AN ABSTRACT OF THE THESIS OF

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(Name) (Degree) (Major)

Date thesis is presented May 14, 1965

Title VECTOR SUMS OF LINE SEGMENTS

Abstract approved [redacted]
(Major professor)

The author shows that a necessary and sufficient condition for a convex polyhedron to be representable as a finite vector sum of line segments is that each of its faces possesses central symmetry.

VECTOR SUMS OF LINE SEGMENTS

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1965

APPROVED:

[redacted]

Professor of Mathematics

In Charge of Major

[redacted]

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Date thesis is presented May 14, 1965

Typed by Carol Baker

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VECTOR SUMS OF LINE SEGMENTS

I. BASIC CONCEPTS

In this section, we present the concepts basic to the understanding of the problem with which this work is concerned. Throughout what follows, we will be dealing with n-dimensional Euclidean space, in which points can be considered as represented by ordered n-tuples of real numbers. These numbers are rectangular coordinates. Addition and multiplication by scalars (which we call scalar multiplication) are defined by the corresponding operations on coordinates. We will call the point, all of whose coordinates are zero, the <u>origin</u>. The usual Euclidean distance between two points will be employed, as will certain other elementary geometric concepts, including that of orthogonal projection of geometric figures. A more thorough discussion of these and related concepts may be found in (3).

The particular geometric figures with which we will deal are point sets of the following kind:

<u>Definition 1.</u> A closed and bounded set of points K is called a <u>convex body</u> if, for every pair P, Q of points of K, the line segment joining P and Q lies entirely in K.

The operations of addition and scalar multiplication defined for points may be generalized to convex bodies as follows:

Definition 2. If K_1 and K_2 are convex bodies, their vector sum, denoted by $K_1 + K_2$, is the set

$$\{P + Q \mid P \varepsilon K_1, \; Q \varepsilon K_2\}$$

If α is a real number and K a convex body, the scalar product of α and K, written αK, is the set

$$\{\alpha P \mid P \varepsilon K\}$$

The scalar product of a positive number α and a convex body K may be interpreted as the magnification of K in the ratio $\alpha:1$, the center of magnification being the origin. The symbol -K means (-1)K, which may be thought of as the reflection of K in the origin. Similarly, $-\alpha K$ means $(-\alpha)K$, and $K_1 - K_2$ is $K_1 + [(-1)K_2]$. It can easily be seen that the set of convex bodies is closed under addition and scalar multiplication, and that the set

$$\alpha_1 K_1 + \alpha_2 K_2 + \cdots + \alpha_m K_m ,$$

which we will also write

$$\sum_{i=1}^{m} \alpha_i K_i \; ,$$

is the convex body

$$\left\{ \sum_{i=1}^{m} \alpha_i P_i \mid P_i \varepsilon K_i \quad \text{for} \quad i = 1, 2, \cdots, m \right\}.$$

The following provides us with a way of forming convex bodies from more general sets:

Definition 3. If K is a bounded point set, then the set $H(K)$, called the convex closure of K, is the convex body satisfying

(i) $K \subseteq H(K)$, and

(ii) if X is any convex body for which $K \subseteq X$, then $H(K) \subseteq X$.

That each bounded set has a unique convex closure is shown by Eggleston (3, p. 21-22).

Definition 4. A convex polytope is the convex closure of a finite collection of points.

A convex polytope is necessarily a convex body, and in two-dimensional space is called a convex polygon, and in three-dimensional space, a convex polyhedron. We use the term "polytope" as a dimensionally independent term.

Minkowski (4, p. 182) has shown that sets of the form

$$\sum_{i=1}^{m} J_i \,, \tag{1}$$

where $J_1, J_2, \cdots, J_m$ are line segments, are convex polytopes. It is the object of this paper to characterize the class of convex polytopes which may be represented in the form (1). This problem is posed in a somewhat different setting by Blaschke (1, p. 154-157) and by Bonnesen and Fenchel (2, p. 29). The solution obtained herein for polytopes involves the following concept:

Definition 5. A convex body K is said to be centrally symmetric with respect to a point P_0 if, for every point P of K, $2P_0 - P \varepsilon K$. K is said to be central if there is a point with respect to which K is centrally symmetric.

