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Title: A PREDICTOR-CORRECTOR METHOD FOR THE TRANSIENT MOTION OF A NONHOMOGENEOUS, INCOMPRESSIBLE, VISCOUS FLUID

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Giles W. Maloof
[redacted]
Larry S. Slotta

The system of partial differential equations which governs the motion of a Newtonian fluid has been known for over a century. Yet, due to the complexity of the equations, an analytical solution is known only for a few simple geometries or a few special cases such as very slow motion. No general analytical solution is known.

The development of the digital computer has led to sophisticated numerical techniques for solving systems of partial differential equations. This paper develops a predictor-corrector numerical method which solves simultaneously the Navier-Stokes equation, the continuity equation, the incompressibility equation and the energy equation --first law of thermodynamics--for a nonhomogeneous, viscous fluid.

As an example of the potential of this method, the problem of finding the transient flow from a density stratified reservoir is solved. In addition a FORTRAN IV listing for this problem is included.

A Predictor-Corrector Method for the Transient Motion of a Nonhomogeneous Incompressible, Viscous Fluid

by

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[redacted]

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[redacted]

Associate Professor of Civil Engineering

in charge of project

[redacted]

Acting Chairman of Department of Mathematics

[redacted]

Dean of Graduate School

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Nomenclature

Independent Variables

x = horizontal distance

y = vertical distance

t = time

Fluid Variables

$\vec{w}$ = $u\vec{i} + v\vec{j}$ = velocity

ρ = density

P = pressure

T = temperature

S = entropy per unit mass

Fluid Parameters

μ = viscosity

k = thermal conductivity

C_v = specific heat per unit mass at constant volume

α = k/C_v

Forces and Stresses Acting on the Fluid

$\vec{F}$ = $X\vec{i} + Y\vec{j}$ = external force per unit volume

Π = stress tensor

σ_n = normal component of the stress at the surface

σ_m = tangential component of the stress at the surface

Numerical Parameters

δx = mesh width in the x direction

δy = mesh width in the y direction

δt = time increment

Auxiliary Variables in the Numerical Solution

ξ = variable used in the u and P calculation

ζ = variable used in the v and P calculation

$B^{(1-4)}$ = pressure coefficients in the pressure relaxation

A = source term in the pressure relaxation

Subscripts

x = x-component of a two dimensional quantity

y = y-component of a two dimensional quantity

i = numerical value at $x = i\delta x$

j = numerical value at $y = j\delta y$

k = numerical value associated with the k^{th} particle

Superscripts

n = numerical value in the n^{th} time cycle

' = indicates normalized independent variable

A PREDICTOR-CORRECTOR METHOD FOR THE TRANSIENT MOTION OF A NONHOMOGENEOUS INCOMPRESSIBLE, VISCOUS FLUID

I. INTRODUCTION

Fluid flow is generally described using one of the following view-points.

Eulerian: Attention can be focused on some point in space and the changes in the fluid can be described as functions of time at this point.

Lagrangian: Attention can be focused on an infinitesimal fluid element and the changes in this fluid element can be expressed as functions of time.

Although the latter method of description bears the name of J. L. Lagrange both were developed by Leonhard Euler. Major analytical works in fluid dynamics use one or both of these viewpoints; correspondingly numerical techniques have developed along these lines.

The early papers on numerical techniques for fluid problems (Harlow, 1955; Evans and Harlow, 1957) used the Lagrangian viewpoint. Instead of considering every infinitesimal fluid element, attention was focused on a finite number of these elements. By marking the elements being considered, the fluid was conveniently represented by an array of particles. This representation by particles is the primary feature of all Lagrangian numerical techniques; the fluid properties such as density and velocity are localized to a finite number of particles which move with the fluid.

Later (Langley, 1959; Welch et al., 1966) an Eulerian technique was developed for fluid problems. Instead of considering the fluid at all spatial points, attention was focused at a finite number of fixed points. Eulerian numerical techniques are characterized by finding the values of the fluid variables at the mesh points of a fixed grid.

It was shown by Welch et al. (1966) that a system containing two discrete fluids could be handled using a mixed Eulerian-Lagrangian scheme. In this scheme the velocity and pressure were considered as Eulerian variables and found at the mesh points of a fixed grid. The density was considered a Lagrangian variable and was localized to fluid particles. With the addition of the Eulerian variable temperature this same mixed Eulerian-Lagrangian scheme is suited to the problem of this thesis: finding the transient motion of a nonhomogeneous fluid with continuous density and temperature profiles.

II. THE SYSTEM OF EQUATIONS

In this chapter the physical laws that govern the motion of a fluid will be presented. The approximations for an incompressible fluid undergoing an adiabatic change will then be derived. For notational and computational convenience it is assumed that the motion is two dimensional so that the independent variables will be x, y, and t.

The Physical Laws

To describe completely the motion of a Newtonian fluid it is necessary to determine the six unknowns:

velocity, $\vec{w} = u\vec{i} + v\vec{j}$;

density, ρ;

pressure, P;

viscosity, μ;

temperature, T; and

entropy per unit mass, S.

Thus, six equations relating these six unknowns are needed. These equations are mathematical expression of the following physical laws:

conservation of mass,

conservation of momentum,

first law of thermodynamics,

second law of thermodynamics,

law of liquid viscosity, and

equation of state.

These laws are given mathematical formulation by:

1. The continuity equation (Schlichting, 1960),

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0; \tag{2.1}$$

2. The Navier-Stokes equation (Welch _et al._, 1966),

$$\rho \frac{D\vec{w}}{Dt} = \rho \vec{F} - \nabla P + 2(\nabla \cdot \mu \nabla)\vec{w} + \nabla x(\mu \nabla x \vec{w}), \quad \underline{1}/ \tag{2.2}$$

where $\vec{F} = X\vec{i} + Y\vec{j}$ is the external force per unit volume acting on the fluid;

3. The first law of thermodynamics (Schlichting, 1960),

$$\rho C_v \frac{DT}{Dt} + P(\nabla \cdot \vec{w}) = \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \mu\phi, \tag{2.3}$$

where C_v is the specific heat per unit mass at constant volume, k is the thermal conductivity, and ϕ is the dissipation expressed as

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[1] The material derivative $\frac{D}{Dt}$ is the time derivative following a fluid particle, namely $\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + (\vec{w} \cdot \nabla)\psi$.

$$\phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 ;$$

4. The second law of thermodynamics (Streeter, 1961),

$$T \frac{D}{Dt}(\rho s) = \frac{\partial}{\partial x}\left(\frac{k \partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{kT}{\partial y}\right) + \mu \phi ; \tag{2.4}$$

5. The liquid viscosity equation (Bird, 1962),

$$\mu = \frac{A}{V(T)} e^{-BT} , \tag{2.5}$$

where $V(T)$ is the volume occupied by a mole of liquid at temperature T and A and B are empirically determined constants for the liquid.

6. The equation of state is a relation between P, ρ, and T. In general for a liquid it must be determined empirically. For an ideal gas it has the familiar form

$$P = \rho R T \tag{2.6}$$

where R is the ideal gas constant.

The Approximations

The general incompressibility equation (Yih, 1965) is

$$\frac{D\rho}{Dt} = 0 \ . \tag{2.7}$$

This can be expressed using the definition of the material derivative as

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0.$$

Similarly, Equation (2. 1) can be expanded to

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

The preceding two equations may be subtracted and the difference divided by ρ (which never vanishes for a real fluid) to obtain an equation which is generally called the continuity equation for incompressible fluids.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla\cdot\vec{w} = 0 \tag{2.8}$$

For an incompressible fluid, temperature variations are small enough that the viscosity, μ, and the thermal conductivity, k, may be considered constant (Schlichting, 1960). The assumption of constant μ allows the following simplification of the last two terms of Equation (2. 2):

$$2(\nabla\cdot\mu\nabla)\vec{w} + \nabla x(\mu\nabla x\vec{w}) = 2\mu\nabla^2\vec{w} + \mu[\nabla(\nabla\cdot\vec{w}) = \nabla^2\vec{w}]$$

$$= \mu\nabla^2\vec{w} + \mu\nabla(\nabla\cdot\vec{w}).$$

This can be further simplified for incompressible fluids $(\nabla \cdot \vec{w} = 0)$ to

$$2(\nabla \cdot \mu \nabla)\vec{w} + \nabla x(\mu \nabla x \vec{w}) = \mu \nabla^2 \vec{w}.$$

Thus, the assumptions of incompressibility and constant viscosity reduce the Navier-Stokes equation to

$$\rho \frac{D\vec{w}}{Dt} = \rho \vec{F} - \nabla P + \mu \nabla^2 \vec{w}.$$

For the finite differencing of the above equation which takes place in the next chapter, it is convenient to use the following vector and tensor identities:

$$\nabla \cdot (k\vec{a}) = (\nabla k) \cdot \vec{a} + k(\nabla \cdot \vec{a})$$

$$\nabla \cdot (\vec{a}\,\vec{b}) = (\vec{a} \cdot \nabla)\vec{b} + (\nabla \cdot \vec{a})\vec{b}$$

to rewrite the first term:

$$\rho \frac{D\vec{w}}{Dt} = \rho \frac{\partial \vec{w}}{\partial t} + \rho(\vec{w} \cdot \nabla)\vec{w}$$

$$= \frac{\partial}{\partial t}(\rho \vec{w}) + \nabla \cdot (\rho \vec{w}\,\vec{w}).$$

Thus, the final form for the Navier-Stokes equation becomes

$$\frac{\partial}{\partial t}(\rho \vec{w}) + \nabla \cdot (\rho \vec{w}\,\vec{w}) = \rho \vec{F} - \nabla P + \mu \nabla^2 \vec{w}. \quad (2.9)$$

The assumption that the thermal conductivity is constant reduces the first law of thermodynamics (2. 3) to

$$\rho C_v \frac{DT}{Dt} + P(\nabla \cdot \vec{w}) = k \nabla^2 T + \mu \phi.$$

The incompressibility assumption reduces it further to

$$\rho C_v \frac{DT}{Dt} = K \nabla^2 T + \mu \phi.$$

In the next chapter an algorithm will be developed to solve the system of partial differential equations now under consideration. One of the conditions for the accuracy of this algorithm will be that $|\vec{w}| << C_s$, C_s being the local sound speed. When $|\vec{w}| << C_s$ it is shown by Welch et al. (1966) that $\mu \phi << k \nabla^2 T$. Thus, the first law of thermodynamics assumes its final form:

$$\rho C_v \frac{DT}{Dt} = k \nabla^2 T. \qquad (2.10)$$

An incompressible fluid undergoing an adiabatic motion has constant entropy. For an incompressible fluid the incompressibility equation (2. 6) replaces the equation of state. Thus, the assumptions of incompressibility, adiabaticity, constant viscosity, and constant thermal conductivity reduce the original six equations in six unknowns (Equations (2. 1) - (2. 6)) to four equations in four unknowns:

$$\frac{D\rho}{Dt} = 0, \tag{2.7}$$

$$\nabla \cdot \vec{w} = 0, \tag{2.8}$$

$$\frac{\partial}{\partial t}(\rho \vec{w}) + \nabla \cdot (\rho \vec{w}\,\vec{w}) = \rho \vec{F} - \nabla P + \mu \nabla^2 \vec{w} \tag{2.9}$$

$$\frac{DT}{Dt} = \frac{k}{\rho C_v} \nabla^2 T \tag{2.10}$$

In the sequel these four partial differential equations will be solved according to the boundary conditions derived below.

