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Title: Triangular Billiards Surfaces and Translation Covers

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We identify all translation covers among triangular billiards surfaces. Our main tools are the J-invariant of Kenyon and Smillie and a property of triangular billiards surfaces, which we call fingerprint type, that is invariant under balanced translation covers.

Triangular Billiards Surfaces and Translation Covers

by

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Major Professor, representing Mathematics

Chair of the Department of Mathematics

Dean of the Graduate School

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Jason Peter Schmurr, Author

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Academic

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TRIANGULAR BILLIARDS SURFACES AND TRANSLATION COVERS

1 INTRODUCTION

1.1 Some History

A billiards problem involves an enclosed planar region ("billiard table") and a point mass ("billiard ball") moving within the region at unit speed. Collisions with the boundary of the region result in the billiard ball changing direction, with the angle of reflection equal to the angle of incidence. In this thesis we shall discuss surfaces which arise from the particular case of billiards in a polygon whose interior angles are all rational multiples of π. Treatments of such a dynamical system go back at least to G.D. Birkhoff in 1927 [2]. Fox and Kershner [5] describe a method of studying such a dynamical system by constructing a flat surface tiled by a finite number of copies of the billiard table. We describe this method in Section 2.1. Katok and Zemlyakov [12] furthered the discussion by proving that most billiard paths are dense in most polygonal billiards systems. The current interest in the field from the algebraic side stems largely from a paper of William Veech in 1989 [15], which proved a relationship between uniform distribution of billiard paths and affine symmetries of billiard surfaces (see Theorem 6.1). Since this discovery, there has been a great deal of attention directed at the affine symmetry groups (often called *Veech groups*) of flat surfaces. Vorobets [16], and independently Gutkin and Judge [6], showed that if two surfaces are related by a certain cover called a *balanced cover*, then the intersection of their Veech groups has finite index in each group. In a series of papers including [8] and [9], Hubert and Schmidt have taken advantage of this work to construct surfaces with certain interesting Veech groups. The covers they use, called *translation covers*, are a generalization of balanced covers.

1.2 Statement of the Main Problem

Results such as those listed at the end of Section 1.1 linking affine symmetry groups with translation covers provide motivation for the classification of all possible translation covers between elements of various sets of translation surfaces. In this thesis we determine all translation covers among triangular billiards surfaces. It is well known (see Section 4.1) that a flat torus admits translation covers of arbitrarily high degree by choosing as covering surfaces appropriate scalar multiples of itself, and that there are three rational triangles which correspond to triangular billiards surfaces of genus 1. However, other translation covers are rare; in fact, our main result is encapsulated in the following lemma and theorem (relevant notation is reviewed in Section 2.1.1).

Lemma 4.1: *Let a_1 and a_2 be relatively prime positive integers, not both equal to one. The right triangular billiards surface $Y := X(a_1 + a_2, a_1, a_2)$ is related to two isosceles triangular billiards surfaces*

$$X_1 = \begin{cases} X(2a_2, a_1, a_1) & a_1 \text{ odd} \\ \\ X(a_2, \frac{a_1}{2}, \frac{a_1}{2}) & a_1 \text{ even} \end{cases}$$

and

$$X_2 = \begin{cases} X(2a_1, a_2, a_2) & a_2 \text{ odd} \\ \\ X(a_1, \frac{a_2}{2}, \frac{a_2}{2}) & a_2 \text{ even} \end{cases}$$

via balanced covers $f_1 : X_1 \to Y$ *and* $f_2 : X_2 \to Y$ *. The maps have degrees*

$$\deg(f_i) = \begin{cases} 2 & a_i \text{ odd} \\ 1 & a_i \text{ even} \end{cases}$$

. Furthermore, at least one of the f_i *has degree 2.*

In fact these are all possible translation covers amongst triangular billiards surfaces, as we assert in the following main theorem.

Theorem 4.1: *Suppose* $f : X \to Y$ *is a translation cover of triangular billiards surfaces of degree greater than 1. Then* f *is of degree 2, and is a composition of one or two of the covers* f_i *described in Lemma 4.1.*

To prove Theorem 4.1, we use two main tools: the J-invariant of Kenyon and Smillie [13], and what we call the *fingerprint* of a point P on a translation surface. The fingerprint of P depends on the configuration of the shortest geodesics connecting P to singularities. We show that every point on a triangular billiards surface which corresponds to a vertex of the triangular billiard table has a fingerprint of one of two distinct types, which we call Type I and Type II (see Chapter 3 for definitions). We establish the following invariance results:

Proposition 3.1: *Suppose the billiards triangulation of a triangular billiards surface* X *contains a point with a Type II fingerprint. Then* X *is uniquely determined by that fingerprint, up to an action of* $O(2,\mathbb{R})$ *.*

Lemma 3.2: *Suppose that* $f : X \to Y$ *is a balanced translation cover, that* $P' \in X$ *and* $P \in Y$ *are vertices of billiards triangulations on their respective surfaces, and that* $f(P') = P$ *. Then either:*

1. P' *and* P *have the same fingerprint, or*

2. *their fingerprints differ only in the cone angle, P has half the cone angle of P', X arises from billiards in an isosceles triangle, and P' corresponds to the apex of that triangle.*

1.3 Organization of this Thesis

In Chapter 2 we review the rational billiards construction. We give some combinatorial formulas for the construction as recorded in [1]. We define translation surfaces and translation covers, and we discuss the J-invariant of Kenyon and Smillie.

In Chapter 3 we introduce the concept of the fingerprint of a point on a translation surface. We give examples, and prove results about the fingerprints of certain points on triangular billiards surfaces.

In Chapter 4, we identify all translation covers among triangular billiards surfaces. In Section 4.1 we give the complete list of possible covers as Lemma 4.1; the remainder of the chapter is devoted to proving that no other covers exist. We first prove the result for balanced covers in Section 4.2, using the fingerprint as the primary tool. Then, using the J-invariant and holonomy field of Kenyon and Smillie, we prove Theorem 4.1 for all translation covers in Section 4.5.

Chapter 5 is devoted to an alternate proof of a result of Calta and Smillie concerning the J-invariant of a triangular billiards surface. In Chapter 6, we demonstrate an application of translation covers to the problem of identifying infinitely generated Veech groups. Finally, in Chapter 7, we give a conclusion and discuss future extensions of this thesis.

2 MATHEMATICAL BACKGROUND

2.1 The Rational Billiards Construction

Let R be a polygonal region whose interior angles are rational multiples of π. Let D_{2Q} be the dihedral group of order $2Q$ generated by Euclidean reflections in the sides of R. Suppose a particle moves within this region at constant speed and with initial direction vector v, changing directions only when it reflects off the sides of R, with the angle of incidence equaling the angle of reflection. Every subsequent direction vector for the particle is of the form $\delta \cdot v$, where $\cdot$ indicates the left action of an element of D_{2Q} on an element of $\mathbb{R}^2$.

The rational billiards construction consists of a compact surface corresponding to this physical system. Consider the set $D_{2Q} \cdot R$ of $2Q$ copies of R transformed by the elements of D_{2Q}. For each edge e of R, we consider the corresponding element $\rho_e \in D_{2Q}$ which represents reflection across e. For each $\delta \in D_{2Q}$, we glue $\rho_e\delta \cdot R$ and $\delta \cdot R$ together along their copies of e. The result is a closed Riemann surface with flat structure induced by the tiling by $2Q$ copies of R. See Figures 2.1-2.4. This construction is described in detail in [12] and [16].

In this thesis we focus on billiards in a rational-angled triangle. We shall term the surface X resulting from the construction above a *triangular billiards surface.* If the billiard table is a right or isosceles triangle, we call X a *right triangular billiards surface* or *isosceles triangular billiards surface*, respectively.

The reflection rule for the billiards dynamical system is not well defined if the point of incidence does not admit a unique tangent line. Occasionally such a difficulty can be resolved by a continuous extension of the dynamical system. In particular, in the case of polygonal billiards, tangents are undefined precisely at vertices of the enclosing polygon, and collisions at such vertices can be resolved if and only if the internal angle is of the form $\frac{\pi}{q}$ for some integer q. As detailed in Remark 2.1.2, if a vertex does not have internal angle of the form $\frac{\pi}{q}$, then it corresponds to points on the billiards surface which are *conical singularities*, which in this setting are points about which the total angle is $2m\pi$ for some integer $m > 1$. In this thesis we refer to conical singularities of billiards surfaces simply as *singular points* or *singularities*, and any point which is not a conical singularity is called a *nonsingular point.*

2.1.1 Elementary Combinatorics

For a given rational-angled triangle T, we can write the angles of T as $\frac{a_1\pi}{Q}$, $\frac{a_2\pi}{Q}$, and $\frac{a_3\pi}{Q}$, where $a_1, a_2, a_3, Q \in \mathbb{N}$ and $\gcd(a_1, a_2, a_3) = 1$. With this notation, we also write $T = T(a_1, a_2, a_3)$. We refer to the billiards surface X corresponding to billiards in $T(a_1, a_2, a_3)$ as $X = X(a_1, a_2, a_3)$. Note that the area and direction of $X(a_1, a_2, a_3)$ depend on the area and direction of $T(a_1, a_2, a_3)$; hence this notation is only well-defined up to an action of $O(2, \mathbb{R})$.

Since a triangular billiards surface X is constructed from copies of T, the surface X admits a natural triangulation by these copies. Given a triangular billiards surface $X = X(a_1, a_2, a_3)$, there is a natural projection map $\pi_X : X \to T$ induced by the billiards triangulation of X by T. This motivates the following definition.

Definition 2.1 *Labeling the vertices of* $T(a_1, a_2, a_3)$ *as* v_1, v_2, v_3*, where* $\angle v_i = \frac{a_i\pi}{Q}$*, we call the three sets* $\pi_X^{-1}(v_i)$ *the* vertex classes *of* X. *Note that all elements of a given vertex class have the same cone angle. Hence we call a vertex class* singular *if all elements are singular and* nonsingular *otherwise.*

Remark 2.1 *As detailed in [1], we have the following formulae concerning* $X(a_1, a_2, a_3)$.

1. *The set* $\pi_X^{-1}(v_i)$ *has cardinality* $\gcd(a_i, Q)$.

2. *Each element of* $\pi_X^{-1}(v_i)$ *has cone angle* $\left(\dfrac{a_i}{\gcd(a_i, Q)}\right) 2\pi$.

3. *The genus of* $X(a_1, a_2, a_3)$ *is* $\dfrac{1}{2}Q + 1 - \dfrac{1}{2}\sum \gcd(a_i, Q)$.

Two immediate consequences of these formulae are that a vertex class $\pi_X^{-1}(v_i)$ *is singular if and only if* $a_i \nmid Q$*, and that the sum of the cone angles of the elements of* $\pi_X^{-1}(v_i)$ *is* $2a_i\pi$.

2.1.2 Examples

As a first example, consider the surface $X(1,1,2)$ generated by an isosceles right triangle. Here $Q = 4$, and the $2Q = 8$ copies of $T(1,1,2)$ glue together to form a square torus (see Figure 2.1).

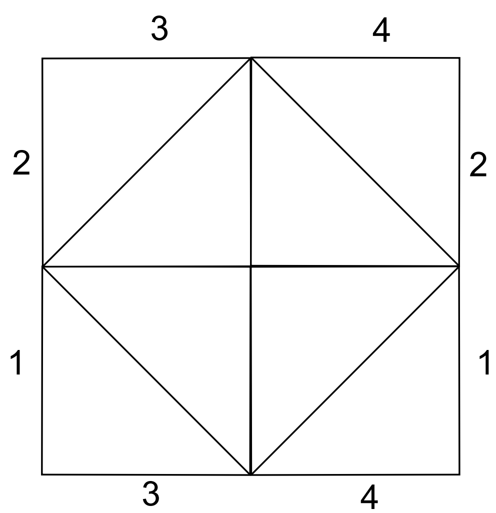

FIGURE 2.1: X(1,1,2) is a square torus.

Next consider the surface $X(1,1,1)$. Here the equilateral triangle $T(1,1,1)$ unfolds to the hexagonal torus. In fact $T(1,2,3)$ also unfolds to the hexagonal torus; this is related to the fact that $T(1,2,3)$ tiles $T(1,1,1)$ via a single flip. See Figure 2.2. We discuss this phenomenon in more detail in Section 4.1. It is a consequence of the third part of Remark 2.1 that these are the only genus 1 triangular billiards surfaces.

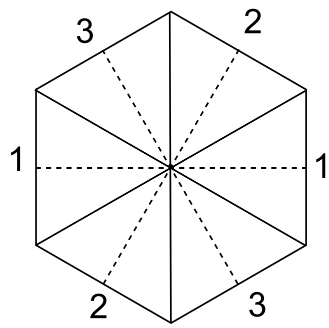

FIGURE 2.2: X(1,1,1) and X(1,2,3) (see dotted lines) are hexagonal tori.

The surface $X(1,1,3)$ has genus two. See Figure 2.3.

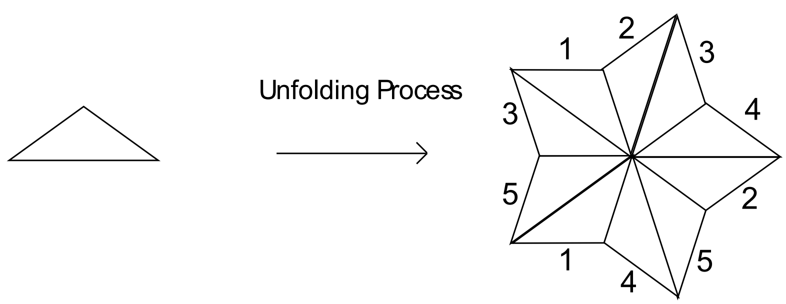

FIGURE 2.3: T(1,1,3) “unfolding” to X(1,1,3).

Another interesting example is the genus 3 surface $X(1,2,4)$, which is a flat representation of Klein's famous quartic curve; see [11] for a detailed exposition of this fact.

The previous examples can all be constructed by taking a single star-shaped polygon whose center corresponds to a vertex of the triangular billiard table and identifying appropriate edges; however in general a triangular billiards surface may have too many singularities for this. For example, $X(8,25,27)$ has three singular vertex classes, and each vertex class has cardinality greater than one. See Figure 2.4 for a diagram of $X(4,7,9)$, which can be realized as a union of four stars with appropriate edges identified.

2.1.3 The Dihedral Group and the Flat Geometry

In this section we show that the flat structure of $X(a_1, a_2, a_3)$ is strongly related to D_{2Q} by using the dihedral group to place an upper bound on the distance between any two points of $X(a_1, a_2, a_3)$.

Let T be a rational-angled triangle, and let r_i be the reflection in the edge e_i of T for $i = 1, 2, 3$. Together the r_i generate the dihedral group D_{2Q}. Define a *generalized star polygon* to be the translation surface (with boundary) obtained from $\langle r_1, r_2 \rangle \cdot T$ by identifying $r_i\delta \cdot e_i$ with $\delta \cdot e_i$ for each $\delta \in \langle r_1, r_2 \rangle$ and each $i \in \{1, 2\}$.

