) - n_2^2(s)} \qquad (3.23)$$

$$\text{Od } F(s) = \frac{m_2(s)n_1(s) - m_1(s)n_2(s)}{m_2^2(s) - n_2^2(s)} \qquad (3.24)$$

$$\text{Arg } F(s) = \tan^{-1} \frac{m_2(s)n_1(s) - m_1(s)n_2(s)}{m_1(s)m_2(s) - n_1(s)n_2(s)} \qquad (3.25)$$

The right-hand side expressions of (3.23), (3.24), and (3.25) are the decompositions of the rational functions Ev $F(s)$, Od $F(s)$, and Arg $F(s)$ respectively and according to theorems (2.1) and (2.2) these decompositions are unique; therefore, within a constant multiplier, $F(s)$ is derived uniquely from all parts of network functions.

As discussed in paragraph (3.1), the network from a driving-point function realized by Foster's or Cauer's method is canonical; therefore, it is concluded that parts of network functions realize canonical networks.

3.6   Synthesis of Driving-Point Function and Transfer Function

Simultaneous realization of a driving-point, $F_1(s)$, and a transfer function, $F_2(s)$, by an RC-network requires that:

1. $F_1(s)$ satisfies the conditions for RC driving-point functions.

2.   The poles of $F_2(s)$ are poles of $F_1(s)$ and the zeros of $F_2(s)$ lie on the negative real axis and are in number equal to or less than

the zeros of $F_1(s)$.

The synthesis of a network from these two functions is accomplished through the known method of zero-shift, pole-removal technique. By this technique the network realization will be dependent upon the sequence in which the zero shifting is accomplished. These equivalent networks may have the same or a definite number of elements depending upon the zero shifting sequence. If the realization is accomplished with a single shift of each zero, the number of elements required is m^2, where m is the number of zeros in $F_2(s)$; the total number of elements required will be higher if multiple shifting of the zeros is required.

To obtain the network with the least number of elements, all possible network configurations have to be realized, since the effect of any zero shift on the other zeros and on the following zero-shifts and pole-removals, cannot be determined in advance.

Due to the amount of work involved in obtaining all the possible configurations a computer program has been written, Appendix 2, to perform this task. The network with the minimum number of elements, if present, is then obtained by scanning the program's output.

To indicate the existence of networks with different number of elements and the work involved, the following simple example is given.

3.4   The transfer function specified by equations (3.25) has

pole and zeros that alternate on the negative real axis.   This transfer

function can therefore be realized as an RC-ladder network.

$$y_{11}(s) = \frac{(s+1)(s+3)(s+5)}{(s+2)(s+4)(s+6)}$$

and

$$(3.25)$$

$$y_{12}(s) = \frac{s(s+1/2)(s+2.72)}{(s+2)(s+4)(s+6)}$$

Two different configurations have been obtained by changing

the order of zero-shifting and zero-producing steps.   The various

zero-shifting and zero-producing steps in the network development

and the corresponding networks are given in Figure (3.14) and

Figure (3.15), having a total of 8 and 9 elements respectively.
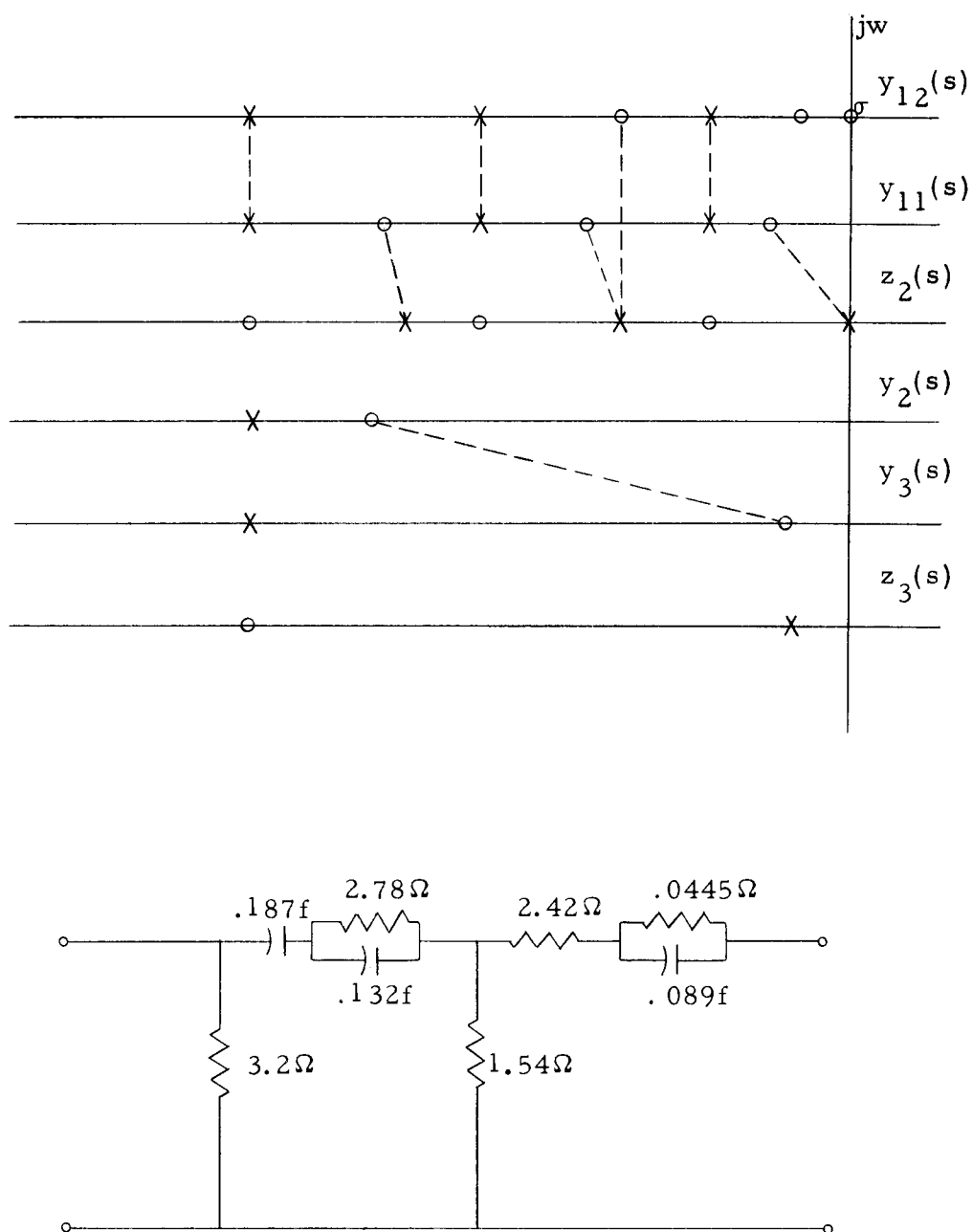
Figure 3. 14. First network development of Example 3. 4.
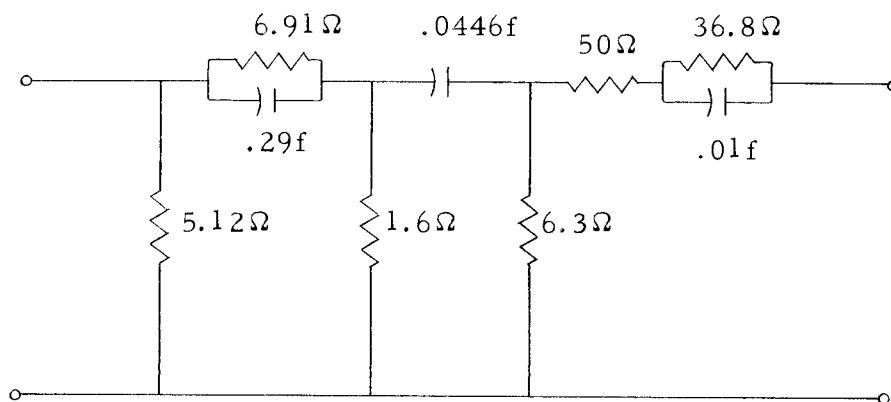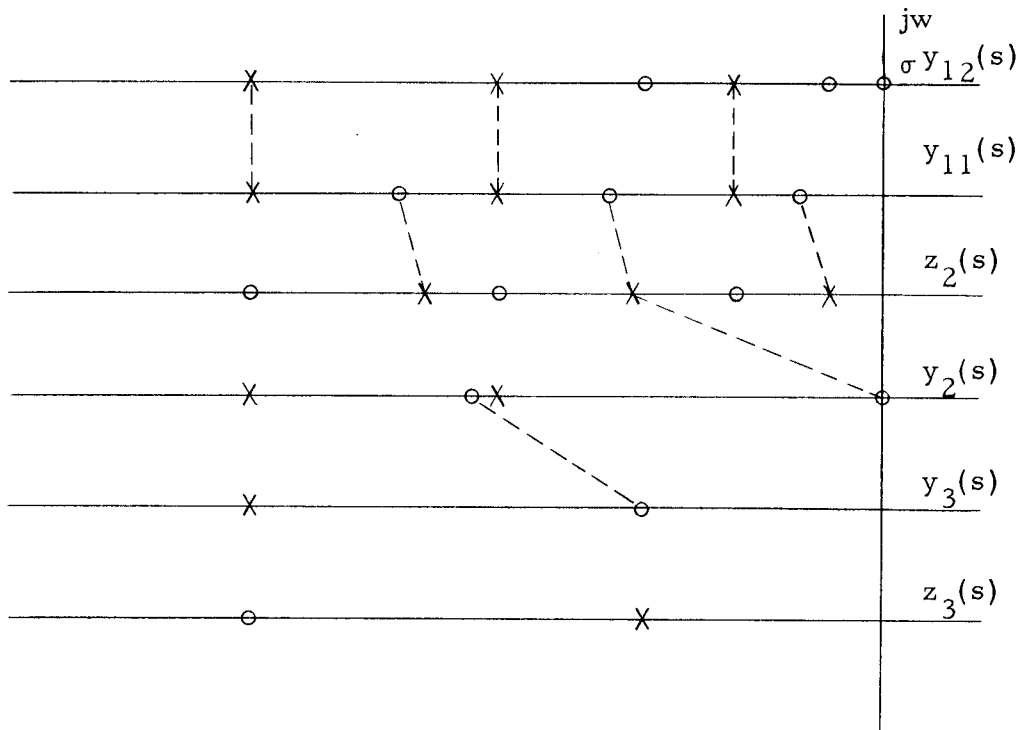
Figure 3. 15.   Second netowrk development of Example 3. 4.

IV. ACTIVE RC-NETWORKS

4.1 Introduction

This chapter deals with the element reduction on active RC-networks realized with a negative impedance converter, NIC. It is pertinent to observe that the major methods of RC-NIC synthesis techniques come about from the different interconnections of two three-terminal networks, one of which consists of a connection in tandem of an NIC and RC network. Such consideration leads to a unification of the field of active RC-networks, in that they can be obtained from the interconnection of this type of networks. Appendix 2 gives the configurations and characteristic functions of all possible interconnections of an RC and an RC-NIC network.

Written in a general form the transfer functions, T(s), and driving-point functions, D(s), have the following expressions.

1. $$T(s) = \frac{f_1(s)f_2(s)}{F_1(s) - F_2(s)}$$

2. $$T(s) = \frac{f_2(s)-f_1(s)}{F_1(s) - F_2(s)}$$

3. $$T(s) = \frac{f_1(s) + f_2(s)}{F_1(s) - F_2(s)}$$

4. $\quad T(s) = \dfrac{f_1(s)F_2(s) - f_2(s)F_1(s)}{F_1(s) - F_2(s)}$

5. $\quad D(s) = F_1(s) - \dfrac{f_1^2(s)}{F_1'(s) - F_2'(s)}$

6. $\quad D(s) = F_1(s) + F_2(s) - \dfrac{f_1^2(s) - f_2^2(s)}{F_1'(s) - F_2'(s)}$

Where $f_1(s)$ and $f_2(s)$ designate the transfer functions of the

RC and RC-NIC network respectively, and where the F(s)'s designate

their driving-point functions; primed in the expressions where both

input and output driving-point functions of the same network are

present to indicate their existence in the expression.

Active-RC-network synthesis, with interconnections of two or

more three-terminal networks leads to an unlimited number of pos-

sible configurations for active RC-network realization. Thus, the

selection of the particular configuration, based on its character,

sensitivity or number of elements of the given function to be realized,

is broadened considerably.

4.2 Transfer Function Synthesis of the Form $f_1(s)f_2(s)/[F_1(s) - F_2(s)]$

Realization of a transfer function of the form,

$$\text{transfer function} = \dfrac{f_1(s)f_2(s)}{F_1(s) - F_2'(s)} \tag{4.1}$$

can be achieved with the configuration of Figure 1, Appendix 2, where

$$Z_{12}(s) = \frac{z_{12a}(s) z_{12b}(s)}{z_{22a}(s) - z_{11b}(s)}$$ (4.2)

For realization of the transfer function, $z_{12}(s) = \frac{p(s)}{q(s)}$, the numerator and denominator are divided by an arbitrary polynomial r(s), having $\underline{n}$ distinct negative real roots, σ_1, σ_2, σ_3, · · · · σ_n; where n is equal to or greater than the degree of p(s) or q(s), whichever is greater.

$$Z_{12}(s) \frac{\frac{p(s)}{r(s)}}{\frac{q(s)}{r(s)}} = \frac{\frac{p(s)}{(s+\sigma_1)(s+\sigma_2)\cdots(s+\sigma_n)}}{\frac{q(s)}{(s+\sigma_1)(s+\sigma_2)\cdots(s+\sigma_n)}}$$ (4.3)

The denominator of equation (4.3) is then expanded into partial fractions yielding terms with positive and negative residues. The sum of the terms with positive residues is associated with z_{22a} while the sum of the terms with negative residues is associated with $z_{11b}(s)$. When $z_{22a}(s)$ and $z_{11b}(s)$ have been determined, the appropriate factors of p(s) may be associated with $z_{12a}(s)$ and $z_{12b}(s)$, and the networks realized by passive network synthesis methods.

The number of elements in the realized network can be reduced through proper selection of the polynomial r(s). Such a selection is made utilizing the theorems of polynomial decomposition in Chapter 2.

Case I:

If the polynomial p(s) has only negative real roots and q(s) is any prescribed real polynomial of degree equal to or smaller than the degree of p(s), then the right-hand side of equation (4. 3) can be decomposed and written in the form,

$$z_{12}(s) = \cfrac{\cfrac{a_1(s)b_1(s)}{a_1(s)b_1(s)}}{\cfrac{a_1(s)a_2(s) - b_1(s)b_2(s)}{a_1(s)b_1(s)}}$$

$$= \cfrac{1}{\cfrac{a_2(s)}{b_1(s)} - \cfrac{b_2(s)}{a_1(s)}}$$

(4. 4)

Comparing equations (4. 2) and (4. 4) yields the following identification of parameters,

$$z_{12a}(s) = 1; \quad z_{22a}(s) = \frac{a_2(s)}{b_1(s)} \qquad (4. 5)$$

$$z_{21b}(s) = 1; \quad z_{11b}(s) = \frac{b_2(s)}{a_1(s)} \qquad (4. 6)$$

Case II:

If p(s) has only negative real roots and q(s) is any prescribed real polynomial of degree greater than the degree of p(s), then the right-hand side of equation (4. 3) can be decomposed and written in

the form,

$$Z_{12}(s) = \cfrac{\cfrac{p(s)}{a_1(s)b_1(s)}}{\cfrac{a_1(s)a_2(s) - b_1(s)b_2(s)}{a_1(s)b_1(s)}}$$

(4.7)

$$= \cfrac{\cfrac{1}{c(s)}}{\cfrac{a_2(s)}{b_1(s)} - \cfrac{b_2(s)}{a_1(s)}}$$

Comparing equations (4. 2) and (4. 7) yields the following identification of parameters,

$$z_{12a}(s) = \frac{1}{c_1(s)} ; \qquad z_{22a}(s) = \frac{a_2(s)}{b_1(s)}$$

(4. 8)

$$z_{12b}(s) = \frac{1}{c_2(s)} ; \qquad z_{11b}(s) = \frac{b_2(s)}{a_1(s)}$$

(4. 9)

where $c(s) = c_1(s)c_2(s)$ and $c_1(s)$ and $c_2(s)$ are factors of $b_1(s)$ and $a_1(s)$ respectively.

Case III:

If $p(s)$ and $q(s)$ are any prescribed real polynomials, the transfer function synthesis will require realization of complex zeros of transmission. Realization of complex zeros of transmission with RC-networks can be accomplished with ladder-networks by the methods of Guillemin [18], Dasher [10], and Hakimi and Seshu [20].

Reduction in the number of elements in these networks can be done through proper decomposition of q(s). The reduction of the number of elements for this case will not be discussed here because the parallel structure of Figures 2 and 3, Appendix 2, will realize complex zeros of transmission with relatively simple network structures.