Scaling the Equations

When solving equations numerically, it is frequently desirable that the variables have magnitudes less than one. The equations (2.7) - (2.10) can be scaled by the tranformation of variables

$$x = Lx'$$

$$y = Ly'$$

$$t = \frac{L}{W}t'$$

$$\vec{w} = W\vec{w}'$$

$$\rho = R\rho'$$

$$P = RW^2P'$$

$$T = \theta T'$$

The incompressibility equation (2.7) becomes

$$\frac{WR}{L}\left(\frac{\partial\rho'}{\partial t'} + u'\frac{\partial\rho'}{\partial x'} + v'\frac{\partial\rho}{\partial y'}\right) = 0.$$

Or,

$$\frac{D'}{Dt'}\rho' = 0. \quad \text{2/} \tag{2. 11}$$

The continuity equation (2. 8) becomes

$$\frac{W}{L}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0.$$

Thus,

$$\nabla'\cdot\vec{w}' = 0\,. \tag{2. 12}$$

Similarly, the Navier-Stokes equation becomes

$$\frac{RW^2}{L}\left[\frac{\partial}{\partial t'}(\rho'\vec{w}') + \nabla'\cdot(\rho'\vec{w}'\vec{w}')\right] = \rho' R\vec{F} - \frac{RW^2}{L}\nabla' P' + \frac{W}{L^2}\mu\,\nabla'^2\vec{w}'.$$

Multiplying through by L/RW^2 and setting $\vec{F}' = L\vec{F}/W^2$ and $\mu' = \mu/LRW$ gives

$$\frac{\partial}{\partial t'}(\rho'\vec{w}') + \nabla'\cdot(\rho'\vec{w}'\vec{w}') = \rho'\vec{F}' - \nabla' P' + \mu'\nabla'^2\vec{w}'. \tag{2. 13}$$

Finally, the temperature equation is expressed as

$$\frac{\partial T'}{\partial t'} + (\vec{w}'\cdot\nabla')T' = \frac{\alpha'}{\rho'}\nabla'^2 T', \tag{2. 14}$$

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[2] $\frac{D'}{Dt'} = \frac{\partial}{\partial t'} + (\vec{w}\cdot\nabla')$, and

$\nabla' = \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}$.

where $\alpha' = k/RL\theta WC_v$. Thus the equations to be solved have the same form before and after scaling. Hereafter, it will be assumed that the equations have been scaled appropriately and the primes on Equations (2.11) - (2.14) will be dropped.

The Boundary Conditions

There are two basic types of boundary conditions to consider:

1. Conditions occurring at a material boundary,
2. Conditions occurring at a free surface.

A boundary of the fluid is a material boundary if no fluid can pass through it. A free surface separates the fluid from a vacuum.

The system of partial differential equations is solved subject to these boundary conditions:

1. At a material boundary the normal and tangential components of the velocity vanish. The normal derivative of the temperature vanishes.
2. At a free surface the normal and tangential components of the stress vanish. The normal derivative of the temperature vanishes.

Let $s(x, y, t) = 0$ be the entire fluid surface. In general s may contain both material boundaries and free surfaces. The unit vector normal to s is defined

$$\vec{n} = \frac{\nabla s}{|\nabla s|} = \frac{\frac{\partial s}{\partial x}\vec{i} + \frac{\partial s}{\partial y}\vec{j}}{\sqrt{(\frac{\partial s}{\partial x})^2 + (\frac{\partial s}{\partial y})^2}} \quad .$$

Thus, we may express $\vec{n}$ as

$$\vec{n} = n_x \vec{i} + n_y \vec{j} \tag{2.15}$$

where $n_x = \frac{\frac{\partial s}{\partial x}}{|\nabla s|}$ and $n_y = \frac{\frac{\partial s}{\partial y}}{|\nabla s|}$.

The unit vector tangent to s is any vector of unit length which is a solution to $\vec{n} \cdot \vec{m} = 0$. In order that $\vec{n}$ and $\vec{m}$ form a right handed coordinate system choose

$$\vec{m} = - n_y \vec{i} + n_x \vec{j} . \tag{2.16}$$

The velocity $\vec{w}$ can be expressed

$$\vec{w} = (\vec{w} \cdot \vec{n})\vec{n} + (\vec{w} \cdot \vec{m})\vec{m} .$$

The normal derivative of T is

$$\frac{\partial T}{\partial n} = \vec{n} \cdot \nabla T .$$

Thus, boundary condition 1 is expressed in equations as:

$$\vec{w} \cdot \vec{n} = 0, \tag{2.17}$$

$$\vec{w} \cdot \vec{m} = 0, \tag{2.18}$$

$$\vec{n} \cdot \nabla T = 0. \tag{2.19}$$

The stress $\vec{\sigma}$ at a point on a free surface with normal $\vec{n}$ is given by (Yuan, 1967)

$$\vec{\sigma} = \Pi \cdot \vec{n}.$$

Here Π is the stress tensor

$$\Pi = \begin{pmatrix} \tau_{xx}\vec{i}\,\vec{i} & \tau_{xy}\vec{i}\,\vec{j} \\ \tau_{yx}\vec{j}\,\vec{i} & \tau_{yy}\vec{j}\,\vec{j} \end{pmatrix}$$

Therefore, $\vec{\sigma}$ is given by

$$\vec{\sigma} = \begin{pmatrix} \tau_{xx}\vec{i}\,\vec{i} & \tau_{xy}\vec{i}\,\vec{j} \\ \tau_{yx}\vec{j}\,\vec{i} & \tau_{yy}\vec{j}\,\vec{j} \end{pmatrix} \begin{pmatrix} n_x\vec{i} \\ n_y\vec{j} \end{pmatrix}$$

$$= (n_x\tau_{xx}+n_y\tau_{xy})\vec{i} + (n_x\tau_{yx}+n_y\tau_{yy})\vec{j}.$$

Since Equations (2.15) and (2.16) can be solved for i and j to yield

$$\vec{i} = n_x\vec{n} - n_y\vec{m},$$

$$\vec{j} = n_y\vec{n} + n_x\vec{m},$$

$\vec{\sigma}$ can be expressed,

$$\vec{\sigma} = (n_x T_{xx} + n_y T_{xy})(n_x \vec{n} - n_y \vec{m}) + (n_x T_{yx} + n_y T_{yy})(n_y \vec{n} + n_x \vec{m})$$

setting

$$\vec{\sigma} = \sigma_n \vec{n} + \sigma_m \vec{m}$$

$$\sigma_n = (n_x^2 T_{xx} + n_x n_y T_{xy} + n_x n_y T_{yx} + n_y^2 T_{yy})$$

$$\sigma_m = (-n_x n_y T_{xx} - n_y^2 T_{xy} + n_x^2 T_{yx} + n_x n_y T_{yy}).$$

In general for a Newtonian fluid (Schlichting, 1960),

$$\Pi = \begin{pmatrix} -P & 0 \\ 0 & -P \end{pmatrix} + \mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} + \mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} - \frac{2}{3}\mu \begin{pmatrix} \nabla \cdot \vec{w} & 0 \\ 0 & \nabla \cdot \vec{w} \end{pmatrix}.$$

For an incompressible fluid the last bracketed term vanishes.

Substituting the components for Π into the equation for σ_n,

$$\sigma_n = n_x^2(-P + 2\mu \frac{\partial u}{\partial x}) + 2n_x n_y \mu (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + n_y^2(-P + 2\mu \frac{\partial v}{\partial y}) .$$

Using the condition that $n_x^2 + n_y^2 = 1$, this may be rewritten as

$$\sigma_n = -P + 2n_x^2 \mu \frac{\partial u}{\partial x} + 2n_x n_y \mu (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2n_y^2 \mu \frac{\partial v}{\partial y} .$$

Similarly, if the components of Π are substituted into the equation for σ_m the result is

$$\sigma_m = 2n_x n_y \mu \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + (n_x^2 - n_y^2)\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) .$$

Thus, boundary condition 2 is expressed by setting σ_n and σ_m equal to zero.

$$P = 2n_x^2 \mu \frac{\partial u}{\partial x} + 2n_x n_y \mu\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + 2n_y^2 \mu \frac{\partial v}{\partial y} ; \qquad (2.20)$$

$$2n_x n_y \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + (n_x^2 - n_y^2)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 . \qquad (2.21)$$

At the free surface the normal derivative of the temperature vanishes so Equation (2. 19) must also be satisfied.

In addition to boundary conditions 1 and 2 the following initial conditions must be satisfied:

1. The initial density field must be given.
2. The initial temperature field must be given.

III. THE NUMERICAL METHOD

The general method of solution of the system of partial differential equations (2.11) - (2.14) will be to represent the continuous variables x, y, and t as multiples of δx, δy, and δt. Then the partial differential equations can be approximated by finite difference equations and solved numerically for $\vec{w}$, ρ, P, and T at $x = i\delta x$, $y = j\delta y$, and $t = n\delta t$ for integer values of i, j, and n.

The Difference Equations

It was found by Welch *et al.* (1966) that some differencing schemes were more accurate than others. The following scheme reportedly gives the most accurate solution of the Navier-Stokes equation (2.13). The fluid is covered by a double Eulerian grid as in Figure 1. The variables P, ρ, and T take values at the mesh points of the grid represented by the dashed lines. The variables u and v are found at the intersection of the dashed and solid grids. The rectangular regions marked off by the solid grid are called cells. Thus, the variables ρ, P, and T are defined at the center of a cell while u is defined at the sides and v is defined at the top and bottom of each cell.

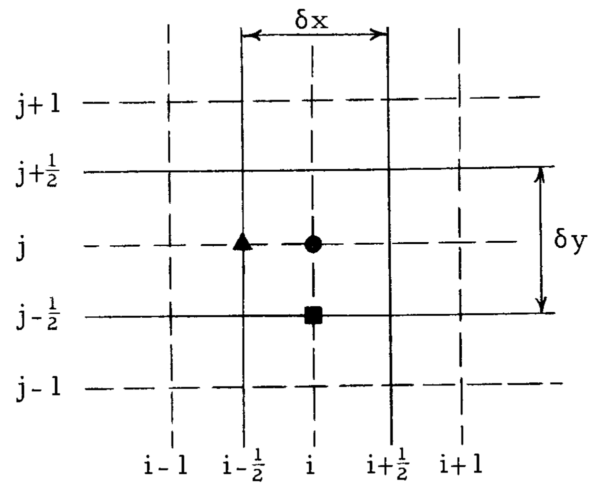

Figure 1. The double Eulerian mesh with the locations of the fluid variables indicated by, ●: ρ, P, T; ▲: u; ■: v.

If $x = i\delta x$, $y = j\delta y$, and $t = n\delta t$ then it is possible to represent the variables as functions of the integers i, j, and n. Thus, in Figure 1 for $t = n\delta t$

$$\rho(x, y, t) = \rho^n_{ij}$$ [3]

$$P(x, y, t) = P^n_{ij}$$

$$u(x, y, t) = u^n_{i-\frac{1}{2}j}$$

$$v(x, y, t) = v^n_{ij-\frac{1}{2}}$$

To express the partial differential equations in finite difference form, the following difference operators are used

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[3] The subscript notation used in this paper is the same as Welch et al. (1966). The comma between the first and second subscript is deleted.

$$\frac{\partial \phi}{\partial x} \longleftrightarrow \frac{\phi_{i+\frac{1}{2}j} - \phi_{i-\frac{1}{2}j}}{\delta x} \quad \text{or} \quad \frac{\phi_{i+1j} - \phi_{ij}}{\delta x}$$

$$\frac{\partial \phi}{\partial y} \longleftrightarrow \frac{\phi_{ij+\frac{1}{2}} - \phi_{ij-\frac{1}{2}}}{\delta y} \quad \text{or} \quad \frac{\phi_{ij+1} - \phi_{ij}}{\delta y}$$

depending on whether ϕ is placed at the center or side of the cell respectively. Moreover,

$$\frac{\partial \phi}{\partial t} \longleftrightarrow \frac{\phi^{n+1} - \phi}{\delta t}$$

where the superscript $n+1$ indicates the value of the variable for $t = (n+1)\delta t$ and the absence of the superscript indicates the value at $t = n\delta t$.

