ERROR BOUNDS FOR ITERATIVE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

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TABLE OF CONTENTS

ERROR BOUNDS FOR ITERATIVE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

INTRODUCTION

Many differential equations of importance in mathematical physics, together with their boundary conditions, may be reformulated in terms of a single linear integral equation of Fredholm type and second kind,

$$y(s) = x(s) - \lambda\int_a^b K(s,t)x(t)dt, \quad a \leq s \leq b, \tag{1.1}$$

where $x(s)$ is the unknown function, $y(s)$ and $K(s,t)$ are known functions which satisfy conditions to be set forth later, and λ is a parameter (12, pp. 514-523). The solution of the integral equation (1.1) in these cases, if it can be obtained, gives at once the solution of the "boundary value problem" associated with the given differential equation, and thus the use of integral equations affords, at least in theory, a method of great convenience for dealing with physical problems.

Unfortunately, Fredholm's general theorem concerning the existence and multiplicity of solutions $x(s)$ of (1.1) does not give a practical method for computing $x(s)$ except in special cases (7, pp. 365-390). For practical purposes, however, a satisfactory approximation to $x(s)$ may be of great value. Many methods have been developed for finding approximate solutions of (1.1) (5, pp. 12-122). The methods considered here will be iterative methods; that is,

procedures which determine a sequence of functions $x_0(s)$, $x_1(s),\ldots,x_n(s),\ldots$ such that $x_n(s)$ converges uniformly to $x(s)$ on the interval $a \leq s \leq b$ with increasing n, and, in general, $x_n(s)$ is computed from the known values of $x_0(s)$, $\ldots,x_{n-1}(s)$.

Most iterative processes appear as modifications of the method due to Neumann (6, pp. 119-120), which converges provided that $|\lambda| < c$, where c is a constant depending on $K(s,t)$. As most equations of the type (1.1) which are of importance in physics have a unique solution $x(s)$ except for a countable set of values of λ, the usefulness of the Neumann process is somewhat limited (6, pp. 107-112). In the consideration of a physical problem involving the loading of an elastic beam, Wiarda obtained an integral equation of the type (1.1) with $|\lambda| \geq c$; in order to deal with this problem, an iterative method was devised which converges provided that $\lambda K(s,t) \leq 0$, $a \leq s,t \leq b$ (15, pp. 119-128). The iterative process devised by Wiarda was further generalized by Bückner (4, pp. 197-206) and Wagner (14, pp. 23-30) to obtain iterative processes which converge under still less restrictive assumptions. More recently, Bückner has stated a process which converges in the physically important case that $K(s,t)$ is <u>symmetric</u>; that is, $K(s,t) = K(t,s)$, $a \leq s,t \leq b$, and that a unique solution of (1.1) exists, (5, pp. 68-71).

A somewhat different approach to the solution of (1.1)

by iteration has been taken by Samuelson (13, pp. 276-286); if (1.1) has a unique solution $x(s)$, then there exists a function $G(s,t;\lambda)$ such that

$$x(s) = y(s) + \int_a^b G(s,t;\lambda)y(t)dt, \quad a \leq s \leq b. \tag{1.2}$$

In case that a sufficiently good approximation $J(s,t;\lambda)$ to $G(s,t;\lambda)$ is known, it is possible to obtain an iterative process which converges to the solution of (1.1). The idea behind all of these iterative processes is the same; the use of the result of substituting an approximation to $x(s)$ into (1.1) to improve the approximation further, and thus a unified treatment of these various methods will be made, from which sufficient conditions for their convergence will be obtained.

For a given iterative process to be useful for practical computation, it is necessary to know more than that the process converges; if $x_n(s)$ denotes the function obtained as the result of n steps of the given iterative process, it is necessary to know to what degree of accuracy (in some sense) that $x_n(s)$ approximates $x(s)$ on the interval $a \leq s \leq b$. Explicit relations for estimating the accuracy of the Neumann process have been given by Lonseth (11, pp. 353-358); similar error bounds will be supplied here for some of the more general methods cited. These results are believed to be new and to make possible the practical application of a wider variety of iterative processes to the

solution of the equation (1.1).

The iterative processes considered follow a set procedure to obtain successive approximations to x(s), and thus they are readily adaptable for use by high-speed computing machines. As the use of such machines generally involves the employment of some method of approximate integration, an additional error is introduced into the iterative process. The error bounds given will be extended to include the use of methods of approximate integration.

PRELIMINARY DEFINITIONS AND RESULTS

The formulation and proofs of the results in this paper, and, in particular, the precise definition of the "degree of accuracy" with which a function $x_n(s)$ "approximates" a function $x(s)$ on the interval $a \leq s \leq b$ will be made to depend on the theory of normed linear vector spaces, and of certain types of operators in these spaces. Only a brief outline of the essential part of the theory will be given, and many results will be stated without proof. Proofs, further elaborations of the theory, and additional references will be found in the works cited.

