AN ABSTRACT OF THE THESIS OF

Arthur Eugene Olson, Jr. for the M.A. in Mathematics
(Name) (Degree) (Major)

Date thesis is presented June 23, 1966

Title NEW THEOREMS AND EXAMPLES FOR FREEDMAN'S DENSITY SPACES

Redacted for Privacy

Abstract approved ______________________________
(Major professor)

Allen Freedman defined a density space to be the ordered pair $(S, \mathfrak{F})$ where S is a certain kind of semigroup called an s-set and $\mathfrak{F}$ is a particular type of family of finite subsets of S called a fundamental family on S.

New theorems for s-sets are presented, with emphasis on the generation of s-sets from subsets of the positive real numbers. Several theorems are proved concerning density inequalities for a special class of fundamental families called discrete fundamental families. A list of density spaces and the density properties they satisfy is included.

NEW THEOREMS AND EXAMPLES FOR FREEDMAN'S DENSITY SPACES

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF ARTS

June 1967

APPROVED:

Redacted for Privacy

Associate Professor of Mathematics

In Charge of Major

Redacted for Privacy

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Date thesis is presented June 23, 1966

Typed by Muriel Davis

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor Robert Stalley for his help in the writing of this thesis.

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CHAPTER I

INTRODUCTION

In 1930, L. Schnirelmann [9, 10] introduced the following density for a subset A of the positive integers: Let $A(n)$ be the number of positive integers in the set A which do not exceed n. Then the Schnirelmann density of A is given by

$$(1) \qquad \alpha = g\ell b\, \{\frac{A(n)}{n} \mid n \geq 1\} .$$

Definition 1.1. Let A and B be two subsets of the positive integers. The sum of A and B, denoted by $A + B$, is the set

$$A \cup B \cup \{a + b \mid a \in A,\ b \in B\} .$$

Now let α, β, and γ denote the Schnirelmann densities of A, B, and $C = A + B$ respectively. Some of the results which have been obtained are:

(2) If $\alpha + \beta \geq 1$, then $\gamma = 1$ (Schnirelmann [10]).

(3) $\gamma \geq \alpha + \beta - \alpha\beta$ (E. Landau [7] and Schnirelmann [10]).

(4) If $\alpha + \beta < 1$, then $\gamma \geq \beta/(1-\alpha)$ (I. Schur [11]).

(5) $\gamma \geq \min\{1, \alpha + \beta\}$ (H. Mann [8] and F. Dyson [2].

In 1965, A. Freedman [3, p. 1] generalized the notion of Schnirelmann density to arbitrary sets as follows:

Definition 1.2. Let S be an arbitrary set. For a subset X of S, and a finite subset D of S, let X(D) denote the number of elements in the set $X \cap D$. Let $\mathcal{G}$ be any family on non-empty finite subsets of S. Then the density of a subset A of S, with respect to $\mathcal{G}$, is

$$\alpha = g\ell b \left\{ \frac{A(G)}{S(G)} \mid G \in \mathcal{G} \right\} .$$

Then Freedman [3, p. 8-15] developed a general theory for density by introducing two sets of axioms. The first set of axioms requires S to be a certain type of abelian semi-group which is called an s-set. The second set of axioms gives structure to the family $\mathcal{G}$, which is then called a fundamental family on S. The pair $(S, \mathfrak{F})$ where S is an s-set and $\mathfrak{F}$ is a fundamental family on S is called a density space. Freedman [3, p. 51-103] was able to extend many of the results which have been obtained for positive integers, including (2), (3), (4), and (5).

In this thesis we present new theorems useful in the construction of s-sets and study a special class of fundamental families called discrete fundamental families. In Chapter II we state those definitions and results of Freedman which are utilized in our work.

Freedman [3, p. 28-40] also provided a list of examples of s-sets and fundamental families. In Chapter III we make a detailed study of methods for constructing s-sets, particularly those which are subsets of the positive real numbers. We also study methods for constructing fundamental families.

In Chapter IV we study a special class of fundamental families which Freedman [3, p. 101-103] calls discrete fundamental families.