Proposition 2.1 *Let $T = T(a_1, a_2, a_3)$, and let $X = X(a_1, a_2, a_3)$. Let v be a vertex of T, and write $\pi_X^{-1}(v) = \{P_0, P_1, ..., P_{n-1}\}$. The surface X admits a decomposition into generalized star polygons $S_0, S_1, ..., S_{n-1}$ such that each S_i has center P_i. Furthermore, it is possible to color each of the $2Q$ triangles in $D_{2Q} \cdot T$ in such a way that the following properties hold:*

1. *Each triangle is colored either black or white.*
2. *Each black triangle shares an edge with three white triangles, and vice versa.*
3. *Each black triangle of S_i shares an edge with a white triangle of S_{i+1}, where indices are calculated modulo n.*

Proof. Let r_1, r_2, and r_3 be the reflections across the sides of T, and let D_{2Q} be the dihedral group $\langle r_1, r_2, r_3 \rangle$. By construction, X can be viewed as the quotient of the set $D_{2Q} \cdot T$ by the relation $\mathcal{R}$ of identifying appropriate edges.

Let v be an endpoint of T such that the reflections across the edges incident on v are r_1 and r_2. Let n be the cardinality of the vertex class $\pi^{-1}(v)$, with elements $P_0, P_1, ..., P_{n-1}$. Let $Id \cdot T$ have P_0 as a vertex. Developing around P_0 gives the set $S_0 := (\langle r_1, r_2 \rangle \cdot T)/\mathcal{R}$, which is a generalized star polygon; in fact, if P_0 is nonsingular, then S_0 is a star-shaped polygon with center P_0 (see Figure 2.4). We have that S_0 is the union of the two sets $(\langle r_1 r_2 \rangle \cdot T)/\mathcal{R}$ (which we color black) and $(\langle r_1 r_2 \rangle r_1 \cdot T)/\mathcal{R}$ (which we color white). If $n > 1$, then S_0 is not all of X and the action of r_3 takes elements of S_0 outside of S_0. Let $S_i := (\langle r_1, r_2 \rangle (r_1 r_3)^i \cdot T)/\mathcal{R}$; again, in Figure 2.4, we have colored

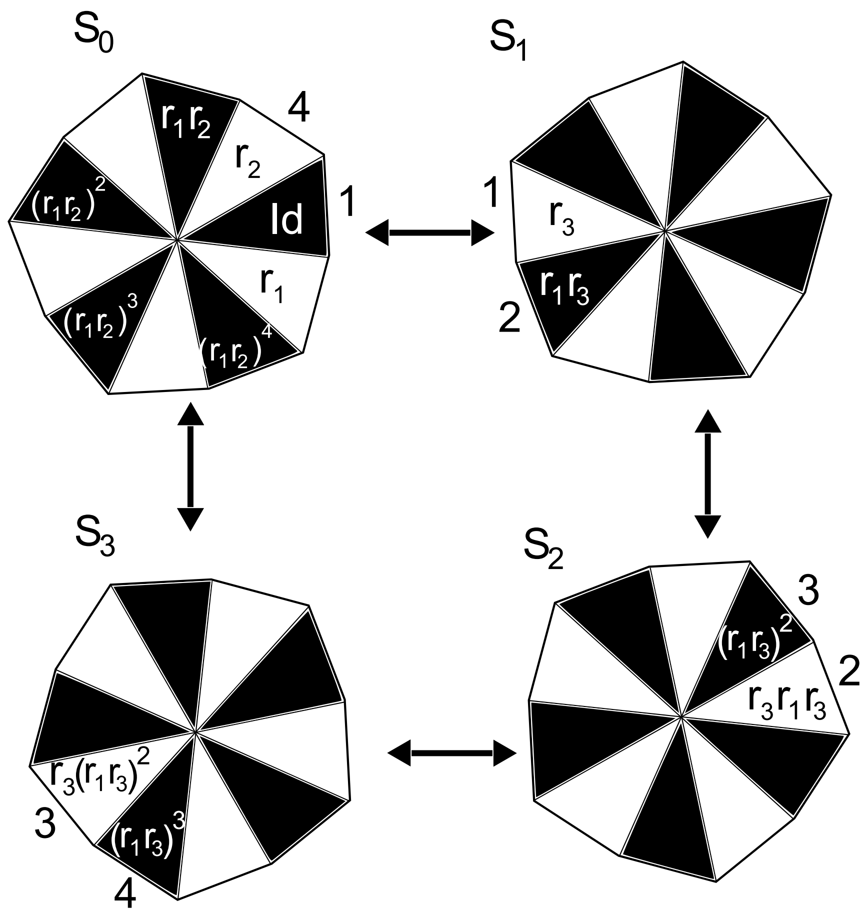

FIGURE 2.4: X(4,7,9) as a union of stars. Note, for example, that in D_{40} we have $r_2 = r_3(r_1 r_3)^3$, accounting for identification 4.

$(\langle r_1 r_2 \rangle (r_1 r_3)^i \cdot T)/\mathcal{R}$ black and $(\langle r_1 r_2 \rangle (r_1 r_3)^i r_1 \cdot T)/\mathcal{R}$ white. We choose a labeling for $P_0, P_1, ..., P_{n-1}$ so that each S_i is realized by development about P_i. The result is that each black triangle of S_i shares an edge with a white triangle of S_{i+1}, where $i+1$ is calculated modulo n. ■

Proposition 2.1 has an interesting consequence for the shortest paths between points in X.

Corollary 2.1 *Let T be a rational triangle, with longest side length L and shortest side length l. Let X be the triangular billiards surface generated by T. For any two points $x, y \in X$, define $\lambda(x,y)$ to be the length of the shortest path connecting x and y. Then* $\max_{x,y \in X}\{\lambda(x,y)\} \leq 2L + l$.

Proof. Using the notation of Proposition 2.1, let C_2 and C_3 be the two vertex classes of X which do not project to v. Any element of $C_2 \bigcup C_3$ must be on the boundary of one of the S_i; but Proposition 2.1 implies that in fact any element of $C_2 \bigcup C_3$ must be on the boundary of each S_i.

Now choose the vertex v so that the shortest edge of T is incident on v. Let C_2 be the vertex class corresponding to the other endpoint of the shortest edge of T. Let x and y be any two points on X. There exist integers i and j so that $x \in S_i$ and $y \in S_j$. The shortest geodesic segment within S_i connecting x to to some point $A \in C_2$ has length at most L. Since A is on the boundary of all the S_i, there is a geodesic segment of length l connecting A to P_j. Finally, the segment within S_j connecting P_j to y has length at most L. The union of these three segments is a path connecting x and y; the length of this path is at most $2L + l$. ■

2.2 Translation Structure

2.2.1 Translation Surfaces

Billiards surfaces are instances of a more general class of surfaces known as *translation surfaces.*

Definition 2.2 *Let S be a topological surface, and let $P_1, ..., P_n$ be a finite subset of S. Let S' be the submanifold of S obtained by deleting the points $P_1, ..., P_n$. If all transition functions of S' are restrictions of Euclidean translations of $\mathbb{R}^2$, then we call S a* translation surface*.*

Given a finite set of disjoint polygons $P_1, P_2, ..., P_n$ in the plane, with the property that each edge e can be associated with a unique parallel edge $e' \neq e$ of the same length, we obtain a translation surface by gluing associated edges via translations as long as the gluing gives a consistent orientation. See, for example, [7]. In fact, it is well known that, up to addition or removal of removable singularities, any compact translation surface can be constructed in such a way.

A second construction of a translation surface is as follows: let S be a Riemann surface, and let ω be a holomorphic 1-form defined on S. For each point $x \in S$, we define coordinates on a neighborhood of x via the map $y \mapsto \int_x^y \omega$. The maximal atlas of such charts defines a translation surface which we denote by (S, ω).

As an example of the translation structure of a triangular billiards surface, consider $X(1,1,3)$. Figure 2.5 demonstrates an application of transition functions which are local translations. Translation structure is invariant under the operation of cutting, translating, and pasting in local coordinates as long as identifications are preserved; thus we can visualize $X(1,1,3)$ as a five-pointed star, or as a union of pentagons, or as a union of two vertical cylinders, each with appropriate side identifications.

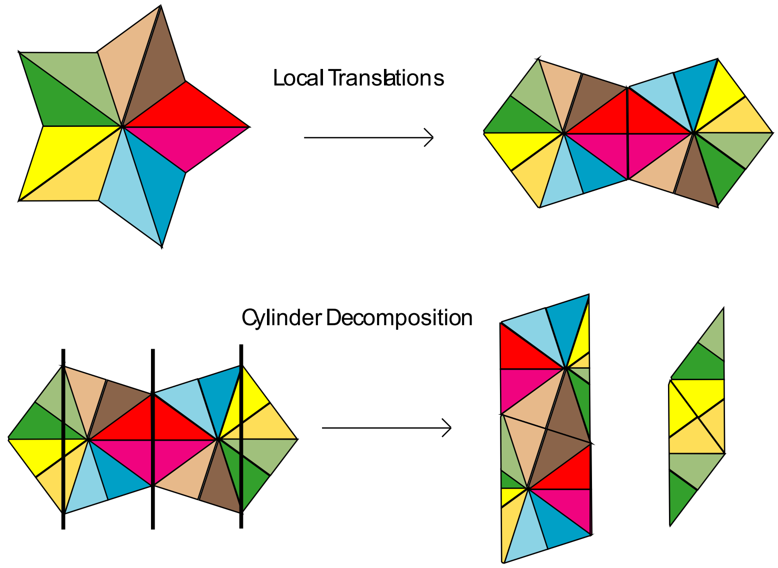

FIGURE 2.5: X(1,1,3) is a translation surface.

2.2.2 Translation Covers

The natural map between translation surfaces is one which respects this translation structure. First we recall the definition of a ramified cover of Riemann surfaces.

Definition 2.3 *Let $f : X \to Y$ be a holomorphic mapping between compact Riemann surfaces X and Y. For each point $x \in X$, there exist local coordinates on X and Y which vanish at x and $f(x)$ respectively, and such that in those coordinates, f has the form $z \mapsto z^{1/m_x}$ for some integer m_x. If $m_x > 1$ then we say that f is* ramified at x, *that f is* ramified above f(x), *and that the* ramification number of f at x *is $m_x - 1$. For each point $y \in Y$, we define the* ramification number of f above y *to be $\sum_{x \in f^{-1}(y)}(m_x - 1)$. We define the* total ramification number *of f to be the sum of the ramification numbers of f above each point $y \in Y$.*

Any holomorphic mapping $f : X \to Y$ between compact Riemann surfaces ramifies at and above at most finitely many points; hence total ramification number is well-defined.

Any ramified cover $f : X \to Y$ has the property that there exists an integer n such that, if f does not ramify above $y \in Y$, then $f^{-1}(y)$ has cardinality n. We say that f *has degree* n, or simply write $\deg f = n$. If f does ramify above a point $y \in Y$ with ramification number r, then $f^{-1}(y)$ has cardinality $n - r$.

An important result about ramified maps between Riemann surfaces is the Riemann-Hurwitz formula:

Theorem 2.1 *(Riemann-Hurwitz Formula) Let $f : X \to Y$ be a ramified map of degree n between Riemann surfaces X and Y. Let g_X and g_Y denote the genera of X and Y, respectively. Let the total ramification number of f be R. Then*

$$g_X = n(g_Y - 1) + 1 + \frac{R}{2}. \tag{2.1}$$

An excellent text for the theory of Riemann surfaces is [4].

Now we define a natural map between translation surfaces.

Definition 2.4 *A* translation cover *is a holomorphic (possibly ramified) cover of translation surfaces $f : X \to Y$ such that, for each pair of coordinate maps ϕ_X and ϕ_Y on X and Y, respectively, the map $\phi_Y \circ f \circ \phi_X^{-1}$ is a translation when ϕ_X and ϕ_Y are restricted to open sets not containing singular points. We say that f is* balanced *if f does not map singular points to nonsingular points.*

If $f : X \to Y$ is a translation cover which ramifies at a point $P' \in X$ above a point $P \in Y$, then for some integer $m > 1$ we have that f is locally of the form $z \mapsto z^{1/m}$, and hence the cone angle at P' is m times the cone angle at P. Therefore the set of f-preimages of singularities of Y are singularities of X. But it may be that f ramifies above a nonsingular point; in this case f is not balanced.

Definition 2.5 *We say that X and Y are* translation equivalent *if there exists a degree one translation cover $f : X \to Y$.*

The following lemma shows how we will use Remark 2.1 to analyze translation covers.

Lemma 2.1 *Suppose $f : X(a_1, a_2, a_3) \to X(b_1, b_2, b_3)$ is a translation cover of triangular billiards surfaces. Let $\pi_X : X(a_1, a_2, a_3) \to T(a_1, a_2, a_3)$ and $\pi_Y : X(b_1, b_2, b_3) \to T(b_1, b_2, b_3)$ be the canonical projections to triangles with vertices v_1, v_2, v_3 and w_1, w_2, w_3 respectively. Suppose that $P \in \pi_Y^{-1}(w_i)$, $P' \in \pi_X^{-1}(v_j)$, and $f(P') = P$ with a ramification index of m at P' . Then*

$$\frac{mb_i}{\gcd(b_i, b_1 + b_2 + b_3)} = \frac{a_j}{\gcd(a_j, a_1 + a_2 + a_3)} .$$

Proof. The cone angle at P' is m times the cone angle at P . Therefore the result follows from the second part of Remark 2.1. ■

As noted in Section 2.1.1, the translation structure of $X(a_1, a_2, a_3)$ depends on the chosen area and direction of $T(a_1, a_2, a_3)$. Suppose that (S, ω) is a triangular billiards surface arising from billiards in some $T(a_1, a_2, a_3)$, and that α is a nonzero complex number. The notation $X(a_1, a_2, a_3)$ does not distinguish the pairs (S, ω) and $(S, \alpha\omega)$. The following lemma shows that this ambiguity will not affect our classification of translation covers.

Lemma 2.2 *Suppose that (S, ω) is a triangular billiards surface of genus greater than one, and let $\alpha \in \mathbb{C}\backslash\{0\}$. Then any translation cover $f : (S, \omega) \to (S, \alpha\omega)$ is of degree 1.*

Proof. This is a simple application of the Riemann-Hurwitz formula. Let (S, ω) have genus g, and let $\deg f = n$. The 1-form ω which gives (S, ω) its translation structure has $2g-2$ zeros (counting multiplicities). Clearly $\alpha\omega$ has the same zeros as ω. The Riemann-Hurwitz formula then gives us that

$$g = n(g-1) + 1 + \frac{R}{2}, \tag{2.2}$$

where R is the total ramification number of f. Since $R \geq 0$, Equation (2.2) is only satisfied if $n = 1$. ■

As a result of this lemma, we shall use the notation $X(a_1, a_2, a_3)$ to refer to any element of the set $\{(S, \alpha\omega) : \alpha \in \mathbb{C}\backslash\{0\}\}$, where (S, α) is a triangular billiards surface arising from billiards in some $T(a_1, a_2, a_3)$. Note that multiplying the 1-form of a translation surface by a nonzero complex number is equivalent to post-composing each coordinate chart of (S, α) by the standard linear action of an element of $O(2, \mathbb{R})$.

2.3 The J-invariant and Holonomy Fields

In [13], Kenyon and Smillie introduce an important property of translation surfaces, called the J-invariant.