Example 4.1

The following example is the realization of a transfer function having only negative real zeros and complex poles.

$$Z_{12}(s) = k \frac{(s+1)(s+2)(s+3)(s+4)}{s^4+5s^3+14s^2+18s+12} \tag{4.10}$$

The degrees of numerator and denominator polynomials of equation (4.10) are the same. Therefore, according to equation (4.4), $Z_{12}(s)$ can be written as:

$$Z_{12}(s)= k \frac{\dfrac{(s+1)(s+2)(s+3)(s+4)}{(s+1)(s+2)(s+3)(s+4)}}{\dfrac{s(s+1.08)(s+2)(s+4)(s+10.85) - (s+1)(s+2.35)(s+3)(s+24.6)}{(s+1)(s+2)(s+3)(s+4)}}$$

$$= \frac{1/2}{\dfrac{(s+1.08)(s+10.85)}{(s+1)(s+3)} - 1/2 \dfrac{(s+2.35)(s+24.6)}{(s+2)(s+4)}}$$

and

$$z_{12a} = 1 \quad ; \quad z_{22a} = \frac{(s+1.08)(s+10.85)}{(s+1)(s+3)} \tag{4.11}$$

$$z_{12b} = 1/2 \quad ; \quad z_{11b} = 1/2\frac{(s+2.35)(s+24.6)}{(s+2)(s+4)} \tag{4.12}$$

For the realization, the removal of the private poles of the driving-point functions, -1 and -3 for z_{22a}, -2 and -4 for z_{11b}, as series impedances will result in the following expressions for $z_{11b}(s)$ and $z_{22a}(s)$,

$$z_{22}a = 1 \quad \text{and} \quad z_{11b} = 1/2 \tag{4.13}$$

Equations (4.13), (4.12), and (4.11) will realize networks containing only a shunt resistor, and the complete realization of equations (4.10) is given by the following network.
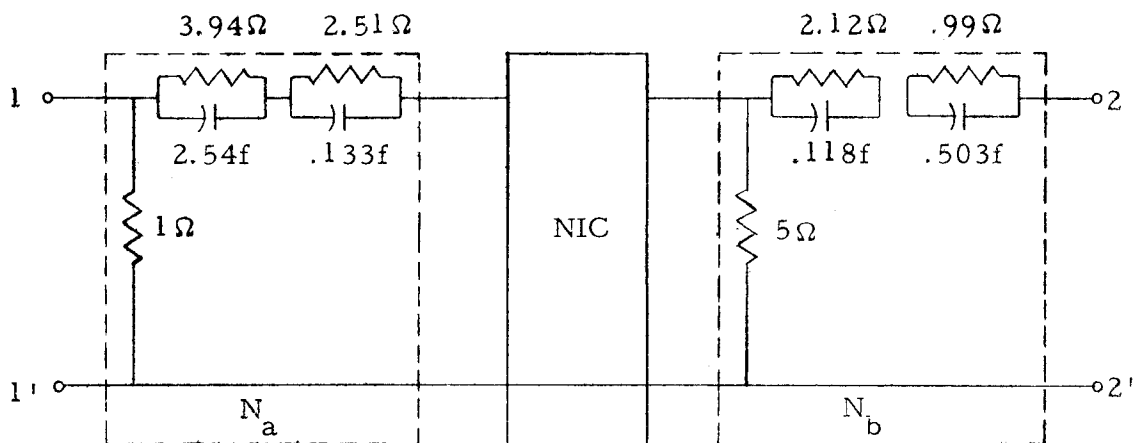
Figure 4.1 Realization with networks in cascade of

$$Z_{12}(s) = \frac{(s+1)(s+2)(s+3)(s+4)}{s^4+5s^3+14s^2+18s+12} \quad \text{with}$$

$$r(s) = (s+1)(s+2)(s+3)(s+4).$$

If an arbitrary selection of r(s) were made, for example, let

$r(s) = s(s+1.5)(s+2.5)(s+4)$, the expressions for $z_{12a}(s)$, $z_{12b}(s)$,

$z_{11b}(s)$, $z_{22a}(s)$, obtained through partial fraction expansion, are:

$$z_{12a}(s) = \frac{(s+1)(s+3)}{s(s+2.5)}$$

$$z_{22a}(s) = \frac{(s+7.25)(s+.475)}{s(s+2.5)}$$

(4.14)

$$z_{12b}(s) = \frac{8(s+1.9)}{(s+1.5)(s+4)}$$

(4.15)

$$z_{11b}(s) = \frac{s+2}{s+1.5}$$

Realization of equations (4.14) and (4.15) by the zero-shift

pole-removal technique, results in the following network, with $\underline{4}$

more elements than the network of Figure 4.1.
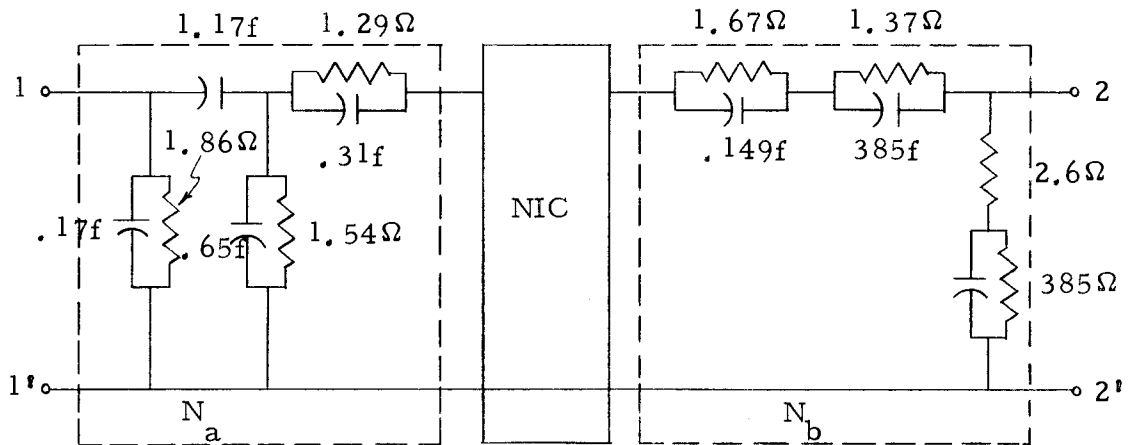
Figure 4.2. Realization with networks in cascade of

$$Z_{12}(s) = \frac{(s+1)(s+2)(s+3)(s+4)}{s^4+5s^3+14s^2+18s+12} \quad \text{with}$$

$r(s) = s(s+1.5)(s+4)$.

4.2 The following example is the realization of a transfer function having negative-real zeros and complex poles, but with zeros of transmission at infinity.

$$Z_{12}(s) = \frac{s+1}{s^4+5s^3+14s^2+18s+12} \qquad (4.16)$$

In equation (4.16) the degree of the numerator polynomial is smaller than the degree of the denominator polynomial. Therefore, according to equation (4.7), $Z_{12}(s)$ can be written as:

$$Z_{12}(s) = \frac{\dfrac{(s+1)}{(s+1)(s+2)(s+3)(s+4)}}{\dfrac{2(s+1.08)(s+2)(s+4)(s+10.85)-(s+1)(s+2.35)(s+24.6)}{(s+1)(s+2)(s+3)(s+4)}}$$

$$= \frac{\dfrac{1/2}{(s+2)(s+3)(s+4)}}{\dfrac{(s+1.08)(s+10.85)}{(s+1)(s+3)} - \dfrac{1/2(s+2.35)(s+24.6)}{(s+2)(s+4)}}$$

from which the following are easily obtained.

$$z_{12a}(s) = \frac{1}{s+3} \quad ; \quad z_{22a}(s) = \frac{(s+1.08)(s+10.85)}{(s+1)(s+3)} \qquad (4.17)$$

$$z_{12b}(s) = \frac{1/2}{(s+2)(s+4)}$$

$$z_{11b}(s) = 1/2 \frac{(s+2.35)(s+24.6)}{(2+2)(s+4)} \qquad (4.18)$$

Equations (4.17) and (4.18) give realization to the network of Figure 4.3.
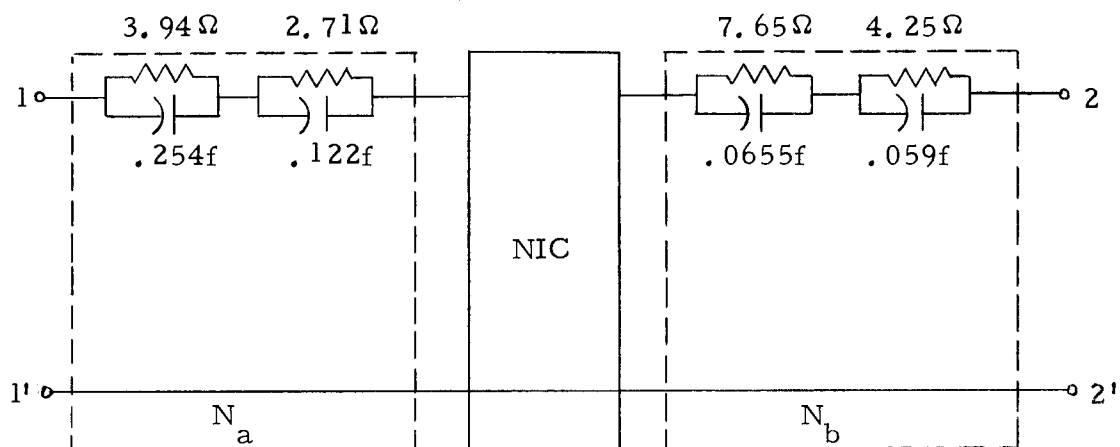
Figure 4. 3.   Realization with networks in cascade of

$$Z_{12}(s) = \frac{s+1}{s^4+5s^3+14s^2+18s+12} \quad \text{with}$$

$$r(s) = (s+1)(s+2)(s+3)(s+4) .$$

Realization of equation (4. 16) with an arbitrary selection of

r(s), r(s) = s(s+5)(s+7)(s+10), gives rise to a network of Figure 4. 4

with <u>2</u> more elements of that of Figure 4. 2.
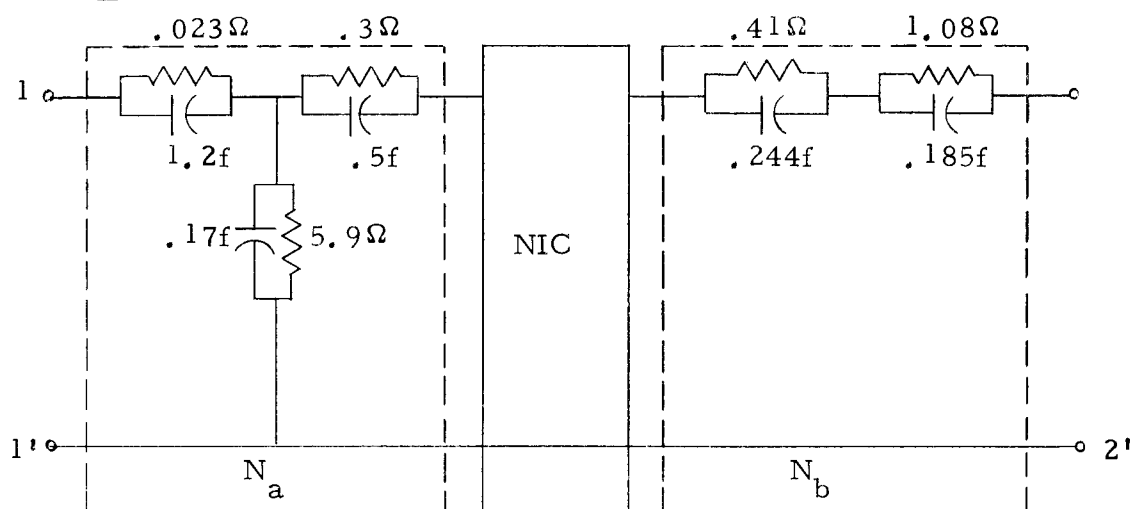
Figure 4. 4.   Realization with networks in cascade of

$$Z_{12}(s) = \frac{s+1}{s^4+5s^3+14s^2+18s+12}$$

$$r(s) = s(s+5)(s+7)(s+10).$$

4. 3 Transfer Function of Form $[f_2(s)-f_1(s)]/[F_1(s)-F_2(s)]$ Synthesis

Realization of a transfer function of the form,

$$\text{transfer function} = \frac{f_2(s) - f_1(s)}{F_1(s) - F_2(s)} \qquad (4.19)$$

is accomplished with the configurations of Figure 2 and Figure 3,

Appendix 2.

In the previous transfer function synthesis only the poles of the

overall transfer function were controlled by the NIC. It is apparent

from equation (4. 19) that in the synthesis with this form the NIC con-

trols the zeros as well as the poles of the transfer function. Conse-

quently, by this technique realization of complex zeros of transmis-

sion is accomplished using relatively simple RC-structures.

For example, consider realizability with the configuration of

Figure 2, Appendix 2, where

$$G_{12}(s) = \frac{y_{12b}(s) - y_{12a}(s)}{y_{22a}(s) - y_{22b}(s)} \qquad (4.20)$$

From equation (4. 20) it is evident that $y_{22a}(s)$ and $y_{12a}(s)$ as

well as $y_{22b}(s)$ and $y_{12b}(s)$ cannot be synthesized separately. The

proposed method of realization of these networks is through the use

of inverted-L circuits as shown in Figure 4. 5a.
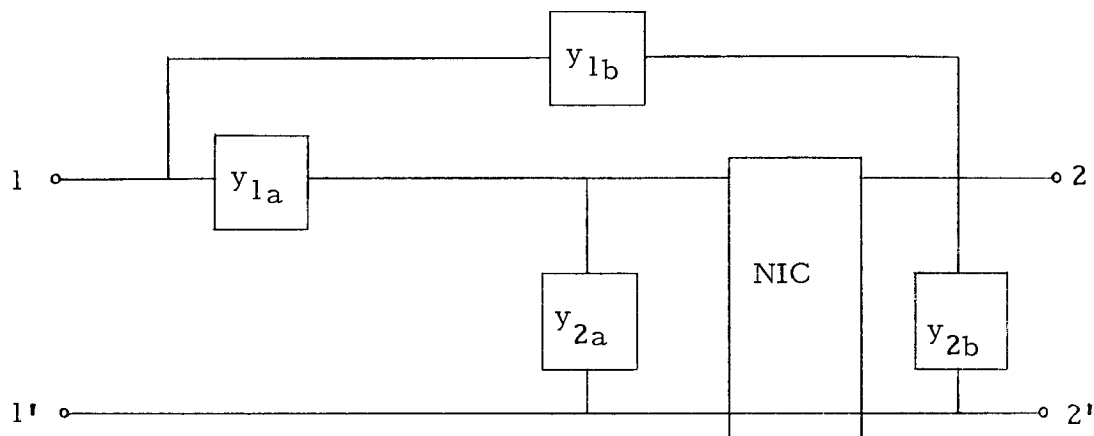
Figure 4. 5a. Realization with inverted-L circuits.

The voltage transfer function of this circuit is

$$G_{12}(s) = \frac{y_{1b} - y_{1a}}{(y_{1b} - y_{1a}) + (y_{2b} - y_{2a})} \qquad (4.21)$$

A rational function $p(s)/q(s)$ to be realized with inverted-L

circuits is first written in the form

$$G_{12}(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{p(s) + q(s) - p(s)} \qquad (4.22)$$

The numerator and denominator are then divided by a polynomial $r(s)$

with $\underline{n}$ distinct negative real roots, $\sigma_1, \sigma_2, \cdots, \sigma_n$. The degree of

$r(s)$ being equal to or greater than the highest degree of $p(s)$ or $q(s)$.

From equations (4. 21) and (4. 22)

$$y_{1b} - y_{1a} = \frac{p(s)}{r(s)} = k_\infty s + k_o + \sum_{i=1}^{n} \frac{k_{is}}{s + \sigma_i} \qquad (4.23)$$

$$y_{2b}-y_{2a} = \frac{q(s) - p(s)}{r(s)} = k_\infty' s + k_o' + \sum_{i=1}^{n} \frac{k_i' s}{s+\sigma_1} \qquad (4.24)$$

In equation (4.23) the terms with negative values of k_i are associated with $y_{1a}(s)$ and those with positive values of k_i with $y_{1b}(s)$. Similarly, in equation (4.24) the terms with negative values of k_i' are associated with $y_{2a}(s)$ and those with positive values with $y_{2b}(s)$. These four RC-admittances can be easily synthesized on two terminal RC-networks by known methods.

Reduction of the number of elements in the realization of this tranfer function can, in certain cases, be made. This reduction is accomplished by using decomposition of the given transfer function for the selection of the common divisor and realizing the network by the basic configuration of Figure 2, Appendix 2.

Any prescribed real rational function $G_{12}(s) = p(s)/q(s)$ can be decomposed and written in the following form.

$$G_{12}(s) = \frac{p(s)}{q(s)} = \frac{b_2(s)a_1(s) - a_2(s)b_1(s)}{a_1(s)a_2(s) - sb_1(s)b_2(s)} \qquad (4.25)$$

Selecting the common divisor $r(s) = a_1(s)b_1(s)$ and dividing the numerator and denominator of equation (4.25) by it, $G_{12}(s)$ can be written as

$$G_{12}(s) = \frac{\dfrac{b_2(s)}{b_1(s)} - \dfrac{a_2(s)}{a_1(s)}}{\dfrac{a_2(s)}{b_1(s)} - \dfrac{sb_2(s)}{a_1(s)}} \qquad (4.26)$$

Comparing equations (4.20) and (4.25) yields the following

identifications of parameters,

$$-y_{12a}(s) = \frac{b_2(s)}{b_1(s)} ; \qquad y_{11a}(s) = \frac{a_2(s)}{b_1(s)} \qquad (4.27)$$

$$-y_{12b}(s) = \frac{a_2(s)}{a_1(s)} ; \qquad y_{11b}(s) = \frac{sb_2(s)}{b_1(s)} \qquad (4.28)$$

Comparison in the number of elements in the resulting networks

using either the basic configuration or the inverted-L configuration

can be made through direct observation of equations (4.23) and (4.24)

and equations (4.27) and (4.28). In the inverted-L realization the

number of elements will depend on the number of roots of r(s) which,

in turn, depends on the power of the highest degree polynomial p(s)

or q(s). If the highest degree is $\underline{n}$, assuming both residues at zero

and infinity are present, the number of elements in the network

realization will be 2(2n+2). In the realization with the basic configura-

tion no positive statement as to the exact number of elements in the

resulting network can be made. In the worst case, if the highest

power polynomial is of degree $\underline{n}$, assuming that every zero of trans-

mission requires one zero shift and since a zero-shift produces one

element and a pole-removal two elements, the number of elements

required for the realization of the given function is $2(\frac{n}{2} + n + 1)$.

Therefore, using the basic configuration, the resulting network is

realized with at least $\underline{n}$ elements less.

4.3 The following example is the realization of a transfer

function having only complex poles and zeros.

$$G_{12}(s) = \frac{2s^2 + s + 3}{s^2 + 3s + 6} \qquad (4.29)$$

with the configuration of Figure 2, Appendix 2.

a. Realization with the Basic Configuration

Equation (4.29) can be put into the form of equation (4.25) by

the use of polynomial decomposition, from which

$$G_{12}(s) = k\frac{2s^2 + s + 3}{s^2 + 3s + 6} = k\frac{6(s+1/2)(s+1) - 4s(s+2)}{5(s+6/5)(s+1) - 4s(s+2)} \qquad (4.30)$$

If $r(s) = 6(s+1)(s+2)$, the admittance expressions according to

equations (4.27) and (4.28) will be

$$-y_{12a}(s) = \frac{s+1/2}{s+2} \; ; \qquad y_{22a}(s) = \frac{5}{3}\frac{s+6/5}{s+2} \qquad (4.31)$$

$$-y_{12b}(s) = \frac{2}{3}\frac{s}{s+1} \; ; \qquad y_{22b}(s) = \frac{4}{3}\frac{s}{s+1} \qquad (4.32)$$

where $k = 1/2$.

Two points of interest are noted in the above example. r(s)

has a constant multiplier 6, equal to the constant multiplier of

$y_{12a}(s)$ in equation (4. 30). This is done to realize $G_{12}(s)$ exactly.

The constant multiplier of $y_{12b}(s)$, equation (4. 32), is realized by

changing the conversion ratio of the NIC from -1 to -4/6. Then the

resulting network will be realized with grounded RC-ladder networks

provided that the Fialkow-Gerst condition is satisfied. If this condi-

tion is not satisfied, $G_{12}(s)$ must be realized within a constant multi-

plier k as shown above.