The system of Equations (2.11) - (2.14) can be written in finite difference form as follows:

$$\frac{\rho_{ij}^{n+1} - \rho_{ij}}{\delta t} + u_{ij} \frac{\rho_{i+1j} - \rho_{ij}}{\delta x} + v_{ij} \frac{\rho_{ij+1} - \rho_{ij}}{\delta y} = 0 ; \tag{3.1}$$

$$\frac{u_{i+\frac{1}{2}j} - u_{i-\frac{1}{2}j}}{\delta x} + \frac{u_{ij+\frac{1}{2}} - u_{ij-\frac{1}{2}}}{\delta y} = 0 ; \tag{3.2}$$

$$(\rho u)^{n+1}_{i+\frac{1}{2}j} = \xi_{i+\frac{1}{2}j} + \frac{\delta t}{\delta x}(P_{ij} - P_{i+1j}) , \tag{3.3}$$

where

$$\xi_{i+\frac{1}{2}j} = (\rho u)_{i+\frac{1}{2}j} - \delta t \left[\frac{(\rho u^2)_{i+1j} - (\rho u^2)_{ij}}{\delta x} + \frac{(\rho uv)_{i+\frac{1}{2}j+\frac{1}{2}} - (\rho uv)_{i+\frac{1}{2}j-\frac{1}{2}}}{\delta y} \right.$$

$$- (\rho X)_{i+\frac{1}{2}j} - \mu \left(\frac{u_{i+\frac{3}{2}j} - 2u_{i+\frac{1}{2}j} + u_{i-\frac{1}{2}j}}{\delta x^2} \right)$$

$$\left. - \mu \left(\frac{u_{i+\frac{1}{2}j+1} - 2u_{i+\frac{1}{2}j} + u_{i+\frac{1}{2}j-1}}{\delta y^2} \right) \right] ;$$

$$(\rho v)^{n+1}_{ij+\frac{1}{2}} = \zeta_{ij+\frac{1}{2}} + \frac{\delta t}{\delta y} (P_{ij} - P_{ij+1}) , \tag{3.4}$$

where

$$\zeta_{ij+\frac{1}{2}} = (\rho v)_{ij+\frac{1}{2}} - \delta t \left[\frac{(\rho v^2)_{ij+1} - (\rho v^2)_{ij}}{\delta y} + \frac{(\rho uv)_{i+\frac{1}{2}j+\frac{1}{2}} - (\rho uv)_{i-\frac{1}{2}j+\frac{1}{2}}}{\delta x} \right.$$

$$\left. - (\rho Y)_{ij+\frac{1}{2}} - \mu \left(\frac{v_{i+1j+\frac{1}{2}} - 2v_{ij+\frac{1}{2}} + v_{i-1j+\frac{1}{2}}}{\delta x^2} \right) - \mu \left(\frac{v_{ij+\frac{3}{2}} - 2v_{ij+\frac{1}{2}} + v_{ij-\frac{1}{2}}}{\delta y^2} \right) \right] ;$$

and

$$\frac{T^{n+1}_{ij} - T_{ij}}{\delta t} + \frac{(uT)_{i+\frac{1}{2}j} - (uT)_{i-\frac{1}{2}j}}{\delta x} + \frac{(vT)_{ij+\frac{1}{2}} - (vT)_{ij-\frac{1}{2}}}{\delta y}$$

$$= \frac{\alpha}{\rho_{ij}} \left(\frac{T_{i+1y} - 2T_{ij} + T_{i-1j}}{\delta x^2} + \frac{T_{ij+1} - 2T_{ij} + T_{ij-1}}{\delta y^2} \right) . \tag{3.5}$$

For computational purposes it is convenient to put this system of equations in a slightly different form. Equation (3. 3) can be solved for $u^{n+1}_{i+\frac{1}{2}j}$;

$$u^{n+1}_{i+\frac{1}{2}j} = \frac{\xi_{i+\frac{1}{2}j}}{\rho^{n+1}_{i+\frac{1}{2}j}} + \frac{\partial t}{\delta x} \frac{(P_{ij} - P_{i+1j})}{\rho^{n+1}_{i+\frac{1}{2}j}} . \tag{3.6}$$

Similarly,

$$u^{n+1}_{i-\frac{1}{2}j} = \frac{\xi_{i-\frac{1}{2}j}}{\rho^{n+1}_{i-\frac{1}{2}j}} + \frac{\delta t}{\delta x}\frac{(P_{i-1j}-P_{ij})}{\rho^{n+1}_{i-\frac{1}{2}j}} \quad . \tag{3.7}$$

For the v components

$$v^{n+1}_{ij+\frac{1}{2}} = \frac{\zeta_{ij+\frac{1}{2}}}{\rho^{n+1}_{ij+\frac{1}{2}}} + \frac{\delta t}{\delta y}\frac{(P_{ij}-P_{ij+1})}{\rho^{n+1}_{ij+\frac{1}{2}}} , \tag{3.8}$$

$$v^{n+1}_{ij-\frac{1}{2}} = \frac{\zeta_{ij-\frac{1}{2}}}{\rho^{n+1}_{ij-\frac{1}{2}}} + \frac{\delta t}{\delta y}\frac{(P_{ij-1}-P_{ij})}{\rho^{n+1}_{ij-\frac{1}{2}}} \quad . \tag{3.9}$$

If Equations (3. 6) - (3. 9) are substituted into the continuity equation (3. 2) for $t = (n+1)\delta t$, the result is

$$\frac{1}{\delta x}\left[\left(\frac{\xi_{i+\frac{1}{2}j}}{\rho^{n+1}_{i+\frac{1}{2}j}} - \frac{\xi_{i-\frac{1}{2}j}}{\rho^{n+1}_{i-\frac{1}{2}j}}\right) + \frac{\delta t}{\delta x}\left(\frac{P_{ij}-P_{i+1j}}{\rho^{n+1}_{i+\frac{1}{2}j}} - \frac{P_{i-1j}-P_{ij}}{\rho^{n+1}_{i-\frac{1}{2}j}}\right)\right]$$

$$+ \frac{1}{\delta y}\left[\left(\frac{\zeta_{ij+\frac{1}{2}}}{\rho^{n+1}_{ij+\frac{1}{2}}} - \frac{\zeta_{ij-\frac{1}{2}}}{\rho^{n+1}_{ij-\frac{1}{2}}}\right) + \frac{\delta t}{\delta x}\left(\frac{P_{ij}-P_{ij+1}}{\rho^{n+1}_{i+\frac{1}{2}j}} - \frac{P_{ij-1}-P_{ij}}{\rho^{n+1}_{ij-\frac{1}{2}}}\right)\right] = 0.$$

This may be put in the form

$$P_{ij} = B^1_{ij}P_{i+1j} + B^2_{ij}P_{i-1j} + B^3_{ij}P_{ij+1} + B^4_{ij}P_{ij-1} + A_{ij} \; . \tag{3.10}$$

The coefficients are given by

$$A_{ij} = \frac{1}{C_{ij}}\left[\frac{1}{\delta x}\left(\frac{\xi_{i+\frac{1}{2}j}}{\rho^{n+1}_{i+\frac{1}{2}j}} - \frac{\xi_{i-\frac{1}{2}j}}{\rho^{n+1}_{i-\frac{1}{2}j}}\right) + \frac{1}{\delta y}\left(\frac{\zeta_{ij+\frac{1}{2}}}{\rho^{n+1}_{ij+\frac{1}{2}}} - \frac{\zeta_{ij-\frac{1}{2}}}{\rho^{n+1}_{ij-\frac{1}{2}}}\right)\right],$$

$$B^1_{ij} = \frac{1}{C_{ij}}\frac{\delta t}{\delta x^2}\frac{1}{\rho^{n+1}_{i+\frac{1}{2}j}},$$

$$B^2_{ij} = \frac{1}{C_{ij}}\frac{\delta t}{\delta x^2}\frac{1}{\rho^{n+1}_{i-\frac{1}{2}j}},$$

$$B^3_{ij} = \frac{1}{C_{ij}}\frac{\delta t}{\delta y^2}\frac{1}{\rho^{n+1}_{ij+\frac{1}{2}}},$$

$$B^4_{ij} = \frac{1}{C_{ij}}\frac{\delta t}{\delta y^2}\frac{1}{\rho^{n+1}_{ij-\frac{1}{2}}},$$

and

$$C_{ij} = \frac{\delta t}{\delta x^2}\left(\frac{1}{\rho^{n+1}_{i+\frac{1}{2}j}} + \frac{1}{\rho^{n+1}_{i-\frac{1}{2}j}}\right) + \frac{\delta t}{\delta y^2}\left(\frac{1}{\rho^{n+1}_{ij+\frac{1}{2}}} + \frac{1}{\rho^{n+1}_{ij-\frac{1}{2}}}\right).$$

Equations (3. 6) and (3. 8) were modified slightly because they led to physically unrealizable results. For fluids at rest they implied that the gravitational and buoyant on a fluid element did not balance. This effect was due to the density gradient in the fluid and was rectified by setting

$$u^{n+1}_{i+\frac{1}{2}j} = \frac{\xi_{i-1\frac{1}{2}j}}{\rho^{n+1}_{i+\frac{1}{2}j}} + \frac{\delta t}{\delta x}\left(\frac{P_{ij}}{\rho^{n+1}_{ij}} - \frac{P_{i+1j}}{\rho^{n+1}_{i+1j}}\right), \tag{3. 11}$$

$$v^{n+1}_{ij+\frac{1}{2}} = \frac{\zeta_{ij+\frac{1}{2}}}{\rho^{n+1}_{ij-\frac{1}{2}}} + \frac{\delta t}{\delta y}\left(\frac{P_{ij}}{\rho^{n+1}_{ij}} - \frac{P_{ij+1}}{\rho^{n+1}_{ij+1}}\right). \tag{3. 12}$$

For computational purposes it is convenient to put Equations (3.1) and (3.5) in a slightly different form.

$$\rho_{ij}^{n+1} = \rho_{ij} - \delta t\left(u_{ij}\frac{\rho_{i+1j}-\rho_{ij}}{\delta x} + v_{ij}\frac{\rho_{ij+1}-\rho_{ij}}{\delta y}\right) \tag{3.13}$$

$$T_{ij}^{n+1} = T_{ij} - \delta t\left[\frac{(uT)_{i+\frac{1}{2}j}-(uT)_{i-\frac{1}{2}j}}{\delta x} + \frac{(vT)_{ij+\frac{1}{2}}-(vT)_{ij-\frac{1}{2}}}{\delta y} + \frac{a}{\rho_{ij}}\left(\frac{T_{i+1j}-2T_{ij}+T_{i-1j}}{\delta x^2} + \frac{T_{ij+1}-2T_{ij}+T_{ij-1}}{\delta y^2}\right)\right]. \tag{3.14}$$

The system of equations which is actually solved is

$$\rho_{ij}^{n+1} = \rho_{ij} - \delta t\left(u_{ij}\frac{\rho_{i+1j}-\rho_{ij}}{\delta x} + v_{ij}\frac{\rho_{ij+1}-\rho_{ij}}{\delta y}\right), \tag{3.13}$$

$$u_{i+\frac{1}{2}j}^{n+1} = \frac{\xi_{i+\frac{1}{2}j}}{\rho_{i+\frac{1}{2}j}^{n+1}} + \frac{\delta t}{\delta x}\left(\frac{P_{ij}}{\rho_{ij}^{n+1}} - \frac{P_{i+1j}}{\rho_{i+1j}^{n+1}}\right), \tag{3.11}$$

$$v_{ij+\frac{1}{2}}^{n+1} = \frac{\zeta_{ij+\frac{1}{2}}}{\rho_{ij+\frac{1}{2}}^{n+1}} + \frac{\delta t}{\delta y}\left(\frac{P_{ij}}{\rho_{ij}^{n+1}} - \frac{P_{ij+1}}{\rho_{ij+1}^{n+1}}\right), \tag{3.12}$$

$$P_{ij} = B_{ij}^{1}P_{i+1j} + B_{ij}^{2}P_{i-1j} + B_{ij}^{3}P_{ij+1} + B_{ij}^{4}P_{ij-1} + A_{ij}, \tag{3.10}$$

and

$$T_{ij}^{n+1} = T_{ij} - \delta t\left[\frac{(uT)_{i+\frac{1}{2}j}-(uT)_{i-\frac{1}{2}j}}{\delta x} + \frac{(vT)_{ij+\frac{1}{2}}-(vT)_{ij-\frac{1}{2}}}{\delta y} + \frac{a}{\rho_{ij}}\left(\frac{T_{i+1j}-2T_{ij}+T_{i-1j}}{\delta x^2} + \frac{T_{ij+1}-2T_{ij}+T_{ij-1}}{\delta y^2}\right)\right]. \tag{3.14}$$

The Algorithm

The following eight steps are the basis of the computer program which solves Equations 3.10 through 3.14. A flow chart is given in Appendix A.

1. The density distribution for $t = (n+1)\delta t$ is calculated from the solution for $t = n\delta t$ using Equation (3.13).

2. The pressure field is calculated roughly by relaxing Equation (3.10) starting from the pressure field for $t = n\delta t$.

3. Provisional values of the new velocities are calculated using Equations (3.11) and (3.12) with the new values of the pressures.

4. The fluid particles are given weighted averages of the four nearest horizontal and the four nearest vertical velocities. The motion of the particles is found according to this provisional velocity field.