II. SOLUTION OF THE PROBLEM IN TWO-DIMENSIONAL SPACE

Theorem 1 is not a new result, but the proof given in this section serves to set the pattern for the proof of the corresponding result in three-dimensional space, given in the next section.

Theorem 1. A convex polygon K is the sum of finitely many line segments if and only if K is central.

Proof. To begin with, assume that K is the sum of the line segments $J_1, J_2, \cdots, J_m$. Let the end points of J_i be denoted by M_i and N_i, and let $B_i = (M_i + N_i)/2$ and $C_i = (M_i - N_i)/2$. Then

$$J_i = \{B_i + \alpha C_i \mid -1 \leq \alpha \leq 1\} \text{ for } i = 1, 2, \cdots, m,$$

and

$$K = \left\{ \sum_{i=1}^{m} (B_i + \alpha_i C_i) \mid -1 \leq \alpha_i \leq 1 \right\}$$

$$= P_0 + \left\{ \sum_{1=1}^{m} \alpha_i C_i \mid -1 \leq \alpha_i \leq 1 \right\},$$

where $P_0 = \sum_{i=1}^{m} B_i$.

If P is a point of K, say

$$P = P_0 + \sum_{i=1}^{m} t_i C_i,$$

where $-1 \leq t_i \leq 1$ for $i = 1, 2, \cdots, m$, then

$$2P_0 - P = P_0 + \sum_{i=1}^{m} (-t_i)C_i \in K$$

since $-1 \leq -t_i \leq 1$ for $i = 1, 2, \cdots, m$. It follows that K is centrally symmetric with respect to P_0.

On the other hand, if we assume that K is a central convex polygon, then it has an even number, say $2m(K)$, of edges. In case $m(K) = 2$, K is a parallelogram and it is easily verified that K is, except for a possible translation, the sum of two of its intersecting edges. Now suppose we have proven the theorem for convex polygons for which $m(K) \leq k$, where k is an integer satisfying $k \geq 2$. Let K be a convex polygon such that $m(K) = k + 1$, and let L_0 be one of its edges. Without loss of generality, we may assume that the center of symmetry of K coincides with the origin, and that L_0 is parallel to the y-axis of a rectangular coordinate system. This follows from the observation that the possible translation and rotation of K required to establish these conditions is a rigid motion of K, and has no effect on its shape.

The orthogonal projection of K onto the x-axis is an interval symmetric about the origin, which we denote by $[-r, r]$. Let L_1 be the edge of K parallel to L_0 and denote the midpoints of L_0 and L_1 by R_0 and R_1, respectively. Now the line segment R_0R_1 passes through the origin and separates K into two halves, each of which is a reflection of the other in the origin. These will be called the upper and lower halves of K, named so that the positive y-axis intersects K in its upper half. Each of the lines $x = \alpha$, for $\alpha \varepsilon [-r, r]$, intersects K in a line segment having one end point in the lower half of K, at $y = f(\alpha) < 0$ say, and the other end point in the upper half of K, at $y = g(\alpha) > 0$. Then K may be represented in terms of the functions f and g as follows:

$$K = \{(x, y) \mid -r \leq x \leq r, \quad f(x) \leq y \leq g(x)\}.$$

The parallelogram $H(L_0 \cup L_1)$ is contained in K, so $g(x) \geq f(x) + \mu$, for all $x \varepsilon [-r, r]$, where μ is the length of L_0. Define

$$K' = \{(x, y) \mid -r \leq x \leq r, \quad f(x) + \frac{\mu}{2} \leq y \leq g(x) - \frac{\mu}{2}\},$$

and $J = \{(0, y) \mid -\frac{\mu}{2} \leq y \leq \frac{\mu}{2}\}$.