Definition 2.6 *Let P be a polygon in the plane. Let $w_1, w_2, ..., w_n$ be the vertices of P. The* J-invariant of P *is the element of $\mathbb{R}^2 \wedge_{\mathbb{Q}} \mathbb{R}^2$ given by $J(P) := w_1 \wedge w_2 + w_2 \wedge w_3 + ... + w_{n-1} \wedge w_n + w_n \wedge w_1$.*

We write $\mathbb{R}^2 \wedge_{\mathbb{Q}} \mathbb{R}^2$ to indicate the exterior product of two copies of $\mathbb{R}^2$ viewed as $\mathbb{Q}$-modules.

It is easily shown that the J-invariant of a polygon is invariant under translations of the polygon, and that it is a "scissors invariant" in the sense that cut-and-paste operations do not affect its J-invariant. Furthermore, it is well known that any compact translation surface can be constructed by identifying parallel edges of a finite set of polygons in the plane. For these reasons the definition naturally extends to translation surfaces.

Definition 2.7 *Let* X *be a compact translation surface. Let* $\{P_1, ..., P_n\}$ *be a collection of planar polygons such that appropriate identification of sides yields the surface* X *. Then the* J-invariant *of* X *is* $J(X) := \sum_{i=1}^{n} J(P_i)$ *.*

Example 2.1 *Suppose that* $X = X(1,1,2)$ *is scaled so that the copies of* $T(1,1,2)$ *in the billiards triangulation of* X *have lengths 1,1, and* $\sqrt{2}$ *. Then* X *can be realized as a square of side length 2 with opposite sides identified. We can assume that the lower lefthand corner of the square lies at the origin. Then the J-invariant of* X *is*
$(0,0) \wedge (2,0) + (2,0) \wedge (2,2) + (2,2) \wedge (0,2) + (0,2) \wedge (0,0) = (2,0) \wedge (2,2) + (2,2) \wedge (0,2) = 4(1,-1) \wedge (1,1)$.

Example 2.2 *Suppose that* $X = X(1,1,3)$ *. Then*

$$J(X) = \sum_{k=0}^{4} \left(\cos \frac{2k\pi}{5}, \sin \frac{2k\pi}{5} \right) \wedge \frac{\sin(3\pi/5)}{\sin(\pi/5)} \left(\cos \frac{(2k+1)\pi}{5}, \sin \frac{(2k+1)\pi}{5} \right).$$

The following lemma, which is presumably well-known, demonstrates the relevance of the J-invariant to the study of translation covers.

Lemma 2.3 *Let* $f : X \to Y$ *be a degree* n *translation cover of translation surfaces. Then* $J(X) = nJ(Y)$ *.*

Proof. We can triangulate Y by Euclidean triangles in such a way that the branch points of f are among the vertices of the triangulation. Let Y' be the set of triangles obtained by cutting open Y along all the edges of our triangulation. Lifting our triangulation to X via f, we let X' be the corresponding decomposition of X . Since J is a scissors invariant, we have $J(Y) = J(Y')$ and $J(X) = J(X')$. Furthermore, since each triangle in Y' lifts to n identical copies in X', we have that $J(X') = nJ(Y')$. Thus $J(X) = J(X') = nJ(Y') = nJ(Y)$. ■

Translation structure gives us a canonical way to associate an element of $\mathbb{C}$ to each element of the first homology group $H_1(X)$. Because it will be advantageous to view the image of $H_1(X)$ in $\mathbb{C}$ as a vector space over $\mathbb{Q}$, we use coefficients in $\mathbb{Q}$ for $H_1(X)$ in the following definition.

Definition 2.8 *The rational* absolute holonomy *of a translation surface X is the image of the map $hol : H_1(X;\mathbb{Q}) \to \mathbb{C}$ defined by $hol : \sigma \mapsto \int_\sigma \omega$, where ω is the 1-form which endows X with a flat structure, as described in Section 2.2.1.*

Now we define a property of translation surfaces which will be useful in classifying triangular billiards surfaces. This definition is due to Kenyon and Smillie [13].

Definition 2.9 *The* holonomy field *of a translation surface X, denoted k_X, is the smallest field k_X such that the absolute holonomy of X is contained in a two-dimensional vector space over k_X.*

Example 2.3 *Consider $X = X(1,1,2)$, scaled so that it is a unit square with opposite sides identified. The absolute holonomy of $H_1(X)$, as a vector space over $\mathbb{Q}$, is generated by 1 and i. Hence the holonomy field of X is $\mathbb{Q}$.*

Example 2.4 *Consider $X = X(1,1,3)$. The surface X can be scaled so that generators for the absolute holonomy of X over $\mathbb{Q}$ are $1, \zeta_5, \zeta_5^2, \zeta_5^3$, where $\zeta_5 = e^{(2\pi i)/5}$. Thus the holonomy field of X properly contains $\mathbb{Q}$. In fact these four elements generate a two-dimensional vector space over $\mathbb{Q}(\sqrt{5})$. Since $\mathbb{Q}(\sqrt{5})$ is a degree 2 extension of $\mathbb{Q}$ there can be no intermediate fields; therefore the holonomy field of X is $\mathbb{Q}(\sqrt{5})$.*

Calta and Smillie [3] discuss the *algebraically periodic directions* of a translation surface, which they define to be those directions in which a certain projection of the J-invariant is zero.

Definition 2.10 *Fix coordinates for a compact translation surface S such that 0, 1, and ∞ are all slopes of algebraically periodic directions. The* periodic direction field *of S is the collection of slopes of algebraically periodic directions in this coordinate system.*

It is shown in [3] that this definition is well-defined, and that the periodic direction field is a number field whose degree is bounded by the genus of S. The following lemma relies on the results of Kenyon and Smillie [13] and Calta and Smillie [3].

Lemma 2.4 *Let $f : X(a_1, a_2, a_3) \to Y$ be a degree n translation cover. Write $Q := a_1 + a_2 + a_3$. Then X and Y have the same holonomy field k, and $k = \mathbb{Q}(\zeta_Q + \zeta_Q^{-1})$, where ζ_Q is a primitive Q^{th} root of unity.*

Proof. By Lemma 2.3, $J(X) = nJ(Y)$. Assume that Y has area 1; thus X has area n. Let X' be the surface of area 1 obtained by uniformly scaling X. We have that $J(X') = \frac{1}{n}J(X) = J(Y)$. Since uniformly scaling a surface clearly does not affect its periodic direction field, X and X' have the same periodic direction field. Calta and Smillie note that their work in Section 6 of [3] implies that the periodic direction field of a surface depends only on the J-invariant of that surface; hence X' and Y have the same periodic direction field. Thus X and Y have the same periodic direction field. Corollary 5.21 of [3] states that a translation surface is completely algebraically periodic if and only if its holonomy field equals its periodic direction field. Furthermore, Theorem 1.4 of [3] states that triangular billiards surfaces are algebraically periodic. Therefore X and Y have the same holonomy field. Finally, Kenyon and Smillie [13] calculate this holonomy field to be $k = \mathbb{Q}(\zeta_Q + \zeta_Q^{-1})$. ■

The proof of the algebraic periodicity of triangular billiards surfaces in [3] contains a small error which could be corrected by applying a normalization outlined in [13]. We also offer a different proof of this result in Chapter 5, where it is listed as Theorem 5.3.

3 THE FINGERPRINT

3.0.1 Definition and Properties

Consider a point P on a translation surface X, along with the set S of all shortest geodesic segments on X which connect P to a singularity. Let s_1 and s_2 be two of these segments. We say that s_1 and s_2 are *adjacent* if s_1 can be rotated continuously about P onto s_2 without first coinciding with any other elements of S.

Definition 3.1 *A* fingerprint *of a point $P \in \tau$ is the data $\{\{\theta_i\}, \phi, L\}$, where $\{\theta_i\}$ contains the distinct angle measures separating adjacent shortest geodesic segments connecting P to singularities, ϕ is the total cone angle at P, and L is the length of each of the shortest geodesic segments. We say that P has a* Type I fingerprint *if $\{\theta_i\}$ has one element, and that P has a* Type II fingerprint *if $\{\theta_i\}$ has two elements. We call $\{\theta_i\}$ the* angle set *of a fingerprint.*

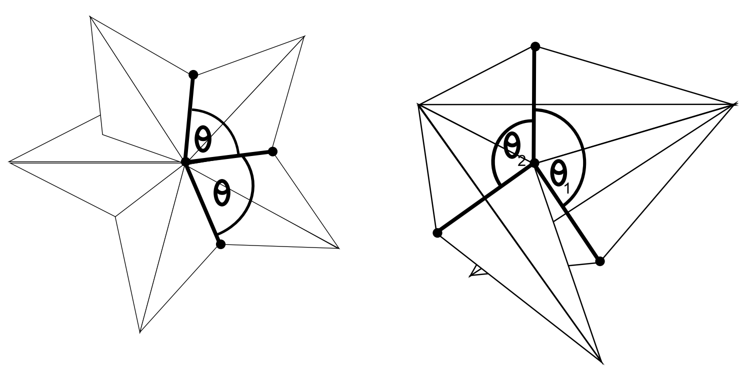

FIGURE 3.1: Parts of a Type I fingerprint (left) and a Type II fingerprint (right).

Note that the angle set (and hence the fingerprint type) of the fingerprint of a point $P \in X$ is invariant under the scaling of the flat structure of X by a nonzero complex

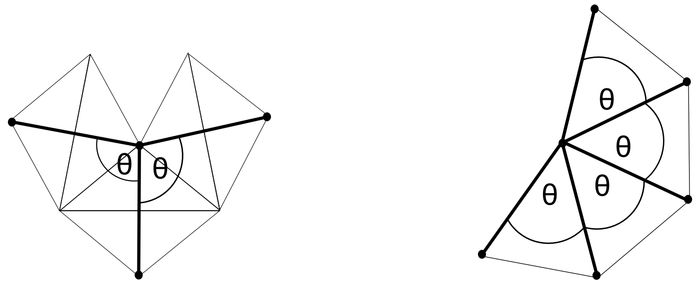

FIGURE 3.2: Type I fingerprints arising from isosceles triangles .

number. Each triangular billiards surface has rotational symmetry about the vertices of its billiards triangulation; this fact places a strong restriction on the angle sets of the fingerprints of vertices. The following lemma illustrates this.

Lemma 3.1 *Let X be a surface of genus greater than one, arising from billiards in a rational triangle T. Fix a billiards triangulation τ of X by T_X. Let P be a vertex of τ. Let s be a shortest geodesic segment connecting P to a singularity of X. Then either s is an edge of τ, or else s is perpendicularly bisected by an edge of τ.*

Proof. Let X, T_X, s, P and τ be as above. Let $\pi_X : X \to T$ be the natural projection induced by τ.

Since singularities in the translation structure of X can only occur at vertices of τ, we only examine geodesics connecting vertices of τ. This is equivalent to considering billiard paths between corners of the triangular billiard table T_X in the original dynamical system.

Let $v = \pi_X(P)$; since P is a vertex of τ, v is a corner of T_X. The shortest billiard path within T_X from v to a different corner w of T cannot be as short as the table edge connecting v and w. This proves the claim if s connects P to a singularity which is not in the vertex class $\pi_X^{-1}(v)$.

Now suppose that s connects P to a singularity in $\pi_X^{-1}(v)$. Then s corresponds to a billiard path from v back to itself. If both of the other two corners of T_X are acute, then the shortest billiard path from v to itself is accomplished via a single reflection by choosing the initial direction to be perpendicular to the side opposite v; hence here an edge of τ bisects s. If one of the two other corners w is obtuse, then $\pi_X^{-1}(w)$ must be a singular vertex class. But the distance from v to an obtuse corner of T_X is less than twice the distance from v to the opposite side of T_X. Thus if w is obtuse then there is a geodesic segment s' in X connecting an element of $\pi_X^{-1}(v)$ to a singular element of $\pi_X^{-1}(w)$ such that s' is shorter than s; this is a contradiction. ■

Lemma 3.1 allows us to relate fingerprints of points on X to the angle measures of vertices of T_X. We summarize these relations in the following Corollary; see Figures 3.1 and 3.2 for illustrations.

Corollary 3.1 *Let τ be a billiards triangulation of a triangular billiards surface X. For a given point $P \in \tau$, let v be the projection of P onto the triangle T generating X. Then one of three situations exists:*

1) P has a Type I fingerprint with angle set $\{\theta\}$, and $\theta = \angle v$.

This occurs if and only if T is isosceles and v is the apex of T.

2) P has a Type I fingerprint with angle set $\{\theta\}$, and $\theta = 2\angle v$.

This occurs if P has a Type I fingerprint and v is not the apex of an isosceles triangle.

3) P has a Type II fingerprint with angle set $\{\theta_1, \theta_2\}$, and $\theta_1 + \theta_2 = 2\angle v$.

Proposition 3.1 *Suppose X is a triangular billiards surface with a point P of Type II fingerprint. Then X is uniquely determined by that fingerprint, up to an action of $O(2, \mathbb{R})$.*

Proof. The proof is evident from Figure 3.0.1, which illustrates the fingerprint of the singularity on $X(3,4,5)$ (since $X(3,4,5)$ is not isosceles and has only one singularity P, it follows that P has a Type II fingerprint. In the figure, the geodesics defining the

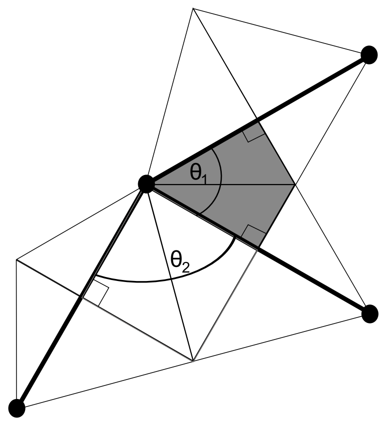

FIGURE 3.3: Part of a Type II fingerprint on X(3,4,5)

fingerprint are the thicker lines, whereas the edges of the billiards triangulation are the thinner lines.) Let the angle set be $\{\theta_1, \theta_2\}$. Each θ_i is an interior angle of a quadrilateral whose other three angles include two right angles and an angle which has twice the measure of an angle of the triangular billiard table T for X. Therefore two of the angles of T have the form $\frac{1}{2}(2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \theta_i) = \frac{\pi - \theta_i}{2}$, and the third angle is $\frac{\theta_1 + \theta_2}{2}$. The length of the geodesics defining the fingerprint of P determines the scaling of T. Thus T (and hence X) is uniquely identified, up to an action of $O(2, \mathbb{R})$. ■

Lemma 3.2 *Suppose that $f : X \to Y$ is a balanced translation cover, that $P' \in X$ and $P \in Y$ are vertices of billiards triangulations on their respective surfaces, and that $f(P') = P$. Then either:*

1. *P' and P have the same fingerprint, or*

2. *their fingerprints differ only in the cone angle, P has half the cone angle of P', X arises from billiards in an isosceles triangle, and P' corresponds to the apex of that triangle.*

Proof. Let d be the length of a shortest geodesic which connects P to a singularity. Let $B \subset Y$ be the set of points of distance less than d from P. Let $B' \subset X$ be the maximal connected component of $f^{-1}(B)$ which contains P'. Since f is a balanced translation cover, B' consists of all points of distance less than d from P', and B' contains no singularities other than possibly P' (P' is singular if and only if P is singular). We have that f is locally an m-to-one cover at P for some integer m.