Consider the set (C) of all functions $x = x(s)$ which are real, single-valued, and continuous on the interval $a \leq s \leq b$. As a consequence of the ordinary properties of these functions, if x is an element of (C), and θ is any real number, then θx is also an element of (C); if x and y are any two elements of (C), so is their <u>sum</u> $x + y$ and <u>product</u> xy. The functions $z(s) = 0$ and $u(s) = 1$, $a \leq s \leq b$, are the elements of (C) such that, for all x in (C), $x + z = x$ and $xu = x$; z and u will be denoted by 0 and 1, respectively. The operations of addition and multiplication obey all of the ordinary algebraic rules. Another type of multiplication, the <u>inner product</u> (x,y) of two elements x and y of (C), which is defined to be the real number

$$(2.1) \qquad (x,y) = \int_a^b x(s)y(s)ds$$

is commutative, and is distributive over addition.

If with every element x of (C) there is associated a finite, non-negative real number $\|x\|$, called the norm of x, which satisfies the following conditions:

(1) $\|x\| > 0$ if $x \neq 0$, $\|0\| = 0$,

(2) $\|\theta x\| = |\theta| \cdot \|x\|$ for any real number θ,

(3) $\|x + y\| \leq \|x\| + \|y\|$ for all x,y in (C),

the set (C) forms what is known as a normed linear vector space (2, p. 33). There are many possible ways to define $\|x\|$, for example, as

$$(2.2) \qquad \|x\| = \max |x(s)| \text{ on } a \leq s \leq b,$$

or, as the square of every element of (C) is integrable, as

$$(2.3) \qquad \|x\| = \left(\int_a^b x^2(s)ds \right)^{1/2} = \sqrt{(x,x)},$$

(10, pp. 206-207). For any real number $\rho \geq 1$, a definition of $\|x\|$ satisfying (1), (2), and (3) is

$$(2.4) \qquad \|x\| = \left(\int_a^b |x(s)|^\rho ds \right)^{1/\rho},$$

(9, pp. 134-150). The practical importance of the abstract idea of the norm of a function is that it makes possible the precise definition of the error involved in the approximation of a function x(s) by a function $x_n(s)$ as the non-negative real number $\|x - x_n\|$.

The error estimates given will thus be estimates of the size of the norm $\|x - x_n\|$. These results will be valid for any definition of the norm satisfying the given conditions, so the computer may choose the one which will give meaningful results for his particular problem. For example, the norm as defined by (2.2) will give the maximum value of the "point-wise" error $|x(s) - x_n(s)|$ on the interval $a \leq s \leq b$, while the norm as defined by (2.3) will give the "global" or "least-squares" error of the approximation $x_n(s)$ to $x(s)$ on the same interval. If, for all definitions of the norm, $\|x - x_n\| \to 0$ as $n \to \infty$, the use of the norm (2.2) shows that the functions $x_n(s)$ converge uniformly to the function $x(s)$ on the interval $a \leq s \leq b$ as $n \to \infty$.

A transformation T which carries each element x of (C) into an element y of (C), in symbols, $Tx = y$, is said to be in (C). T is said to be <u>additive</u> if $T(x+y) = Tx + Ty$ for all x,y in (C), and to be <u>continuous</u> if $\|x - x_n\| \to 0$ as $n \to \infty$ implies that $\|Tx - Tx_n\| \to 0$ as $n \to \infty$. A continuous and additive T is called <u>linear</u>, and the non-negative real numbers

$$M(T) = \text{l.u.b.}(\|Tx\|/\|x\|), \quad \|x\| \neq 0, \tag{2.5}$$

and

$$m(T) = \text{g.l.b.}(\|Tx\|/\|x\|), \quad \|x\| \neq 0, \tag{2.6}$$

exist for a linear T (3, p. 54). M(T) and m(T) are called the <u>upper bound</u> and <u>lower bound</u> of T, respectively. From their definitions,

(2.7) $$m(T)\|x\| \leq \|Tx\| \leq M(T)\|x\|$$

for all x in (C). A linear T is also homogeneous, (3, p. 36), so for any real number θ,

(2.8) $$M(\theta T) = |\theta| M(T).$$

The transformation I such that $Ix = x$ for all x in (C) is defined to be the identity transformation in (C). From (2.5) and (2.6),

(2.9) $$m(I) = M(I) = 1.$$

The sum $T_1 + T_2$ of two linear transformations T_1 and T_2 in (C) is defined to be the transformation T such that $Tx = T_1x + T_2x$ for all x in (C), and is written $T = T_1 + T_2$. The product T_1T_2 of two linear transformations T_1 and T_2 in (C) is defined to be the transformation T such that $Tx = T_1(T_2x)$ for all x in (C), and is written $T = T_1T_2$. Furthermore,

(2.10) $$M(T_1 + T_2) \leq M(T_1) + M(T_2),$$

(2.11) $$M(T_1T_2) \leq M(T_1)M(T_2)$$

for all linear transformations T_1, T_2 in (C) (10, p. 194).