In Chapter V we investigate the relationships between inequalities (3) and (4) for density spaces and two transformation properties defined by Freedman [3, p. 43-44].

In Chapter VI we list ten examples of density spaces and summarize the results which have been obtained for each of them.

Throughout this thesis various unsolved problems are stated. Such problems are clearly marked.

CHAPTER II

BACKGROUND

In this chapter we state those definitions and theorems from Freedman's thesis [3, p. 8-78] which are used in the remainder of the thesis. Proofs of these theorems are found in Freedman's thesis.

2.1. s-sets

Throughout this section, unless otherwise indicated, S is a non-empty subset of an abelian group G. The operation in G is denoted by $+$ and the identity element by 0.

Definition 2.1. For x and y in G, we write $x \prec y$ (or $y \succ x$) whenever $y - x \in S$.

Definition 2.2. For $x \in S$, let $L(x)$ denote the set of all $y \in S$ for which $y \prec x$ or $y = x$. We call $L(x)$ the lower set of x with respect to S.

The set S is called an s-set whenever the following three axioms are satisfied.

Axiom s-1. S is closed under $+$.

Axiom s-2. $0 \notin S$.

Axiom s-3. $L(x)$ is finite for each $x \in S$.

It is easy to see the set I of positive integers is an s-set.

Now we look at some theorems which enable us to construct new s-sets from given ones.

Theorem 2.1. If T is a closed subset of an s-set S, then T is an s-set.

For example, the set of even positive integers is an s-set.

Definition 2.3. For a set X contained in a group G, we denote by X^o the set $X \cup \{0\}$ where 0 is the identity element of G.

Definition 2.4. Let Δ be a non-empty index set. Consider, for each $\delta \in \Delta$, a set X_δ contained in an abelian group G. Let X be the set of all functions f defined on Δ which satisfy the following two properties:

(i) $f(\delta) \in X_\delta^o$ for each $\delta \in \Delta$,

(ii) the set of δ for which $f(\delta) \neq 0$ is non-empty and finite.

We call X the product of the X_δ and write $X = \overline{\Pi} \{X_\delta \mid \delta \in \Delta\}$.

Theorem 2.2. If S_δ is an s-set for each $\delta \in \Delta$, then $S = \overline{\Pi} \{S_\delta \mid \delta \in \Delta\}$ is an s-set. Addition is defined by

$$(f_1 + f_2)(\delta) = f_1(\delta) + f_2(\delta).$$

Theorem 2.3. Let T be an s-set and G be a finite abelian group. Then the set $S = G \times T = \{(x, y) \mid x \in G,\ y \in T\}$ is an s-set where addition on S is defined by

$$(x, y) + (x', y') = (x + x', y + y').$$

2.2 Fundamental Families

Before listing the fundamental family axioms we must define some terminology used in the axioms.

Definition 2.5. Let S be an s-set and let $x \in S$. Denote by $U(x)$ the set of all $y \in S$ such that $x \prec y$.

Note that x is never a member of $U(x)$.

Definition 2.6. Let X be a subset of an s-set S. An element $x \in X$ is called a *maximal element* of X if $X \cup U(x) = \phi$. The set of all maximal elements of X is denoted by $Max(X)$.

Definition 2.7. For an arbitrary set S, let $\mathfrak{D} = \mathfrak{D}(S)$ denote the family of all non-empty finite subsets of S.

We write $F \setminus A$ for the set of all elements in F and not in A.

Definition 2.8. Let $\mathfrak{F}$ be an arbitrary subfamily of $\mathfrak{D}$ and let F be a set in $\mathfrak{F}$. An element $x \in F$ is called a corner element of F if either $F = \{x\}$ or $F \setminus \{x\} \in \mathfrak{F}$. The set of all corner elements of F is denoted by F^*.

Let S be an s-set. A non-empty family $\mathfrak{F} \subseteq \mathfrak{D}(S)$ is called a fundamental family on S if the following four axioms are satisfied:

Axiom f-1. For each $x \in S$ there is an $F \in \mathfrak{F}$ with $x \in F$.