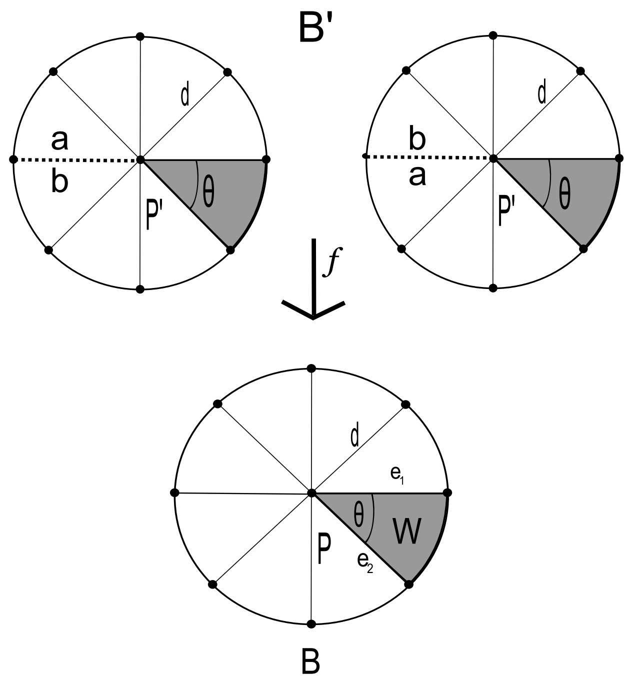

FIGURE 3.4: A balanced cover ramified above P. Here, $m = 2$.

Now consider a pair of adjacent geodesics e_1 and e_2, each of length d, connecting P to singularities. Label the angle between them θ. The union of these two edges with a portion of the boundary of B bounds a wedge-shaped region W which contains singularities only at the endpoints of e_1 and e_2 (see Figure 3.4). Since f is a translation cover, the f-preimage of W is m copies of W, each of which is bounded by part of the boundary of B' and two shortest geodesics e_1' and e_2' of length d connecting P' to singularities of X. The interior angle measure between e_1' and e_2' is θ. Because f is balanced, we know that e_1' and e_2' are adjacent; otherwise, the wedge they bound would have a geodesic e' in its interior such that $f(e')$ lies in the interior of W and connects P to a singularity, a contradiction to the adjacency of e_1 and e_2. Therefore we have established that the fingerprints of P and P' have the same angle sets.

Because f is a translation cover, the cone angle at P' is m times the cone angle at P. We claim that $m \leq 2$. Let v and v' be the vertices of the triangles T and T' corresponding to P and P'. By Remark 2.1, the cone angle at P is completely determined by $\angle v$. But Corollary 3.1 tells us that $\angle v$ is determined, up to a factor of 2, by the angle set of the fingerprint of P. Hence, since the fingerprints of P and P' have the same angle set, we see that $m \in \{1, 2\}$, and our claim is proven.

Furthermore, note that if $m = 2$, then since the cone angle at P' is greater than the cone angle at P and cone angle is completely determined by the corresponding vertex of the triangular billiard table, Corollary 3.1 implies that T_X is isosceles and v' is the apex of T_X. ■

Corollary 3.2 *Fingerprint type is invariant under balanced translation covers.*

Corollary 3.3 *Any rational triangular billiards surface with a Type II singularity cannot be a part of any composition of nontrivial balanced covers.*

Proof. This follows directly from Proposition 3.1. Suppose we have $f : X \to Y$ a balanced cover with either X or Y possessing a singularity with a Type II fingerprint. By Corollary 3.2, X and Y must both have singularities with Type II fingerprints. Since a Type II fingerprint identifies the triangular billiards table of a surface, X and Y must be the same surface. ■

3.0.2 Examples

Example 3.1 *The surface* $X = X(1,1,3)$ *has exactly one singularity* P *. Thus all geodesics connecting* P *to a singularity connect* P *to itself. By Lemma 3.1, the shortest such geodesics must be those which correspond to a billiard path with a single reflection. The angle between any two such adjacent shortest geodesics is* $\frac{3\pi}{5}$ *. Thus* P *has fingerprint* $\{\{\frac{3\pi}{5}\}, 6\pi, L\}$*, where the length* L *depends on the scaling of* X *. Let* R *be the only element of one of the nonsingular vertex classes of* X *. The shortest geodesics connecting* R *to* P *are edges of the billiards triangulation of* X *by* $T(1,1,3)$ *. Then the angle between any two such geodesics which are adjacent is* $\frac{2\pi}{5}$ *.*

Next we give an example of a surface with both Type I and Type II fingerprints.

Example 3.2 *Consider the surface* $X = X(1,2,12)$ *. Let* τ *be the billiards triangulation of* X *. Label the vertices of* $T = T(1,2,12)$ *as* v_1, v_2, v_3 *such that* $\angle v_1 = \frac{\pi}{15}$*,* $\angle v_2 = \frac{2\pi}{15}$*, and* $\angle v_3 = \frac{12\pi}{15}$ *. The vertex class corresponding to* v_1 *is nonsingular and has a single element* P_1 *. The vertex class corresponding to* v_2 *is singular and has a single element* P_2 *of cone angle* 4π *. The vertex class corresponding to* v_3 *is singular and has three elements* P_3, P_3', P_3''*; each of these points has cone angle* 8π *. The shortest geodesics connecting* P_1 *to singularities are those which connect* P_1 *to* P_3*,* P_3'*, and* P_3'' *via edges of* τ *. So* P_1 *has a Type I fingerprint* $\{\{\frac{\pi}{6}\}, 2\pi, L\}$ *. Similarly, the shortest geodesics connecting* P_2 *to singularities are those which connect* P_2 *to* P_3*,* P_3'*, and* P_3'' *via edges of* τ *. So* P_2 *has a Type I fingerprint* $\{\{\frac{\pi}{3}\}, 4\pi, \frac{\sin(\pi/15)}{\sin(2\pi/15)}L\}$ *(the length can be calculated by the Law of Sines). Finally, the shortest geodesics connecting* P_3 *to elements of its own vertex class are via a single reflection and are shorter than the shortest geodesics connecting* P_3 *to* P_2*; hence* P_3 *has Type II fingerprint* $\{\{\frac{11\pi}{15}, \frac{13\pi}{15}\}, 8\pi, 2\sin(\pi/15)L\}$ *. The calculation of the angle set of a Type II fingerprint is given in Proposition 3.1 .*

Definition 3.2 *A* saddle connection *on a translation surface is a geodesic with singular*

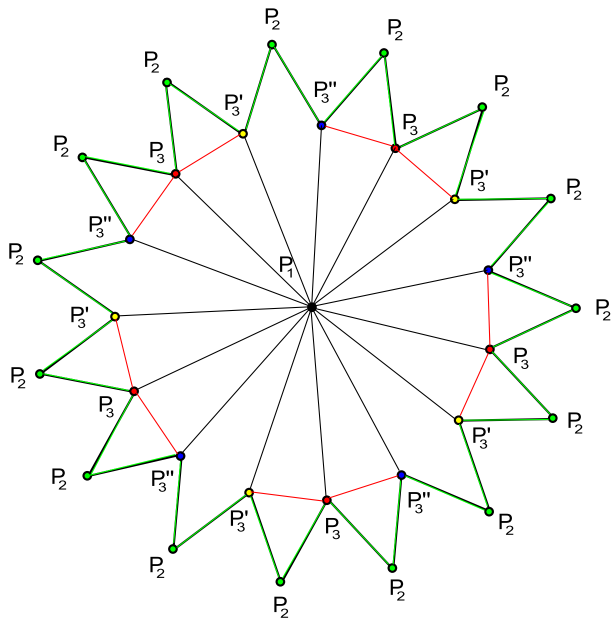

FIGURE 3.5: Fingerprints on X(1,2,12).

endpoints and no singularities in its interior.

As we shall see, the preceding results allow us to quickly classify all balanced covers in the category of triangular billiards surfaces. However, to extend our results to unbalanced covers, we shall refine our use of the fingerprint with the following lemma.

Lemma 3.3 *Let X be a triangular billiards surface with more than one singular vertex class. Let $\tilde{X}$ be the surface obtained from X by puncturing either one entire singular vertex class or two entire singular vertex classes such that neither deleted class corresponds to an obtuse angle of the triangular billiard table and such that at least one singular vertex class remains. Let $\pi_X^{-1}(v_i)$ be a singular vertex class not deleted. Let $P \in \pi_X^{-1}(v_i)$. If P has Type II fingerprint on $\tilde{X}$ with angle set $\{\theta_1, \theta_2\}$, then X arises from billiards in the triangle with angles $\dfrac{\pi-\theta_1}{2}$, $\dfrac{\pi-\theta_2}{2}$, and $\dfrac{\theta_1+\theta_2}{2}$. If P has a Type I fingerprint on $\tilde{X}$ with* *angle set* $\{\theta_1\}$, *then* $\angle v_i \in \{\theta_1, \frac{\theta_1}{2}\}$.

Proof. If none of the punctured points are endpoints of shortest geodesics connecting P to singularities, then P has the same fingerprint on $\tilde{X}$ as on X, and we are done.

Suppose a singular vertex class has been punctured which contained endpoints of shortest separatrices through P. Then there is a new "closest" vertex class to P; call it C. If C does not contain P then the shortest geodesics connecting P to C are edges of the billiards triangulation of X. If C does contain P then, since a vertex class corresponding to an obtuse angle of the billiard table must be singular (by Remark 2.1) and we have assumed that no such classes have been deleted, it follows that the shortest geodesics from P to C correspond to a single reflection in the original dynamical system. Thus the same reasoning holds as in Lemma 3.1.

The only potential difficulty would be if the new "closest" vertex class was the one containing P, for in that case, since the shortest geodesics from P to elements of its own class pass through more than one triangle, we must consider the possibility that our punctures obstruct these geodesics. However, since the shortest geodesics are perpendicular to the sides of the triangles opposite P, this is only a problem if the vertex class punctured is $\pi_X^{-1}(v_j)$ with $\angle v_j = \frac{\pi}{2}$. But such a class is nonsingular. ∎

4 IDENTIFYING ALL TRANSLATION COVERS

4.1 The Possible Covers

Any isosceles triangle is naturally "tiled by flips" by a right triangle. The following lemma demonstrates how to use this tiling to create nontrivial translation covers in the category of triangular billiards surfaces. In fact, our main theorem is that the covers of Lemma 4.1 are the only nontrivial translation covers among triangular billiards surfaces.

Lemma 4.1 *Let a_1 and a_2 be relatively prime positive integers, not both equal to one. The right triangular billiards surface $Y := X(a_1 + a_2, a_1, a_2)$ is related to two isosceles triangular billiards surfaces*

$$X_1 = \begin{cases} X(2a_2, a_1, a_1) & a_1 \text{ odd} \\ \\ X(a_2, \frac{a_1}{2}, \frac{a_1}{2}) & a_1 \text{ even} \end{cases}$$

and

$$X_2 = \begin{cases} X(2a_1, a_2, a_2) & a_2 \text{ odd} \\ \\ X(a_1, \frac{a_2}{2}, \frac{a_2}{2}) & a_2 \text{ even} \end{cases}$$

via balanced covers $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$. The maps have degrees

$$\deg(f_i) = \begin{cases} 2 & a_i \text{ odd} \\ \\ 1 & a_i \text{ even} \end{cases}$$

. Furthermore, at least one of the f_i has degree 2.

Proof. It suffices to prove the result for X_1 and f_1. Write $Q := 2a_1 + 2a_2$. We reflect the triangle $T = T(a_1 + a_2, a_1, a_2)$ across the edge connecting the a_2 and $a_1 + a_2$ vertices, to obtain its mirror image T'. By joining T and T' along the edge of reflection we create an isosceles triangle $\tilde{T}$ which can be written as either $T(2a_2, a_1, a_1)$ (if a_1 is odd) or $T(a_2, \frac{a_1}{2}, \frac{a_1}{2})$ (if a_1 is even). Note that since $(a_1 + a_2, a_1, a_2)$ must be a reduced triple, a_1 and a_2 cannot both be even. It also follows that $\gcd(a_i, Q) \leq \gcd(2a_i, Q) = 2$.

Suppose a_1 is even. Consider the translation surface S (with boundary) obtained by developing T around its a_2 vertex. Since a_2 is odd we have $\gcd(a_2, Q) = 1$, so S is tiled (by reflection) by $2Q$ copies of T, and hence after appropriate identifications along the boundary we will have $X(a_1 + a_2, a_1, a_2)$. Let $\tilde{S}$ be the surface obtained by developing $\tilde{T}$ around the corresponding vertex; it is tiled via reflection by Q copies of $\tilde{T}$, so appropriate boundary identifications will yield Y_1. Because $\tilde{T}$ is tiled via reflection by two copies of T, it follows that S and $\tilde{S}$ are translation equivalent. Finally, note that the boundary identifications are the same for S and $\tilde{S}$. Therefore Y and X_1 are translation equivalent.

Now suppose that a_1 is odd and a_2 is even. We then have $\tilde{T} = T(2a_2, a_1, a_1)$. Since $\gcd(2a_2, Q) = 2$, we again have that $\tilde{S}$ is tiled by Q copies of $\tilde{T}$. Since a_2 is even, $\gcd(a_2, Q) = 2$, implying that S is tiled by Q copies of T. Thus if a_2 is even then there exists a degree two cover $f : \tilde{S} \to S$, ramified over a single point. Furthermore, in this case X_1 and Y are obtained by identifying appropriate edges of *two* copies of $\tilde{S}$ and S, respectively. It follows that if a_2 is even then there exists a ramified degree two cover $f : X_1 \to Y$.

Finally, suppose that a_1 and a_2 are both odd. We have that $\tilde{T} = T(2a_2, a_1, a_1)$, $\gcd(2a_2, Q) = 2$, and $\gcd(a_2, Q) = 1$. In this case we have that S and $\tilde{S}$ are translation equivalent surfaces; however, X_1 is obtained from two copies of $\tilde{S}$ whereas Y is obtained from a single copy of S. Thus again we have a double cover $f : X_1 \to Y$, this time unramified. ■

Remark 4.1 *Note that in addition to relating right and isosceles triangles, Lemma 4.1 also gives a way to construct covers between isosceles triangular billiards surfaces. In the language of Lemma 4.1, if a_2 is even, then $f_2^{-1} \circ f_1$ is a degree two translation cover of X_2 by X_1 .*

Remark 4.2 *If we allow $a_1 = a_2 = 1$ in the statement of Lemma 4.1, then we arrive at $Y = X_1 = X_2 = X(1,1,2)$. This is because $T(1,1,2)$ is the unique right isosceles triangle.*

Because the location of singularities is such a major tool in analyzing translation surfaces, it is worth identifying the triangular billiards surfaces which have no singularities. As detailed in [1], there are only three of these surfaces: $X(1,1,2)$, $X(1,2,3)$, and $X(1,1,1)$. These are also the only three triangular billiards surfaces of genus 1; furthermore $X(1,2,3)$ and $X(1,1,1)$ are actually translation equivalent. Each of these surfaces admits balanced translation covers of itself by itself of arbitrarily high degree; this fact is related to the fact that $T(1,1,2)$, $T(1,2,3)$, and $T(1,1,1)$ are the only Euclidean triangles which tile the Euclidean plane by flips. Note that any such cover must be unramified, since flat ramified covers are locally of the form $z \mapsto z^{1/n}$ for some $n > 1$, implying that the cone angle of the ramification point is greater than 2π .