5. A corrected value for the density for $t = (n+1)\delta t$ is calculated from

$$\rho_{ij} = \frac{\sum_k n_{ij}^k \rho_k}{\sum_k n_{ij}^k}$$

where n_{ij}^k is the number of particles of density ρ_k in the ij^{th} cell according to the trajectories from step 4.

6. The corrected value of the density from step 5 is compared

with the prediction of step 1. If there is any difference the new density field is introduced into step 2. The process is repeated until there is no difference.

7. Now the pressures are calculated more precisely. Final values for the velocities are calculated and the particles are moved.

8. After u, v, and ρ are found, the temperature equation (3.14) is solved.

These eight steps relate all the essential features of the algorithm. Steps 1, 5, and 6 are the predictor-corrector portion. The calculation cycle continues until the density remains unchanged. Steps 2, 3, 7, and 8 are the Eulerian calculation of the variables P, u, v, and T. Steps 4 and 5 are the Lagrangian calculation of the particle positions and the density in each cell.

In the Lagrangian calculation of the particle positions, the velocity used to move each particle is a weighted average of nearby velocities. The calculation of these weights is given below for the horizontal velocity, u.

A rectangle of dimension δx by δy is centered over the four nearest horizontal components of the velocity field. A similar rectangle is centered over the k^{th} particle. The particle rectangle and the velocity rectangles overlap (see Figure 2). Each velocity's weight is the percentage of the particle's rectangle that it covers.

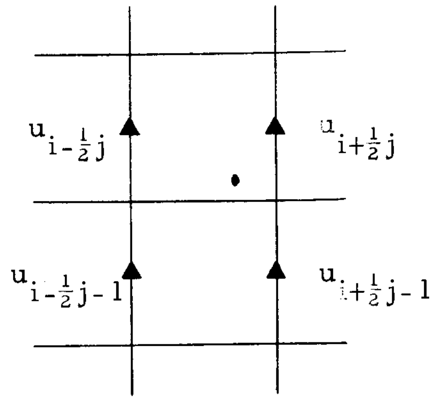

Figure 2a. A particle and the four nearest horizontal velocities.

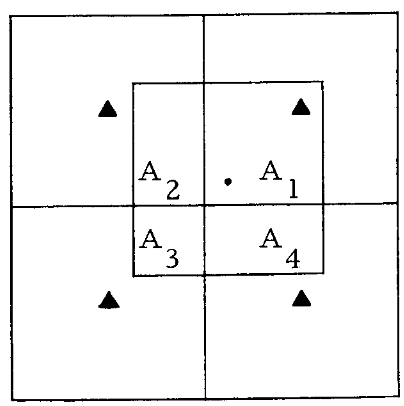

Figure 2b. The velocities and their weights.

Thus, the particle's horizontal velocity is given by

$$u_k = \frac{1}{\delta x \delta y}(A_1 u_{i+\frac{1}{2}j} + A_2 u_{i-\frac{1}{2}j} + A_3 u_{i-\frac{1}{2}j-1} + A_4 u_{i-\frac{1}{2}j-1}) .$$

The particle's new x-coordinate is given by

$$x_k^{n+1} = x_k + u_k \delta t.$$

Similar calculations are performed for the vertical velocities and the y-coordinate.

Boundary Conditions for the Algorithm

The region in which the fluid motion occurs has been covered with a mesh. It is necessary to approximate the boundary of the fluid, s, in terms of line segments from the mesh. The algorithm requires quantities from surrounding cells for the calculations at any particular cell. Thus it is necessary to create a layer of fictitious cells outside the boundary of the fluid. The quantities for these cells are determined by the boundary conditions at the interface of the fictitious and actual cells. In this way the boundary conditions are accounted for in the algorithm.

Figure 3 depicts a boundary between a cell and its fictitious image. The problem is to determine $u_{i-\frac{3}{2}j}$, $u_{i-\frac{1}{2}j}$, $v_{i-1j-\frac{1}{2}}$, ρ_{i-1j}, P_{i-1j}, and T_{i-1j} from the boundary conditions. For all types of boundary cells, $\rho_{i-1j} = \rho_{ij}$.

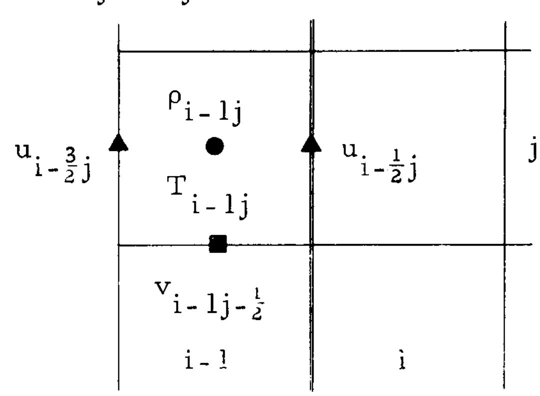

Figure 3. Cell i-1, j is a boundary cell.

Suppose the boundary is a material boundary. The boundary conditions to be satisfied are

$$\vec{w}\cdot\vec{n} = 0,$$

$$\vec{w}\cdot\vec{m} = 0,$$

$$\vec{n}\cdot\nabla T = 0.$$

For a boundary oriented as in Figure 3 these equations become

$$u = 0,$$

$$v = 0,$$

$$\frac{\partial T}{\partial x} = 0.$$

In finite difference form

$$u_{i-\frac{1}{2}j} = 0,$$

$$v_{i-\frac{1}{2}j-\frac{1}{2}} = 0,$$

$$\frac{T_{ij} - T_{i-1j}}{\delta x} = 0 \; .$$

The boundary value of v, namely $v_{i-\frac{1}{2}j-\frac{1}{2}}$, is equal to the average of the values at either side;

$$v_{i-\frac{1}{2}j-\frac{1}{2}} = \frac{v_{i-1j-\frac{1}{2}} + v_{ij-\frac{1}{2}}}{2} \; .$$

Thus, at a material boundary

$$v_{i-1j-\frac{1}{2}} = - v_{ij-\frac{1}{2}} .$$

Since

$$\frac{T_{ij} - T_{i-1j}}{\delta x} = 0,$$

$$T_{i-1j} = T_{ij}.$$

Applying the continuity equation (2. 12) to the $i-1j^{th}$ and ij^{th} cells respectively gives

$$\frac{u_{i-\frac{1}{2}j} - u_{i-\frac{3}{2}j}}{\delta x} + \frac{v_{i-1j+\frac{1}{2}} - v_{i-1j-\frac{1}{2}}}{\delta y} = 0 ,$$

$$\frac{u_{i+\frac{1}{2}j} - u_{i-\frac{1}{2}j}}{\delta x} + \frac{v_{ij+\frac{1}{2}} - v_{ij-\frac{1}{2}}}{\delta y} = 0 .$$

Adding the preceding equations gives

$$\frac{u_{i+\frac{1}{2}j} - u_{i-\frac{3}{2}j}}{\delta x} + \frac{v_{i-1j+\frac{1}{2}} + v_{ij+\frac{1}{2}}}{\delta y} - \frac{v_{i-1j-\frac{1}{2}} + v_{ij-\frac{1}{2}}}{\delta y} = 0.$$

At a material boundary the last two terms of the preceding equation vanish. Thus,

$$u_{i-\frac{3}{2}j} = u_{i+\frac{1}{2}j}.$$

Since $u^n_{i-\frac{1}{2}j} = 0$ for all n at a material boundary,

$$(\rho u)^{n+1}_{i-\frac{1}{2}j} = (\rho u)_{i-\frac{1}{2}j} = 0.$$

Substituting these values into the Navier-Stokes equation (3.3) the resulting equation can be solved for P_{i-1j}.

Summarizing, at a material boundary

$$\rho_{i-1j} = \rho_{ij},$$

$$u_{i-\frac{1}{2}j} = 0,$$

$$u_{i-\frac{3}{2}j} = u_{i+\frac{1}{2}j},$$

$$v_{i-1j-\frac{1}{2}} = -v_{ij-\frac{1}{2}},$$

$$P_{i-1j} = \rho_{ij} - \delta x(\rho X)_{i-\frac{1}{2}j} - \frac{2\mu}{\delta x} u_{i-\frac{1}{2}j},$$

$$T_{i-1j} = T_{ij}.$$

Suppose now, that the boundary in Figure 3 is a free surface. The boundary conditions to be satisfied are

$$P = 2n_x^2 \mu \frac{\partial u}{\partial x} + 2n_x n_y \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2n_y^2 \mu \frac{\partial v}{\partial y},$$

$$2n_x n_y \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + (n_x^2 - n_y^2)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,$$

$$\vec{n} \cdot \nabla T = 0.$$

For the vertical surface being considered $n_x = -1$ and $n_y = 0$. Thus the equations reduce to

$$P = 2\mu \frac{\partial u}{\partial x},$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \; ,$$

$$\frac{\partial T}{\partial x} = 0 \; .$$

In finite difference form

$$P_{i-1j} = 2\mu \frac{u_{i-\frac{1}{2}j} - u_{i-\frac{3}{2}j}}{\delta x} \; ,$$

$$\frac{u_{i-\frac{1}{2}j} - u_{i-\frac{1}{2}j-1}}{\delta x} + \frac{v_{ij-\frac{1}{2}} - v_{i-1j-\frac{1}{2}}}{\delta y} = 0,$$

$$\frac{T_{ij} - T_{i-1j}}{\delta x} = 0 \; .$$

The second equation above can be solved for $v_{i-1j-\frac{1}{2}}$;

$$v_{i-1j-\frac{1}{2}} = v_{ij-\frac{1}{2}} + \frac{\delta y}{\delta x}(u_{i-\frac{1}{2}j} - u_{i-\frac{1}{2}j-1}) \; .$$

The third equation reduced to

$$T_{i-1j} = T_{ij} \; .$$

As above, applying the continuity equation to the $i-1j^{th}$ cell yields

$$u_{i-\frac{1}{2}j} = u_{i+\frac{1}{2}j} + \frac{\delta x}{\delta y}(v_{ij+\frac{1}{2}} - v_{ij-\frac{1}{2}}) \; .$$

For a free surface the normal component of the velocity vanishes, namely

$$u_{i-\frac{3}{2}j} = 0.$$

Summarizing again, at a free surface

$$\rho_{i-1j} = \rho_{ij} ,$$

$$u_{i-\frac{3}{2}j} = 0 ,$$

$$u_{i-\frac{1}{2}j} = u_{i+\frac{1}{2}j} + \frac{\delta x}{\delta y}(v_{ij+\frac{1}{2}} - v_{ij-\frac{1}{2}}) ,$$

$$v_{i-1j-\frac{1}{2}} = v_{ij-\frac{1}{2}} + \frac{\delta y}{\delta x}(u_{i-\frac{1}{2}j} - u_{i-\frac{1}{2}j-1}) ,$$

$$P_{i-1j} = \frac{2\mu}{\delta x} u_{i-\frac{1}{2}j} ,$$

$$T_{i-1j} = T_{ij} .$$

The variety of problems which can be handled using this numerical method can be greatly increased if fluid is allowed to enter or leave the region in which the calculations are being made. Following is a development of conditions for an in boundary and an out boundary.

Frequently some of the boundary values of the variables are known from the problem. For instance, the input velocity may be specified or the input or output pressure may be held at some constant value. In the absence of other information, the values may be calculated as follows.

To simplify calculations, only problems are considered for which the fluid enters normally to the in boundary. Thus, at the boundary in Figure 4 the horizontal velocity $v_{i-\frac{1}{2}j-\frac{1}{2}}$ must be zero.

This yields, as in the case of the material boundary,

$$v_{i-1j-\frac{1}{2}} = -v_{ij-\frac{1}{2}} ,$$

$$u_{i-\frac{1}{2}j} = u_{i+\frac{1}{2}j} .$$

Also like a material boundary a pressure boundary condition can be derived if necessary by solving the Navier-Stokes equation (3.3) for P_{i-1j}. A temperature boundary condition can be derived by requiring that the initial temperature profile is maintained at the in boundary.

At an out boundary it is assumed that the fluid is not accelerated. With this assumption--valid for problems where $|\vec{w}| << C_s$ [4] if the cell size is small--the boundary conditions for an out boundary are as follows.