Axiom f-2. The union of any non-empty finite subfamily of $\mathfrak{F}$ is a set in $\mathfrak{F}$.

Axiom f-3. The intersection of any non-empty subfamily of $\mathfrak{F}$ is a set in $\mathfrak{F}$, provided the intersection is non-empty.

Axiom f-4. If $F \in \mathfrak{F}$, then $\mathrm{Max}(F) \subseteq F^*$.

Definition 2.9. The ordered pair $(S, \mathfrak{F})$ is called a density space whenever S is an s-set and $\mathfrak{F}$ is a fundamental family on S.

The following theorem is useful in the actual construction of fundamental families for s-sets.

Theorem 2.4. Let S be an arbitrary s-set. Corresponding to each $x \in S$, let $B(x)$ be a subset of S satisfying the following three conditions:

(i) $x \in B(x)$,

(ii) $B(x) \subseteq L(x)$,

(iii) if $y \in B(x)$, then $B(y) \subseteq B(x)$.

Let $\mathfrak{F}_B = \{F \mid F \in \mathfrak{D}(S),\ x \in F$ implies $B(x) \subseteq F\}$. Then $\mathfrak{F}_B$ is a fundamental family on S. Conversely, given any fundamental family $\mathfrak{F}$ on S, there exists a function B(x) satisfying conditions (i), (ii), and (iii) such that $\mathfrak{F}_B = \mathfrak{F}$.

Definition 2.10. Let $x \in S$. Denote by $[x]$ the intersection of all F such that $x \in F \in \mathfrak{F}$. Then $[x]$ is called the Cheo set of $\mathfrak{F}$ determined by x.

Theorem 2.5. Let S be an s-set and let B(x) satisfy conditions (i), (ii), and (iii) of Theorem 2.4. Then $[x]$, the Cheo set of $\mathfrak{F}_B$ determined by x, is equal to B(x).

We define a special fundamental family as follows:

Definition 2.11. Let S be an s-set. We denote by $\mathcal{K} = \mathcal{K}(S)$ the fundamental family $\mathfrak{F}_B$ with $B(x) = L(x)$ for each $x \in S$.

Theorem 2.6. For any fundamental family $\mathfrak{F}$ on an s-set S we have $\mathcal{K} \subseteq \mathfrak{F} \subseteq \mathfrak{D}$.

The following theorem enables us to construct new fundamental families from given ones. We will consider this problem in more detail in the next chapter.

Theorem 2.7. Let $\{\mathfrak{F}_\delta \mid \delta \in \Delta\}$ be a non-empty class of fundamental families on an s-set S. Then $\cap\{\mathfrak{F}_\delta \mid \delta \in \Delta\}$ is a fundamental family on S.

2.3 Density

In the remainder of this chapter let $(S, \mathcal{F})$ be an arbitrary density space.

Definition 2.12. Let X be a subset of S. For any finite (possibly empty) subset D of S we let X(D) be the number of elements in the set $X \cap D$. If D is non-empty, let q(X, D) be the quotient X(D)/S(D).

Definition 2.13. Let A be an arbitrary subset of S. The K-density of A with respect to $\mathcal{F}$ is

$$d(A, \mathcal{F}) = g\ell b \{ q(A, F) \mid F \in \mathcal{F} \}.$$

Definition 2.14. Let A be an arbitrary subset of S. The C-density of A with respect to $\mathcal{F}$ is

$$d_c(A, \mathcal{F}) = g\ell b \{ q(A, [x]) \mid x \in S \},$$

where [x] is the Cheo set of $\mathcal{F}$ determined by x.

K-density generalizes the density defined by B. Kvarda [6] and C-density generalizes the density used by L. Cheo [1] and F. Kasch [5]. Both of these densities reduce to Schnirelmann density when the density space is $(I, \mathcal{K})$, where we recall that I denotes the s-set of positive integers. It is immediate from the definitions that $0 \leq d(A, \mathfrak{F}) \leq 1$ and $0 \leq d_c(A, \mathfrak{F}) \leq 1$.