The inverse of the transformation T in (C) is defined to be the transformation T^{-1} such that $T^{-1}T = TT^{-1} = I$ if such exists. The inverse of a linear transformation T, if it exists, is likewise linear, and if also $m(T) > 0$, then

(2.12) $$m(T)M(T^{-1}) = 1,$$

(10, p. 194). The nth power T^n of a linear transformation T in (C) is defined to be $T^n = TT^{n-1}$ for all positive integers n, with $T^0 = I$ by definition. Addition of linear

transformations is associative and commutative; multiplication is associative and distributive over addition, but is not commutative in general, although multiplication of powers of the same linear transformation is commutative (10, p. 194).

A linear transformation T in (C) is said to be completely continuous if $([x - x_n],y) \to 0$ as $n \to \infty$ for all y in (C) implies that $\|Tx - Tx_n\| \to 0$ as $n \to \infty$ (1, pp. 2-3). A transformation in (C) is sometimes referred to as an operator in (C). In what follows, all operators considered are completely continuous linear transformations in (C), unless the contrary is explicitly stated.

If Q is an operator in (C), the transformation (I - Q) has the unique inverse

$$(2.13) \qquad (I - Q)^{-1} = \sum_{j=0}^{\infty} Q^j,$$

provided that $M(Q) < 1$ (10, pp. 194-195). Thus

$$(2.14) \qquad 1 - M(Q) \leq m(I-Q) \leq M(I-Q) \leq 1 + M(Q)$$

from (2.10), (2.11), and (2.12). An operator Q in (C) is said to be positive definite if $(Qx,x) > 0$ for all $x \neq 0$ in (C), and to be positive semi-definite if $(Qx,x) \geq 0$ for all x in (C). The sum and product of two positive (semi-)definite operators in (C) is likewise positive (semi-)definite, and if Q is positive definite,

$$(2.15) \qquad \|x + Qx\| > \|x\|$$

for all $x \neq 0$ in (C), while if Q is positive semi-definite,

(2.16) $$\|x + Qx\| \geq \|x\|$$

for all x in (C).

If for some operator K in (C) there exists a number λ such that for some $x \neq 0$ in (C), $\lambda Kx = x$, λ is said to be a characteristic value of K, and x is called a characteristic function belonging to λ. If the function K(s,t) is real, single-valued, and continuous on the square $a \leq s,t \leq b$, the linear integral operator K defined by

(2.17) $$Kx = \int_a^b K(s,t)x(t)dt, \quad a \leq s \leq b,$$

for all x in (C) satisfies the conditions of linearity and complete continuity imposed on operators in (C), as the consequence of well-known theorems concerning Riemann integrals (16, pp. 126-154). The function K(s,t) is called the kernel of the linear integral operator K. The sum $K = K_1 + K_2$ of two linear integral operators K_1, K_2 with kernels $K_1(s,t)$ and $K_2(s,t)$, respectively, is a linear integral operator with the kernel

(2.18) $$K(s,t) = K_1(s,t) + K_2(s,t),$$

and the product $K = K_1K_2$ of two linear integral operators is a linear integral operator with the kernel

(2.19) $$K(s,t) = \int_a^b K_1(s,r)K_2(r,t)dr, \quad a \leq s,t \leq b,$$

where $K_1(s,t)$ and $K_2(s,t)$ are the kernels of K_1 and K_2, respectively (5, p. 3).

A GENERAL ITERATIVE METHOD

The integral equation (1.1) may be put in the form

(3.1) $$y = x - \lambda Kx,$$

where it is assumed that y is in (C) and K is a linear integral operator in (C). Fredholm's general theorem (7, pp. 365-390) guarantees the existence of a unique solution x in (C) of (3.1), provided that λ is not a characteristic value of K, in particular, $x = 0$ if $y = 0$. In what follows, it will be assumed that $y \neq 0$ and λ is not a characteristic value of K, unless the contrary is explicitly stated.

All iterative processes considered are of the form

(3.2) $$x_n = x_{n-1} - P(I - \lambda K)x_{n-1} + Py,$$

where P is an operator in (C). To solve (3.1) by the iterative process (3.2), an initial function x_0 in (C) is chosen, and repeated applications of (3.2) result in a sequence of functions $x_1, x_2, \ldots, x_n, \ldots$ in (C). If for some x_0 in (C), $\|x - x_n\| \to 0$ as $n \to \infty$, the given iterative process is said to be <u>convergent</u>. If the process converges for all x_0 in (C), it is defined to be <u>totally convergent</u>, a definition originating with Hans Bückner (5, p. 68). For convenience of notation, take

(3.3) $$Q = P(I - \lambda K),$$

so Q is also an operator in (C).