Since the fluid is not accelerated

$$v_{i-1j-\frac{1}{2}} = v_{ij-\frac{1}{2}} .$$

The horizontal velocity $u_{i-\frac{1}{2}j}$ is calculated from Equation (3.6). Since the continuity equation (3.2) is satisfied in the $i-1j^{th}$ cell,

$$u_{i-\frac{3}{2}j} = u_{i-\frac{1}{2}j} + \frac{\delta t}{\delta y}(v_{i-1j+\frac{1}{2}} - v_{i-1j-\frac{1}{2}}) .$$

---

[4] See page 8.

The pressure boundary condition was found by Welch *et al.* (1966) to be

$$P_{i-1j} = \rho_{ij} - \frac{\delta x}{\delta y}[(\rho uv)_{i-\frac{1}{2}j+\frac{1}{2}} - (\rho uv)_{i-\frac{1}{2}j-\frac{1}{2}}].$$

The adiabaticity requirement is as before

$$T_{i-1j} = T_{ij}.$$

IV. APPLICATIONS

A computer program for the flow of a fluid with a continuously varying density stratification has immediate applications in meteorology, oceanography, and hydraulics.

The infinite reservoir problem offers an interesting example of the problems and potential of this numerical method. This problem is interesting because it utilizes all four types of boundaries: in, out, material, and free surface. It also is typical of a particular problem in numerical solutions: providing a finite approximation to an infinite problem. Before attempting this problem some attempt should be made to check the algorithm to determine whether it is, in fact, solving the system of partial differential equations (2. 11) - (2. 14).

Checking the Algorithm

As a check, the algorithm was applied to various problems for which the analytical solution was known. Simplest of these was a tank of water undergoing no motion. The algorithm gave, to within the error of the numerical approximations, no change in any of the variables. Another problem whose analytical solution is known is uniform horizontal motion. The algorithm provided the same solution as the analytical methods. Although there are other checks that could have been run, success in these two cases was deemed sufficient to try a more sophisticated problem.

The Infinite Reservoir Problem

The infinite reservoir problem is the problem of determining the motion of a semi-infinite strip of fluid as it flows into a point sink as shown in Figure 4. Any continuous density profile may be used. To handle an infinite problem such as this it is necessary to simulate the infinite reservoir on the left with an in boundary. Sufficiently for downstream, i.e., to the left, the motion in the reservoir will be uniformly to the right and the pressure will be hydrostatic. Fluid can be drawn into the finite reservoir by setting the value of the horizontal velocity at the in boundary equal to the velocity of the fluid immediately to the right of the boundary.

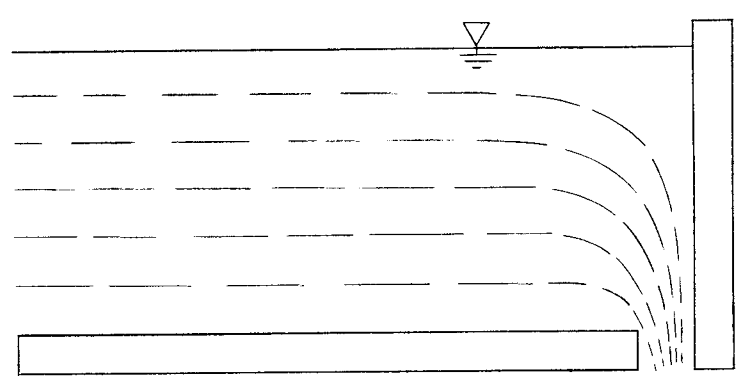

Figure 4. An infinite reservoir with a point sink.

The model, then, is a rectangular region partially filled with fluid particles. On the left is an in boundary. The velocity is equal to the velocity of the column of cells to the right and the pressure is hydrostatic. On the right is a material boundary, on the top is a free surface, and on the bottom is a material boundary with a small out boundary to the right.

The primary feature of this numerical technique is that plotting the positions of the fluid particles provides a visual display of the motion of the fluid. The plots in Figure 5 are from a computer run using a hyperbolic tangent curve as the initial density and temperature profile.

The plots were photographically obtained from cathode ray tube displays of the particle positions. Successive photographs were taken to provide sequences for a motion picture of the flow. In addition to the photographs, which were taken for each time increment, numerical values of all the variables were printed at regular intervals.

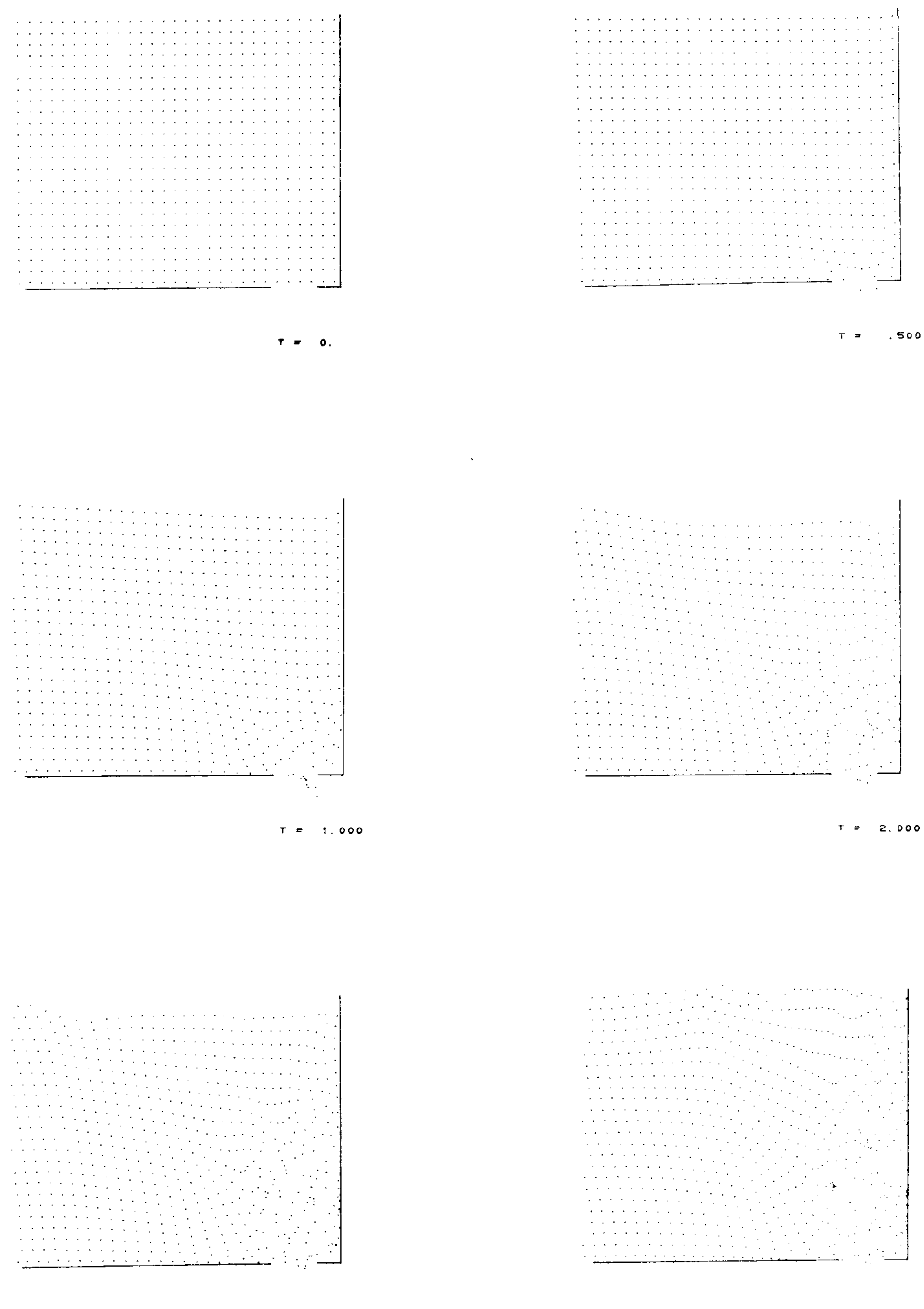

Figure 5. The above results were obtained using a mesh size $\delta x = .1$, $\delta y = .1$, and $\delta t = .05$. The mesh was a square array of 16 cells by 16 cells. Initially there were four particles per cell.

V. CONCLUSIONS

The goal of this study was to develop a numerical technique to find the transient motion of a nonhomogeneous fluid. The general success in solving all such problems to which this algorithm has been applied indicates that the algorithm devised by Welch et al. (1966) has been successfully modified and extended to handle nonhomogeneous fluids. The test problems mentioned previously, reservoir problems, and jets of fluid into tanks filled with a similar fluid are among those successfully run.

Recommendations

Following are questions whose answers could be of great benefit in the applications of this numerical technique.

1. Can the requirement of strictly adiabatic flow be weakened to allow heat transfer problems?
2. Can the requirement of perpendicular flow across an in boundary be removed?
3. How does the scaling of the problem effect the simulated motion of the fluid?

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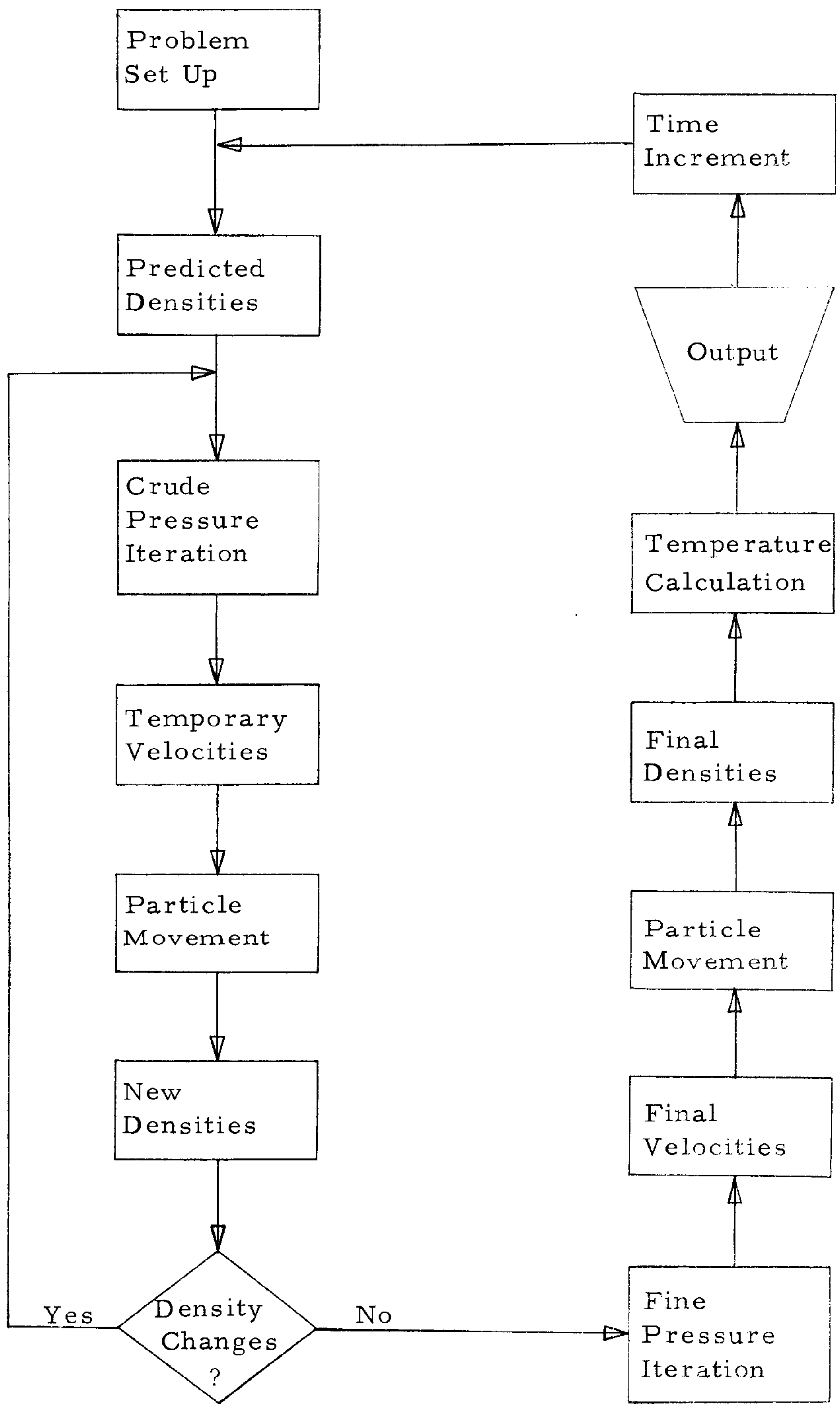
APPENDICES

A. Flow Chart

Following is a flow chart of the algorithm developed on pages 23 and 24.