Synthesis of equations (4. 31) and (4. 32) gives the network con-

figuration of Figure 4. 5b.
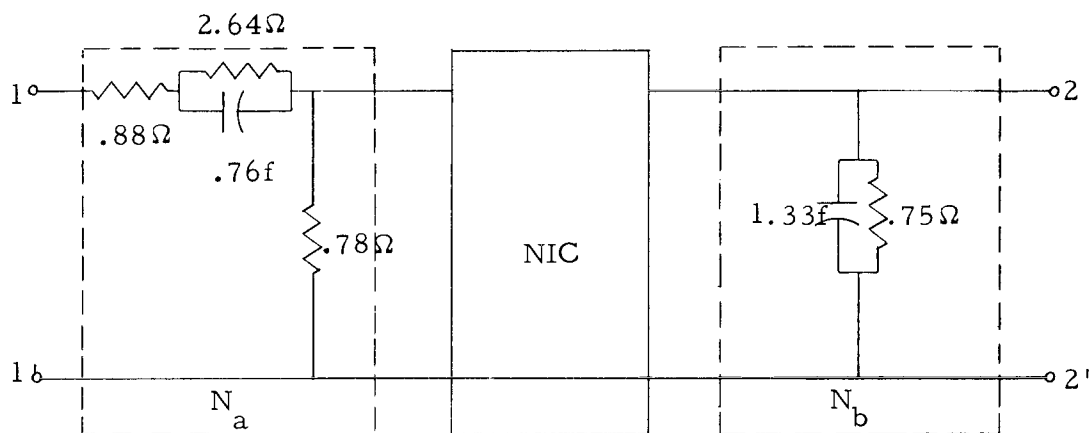
Figure 4. 5b. Parallel realization of
$$G_{12}(s) = \frac{2s^2+s+3}{s^2+3+6}$$ using the basic configuration
and r(s) = 6(s+1)(s+2).

b. Realization with Inverted-L Sections

From equation (4. 21) and for $r(s) = (s+1)(s+2)$ one obtains the

following expressions for $y_{1b} - y_{1a}$ and $y_{2b} - y_{2a}$ if $G_{12}(s)$ is to be

realized within a constant multiplier of $1/2$.

$$y_{1b} - y_{1a} = \frac{3/2}{s} + \frac{9/2}{s+2} - \frac{4}{s+1} \tag{4.33}$$

$$y_{2b} - y_{2a} = \frac{3/2}{s} + \frac{2}{s+1} - \frac{7/2}{s+2} \tag{4.34}$$

The resulting network will therefore have the element values

and configurations of Figure 4. 6. The number of elements in this

realization is $\underline{10}$ as compared to $\underline{6}$ of the previous realization.
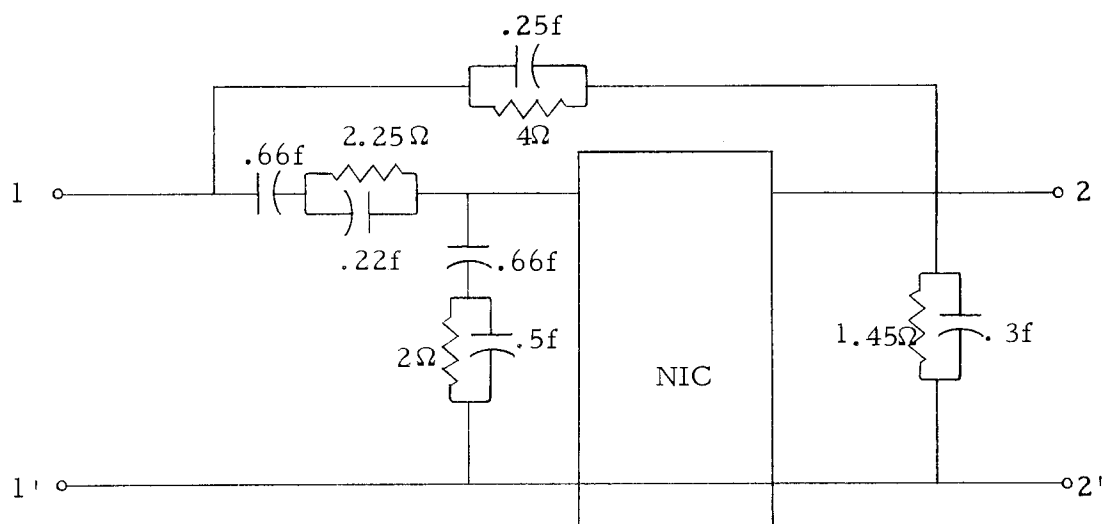
Figure 4. 6. Parallel realization of

$$G_{12}(s) = \frac{2s^2+s+3}{s^2+3s+6} \quad \text{using the inverted-L}$$

configuration.

4.4 Transfer Function of Form $[f_1(s)+f_2(s)]/[F_1(s)-F_2(s)]$ Synthesis

Realization of a transfer function of the form,

$$\text{transfer function} = \frac{f_1(s) + f_2(s)}{F_1(s) - F_2(s)} \tag{4.35}$$

is accomplished with the configuration of Figure 2 and Figure 3,

Appendix 2.

As in the previous realization form, the NIC controls the zeros

as well as the poles of the transfer function. Consequently, by this

technique realization of complex zeros of transmission is accom-

plished using relatively simple RC-structures.

Realization of a transfer function, with this new form, is pos-

sible provided that the function has a numerator and denominator

which satisfy the conditions for Calahan and Horowitz decomposition

respectively. For example, consider the realizability with the con-

figuration of Figure 2, Appendix 2, where

$$G_{12}(s) = \frac{y_{12a}(s) + y_{12b}(s)}{y_{22a}(s) - y_{22b}(s)} \tag{4.36}$$

From equation (4.36) it is evident that $y_{22a}(s)$ and $y_{12a}(s)$ as

well as $y_{22b}(s)$ and $y_{12b}(s)$ cannot be synthesized separately. The

proposed method of realization of these networks is as follows:

If the function to be realized is of the $G_{12}(s) = p(s)/q(s)$, and

p(s) and q(s) are polynomials containing both real and imaginary

roots, they can be decomposed and written into the form

$$p(s) = C(s) + D(s)V(s) \tag{4.37}$$

and

$$q(s) = a_1(s)a_2(s) - b_1(s)b_2(s) \tag{4.38}$$

Where V(s) is a polynomial containing all the real roots of

p(s) and where C(s) and D(s) are the decomposed polynomials of

some function V(s) containing all the complex roots of p(s).

To proceed with the realization, the decomposition of p(s) is

obtained and is written in the following form

$$p(s) = [a_3(s)C_3(s) + b_3(s)C_4(s)] V(s) \tag{4.39}$$

The decomposition of q(s) is then obtained according to the

procedure discussed in Chapter II using as a divisor the polynomial

$a_3(s)b_3(s)V(s)$. Therefore,

$$\frac{q(s)}{a_3(s)b_3(s)V(s)} = \frac{a_1(s)a_2(s) - b_1(s)b_2(s)}{a_3(s)b_3(s)V(s)}$$

or

$$q(s) = a_1(s)a_2(s) - b_1(s)b_2(s) \tag{4.40}$$

In view of equations (4.39) and (4.40), $G_{12}(s)$ can be written

as

$$G_{12}(s) = \frac{\dfrac{C_3(s)}{a_4(s)} + \dfrac{C_4(s)}{b_4(s)}}{\dfrac{a_2(s)}{b_1(s)} - \dfrac{b_2(s)}{a_1(s)}} \qquad \text{if n even} \qquad (4.43)$$

or

$$G_{12}(s) = \frac{\dfrac{C_3(s)}{b_4(s)} + \dfrac{C_4(s)}{a_4(s)}}{\dfrac{a_2(s)}{b_1(s)} - \dfrac{b_2(s)}{b_1(s)}} \qquad \text{if n odd} \qquad (4.44)$$

From equations (4.43) and (4.44) the expression for admittance functions, $y_{12a}(s)$, $y_{12b}(s)$, $y_{22a}(s)$, and $y_{22b}(s)$ can be easily obtained.

The use of decomposition in obtaining the functions for the subnetwork of equations (4.42) and (4.43) results in a network with a minimum number of elements for a network with the configuration of Figure 2, Appendix 2.

4.4 The following example is an alternate realization of a transfer function having negative-real zeros and complex with zeros of transmission at infinity.

$$G_{12}(s) = k\frac{(s+2.5)}{s^2+2s+2} \qquad (4.45)$$

Decompose equation (4.45) into the form of equation (4.39)

$$G_{12}(s) = \frac{(s+3) + (s+2)}{s^2+2s+2} \qquad (4.46)$$

Setting $a_3(s)b_3(s)V(s) = (s+2)(s+3)$, the denominator of equation (4.45) is decomposed as

$$s^2+2s+2 = 3(s+4/3)(s+2) - 2(s+1)(s+3) \qquad (4.47)$$

From equations (4.46) and (4.47) and with $a_3(s)b_3(s)V(s)$ on a common divisor $G_{12}(s)$ takes the following form.

$$G_{12}(s) = \frac{\dfrac{1}{s+2} + \dfrac{1}{s+3}}{\dfrac{3(s+4/3)}{(s+3)} - \dfrac{2(s+1)}{(s+2)}} \qquad (4.48)$$

By consequence of equation (4.48)

$$-y_{12a}(s) = \frac{1}{s+2} \quad ; \quad y_{22a}(s) = \frac{2(s+1)}{(s+2)}$$

$$-y_{12b}(s) = \frac{1}{s+3} \quad ; \quad y_{22b}(s) = \frac{3(s+4/3)}{(s+3)}$$

and the related network is given in Figure (4.7).
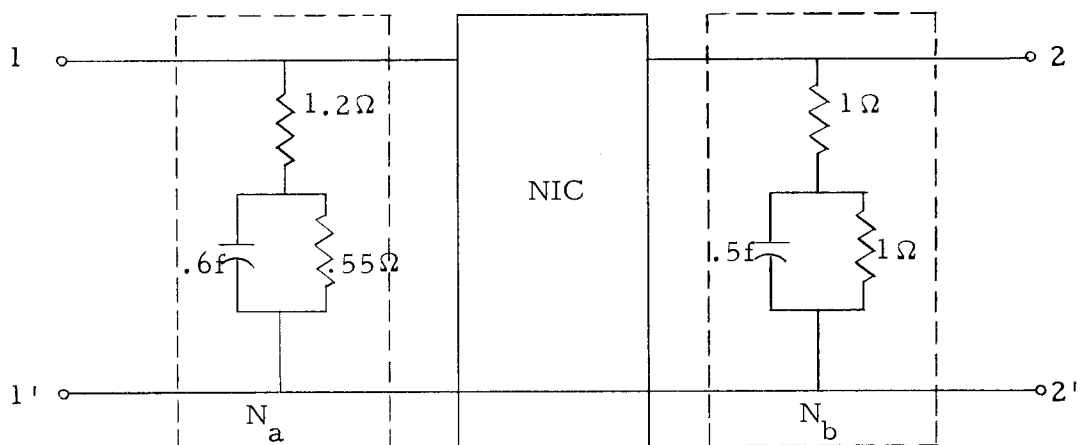
Figure 4.7. Parallel realization of

$$G_{12}(s) = \frac{2s^3+12s^2+24s+14}{s^2+2s+2}$$

4.5 The following example is an alternate realization of a

transfer function having only complex poles and zeros.

$$G_{12}(s) = \frac{2s^2 + s + 3}{s^2 + 3s + 6} \qquad (4.49)$$

with the configuration of Figure 2, Appendix 2.

Decompose equation (4.49) into the form of equation (4.39)

$$G_{12}(s) = \frac{s(s+5.94) + 2s(s+.25)}{s^2 + 3s + 6}$$

Setting $a_3(s)b_3(s)V(s) = s(s+.25)(s+5.94)$ the denominator of

equation (4.49) is decomposed as

$$s^2 + 3s + 6 - 4.73(s+5.07)(s+.25) - 3.73s(s+5.94)$$

as in the previous example

$$G_{12}(s) = \frac{\dfrac{.5}{s+.25} + \dfrac{2s}{s+5.94}}{\dfrac{4.73(s+5.07)}{s+5.94} - \dfrac{3.73s}{s+.25}}$$

from where

$$-y_{12a}(s) = \frac{.5}{s+.25} \quad ; \quad y_{22a}(s) = \frac{3.73s}{s+.25} \qquad (4.50)$$

$$-y_{12b}(s) = \frac{2s}{s+5.94} \quad ; \quad y_{22b}(s) = \frac{4.73(s+5.07)}{s+5.94} \qquad (4.51)$$
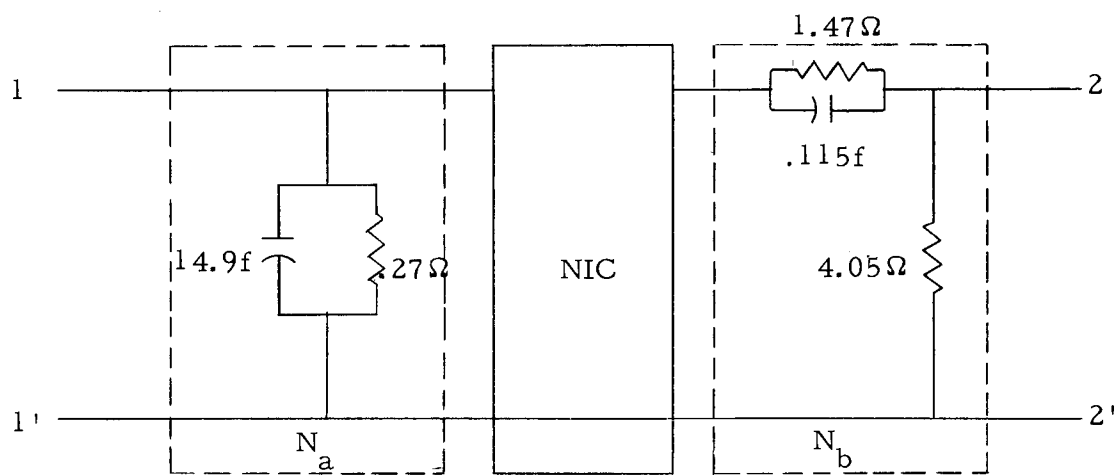
Figure 4.8. Parallel realization of

$$G_{12}(s) = \frac{2s^2+s+3}{s^2+3s+6} \; .$$

4.5 Comparison of the Methods of Transfer Function Synthesis

From the above it is concluded that if the numerator polynomial

of the given transfer function has only negative real roots or it is of

degree zero, the cascade method of synthesis will realize a network

with a minimum number of elements. This is because the transfer

functions of the subnetworks can be made to have all their zeros at

infinity, a condition that cannot be attained with either one of the

methods.

If the numerator polynomial has complex roots, then the syn-

thesis by the methods of paragraphs (4. 3) and (4. 4) will yield a

realization with a minimum number of elements. In general, both

methods will realize a given function with the same number of elements in the resulting network except in the case when one zero-shift will allow more than one pole-removal in one method but not in the other. This chance occurrence cannot be controlled by the decomposition or be determined in advance.

4.6 Driving-Point of Form $F_1(s)+F_2(s) - [f_1^2(s) - f_2^2(s)]/[F_1'(s) - F_2'(s)]$ Synthesis

Realization of a driving-point function of the form

$$\text{Driving-point function} = F_1(s)+F_2(s) - \frac{f_1^2(s) - f_2^2(s)}{F_1'(s) - F_2'(s)}$$

is accomplished through configurations of Figure 2 and Figure 3, Appendix 2.

Sipress [41] introduced this realization with Figure 2, for which the realized function is

$$Y(s) = y_{11a}(s)+y_{11b}(s) - \frac{y_{12a}^2(s) - y_{12b}^2(s)}{y_{22a}(s) - y_{22b}(s)} \tag{4.52}$$

or if negative impedance conversion ratio $\underline{k}$ is different than unity,

$$Y(s) = y_{11a}(s) + y_{11b}(s)$$
$$- \frac{[y_{12a}(s) - y_{12b}(s)][y_{12a}(s) - ky_{12b}(s)]}{y_{22a}(s) - ky_{22b}(s)} \tag{4.53}$$

and indicated that $F_1(s) - f_1^2(s)/[F_1'(s) - F_2'(s)]$ is a special case of

this realization. This realization is of special interest because it

realizes any driving-point function. To carry out the realization of

a given $Y(s)$ an arbitrary selection for the expression $y_{11a}(s)$

$+y_{11b}(s)$ is made. The degree of the numerator and denominator of

$y_{11a}(s) + y_{11b}(s)$ must be equal to the highest degree of the numerator

or denominator of $Y(s)$.

Let the given function to be realized be $Y(s) = N(s)/D(s)$, then

if $y_{11a}(s) = y_{11b}(s) = A(s)/B(s)$ from equation (4.52) through simple

algebraic manipulations one obtains

$$\frac{y_{12a}^2(s) - y_{12b}^2(s)}{y_{22a}(s) - y_{22b}(s)} = \frac{k_1 A(s)D(s) - N(s)B(s)}{B(s)D(s)} \qquad (4.54)$$

where k_1 is a constant, as yet undetermined.

From root locus considerations and properties of $A(s)$ and $B(s)$,

a positive value of k_1 can be determined such that

$$k_1 A(s)D(s) - N(s)B(s) = KU(s)V(s)$$

$$= 1/2[KU(s) + V(s)]^2 - 1/2[KU(s) - V(s)]^2 \qquad (4.55)$$

where $U(s)$ is a polynomial containing the complex roots and $V(s)$ a

polynomial containing the real roots of the expression $k_1 A(s)D(s)$

$- N(s)B(s)$. If $k = 1$, $V_a(s) = 1/2[KV(s) + V(S)]$ and $U_b(s) = 1/2[KU(s)$

$-V(s)]$ but if $k \neq 1$, $U_a(s) = \dfrac{kKV(s) + V(s)}{1+k}$ and $U_b(s) = \dfrac{KU(s) - V(s)}{1+k}$.

Substituting equation (4. 55) into (4. 54) and dividing the numerator and denominator by $B^2(s)$, the following expression is obtained

$$\frac{y_{12a}^2(s) - y_{12b}^2(s)}{y_{22a}(s) - y_{22b}(s)} = \frac{\dfrac{K_a U_a^2(s) - U_b^2(s)}{B^2(s)}}{\dfrac{D(s)}{B(s)}} \tag{4. 56}$$

where K_a is determined at the end so that the Fialkow-Gerst condition is satisfied for each network.

From the above it can be concluded that

$$y_{12a}^2(s) - y_{12b}^2(s) = K_a U_a^2(s) - U_b^2(s) \tag{4. 57}$$

$$y_{22a}(s) - y_{22b}(s) = \frac{D(s)}{B(s)} \tag{4. 58}$$

$$y_{11a}(s) + y_{11b}(s) = \frac{A(s)}{B(s)} \tag{4. 59}$$

from which the expressions for all the admittance functions may be easily obtained by proper partitioning.

Note that expressions for the admittance functions are given for $k = 1$, but similar expressions for $k \neq 1$ can be easily derived [41, p. 262]. A value of $k \neq 1$ is used in many cases when different values of K are needed to make both $U_a(s)$ and $U_b(s)$ have roots on the negative real axis.