Convergence Theorem: The iterative process defined by (3.2) is totally convergent, provided that $M(I - Q) < 1$, which is always the case if Q is positive definite and $M(Q) < 1$.

Proof: From (3.2),

$$x - x_n = x - x_{n-1} + P(I - \lambda K)x_{n-1} - Py \tag{3.4}$$
$$= x - x_{n-1} + Qx_{n-1} - Qx$$
$$= (I - Q)(x - x_{n-1}),$$

as $y = (I - \lambda K)x$, and thus $Py = Qx$. By induction on n,

$$x - x_n = (I - Q)^n(x - x_0) \tag{3.5}$$

for all positive integers n, and thus from (2.7) and (2.11),

$$\|x - x_n\| \leq M^n(I - Q)\|x - x_0\|, \tag{3.6}$$

from which it follows that $M(I - Q) < 1$ implies that $\|x - x_n\| \to 0$ as $n \to \infty$. This proves the first part of the theorem. For $M(Q) < 1$, by (2.13),

$$(I - Q)^{-1} = \sum_{j=0}^{\infty} Q^j. \tag{3.7}$$

If Q is positive definite, from (2.6) and (2.15),

$$0 < \mu = m(Q) < 1, \tag{3.8}$$

provided that $M(Q) < 1$, and from (2.15) and (3.7),

$$m[(I - Q)^{-1}] \geq 1/(1 - \mu). \tag{3.9}$$

Therefore, from (2.12),

$$M(I - Q) \leq 1 - \mu < 1, \tag{3.10}$$

which completes the proof of the theorem.

General expressions for error estimates for the

process (3.2) will be obtained on the assumption that the conditions for total convergence are satisfied. First, for Q positive definite and $M(Q) < 1$, from (3.6) and (3.10),

$$\|x - x_n\| \leq (1 - \mu)^n \|x - x_0\|, \tag{3.11}$$

an error estimate which is useful if a bound is known for the value of $\|x - x_0\|$. In case that nothing is known concerning the size of $\|x - x_0\|$, error estimates may be derived in the following fashion: From (3.2),

$$x_n - x_{n-1} = Q(x - x_{n-1}), \tag{3.12}$$

which implies that

$$\|x_n - x_{n-1}\| \geq \mu \|x - x_{n-1}\|. \tag{3.13}$$

From (3.4) and (3.10),

$$\|x - x_n\| \leq (1 - \mu)\|x - x_{n-1}\|, \tag{3.14}$$

and thus

$$\|x - x_n\| \leq \frac{1 - \mu}{\mu} \|x_n - x_{n-1}\|. \tag{3.15}$$

From (3.11) and (3.13), if k is a positive integer,

$$\|x - x_{n+k}\| \leq \frac{(1-\mu)^k}{\mu} \|x_n - x_{n-1}\|. \tag{3.16}$$

If the choice $x_0 = y$ is made, from (3.2),

$$x_1 - y = P\lambda K y, \tag{3.17}$$

and thus from (3.16),

$$\|x - x_n\| \leq \frac{(1 - \mu)^n}{\mu} M(P)M(\lambda K)\|y\|, \tag{3.18}$$

all of the above inequalities holding for all positive

integers n.

Now, if $M(I - Q) < 1$, from (2.4),

(3.19) $$\mu = m(Q) \geq 1 - M(I - Q) > 0,$$

and thus in all of the inequalities given, μ may be replaced by $1 - M(I - Q)$ by (3.6) and (3.19), as (3.19) states that

(3.20) $$1/\mu \leq 1/[1 - M(I - Q)],$$

provided that $M(I - Q) < 1$.

EXPLICIT ERROR BOUNDS

The results found for the iterative process (3.2) were derived without actually specifying the operator P. This operator will now be specified to obtain explicit error bounds for iterative processes of practical importance.

The Neumann process

(4.1) $$x_n = y + \lambda K x_{n-1}$$

is totally convergent, provided that $M(\lambda K) < 1$. This process is (3.2) with $P = I$, and thus $Q = (I - \lambda K)$. As $(I - Q) = \lambda K$, it follows that $M(\lambda K) < 1$ implies $M(I-Q) < 1$, and the total convergence of the Neumann process is thus a consequence of the Convergence Theorem. Error bounds for the Neumann process are obtained from the general expressions by substituting $1 - M(\lambda K)$ for μ, which is valid by (3.19) and (3.20). From (3.11),

(4.2) $$\|x - x_n\| \leq M^n(\lambda K)\|x - x_0\|;$$

from (3.15),

(4.3) $$\|x - x_n\| \leq \frac{M(\lambda K)}{1 - M(\lambda K)} \|x_n - x_{n-1}\|;$$

from (3.16), for any positive integer k,

(4.4) $$\|x - x_{n+k}\| \leq \frac{M^k(\lambda K)}{1 - M(\lambda K)}\|x_n - x_{n-1}\|;$$

and from (3.18), as $M(P) = M(I) = 1$, for $x_0 = y$,

$$(4.5) \qquad \|x - x_n\| \leq \frac{M^{n+1}(\lambda K)}{1 - M(\lambda K)} \|y\|,$$

the above inequalities holding for all positive integers n.