4.2 Balanced Covers

Balanced translation covers $f : X \to Y$ of translation surfaces are of interest because they imply an especially strong relationship between the affine symmetry groups of X and Y; in particular, these groups must have finite-index subgroups which are $SL(2,\mathbb{R})$-conjugate. We shall prove Theorem 4.1 for balanced covers using only the machinery built up thus far.

Lemma 4.2 *Let $X = X(a_1, a_2, a_3)$ be part of a composition of nontrivial balanced covers.*

If X has exactly one singular vertex class, then either X is an isosceles triangular billiards surface or $X = X(1, n, n+1)$ with $n > 2$ an odd integer.

Proof. Let v be the vertex of $T(a_1, a_2, a_3)$ that unfolds to a singular vertex class. Let $P \in \pi_X^{-1}(v)$. Since X is part of a composition of nontrivial balanced covers, Corollary 3.3 implies that P has a Type I fingerprint. All saddle connections on X have endpoints in $\pi_X^{-1}(v)$, so by Lemma 3.1 the geodesics defining the fingerprint of P are realized via single reflections of P across the opposite sides of the copies of $T(a_1, a_2, a_3)$ of which P is a vertex. Thus T_X is either a right triangle or an isosceles triangle. Suppose T_X is a right triangle, and write $T_X = T(a_1, a_2, a_1 + a_2)$. Since X has only one singular vertex class we can assume that $a_1 | 2(a_1 + a_2)$ and $a_2 \nmid 2(a_1 + a_2)$. By Lemma 4.1, X is also (translation equivalent to) an isosceles triangular billiards surface unless a_1 and a_2 are both odd. Thus either X is an isosceles billiards surface or $a_1 = 1$. ■

Lemma 4.3 *Let X and Y be triangular billiards surfaces such that the genus of X is greater than 1. Suppose that $f : X \to Y$ is a nontrivial balanced translation cover. Then f is of the form described in Lemma 4.1.*

Proof.

Let P' be a singular point of X, and write $f(P') = P$. Since f is balanced, Lemma 3.2 guarantees that the fingerprints of P' and P have the same angle sets. By Corollary 3.1, $\angle\pi_X(P') = \angle\pi_Y(P)$ unless $\pi_X(P')$ or $\pi_Y(P)$ is the apex of an isosceles triangle. With this reasoning in mind, we split the proof into cases.

Case 1 *Neither T_X nor T_Y are isosceles triangles.*

By Lemma 4.2, if X has only one singular vertex class then $X = X(1, n, n+1)$ for $n > 2$ an odd integer. But $\gcd(n, 2n+2) = 1$, so P' is the only singularity on X, and hence P is the only singularity on Y. Thus by Lemma 4.2, $Y = X(1, m, m+1)$, and since P and

P' have the same angle set, $m = n$. Thus $Y = X$. But this is impossible by Lemma 2.2. Therefore we may assume that X has at least two singular vertex classes. Let $R' \in X$ be in a vertex class distinct from the vertex class of P', and write $f(R') = R$. If R and P are in distinct vertex classes then since $\angle\pi_X(P') = \angle\pi_Y(P)$ and $\angle\pi_X(R') = \angle\pi_Y(R)$, in fact $T_X = T_Y$ and f must be trivial. If R and P share a vertex class then we have $\angle\pi_X(P') = \angle\pi_Y(P) = \angle\pi_X(R')$; but then T_X is isosceles, contradicting the hypothesis of this case.

Case 2 *The triangle T_X is isosceles, with its apex unfolding to a singular vertex class.*

Let P' be in the singular vertex class which projects to the apex of T_X. Since $T_X \neq T_Y$, we must have that $\pi_Y(P)$ is not the apex of an isosceles triangle. Thus $\angle\pi_X(P') = 2\angle\pi_Y(P)$. Furthermore, since the fingerprints of P and P' have the same angle set, Y must have a second singular vertex class. Let $R \in Y$ be a member of a singular vertex class not containing P. Let R' be a singularity of X with $f(R') = R$. If R' is in the same vertex class as P', then it follows that $\angle\pi_Y(P) = \angle\pi_Y(R)$, T_Y is isosceles, and f is a composition of covers from Lemma 4.1. If R' and P' are in distinct vertex classes, then either $\angle\pi_Y(R) = \angle\pi_X(R')$, in which case T_Y is a right triangle as described in Lemma 4.1; or else $\pi_Y(R)$ is the apex of an isosceles triangle, and again f is a composition of covers from Lemma 4.1.

Case 3 *The triangle T_X is isosceles, with its apex unfolding to a nonsingular vertex class.*

Here, X must have exactly one other singularity R' corresponding to the other vertex of T_X which is not the apex. Write $f(R') = R$. If $\pi_Y(R)$ or $\pi_Y(P)$ is the apex of an isosceles T_Y, then f is a composition of covers from Lemma 4.1. Suppose not. Then $\angle\pi_Y(P) = \angle\pi_Y(R)$. If $R \neq P$ then $T_X = T_Y$, which is ruled out by Lemma 2.2. So we are left with $R = P$ as the only singularity on Y, and thus by Lemma 4.2 f must be a composition of covers from Lemma 4.1.

Case 4 *The triangle T_Y is isosceles, with its apex unfolding to a singular vertex class.*

Let P be such that $\pi_Y(P)$ is the apex of T_Y. Then $\angle\pi_Y(P) = 2\angle\pi_X(P')$. Furthermore, X must have a singular vertex class not containing P'. Let $R' \in X$ be in this second singular vertex class, and write $f(R') = R$.

Subcase 4A. *R and P share a vertex class.*

Then either $\angle\pi_X(R') = \frac{1}{2}\angle\pi_Y(P) = \angle\pi_X(P')$, in which case T_X is isosceles (see previous cases), or else $\angle\pi_X(R') = \angle\pi_Y(P)$, in which case $T_X = T_Y$, which is impossible.

Subcase 4B. *R and P are in distinct vertex classes.*

Since R does not project to the apex of T_Y, $\angle\pi_X(P') \neq \frac{1}{2}\angle\pi_Y(R)$. If $\angle\pi_X(R') = 2\angle\pi_Y(P)$ then T_X is isosceles and f is a composition of covers from Lemma 4.1. Finally, if $\angle\pi_X(R') = \angle\pi_Y(R)$ then we see that $\angle\pi_X(R') + \angle\pi_X(P') = \angle\pi_Y(R) + \frac{1}{2}\angle\pi_Y(P) = \frac{\pi}{2}$, so T_X is a right triangle which tiles T_Y by a single flip. Thus by Lemma 4.1 there exists a translation cover $g : Y \to X$. If $\deg g = 1$ then Y and X are translation equivalent so $\deg f = 1$. If $\deg g > 1$ then an easy application of the Riemann-Hurwitz formula shows that f cannot exist.

Case 5 *The triangle T_Y is isosceles, with its apex unfolding to a nonsingular vertex class.*

In this case Y has two singular vertex classes, each consisting of one point. Let the singularities be P and R, and as before let $f(P') = P$, $f(R') = R$. By cases 2 and 3, we can assume that T_X is not isosceles; thus $\angle\pi_X(R') = \angle\pi_Y(R) = \angle\pi_Y(P) = \angle\pi_X(P')$. Therefore (since T_X is not isosceles) X must have only one singular vertex class. Thus by Lemma 4.2, $X = X(1, n, n+1)$ with $n > 2$ odd. But this surface has only one singularity, and X must have at least two singularities to form the f-preimage of P and R.

Cases 1-5 exhaust the possibilities; the proof is complete. ∎

4.3 Some Elementary Number Theory

Note that the holonomy field $k_X := \mathbb{Q}(\zeta_Q + \zeta_Q^{-1})$ is a degree two subfield of the cyclotomic field $\mathbb{Q}(\zeta_Q)$, since it is the maximal subfield fixed by complex conjugation. In light of this, we list some classical results about these two fields as recorded in Washington's text[18].

Lemma 4.4 *If Q is odd then $\mathbb{Q}(\zeta_Q) = \mathbb{Q}(\zeta_{2Q})$.*

Lemma 4.5 *(Prop 2.3 in [18]) Assume that $Q \not\equiv 2 \bmod 4$. A prime p ramifies in $\mathbb{Q}(\zeta_Q)$ if and only if $p|Q$.*

Lemma 4.6 *(Prop 2.15 in [18]) Let p be a prime, and assume that $n \not\equiv 2 \bmod 4$. If $n = p^m$ then $\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ is ramified only at the prime above p and at the archimedean primes. If n is not a prime power, then $\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ is unramified except at the archimedean primes.*

Remark 4.3 *Washington's proofs of Lemmas 4.5 and 4.6 make clear that the results carry through to the case $Q \equiv 2 \bmod 4$ except that in that case, the prime 2 does not ramify in $\mathbb{Q}(\zeta_Q)$.*

For a triangular billiards surface $X = X(a_1, a_2, a_3)$, it is tempting to define a "Q-value" for the surface by $Q_X := a_1 + a_2 + a_3$. Unfortunately this notion is not quite well-defined up to translation equivalence; as demonstrated in Lemma 4.1, the triangles $T(a, a, b)$ and $T(2a, b, 2a + b)$ unfold to translation equivalent translation surfaces if (and only if) b is odd. However, the following lemma and its corollary show that this notion is well-defined up to a factor of 2.

Lemma 4.7 *If* $\mathbb{Q}(\zeta_m) \neq \mathbb{Q}(\zeta_n)$ *then* $\mathbb{Q}(\zeta_m + \zeta_m^{-1}) \neq \mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

Proof. This is an exercise in elementary algebraic number theory, and is presumably well known. Let k be the maximal totally real subfield of the cyclotomic fields $\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_n)$ for positive integers $m, n > 2$.

The degrees of $\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_n)$ as field extensions of $\mathbb{Q}$ are $\phi(m)$ and $\phi(n)$ respectively, where ϕ is the Euler totient function. Since $\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_n)$ are each degree 2 extensions of k, we have that $\phi(m) = \phi(n)$.

Let p be an odd prime dividing m. By Lemma 4.5, p ramifies in $\mathbb{Q}(\zeta_m)$. If m is a power of p, then p is totally ramified in $\mathbb{Q}(\zeta_m)$. Since $\mathbb{Q} \subset k \subset \mathbb{Q}(\zeta_m)$, if m is a power of p then p must ramify in k. If m is not a power of p, then Lemma 4.6 tells us that the extension $\mathbb{Q}(\zeta_m)/k$ is not ramified at the prime above p ; thus again p must ramify in k. But also $\mathbb{Q} \subseteq k \subset \mathbb{Q}(\zeta_n)$, so p must ramify in $\mathbb{Q}(\zeta_n)$. By Lemma 4.5, this implies that p divides n. Therefore m and n have the same odd prime divisors; furthermore, by Remark 4.3, these arguments extend to show that either 4 divides both m and n or it divides neither.

First suppose that m and n are congruent modulo 2. Let $m = \Pi p_i^{e_i}$ and $n = \Pi p_i^{f_i}$ be the prime factorizations of m and n. Then we have

$$1 = \frac{\phi(m)}{\phi(n)} = \frac{\prod (p_i - 1) p_i^{e_i - 1}}{\prod (p_i - 1) p_i^{f_i - 1}} = \prod p_i^{e_i - f_i}. \tag{4.1}$$

Therefore $e_i = f_i$ for each i, and $m = n$. Hence in this case $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_n)$.

If m and n are not congruent modulo 2, then we may assume that m is odd and n is congruent to 2 modulo 4. Since $\phi(m) = \phi(2m)$ when m is odd, we can repeat the calculation (4.1) with $2m$ and n, and get that $2m = n$. But it is well known that for any odd m, $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{2m})$. Therefore in fact k is the maximal totally real subfield of only one cyclotomic field. ■

Corollary 4.1 *Suppose that $X(a_1, a_2, a_3)$ and $X(b_1, b_2, b_3)$ have the same holonomy field,* *and that* $b_1+b_2+b_3 < a_1+a_2+a_3$ *. Then* $b_1+b_2+b_3$ *is odd, and* $a_1+a_2+a_3 = 2(b_1+b_2+b_3)$ *.*

Proof. Suppose $X(a_1, a_2, a_3)$ and $X(b_1, b_2, b_3)$ have the same holonomy field k . Write $Q_X = a_1 + a_2 + a_3$ and $Q_Y = b_1 + b_2 + b_3$. Then by Lemma 2.4, we have that k is the maximal totally real subfield of $\mathbb{Q}(\zeta_{Q_X})$ and of $\mathbb{Q}(\zeta_{Q_Y})$. The result then follows directly from Lemma 4.7. ■

4.4 Combinatorial Lemmas

Lemma 4.8 *Let* $f : X(a_1, a_2, a_3) \to X(b_1, b_2, b_3)$ *be a translation cover of triangular billiards surfaces. Then*

$$\sum_{a_i \nmid (a_1+a_2+a_3)} a_i \geq n \sum_{b_i \nmid (b_1+b_2+b_3)} b_i \ . \tag{4.2}$$

Proof. The sum of the cone angles of the singular points of $X(a_1, a_2, a_3)$ is at least n times the sum of the cone angles of the singular points of $X(b_1, b_2, b_3)$. By Remark 2.1, the result follows. ■

Lemma 4.9 *Let* $f : X(a_1, a_2, a_3) \to X(b_1, b_2, b_3)$ *be a translation cover of triangular billiards surfaces such that the genus of* $X(a_1, a_2, a_3)$ *is greater than 1. If* $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ *and* f *is not a composition of covers from Lemma 4.1, then* f *is of degree 1.*

Proof. Write $Q := a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. Let n be the degree of f, and suppose that $n \geq 2$. Lemma 4.8 then gives $\sum_{b_i \nmid Q} b_i \leq \dfrac{Q}{n}$. Hence, since $n \geq 2$, we have

$$\sum_{b_i | Q} b_i \geq \frac{Q}{2}. \tag{4.3}$$

Writing $q_i = \dfrac{Q}{b_i}$, we have the equivalent expression

$$\sum_{b_i|Q} \frac{1}{q_i} \geq \frac{Q}{2}. \tag{4.4}$$

Note that if $b_i|Q$ then q_i is an integer. Of course, Equation (4.3) is always satisfied if $T(b_1, b_2, b_3)$ is a right triangle. If $T(b_1, b_2, b_3)$ is not a right triangle, the equation is rarely satisfied. Thus we will reduce the problem to three cases (up to permutation of vertices).

Case 1 *The triangle $T(b_1, b_2, b_3)$ is not a right triangle.*

In this case, recalling that $\gcd(b_1, b_2, b_3) = 1$, we show that there are only three possibilities for the b_i which satisfy Equation (4.3).