Definition 2.15. Let A and B be subsets of S. The sum $A + B$ of A and B is the set

$$A \cup B \cup \{a + b \mid a \in A,\ b \in B\}.$$

In all that follows A and B are arbitrary subsets of S. We write $C = A + B$, $\alpha = d(A, \mathfrak{F})$, $\beta = d(B, \mathfrak{F})$, and $\gamma = d(C, \mathfrak{F})$. We also write $\alpha_c = d_c(A, \mathfrak{F})$, $\beta_c = d_c(B, \mathfrak{F})$, and $\gamma_c = d_c(C, \mathfrak{F})$.

Theorem 2.8. $\gamma \geq \max\{\alpha, \beta\}$.

Theorem 2.9. If $\alpha + \beta \geq 1$, then $\gamma = 1$.

Theorem 2.10. $\gamma_c \geq \max\{\alpha_c, \beta_c\}$.

Theorem 2.11. If $\mathfrak{F} = \mathcal{K}(S)$ and $\alpha_c + \beta_c \geq 1$, then $\gamma_c = 1$.

Definition 2.16. A fundamental family $\mathfrak{F}$ is separated if, whenever x and y are elements of S such that $x \notin [y]$ and $y \notin [x]$, then $[x] \cap [y]$ is empty.

Theorem 2.12. If $\mathfrak{F}$ is a separated fundamental family, then K-density and C-density are identical. That is, $\alpha = \alpha_c$ for each $A \subseteq S$.

Freedman [3, p. 43-44] defines two transformation properties, which we call T-1 and T-2 respectively.

Definition 2.17. For each $F \in \mathfrak{F}$ and $x \in F^o$, let $D = F \cap U(x)$ and $T_1[D] = \{y - x \mid y \in D\}$. Then $\mathfrak{F}$ is T-1 if $T_1[D]$ is in $\mathfrak{F} \cup \{\phi\}$ for every F and $x \in F^o$.

Definition 2.18. For each $x \in S$ and $F \in \mathfrak{F} \cup \{\phi\}$, let $D = [x] \setminus F$ and $T_2[D] = \{x - y \mid y \in D \setminus \{x\}\}$. Then $\mathfrak{F}$ is T-2 if $T_2[D]$ is in $\mathfrak{F} \cup \{\phi\}$ for every x and F.

Theorem 2.13. If S is an s-set, then $\mathcal{K}(S)$ is both T-1 and T-2.

Freedman [3, p. 65-78] uses properties T-1 and T-2 to obtain several results including the following:

Theorem 2.14. If $\mathfrak{F}$ is T-1, then $\gamma \geq \alpha + \beta - \alpha\beta$.

Theorem 2.15. If $\mathfrak{F}$ is T-2 and $\alpha + \beta < 1$, then $\gamma \geq \beta/(1 - \alpha)$.

Theorem 2.16. If $\mathfrak{F}$ is T-1, then $\gamma_c \geq \alpha_c + \beta - \alpha_c \beta$.

CHAPTER III

DENSITY SPACES: THEOREMS AND EXAMPLES

In Section 2.1 we saw several ways to obtain new s-sets from given ones. In this chapter we develop theorems which are useful in constructing s-sets, particularly s-sets which are subsets of the set of positive real numbers. We also study the problem of constructing fundamental families.

3.1. Definitions and Preliminary Theorems for Some s-sets of Positive Real Numbers

Let R^+ denote the set of all positive real numbers.

Definition 3.1. Let A be any subset of R^+. Let S be the set of all finite linear combinations of elements of A with positive integral coefficients. Then we say S is the set generated by A.

Definition 3.2. Let $A \subseteq R^+$ and $x \in S$, where S is the set generated by A. Each way in which x is generated by A is called a representation of x in A. We say x is finitely represented in A if it can be generated by A in only a finite number of ways. The sum of the coefficients of a representation is called the length of the representation.

Theorem 3.1. Let $A \subseteq R^+$ and $x, y \in S$, where S is the set generated by A. Then $y \prec x$ iff a representation of y in A can be extended to a representation of x in A by adding a finite, non-zero number of elements of A to it.