4. 7 Simplifications

The number of elements in the resulting network will depend greatly on the expression selected for $y_{11a}(s) + y_{11b}(s)$. For example, since realization of only one of the given admittances, (either input driving-point or output driving-point admittance) is obtained, to realize the other admittance it may be necessary to use compensating networks such as those shown in Figure 8.

In general, element reduction in the resulting network is achieved if a selection of $y_{11a}(s) + y_{11b}(s)$ is made such that one or more of the following simplifications is attained.

1. Selection of $y_{11a}(s) + y_{11b}(s)$ such that the compensating networks are simplified or are completely eliminated.

2. Selection of $y_{11a}(s) + y_{11b}(s)$ such that the zeros of transmission of network "a" and "b" are the same as those of $y_{11a}(s)$ and $y_{11b}(s)$ respectively.

3. Selection of $y_{11a}(s) + y_{11b}(s)$ such that either $y_{12a}(s)$ or $y_{12b}(s)$ is equal to zero.

4. Application of polynomial decomposition techniques in certain cases greatly reduces the number of elements.

—

Simplification 1

Because there is no direct relation between the two driving-point functions and the transfer function of a network, an expression for $y_{11a}(s) + y_{11b}(s)$ which will simplify the compensating networks is not possible and cannot be obtained through a relation of these functions. However, an alternate approach can be used utilizing the principle of simplification $\underline{4}$.

Given a function $Y(s) = N(s)/D(s)$, obtain an arbitrary decomposition of $D(s)$ that is,

$$D(s) = a_1(s) - sb_1(s). \tag{4.60}$$

If $A(s)$ is formed as

$$A(s) = a_1(s) + sb_1(s) \tag{4.61}$$

a function $B(s)$ can always be found [Chapter II], such that $A(s)/B(s)$, $a_1(s)/B(s)$, $sb_1(s)/B(s)$ are RC driving-point admittance functions.

Equations (4.60) and (4.61) reveal that the input and output open-circuit admittances are the same in each subnetwork. This suggests that if the function is realized with a symmetrical network no compensating networks will be needed.

The realization will be done from the transfer function defined as

$$G_{12}(s) = - \frac{y_{12}(s)}{y_{22}(s)} \tag{4.62}$$

where the expressions for $y_{12}(s)$ and $y_{22}(s)$ are obtained as discussed above with the stipulation that $A(s)$ and $D(s)$ satisfy equations (4. 60) and (4. 61).

4.7 The following example is the realization of a Butterworth function without the use of compensating networks.

$$Y(s) = \frac{1}{s^2+2s+2}$$

Decompose the polynomial $D(s)$ as

$$D(s) = s^2+2s+2 = 3/2 \, (s+1)(s+. 75) - 1/2 \, s(s+3) \qquad (4. 63)$$

Form

$$A(s) = 3/2(s+1)(s+. 75) + 1/2 \, s(s+3) = 1/8(16s^2+33s+9)$$

$$= 1/8(s+. 322)(s+1. 74)$$

The selection of polynomial $B(s)$ is made such that

$$\frac{3/2(s+1)(s+. 75)}{B(s)} \quad \text{and} \quad \frac{1/2 \, s(s+3)}{B(s)} \quad \text{are RC impedance functions.}$$

Therefore $B(s) = (s+. 8)(s+4)$.

The expressions for $y_{12a}(s)$, $y_{12b}(s)$ are obtained as in the previous example.

$$KA(s)D(s) - N(s)B(s) = (2s^4+8. 15s^3+13. 4s^2+10. 5s+2. 25)$$

$$- s^2+4. 8s+3. 2$$

Let $K = \dfrac{3.2}{2.25}$, then

$$KA(s)D(s) - N(s)B(s) = s(2.84s^3 + 11.6s^2 + 18s + 10.1)$$

$$= s(s+1.3)(2.84s^2 + 7.9s + 7.75) \qquad (4.64)$$

and consequently

$$U_a(s) = K_1(2.84s^2 + 7.9s + 7.75) + s(s+1.3)$$

$$U_b(s) = K_2(2.84s^2 + 7.9s + 7.75) - s(s+1.3) \qquad (4.65)$$

Note here that this is a case in which a different value of K for $U_a(s)$ and $U_b(s)$ must be selected to force all their roots to be on the negative real axis. Setting $K_1 = .05$ and $K_2 = .5$ the following expressions are obtained.

$$U_a(s) = 1.142s^2 + 1.7s + .388 = 1.142(s+1.2)(s+.285) \qquad (4.66)$$

$$U_b(s) = .42s^2 + 2.65s + 3.88 = .42(s+2.32)(s+4) \qquad (4.67)$$

From equations (4.66) and (4.67) it is easily derived that the negative impedance converter must have an amplification factor of 100.

To satisfy the Fialkow-Gerst condition equations (4.66) and (4.67) are realized within a constant multiplier $k_a = 1/10$. The expressions for $y_{12a}(s)$, $y_{12b}(s)$, $y_{11a}(s)$, $y_{11b}(s)$, $y_{22a}(s)$, and $y_{22b}(s)$ can now be easily obtained from equations (4.63), (4.66), and (4.67) and

$$y_{12a}(s) = \frac{.11(s+1.2)(s+.285)}{(s+.8)(s+4)} \quad ;$$

$$y_{11a}(s) = \frac{.75(s+.75)(s+1)}{(s+.8)(s+4)} \quad ; \qquad (4.68)$$

$$y_{22a}(s) = \frac{.75(s+.75)(s+1)}{(s+.8)(s+4)}$$

and

$$y_{12b}(s) = \frac{.42(s+2.32)(s+4)}{(s+.8)(s+4)} \quad ;$$

$$y_{11b}(s) = \frac{.5s(s+3)}{(s+.8)(s+4)} \quad ; \qquad (4.69)$$

$$y_{22b}(s) = \frac{.5s(s+3)}{(s+.8)(s+4)}$$

Forming the open-circuit transfer-voltage ratio equation (4.62), for each of the subnetworks, the realization is achieved through the known methods of lattice-network synthesis techniques.

$$G_{12a}(s) = .0825 \frac{(s+1.2)(s+.285)}{(s+.75)(s+1)} \quad ;$$

$$G_{12b}(s) = .84 \frac{(s+2.32)(s+4)}{s(s+3)}$$

and the resulting network structure is given in Figure 4.9 where $G_{12a}(s)$ and $G_{12b}(s)$ are realized within a constant multiplier.
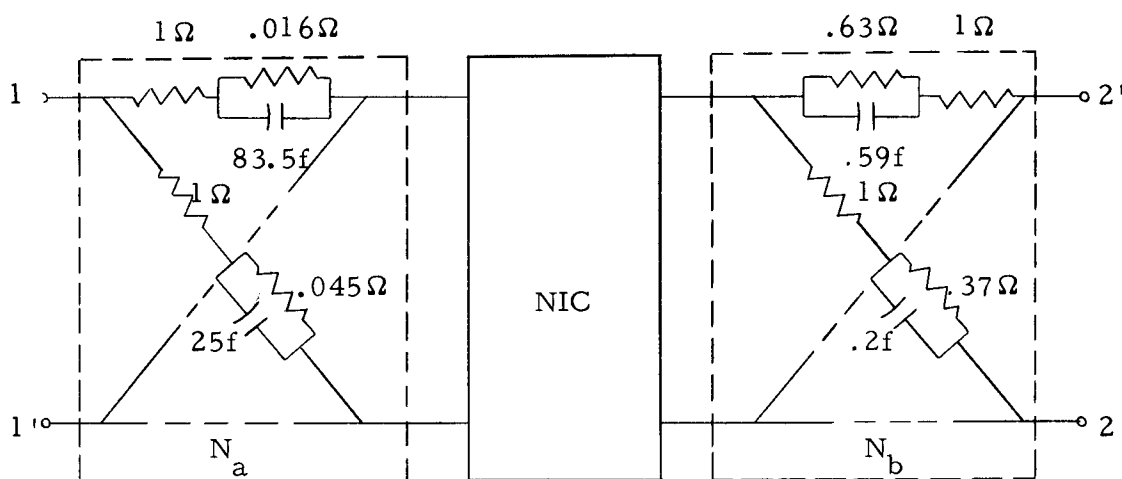
Figure 4. 9.   Realization of $Y(s) = \dfrac{1}{s^2+2s+2}$ using symmetrical network structures.

Simplification 2

This simplification reduces the number of elements in the realized function because the synthesis of network "a" and "b" from $y_{11a}(s)$, $y_{12a}(s)$, and $y_{11b}(s)$, $y_{12b}(s)$, respectively will be done without zero shifting. In such a case, for a given $Y(s) = N(s)/D(s)$, equation (4. 52) will have the form:

$$Y(s) = \frac{N(s)}{d(s)} = \frac{a_1(s)}{B(s)} + \frac{a_2(s)}{B(s)} - \frac{\dfrac{a_1^2(s)}{B^2(s)} - \dfrac{a_2^2(s)}{B^2(s)}}{\dfrac{D(s)}{B(s)}} \qquad (4.70)$$

Manipulating equation (4.70) results in,

$$\frac{N(s)}{D(s)} = \frac{[a_1(s)+a_2(s)][D(s)-(a_1(s)-a_2(s))]}{D(s)B(s)} \tag{4.71}$$

from where

$$N(s) = \left[\frac{a_1(s)+a_2(s)}{B(s)}\right][D(s)-(a_1(s)-a_2(s))] \tag{4.72}$$

Equation (4.72) will give the conditions that either,

$$\frac{a_1(s)+a_2(s)}{B(s)} = 1 \text{ and therefore,}$$

$$N(s) = D(s) - [a_1(s)-a_2(s)] \tag{4.73}$$

or that, $\dfrac{D(s) - (a_1(s) - a_2(s))}{B(s)} = \dfrac{Q(s)B(s)}{B(s)} = Q(s)$

and therefore,

$$N(s) = Q(s)[a_1(s)+a_2(s)]. \tag{4.74}$$

Theorem I.

If $a_1(s)/B(s)$ and $a_2(s)/B(s)$ are RC-admittance functions and if $a_1(s)+a_2(s) = B(s)$, than $a_1(s) = a_2(s) = B(s)/2$.

Proof:

Figure 4.10 indicates the location of the zeros of $a_1(s)$ and $a_2(s)$ with respect to those of B, and satisfying the restriction that $\dfrac{a_1(s)}{B(s)}$ and $\dfrac{a_2(s)}{B(s)}$ are driving-point admittance functions. Figure 4.10 shows the root-locus of $a_1(s)+ka_2(s)$ and indicates the regions of the

zeros of this function. It can, therefore, be seen that the realization

of $a_1(s)+a_2(s) = B(s)$ will hold only if $a_1(s) = a_2(s) = B(s)/2$.

Application of the results of Theorem I to equation (4.73)

shows that this condition can be considered if, and only if, $N(s)=D(s)$

which is the trivial case $Y(s) = 1$.

Condition (4.74) is then the only one which, if satisfied, will

lead to representation of the given function $Y(s)$ in the form pre-

scribed by equation (4.70). The fact that $a_1(s)/B(s)$ and $a_2(s)/B(s)$

are RC driving-point admittances requires that $a_1(s)$ and $a_2(s)$ be of

degree equal to, or one greater than, the degree "m" of $D(s)$;

therefore $N(s)$ must contain at least $(m-1)$ real roots. This is a

necessary condition. The sufficiency is secured if condition (4.74)

is satisfied. In conclusion, any given rational real function $Y(s)$ of

numerator and denominator polynomial degree n and m, respectively,

can be written in the form prescribed by equation (4.70) if

 a. The numerator polynomial contains at least m-1 real

  roots.

 b. The condition (4.74) is satisfied.

The feasibility of this method comes from the fact that for a

given polynomial p(s) with roots on the real axis there is only a

finite number (if any) of decompositions of the form $p(s) = a_1(s)$

$+ a_2(s)$, such that $a_1(s)/B(s)$ and $a_2(s)/B(s)$ are RC driving-point

functions, $B(s)$ being an arbitrary polynomial. These decompositions

can only be obtained by trial-and-error procedure.
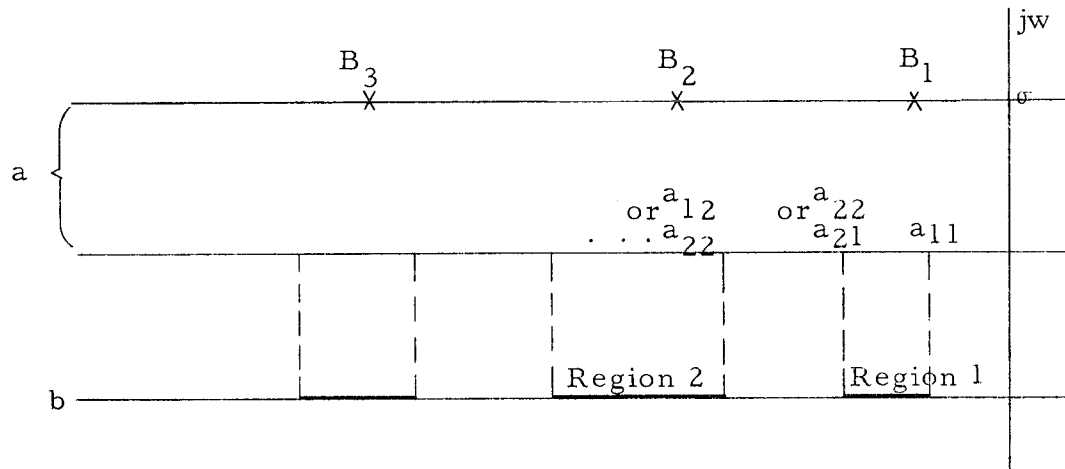
Figure 4.10. Roots of $a_1(s)$, $a_2(s)$, and $B(s)$ and regions of root location of $a_1^2(s) + a_2(s)$.

To illustrate the simplification and to outline its procedure, the following example is given.

4.8   The following example is the realization of a driving-point function with negative-real zeros and complex poles by the method of Sipress.

$$Y(s) = \frac{s^2 + 1.5s + .5}{s^2 + 2s + 2} = \frac{(s + .5)(s + 1)}{s^2 + 2s + 2} \tag{4.75}$$

Since the numerator and denominator polynomials of equation (4.75) are of the same degree $Q(s) = 1$. Therefore, from equation (4.74),

$$KN(s) = K(s^2 + 1.5s + .5) = a_1(s) + a_2(s) \tag{4.76}$$

and

$$\frac{K_a D(s) - a_1(s) - a_2(s)}{B(s)} = \frac{(s^2+2s+2) - a_1(s) - a_2(s)}{B(s)} \qquad (4.77)$$

The next step is the selection of a polynomial for $a_1(s)$ or $a_2(s)$, and here is where the trial-and-error procedure enters. As a rule select a polynomial $a(s)$ such that $N(s)/a(s)$ is an RC admittance function. The following steps follow directly from equations (4.76) and (4.77) and are explained here with the help of the root-locus diagram, Figure (4.11).

location of roots of

$a_1(s) + a_2(s)$.

locus of $KN(s)-a_2(s)$,

allowable regions. . .,

for roots of $a_1(s)$.

Representative locations for roots of $a_1(s)x$, locus of $a_1(s)-a_2(s)$ and allowable regions. . ., for roots of $a_1(s)-a_2(s)$.

Representative locations for roots of $a_1(s)-a_2(s)x$ locus of $K_a D(s)-(a_1(s)-a_2(s))$ and desirable locations of roots of $K_a D(s)-(a_1(s)-a_2(s))$.

jw

σ

-1

-.5

K=∞   K=0   K=1.32

K=∞   K=0   K=2.34
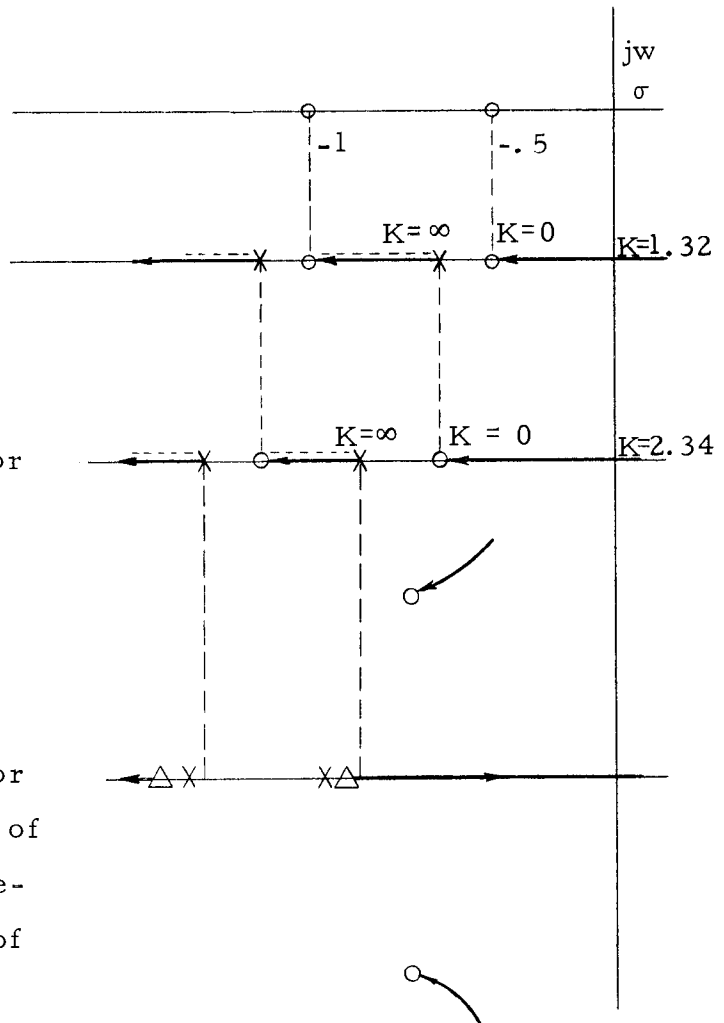
Figure 4.11.   Loci of expressions of Examples 4.8.