J is a line segment, and

$$K' + J = \{(x, y_1 + y_2) \mid -r \leq x \leq r, [f(x) + \frac{\mu}{2}] - \frac{\mu}{2} \leq y_1 + y_2 \leq [g(x) - \frac{\mu}{2}] + \frac{\mu}{2}\}$$

$$= K.$$

Geometrically, K' is the figure obtained from K by removing the parallelogram $H(L_0 \cup L_1)$ and moving the two remaining parts together. Thus K' is a central convex ploygon with $2k$ edges. But we have assumed for induction purposes that the theorem holds for K' since $m(K') = k$. Therefore, K' is the sum of finitely many line segments, and so is $K = K' + J$. Since K is any convex polygon for which $m(K) = k+1$, we conclude by the principle of mathematical induction that the theorem holds for all convex polygons.

III. SOLUTION OF THE PROBLEM IN THREE-DIMENSIONAL SPACE

For a convex polyhedron to be represented as a finite sum of line segments, the simple condition of centrality does not suffice in general. For example, the regular octahedron is a central convex polyhedron which cannot be so represented, as the following argument shows. Any sum of line segments having three pairwise distinct directions is a parallelopiped, which has eight vertices. Adding more line segments to this sum does not decrease the number of vertices of the resultant polytope. But the regular octahedron has only six vertices, so it is not the sum of a set of line segments having three or more distinct directions. On the other hand, a sum of line segments, where there are two or fewer distinct directions, is a plane figure. In no case, then, are we able to construct a regular octahedron by summing a finite number of line segments. The reason for this, as will be shown by Theorem 2, is that the triangular faces of the regular octahedron are not central figures.

The following definitions serve to formalize our notions of faces, edges, and vertices of polyhedra and polygons.

Definition 6. If K is a convex polyhedron, if G is a plane containing three non-collinear points of K, and if K lies entirely on one side of G, then $G \cap K$ is called a face of K.

Definition 7. If K is a convex polygon, if λ is a line containing two distinct points of K, and if K lies entirely on one side of λ, then $\lambda \cap K$ is called an edge of K.

That each face of a convex polyhedron is a convex polygon, and that each edge of a convex polygon is a line segment, is not difficult to verify.

Definition 8. If K is a convex polyhedron, if F is a face of K, and if L is an edge of F, then L is called an edge of K.

Definition 9. If L and L' are distinct edges of a convex polyhedron (or polygon) K, and if L and L' intersect, then $L \cap L'$ is a vertex of K.

It follows from these definitions that, if K is a convex polyhedron or a convex polygon, the edges of K are line segments, and the vertices of K are points. Another fact we will employ is that, if K is a convex polygon and Z the set of vertices of K, then

$$K = H(Z) .$$

This is shown by Eggleston (3, p. 29).

With this preparation, we state our main result:

Theorem 2. For a convex polyhedron K to be a sum of finitely many line segments, it is necessary and sufficient that all of the faces of K be central.

Proof. To demonstrate the necessity of the condition, suppose that

$$K = \sum_{i=1}^{m} J_i ,$$

where, for $i = 1, 2, \cdots, m$, J_i is a line segment. Denote the end points of J_i by M_i and N_i, and let $B_i = (M_i + N_i)/2$ and $C_i = (M_i - N_i)/2$. Then

$$J_i = \{B_i + \alpha C_i \mid -1 \leq \alpha \leq 1\} .$$

Now let F be a face of K, and G the plane of F. F contains a point P of K, say

$$P = \sum_{i=1}^{m} (B_i + t_i C_i) ,$$

where $-1 \leq t_i \leq 1$ for $i = 1, 2, \cdots, m$. Without loss of generality, we can suppose that the J_i are labelled so that, for some positive integer $r \leq m$, $J_1, J_2, \cdots, J_r$ are parallel to G and $J_{r+1}, J_{r+2}, \cdots, J_m$ are not. Then the set

$$G' = \left\{ \sum_{i=1}^{r} \alpha_i C_i \mid \text{all real } \alpha_i \right\}$$

is a plane parallel to G through the origin, whence

$$G = G' + P = \sum_{i=1}^{m} B_i + \sum_{i=r+1}^{m} t_i C_i + \left\{ \sum_{i=1}^{r} (t_i + \alpha_i) C_i \,\middle|\, \text{all real } \alpha_i \right\} .$$