Proof: By Definition 2.1, we have $y \prec x$ iff $x - y \in S$. But $x - y \in S$ iff there is a representation of $x - y$ in A, which is equivalent to saying that a representation of y in A can be extended to a representation of x in A by adding a finite number of elements of A to it.

Theorem 3.2. Let $A \subseteq R^+$ and $x \in S$, where S is the set generated by A. Then $L(x)$ is finite iff x is finitely represented in A.

Proof: Assume x is finitely represented in A. Let n be the number of distinct representations of x in A and let h be the length of the longest one. Then there can be at most nh distinct elements of A occurring in representations of x in A. Let A' be the set of elements of A occurring in representations of x in A. Let m be the number of elements in A'. Of course $m \leq nh$. Let

$$B = \{b \mid b = \sum_{i=1}^{m} b_i a_i \text{ , } a_i \in A' \text{ , } b_i \text{ non-negative integers, } \sum_{i=1}^{m} b_i \leq h\} .$$

Then B is a finite set. Now let $y \in L(x)$. If $y \neq x$, then $y \prec x$ and $y \in S$, and so by Theorem 3.1, any representation of y in A is of length $h-1$ or less and consists entirely of elements of A'. Thus $y \in B$. If $y = x$, we also have $y \in B$. Hence, $L(x) \subseteq B$ and $L(x)$ is finite.

Assume x is not finitely represented in A. We show that the set A' of elements occurring in representations of x in A is infinite by supposing that A' is finite and obtaining a contradiction. If A' is finite there is a least element $a_o \in A'$. Let h be the greatest integer part of $\frac{x}{a_o}$. Each representation of x in A must be of length h or less. Therefore, we can construct only a finite number of different representations of x in A. This contradicts our assumption that x is not finitely represented in A. Therefore, A' is infinite. If $a \in A'$ and $a \neq x$, then a occurs in some representation of x in A. By adding the rest of that representation to the element a, we can extend a to a representation of x in A by adding only a finite number of elements of A to a. Hence, by Theorem 3.1, we have $a \prec x$. Therefore, $a \in L(x)$ and we have $L(x)$ infinite. Hence, $L(x)$ finite implies x is finitely represented in A and the proof is complete.

<u>Definition 3.3.</u> Let $A \subseteq R^+$. If there exists a sequence $\{a_n\}$ of distinct elements of A such that $\lim_{n \to \infty} a_n = a$, then a is called an <u>accumulation point</u> of A.

Theorem 3.3. Let $A \subseteq R^+$. Let S be the set generated by A. If A has no accumulation points, then S is an s-set.

Proof: Axioms s-1 and s-2 are immediate. To verify Axiom s-3 we must show that $L(x)$ is finite for all $x \in S$. Consider any $x \in S$. Since A has no accumultion points, there are only a finite number of elements of A in the interval $(0, x]$. Therefore, the set $A' = \{a \mid a \in A, a \prec x \text{ or } a = x\}$ is finite. Let a_o be the smallest element (real number) in A'. Let h be the greatest integer part of $\frac{x}{a_o}$. Then each representation of x in A has length h or less. However, we can construct only a finite number of different linear combinations of elements of A' with coefficient sums not exceeding h, because A' is a finite set and h is finite. Therefore, x is finitely represented in A, and by Theorem 3.2, we have $L(x)$ finite. Therefore, S is an s-set and the proof is complete.

If A is finite, it has no accumulation point, and so by Theorem 3.3, we have S is an s-set. Hence, we have proved the following corollary.

Corollary 3.4. Let $A \subseteq R^+$. Let S be the set generated by A. If A is finite, then S is an s-set.

3.2. Some s-sets Generated by Strictly Increasing Sequences of Positive Real Numbers

Lemma 3.5. Let $\{a_n\}$ be a strictly increasing convergent sequence of positive real numbers. Let A denote the set of elements in $\{a_n\}$ and let S be the set generated by A. Then there are no strictly decreasing convergent sequences in S.