Problem
Set Up
Time
Increment
Predicted
Densities
Output
Crude
Pressure
Iteration
Temperature
Calculation
Temporary
Velocities
Final
Densities
Particle
Movement
Particle
Movement
New
Densities
Final
Velocities
Yes
Density
Changes
?
No
Fine
Pressure
Iteration

B. A Sample Listing

Following is a computer listing for the infinite stratified reservoir problem discussed on pages 35-36. The primary variables are designated in the program as follows:

$u \longleftrightarrow$ U

$v \longleftrightarrow$ V

$\rho \longleftrightarrow$ R

$P \longleftrightarrow$ P

$T \longleftrightarrow$ T

$x \longleftrightarrow$ X

$y \longleftrightarrow$ Y

$t \longleftrightarrow$ TIME

$\delta x \longleftrightarrow$ DX

$\delta y \longleftrightarrow$ DY

$\delta t \longleftrightarrow$ DT

To identify the cells, each cell is labeled:

IN if it is a boundary cell at an in boundary,

OUT if it is a boundary cell at an out boundary,

NOSLIP if it is a boundary cell at a material boundary,

SUR if it is a free surface,

EMP if it contains no fluid particles,

REG if it contains fluid particles and is not a boundary cell.

Similarly each particle is labeled:

IN if it is in an IN cell,

OUT if it is in an OUT cell,

REG if it is in a REG cell,

AVAIL if it is not in the computing region.