Let

$$P_0 = \sum_{i=1}^{m} B_i + \sum_{i=r+1}^{m} t_i C_i \;.$$

Then, since $F = G \cap K$,

$$F = P_0 + \left\{ \sum_{i=1}^{r} (t_i + \alpha_i) C_i \,\middle|\, -1 \le \alpha_i + t_i \le 1 \right\}$$

$$= P_0 + \left\{ \sum_{i=1}^{r} \beta_i C_i \,\middle|\, -1 \le \beta_i \le 1 \right\} .$$

Thus F is a finite sum of line segments and so, by Theorem 1, is a central polygon.

We come now to the proof of the sufficiency of the facial centrality condition. That is, given that K is a convex polyhedron, all of whose faces are central, we are to show that K is a sum of finitely many line segments.

Let $m(K)$ be the number of edges of K having pairwise distinct directions. Thus we assume that there are exactly $m(K)$ edges $L_1, L_2, \cdots, L_{m(K)}$ of K such that, for $i \neq j$, L_i is not parallel to L_j, and if L is any edge of K, then L is parallel to L_i for some $i = 1, 2, \cdots, m(K)$. In case $m(K) = 3$, K is a parallelopiped, and it is easily shown that K is the sum of the three edges intersecting in some vertex of K. If we assume that, for some integer $k \geq 3$, the theorem holds for all convex polyhedra for which $m(K) \leq k$, then we have to prove that the theorem holds for convex polyhedra satisfying $m(K) = k+1$. When this has been shown, it will follow by induction that the theorem is true for all convex polyhedra.

Let K be a convex polyhedron, all of whose faces are central, and for which $m(K) = k+1$. Let L_0 be an edge of ~~Q~~. K. Employing three-dimensional rectangular coordinates (x, y, z), we can assume, without loss of generality, that L_0 is parallel to the z-axis. For each point set X let $T(X)$ denote the orthogonal projection of X onto the xy-plane; for each line segment J, let $\delta(J)$ be the length of J; for each point P, let $\Gamma(P)$ be the line segment through P parallel to L_0; and for each point P let $W(P) = \delta(K \cap \Gamma(P))$. Finally, let

$$K_0 = T(K), \quad \text{and}$$

$$\mu = \delta(L_0) .$$

The proof of the induction step will consist of the following results:

(i) For every edge L of K parallel to L_0, $T(L)$ is a vertex, V, of K, and $W(V) = \mu$. That is, all edges of K parallel to L_0 have the same length.

(ii) The set $E = \{P | P \varepsilon K_0, \ W(P) \geq \mu\}$ is a convex body. Since $E \subseteq K_0$, and since, by (i), E contains all the vertices of K_0, it follows that $E = K_0$, when $W(P) \geq \mu$ for all $P \varepsilon K_0$.

(iii) There is a convex polyhedron K' and a line segment J such that $K = K' + J$, K' has all its faces central, and $m(K') \leq k$.

(iv) By the induction hypothesis, the theorem holds for K', whence K' is the sum of a finite number of line segments and so, therefore, is K.

The demonstration of these results follow.