Assume $a_2(s) = (s+1.1)(s+.6)$; then from equation (4.76)

$$a_1(s) = (k-1)s^2 + (1.5k-1.7)s+.5k-.66 \qquad (4.78)$$

where k is, as yet undetermined, but as easily derived from Figure (4.11), k must satisfy the restriction

$$k \geq 1.32 \qquad (4.79)$$

The expression for $a_1(s) - a_2(s)$ can now be written as

$$a_1(s) - a_2(s) = (k-2)s^2 + (1.5k-3.4)s+.5k-1.32 \qquad (4.80)$$

and again from Figure (4.11) it is seen that k must satisfy the following restriction

$$k < 2.24 \qquad (4.81)$$

From equations (4.45) and (4.47) the value of k must therefore be

$$1.35 \leq k < 2.24 \qquad (4.82)$$

If from equation (4.82) k = 1.32, the expressions for $a_1(s)$ and $a_1(s) - a_2(s)$ will be equations (4.78) and (4.80),

$$a_1(s) = .32s^2 + .28s = .32s(s+.875)$$

$$a_1(s) - a_2(s) = -(.68s^2 + 1.42s + .66)$$

$$= -.68(s+.698)(s+1.39)$$

and for $K_a = .066$,

$$K_a D(s) - (a_1(s) - a_2(s)) = 1.93(s+.9)(s+1.18)$$

Therefore, from equation (4.70)

$$y_{11a}(s) = \frac{(s+.6)(s+1.1)}{1.93(s+.9)(s+1.18)} \quad ;$$

$$y_{12a}(s) = \frac{(s+.6)(s+1.1)}{1.93(s+.9)(s+1.18)} \quad ;$$

$$y_{22a}(s) = \frac{c(s)}{1.93(s+.9)(s+1.18)} \quad .$$

$$y_{11b}(s) = \frac{.32s(s+.875)}{1.93(s+.9)(s+1.18)} \quad ;$$

$$y_{12b}(s) = \frac{.32s(s+.875)}{1.93(s+.9)(s+1.18)} \quad ;$$

$$y_{22b}(s) = \frac{d(s)}{1.93(s+.9)(s+1.18)} \quad .$$

where $c(s) - d(s) = s^2 + 2s + 2$, and the realization of $Y(s)$ can be attained from $y_{11a}(s)$, $y_{12a}(s)$, and $y_{11b}(s)$, $y_{12b}(s)$, without zero shifting.

Simplification 3

Going back to equation (4.78), if $y_{11a}(s) + y_{11b}(s)$ is set equal to $KA(s)/B(s)$ for a given $Y(s) = \frac{N(s)}{D(s)}$, we will have

$$\frac{KA(s)}{B(s)} = \frac{N(s)}{D(s)} = \frac{y_{12a}^2(s) - y_{12b}^2(s)}{y_{22a}(s) - y_{22b}(s)} \qquad (4.83)$$

from where $y_{12a}(s) - y_{12b}(s) = KA(s)D(s) - N(s)B(s)$. $\qquad (4.84)$

For simplification 3, it is desirable for $KA(s)D(s) - N(s)B(s)$

be a perfect square and have roots on the negative real axis. If we

arbitrarily select any polynomial for N(s) or D(s) from the root-

locus configuration of equation (4. 84), it is seen that one necessary

condition is that one of the polynomials N(s) or D(s) must have only

negative real roots. If either N(s) or D(s) has complex roots, i. e.,

N (s), then, from the root-locus, B(s) must be unity and the degree

of A(s)D(s) must be one less than the degree of N (s). As shown in

Figure 4. 13, this is restricted to the case when either N(s) or D(s)

is of degree 2 and the other of degree zero.

For the case where both N(s) and D(s) have real roots, condi-

tions for KA(s)D(s) - N(s)B(s) to be a perfect square with roots on

the negative real axis can be found from the following considerations.

Let

$$KA(s)D(s) - N(s)B(s) = V^2(s) \tag{4.85}$$

The polynomial V(s) can now be thought of as being a summa-

tion or a difference of two polynomials a(s) and b(s).

The former assumption gives

$$V^2(s) = [a(s)+k_1 b(s)]^2 = a^2(s)+k_1^2 b^2(s)+2k_1 a(s)b(s)$$

$$\tag{4.86}$$

$$= a^2(s) + K_1 b(s)[k_1 b(s) + 2a(s)]$$

where k_1 is a positive constant, the value of which is to be deter-

mined later.

No direct association of the terms of (4. 85) and (4. 86) can be
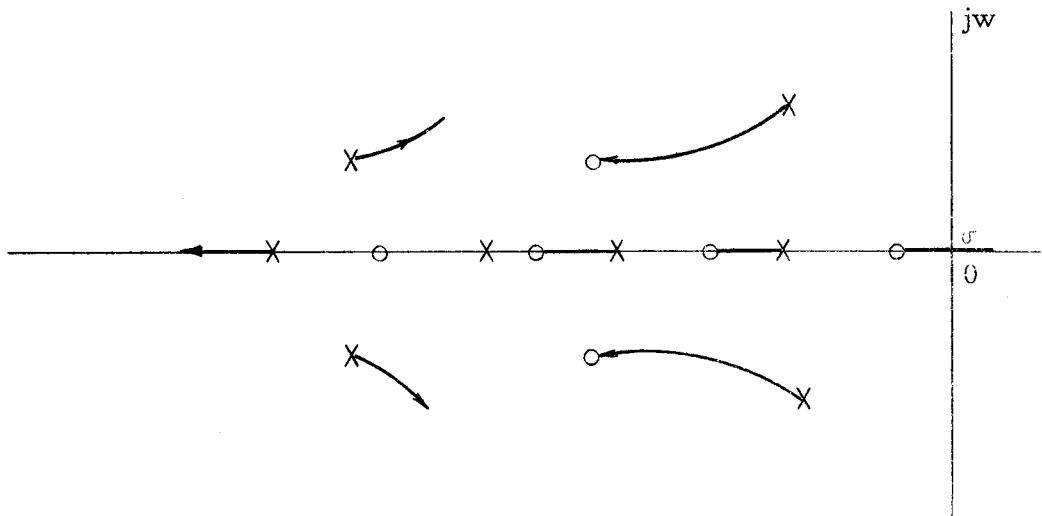
Figure 4.12. Root-locus plot of $\dfrac{KA(s)D(s)}{N(s)B(s)} - 1 = 0$.
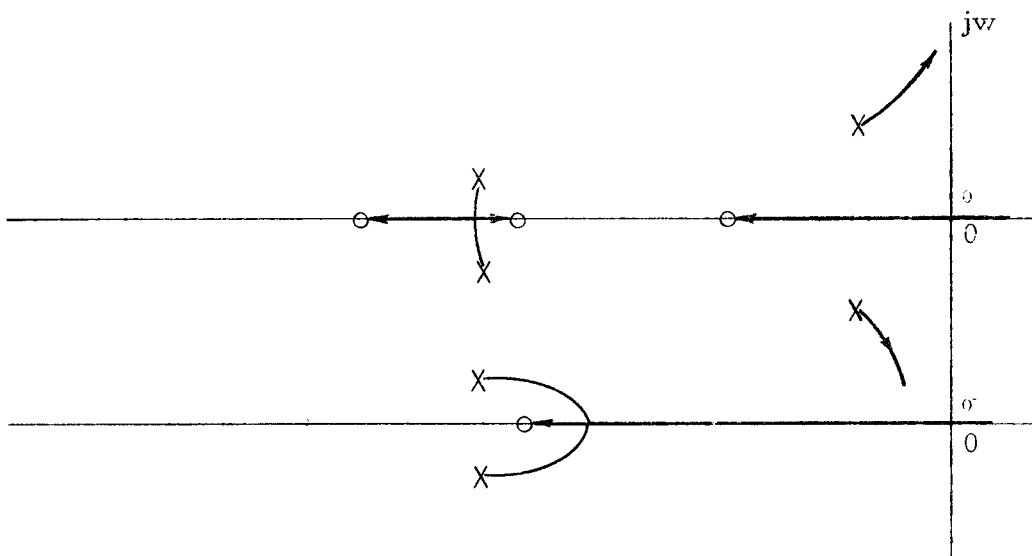
Figure 4.13. Root-locus of $\dfrac{KA(s)D(s)}{N(s)B(s)} - 1 = 0$.

made and such a representation is being rejected.

Considering the difference of polynomials

$$V^2(s) = [a(s) - k_1 b(s)]^2$$

$$= a^2(s) + k_1^2 b^2(s) - 2k_1 a(s)b(s) \tag{4.87}$$

$$= a^2(s) - k_1 b(s)[2a(s) - k_1 b(s)] \tag{4.88}$$

where k_1 is as defined as above.

Comparison of equations (4.85) and (4.87), direct association of the similar terms will give,

for $V^2(s)$ positive

$$KA(s)D(s) = a^2(s)$$

$$N(s)B(s) = k_1 b(s)[2a(s) - k_1 b(s)]$$

for $V^2(s)$ negative

$$KA(s)D(s) = k_1 b(s)[2a(s) - k_1 b(s)]$$

$$N(s)B(s) = a^2(s)$$

from where (consider only case, $V^2(s)$ positive,)

$$KA(s) = a(s), \qquad D(s) = a(s)$$

$$B(s) = k_1 b(s), \qquad N(s) = 2a(s) - k_1 b(s) \tag{4.89}$$

or

$$KA(s) = a(s), \qquad D(s) = a(s)$$

$$B(s) = 2a(s) - k_1 b(s), \qquad N(s) = k_1^2 b(s) \tag{4.90}$$

For the comparison the other possibility of grouping, $V^2(s) = a^2(s) + b^2(s) - 2a(s)b(s)$, is not considered because the term $a^2(s) + b^2(s)$ contains only complex roots while each term to the left of equation (4.85) contains real roots at least equal to the number of roots of either $A(s)$ or $B(s)$ depending on the term.

Returning to equations (4.89) and (4.90) it can be concluded that since $A(s)/B(s)$ must be an RC driving-point admittance function $N(s)/D(s)$ is either an admittance or an impedance driving-point function, depending on the choice of the value of k_1. In such a case no active-network realization is necessary. Similar conclusions can be drawn for the case $V^2(s)$ negative. In summary, simplification 3 can only be applied to the case in which either $N(s)$ or $D(s)$ is of degree two with complex roots only, whereas the other is of degree zero.

Simplification 4

Polynomial decomposition in the realization of $Y(s)$ by equation (4.52), under the constraint that the degree of numerator and denominator polynomials are equal, can be used if certain conditions are satisfied. Decomposition results in a reduction of elements because it realizes networks "a" and "b" with describing polynomials having degrees one-half of that of the polynomials of $Y(s)$.

By algebraic manipulations equation (4.52) can be expressed

in the following form,

$$Y(s) = \frac{N(s)}{D(s)} = y_{11a}(s) \frac{1/z_{22a}(s) - y_{22b}(s)}{y_{22a}(s) - Y_{22b}(s)}$$

$$+ y_{11b} \frac{y_{22a}(s) - 1/z_{22b}(s)}{y_{22a}(s) - y_{22b}(s)}$$

$$(4.95)$$

Note here that each term on the right-hand side of equation (4.95) is of the form $F_1(s) - f_1^2(s)/[F_1'(s) - F_2'(s)]$.

The application of polynomial decomposition requires at least one of the polynomials, $N(s)$, $D(s)$, to have only complex roots, which will give us the following cases.

Case 1: If $N(s)$ and $D(s)$ have only complex roots, then application of Horowitz decomposition will allow the following expressions for the synthesis of $Y(s)$ within a constant multiplier,

$$Y(s) = \frac{N(s)}{D(s)} = \frac{a_1(s)a_2(s) - sb_1(s)b_2(s)}{a_1^2(s) - sb_1^2(s)}$$

$$+ \frac{a_1(s)a_2(s) - sb_1(s)b_2(s)}{a_1^2(s) - sb_1^2(s)}$$

from which

$$Y(s) = \frac{b_2(s)}{b_1(s)} \frac{a_2(s)/b_2(s) - sb_1(s)/a_1(s)}{a_1(s)/b_1(s) - sb_1(s)/a_1(s)}$$

$$(4.96)$$

$$+ \frac{a_2(s)}{a_1(s)} \frac{a_1(s)/b_1(s) - sb_2(s)/a_2(s)}{a_1^2(s) - sb_1^2(s)}$$

where the ratios of $a_i(s)/bj(s)$ and $sb_i(s)/q(s)$ ($i = 1, 2$ and $j = 1, 2$)

satisfy the conditions for RC driving-point admittance functions.

Furthermore, if $b_2(s)/b_1(s)$ and $a_2(s)/a_1(s)$, satisfy also the condi-

tions for RC driving-point functions such a decomposition can be

applied. From (4.95) and (4.96) the following associations can be

made.

$$y_{11a}(s) = \frac{b_2(s)}{b_1(s)} \quad ; \qquad 1/z_{22a}(s) = \frac{a_2(s)}{b_2(s)}$$

$$y_{11b}(s) = \frac{a_2(s)}{a_1(s)} \quad ; \qquad 1/z_{22b}(s) = \frac{sb_2(s)}{a_2(s)}$$

$$y_{22a}(s) = \frac{a_1(s)}{b_1(s)} \qquad\qquad y_{22b}(s) = \frac{sb_1(s)}{a_1(s)}$$

Case 2: If $N(s)$ is any polynomial with one root at the origin

and $D(s)$ is a polynomial with only complex roots, the application of

Horowitz decomposition gives the following expression:

$$\frac{Y(s)}{s} = \frac{a_1(s)b_3(s) - a_3(s)b_1(s)}{a_1^2(s) - sb_1^2(s)}$$

$$+ \frac{a_1(s)b_3(s) - a_3(s)b_1(s)}{a_1^2(s) - sb_1^2(s)}$$

$$= \frac{a_3(s)}{b_1(s)} \frac{b_3(s)/a_3(s) - b_1(s)/a_1(s)}{a_1(s)/b_1(s) - sb_1(s)/a_1(s)}$$

$$+ \frac{b_3(s)}{a_1(s)} \frac{a_1(s)/b_1(s) - a_3(s)/b_3(s)}{a_1(s)/b_1(s) - sb_1(s)/a_1(s)}$$

(4.97)

Multiplying through by "s"

$$Y(s) = \frac{a_3(s)}{b_1(s)} \frac{sb_3(s)/a_3(s) - sb_1(s)/a_1(s)}{a_1(s)/b_1(s) - sb_1(s)/a_1(s)}$$

$$+ \frac{sb_3(s)}{a_1(s)} \frac{a_1(s)/b_1(s) - a_3(s)/b_3(s)}{a_1(s)/b_1(s) - sb_1(s)/a_1(s)}$$

(4.98)

where the ratios of $a_i(s)/sb(s)$ and $sb_i(s)/a_j(s)$ satisfy the conditions

for RC driving-point admittance functions.

Comparison of (4.98) and (4.95) yields the following parameter

identification:

$$y_{11a}(s) = \frac{a_3(s)}{b_1(s)} \quad ; \qquad 1/z_{22a}(s) = \frac{sb_3(s)}{a_3(s)} \quad ;$$

$$y_{11b}(s) = \frac{sb_2(s)}{a_1(s)} \quad ; \qquad 1/z_{22b}(s) = \frac{a_3(s)}{b_3(s)} \quad ;$$

$$y_{22a}(s) = \frac{a_1(s)}{b_1(s)} \quad ; \qquad y_{22b}(s) = \frac{sb_1(s)}{a_1(s)}$$

No other cases exist because the numerator of each term of

the right-hand side of equation (4.95) contains a term which appears

in its denominator. For the above decompositions to be applicable

it must be shown that

1. $y_{12a}(s)$ and $y_{12b}(s)$ are rational and they can always be

    obtained.

2. The residue condition is always satisfied.

Requirement 1

To show that a rational $y_{12}(s)$ can always be obtained, it is helpful to solve $y_{12}(s)$ in terms of the decompositions present. In general

$$y_{12}^2(s) = y_{11}(s)(y_{22}(s) - 1/z_{22}(s)) \qquad (4.99)$$

For this problem $y_{12a}(s)$ and $y_{12b}(s)$ become:

$$y_{12a}(s) = \frac{a_1(s)b_2(s) - a_2(s)b_1(s)}{b_1^2(s)} \quad ;$$

$$y_{12b}(s) = \frac{sa_2(s)b_1(s) - sa_1(s)b_2(s)}{a_1^2(s)} \qquad (4.100)$$

for Case 1, and

$$y_{12a}(s) = \frac{a_1(s)a_3(s) - sb_1(s)b_3(s)}{b_1^2(s)}$$

$$y_{12b}(s) = \frac{s^2 b_1(s)b_3(s) - sa_1(s)a_3(s)}{a^2(s)} \qquad (4.101)$$

for Case 2.

Clearly the polynomials appearing in the numerators of expressions (4.100) and (4.101) must be full squares in order that y_{12}'s to be rational. The proof that the y_{12}'s are rational is similar to all expressions of (4.100) and (4.101). Therefore, only the proof for $y_{12}(s) = [a_1(s)b_2(s) - a_2(s)b_1(s)]/b^2(s)$ is shown. The rest will be

similar. Before proceeding, recall here the proof of Theorem 2,

Chapter 2. In proving this theorem, given a rational function

$p_1(s)/p_2(s)$, $sp_1(s^2)/p_2(s^2)$ was formed, which in turn was treated

as the odd part of a positive real admittance. This admittance has

the form,

$$\frac{A_2(s) + sB_2(s)}{A_1(s) + sB_1(s)} \qquad\qquad (4.102)$$

where s was substituted for s^2, and the theorem was proven. If,

in (4.102), the odd part is formed and s is substituted for s^2, ex-

pression (4.103) results

$$\frac{a_1(s)b_2(s) - a_2(s)b_1(s)}{a_1^2(s) - sb_1^2(s)} \qquad\qquad (4.103)$$

Thus, the polynomial to be made a full square has as its zeros the

left-half plane zeros of the odd part of the constructed admittance

(4.102). The odd part zeros of a positive real admittance need not

be of even multiplicity and so $a_1(s)b_2(s) - a_2(s)b_1(s)$ is not neces-

sarily full square.

To show that the polynomial in question can always be made

full square proceed as follows. Suppose that constructed admittance

(4.102) is augmented by a Hurwitz polynomial, $A_o(s) - sB_o(s)$. This

augmentation propagates into the even and odd parts of the constructed

admittance as $A_o^2(s) - s^2B_o^2(s)$. The substitution of s^2 by s reveals

that the numerator of $y_{12}(s)$ has been multiplied by the polynomial $a_o^2(s) - sb_o^2(s)$, while the admittance function to be realized is augmented by a surplus factor $a_o^2(s) - s^2 b_o^2(s)$. In other words the given admittance function can always be augmeted by a surplus factor which makes $y_{12}(s)$ full square and so rational.

Requirement 2

To show that the residue condition is always satisfied, form $y_{11}(s)y_{22}(s)$ in terms of the decomposition present,

$$y_{11a}(s)y_{22a}(s) = \frac{b_2(s)a_1(s)}{b_1^2(s)} \; ;$$

$$\text{(4.104)}$$

$$y_{11b}(s)y_{22b}(s) = \frac{Sa_2(s)b_1(s)}{a_1^2(s)}$$

for Case 1.

$$y_{11a}(s)y_{22a}(s) = \frac{a_3(s)a_1(s)}{b_1^2(s)};$$

$$\text{(4.105)}$$

$$y_{11b}(s)y_{22b}(s) = \frac{s^2 b_3(s)b_1(s)}{a_1^2(s)}$$

for Case 2.

Since the poles in the expressions of (4.104) and (4.105) appear as zeros of the negative part in the corresponding expressions of the y_{12}'s, equations (4.100) and (4.101), the residue condition at the finite poles is satisfied with an equal sign. Furthermore, it can be seen that the poles (if any) at infinity also satisfy the residue condition.