The Wiarda process

$$(4.6) \qquad x_n = (1 - \theta)x_{n-1} + \theta\lambda K x_{n-1} + \theta y$$

is totally convergent, provided that $-\lambda K$ is positive semi-definite and θ is a real number such that

$$(4.7) \qquad 0 < \theta < 1/[1 + M(\lambda K)].$$

The process (4.6) is (3.2) with $P = \theta I$, and thus $Q = \theta(I - \lambda K)$. If $-\lambda K$ is positive semi-definite, Q is positive definite, and the condition (4.7) insures that $M(Q) < 1$; the total convergence of the Wiarda process thus follows from the Convergence Theorem. From (2.16),

$$(4.8) \qquad \mu = m(Q) = m[\theta(I - \lambda K)] \geq \theta$$

for $-\lambda K$ positive semi-definite, and as $0 < \theta < 1$,

$$(4.9) \qquad 1 - \mu \leq 1 - \theta.$$

Error bounds for the Wiarda process, therefore, may be obtained by substituting these values into the general expressions. From (3.11),

$$(4.10) \qquad \|x - x_n\| \leq (1 - \theta)^n \|x - x_0\|;$$

from (3.15),

$$(4.11) \qquad \|x - x_n\| \leq \frac{1 - \theta}{\theta} \|x_n - x_{n-1}\|;$$

from (3.16), for any positive integer k,

$$(4.12) \qquad \|x - x_{n+k}\| \leq \frac{(1 - \theta)^k}{\theta} \|x_n - x_{n-1}\|;$$

and from (3.18), as $M(P) = M(\theta I) = \theta$, for $x_0 = y$,

$$\|x - x_n\| \leq (1 - \theta)^n M(\lambda K)\|y\|, \tag{4.13}$$

the above inequalities holding for all positive integers n.

An iterative process due to Bückner,

$$x_n = (1 + \theta)v_{n-1} - \theta\lambda K v_{n-1} - \theta y, \tag{4.14}$$

where

$$v_{n-1} = (1 - \theta)x_{n-1} + \theta\lambda K x_{n-1} + \theta y, \tag{4.15}$$

is totally convergent, provided that θ satisfies (4.7) and the kernel K(s,t) of K is symmetric. From (4.14) and (4.15),

$$x_n = x_{n-1} - \theta^2(I - \lambda K)^2 x_{n-1} + \theta^2(I - \lambda K)y, \tag{4.16}$$

so the iterative process defined by (4.14) and (4.15) is (3.2) with $P = \theta^2(I - \lambda K)$, $Q = \theta^2(I - \lambda K)^2$. If the kernel K(s,t) of K is symmetric, direct calculation from (2.1) verifies that

$$(\lambda^2 K^2 x, x) = (\lambda K x, \lambda K x) \tag{4.17}$$

for all x in (C), and thus

$$\begin{aligned} ([I-\lambda K]^2 x, x) &= (x,x) - 2(\lambda K x, x) + (\lambda^2 K^2 x, x) \\ &= (x,x) - 2(\lambda K x, x) + (\lambda K x, \lambda K x) \\ &= ([I-\lambda K]x, [I-\lambda K]x), \end{aligned} \tag{4.18}$$

which, for all $x \neq 0$ in (C) is the integral of the square of a function which is not identically zero over the range of integration, and thus is positive. The operator $(I-\lambda K)^2$ is therefore positive definite, from which it follows that Q is positive definite, and the condition (4.7) insures that $M(Q) < 1$. The total convergence of the process

defined by (4.14) and (4.15) thus follows from the Convergence Theorem.

It is known that all of the characteristic values of a linear integral operator K with a symmetric kernel are real (6, pp. 107-112), and that every such operator has at least one characteristic value (15, pp. 79-84). If K has only one characteristic value λ^*, then

$$m[(I - \lambda K)^2] = \rho = (1 - \lambda/\lambda^*)^2, \tag{4.19}$$

while if λ_L and λ_R are two characteristic values of K such that $\lambda_L < \lambda < \lambda_R$, and there are no characteristic values of K lying between λ_L and λ_R, then

$$\rho = m[(I - \lambda K)^2] = \min[(1 - \lambda/\lambda_L)^2, (1 - \lambda/\lambda_R)^2], \tag{4.20}$$