If all three b_i divide Q then Y is nonsingular. The only non-right triangle which unfolds to a nonsingular surface is $T(1,1,1)$; but since this is also the only triangle with $Q = 3$, if $Y = X(1,1,1)$ then $X = X(1,1,1)$, contradicting our assumption that X has a singularity.

Hence we can assume for this case that $b_3 \nmid Q$. Therefore to satisfy Equation 4.4 we seek integers $q_1, q_2 > 2$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} > \frac{1}{2} \tag{4.5}$$

Without loss of generality we assume $q_1 \leq q_2$. If $q_1 \geq 4$, Equation (4.5) is impossible. If $q_1 = 3$ then Equation (4.5) is satisfied if $q_2 \leq 5$. Thus the remaining candidates for Y are $X(3,4,5)$ and $X(3,5,7)$. By Lemma 4.8, $X(3,4,5)$ admits at most a degree two cover; by Lemma 2.1 the degree two covers satisfying the hypotheses of the lemma could only be $f : X(2,5,5) \to X(3,4,5)$ or $X(1,1,10) \to X(3,4,5)$. However, these maps would have to be balanced covers, and $X(3,4,5)$ has a singularity with a Type II fingerprint. Thus by Corollary 3.3 these maps do not exist. Similarly, the only feasible cover of $X(3,5,7)$ of degree greater than 1 is $f : X(1,7,7) \to X(3,5,7)$; again, this would be a balanced cover, and $X(3,5,7)$ has a singularity with a Type II fingerprint.

Case 2 *The triangle $T(b_1,b_2,b_3)$ is a right triangle, with $b_1 = \dfrac{Q}{2}$ and neither b_2 nor b_3 dividing Q.*

Here Lemma 4.8 implies that the degree of f is at most two. The sum of the cone angles of the singularities of Y is $b_2 + b_3$. Thus if $n = 2$ then the sum of the cone angles of the singularities of X is $2(b_2 + b_3) = Q = a_1 + a_2 + a_3$. Therefore $T(a_1,a_2,a_3)$ must be either $T(b_2,b_2,2b_3)$ or $T(2b_2,b_3,b_3)$. Both these possibilities are accounted for by the covers of Lemma 4.1.

Case 3 *The triangle $T(b_1,b_2,b_3)$ is a right triangle, with $b_1 = \dfrac{Q}{2}$ and $b_2|Q$.*

Hence the triangle has angles $\dfrac{\pi}{2}$, $\dfrac{\pi}{q}$, and $\dfrac{q-2}{2q}$ for some integer q dividing Q. We have

$$T(b_1,b_2,b_3) = \begin{cases} T(2,q-2,q) & \text{if} \quad q \text{ odd} \\ & \text{if} \\ T(1,\frac{q}{2}-1,\frac{q}{2}) & \text{if} \quad q \text{ even} \end{cases}$$

First suppose that q is odd. Then $Y = X(2,q-2,q)$. If $q = 3$ then $Y = X(1,2,3)$ and X is either $X(1,2,3)$ (ruled out because it is genus 1) or $X(1,1,4)$ (already listed in Lemma 4.1). If $q = 5$ then by Lemma 2.1 X is either $X(3,3,4)$ (already listed in Lemma 4.1) or $X(1,3,6)$. A translation cover $f : X(1,3,6) \to X(2,3,5)$ would have to be a balanced triple cover, and the fingerprints would not match. For $q \geq 7$, only double covers are possible, by Lemma 4.8. Since $\gcd(q-2,q) = 1$, there is only one singularity on Y and it has cone angle $2(q-2)\pi$. Thus by Lemma 2.1 possible double covers are $f : X(4,q-2,q-2) \to Y$ and $f : X(1,3,2q-4) \to Y$. The covering surfaces

$X(4, q-2, q-2)$ are accounted for by Lemma 4.1. The covering surfaces $X(1, 3, 2q-4)$ have one singular vertex class when $3|q$; in this case f must be balanced. But if $3 \nmid q$ then X would have a conical singularity with cone angle 6π mapping to a nonsingular point of Y, which is impossible since the degree of the cover is at most 2. Now suppose that q is even. If $q = 4$ then $Y = X = X(1, 1, 2)$, but the lemma assumes that X has a singularity. If $q = 6$ then $Y = X(1, 2, 3)$, but we have already dealt with this surface. If $q \geq 8$ then $\gcd(q, \frac{q}{2} - 1) < \frac{q}{2} - 1$, so Y has a singular vertex class and the total cone angle of the singularities in that class is $2(q-2)\pi$. Thus the only possible covering surfaces are $X(2, \frac{q}{2} - 1, \frac{q}{2} - 1,)$ and $X(1, 1, q-2)$; but both these possibilities are accounted for by Lemma 4.1. ■

Lemma 4.10 *Let $f : X \to Y$ be a translation cover of triangular billiards surfaces. Let m be the smallest integer such that all singularities of Y have cone angle at least $2m\pi$. Suppose that $\deg f < m$. Then for each vertex class C_i on X, $f(C_i)$ consists entirely of singular points or entirely of nonsingular points.*

Proof. Let m be as above and assume that $\deg(f) < m$. Suppose for contradiction that for some j, $f(C_j)$ contains singular points and nonsingular points. Each member of C_j has the same cone angle, and this cone angle must be at least $2m\pi$, since some of the members are mapped by a translation cover to a singularity of cone angle $2m\pi$. Thus, for those elements of C_j which are mapped to nonsingular points, the definition of a ramified cover requires that f be locally of degree at least m, which contradicts our assumption that $\deg(f) < m$. This completes the proof. ■

4.5 Proof of the Main Theorem

Now we can prove Theorem 4.1.

Theorem 4.1 *Suppose $f : X \to Y$ is a translation cover of triangular billiards surfaces of degree greater than 1. Then f is of degree 2, and is a composition of one or two of the covers f_i described in Lemma 4.1.*

Proof. Suppose $X := X(a_1, a_2, a_3)$, $Y := X(b_1, b_2, b_3)$, and $f : X \to Y$ is a translation cover of degree $\deg f > 1$. Assume that the genus of X is greater than 1. Write $Q_X := a_1 + a_2 + a_3$ and $Q_Y := b_1 + b_2 + b_3$. Let v_1, v_2, v_3 and w_1, w_2, w_3 be the corresponding vertices of $T(a_1, a_2, a_3)$ and $T(b_1, b_2, b_3)$ respectively. By Corollary 1, X and Y have the same holonomy field k. By Corollary 4.1, we have $Q_Y \in \{2Q_X, Q_X, \frac{1}{2}Q_X\}$. If $Q_Y = 2Q_X$, then by Lemma 4.8, we must have $\sum_{b_i \nmid Q_Y} b_i \leq \frac{Q_X}{2} = \frac{Q_Y}{4}$. But then we would have $\sum_{b_i | Q_Y} b_i \geq \frac{3}{4}Q_Y$, which is only the case for the following surfaces with even Q-value: $X(1,1,2)$, $X(1,2,3)$, $X(3,4,5)$. Of course, $Q_X \geq 3$, so $Y \neq X(1,1,2)$. If $Y = X(1,2,3)$ then $X = X(1,1,1)$, which is of genus 1, a contradiction. If $Y = X(3,4,5)$, then Y has a singularity with cone angle 10π. But, no surface X with $Q_X = 6$ could have a cone angle of at least 10π.

If $Q_Y = Q_X$, then we are done by Lemma 4.9. Thus, appealing to Corollary 4.1, *we shall assume for the remainder of the proof that $Q_X = 2Q_Y$.*

If Y has no singular vertex classes, then since Q is odd, we must have $Y = X(1,1,1)$. There are only two surfaces with a Q-value of 6: they are $X(1,1,4)$ and $X(1,2,3)$, and each of these surfaces covers $X(1,1,1)$ as described in Lemma 4.1. If Y has three singular vertex classes, then Lemma 4.8 implies that f can only be a degree two balanced cover. Thus we are done by Lemma 4.3.

There are two cases remaining: Y may have either one or two singular vertex classes.

Case 1 *The surface Y has one singular vertex class.*

In this case we have, without loss of generality, $b_1 | Q_Y$, $b_2 | Q_Y$, and $b_3 \nmid Q_Y$. Since b_1 and b_2 are divisors of the odd number $Q_Y := b_1 + b_2 + b_3$, b_3 must also be odd. Therefore

$\dfrac{b_3}{\gcd(b_3, Q)} \geq 3$. The cone angle at each of the singularities of Y corresponding to b_3 is $\dfrac{b_3}{\gcd(b_3, Q)} 2\pi \geq 6\pi$.

Lemma 4.8 eliminates all possible Y for $\deg f \geq 4$ except $Y = X(3,5,7)$. But, again by Lemma 4.8, the only possible degree four covering surface would be $X(1,1,28)$, and such a cover would have to be balanced, contradicting Lemma 4.3.

<u>If $\deg f = 2$:</u> Lemma 4.10 tells us that if $\deg f = 2$ then for each $j = 1, 2, 3$, we have that $f(\pi_X^{-1}(v_j)) \cap \pi_Y^{-1}(w_3)$ is either empty or all of $f(\pi_X^{-1}(v_j))$.

Suppose that $Y = X(3,5,7)$. Lemma 4.10 restricts the possible degree two covers to surfaces of the form $f : X(14, a_2, a_3) \to Y$, where each of a_2 and a_3 is either a divisor of 30 or twice a divisor of 30. The only possible covering surface this leaves is $X(15,14,1)$. But any translation cover $f : X(15,14,1) \to X(3,5,7)$ would have to be balanced, so Lemma 4.3 applies.

Now suppose that $Y \neq X(3,5,7)$. Let C be the singular vertex class of Y. We must have $\dfrac{b_3}{Q} > \dfrac{1}{2}$, and so by Remark 2.1 C must correspond to an obtuse angle θ of the billiard table. Let $\tilde{X}$ be the surface obtained from X by puncturing all singular vertex classes of X which are not contained in $f^{-1}(C)$. Since $\dfrac{b_3}{Q} > \dfrac{1}{2}$ and f is degree 2, the sum of the angles of the billiard table corresponding to the vertex classes in the f-preimage of C must be obtuse. Thus we can apply Lemma 3.3 to $\tilde{X}$. The restriction of f to $\tilde{X}$ is balanced. Since Y has only one singular vertex class, elements of C must have Type II fingerprints unless $T(b_1, b_2, b_3)$ is isosceles. If the fingerprints are Type II, then Proposition 3.1 and Lemma 3.3 demonstrate that X and Y are translation equivalent. So the only possibility is that the fingerprints are Type I. In that case Y is an isosceles triangular billiards surface. Let C' be a vertex class on X that is in $f^{-1}(C)$, and write $\theta = \dfrac{b_3\pi}{Q}$. The billiard table angle that C' corresponds to is either θ or $\dfrac{\theta}{2}$. If the angle is θ, then X and Y are translation equivalent. If the angle is $\dfrac{\theta}{2}$, then there is another vertex class on X which is also mapped to C. But then that vertex class would also correspond to an angle of $\dfrac{\theta}{2}$, and we would have that X is an isosceles triangular billiards surface, implying that $f : X \to Y$ is of the form described in Lemma 4.1.

If $\deg(f) = 3$: Then Lemma 4.8 allows only the following possibilities for Y: the surfaces

$$Y_n = \begin{cases} X(3, n, 2n-3) & 3 \nmid n \\ \\ X(1, \frac{n}{3}, \frac{2n}{3} - 1) & 3 | n \end{cases}.$$

Note that $\gcd(2n-3, 3n) \in \{1, 3\}$. First suppose that $\gcd(2n-3, 3n) = 1$. Then $Q = 3n$ (thus n is odd), $3 \nmid n$, and we have $Y_n = X(3, n, 2n-3)$. We have that $n \geq 5$ and hence that $2n - 3 \geq 7$. On Y_n, there is only one singular vertex class and the cone angle of each singular point is $(2n-3)2\pi$. Thus Lemma 4.10 applies here. Since Y_n is never isosceles, each singular point has a Type II fingerprint. Let $\tilde{X}$ be the surface obtained from X by deleting all singularities of X which f maps to nonsingular points, and let $\tilde{f}$ be the restriction of f to $\tilde{X}$. By Lemma 4.10, the elements of $X - \tilde{X}$ are the union of entire vertex classes. Thus a Type II fingerprint on $\tilde{X}$ will uniquely identify the triangular billiards table used to generate X, by Lemma 3.3. Because $\tilde{f}$ is a balanced map, each singular point of $\tilde{X}$ must have the same Type II fingerprint (on $\tilde{X}$) as its $\tilde{f}$-image on Y. But, a Type II fingerprint uniquely identifies the triangle used to generate the surface (this works for $\tilde{X}$ as well); hence X and Y_n are the same billiards surface, and Lemma 2.2 says that a triple cover is impossible.

Now suppose that $\gcd(2n-3, 3n) = 3$. Then the cone angle of each singular point on Y_n is $\frac{2n-3}{3}2\pi$. If $n > 6$ then $\dfrac{2n-3}{3} > 3$, so that again we can apply Lemma 4.10 and Lemma 3.3, and the same fingerprint argument goes through. The remaining cases are $n = 3, 6$. We have $Y_3 = X(1, 1, 1)$ and $Y_6 = X(1, 2, 3)$, neither of which have singularities.

Case 2 *The surface Y has two singular vertex classes.*

Assume $b_1 | Q$ and $b_2, b_3 \nmid Q$. Since Q is odd, $\dfrac{b_1}{Q} \leq \dfrac{1}{3}$, so Lemma 4.8 implies that $\deg(f) \leq 3$. But, if $\deg(f) = 3$, Lemma 4.8 also implies that f is balanced, contradicting the result of Lemma 4.3 that balanced covers are of degree at most 2. Thus $\deg(f) = 2$.

Note that b_2 and b_3 must have the same parity.

Subcase 2A. *Both b_2 and b_3 are odd.*

Then $\dfrac{b_i}{\gcd(b_i, Q)} \geq 3$, so by Lemma 4.10, each vertex class of X maps to all singular points or all nonsingular points.

<u>If one vertex class of X maps to nonsingular points:</u> Say the vertex class C_1 corresponding to a_1 maps to nonsingular points. Then $a = 2b_1$, and $2b_1 | 2Q$, so C_1 is nonsingular, so f is balanced.

<u>If two vertex classes of X map to nonsingular points:</u> Let them be C_1 and C_2, corresponding to a_1 and a_2. If C_1 is singular, then by Lemma 4.10 we have $a_1 = 2d$ for some $d | Q$. But since $a_3 = 2(b_2 + b_3)$, this would mean that all the a_i are even, contradicting the fact that $\gcd(a_1, a_2, a_3) = 1$.

Subcase 2B. *Both b_2 and b_3 are even.*

<u>If one vertex class of X maps to nonsingular points:</u> Let it be C_1. We have $a_2 + a_3 = 2(b_2 + b_3)$, so a_1 must be even. But also a_2 and a_3 must be even, since $2 | \dfrac{b_i}{\gcd(b_i, Q)}$ and $\dfrac{b_i}{\gcd(b_i, Q)} | \dfrac{a_j}{\gcd(a_j, Q)}$ for each $i, j \in \{2, 3\}$. Again, this is a contradiction.