V. PRACTICAL CONSIDERATIONS

The problems selected for the examples in this work are in a normalized form. The quantities most often normalized are the level of the network function and the frequency. Normalization in the synthesis of a network is desirable because it offers the advantages of making the calculations easier and allowing the use of the so-called universal curves. Denormalization can be done at any convenient stage in the calculations or it may be performed on the network. For example, consider a case where both the level and the frequency have been scaled down by the factors H and W respectively. The removal of the normalization on the network is then accomplished by multiplying every resistance by H, every inductance by H/W, and every capacitance by 1/HW [41, p. 40].

In practice the network functions are commonly given as plots in the frequency domain and their mathematical expressions are obtained either by polynomial (Butterworth, Chebyshev, Bessel) or break-point approximation. Normalization is then applied to these expressions.

Examples

Figure 5. 1 gives the magnitude characteristic of a compensation network. It is required to find a network function which

approximates this curve.

Figure 5. 1 shows the break-point approximation of the charac-
teristic curve with low-frequency asymptote of 12 db/octave and
high-frequency asymptote of 0 db/octave.   There is also a "resonant
peak" indicating a second degree factor, at $W = 14000$ radians/sec.
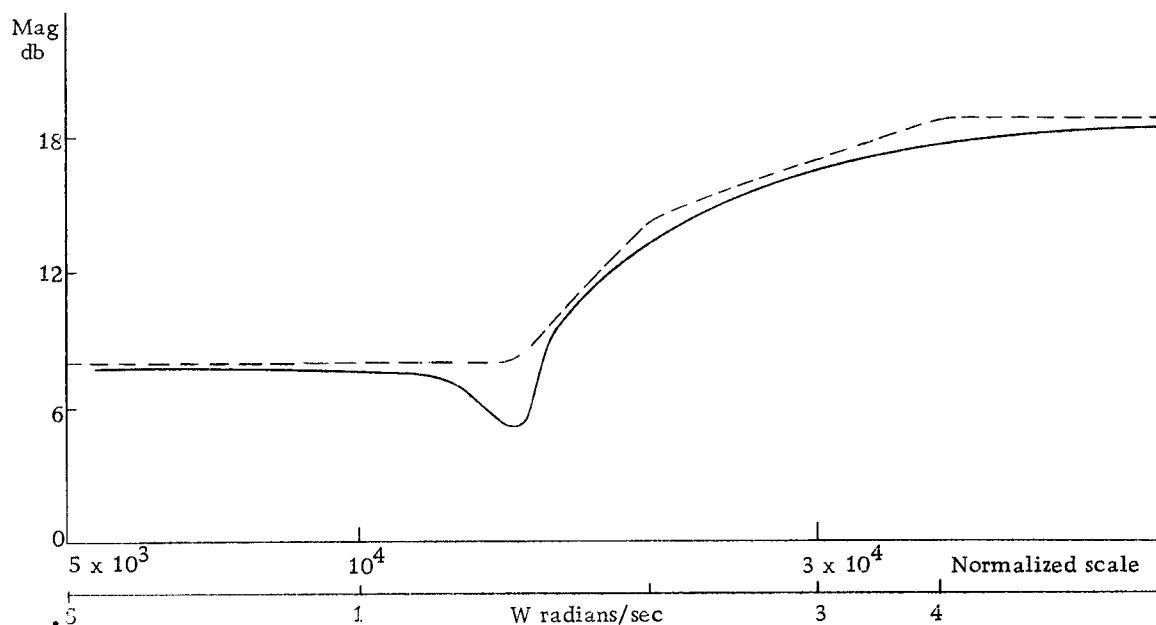
Figure 5. 1.   Magnitude characteristic for Example 5. 1

For these asymptotes, the break frequencies are 14000,
20, 000, and 40, 000 radians/sec.   The second degree factor was
obtained with the use of the universal curves and has the form
s^2+2s+2.   The frequency-normalized gain function is then,

$$G(s) = 10^2 \frac{s^2+2s+2}{s^2+6s+8} \qquad (5.1)$$

The network corresponding to Equation (5. 1) has been realized,

within the constant multiplier 10^2, on Chapter III and is shown on

Figure 3. 13. The removal of the normalization from the network of

Figure 3. 13 (level normalization constant 10^2, frequency normaliza-

tion constant 10^4) will result in a network with element values indi-

cated on Figure 5. 2.
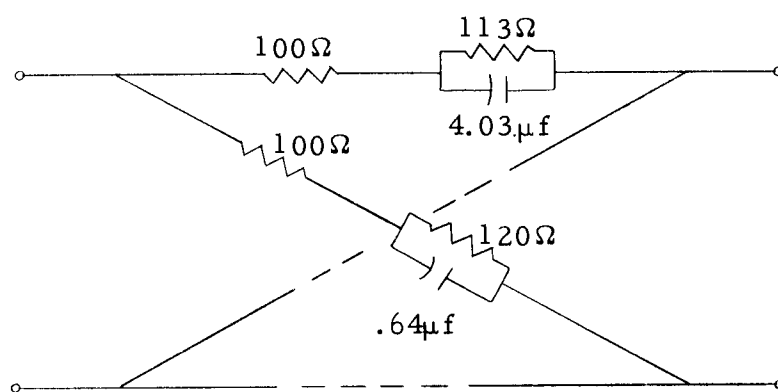
Figure 5. 2.   Network with magnitude characteristic is
given by Figure 5. 1.

VI. SUMMARY

The procedure for minimal RC and RC-NIC network realization has been systematically developed and can be applied to a variety of problems in different realizations.

In the synthesis of RC network it was noted that the realization of a driving-point function or part of a network function results in a canonical network. Element reduction, therefore, can be made only in the realization of a transfer function or in the simultaneous realization of a driving-point function and a transfer function. A synthesis method has been introduced, which will permit a reduction in the number of elements in the synthesis of transfer functions. Furthermore, this new method of synthesis broadens the class of functions that can be realized with RC-networks. The ladder realization of RC transfer functions requires zero shifting. The effect of this zero shift on the remaining zeros cannot be determined in advance. The reduction in the number of elements in the RC ladder network was therefore obtained through a trial-and-error procedure. A FORTRAN program was written to assist in this direction.

The application of the concept of interconnection of two-three terminal networks (in this case an RC network and an RC-NIC network) lead to the unification of the field of RC-NIC network synthesis and introduce a variety of configurations through which RC-NIC

realization can be attained. Element reduction in these networks has been shown in this work that can be achieved proper selection of the real-root-polynomials. The selection of such polynomials which will render minimal RC-NIC networks can be obtained through proper decomposition of the numerator and denominator expressions of the function to be realized. The rules and procedures for the selection of the polynomials are given in Chapter IV.

SUGGESTIONS FOR FURTHER STUDY

Variations of a network characteristic function with small modification of their pole or zero location has been given considerable attention in the literature today. A need, however, was felt in this work for variations of these functions with arbitrary modification of their poles or zeros. The study of this problem may be of interest.

Continuously equivalent networks were thought as an approach to be used in the element reduction of realized network structures. Its feasibility, however, was not studied and could present another area of investigation.

Finally, a more sophisticated unification of the field of RC-NIC network synthesis can be obtained through topological considerations. This may be of special interest in that it may open new ways to active network synthesis which avoids the problems inherent in dividing active network synthesis into the synthesis of separate, purely passive and purely active networks.

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APPENDICES

APPENDIX I

The simultaneous realization of a driving-point function and a transfer function has been programmed and its flow chart and FORTRAN listing is given in the following pages.

The program has been written so as the user has only to enter as the input the coefficients of the numerator and denominator polynomials of the driving-point function and transfer function or their roots. The program will perform the realization of the network and will write out the value of the elements of each branch of the realized ladder network.

SIMULTANEOUS RC—NETWORK REALIZATION
OF A DRIVING-POINT FUNCTION $\left(Y_{11}(s) = \frac{Q_1(s)}{P_1(s)}\right)$
AND A TRANSFER FUNCTION $\left(Y_{12}(s) = \frac{Q_2(s)}{P_2(s)}\right)$

```
                    ┌──────────┐
                    │    IF    │
          YES       │  SENSE   │      NO
       ┌────────────│  EQUAL   │────────────┐
       │            │    1     │            │
       │            └──────────┘            │
┌──────────────┐              ┌──────────────┐
│ READ ROOTS   │              │    READ      │
│ OF POLYNOMIALS│             │ COEFFICIENTS │
│ Q₁, P₁, Q₂,P₂ │             │ OF POLYNOMIALS│
└──────────────┘              │ Q₁, P₁, Q₂,P₂ │
                              └──────────────┘
┌──────────────┐              ┌──────────────┐
│    FORM      │              │CALL SUBROUTINE│
│ POLYNOMIALS  │              │  AND FIND    │
│ FROM THEIR   │              │ THE ROOTS OF │
│   ROOTS      │              │ POLYNOMIALS  │
└──────────────┘              └──────────────┘
```

IF P_1 HAS OTHER ROOTS (P_i) BESIDES THOSE OF P_2

FIND RESIDUES (K_i) OF "$S.Y_{11}$" AT THE EXTRA ROOTS OF P_i

ARE RESIDUES POSITIVE — NO — PRINT Y_{11} NOT PR-FUNCTION → END

SUBTRACT FROM Y_{11} THE FUNCTION $\sum_i \frac{K_i S}{S+P_i}$

PRINT K_i, $\frac{K_i}{P_i}$

①

(I)

EVALUATE Y_{11} AT ONE OF THE ZEROS OF Y_{12} CALL IT $\bar{Y}$

IF "$\bar{Y}$"

POSITIVE

NEGATIVE

(VII)

GO TO (I)

ARE ALL ROOTS OF Y_{12} EXAUSTED

YES

NO

PRINT $\bar{Y}$

SUBTRACT FROM Y_{11} $\bar{Y}$ FORM Y_1 AS: $Y = Y_{11} \bar{Y}$

ARE ALL COEFFICIENTS OF Y_1 POSITIVE

YES

NO

GO TO VII

CALL SUBROUTINE FACTOR NUMER. OF Y_1

ARE ANY ZEROS OF WITH THOSE OF Y_{12}

IF NONE

IF ALL

IF SOME

GO TO (VII)

INVERT Y_1 FORM Z

(II)

INVERT Y_1 FORM Z $\frac{1}{Y_1}$ WHERE $Z = \frac{1}{Y_1}$

ARE NUM. AND DEN. POLYNOMIALS OF Z OF THE SAME POWER

YES

NO

FIND K(0)WHICH WILL BE EQUAL TO THE RATIO OF THE COEFFICIENTS OF THE HIGHEST POWER TERMS OF NUM. AND DEN. POLYNOMIALS

(V)

SELECT A POLE OF Y_{11} THE ABS. VALUE OF WHICH IS GREATER FROM THE ABS. VALUE OF AT LEAST ONE ZERO OF Y_{12} AND ONE ZERO OF Y_{11}. UNDER THE RESTRICTION THAT THERE IS NO OTHER POLE BETWEEN THIS POLE THESE ZEROS. START WITH A POLE WITH THE SMALLEST ABS. VALUE.

IF B IS VALUE OF SELECTED POLE FORM EQUATION $Y_{11} - \frac{AS}{S+B} = Y$ WHERE A = CONST.

EVALUATE Y AT THE ZERO (0) OF Y_{12} CLOSEST TO THIS POLE. SET Y TO ZERO, SOLVE FOR A; $\left\{ Y_{11} - \frac{AS}{S+B} \Big|_{S=D}^{=0} \right\}$

FIND RESIDUE (K) OF S.Y_{11} AT THE SELECTED POLE (B) FORM RATIO R AS $R = \frac{A}{K}$

IS R POSITIVE

NO

YES

GO TO (V)

(2) (3) (4) (5)

(2)

FIND RESIDUES OF Z, AT THESE POLES (P₁)

SUBTRACT FROM Z₁ THE FUNCTION $\sum \frac{K_i}{S+P_i}$

PRINT $\frac{1}{K_i}, \frac{K_i}{P_i}$

FORM Z₂ AS $Z_2 = Z - \sum \frac{K_i}{S+P_i}$

(III)

CALL SUBROUTINE FACTOR NUMER. OF Z₂

ARE ANY ZEROS OF Z₂ THE SAME AS THOSE OF Y₁₂

IF NONE — (IV) — ARE ALL ZEROS OF Y EXAUSTED

NO

END

YES

IF SOME

SELECT ANOTHER ZERO OF Y₁₂ INVERT Z₂ TO FORM Y₃

GO TO (I)

INVERT Z₂ FORM Y₁₃ MULTIPLY Y₁₃ BY S

FIND RESIDUE (K₁) OF S.Y₁₃ AT THESE ZEROS (A

(6)

IF ALL

INVERT Z₂ FORM Y₂ MULTIPLY Y₂ BY S

(7)

(3)

PRINT K ∞

(4)

DOES EITHER NUMER OR DEN POLYNOM HAVE A VALUE OTHER THAN ZERO AT S=0

IF BOTH   IF NONE

SET VALUE OF THAT POLYNOM. AT S=0 TO BE EQUAL TO ONE

FORM K₀ AS $K_0 = \frac{VAL. OF NUM. AT S=0}{VAL. OF DEN. AT S=0}$

PRINT K₀

FIND RESIDUES (K₁) OF Z AT ALL IT'S POLES

PRINT $\frac{1}{K_i}, \frac{P_i}{K_i}$

END

(5)

IS "R" LESS THAN ONE

YES   NO

FORM YR AS $YR = Y_{11} - \frac{AS}{S+B}$

PRINT $A, \frac{B}{A}$

GO TO (I)

HAVE ALL POLES BEEN EXAUSTED

YES

NO

GO TO (V)

ARE NUMER. AND DENOM POLYNOM OF SAME DEGREE

YES   NO

FIND K ∞

SELECT A POLE OF Yₙ THE ABS VALUE OF WHICH IS GREATER OF THE ABS. VALUE OF AT LEAST ONE ZERO OF Y₁₂ AND ONE OF Y₁₁. RESTRICTION NO OTHER POLE BETWEEN THIS POLE AND THE ZEROS.

(8)

(6)

PRINT
K_i , $\dfrac{K_i}{A_i}$

FORM Y_3 AS
$\bar{Y}_3 = Y_{13} - \sum \dfrac{K_i S}{S + A_i}$

ARE ANY ZEROS OF Y_3 THE SAME AS THOSE OF Y_{12}

IF NONE → GO TO IV

IF ALL → GO TO II

INVERT $\bar{Y}_3$
FORM Z_3

FIND RESIDUES (K_i) OF Z_3 AT THESE ZEROS

PRINT
$\dfrac{1}{K_i}$ $\dfrac{K_i}{A_i}$

FORM Z_4 AS
$Z_4 = Z_3 - \sum \dfrac{K_i}{S + A_i}$

GO TO III

(7)

ARE NUMER & DENOM. POLYNOM. OF THE SAME POWER

YES

FIND $K\infty$

NO

DOES EITHER NUMER. OR DEN POLYNOM. HAVE A VALUE OTHER THAN ZERO AT $S = 0$

IF BOTH

IF NONE

IF ONE

FORM K_0
$K_o = \dfrac{\text{VALUE OF NUM. AT } S=0}{\text{VALUE OF DEN. AT } S=0}$

FIND RESIDUE (K_i) OF Y_2 AT

PRINT
$K\infty$, KO, K_i , $\dfrac{P_i}{K_i}$

END

(8)