(5, pp. 10-11 and 6, pp. 112-113). Thus

$$\mu = m(Q) = \theta^2\rho, \tag{4.21}$$

and explicit error bounds for the process defined by (4.14) and (4.15) may be obtained by substituting this value into the general expressions. From (3.11),

$$\|x - x_n\| \leq (1 - \theta^2\rho)^n \|x - x_0\|; \tag{4.22}$$

from (3.15),

$$\|x - x_n\| \leq \frac{(1 - \theta^2\rho)}{\theta^2\rho} \|x_n - x_{n-1}\|; \tag{4.23}$$

from (3.16), for any positive integer k,

$$\|x - x_{n+k}\| \leq \frac{(1 - \theta^2\rho)^k}{\theta^2\rho} \|x_n - x_{n-1}\|; \tag{4.24}$$

and from (3.18), as $M(P) = M[\theta^2(I - \lambda K)] \leq \theta$, for $x_0 = y$,

$$(4.25) \qquad \|x - x_n\| \leq \frac{(1 - \theta^2\rho)^n}{\theta\rho} M(\lambda K)\|y\|,$$

the above inequalities holding for all positive integers n.

In the more general iterative processes suggested by Bückner, P, and hence Q, has the form of a polynomial in K (5, pp. 68-71); that is, an operator obtained from I and K by a finite number of additions, multiplications, and multiplications by constants. Any iterative process obtained in this fashion is totally convergent, provided that Q satisfies the conditions of the Convergence Theorem. The characteristic values of Q may be obtained in terms of the characteristic values of K (5, pp. 10-11), from which $\mu = m(Q)$ can be obtained by use of the maximum-minimum properties of the characteristic values of Q (6, pp. 112-113), and thus explicit evaluations of the general error bounds may be made.

Samuelson (13, pp. 278-279) takes $P = (I + J)$, where J is an approximation to the resolvent operator G which gives the solution of (3.1) as

$$(4.26) \qquad x = (I + G)y.$$

For $P = (I + G)$, from (3.2),

$$(4.27) \qquad x_1 = x_0 - (I + G)(I - \lambda K)x_0 + (I + G)y,$$

and thus $x_1 = x$ for all x_0 in (C), as $(I + G)$ is the inverse of $(I - \lambda K)$, the exact solution of (3.1) being obtained in one step. In case that some expression for G

is known, the Samuelson process

(4.28) $$x_n = x_{n-1} - (I + J)(I - \lambda K)x_{n-1} + (I + J)y$$

offers the possibility of attaining a satisfactory approximation to x in a small number of steps. The process defined by (4.28) is totally convergent, provided that

(4.29) $$M(G - J) < 1/[1 + M(\lambda K)],$$

as

(4.30) $$\begin{aligned} Q &= (I + J)(I - \lambda K) \\ &= [(I + G) - (G - J)](I - \lambda K) \\ &= I - (G - J)(I - \lambda K), \end{aligned}$$

and thus $(I - Q) = (G - J)(I - \lambda K)$, and $M(I - Q) < 1$ by (4.29); therefore, the total convergence of the process defined by (4.28) is a consequence of the Convergence Theorem. As

(4.31) $$1 > \theta = M(G - J)[1 + M(\lambda K)] \geq M(I - Q),$$

the quantity $1 - \theta$ may be substituted for μ in the general expressions, by (3.19) and (3.20), to obtain explicit error bounds for the Samuelson process. From (3.11),

(4.32) $$\|x - x_n\| \leq \theta^n \|x - x_0\|;$$

from (3.15),

(4.33) $$\|x - x_n\| \leq \frac{\theta}{1 - \theta}\|x_n - x_{n-1}\|;$$

from (3.16), for any positive integer k,

(4.34) $$\|x - x_{n+k}\| \leq \frac{\theta^k}{1 - \theta}\|x_n - x_{n-1}\|;$$

and from (3.18), as $M(P) = M(I + J) \leq 1 + M(J)$, for $x_0 = y$,

$$(4.35) \qquad \|x - x_n\| \leq \frac{\theta^n}{1 - \theta}\left[1 + M(J)\right]M(\lambda K)\|y\|,$$

the above inequalities holding for all positive integers n. Estimates for $M(G - J)$ for a certain method of obtaining J from a known expression for G have been given (10, pp. 207-208).

Wagner (14, pp. 23-24) considers the process (3.2) with

$$(4.36) \qquad P = \theta\left(1 - \lambda\int_a^b K(s,t)dt\right)^{-1} I,$$

$0 < \theta < 1$, so that

$$(4.37) \qquad Q = \frac{\theta(I - \lambda K)}{(I - \lambda K)u},$$

where $u = u(s) = 1$, $a \leq s \leq b$. If $K(s,t)$ is such that

$$(4.38) \qquad f(s) = 1 - \lambda\int_a^b K(s,t)dt \neq 0, \quad a \leq s \leq b,$$

then, as $f(s)$ is continuous, $f(s) > 0$ or $f(s) < 0$, $a \leq s \leq b$, and thus Q is positive definite. If

$$(4.39) \qquad \phi = \min|f(s)|, \quad a \leq s \leq b,$$

$K(s,t)$ satisfies (4.38), and θ is chosen so that

$$(4.40) \qquad 0 < \theta < \phi/[1 + M(\lambda K)],$$

then $M(Q) < 1$, and the total convergence of the Wagner process

$$(4.41) \qquad x_n = x_{n-1} - (\theta/f)(I - \lambda K)x_{n-1} + (\theta/f)y$$

is assured by the Convergence Theorem. As

(4.42) $\qquad 1 > \mu = m(Q) \geq \theta\phi/[1 + M(\lambda K)] = \rho,$

this value may therefore be substituted into the general expressions to obtain explicit error bounds for the Wagner process. From (3.11),