Proof: Since A has a least element a_1, which is positive, the maximum length of the representations of x in A, for any $x \in S$, is finite. Hence, the proof is by induction over the maximum of the sum of the coefficients of linear combinations of elements of A. Let S_p denote the set of elements generated from A by linear combinations with coefficient sum not exceeding p.

When $p = 1$, we have $S_1 = A$. Any strictly decreasing sequence in S_1 must begin with an element in $\{a_n\}$, say a_j. However, $\{a_n\}$ is strictly increasing and hence exactly $j-1$ elements of A are less than a_j. Therefore, there exist no strictly decreasing infinite sequences in $S_1 = A$.

For our induction step, we assume that there are no strictly decreasing convergent sequences in S_k, for some $k(k \geq 1)$.

Let $x \in R^+$. We must show that there are no strictly decreasing sequences in S_{k+1} which converge to x. Let $\lim_{n\to\infty} a_n = a$. From our induction hypothesis, we know that there are no strictly decreasing sequences in S_k which converge to x, none converging to $x - a$, and none converging to $x - a_i$ for each positive integer i. Therefore there exist positive numbers ε_x, ε_a, and ε_i for each positive integer i such that the open intervals $(x, x + \varepsilon_x)$, $(x - a, x - a + \varepsilon_a)$, and $(x - a_i, x - a_i + \varepsilon_i)$ for each positive integer i contain no elements of S_k.

Since $\{a_n\}$ converges to a, we can find an integer $J > 0$ such that $a - a_i < \frac{1}{2}\varepsilon_a$ for all $i > J$. Let

$$(1) \qquad \varepsilon = \min\{\varepsilon_x, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_J, \tfrac{1}{2}\varepsilon_a\}.$$

We proceed to show that there are no elements of S_{k+1} in the open interval $(x, x + \varepsilon)$.

By (1), we have $\varepsilon \leq \varepsilon_x$. Therefore, there are no elements of S_k in $(x, x + \varepsilon)$, since there are no elements of S_k in $(x, x + \varepsilon_x)$. Hence, in order for an element of S_{k+1} to be in $(x, x + \varepsilon)$, there must be an element of S_k in at least one of the intervals $(x - a_i, x - a_i + \varepsilon)$, because all elements in S_{k+1} not also in S_k are of the form $y + a_i$ where $y \in S_k$ and $a_i \in A$.

There are no elements of S_k in $(x - a_i, x - a_i + \varepsilon)$ for $i (1 \leq i \leq J)$, because by (1), $\varepsilon \leq \varepsilon_i$ for all $i (1 \leq i \leq J)$. If $i > J$, then $a - a_i < \frac{1}{2}\varepsilon_a$. Therefore, $x + a - a_i < x + \frac{1}{2}\varepsilon_a$, and so $x - a_i + \varepsilon < x - a + \frac{1}{2}\varepsilon_a + \varepsilon$. Since $\varepsilon \leq \frac{1}{2}\varepsilon_a$ by (1), we have $x - a_i + \varepsilon < x - a + \varepsilon_a$. Hence $(x - a_i, x - a_i + \varepsilon) \subset (x - a, x - a + \varepsilon_a)$, which contains no elements of S_k. Therefore, there are no elements of S_k in $(x - a_i, x - a_i + \varepsilon)$ for $i > J$.

We have shown that there are no elements of S_{k+1} in $(x, x + \varepsilon)$. Hence, there are no strictly decreasing sequences in S_{k+1} which converge to x. But x is arbitrary, so the proof is complete.

Theorem 3.6. Let $\{a_n\}$ be a strictly increasing convergent sequence of positive real numbers. Let A denote the set of elements in $\{a_n\}$ and let S be the set generated by A. Then S is an s-set.

Proof: Axioms s-1 and s-2 are immediate. To verify Axiom s-3, we must show that $L(x)$ is finite for each $x \in S$. Assume that there is an $x_o \in S$ such that $L(x_o)$ is not finite. By Theorem 3.2, we know that x_o is not finitely represented in A. Moreover, since $\{a_n\}$ is a strictly increasing sequence, we have $a_1 < a_i$ for all $i (i > 1)$. Let h be the greatest integer part of $\frac{x_o}{a_1}$. Then each representation of x_o in A has length h or less.