TAKE DISTANCE BETWEEN THE POLE OF Y_{11} AND THE ZERO OF Y_{12}

DIVIDE DISTANCE BY A GIVEN VALUE CALL QUANTITY X

TO THE VALUE OF THE ZERO OF Y_{12} ADD X_1 CALL NEW VALUE X_1

FIND RESIDUE OF Y_{11} AT X_1 CALL SIDUE XR

IS $XR - K\infty$ POSITIVE

YES

REDEFINE X_1 AS $X_1 + X$

PRINT
$\dfrac{1}{XR}$

EVALUATE Y_{11} AT XR
FORM $Y_1 = Y_{11} - XR$
INVERT Y_1 FORM
$Z_1 = 1/Y_1$

OF Z_1 AT $S = XR$
FORM Z_2
$Z_2 = Z_1 - \dfrac{K_i}{S + XR}$

PRINT
$\dfrac{1}{K_i}$, $\dfrac{K_i}{XR}$

GO TO (I)

```
$JOBLH/05052-00    GEP ROOM447A    301 IX 08 11        2 D100111       LVS
$EXECUTE        IBJOB
$IBJOB
$IBFIC MAIN      NODECK, NOLIST
      DIMENSIONL (4), C (11), D (11), E (11), F (11), ALPHA (10), BETA (10), THETA (10)
     1, DELTA (10), A (11), B (10), NN (5)
 71   FORMAT (11F7. 0)
 51   FORMAT (11, 412)
 61   FORMAT (10F8. 0)
 53   FORMAT (1HO, 2F12. 3)
 55   FORMAT (7HL1. /Y =, F12.5)
 57   FORMAT (6HOK-8 =, F12.4)
 58   FORMAT (6HOK-0 =, F12.4)
      X=0.
      READ (5, 51) ISENSE, (L(I), I=1, 4)
      IF (ISENSE. EQ. 1) GO TO 100
C
C     FIND COEFFICIENTS OF POLYNOMIALS
C
      K=1
      DO 205 I=1, 4
      M=L (I)
      MM=M+1
      READ (5, 61) (B (J), J=1, M)
      DO 211 J=MM, 10
      B (J)=0.
211   A (J) = 0.
      CALL COEFIN (K, M, B, A)
      WRITE (6, 53) B (1), A (1)
      DO 205 J = 2, M
      N = M - (J- 2)
      IF (I- 2) 206, 207, 208
206   ALPHA (J) = B (J)
      C (J) = A (N)
      C (1) = A (1)
      ALPHA (1) = B (1)
      GO TO 205
207   BETA (J) = B (J)
      D (J) = A (N)
      D (1) = A (1)
      BETA (1) = B (1)
      GO TO 205
208   IF (I. EQ. 4) GO TO 210
      THETA (J) = B (J)
      E (J) = A (N)
      THETA (1) = B (1)
      E (1) = A (1)
      GO TO 205
210   DELTA (J) = B (J)
      F (J) = A (N)
      F (1) = A (1)
      DELTA (1) = B (1)
```

```
  205  WRITE  (6, 53)  B(J), A(J)
       GO TO 101
C
C      FIND ROOTS OF POLYNOMIALS
C
  100  XE = . 1
       DO 351 I = 1, 4
       M = L (I) + 1
       READ(5, 71)  (B(J), J = 1, M)
       CALL ROOTFN  (M, XE, B, A)
       DO 351  J = 1, M
       IF  (I - 2)  317, 318, 319
  317  C(J) = B(J)
       ALPHA(J) =  A(J)
       GO TO 351
  318  D(J) = B(J)
       BETA(J) = A(J)
       GO TO 351
  319  IF (I. EQ. 4) GO TO 320
       E(J) = B(J)
       THETA(J) = A(J)
       GO TO 351
  320  F(J) = B(J)
       DELTA(J) = A(J)
  351  CONTINUE
  101  IF(L(2). LE. L(4)) GO TO 500
C
C      P1 HAS OTHER ROOTS THAN THOSE OF P2
C
       K = L(2) - L(4)
       DO 410  I = 1, K
       M = L(2)
       N = L(4)
       DO 410  J = 1, M
       IODD = 0
       DO 411  JJ = 1, N
       IF (BETA(J). NE. DELTA(JJ)) GO TO 411
       IODD = 1
  411  CONTINUE
       IF (IODD. EQ. 1) GO TO 410
       NT = L(1) + 1
       CALL RESID (J, M, NT, D, C, BETA, RES, X)
       IF (RES. LT. 0) GO TO 1000
       JODD = 0
       CALL SUBTRA(J, NT, RES, JODD, D, C, ALPHA, BETA, X)
       L(1) = L(1) - 1
       L(2) = L(2) - 1
  410  CONTINUE
C
C      EVALUATE Y11 AT ONE OF THE ROOTS OF THE NUM. OF Y12
C
```

```
500  LJ = 0
510  UBAR = D(1)
     YBAR = C(1)
     LJ = LJ + 1
     U = THETA(LJ)
     M = L(1) + 1
     DO 501  J = 2, M
501  YBAR = YBAR + C(J)*U**(J - 1)
     MM = L(2) + 1
     DO 502  J = 2, MM
502  UBAR = UBAR + D(J)*U**(J - 1)
     YBAR = YBAR/UBAR
     IF (YBAR. LT. 0) GO TO 600
     UBAR = 1. /YBAR
     WRITE (6, 55) UBAR
     DO 503  J = 1, MM
     C(J) = C(J) - YBAR*D(J)
     IF (C(J). GE. 0.) GO TO 503
     IODD = 1
503  CONTINUE
     IF (IODD. EQ. 1) GO TO 600
C
C       FACTOR NUMERATOR OF Y1
C
     CALL ROOTFN  (M, XE, C, ALPHA)
     LL = 0
     MM = L(1)
     MN = L(3)
     DO 504  J = 1, MM
     DO 504  K = 1, MN
     IF (ALPHA(J) - THETA(K)) 504, 505, 504
505  LL = LL + 1
     NN(LL) = J
504  CONTINUE
     IF (LL. EQ. 0) GO TO 600
     IF (LL. EQ. MM) GO TO 700
C
C       SOME ROOTS OF NUM. OF Y1 ARE THE SAME AS THOSE OF Y12
C       COMPUTE RESIDUES AT THESE ROOTS OF Z1
C       NN(LL) = INDICES OF THESE ROOTS
C       FORM Z2 BY SUBTRACTING RESIDUALS FROM Z1
C
     DO 506  J = 1, LL
     M = NN(J)
     NT = L(2) + 1
     CALL RESID (M, MM, NT, C, D, ALPHA, RES, X)
     JODD = 1
     CALL SUBTRA (M, NT, RES, JODD, C, D, BETA, ALPHA, X)
     L(1) = L(1) - 1
506  L(2) = L(2) - 1
C
```

```
C       FACTOR NUMERATOR OF Z2
C
  513 CALL ROOTFN (NT, XE, D, BETA)
       LL = 0
       DO 507 J = 1, MM
       DO 507 KK = 1, MM
       IF (BETA(J). NE. THETA(KK)) GO TO 507
       LL = LL + 1
       NN(LL) = J
  507 CONTINUE
       IF (LL. EQ. 0) GO TO 800
       IF (LL. EQ. L(3)) GO TO 900
C
C       SOME ROOTS OF Z2 ARE THE SAME AS THOSE OF Y12
C
       MM = MM + 1
       DO 508 J = 1, MM
  508 C(J + 1) = C(J)
       C(1) = 0.
       CALL ROOTFN(MM, XE, C, ALPHA)
       DO 509 J = 1, LL
       M = NN(J)
       NT = L(1) + 1
       CALL RESID (M, MM, NT, D, C, BETA, RES, X)
       JODD = 0
       CALL SUBTRA (M, NT, RES, JODD, D, C, ALPHA, BETA, X)
       L(1) = L(1) - 1
  509 L(2) = L(2) - 1
C
C       FACTOR NUMERATOR OF YBAR 3
C
       CALL ROOTFN (NT, XE, C, ALPHA)
       LL = 0
       DO 511 J = 1, MM
       DO 511 JJ = 1, MM
       IF (ALPHA(J). NE. THETA(JJ)) GO TO 511
       LL = LL + 1
       NN(LL) = J
  511 CONTINUE
       IF (LL. EQ. 0) GO TO 800
       IF (LL. EQ. L(3)) GO TO 700
C
C       SOME ROOTS OF YBAR 3 ARE THE SAME AS THOSE OF Y12
       DO 512 J = 1, LL
       M = NN(J)
       NT = L(2) + 1
       CALL RESID (M, MM, NT, C, D, ALPHA, X)
       JODD = 1
  512 CALL SUBTRA (M, NT, RES, JODD, C, D, BETA, ALPHA, X)
       GO TO 513
  800 IF (LJ. EQ. L(3)) GO TO 1000
```

```
        GO TO 510
C
C       ALL ROOTS OF Y1 ARE THE SAME AS Y12
C
   700  IF (L(1). NE. L(2)) GO TO 701
        M = L(2)
        EIGHT  = D(M)/C(M)
        WRITE (6, 57) EIGHT
   701  IF (D(1). EQ.0.) GO TO 702
        GO TO 704
   702  IF (C(1). EQ. 0. ) GO TO 703
   704  OK  = D(1)/C(1)
        WRITE (6, 58) OK
   703  M = L(1)
        DO 705  J = 1, M
        NT = L(2) + 1
        CALL RESID (J, M, NT, C, D, ALPHA, RES, X)
        YBAR = ALPHA(J)/RES
        RES = 1. /RES
   705  WRITE (6, 53) RES, YBAR
        GO TO 1000
C
C       ALL ROOTS OF Z2 ARE THE SAME AS Y12
C
   900  IF (L(1). NE. L(2)) GO TO 901
        M = L(1)
        EIGHT = C(M)/D(M)
        WRITE (6, 57) EIGHT
   901  IF (C(1). EQ.0.)GO TO 902
        GO TO 904
   902  IF (D(1). EQ.0.)GO TO 903
   904  OK = C(1)/D(1)
        WRITE (6, 58) OK
   903  M = L(2)
        DO 905  J = 1, M
        CALL RESID (J, M, NT, D, C, BETA, RES, X)
        YBAR = BETA(J)/RES
        RES = 1. /RES
   905  WRITE (6, 53) RES, YBAR
        GO TO 1000
   640  DO 651  I = 1, M
        X = ABS(T(1))
        X1 = -1.
        X2 = -1.
        DO 649  J = 1, MM
        DO 649  K = 1, MN
        IF (X. LE. ABS(B(J))) GO TO 649
        IF (X. LE. ABS(A(K))) GO TO 649
        X1 = ABS(B(J))
        X2 = ABS(A(K))
   649  CONTINUE
```

```
          IF(X1. EQ. -1. ) GO TO 651
          DO 650 II = 1, M
          IF (X - X1. GT. ABS(T(II)) - X1) GO TO 651
          IF (X - X2. GT. ABS(T(II)) - X2) GO TO 651
      650 CONTINUE
          GO TO 652
      651 CONTINUE
          GO TO 655
      652 U = 0. 1
          V = 0.  M = M + 1
          DO 653  K = 1, M
      653 M = M + E(K)*X1**(K - 1)
          DO 654  J = 1, M
      654 V = V + F(J)*X1**(J - 1)
          AX = M*(X1 + X)/(V*X)
C
C         EVALUATE Y AT THE ZERO OF Y12 CLOSEST TO THE POLE, SET Y TO 0. AND
C         SOLVE FOR AX.
C
          NT = L(1) + 1
          M = M - 1
          CALL RESID (I, M, NT, D, C, BETA, RES, X)
          R = AX/RES
          WRITE (3, 53) X, RES, AX, R
          IF (R. LT.0.) GO TO 651
          IF (R. GE. 1.) GO TO 651
          JODD = 0
          CALL SUBTRA (I, M, RES, JODD, D, C, ALPHA, X)
          R = BETA(I)/AX
          WRITE (3, 53) AX, R
          GO TO 510
      655 CONTINUE
          M = M + 1
          EIGHT = C(M)/D(M)
          WRITE (3, 57) EIGHT
          M = M - 1
          DO 656  I = 1, M
          X = ABS(BETA(I))
          X1 = -1.
          X2 = -1.
          DO 657  J = 1, MM
          DO 657  K = 1, MN
          IF (X. LE. ABS(DELTA(J))) GO TO 657
          IF (X. LE. ABS(THETA(K))) GO TO 657
          X1 = ABS(DELTA(J))
          X2 = ABS(THETA(K))
      657 CONTINUE
          IF(X1. EQ. -1. ) GO TO 656
          DO 658 II = 1, M
          IF (X - X1. GT. ABS(BETA(II)) - X1) GO TO 656
          IF (X - X2. GT. ABS(BETA(II)) - X2) GO TO 656
```

```
  658  CONTINUE
       GO TO 659
  656  CONTINUE
  659  XX = X - X1
       X1 = X1 + X
  660  CALL RESID (I, M, NT, D, C, BETA, RES, X)
       IF(RES-EIGHT. LT. 0.)GO TO 661
       X1 = X1 + X
       GO TO 660
  661  EIGHT = 1. /(RES-EIGHT)
       WRITE (3, 53) RES, EIGHT
       M = 0.
       V = 0.   M = M + 1
       DO 662  J = 1, M
       V = V + C(J)*RES**(J - 1)
  662  C(J) = C(J) - RES
       M = L(2) + 1
       DO 663  J = 1, M
       M = M + D(J)*RES**(J - 1)
  663  D(J) = D(J) - RES
       V = V/M
       WRITE (3, 53) V
C
C      INVERT Y1 FORM Z1 AND FIND RESIDUAL AT XR.
C
       M = L(1)
       CALL (I, M, NT, C, D, ALPHA, RES, X)
       X1 = 1. /RES
       X2 = RES/ALPHA(11)
       GO TO 510
 1000  STOP
       END
$IBFTC ROOTF    NODECK, NOLIST
       SUBROUTINE ROOTFN(M, XE, B, A)
       DIMENSION A(11), B(10)
   72  FORMAT (44HLPOLYNOMIAL IS NOT REAL OR HAS COMPLEX ZEROS, 15)
   53  FORMAT (1HO, 2F12. 3)
       XINCRE = XE
       DO 306  J = 1, 10
  306  A(J) = 0.
       N = 0
       X = 0.
       IF (B(1). LE. . 000001) GO TO 307
  302  X = X + XINCRE
  303  V = 0.
       AX = -X
       DO 301  J = 1, M
  301  V = V + B(J)*AX**(J - 1)
       IODD = 0
       DO 304  J = 1, 9, 2
       IF (N. NE. J) GO TO 304
```

```
          IODD = 1
  304  CONTINUE
          IF(IODD. EQ. 0) GO TO 323
          V = V*(-1. )
  323  IF (V. LT. 0. ) GO TO 321
          IF (XINCRE. LE. . 0000001) GO TO 305
          IF (V. GT. . 0001) GO TO 302
          GO TO 305
  307  B(1) = 0.
          GO TO 305
  321  X = X - XINCRE
          XINCR= XINCRE/2.
          XINCRE = XINCR
          X = X + XINCR
          GO TO 303
  305  N = N + 1
          A(N) = -X
          IF (N. EQ. (M - 1)) GO TO 316
          IF (B(1). EQ. 0. ) GO TO 312
          IF (X. GT. B(1)) GO TO 315
  311  X = X + XE
          XINCR = XE
          GO TO 302
  312  IF (X. LE. B(2)) GO TO 311
  315  WRITE (6, 72) M
          STOP
  316  DO 317  J = 1, M
  317  WRITE (6, 53) A(J), B(J)
          RETURN
          END
$IBFTC COEFI     NODECK, NOLIST
          SUBROUTINE COEFIN  (K, M, B, A)
          DIMENSION A(11), B(10)
          T1 = 0.
          S1 = 0.
          DO 178  J = 1, 10
  178  B(J) = ABS(B(J))
          DO 177  J = 1, 11
  177  A(J) = ABS(A(J))
          DO 222  I = 1, 5
          S1 = S1 + B(I)
  222  T1 = T1 + B(I + 5)
          T2 = 0.
          S2 = 0.
          DO 220  I = 1, 4
          JJ = I + 1
          DO 220J = JJ, 5
          S2 = S2 + B(I)*B(J)
  220  T2 = T2 + B(I + 5)*B(J + 5)
          T3 = 0.
          S3 = 0.
```

```
        IODD = 1
304  CONTINUE
        IF(IODD. EQ. 0) GO TO 323
        V = V*(-1.)
323  IF (V. LT. 0.) GO TO 321
        IF (XINCRE. LE..0000001) GO TO 305
        IF (V. GT..0001) GO TO 302
        GO TO 305
307  B(1) = 0.
        GO TO 305
321  X = X - XINCRE
        XINCR= XINCRE/2.
        XINCRE = XINCR
        X = X + XINCR
        GO TO 303
305  N = N + 1
        A(N) = -X
        IF (N. EQ. (M - 1)) GO TO 316
        IF (B(1). EQ. 0.) GO TO 312
        IF (X. GT. B(1)) GO TO 315
311  X = X + XE
        XINCR = XE
        GO TO 302
312  IF (X. LE. B(2)) GO TO 311
315  WRITE (6, 72) M
        STOP
316  DO 317  J = 1, M
317  WRITE (6, 53) A(J), B(J)
        RETURN
        END
$IBFTC COEF1    NODECK, NOLIST
        SUBROUTINE COEFIN  (K, M, B, A)
        DIMENSION A(11), B(10)
        T1 = 0.
        S1 = 0.
        DO 178  J = 1, 10
178  B(J) = ABS(B(J))
        DO 177  J = 1, 11
177  A(J) = ABS(A(J))
        DO 222  I = 1, 5
        S1 = S1 + B(I)
222  T1 = T1 + B(I + 5)
        T2 = 0.
        S2 = 0.
        DO 220  I = 1, 4
        JJ = I + 1
        DO 220J = JJ, 5
        S2 = S2 + B(I)*B(J)
220  T2 = T2 + B(I + 5)*B(J + 5)
        T3 = 0.
        S3 = 0.
```

```
            DO 221  I = 1, 3
            JJ = I + 1
            DO 221  J = JJ, 4
            KK = J + 1
            DO 221  K1 = KK, 5
            S3 = S3 + B(I)*B(J)*B(K1)
      221   T3 = T3 + B(I + 5)*B(J + 5)*B(K1 + 5)
            S4 = 0.
            T4 = 0.
            DO 223  I = 1, 2
            JJ = I + 1
            DO 223  J = JJ, 3
            KK = J + 1
            DO 223  J1 = KK, J
            II = J1 + 1
            DO 223  K1 = II, 5
            S4 = S4 + B(I)*B(J)*B(J1)*B(K1)
      223   T4 = T4 + B(I + 5)*B(J + 5)*B(J1 + 5)*B(K1 + 5)
            S5 = B(1)*B(2)*B(3)*B(4)*B(5)
            T5 = B(6)*B(7)*B(8)*B(9)*B(10)
            A(1) = B(1)
            IF (M. EQ. 1) GO TO 206
            DO 204  J = 2, M
      204   A(1) = A(1)*B(J)
      206   A(2) = S1 + T1
            A(3) = S2 + T1*S1 + T2
            A(4) = S3 + T1*S2 + T2*S1 + T3
            A(5) = S4 + T1*S3 + T2*S2 + T3*S1 + T4
            A(6) = S5 + T1*S4 + T2*S3 + T3*S2 + T4*S1 + T5
            A(7) = T1*S5 + T2*S4 + T3*S3 + T4*S2 + T5*S1
            A(8) = T2*S5 + T3*S4 + T4*S3 + T5*S2
            A(9) = T3*S5 + T4*S4 + T5*S3
            A(10) = T4*S5 + T5*S5
            MM = M + 1
            DO 205  I = 1, M
            J = MM - 1
            P = A(J)
            A(J) = A(I)
      205   A(I) = P
            A(MM) = K
            RETURN
            END
      $IBFTC RESI    NODECK, NOLIST
            SUBROUTINE RESID (MM, M, NT, B, A, T, RES, X)
      C
      C     MM = POWER OF ROOT OF DEN. AT WHICH RESIDUAL IS BEING EVALUATED
      C     M = POWER OF POLYNOMIALS
      C     B = COEFFICIENTS OF DENOM
      C     A = COEFFICIENTS OF NUM
      C     T = ROOTS OF DENOM
      C     RES = RESIDUAL
```

```
C       NT = NO. OF COEFFICIENTS OF NUM.
C
        DIMENSION A(11), B(11), T(10)
   81   FORMAT (1HO, 215, 11F10. 3, /, 1H , 10X, 11F10. 3)
   80   FORMAT (1HO, 4F15. 5)
        X = T(MM)
        DO 514 J = 1, M
        IF (J. LE. MM) GO TO 514
        T(J - 1) = T(J)
  514   CONTINUE
        T(M) = 0.
        K = 1
        KK = M + 1
        CONST = B(KK)
        DO 1006 J = 1, KK
 1006   B(J) = B(J)/B(KK)
        WRITE (6, 81) K, M, T,B
        CALL COEFIN (K, M, T, B)
        WRITE (6, 81) K, M, T, B
        YBAR = A(1)
        DO 1001 J = 2, NT
 1001   YBAR = YBAR + A(J)*X**(J - 1)
        IF (B(1). NE. 0.)GO TO 1003
        DO 1004 J = 2, KK
 1004   B(J - 1) = B(J)
 1003   DO 1005 J = 1, M
 1005   B(J) = B(J)*CONST
        UBAR = B(1)
        B(KK) =0.
        WRITE (6, 81) K, M, T, B
        IF (X. EQ. 0. ) GO TO 1002
        DO 1000 J = 2, M
        YBAR = YBAR + A(J)*X**(J - 1)
 1000   UBAR = UBAR + B(J)*X**(J - 1)
 1002   RES = YBAR/UBAR
        WRITE (6, 80) RES, YBAR, UBAR, X
        RETURN
        END
$IBFTC SUBTR    NODECK, NOLIST
        SUBROUTINE SUBTRA (MM, M, RES, JODD, T, B, A, AA, X)
C
C       MM = INDICE OF ROOT AT WHICH RESIDUAL IS BEING COMPUTED
C       M = POWER OF POLYNOMIAL
C       RES = RESIDUAL
C       T = COEFF. OF DEN.
C       B = COEFFICIENTS OF NUM.
C       A = ROOTS OF NUM. THAT ARE RETURNED
C
        DIMENSION T(11), B(11), A(10), AA(10)
   81   FORMAT (1HO, 215, 11F10. 3, /, 1H , 10X, 11F10. 3)
   53   FORMAT (1HO, 2F12. 5)
```

```
        YBAR = RES/X
        WRITE (6, 53) RES, YBAR
        XE = . 1
        M = M + 1
        DO 422 KK = 1, M
        IF (JODD. EQ. 0) GO TO 423
        JJ = KK
        GO TO 422
423  JJ = KK + 1
422  B(JJ) = B(JJ) - T(KK)*RES
        M = M - 1
        K = 1
        WRITE (6, 81) K, M, B, A
        CALL ROOTFN (M, XE, B, A)
        WRITE (6, 81) K, M, B, A
        M = M - 1
        AA(M) = X
        DO 425 I = 1, M
        DO 425 J = 1, M
        IF (AA(I). NE. A(J)) GO TO 425
        II = I
        JI = J
425  CONTINUE
        DO 427 J = 1, M
        IF (J. LE. II) GO TO 426
        AA(J - 1) = AA(J)
426  IF (J. LE. JI) GO TO 427
        A(J - 1) = A(J)
427  CONTINUE
        AA(M) = 0.
        A(M) = 0.
        X = B(M + 1)
        DO 429 J = 1, M
429  B(J) = B(J + 1)
        B(M + 1) = 0.
        M = M - 1
        WRITE (6, 81) K, M, A, B
        CALL COEFIN (K, M, A, B)
        WRITE (6, 81) K, M, A, B
        M = M + 1
        DO 428 I = 1, M
428  B(I) = B(I)*X
        WRITE (6, 81) K, M, A, B
        WRITE (6, 81) K, M, AA, T
        RETURN
        END
```