(4.43) $\qquad \|x - x_n\| \leq (1 - \rho)^n \|x - x_0\|;$

from (3.15),

(4.44) $\qquad \|x - x_n\| \leq \frac{1 - \rho}{\rho} \|x_n - x_{n-1}\|;$

from (3.16), for any positive integer k,

(4.45) $\qquad \|x - x_{n+k}\| \leq \frac{(1 - \rho)^k}{\rho} \|x_n - x_{n-1}\|;$

and from (3.18), as $M(P) = M(\theta I/f) \leq \theta/\phi$, for $x_0 = y$,

(4.46) $\qquad \|x - x_n\| \leq \frac{(1 - \rho)^n}{\rho\phi} \theta M(\lambda K) \|y\|,$

the above inequalities holding for all positive integers n.

PRACTICAL APPLICATION OF ITERATIVE PROCESSES

As the results given to this point were obtained on the assumption that all integrations were carried out exactly, some modification of them is necessary to make allowance for the use of some method of approximate integration in the actual computation. The process of approximate integration can be thought of as resulting in the replacement of x_n by x_n^*, so that

$$\|x - x_n^*\| \leq \|x - x_n\| + \|x_n - x_n^*\|, \tag{5.1}$$

where $\|x_n - x_n^*\|$ is a measure of the error of the approximate integration. From (3.5),

$$\|x - x_n^*\| \leq \|x - x_n\| + \sum_{j=0}^{n} M^j(I-Q)\|x_{n-j} - x_{n-j}^*\|. \tag{5.2}$$

By (3.10), $M(I - Q)$ may be replaced by $(1 - \mu)$ in (5.2) to obtain the equivalent expression

$$\|x - x_n^*\| \leq \|x - x_n\| + \sum_{j=0}^{n} (1 - \mu)^j\|x_{n-j} - x_{n-j}^*\|. \tag{5.3}$$

As $0 < (1 - \mu) < 1$,

$$\sum_{j=0}^{n} (1 - \mu)^j < \sum_{j=0}^{\infty} (1 - \mu)^j = 1/\mu, \tag{5.4}$$

hence, for

$$\varepsilon = \max_{(n)}\|x_n - x_n^*\|, \tag{5.5}$$

$$\|x - x_n^*\| < \|x - x_n\| + \varepsilon/\mu, \tag{5.6}$$

where the estimate for $\|x - x_n\|$ is obtained from the

explicit error bounds for the iterative process used.

Most of the error bounds stated involve the number $M(\lambda K)$, which is rarely known exactly. For a given choice of the definition of the norm, however, it is usually possible to obtain a satisfactory estimate for $M(\lambda K)$. For example, for the norm as defined by (2.2),

$$M(\lambda K) \leq |\lambda| \max \int_a^b |K(s,t)|\,dt, \quad a \leq s \leq b, \tag{5.7}$$

and for the norm as defined by (2.3),

$$M(\lambda K) \leq |\lambda| \left(\int_a^b \int_a^b K^2(s,t)\,ds\,dt \right)^{1/2}, \tag{5.8}$$

(11, p. 357). In order to make use of the error bounds for the iterative process defined by (4.14) and (4.15), the computer must know the values of λ_L and λ_R, or at least values $\underline{\lambda}$, $\overline{\lambda}$, such that $\lambda_L \leq \underline{\lambda} < \lambda < \overline{\lambda} \leq \lambda_R$. Methods for obtaining $\underline{\lambda}$ and $\overline{\lambda}$ are known (5, pp. 50-55 and 17, pp. 15-38), and thus it is possible to obtain a lower bound for μ.

Generally speaking, the iterative processes considered are rather slowly convergent, which may be of little consequence if the computation is carried out by machine. However, methods for accelerating the convergence of iterative processes have been given by Lonseth (11, pp. 354-355) and Samuelson (13, pp. 279-281); these methods may prove valuable in case that they materially shorten the computation, even though they obtain more rapid convergence at the cost of an increase in complexity.