<u>If two vertex classes of X map to nonsingular points:</u> Let them be C_1 and C_2. We have that $a_3 = 2(b_2 + b_3)$ is even. If C_1 is singular then again we have that a_1 (and hence a_2) is even, once more contradicting that $\gcd(a_1, a_2, a_3) = 1$. Hence C_1 and C_2 are nonsingular, and f is balanced. ∎

5 ALGEBRAIC PERIODICITY

The purpose of this note is to provide an alternate proof of the claim, originally stated in [3], that surfaces arising from billiards in a rational triangle are algebraically periodic. The proof of the claim there contains two minor errors. First, letting ζ_Q denote $e^{2\pi i/Q}$, it assumes that the coordinates of the vertices of the $\left(\frac{a_1\pi}{Q}, \frac{a_2\pi}{Q}, \frac{a_3\pi}{Q}\right)$ triangles in its construction are contained in $\mathbb{Q}(\zeta_Q)$, when in fact they are only guaranteed to be in $\mathbb{Q}(\zeta_{2Q})$. Second, on a related note, it assumes that the real and imaginary parts of elements of the field $\mathbb{Q}(\zeta_Q)$ lie in the field $\mathbb{Q}(\zeta_Q + \zeta_Q^{-1})$. In general, the imaginary parts may lie in a degree 2 extension of $\mathbb{Q}(\zeta_Q + \zeta_Q^{-1})$. Examples of this already occur when $Q = 3, 5$. However, these issues can be resolved by a simple geometric argument, as we show in the proof of Lemma 5.1.

Remark 5.1 *We let U_n denote the n^{th} Chebyshev polynomial of the second kind. We will use the following properties of Chebyshev polynomials.*

1. $\frac{\sin((n+1)\theta)}{\sin\theta} = U_n(\cos\theta)$
2. *If n is even, then U_n is an even polynomial of degree n. If n is odd, then U_n is an odd polynomial of degree n.*

Remark 5.2 *Let ϕ be the Euler totient function. It is well known that, for any positive integer Q, the degree of the number field $\mathbb{Q}(\cos(\frac{2\pi}{Q}))$ is equal to $\frac{1}{2}\phi(Q)$. Note that if Q is odd, then $\phi(Q) = \phi(2Q)$. It follows that, when Q is odd, we will have $\mathbb{Q}(\cos(\frac{2\pi}{Q})) = \mathbb{Q}(\cos(\frac{\pi}{Q}))$.*

The following is Theorem 2.5 of [3].

Theorem 5.1 *(Calta-Smillie) If a translation surface X is obtained by identifying the edges of polygons in the plane by maps which are restrictions of translations, and if all the* *vertices of these polygons lie in a subgroup $\Lambda \subset \mathbb{R}^2$, then the holonomy of S is contained in Λ.*

Lemma 5.1 *The holonomy field of $X(a_1, a_2, a_3)$ is contained in $\mathbb{Q}(\cos(\frac{2\pi}{Q}))$, where $Q = \gcd(a_1, a_2, a_3)$.*

Proof. Let $\alpha = \frac{\pi}{Q}$. Let $T := T(a_1, a_2, a_3)$. Since $\gcd(a_1, a_2, a_3) = 1$, we can and do assume that a_1 is odd. Label the vertices of T corresponding to the angles $a_1\alpha$, $a_2\alpha$, and $a_3\alpha$ as P_1, P_2, and P_3. We scale and rotate T so that the $\overline{P_1P_2}$ side has edge vector $v = (1, 0)$, and so that the $\overline{P_1P_3}$ side has edge vector $w = (t\cos(a_1\alpha), t\sin(a_1\alpha))$, where by the Law of Sines we have $t = \dfrac{\sin(a_2\alpha)}{\sin(a_3\alpha)}$. The dihedral group D generated by reflections in the sides of T acts on the set $D \cdot T$ of $2Q$ distinct oriented triangles arising from billiards in T. We can construct X from this set by identifying the appropriate edges of the elements of $D \cdot T$. We may also view D as acting on the edge vectors of T. Let $v_n = (\cos(2n\alpha), \sin(2n\alpha))$ and $w_n = (t\cos((2n+1)\alpha), t\sin((2n+1)\alpha))$. With this notation, we see that $D \cdot v$ is the set $\{v = v_0, v_1, ..., v_{Q-1}\}$. Recalling that a_1 is odd, we also see that $D \cdot w$ is the set $\{w_0, w_1, ..., w_{Q-1}\}$. Note that $w = w_{a_1/2-1}$.

Let Λ be the subgroup of $\mathbb{R}^2$ generated by the v_n and w_n. Theorem 5.1 implies that the entire holonomy of S is contained in Λ.

Let $L = \mathbb{Q}(\cos 2\alpha)$. We will show that all the v_n and w_n are L-linear combinations of v_0 and v_1, and that furthermore L is the smallest such field.

Let l and l' be the real numbers such that $lv_0 + l'v_1 = w_0$. Since v_0 and v_1 are reflections of each other across the line generated by w_0, we see that $v_0 + v_1$ is a real multiple of w_0. Hence $l' = l$.

Projecting v_0 and v_1 onto w_0, we see that

$$l = \frac{||w_0||}{||v_0 + v_1||} = \frac{t}{2\cos\alpha} = \frac{\sin(a_2\alpha)\sin\alpha}{\sin(a_3\alpha)\sin(2\alpha)} = \frac{\sin(a_2\alpha)}{\sin\alpha}\frac{\sin\alpha}{\sin(a_3\alpha)}\frac{\sin\alpha}{\sin(2\alpha)}. \tag{5.1}$$

Applying Remark 5.1 to the last expression, we get

$$l = \frac{U_{a_2-1}(\cos\alpha)}{U_{a_3-1}(\cos\alpha)U_1(\cos\alpha)}. \tag{5.2}$$

If Q is even, we have that $(a_2 - 1)$ and $(a_3 - 1)$ have opposite parity, and thus by our Remark 5.1, $\frac{U_{a_2-1}(\cos\alpha)}{U_{a_3-1}(\cos\alpha)U_1(\cos\alpha)}$ is a rational function in $\cos^2\alpha$. Thus $l \in \mathbb{Q}(\cos^2\alpha) = \mathbb{Q}(\cos(2\alpha))$. If Q is odd, then already by Remark 5.2, $\mathbb{Q}(\cos\alpha) = L$, and since $\frac{U_{a_2-1}(\cos\alpha)}{U_{a_3-1}(\cos\alpha)U_1(\cos\alpha)}$ is a rational function in $\cos\alpha$, we again have that $l \in L$.

Similarly, for some real number k, we have $k(w_0+w_1)=v_1$; by projection we calculate

$$k = \frac{||v_1||}{||w_0 + w_1||} = \frac{1}{t\cos(2\alpha)} = \frac{1}{t^2}l = \frac{\sin^2(a_3\alpha)}{\sin^2(a_2\alpha)}l = \frac{U^2_{a_3-1}}{U^2_{a_2-1}}l. \tag{5.3}$$

Since $U^2_{a_3-1}$ and $U^2_{a_2-1}$ are both polynomials in $\cos^2\alpha$, we get $k \in L$.

Let R be the element of D that acts on the plane as counterclockwise rotation by 2α. Note that $R \cdot v_n = v_{n+1}$ and $R \cdot w_n = w_{n+1}$. Thus for all integers n,

$$lv_n + lv_{n+1} = R^n \cdot (lv_0 + lv_1) = R^n \cdot w_0 = w_n \tag{5.4}$$

and

$$kw_n + kw_{n+1} = R^n \cdot (kw_0 + kw_1) = R^n \cdot v_1 = v_{n+1}. \tag{5.5}$$

Thus we have the relations $w_n = (\frac{1}{k} - l)v_n - lv_{n-1}$ and $v_{n+1} = \frac{1}{l}w_n - v_n$. These two relations demonstrate that w_n and v_n are in $\mathrm{span}_L\{v_0, v_1\}$ for all n. Hence $\mathrm{span}_L\{v_0, v_1\}=\Lambda$.

Theorem 5.1 says that Λ contains the absolute holonomy of S. Hence L contains the holonomy field of X. ∎

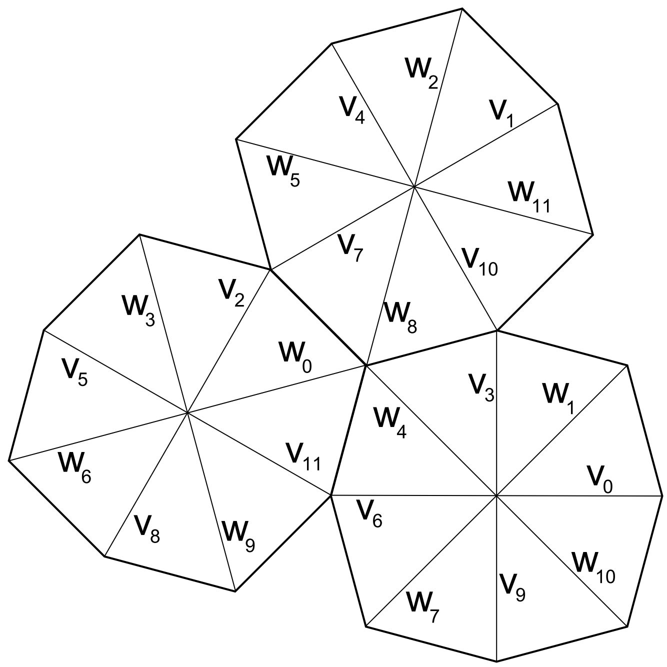

FIGURE 5.1: The sets $\{v_n\}$ and $\{w_n\}$ for $X(3,4,5)$, with $a_1 = 3$.

The following is a slight strengthening of Theorem 9.1 in [Calta-Smillie], which follows from the proof of Theorem 1.2 of [Calta-Smillie], which they in turn attribute to McMullen.

Theorem 5.2 *(Calta-Smillie) If there is an affine automorphism of S with trace θ and the holonomy field of S is contained in a field generated by θ, then S is completely algebraically periodic.*

The following theorem is stated as Theorem 1.4 in [Calta-Smillie].

Theorem 5.3 *(Calta-Smillie) If X is a triangular billiards surface then X is completely algebraically periodic.*

Proof. The surface X admits rotation by $\frac{2\pi}{Q}$ as an affine automorphism. This automorphism has trace $2\cos(\frac{2\pi}{Q})$. In Lemma 5.1 we showed that the holonomy field of X is contained in the field generated by $\cos(\frac{2\pi}{Q})$. Hence, by Theorem 5.2, X is algebraically periodic. ■

6 INFINITELY GENERATED VEECH GROUPS VIA TRANSLATION COVERS

In this chapter we discuss the use of translation covers in constructing translation surfaces with infinitely generated affine symmetry groups (called Veech groups). We review the relevant definitions, present results of Hubert and Schmidt, then demonstrate that members of a special class of surfaces identified by Aurell and Itzykson in [1] have infinitely generated Veech groups. Throughout this chapter, we shall use the notation (S, ω) to refer to a translation surface, where S is the underlying Riemann surface and ω is the holomorphic one-form which endows X with a translation structure, as described in Section 2.2.1.

6.1 Veech Groups and Veech Surfaces

The matrix group $SL_2\mathbb{R}$ acts on the set of all translation surfaces in the following way: for each $A \in SL_2\mathbb{R}$, $A \cdot X$ is the result of post-composing the coordinate charts of X with the standard linear action of A on $\mathbb{R}^2$. See, for example, [10]. Note that, since A acts linearly on the charts of (X, ω), the change-of-coordinate functions of $A \cdot X$ will be translations, so $SL_2\mathbb{R}$ really does act on the set of translation surfaces.

Definition 6.1 *Let $SL(X)$ be the $SL_2\mathbb{R}$-stabilizer of X . The* Veech group *of X is the image of $SL(X)$ in $PSL_2\mathbb{R}$, denoted $PSL(X)$.*

A diffeomorphism of X whose image is a translation surface is called an *affine diffeomorphism*. Elements of $SL(X)$ can also be viewed as the differentials of those affine diffeomorphisms whose images are translation equivalent to X . It is a common abuse of notation to let a matrix $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ refer to both an element of $SL(X)$ and its image in $PSL(X)$.

The hyperbolic upper half plane $\mathcal{H}$ admits an action by $PSL_2\mathbb{R}$ via Möbius transformations. If the quotient of $\mathcal{H}$ by the action of $PSL(X)$ has finite hyperbolic area, then we say that X is a *Veech surface*. Veech [15] gave the following result, known as the Veech Dichotomy:

Theorem 6.1 *(Veech) If X is a Veech surface, then for each direction θ, either:*

(1) X decomposes into a finite number of cylinders in the direction θ with commensurable moduli; or

(2) Each geodesic path in the direction θ is uniformly distributed in X .

Here, the *modulus* of a cylinder refers to the ratio of its width to its height.

6.2 Techniques of Hubert and Schmidt

In [9], Hubert and Schmidt use the fact that there exists a translation cover $f : X(3,3,4) \to X(1,1,3)$ to prove that the Veech group of $X(3,3,4)$ is infinitely generated. In this section we review their tools, so that we can apply them to a different surface in Section 6.4.

Definition 6.2 *A point P on a translation surface X is a* connection point *if every geodesic connecting P to a singularity of X extends to be a saddle connection on X .*

A direction is said to be a *periodic direction* on X if every geodesic on X in that direction is closed.

A direction is called a *parabolic direction* of X if there exists an affine diffeomorphism of X which preserves the set of geodesics in this direction and whose differential is parabolic (has trace equal to 2). A consequence of the Veech Dichotomy is that, on a Veech surface, the parabolic directions coincide with the periodic directions.

Definition 6.3 *A translation surface X is of* strong holonomy type *if the following conditions hold:*

(1) Every holonomy vector and every saddle connection vector of X has its x-and y-coordinates in the holonomy field of X.

(2) The periodic directions of X are exactly the vertical and those directions whose slopes are in the holonomy field of X.

Definition 6.4 *A point P on a translation surface X is a* rational point *if there exist two distinct parabolic directions for X with corresponding parabolic elements of $SL(X)$ that fix P.*

Lemma 6.1 *(Hubert-Schmidt) For P a nonsingular point on a Veech surface X of strong holonomy type, the following are equivalent:*

1. *P is a connection point;*
2. *P is a rational point;*
3. *after the development of a singular point has been fixed as the origin, every developed image of P is of coordinates in the holonomy field.*

Hubert and Schmidt mark certain nonsingular points $\{P_1, ..., P_n\}$ on a translation surface X, call the resulting marked surface $(X; P_1, ..., P_n)$, and then define the Veech group of the resulting surface to be those elements of $PSL(X)$ which stabilize the set of marked points. On the marked surface, the points $\{P_1, ..., P_n\}$ are considered to be (removable) singularities. Note that $(X; P_1, ..., P_n)$ is still a translation surface.

Proposition 6.1 *(Hubert-Schmidt) Let P be a nonperiodic connection point on a Veech surface X. Then $PSL(X; P)$ is infinitely generated.*

Sketch of Proof. Hubert and Schmidt show in [9] that it suffices to prove that the parabolic directions of $PSL(X; P)$ are dense in the unit circle S^1. It is well known that the set of directions of geodesics connecting any point on X to singularities on X is dense in S^1. Since P is a connection point, the set of directions of saddle connections through P must be dense in S^1. Because X is a Veech surface, each such direction is a periodic direction on X and hence there exists a parabolic element $\tau \in SL(X)$ which is the differential of an affine automorphism of X fixing P. Since τ fixes P, $\tau \in SL(X; P)$. Therefore the parabolic directions of $PSL(X; P)$ are dense in S^1. ■

Proposition 6.1 has the following immediate corollary, which we will use in Section 6.4. Although Hubert and Schmidt do not explicitly state this corollary, they do implicitly use it in [9].