The following program is in the FORTRAN IV language and was run on a CDC 6600.

```
      PROGRAM STRATRES
C     PROGRAM STRATRES SOLVES THE INFINITE STRATIFIED RESERVOIR PROBLEM
C     WITH HYPERBOLIC TANGENT INITIAL DENSITY AND TEMPERATURE DISTRIBU-
C     TIONS
      COMMON X,XN,Y,YN,F,DENS,U,V,P,PN,R,RN,RO,A,B1,B2,B3,B4,XI,ZETA,S,N
     1,IC,T,TN,TO
      DIMENSION X(800),XN(800),Y(800),YN(800),F(800),DENS(800)
      DIMENSION U(17,16),V(16,17),P(16,16),PN(16,16),R(16,16),RN(16,16),
     1RO(32),PO(16)
      DIMENSION A(16,16),B1(16,16),B2(16,16),B3(16,16),B4(16,16)
      DIMENSION XI(16,16),ZETA(16,16),S(16,16),N(16,16),IC(16,16)
      DIMENSION T(16,16),TN(16,16),TO(16)
      DIMENSION D(16,16),DR(16,16)
      DIMENSION XP(2),YP(2),XA(800),YA(800)
      INTEGER F
      REAL MU
C
C     INITIAL PARAMETER SET UP
C
      DX=.1
      DY=.1
      DT=.05
      TIME=0.0
      TIMELIM=10.0
      ITIME=10
      MU=.000114
      ALPHA=.0000141
      G=-.98
      MMM=ITIME
C     NI IS THE NUMBER OF CELLS WIDE, NJ IS THE NUMBER OF CELLS DEEP,
C     NK IS THE NUMBER OF PARTICLES, NOUT IS THE ITH COORDINATE OF THE
C     LAST NOSLIP CELL ALONG THE BOTTOM, NSUR IS THE JTH COORDINATE OF
C     THE LAST CELL CONTAINING PARTICLES, ND IS THE NUMBER OF ROWS CON-
C     TAINING PARTICLES.
      NI=16
      NJ=16
      NK=800
      NOUT=NI-3
      NSUR=NJ-3
      ND=NSUR-1
      NIMNS1=NI-1
      NJMNS1=NJ-1
      NIPLS1=NI+1
      NJPLS1=NJ+1
      H=(NSUR-.5)*DY
      WIDTH=(NI-2)*DX
      DEPTH=(NJ-2)*DY
      DEEP=ND*DY
C
C      INITIAL CELL SET UP
C
C     THE VARIABLE ARRAYS ARE PRESET TO ZERO
      DO 9 I=1,NIPLS1
      DO 9 J=1,NJ
    9 U(I,J)=0.0
      DO 10 I=1,NI
      DO 10 J=1,NJPLS1
   10 V(I,J)=0.0
      DO 11 I=1,NI
      DO 11 J=1,NJ
```

```
      IC(I,J)=0
      D(I,J)=0.0
      DR(I,J)=0.0
      RN(I,J)=0.0
      R(I,J)=0.0
      T(I,J)=0.0
   11 P(I,J)=0.0
      DO 25 J=1,NJ
      TO(J)=0.0
   25 PO(J)=0.0
      NN=2*NI
      DO 23 J=1,NN
   23 RO(J)=0.0
C     IC...BND=1,IN=2,OUT=3,FULL=10,SUR=11,EMP=12
      DO 1 I=2,NIMNS1
      DO 1 J=2,NSUR
    1 IC(I,J)=10
C     FREE SURFACE IS THE EIGHTH ROW
      DO 2 I=2,NIMNS1
      IC(I,NJ-2)=11
    2 IC(I,NJ-1)=12
C     BOTTOM WALL IS NOSLIP WITH TWO OUT CELLS
      DO 3 I=1,NI
    3 IC(I,1)=1
      IC(NOUT,1)=3
      IC(NOUT+1,1)=3
C     TOP WALL IS NOSLP
      DO 4 I=1,NI
    4 IC(I,NJ)=1
C     LEFT WALL IS IN
      DO 5 J=2,NJMNS1
    5 IC(1,J)=2
C     RIGHT WALL IS NOSLP
      DO 6 J=1,NJ
    6 IC(NI,J)=1
C     INITIAL DENSITY DISTRIBUTION
      NN=ND*2
      DO 12 L=1,NN
      Y=.025+.5*DY*(L-1)
   12 RO(L)=.9985-.0015*TANH(Y-.5*DEEP)/TANH(.5*DEEP)
      DO 13 I=1,NIMNS1
      DO 13 J=2,NSUR
   13 R(I,J)=.5*(RO(2*J-3)+RO(2*J-2))
C     DENSITY BOUNDARY CONDITIONS
      DO 7 I=2,NIMNS1
    7 R(I,NSUR+1)=R(I,NSUR)
C     INITIAL TEMPERATURE DISTRIBUTION
      DO 16 J=2,NSUR
      Y=.05+(J-2)*DY
   16 TO(J)=15.0+10.0*TANH(Y-.5*DEEP)/TANH(.5*DEEP)
      DO 17 I=1,NIMNS1
      DO 17 J=2,NSUR
   17 T(I,J)=TO(J)
C     INITIAL  PRESSURE DISTRIBUTION
      DO 24 J=2,NSUR
      Y=(J-2)*DY
      COSH1=.5*(EXP(.5*DEEP)+EXP(-.5*DEEP))
      COSH2=.5*(EXP( Y-.5*DEEP)+EXP(-Y+.5*DEEP))
   24 PO(J)=.98*(.9985*(DEEP-Y)-.0015/TANH(.5*DEEP)*ALOG(COSH1/COSH2))
      DO 18 I=1,NIMNS1
```

```
      DO 18 J=2,NSUR
   18 P(I,J)=PO(J)
C
C     INITIAL PARTICLE SET UP
C
C     F...AVAIL=1,IN=2,OUT=3,REG=4
      KL=2*ND
      DO 19 K=1,KL
      F(K)=2
      ZK=K
      X(K)=-.25*DX
      Y(K)=(.5*ZK-.25)*DY
      DENS(K)=RO(K)
      XN(K)=0
      YN(K)=0
   19 CONTINUE
      IN=1
      KM=KL+1
      KN=4*(NI-2)*ND+KL
      DO 20 K=KM,KN
      IK=K-KL-(IN-1)*2*(NI-2)
      ZIK=IK
      ZIN=IN
      F(K)=4
      X(K)=(.5*ZIK-.25)*DX
      Y(K)=(.5*ZIN-.25)*DY
      DENS(K)=RO(IN)
      XN(K)=0
      YN(K)=0
      IF(IK.LT.2*(NI-2))20,21
   21 IN=IN+1
   20 CONTINUE
      KL=KN+1
      DO 22 K=KL,NK
      F(K)=1
      X(K)=0
      Y(K)=0
      DENS(K)=0
      XN(K)=0
      YN(K)=0
   22 CONTINUE
      GO TO 600
  100 M=0
      MM=0
      ITNUM=0
      DO 123 I=2,NIMNS1
  123 R(I,1)=R(I,2)
      DO 124 J=2,NJMNS1
      R(1,J)=R(2,J)
  124 R(NI,J)=R(NI-1,J)
      DO 101 I=2,NI
      DO 101 J=2,NJMNS1
      RL=.5*(R(I-1,J)+R(I,J))
      UR=.5*(U(I+1,J)+U(I,J))
      UL=.5*(U(I-1,J)+U(I,J))
      VB=.5*(V(I-1,J)+V(I,J))
      IF(VB.GE.0)110,111
  110 RUVB=.5*(R(I,J-1)+R(I-1,J-1))*U(I,J-1)*VB
      GO TO 115
  111 RUVB=.5*(R(I,J)+R(I-1,J))*U(I,J)*VB
```

```
  115  VT=.5*(V(I-1,J+1)+V(I,J+1))
       IF(VT.GE.0)113,114
  113  RUVT=.5*(R(I,J)+R(I-1,J))*U(I,J)*VT
       GO TO 116
  114  RUVT=.5*(R(I,J+1)+R(I-1,J+1))*U(I,J+1)*VT
  116  DUX2=(U(I+1,J)-2.0*U(I,J)+U(I-1,J))/(DX*DX)
       DUY2=(U(I,J+1)-2.0*U(I,J)+U(I,J-1))/(DY*DY)
  101  XI(I,J)=RL*U(I,J)-DT*((R(I,J)*UR**2-R(I-1,J)*UL**2)/DX+(RUVT-RUVB)
      1/DY-MU*(DUX2+DUY2))
       DO 102 I=2,NIMNS1
       DO 102 J=2,NJ
       RB=.5*(R(I,J-1)+R(I,J))
       VT=.5*(V(I,J+1)+V(I,J))
       VB=.5*(V(I,J-2)+V(I,J))
       UR=.5*(U(I+1,J)+U(I+1,J-1))
       IF(UR.GE.0)117,118
  117  RUVR=.5*(R(I,J)+R(I,J-1))*UR*V(I,J)
       GO TO 119
  118  RUVR=.5*(R(I+1,J)+R(I+1,J-1))*UR*V(I+1,J)
  119  UL=.5*(U(I,J)+U(I,J-1))
       IF(UL.GE.0)120,121
  120  RUVL=.5*(R(I-1,J)+R(I-1,J-1))*UL*V(I-1,J)
       GO TO 122
  121  RUVL=.5*(R(I,J)+R(I,J-1))*UL*V(I,J)
  122  DVX2=(V(I+1,J)-2.0*V(I,J)+V(I-1,J))/(DX*DX)
       DVY2=(V(I,J+1)-2.0*V(I,J)+V(I,J-1))/(DY*DY)
  102  ZETA(I,J)=RB*V(I,J)-DT*((R(I,J)*VT**2-R(I,J-1)*VB**2)/DY+(RUVR-RUV
      1L)/DX-MU*(DVX2+DVY2)-RB*G)
C
C      INITIAL DENSITY PREDICTION
C
       DO 103 I=2,NIMNS1
       DO 103 J=2,NJMNS1
  103  RN(I,J)=R(I,J)-DT*(.5*(U(I,J)+U(I+1,J))*(R(I+1,J)-R(I,J))/DX+.5*(V
      1(I,J)+V(I,J+1))*(R(I,J+1)-R(I,J))/DY)
C
C      DENSITY BOUNDARY CONDITIONS
C
  106  DO 109 I=2,NIMNS1
  109  R(I,1)=R(I,2)
       DO 107 J=2,NJMNS1
       R(1,J)=R(2,J)
  107  R(NI,J)=R(NI-1,J)
C
C       PRESSURE COEFFICIENTS
C
       DO 112 I=2,NIMNS1
       DO 112 J=2,NJMNS1
       IF(IC(I,J).LE.10)108,112
  108  RL=.5*(R(I-1,J)+R(I,J))
       RR=.5*(R(I+1,J)+R(I,J))
       RT=.5*(R(I,J+1)+R(I,J))
       RB=.5*(R(I,J-1)+R(I,J))
       A(I,J)=((XI(I,J)/RL-XI(I+1,J)/RR)/DX+(ZETA(I,J)/RB-ZETA(I,J+1)/RT)
      1/DY)/(DT*(1.0/RL+1.0/RR)/(DX*DX)+DT*(1.0/RT+1.0/RB)/(DY*DY))
       B1(I,J)=(DT/(DX*DX*RR))/(DT*(1.0/RL+1.0/RR)/(DX*DX)+DT*(1.0/RT+1.0
      1/RB)/(DY*DY))
       B2(I,J)=(DT/(DX*DX*RL))/(DT*(1.0/RL+1.0/RR)/(DX*DX)+DT*(1.0/RT+1.0
      1/RB)/(DY*DY))
       B3(I,J)=(DT/(DY*DY*RT))/(DT*(1.0/RL+1.0/RR)/(DX*DX)+DT*(1.0/RT+1.0
```

```
     1/RB)/(DY*DY))
      B4(I,J)=(DT/(DY*DY*RB))/(DT*(1.0/RL+1.0/RR)/(DX*DX)+DT*(1.0/RT+1.0
     1/RB)/(DY*DY))
  112 CONTINUE
C
C     PRESSURE BOUNDARY CONDITIONS
C
      L=1
      LL=1
      IF=1
      IFF=1
C     LEFT WALL IS IN
  200 CONTINUE
      I=2
      DO 201 J=2,NJMNS1
  201 P(I-1,J)=PO(J)
      I=NI
      DO 202 J=2,NJMNS1
C     RIGHT WALL IS NOSLIP
      USQ1=.25*(U(I,J)+U(I+1,J))**2
      USQ2=.25*(U(I-1,J)+U(I,J))**2
      UV1=.25*(U(I,J+1)+U(I,J))*(V(I,J+1)+V(I-1,J+1))
      UV2=.25*(U(I,J)+U(I,J-1))*(V(I,J)+V(I-1,J))
      R1=.25*(R(I-1,J)+R(I,J)+R(I-1,J+1)+R(I,J+1))
      R2=.25*(R(I-1,J)+R(I,J)+R(I-1,J-1)+R(I,J-1))
      DUX2=(U(I+1,J)-2.0*U(I,J)+U(I-1,J))/(DX*DX)
      DUY2=(U(I,J+1)-2.0*U(I,J)+U(I,J-1))/(DY*DY)
      DPX=DX*((R(I,J)*USQ1-R(I-1,J)*USQ2)/DX+(R1*UV1-R2*UV2)/DY- MU*(DUX
     12+DUY2))
  202 P(I,J)=P(I-1,J)-DPX
C     TOP WALL IS NOSLIP
      J=NJ
      DO 203 I=2,NIMNS1
      VSQ1=.25*(V(I,J)+V(I,J+1))**2
      VSQ2=.25*(V(I,J)+V(I,J-1))**2
      UV1=.25*(U(I+1,J)+U(I+1,J-1))*(V(I+1,J)+V(I,J))
      UV2=.25*(U(I,J)+U(I,J-1))*(V(I,J)+V(I-1,J))
      R1=.25*(R(I+1,J)+R(I,J)+R(I+1,J-1)+R(I+1,J-1))
      R2=.25*(R(I-1,J)+R(I,J)+R(I-1,J-1)+R(I,J-1))
      DVX2=(V(I+1,J)-2.0*V(I,J)+V(I-1,J))/(DX*DX)
      DVY2=(V(I,J+1)-2.0*V(I,J)+V(I,J-1))/(DY*DY)
      DPY=DY*((R(I,J)*VSQ1-R(I,J-1)*VSQ2)/DY+(R1*UV1-R2*UV2)/DX- MU*(DVX
     12+DVY2)-.5*(R(I,J)+R(I,J-1))*G)
  203 P(I,J)=P(I,J-1)-DPY
C     BOTTOM WALL IS NOSLIP WITH TWO OUT CELLS
      J=2
      DO 204 I=2,NIMNS1
      VSQ1=.25*(V(I,J)+V(I,J+1))**2
      VSQ2=.25*(V(I,J)+V(I,J-1))**2
      UV1=.25*(U(I+1,J)+U(I+1,J-1))*(V(I+1,J)+V(I,J))
      UV2=.25*(U(I,J)+U(I,J-1))*(V(I,J)+V(I-1,J))
      R1=.25*(R(I+1,J)+R(I,J)+R(I+1,J-1)+R(I+1,J-1))
      R2=.25*(R(I-1,J)+R(I,J)+R(I-1,J-1)+R(I,J-1))
      DVX2=(V(I+1,J)-2.0*V(I,J)+V(I-1,J))/(DX*DX)
      DVY2=(V(I,J+1)-2.0*V(I,J)+V(I,J-1))/(DY*DY)
      DPY=DY*((R(I,J)*VSQ1-R(I,J-1)*VSQ2)/DY+(R1*UV1-R2*UV2)/DX- MU*(DVX
     12+DVY2)-.5*(R(I,J)+R(I,J-1))*G)
  204 P(I,J-1)=P(I,J)+DPY
C     PRESSURE BOUNDARY CONDITIONS FOR THE OUT CELLS
      I=NOUT
```

```
      RUVL=.5*R(I-1,2)*U(I,2)*(V(I-1,2)+V(I,2))
      RUVR=.5*R(I,2)*U(I+1,2)*(V(I,2)+V(I+1,2))
      P(I,1)=P(I,2)+DY/DX*(RUVR-RUVL)
      I=NOUT+1
      RUVL=.5*R(I-1,2)*U(I,2)*(V(I-1,2)+V(I,2))
      RUVR=.5*R(I,2)*U(I+1,2)*(V(I,2)+V(I+1,2))
      P(I,1)=P(I,2)+DY/DX*(RUVR-RUVL)
C     FREE SURFACE PRESSURE BOUNDARY CONDITIONS
      SQR=1.0/SQRTF(2.0)
      DO 205 I=2,NIMNS1
      DO 205 J=2,NJMNS1
      IF(IC(I,J).EQ.11)206,205
  206 IF(IC(I,J+1).EQ.12)207,208
  207 IF(IC(I+1,J).EQ.12)209,210
  209 PX=SQR
      PY=SQR
      GO TO 211
  210 IF(IC(I-1,J).EQ.12)212,213
  212 PX=-SQR
      PY=SQR
      GO TO 211
  213 PX=0.0
      PY=1.0
      GO TO 211
  208 IF(IC(I+1,J).EQ.12)214,215
  214 IF(IC(I,J-1).EQ.12)216,217
  216 PX=SQR
      PY=-SQR
      GO TO 211
  217 PX=1.0
      PY=0.0
      GO TO 211
  215 IF(IC(I-1,J).EQ.12)218,219
  218 IF(IC(I,J-1).EQ.12)220,221
  220 PX=-SQR
      PY=-SQR
      GO TO 211
  221 PX=-1.0
      PY=0.0
      GO TO 211
  219 IF(IC(I,J-1).EQ.12)222,205
  222 PX=0.0
      PY=-1.0