NUMERICAL EXAMPLE

In order to illustrate the actual application of some of the iterative processes and error bounds given, an approximate solution of the integral equation

$$(6.1) \qquad s^2 = x(s) - \lambda\int_0^1 K(s,t)x(t)dt, \quad 0 \leq s \leq 1,$$

where

$$(6.2) \qquad K(s,t) = s(1 - t), \quad 0 \leq s \leq t \leq 1,$$
$$= t(1 - s), \quad 0 \leq t \leq s \leq 1,$$

will be sought for values -1, -10, 25 of λ. For each value of λ considered, it is desired to find an approximation $x_n(s)$ to the solution $x(s)$ of (6.1) such that, for the norm as defined by (2.3),

$$(6.3) \qquad \|x - x_n\| < 0.01.$$

The characteristic values of K are known to be $\lambda_n = n^2\pi^2$, $(n = 1,2,\ldots)$, and thus $M(K) = 1/\pi^2$ (8, p. 8 and 6, pp. 112-114); no use of the exact solution of (6.1) will be made.

For $\lambda = -1$, $M(\lambda K) = 1/\pi^2 < 1$, and thus the Neumann process may be used. As no approximation to $x(s)$ is assumed to be known, take

$$(6.4) \qquad x_0(s) = s^2, \quad 0 \leq s \leq 1.$$

Here $\|s^2\| = 1/\sqrt{5}$. In order to estimate the value of n for which (6.3) will hold, (4.5) is used, n being taken

to be the smallest positive integer for which

(6.5) $\|x - x_n\| \leq (1/\pi^2)^{n+1}/[(1 - 1/\pi^2)\sqrt{5}] < 0.01.$

The solution of the inequality (6.5) for n yields $n \geq 1$, and thus

(6.6) $x_1(s) = s^2 - s(1 - s^3)/12$

is a satisfactory approximation to x(s) on $0 \leq s \leq 1$.

For $\lambda = -10$, $M(\lambda K) = 10/\pi^2 > 1$, so that the condition for the convergence of the Neumann process is no longer satisfied; however, $-\lambda K$ is positive semi-definite, and thus the Wiarda process may be used. Take

(6.7) $\theta = 0.49650 < 1/(1 + 10/\pi^2),$

and $x_0(s) = s^2$; hence $(1 - \theta) = 0.50350$, and (4.13) will be used to obtain an estimate for n. The solution of the inequality

(6.8) $\|x - x_n\| \leq (10)(0.50350)^n/(\pi^2\sqrt{5}) < 0.01$

for n gives $n \geq 6$. From (4.6),

(6.9) $x_1(s) = s^2 - (0.41375)s(1-s^3),$

and as

(6.10) $\|x_1 - x_0\| = 0.13792,$

from (4.11),

(6.11) $\|x - x_1\| < 0.14.$

The next application of (4.6) gives

(6.12) $x_2(s) = s^2 + (0.34238)s(1-s^2) - (0.62207)s(1-s^3) - (0.06848)s(1-s^5),$

and thus from (4.11),

(6.13) $\|x - x_2\| < 0.0116,$

as

(6.14) $\|x_2 - x_1\| = 0.01140.$

The third step of the process gives

(6.15) $$x_3(s) = s^2 + (0.46050)s(1-s^2) - (0.72696)s(1-s3) + (0.08500)s(1-s^4) - (0.13542)s(1-s^5) - (0.00607)s(1-s^7),$$

with

(6.16) $\|x - x_3\| < 0.006$

by (6.13) and (4.12), so $x_3(s)$ is the desired approximation. In practice, error bounds of the same type as (3.11), (3.16), and (3.18) seem to exhibit a considerable degree of "pessimism", so that the "step-wise" error bounds such as (3.15) are of perhaps greater practicality, even with regard to the additional labor of calculating $\|x_n - x_{n-1}\|$ at each step. In the example just given, a satisfactory approximation to x(s) was obtained in half the number of steps predicted by the use of an error bound of the type (3.18).

For $\lambda = 25$, $M(\lambda K) = 25/\pi^2 > 1$, and the Neumann process fails to converge in this case; also

(6.17) $$(-\lambda Ks, s) = -(25/6)\int_0^1 s^2(1-s^2)ds = -5/9 < 0,$$

so that $-\lambda K$ is not positive semi-definite, and the Wiarda process also fails to converge. However, as K(s,t) is

symmetric, and $\pi^2 < \lambda < 4\pi^2$, the process defined by (4.14) and (4.15) is applicable. Choose

(6.18) $\theta = 0.28304 < 1/[1 + 25/\pi^2]$;

from (4.20),

(6.19) $\rho = (1 - 25/4\pi^2)^2 = 0.13450.$

For $x_0(s) = s^2$, the number of steps necessary may be estimated from (4.25), n being taken to be the smallest positive integer satisfying

(6.20) $\|x - x_n\| \leq (2.53303)(0.98922)^n/(0.08513) < 0.01,$

which is n = 727. The actual computation in this case will be very laborious to carry out without the use of a computing machine. However, as

(6.21) $Ks^j = s(1 - s^{j+1})/[(j+1)(j+2)]$

for all non-negative integers j, the "programming" of the computation for a high-speed computing machine in this case is relatively simple, provided that $x_0(s)$ is taken to be a polynomial in s.

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