Corollary 6.1 *Let $P_1, ..., P_n$ be nonperiodic connection points on a Veech surface X such that the set of directions of saddle connections through P_i is the same for each i. Then $PSL(X; P_1, ..., P_n)$ is infinitely generated.*

Proof. Hubert and Schmidt's proof of Proposition 6.1 goes through for this additionally marked surface as long as we can show that the set of parabolic directions of $(X, P_1, ..., P_n)$ is dense in S^1. But since the set of directions of saddle connection through P_i is the same for each i, each direction in this set corresponds to a parabolic element of $PSL(X)$ which is the differential of an automorphism fixing the set $\{P_1, ..., P_n\}$. ■

We will use the following result of McMullen, which he proves in [14] and which Hubert and Schmidt [9] restate in the following way.

Lemma 6.2 *(McMullen) If the holonomy field of a translation surface X is a real quadratic extension of $\mathbb{Q}$, then X is in the $GL_2(\mathbb{R})$-orbit of a nonarithmetic surface of strong holonomy type.*

The following lemma, proven independently by Vorobets [16] and Gutkin and Judge [6], demonstrates a connection between translation covers and Veech groups.

Lemma 6.3 *(Vorobets, Gutkin-Judge) If $f : X \to Y$ is a balanced cover of translation surfaces, then there exist subgroups $H \in PSL(X)$ and $G \in PSL(Y)$ such that H and G are $PSL_2(\mathbb{R})$-conjugate.*

This lemma has the following corollary which will be important in the next section:

Corollary 6.2 *Let Y be a Veech surface and let $\{P_1, ..., P_n\} \subset Y$ be a set of nonperiodic connection points such that the set of directions of saddle connections through P_i is the same for each i. Let $f : X \to Y$ be a translation cover which is ramified above each P_i and is not ramified above any other nonsingular points of Y. Then $PSL(X)$ is infinitely generated.*

Proof. Because f ramifies only above the points $\{P_1, ..., P_n\}$ as well as possibly above the singular points of Y, f induces a balanced translation cover $f' : X \to (Y; P_1, ..., P_n)$. By Corollary 6.1, $PSL(Y; P_1, ..., P_n)$ is infinitely generated. By Lemma 6.3, there must exist subgroups $H_1 \in PSL(Y; P_1, ..., P_n)$ and $H_2 \in PSL(X)$ which are $PSL_2\mathbb{R}$-conjugate. But a finite-index subgroup of an infinitely generated group must itself be infinitely generated; hence H_1 is infinitely generated, and its conjugate H_2 is therefore also infinitely generated. Likewise, a finite group extension of an infinitely generated group must also be infinitely generated. Thus $PSL(Y)$ is infinitely generated. ∎

Hubert and Schmidt [9] implicitly use the preceding corollary, along with the fact that $f : X(3, 3, 4) \to X(1, 1, 3)$ is a translation cover ramified over nonperiodic connection points of the Veech surface $X(1,1,3)$, to prove that $PSL(X(3,3,4))$ is infinitely generated. We shall prove something similar for a special collection of surfaces in the following section.

6.3 The Aurell-Itzykson Construction

In [1], Aurell and Itzykson show that for a given triangular billiards surface (S,ω) of genus g, there exists a basis $\{\omega = \omega_1, \omega_2, ..., \omega_g\}$ for $H^1(X;\mathbb{C})$ such that each (X,ω_i) is either a triangular billiards surface or a covering surface of a triangular billiards surface via a nontrivial translation cover. The various ω_i are called the *associates* of ω, and by analogy we call the surfaces (X,ω_i) *associate surfaces* of (S,ω). Using translation cover techniques of Hubert and Schmidt [9], as well as results of Ward [17] and McMullen [14], we can show that certain of these surfaces have an infinitely generated Veech group.

For our purposes, the results of Aurell and Itzykson in [1] regarding associates can be summarized as follows:

Proposition 6.2 *(Aurell-Itzykson) Let $(S,\omega) := X(a_1,a_2,a_3)$ be a triangular billiards surface of genus g, with $Q := a_1+a_2+a_3$. For any integer m, let $\overline{m}$ denote the nonnegative remainder when dividing m by Q. Let $n \in \{1,2,...,Q\}$ such that $\overline{na_1}+\overline{na_2}+\overline{na_3} = Q$. Let $t = \gcd(\overline{na_1},\overline{na_2},\overline{na_3})$. Then there exists a 1-form γ defined on X such that there is a degree t translation cover $f:(X,\gamma) \to X(\frac{\overline{na_1}}{t},\frac{\overline{na_2}}{t},\frac{\overline{na_3}}{t})$. Each such γ is called an associate of ω. Furthermore, there are exactly g such values of n.*

For each $n \in \{1,2,...,Q\}$ such that $\overline{na_1}+\overline{na_2}+\overline{na_3} = Q$, we shall refer to the associate surface (S,γ) as $X(\overline{na_1},\overline{na_2},\overline{na_3})$.

6.4 Aurell-Itzykson Surfaces With Infinitely Generated Veech Group

In this section, we use techniques of Hubert and Schmidt to show that $X(n, 2n, 7n)$ is infinitely generated. Note that the surface $X(1, 2, 7)$ can be realized as the union of two pentagons and a decagon with appropriate sides identified, as illustrated in Figure 6.1.

In [17], Ward calculates that (the images in $PSL_2(\mathbb{R})$ of) the matrices

$$\tau := \begin{pmatrix} 1 & \cot\frac{\pi}{10} + \cot\frac{\pi}{5} \\ 0 & 1 \end{pmatrix} \text{ and } \rho := \begin{pmatrix} \cos\frac{\pi}{5} & -\sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & \cos\frac{\pi}{5} \end{pmatrix}$$

form a generating set for the Veech group of $X(1, 2, 7)$. The presence of τ in the Veech group reflects the fact that $X(1, 2, 7)$ admits a "Dehn twist" along each maximal vertical cylinder which fixes the boundaries of the cylinders. In Figure 6.1, the maximal cylinder containing one of the pentagon centers P is shaded. By an argument involving Dehn twists, if the width of this cylinder is not rationally related to the distance from P to the left edge of the enclosing cylinder, then P has infinite orbit under the action of τ, and hence P is a nonperiodic point. Here, a quick application of trigonometry reveals that the ratio of these two quantities is

$$\frac{\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5}}{\cos\frac{2\pi}{5}} = 1 - \frac{\cos\frac{4\pi}{5}}{\cos\frac{2\pi}{5}} = 1 + \frac{\frac{1}{4}(\sqrt{5}+1)}{\frac{1}{4}(\sqrt{5}-1)} = 2 + \frac{1}{3}\sqrt{5}. \tag{6.1}$$

Hence, the ratio is not rational and P is a nonperiodic point.

Next we show that P is a rational point of X . A consequence of the Veech Dichotomy is that the direction θ of any saddle connection on a Veech surface is the direction of a decomposition of the surface into cylinders with commensurable moduli. The saddle connection must be on the boundary of one of the cylinders, and hence it will be fixed by an element of the Veech group which corresponds to a Dehn twist in the direction θ . Now consider the horizontal geodesics on $X(1, 2, 7)$; there is clearly a horizontal saddle connection on $X(1,2,7)$ which runs through P and the center of the other pentagon. Furthermore, because of the 10-fold rotational symmetry of $X(1,2,7)$, there must be at least four other saddle connections running through the pentagon centers. Therefore there are at least five unique parabolic elements of the Veech group which fix P; we conclude that P is a rational point on $X(1,2,7)$.

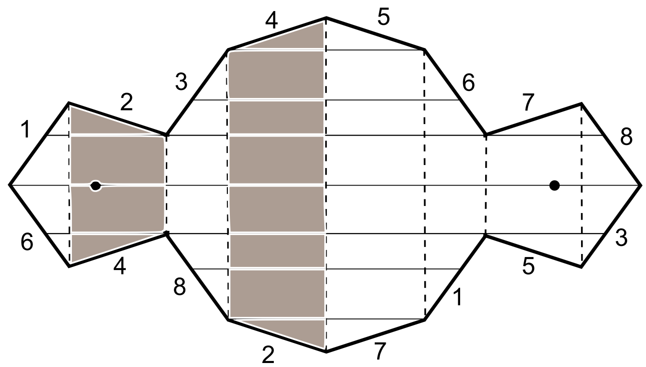

FIGURE 6.1: Horizontal (solid) and vertical (dotted) cylinders for X(1,2,7). A vertical cylinder containing a pentagon center is shaded.

Finally, we show that P is a connection point. The holonomy field of $X(1,2,7)$ is $\mathbb{Q}(\sqrt{5})$, so Lemma 6.2 implies that there exists an $A \in GL_2(\mathbb{R})$ such that $A \cdot X(1,2,7)$ is of strong holonomy type. Let $\tau_1, \tau_2 \in SL(X(1,2,7))$ be (derivatives of) Dehn twists in distinct directions such that both twists fix P . Then $A\tau_1A^{-1}$ and $A\tau_2A^{-1}$ are elements of $SL(A \cdot X(1,2,7))$ which fix $A \cdot P$. Hence $A \cdot P$ is a rational point on $A \cdot X(1,2,7)$. Thus, by Lemma 6.1, $A \cdot P$ is a connection point on $A \cdot X(1,2,7)$. Since A acts linearly on the charts of $X(1,2,7)$, its action is a bijection between the set of saddle connections on $X(1,2,7)$ and the set of saddle connections on $A \cdot X(1,2,7)$. Thus, P must be connection point on $X(1,2,7)$.

We summarize this discussion in the following lemma:

Lemma 6.4 *Viewing $X(1,2,7)$ as the union of two pentagons and a decagon with appro-*

priate edges identified, as in Figure 6.1, the centers of the two pentagons are nonperiodic connection points.

Proposition 6.3 *For each integer $n > 1$, the surface $X(1,2,10n-3)$ has an associate surface $X(n,2n,7n)$ which admits a ramified n-fold translation cover of $f : X(n,2n,7n) \to X(1,2,7)$, and which has an infinitely generated Veech group. The genus of each $X(n,2n,7n)$ is*

$$g_n = \begin{cases} 5n-1 & , \quad 3 \nmid n \\ 5n-2 & , \quad 3|n \end{cases}$$

Proof. The triple $(1,2,10n-3)$ has, via multiplication by n modulo $10n$, the associate triple $(n,2n,\overline{10n^2-3n}) = (n,2n,7n)$. Therefore, by Proposition 6.2, the triangular billiards surface $X(1,2,10n-3)$ has an associate surface $X(n,2n,7n)$ which admits a degree n translation cover of $X(1,2,7)$. Since $X(1,2,10n-3)$ and $X(n,2n,7n)$ are translation surfaces with the same underlying topological space, they have the same genus. Therefore, by Remark 2.1, the genus is

$$5n+1-\frac{1}{2}(\gcd(1,10n)+\gcd(2,10n)+\gcd(10n-3,10n) = 5n-\frac{1}{2}(1-\gcd(10n-3,10n)),$$

which is either $5n-2$ or $5n-1$ depending on whether or not 3 divides n. We write $X(1,2,7) = (Y,\alpha)$ and $X(n,2n,7n) = (S,\omega)$. Let $f : X(n,2n,7n) \to X(1,2,7)$ be the translation cover given in [1]. Let $p : X(1,2,7) \to \mathbb{C} \cup \{\infty\}$ and $p' : X(n,2n,7n) \to \mathbb{C} \cup \{\infty\}$ be the covers of the Riemann sphere guaranteed by the Aurell-Itzykson construction. We have that $p' = p \circ f$. A consequence of the construction is that p can only ramify at vertices of the triangular billiards triangulations of $X(1,2,7)$, and that p' can only ramify at f-preimages of these vertices. But, if f ramifies above a point $P \in X(1,2,7)$, then since $p' = p \circ f$, p' must ramify above $p(P)$; hence P must be a vertex of the billiards triangulation of $X(1,2,7)$.

Suppose $n = 2$. Then applying the Riemann-Hurwitz formula to the translation cover $f : X(2,4,14) \to X(1,2,7)$, we have that $9 = 2(3) + 1 + \frac{R}{2}$, where R is the total ramification number of f. Hence $R = 4$. Since the ramification number of f above a single point of $X(1,2,7)$ cannot exceed $n-1$, we see that f must ramify above all four elements of the vertex classes of $X(1,2,7)$. Thus, in particular, f must ramify above the centers of the pentagons in the flat diagram of $X(1,2,7)$ in Figure 6.1.

Now suppose $n > 2$. The genus of $X(n,2n,7n)$ is at least $5n-2$, so this time the Riemann-Hurwitz formula tells us that f has a total ramification number at least $4n-6$. For $n > 2$, we thus have $4n-6 > 2(n-1)$, so again f must ramify above at least one of the pentagon centers.

Ward shows in [17] that $X(1,2,7)$ is a Veech surface. Since $X(1,2,7)$ is Veech, and the pentagon centers are nonperiodic connection points, it now follows from Corollary 6.2 that $PSL(X(n,2n,7n))$ is infinitely generated. ■

7 CONCLUSION

The guiding problem for Chapters 2 through 5 in this thesis was the classification of all translation covers between triangular billiards surfaces. We solved this problem by identifying two types of data about such surfaces: the fingerprint of a point, which is essentially local data; and the holonomy field of a surface, which is a more global piece of information. The fingerprint was sufficient to complete the smaller classification of all balanced covers; uniqueness and invariance results such as Lemma 3.2, Corollary 3.2 and Proposition 3.1 were key there. We finished the complete classification by also considering the holonomy fields of surfaces.

Hubert and Schmidt used the existence of a translation cover $f : X(3,3,4) \to X(1,1,3)$ to prove that the Veech group of $X(3,3,4)$ is infinitely generated; Theorem 4.1 shows that such covers are fairly rare.

An obvious extension of this work would be to apply the same two tools to the consideration of translation covers among larger families of translation surfaces. For example, any rational polygonal billiards surface possesses rotational symmetry with respect to any vertex of its billiards triangulation; hence, the fingerprints of such vertex points will give nontrivial data about the surfaces involved. Note that the cardinalities of the angle sets may be larger than two, unlike the triangular case. Therefore combinatorial arguments along the lines of this thesis would be more complicated.

Similarly, the calculation of the holonomy field of a billiards surface of a rational polygon is more difficult, in general, than the triangular case, and yields more generic results. Indeed, such a field need not even be a number field; this is connected to the fact that such surfaces need not be completely algebraically periodic.

A slightly different extension of this thesis would be to classify all translation covers of triangular billiards surfaces. As we demonstrated in Chapter 6, ramified translation covers $f : X \to Y$ in which Y is a triangular billiards surface but X is not can yield examples of interesting Veech groups. It could be interesting to know if the Aurell-Itzykson surfaces described in Chapter 6 give a special subset of these covers.

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