  211 P(I,J)=2.0*MU*(PX*PX*(U(I+1,J)-U(I,J))/DX+PX*PY*(.25*(U(I+1,J+1)+U
     1(I,J+1)-U(I,J-1)-U(I+1,J-1))/DY+.25*(V(I+1,J+1)+V(I+1,J)-V(I-1,J+1
     2)-V(I-1,J))/DX)+PY*PY*(V(I,J+1)-V(I,J))/DY)
  205 CONTINUE
C
C     PRESSURE ITERATION
C
      DO 223 I=1,NI
      DO 223 J=1,NJ
  223 PN(I,J)=P(I,J)
      DO 224 I=2,NIMNS1
      DO 224 J=2,NJMNS1
      IF(IC(I,J).EQ.10)225,224
  225 PN(I,J)=B1(I,J)*PN(I+1,J)+B2(I,J)*PN(I-1,J)+B3(I,J)*PN(I,J+1)+B4(I
     1,J)*PN(I,J-1)+A(I,J)
  224 CONTINUE
      IF(M.EQ.0)226,227
```

```
C       CRUDE PRESSURE TEST
 226    IF(L.EQ.IF*4)228,229
 228    IF=IF+1
        MM=0
        DO 230 I=2,NIMNS1
        DO 230 J=2,NJMNS1
        IF(IC(I,J).EQ.10)231,230
 231    DP=ABSF((PN(I,J)-P(I,J))/(R(I,J)*G*DEPTH))
        IF(DP.LT..0008)230,229
 230    CONTINUE
        MM=1
        WRITE(61,233)
 233    FORMAT(5X,17HCRUDE TEST PASSED)
        WRITE(61,234) L
 234    FORMAT(5X,4HL = ,I3)
        GO TO 235
 229    IF(L.LT.500)232,236
 232    L=L+1
 235    DO 237 I=1,NI
        DO 237 J=1,NJ
 237    P(I,J)=PN(I,J)
        IF(MM-1)200,248,331
 248    MM=0
        GO TO 331
C       FINE PRESSURE TEST
 227    IF(LL.EQ.IFF*13)238,239
 238    IFF=IFF+1
        MM=0
        DO 240 I=2,NIMNS1
        DO 240 J=2,NJMNS1
        IF(IC(I,J).EQ.10)241,240
 241    DP=ABSF((PN(I,J)-P(I,J))/(R(I,J)*G*DEPTH))
        IF(DP.LT..0002)240,239
 240    CONTINUE
        MM=2
        WRITE(61,243)
 243    FORMAT(5X,16HFINE TEST PASSED)
        WRITE(61,244) LL
 244    FORMAT(5X,5HLL = ,I3)
 239    IF(LL.LT.500)242,245
 242    LL=LL+1
        GO TO 235
 236    WRITE(61,246)
 246    FORMAT(5X,33HTOO MANY ITERATIONS IN CRUDE TEST)
        STOP
 245    WRITE(61,247)
 247    FORMAT(5X,32HTOO MANY ITERATIONS IN FINE TEST)
        STOP
C
C       NEW VELOCITY FIELD
C
 331    DO 300 I=1,NIPLS1
        DO 300 J=1,NJ
 300    U(I,J)=0.0
        DO 301 I=1,NI
        DO 301 J=1,NJPLS1
 301    V(I,J)=0.0
        DO 302 I=3,NIMNS1
        DO 302 J=2,NJMNS1
        IF(IC(I,J).LE.11)351,302
```

```
 351  IF(IC(I-1,J).LE.10)303,302
 303  U(I,J)=2.0*XI(I,J)/(R(I,J)+R(I-1,J))+DT*(P(I-1,J)/R(I-1,J)-P(I,J)/
     1R(I,J))/DX
 302  CONTINUE
      DO 304 I=2,NIMNS1
      DO 304 J=2,NJMNS1
      IF(IC(I,J).EQ.10)350,304
 350  IF(IC(I,J-1).LE.10)305,304
 305  V(I,J)=2.0*ZETA(I,J)/(R(I,J)+R(I,J-1))+DT*(P(I,J-1)/R(I,J-1)-P(I,J
     1)/R(I,J))/DY
 304  CONTINUE
C
C     VELOCITY BOUNDARY CONDITIONS
C
C     LEFT WALL IS IN
      DO 311 J=2,NSUR
 311  U(2,J)=U(3,J)
      DO 306 J=2,NJMNS1
 306  U(1,J)=U(3,J)
      DO 307 J=2,NJ
 307  V(1,J)=-V(2,J)
C     RIGHT WALL IS NOSLP
      DO 308 J=2,NJ
      U(NI+1,J)=U(NI-1,J)
 308  V(NI,J)=-V(NI-1,J)
C     BOTTOM WALL IS NOSLP
      DO 309 I=2,NIMNS1
      V(I,1)=V(I,3)
 309  U(I,1)=-U(I,2)
C     TOP WALL IS NOSLP
      DO 312 I=2,NI
      V(I,NJ+1)=V(I,NJ-1)
 312  U(I,NJ)=-U(I,NJ-1)
C     VELOCITY BOUNDARY CONDITIONS FOR THE TWO OUT CELLS
      I=NOUT
      IOUT=0
 334  J=2
      V(I,J)=2.0*ZETA(I,J)/(R(I,J)+R(I,J-1))+DT*(P(I,J-1)/R(I,J-1)-P(I,J
     1)/R(I,J))/DY
      IF(IOUT.EQ.0)332,333
 332  I=NOUT+1
      IOUT=1
      GO TO 334
 333  U(NOUT,1)=U(NOUT,2)
      U(NOUT+1,1)=U(NOUT+1,2)
      U(NOUT+2,1)=U(NOUT+2,2)
      V(NOUT,1)=V(NOUT,2)+DY/DX*(U(NOUT+1,1)-U(NOUT,1))
      V(NOUT+1,1)=V(NOUT+1,2)+DY/DX*(U(NOUT+2,1)-U(NOUT+1,1))
C     FREE SURFACE VELOCITY BOUNDARY CONDITIONS
      DO 313 I=2,NIMNS1
      DO 313 J=2,NJMNS1
      IF(IC(I,J).EQ.10)314,313
 314  IF(IC(I,J+1).EQ.11)315,316
 315  IF(IC(I+1,J).EQ.11)317,318
 317  V(I,J+1)=V(I,J)-DY/DX*(U(I+1,J)-U(I,J))
      U(I+1,J+1)=U(I+1,J)
      GO TO 313
 318  IF(IC(I-1,J).EQ.11)319,320
 319  V(I,J+1)=V(I,J)-DY/DX*(U(I+1,J)-U(I,J))
      U(I+1,J+1)=U(I+1,J)
```

```
      GO TO 313
 320  V(I,J+1)=V(I,J)-DY/DX*(U(I+1,J)-U(I,J))
      U(I+1,J+1)=U(I+1,J)
      GO TO 313
 316  IF(IC(I+1,J).EQ.11)321,322
 321  IF(IC(I,J-1).EQ.11)323,324
 323  U(I+1,J)=U(I,J)
      V(I,J)=V(I,J+1)
      GO TO 313
 324  U(I+1,J)=U(I,J)-DX/DY*(V(I,J+1)-V(I,J))
      GO TO 313
 322  IF(IC(I-1,J).EQ.11)325,326
 325  IF(IC(I,J-1).EQ.11)327,328
 327  U(I,J)=U(I+1,J)
      V(I,J)=V(I,J+1)
      GO TO 313
 328  U(I,J)=U(I+1,J)+DX/DY*(V(I,J+1)-V(I,J))
      GO TO 313
 326  IF(IC(I,J-1).EQ.11)329,313
 329  V(I,J)=V(I,J+1)+DY/DX*(U(I+1,J)-U(I,J))
 313  CONTINUE
C
C     PARTICLE MOVEMENT
C
      DO 400 K=1,NK
      IF(F(K).EQ.1)400,401
 401  I=X(K)/DX+2
      J=Y(K)/DY+2
      XCI=(I-1.5)*DX
      YCJ=(J-1.5)*DY
      FX=X(K)/DX+2.0-I
      FY=Y(K)/DY+2.0-J
      IF(F(K).EQ.3)453,403
 453  UOUT=.5*(U(I+1,J)+U(I,J))
      XN(K)=X(K)+UOUT*DT
      GO TO 454
 403  IF(FY.LE..5)407,408
 407  JJ=J-1
      GO TO 409
 408  JJ=J
 409  YCJJ=(JJ-1.5)*DY
      SX=(XCI-X(K))/DX
      SY=(YCJJ-Y(K))/DY+.5
      W1=ABSF((.5-SX)*(.5-SY))
      W2=ABSF((.5+SX)*(.5-SY))
      W3=ABSF((.5+SX)*(.5+SY))
      W4=ABSF((.5-SX)*(.5+SY))
      UK=W1*U(I+1,JJ+1)+W2*U(I,JJ+1)+W3*U(I,JJ)+W4*U(I+1,JJ)
      XN(K)=X(K)+UK*DT
      IF(F(K).EQ.2)452,454
 452  YN(K)=Y(K)
      GO TO 400
 454  IF(FX.LE..5)410,411
 410  II=I-1
      GO TO 412
 411  II=I
 412  XCII=(II-1.5)*DX
      SX=(XCII-X(K))/DX+.5
      SY=(YCJ-Y(K))/DY
      W1=ABSF((.5-SX)*(.5-SY))
```

```
      W2=ABS(F(I,[illegible])-[illegible])
      W3=ABS((I,[illegible]X)*([illegible]Y))
      W4=ABS((J5-[illegible])*([illegible]Y))
      VK=W1*V(I[illegible]+1,J+1)+[illegible]*V(II,J+1)+[illegible]*V(II,J)+W4*V(II+1,J)
      YN(K)=[illegible]
 401  CONTINUE
      WRITE(61,413)
 413  FORMAT(5X,15HPARTICLES MOVED)
C
C     DENSITY CALCULATION
C
      DO 414 I=1,NI
      DO 414 J=1,NJ
      S(I,J)=0.0
 414  N(I,J)=0
      DO 415 K=1,NK
      IF(F(K).EQ.1)415,417
 417  I=XN(K)/DX+2
      J=YN(K)/DY+2
      N(I,J)=N(I,J)+1
      S(I,J)=S(I,J)+DENS(K)
 415  CONTINUE
C     REFLAGGING THE CELLS
      DO 418 I=2,NIMNS1
      DO 418 J=2,NJMNS1
      IF(N(I,J).EQ.0)420,419
 419  IC(I,J)=10
      GO TO 418
 420  IF(N(I,J+1).NE.0.OR.N(I,J-1).NE.0.OR.N(I+1,J).NE.0.OR.N(I-1,J).NE.
     10)421,422
 421  IC(I,J)=11
      GO TO 418
 422  IC(I,J)=12
 418  CONTINUE
      M=1
      DO 423 I=2,NIMNS1
      DO 423 J=2,NJMNS1
      IF(N(I,J).EQ.0)424,425
 424  R(I,J)=0.0
      GO TO 423
 425  IF(S(I,J)/N(I,J).EQ.R(I,J))423,426
 426  R(I,J)=S(I,J)/N(I,J)
      M=0
 423  CONTINUE
      DO 458 I=2,NIMNS1
      DO 458 J=2,NJMNS1
      IF(IC(I,J).EQ.11)459,458
 459  R(I,J)=R(I,J-1)
 458  CONTINUE
      IF(MM.EQ.2)432,451
 451  ITNUM=ITNUM+1
      IF(M.EQ.0)427,430
 427  IF(ITNUM.GE.10)428,106
 428  M=1
      WRITE(61,429)
 429  FORMAT(5X,20HOSCILLATING PARTICLE)
      GO TO 106
 430  WRITE(61,431)
 431  FORMAT(5X,17HNO DENSITY CHANGE)
      GO TO 106
```

```
C
C     REFLAGGING OF PARTICLES
C
  432 DO 433 K=1,NK
      IF(F(K).EQ.1)433,455
  455 IF(F(K).EQ.2)434,440
  434 I=XN(K)/DX+2
      J=YN(K)/DY+2
      IF(I.GE.2)436,433
  436 F(K)=4
      DO 437 KK=1,NK
      IF(F(KK).EQ.1)438,437
  438 F(KK)=2
      XN(KK)=XN(K)-.5*DX
      YN(KK)=YN(K)
      DENS(KK)=DENS(K)
      GO TO 433
  437 CONTINUE
      WRITE(61,439)
  439 FORMAT(5X,16HOUT OF PARTICLES)
      STOP
  440 I=XN(K)/DX+2
      J=YN(K)/DY+2
      IF(F(K).EQ.3)441,443
  441 IF(I.EQ.NI-1.OR.I.EQ.NI-2)456,445
  456 IF(J.EQ.1)443,445
  443 IF(F(K).EQ.4)449,433
  449 IF(I.EQ.NI-1.OR.I.EQ.NI-2)457,450
  457 IF(J.EQ.1)442,450
  442 F(K)=3
      WRITE(61,444) DENS(K)
  444 FORMAT(5X,19HPARTICLE OF DENSITY,F18.8,3HOUT)
      GO TO 433
  450 IF(I.GT.NIMNS1.OR.I.LT.2.OR.J.GT.NJMNS1.OR.J.LT.2)445,433
  445 F(K)=1
      X(K)=0
      Y(K)=0
      DENS(K)=0.
      XN(K)=0
      YN(K)=0
  433 CONTINUE
      WRITE(61,446)
  446 FORMAT(5X,19HPARTICLES REFLAGGED)
      DO 447 K=1,NK
      IF(F(K).EQ.1)447,448
  448 X(K)=XN(K)
      Y(K)=YN(K)
  447 CONTINUE
C
C     TEMPERATURE BOUNDARY CONDITIONS
C
      DO 500 I=1,NI
      DO 500 J=1,NJ
      IF(IC(I,J).EQ.12)501,500
  501 T(I,J)=0.0
  500 CONTINUE
C     BOTTOM WALL IS ADIABATIC
      DO 502 I=2,NIMNS1
  502 T(I,1)=T(I,2)
C     TOP WALL IS ADIABATIC
```

```
      DO 503 I=2,NIMNS1
  503 T(I,NJ)=T(I,NJ-1)
C     RIGHT WALL IS ADIABATIC
      DO 504 J=2,NJMNS1
  504 T(NI,J)=T(NI-1,J)
C     LEFT WALL IS CONSTANT PROFILE
      DO 505 J=2,NSUR
  505 T(1,J)=TO(J)
C     FREE SURFACE IS ADIABATIC
      DO 506 I=2,NIMNS1
      DO 506 J=2,NJMNS1
      IF(IC(I,J).EQ.11)507,506
  507 IF(IC(I,J+1).EQ.10)508,509
  508 T(I,J)=T(I,J+1)
  509 IF(IC(I+1,J).EQ.10)510,511
  510 T(I,J)=T(I+1,J)
  511 IF(IC(I-1,J).EQ.10)512,513
  512 T(I,J)=T(I-1,J)
  513 IF(IC(I,J-1).EQ.10)514,506
  514 T(I,J)=T(I,J-1)
  506 CONTINUE
C
C     TEMPERATURE CALCULATION
C
      DO 515 I=2,NIMNS1
      DO 515 J=2,NJMNS1
      IF(IC(I,J).EQ.10)516,515
  516 TL=.5*(T(I-1,J)+T(I,J))
      TR=.5*(T(I+1,J)+T(I,J))
      TT=.5*(T(I,J+1)+T(I,J))
      TB=.5*(T(I,J-1)+T(I,J))
      DTX2=(T(I+1,J)-2*T(I,J)+T(I-1,J))/(DX*DX)
      DTY2=(T(I,J+1)-2*T(I,J)+T(I,J-1))/(DY*DY)
      TN(I,J)=T(I,J)+DT*((U(I,J)*TL-U(I+1,J)*TR)/DX+(V(I,J)*TB-V(I,J+1)*
     1TT)/DY+ALPHA*MU/R(I,J)*(DTX2+DTY2))
  515 CONTINUE
      DO 517 I=2,NIMNS1
      DO 517 J=2,NJMNS1
      IF(IC(I,J).EQ.10)518,517
  518 T(I,J)=TN(I,J)
  517 CONTINUE
C
C     OUTPUT ROUTINES
C
  600 CONTINUE
      IF(MMM.EQ.ITIME)601,602
  602 MMM=MMM+1
      GO TO 603
  601 WRITE(61,604)
  604 FORMAT(1H1)
      WRITE(61,605) TIME
  605 FORMAT(5X,6HTIME =,F6.3,///)
      DO 624 I=1,NI
      DO 624 J=1,NJ
  624 D(I,J)=(U(I+1,J)-U(I,J))/DX+(V(I,J+1)-V(I,J))/DY
      DO 629 I=2,NI
      DO 629 J=1,NJ
  629 DR(I,J)=R(I,J)-RN(I,J)
      DO 703 I=1,NI
      DO 703 J=1,NJ
```

```
703  WRITE(61,704) I,J,IC(I,J),U(I,J),V(I,J),R(I,J),P(I,J),T(I,J),D(I,J
    1),DR(I,J)
704  FORMAT(5X,3I8,7F11.6)
     MMM=1
603  TIME=TIME+DT
     IF(TIME.LE.TIMELIM)100,623
623  CALL TVEND
     STOP
     END
```