(i)

Let L be any edge of K parallel to L_0. L is the intersection of two faces F and F' of K. Let G and G' be the planes of F and F', respectively, and let λ and λ' be the lines $T(G)$ and $T(G')$, respectively. Since K lies entirely on one side of G, K_0 lies entirely on one side of λ. Furthermore, G contains at least three non-collinear points R_1, R_2, and R_3 of K. Therefore, at least two of the lines $\Gamma(R_1)$, $\Gamma(R_2)$, and $\Gamma(R_3)$ are distinct, and thus project onto K_0 as two distinct points. But these latter points are in $\lambda \cap K_0$ since R_1, R_2, and R_3 are in $G \cap K$. Therefore, $\lambda \cap K_0$ is an edge of K_0. Similarly, $\lambda' \cap K_0$ is an edge of K_0. Moreover, λ and λ' intersect at $T(L)$, which is in K_0. Therefore, for any edge L of K parallel to L_0, $T(L)$ is a vertex of K_0.

Now let $V_0 = T(L_0)$. Then V_0 is a vertex of K_0. Starting at V_0, label the remaining vertices of K_0 $V_1, V_2, \cdots V_r$. We have $W(V_0) = \delta(L_0) = \mu$. Suppose we have shown that $W(V_j)=\mu$, for some non-negative integer $j < r$. We demonstrate next that $W(V_{j+1}) = \mu$, and when this has been done, we will have shown by induction that $W(V) = \mu$ for all vertices V of K_0.

Let L be the edge of K_0 joining the vertices V_j and V_{j+1}, and let P be any point of $K \cap \Gamma(V_{j+1})$. By the induction assumption, $K \cap \Gamma(V_j)$ is a line segment, J' say, parallel to L_0 with length μ. Let G be the plane containing J' and P, and let $F = G \cap K$. K lies entirely on one side of G since K_0 lies entirely on one side of the line $T(G)$. Thus F is a face of K, and therefore a central convex polygon. Furthermore, $T(F) = L$, so F lies between the lines $\Gamma(V_j)$ and $\Gamma(V_{j+1})$. Hence J' is an edge of F, and P lies on the edge of F opposite J', which we denote by J''. By the symmetry of F, $\delta(J'') = \delta(J') = \mu$. But $J'' = K \cap \Gamma(V_{j+1})$ so $W(V_{j+1}) = \mu$, as was to be shown.

(ii)

Let $E = \{P | P \varepsilon K_0, \ W(P) \geq \mu\}$, and let Z be the collection of vertices of K_0: $Z = \{V_j | j = 0, 1, \cdots, r\}$. We are to show that E is a convex body. Let Q and R be arbitrary points of E. Then Q and R are in K_0. Denote by Q_1 and Q_2 the end points of the line segment $K \cap \Gamma(Q)$, and by R_1 and R_2 the end points of $K \cap \Gamma(R)$, labelled so that the line segment Q_1R_1 does not intersect Q_2R_2. Then $Q_1R_1R_2Q_2$ is a plane quadrilateral with two parallel edges Q_1Q_2 and R_1R_2. We note that

$$\delta(Q_1Q_2) \geq \mu \quad \text{and} \quad \delta(R_1R_2) \geq \mu \tag{2}$$

because Q and R are in E.

Suppose P is a point on the line segment QR. We may represent P in the form

$$P = tQ + (1-t)R \quad \text{for some} \quad t \varepsilon [0, 1].$$

We define $P_1 = tQ_1 + (1-t)R_1$, and

$$P_2 = tQ_2 + (1-t)R_2.$$

Then $P_1 \varepsilon K$ and $P_2 \varepsilon K$ since K is convex. Furthermore,

$\Gamma(P)$ intersects the quadrilateral $Q_1R_1R_2Q_2$ in the line segment P_1P_2 parallel to Q_1Q_2 and R_1R_2. Thus

$$\begin{aligned} W(P) &\geq \delta(P_1P_2) \\ &\geq \min\{\delta(Q_1Q_2),\ \delta(R_1R_2)\} \\ &\geq \mu, \end{aligned}$$

by (2), so that $P \varepsilon E$. But P was assumed to be an arbitrary point of QR, and Q and R were arbitrary points of E. Hence E is a convex body. We have shown that $Z \subseteq E$, so we deduce that $H(Z) \subseteq E$. But $H(Z) = K_0$ because a convex polygon is the convex closure of its vertices, and, from its definition, $E \subseteq K_0$, whence $E = K_0$. Therefore

$$W(P) \geq \mu \quad \text{for all} \quad P \varepsilon K_0.$$

(iii)