NO. OF CARDS IN FILE = 557

APPENDIX 2

Appendix 2 gives a tabulation of the describing matrices and characteristic functions of the different interconnections of an RC two-port network and RC-NIC two-port network. Note that from each interconnection only the characteristic function that could be used for RC-NIC synthesis are given.
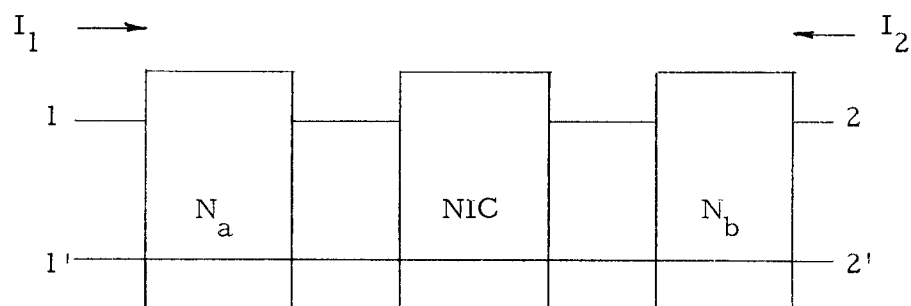
Connection in Tandem

Figure 1. Connection in cascade

Describing Matrix

$$
\begin{bmatrix} A \end{bmatrix}_{all} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}
$$

in terms of z's

$$
\left[A \right]_{all} =
\begin{bmatrix}
z_{11a} + \dfrac{z_{12a}z_{21a}}{z_{11b}-z_{22a}} & \dfrac{z_{12a}z_{22a}}{z_{11b}-z_{22a}} \\[4ex]
\dfrac{z_{12a}z_{12b}}{z_{11b}-z_{22a}} & z_{22b} - \dfrac{z_{12b}z_{21b}}{z_{11b}-z_{22a}}
\end{bmatrix}
$$

in terms of y's

$$
\left[A \right]_{all} =
\begin{bmatrix}
y_{11a} + \dfrac{y_{21a}y_{12a}}{y_{11b}-y_{22a}} & \dfrac{y_{12b}y_{21b}}{y_{11b}-y_{22a}} \\[4ex]
\dfrac{y_{12a}y_{12b}}{y_{11b}-y_{22a}} & y_{22b} - \dfrac{y_{12b}y_{21b}}{y_{11b}-y_{22a}}
\end{bmatrix}
$$

Characteristic Functions

$$
z_{12} = \frac{z_{12a}z_{12b}}{z_{11b}-z_{22a}} \quad , \qquad y_{12} = \frac{y_{12a}y_{12b}}{y_{11b}-y_{22a}}
$$

$$
y_{in} = y_{11} - \frac{y_{12}y_{21}}{y_{22}-y_{L}} = y_{11a} - \frac{y_{12}y_{21a}}{y_{22a}-y_{11}}
$$

$$
Z_{in} = z_{11} - \frac{z_{12}z_{21}}{z_{22}-Z_{2}} = z_{11a} - \frac{z_{12a}z_{21a}}{z_{22a}-z_{11b}}
$$

For ENIC

$$\begin{bmatrix} A \end{bmatrix}_{all} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

in terms of z's

$$\begin{bmatrix} A \end{bmatrix}_{all} \rightarrow \begin{bmatrix} z_{11a} - \dfrac{z_{12a}z_{21a}}{z_{22a}-z_{11b}} & \dfrac{z_{21a}z_{21b}}{z_{22a}-z_{11b}} \\ \dfrac{z_{21a}z_{12b}}{z_{22a}-z_{11b}} & z_{22b} + \dfrac{z_{22b}z_{21b}}{z_{22a}-z_{11b}} \end{bmatrix}$$

in terms of y's

$$\begin{bmatrix} A \end{bmatrix}_{all} \rightarrow \begin{bmatrix} y_{11a} - \dfrac{y_{12a}y_{21b}}{y_{22a}-y_{11b}} & \dfrac{y_{21a}y_{21b}}{y_{22a}-y_{11b}} \\ \dfrac{y_{21a}y_{21b}}{y_{22a}-y_{11b}} & y_{22b} + \dfrac{y_{12b}y_{21b}}{y_{22a}-y_{11b}} \end{bmatrix}$$

Characteristic Functions

$$Z_{12} = \frac{z_{22a}z_{21b}}{z_{22a}-z_{11b}} \quad , \quad Y_{12} = \frac{y_{21a}y_{21b}}{y_{22a}-y_{11b}}$$

$$Y_{in} = y_{11a} - \frac{y_{12a}y_{21a}}{y_{22a} - y_{11b}} \quad , \quad Z_{in} = z_{11a} - \frac{z_{12a}z_{21a}}{z_{22a} - z_{11b}}$$
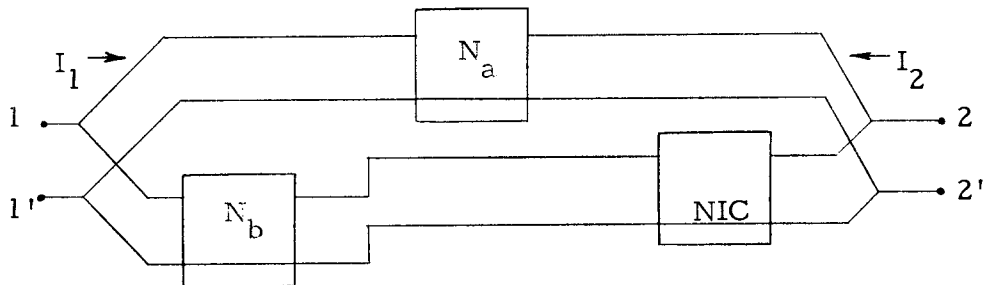
Parallel-Parallel Connections

Figure 2. Parallel-parallel connection

Describing Matrix

For INIC

$$\begin{bmatrix} y \end{bmatrix}_{all} = \begin{bmatrix} y_{11a} & y_{12a} \\ \\ y_{21a} & y_{22a} \end{bmatrix} + \begin{bmatrix} y_{11c} & y_{12c} \\ \\ y_{21c} & y_{22c} \end{bmatrix}$$

in terms of y_a's and y_b's

$$\left[y \right]_{all} = \begin{bmatrix} y_{11a} + y_{11b} & y_{12a} - y_{12b} \\ y_{21a} + y_{21b} & y_{22a} - y_{22b} \end{bmatrix}$$

Characteristic Functions

$$G_{12} = - \frac{y_{12a} - y_{12b}}{y_{22a} - y_{22b}}$$

$$Y_{in} = y_{11a} + y_{11b} - \frac{(y_{12a} - y_{12b})(y_{21a} - y_{21b})}{y_{22a} - y_{22b}}$$

For ENIC

$$\left[y \right]_{all} = \begin{bmatrix} y_{11a} & y_{12a} \\ y_{21a} & y_{22a} \end{bmatrix} + \begin{bmatrix} y_{11c} & y_{12c} \\ y_{21c} & y_{22c} \end{bmatrix}$$

in terms of y_a's and y_b's

$$\left[y \right]_{all} = \begin{bmatrix} y_{11a} + y_{11b} & y_{12a} + y_{12b} \\ y_{21a} + y_{21b} & y_{22a} + y_{22b} \end{bmatrix}$$

Characteristic Functions

$$G_{12} = -\frac{y_{12a} + y_{12b}}{y_{22a} - y_{22b}}$$

$$Y_{in} = y_{11a} + y_{11b} - \frac{(y_{12a} + y_{12b})(y_{21a} - y_{21b})}{y_{22a} - y_{22b}}$$
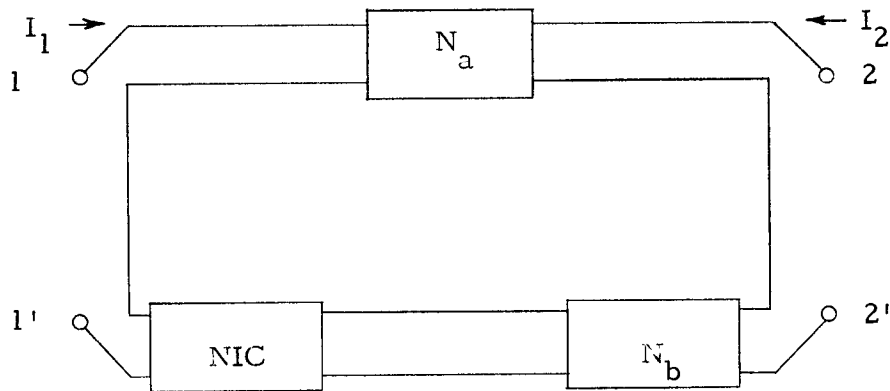
Series-Series Connection

Figure 3. Series-series connection

Describing Matrix

For INIC

$$\begin{bmatrix} z \end{bmatrix}_{all} = \begin{bmatrix} z_{11a} & z_{12a} \\ z_{21a} & z_{22a} \end{bmatrix} + \begin{bmatrix} z_{11c} & z_{12c} \\ z_{21c} & z_{22c} \end{bmatrix}$$

in terms of z_a's and z_b's

$$\left[z \right]_{all} = \begin{bmatrix} z_{11a} - z_{11b} & z_{12a} + z_{12b} \\ z_{21a} - z_{21b} & z_{22a} + z_{22b} \end{bmatrix}$$

Characteristic Functions

$$G_{12} = \frac{z_{12a} + z_{12b}}{z_{11a} - z_{11b}}$$

For ENIC

$$\left[z \right]_{all} = \begin{bmatrix} z_{11a} & z_{12a} \\ z_{21a} & z_{22a} \end{bmatrix} + \begin{bmatrix} z_{11c} & z_{12c} \\ z_{21c} & z_{22c} \end{bmatrix}$$

in terms of z_a's and z_b's

$$\left[z \right]_{all} = \begin{bmatrix} z_{11a} - z_{11b} & z_{12a} - z_{12b} \\ z_{21a} + z_{21b} & z_{22a} + z_{22b} \end{bmatrix}$$

Characteristic Functions

$$G_{12} = \frac{z_{12a} - z_{12b}}{z_{22a} - z_{22b}}$$
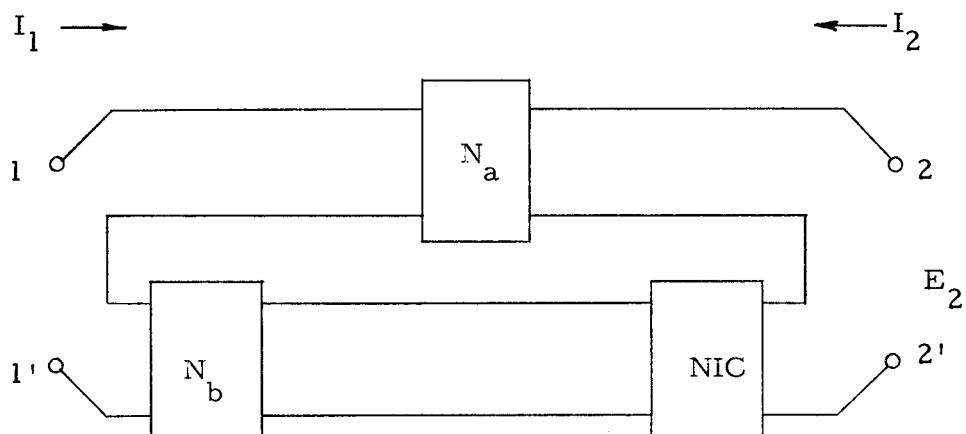
Changing the order of N_b and NIC

Figure 4. Series-series connection

In terms of z_a's and z_b's for INIC

$$
\begin{bmatrix} z \end{bmatrix}_{all} = \begin{bmatrix} z_{11a} + z_{11b} & z_{12a} - z_{12b} \\ z_{21a} + z_{21b} & z_{22a} - z_{22b} \end{bmatrix}
$$

Characteristic Functions

$$
Z_{in} = z_{11a} + z_{11b} - \frac{(z_{12a} - z_{12b})(z_{21a} + z_{21b})}{z_{22a} - z_{22b}}
$$

In terms of z_a's and z_n's for ENIC

$$
\begin{bmatrix} z \end{bmatrix}_{all} = \begin{bmatrix} z_{11a} + z_{11b} & z_{12a} + z_{12b} \\ z_{21a} - z_{21b} & z_{22a} - z_{22b} \end{bmatrix}
$$

Characteristic Functions

$$
Z_{in} = z_{11a} + z_{11b} - \frac{(z_{12a} + z_{12b})(z_{21a} - z_{21b})}{z_{22a} - z_{22b}}
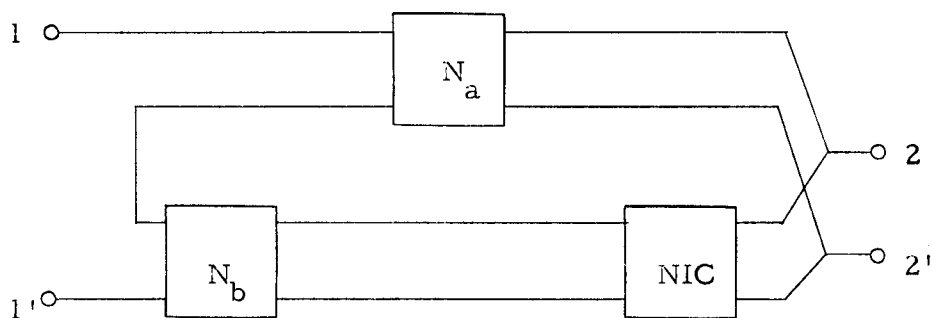$$

Series-Parallel Connection

Figure 5. Series-parallel connection

Describing Matrix

For INIC

$$
\begin{bmatrix} H \end{bmatrix}_{all} = \begin{bmatrix} H_{11a} & H_{12a} \\ H_{21a} & H_{22a} \end{bmatrix} + \begin{bmatrix} H_{11c} & -H_{12c} \\ -H_{21c} & H_{22c} \end{bmatrix}
$$

$$
= \begin{bmatrix} H_{11a} + H_{11b} & H_{12a} + H_{12b} \\ H_{21a} - H_{21b} & H_{22a} - H_{22b} \end{bmatrix}
$$

In terms of z's

$$
\begin{bmatrix} H \end{bmatrix}_{all} \longrightarrow \begin{bmatrix} f(z) & \dfrac{z_{12a}z_{22b}+z_{12b}z_{22a}}{z_{22b}-z_{22a}} \\[4mm] \dfrac{z_{22a}z_{22b}-z_{21b}z_{22a}}{z_{22b}-z_{22a}} & \dfrac{z_{22a}z_{22b}}{z_{22b}-z_{22a}} \end{bmatrix}
$$

Characteristic Functions

$$
Z_{21} = \frac{z_{21a}z_{22b} - z_{21b}z_{22a}}{z_{22b} - z_{22a}} \; ; \qquad Z = \frac{z_{22a}z_{22b}}{z_{22b}-z_{22a}}
$$
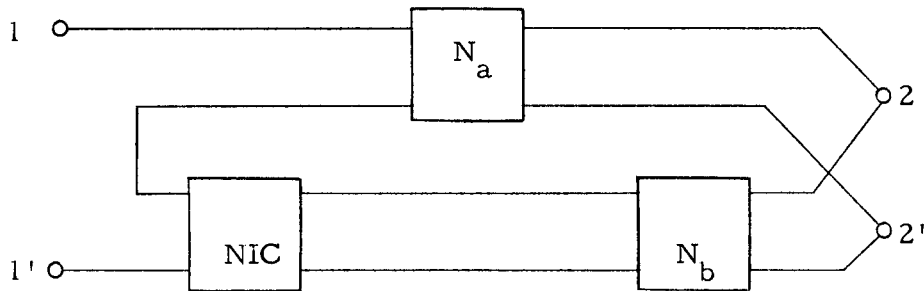
Changing the order of N_b and NIC

Figure 6. Series-parallel connection

Describing Matrix

For INIC

$$
\begin{bmatrix} H \end{bmatrix}_{all} = \begin{bmatrix} H_{11a} & H_{12a} \\ H_{21a} & H_{22a} \end{bmatrix} + \begin{bmatrix} H_{11c} & -H_{12c} \\ -H_{21c} & H_{22c} \end{bmatrix}
$$

$$
= \begin{bmatrix} H_{11a} - H_{11b} & H_{12a} - H_{12b} \\ H_{21a} + H_{21b} & H_{22a} + H_{22b} \end{bmatrix}
$$

in terms of y's

$$
\begin{bmatrix} H \end{bmatrix} \rightarrow \\ \text{all}
\begin{bmatrix}
\dfrac{y_{11a}y_{11b}}{y_{11b}-y_{11a}} & \dfrac{y_{12a}y_{11b}-y_{12b}y_{11a}}{y_{11b}-y_{11a}} \\
\\
\dfrac{y_{12a}y_{11b}+y_{12b}y_{11a}}{y_{11b}-y_{11a}} & f(y)
\end{bmatrix}
$$

Characteristic Functions

$$
Y_{12} = \frac{y_{12a}y_{11b} - y_{12b}y_{11a}}{y_{11b} - y_{11a}}
$$