Let $C = (0, \rho, 1)$ and

$$J = \{\alpha C \mid 0 \leq \alpha \leq \mu\} \quad . \tag{3}$$

As we did in Theorem 1, we represent K in terms of the lower and upper end points of the line segments $K \cap \Gamma(P)$ for $P \varepsilon K_0$:

$$K = \{P + \alpha C \mid P \varepsilon K_0, \quad f(P) \leq \alpha \leq g(P)\} . \tag{4}$$

For each $P \varepsilon K_0$, we see from (4) that

$$\begin{aligned} g(P) - f(P) &= \delta(K \cap \Gamma(P)) \\ &= W(P) \\ &\geq \mu . \end{aligned}$$

Now define
$$K' = K \cap (K - \mu C) , \qquad (5)$$

where, as mentioned in section I, $K - \mu C$ is K translated through $-\mu C = (0, 0, -\mu)$. Then

$$K - \mu C = \{P + \alpha C \mid P \varepsilon K_0, \quad f(P) - \mu \leq \alpha \leq g(P) - \mu\} ,$$

and

$$K' = \{P + \alpha C \mid P \varepsilon K_0, \quad f(P) \leq \alpha \leq g(P) - \mu\}. \qquad (6)$$

Since the inequality $f(P) \leq g(P) - \mu$ holds for all $P \varepsilon K_0$, we see that $T(K') = K_0$. Hence, from (3) and (6), we get:

$$\begin{aligned} K' + J &= \{P + \alpha C \mid P \varepsilon K_0, \quad f(P) \leq \alpha \leq g(P) - \mu\} \\ &\quad + \{\beta C \mid 0 \leq \beta \leq \mu\} \\ &= \{P + \alpha C \mid P \varepsilon K_0, \quad f(P) \leq \alpha \leq [g(P) - \mu] + \mu\} \\ &= K. \end{aligned}$$

The upper surface of K' (that is, the set of upper end points of the line segments $K' \cap \Gamma(P)$ for $P \varepsilon K_0$) is

$$\{P + [g(P) - \mu] C \mid P \varepsilon K_0\}$$

$$= \{P + g(P)C \mid P \varepsilon K_0\} - \mu C$$

which is a translate of the upper surface of K. Thus, if F is a face of K' in its upper surface, it is a translate of a face of K, and it is therefore central. Similarly for faces of K' in its lower surface. In the case of faces F of K' parallel to L_0, $F + J$ is a face of K with L_0 as an edge. Thus $F + J$ is central, and by Theorem 1, F is central. Hence all of the faces of K' are central.

We have left to show that $m(K') \leq k$. First observe that there are no edges of K' parallel to L_0; for, if L were such an edge, then $T(L)$ would be a vertex V of K_0, by the argument employed in (i) for K. Then

$$\delta(L) = \delta(K' \cap \Gamma(V)) = [g(V) - \mu] - f(V) = 0 ,$$

since $g(V) - f(V) = W(V) = \mu$. Moreover, if L is an edge of K', then L is an edge of some face F of K', and one of the following holds:

case a : F is in the upper surface of K' .

case b : F is in the lower surface of K' .

case c : F is parallel to L_0 .

In cases (a) and (b), F is a translate of a face of K, so that L is parallel to an edge of K. In case (c), L is parallel to an edge of $F+J$, which is a face of K. Hence in all three cases, L is parallel to an edge of K. But $m(K')$ is, by its definition, the number of edges of K' with pairwise distinct directions, so $m(K') \leq m(K) = k+1$. We have also observed that K' has no edge parallel to the edge L_0 of K. We conclude that $m(K') < m(K)$, or $m(K') \leq k$.

This completes the induction step outlined previously in this section, and so completes the proof of the theorem.

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