Let $\{a_n^*\}$ be the set of all elements of A, listed in strictly increasing order, which occur in at least one representation of x_o. Then $\{a_n^*\}$ contains an infinite number of elements. If not, x_o would be finitely represented in A, since each representation has length h or less and there would be only finitely many elements of A available.

Consider the sequence $\{x_o - a_n^*\}$. Each $x_o - a_n^* \in S$ because each a_n^* is in at least one representation of x_o. The sequence $\{x_o - a_n^*\}$ is strictly decreasing, because $\{a_n^*\}$ is strictly increasing. Let $\lim_{n\to\infty} a_n = a$. Then $\lim_{n\to\infty} (x_o - a_n^*) = x_o - a$, since $\{a_n^*\}$ is an infinite subsequence of the convergent sequence $\{a_n\}$ and hence converges to the same limit. But now we have a strictly decreasing convergent sequence in S, which contradicts Lemma 3.5. Therefore, L(x) is finite for each $x \in S$ and Axiom s-3 holds. Hence, S is an s-set and the proof is complete.

Theorem 3.6 can be generalized to the following theorems.

Theorem 3.7. Given a finite number of strictly increasing convergent sequences of positive real numbers, let A be the set of elements which occur in at least one of the sequences. Let S be the set generated by A. Then if S is non-empty, it is an s-set.

Theorem 3.8. Given a countably infinite collection of strictly increasing convergent sequences of positive real numbers, let A be the set of elements which occur in at least one of the sequences. Let S be the set generated by A. Then S is an s-set if the following condition holds:

(*) for each $x \in R^+$, the interval $(0, x]$ contains elements from only a finite number of the sequences.

The proof of Theorems 3.7 and 3.8 parallel that of Theorem 3.6. We omit the details of these proofs but indicate how they may be constructed from the proof of Theorem 3.6. In each case we first show that there are no strictly decreasing convergent sequences in S, which uses a proof paralleling that of Lemma 3.5. Then we show that S is an s-set. The basic difference is that at each step we must work with a finite number of sequences instead of just one. We only need to work with a finite number of sequences, even in the proof of Theorem 3.8, because given any $x \in R^+$, all but a finite number of the sequences occur completely to the right of x on the real line. In Theorem 3.8, this follows immediately from condition (*). In fact, if we do not have (*), the set S is not always an s-set. For example, if the first elements of the sequences form the strictly decreasing sequence $\{\frac{1}{n}\}$, an s-set is not generated, as we shall see in Theorem 3.15.

We can generalize Theorem 3.7 to the following theorem.

<u>Theorem 3.9.</u> Given a finite number of strictly increasing convergent sequences of positive real numbers, let A be the set of elements which occur in at least one of the sequences. Let a

be the maximum of the set of limits of the given sequences. Let B be a finite or countably infinite set of positive real numbers having no accumulation point and subject to the condition that $b > a$ for each $b \in B$. Let S be the set generated by $A \cup B$. Then S is an s-set.

Proof: By properly placing more elements in the set B, we can create a finite or countably infinite number of strictly increasing convergent sequences satisfying condition (*) of Theorem 3. 8. Call such a set of additional elements B'. Let S' be the set generated by $A \cup B \cup B'$. By Theorem 3. 8, or Theorem 3. 7, we know S' is an s-set. Therefore, by Theorem 2. 1, S is an s-set, because S is a closed subset of S'.

3. 3. Some s-sets Generated by Strictly Decreasing Sequences of Positive Real Numbers

Now we are ready to examine strictly decreasing convergent sequences of positive real numbers. The word convergent can be omitted, because all strictly decreasing sequences of positive real numbers converge. The next lemma is similar to Lemma 3. 5.

Lemma 3. 10. Let $\{a_n\}$ be a strictly decreasing (convergent) sequence of positive real numbers. Let A denote the set of elements in $\{a_n\}$ and let S be the set generated by A. Let

$\lim_{n\to\infty} a_n = a$. If $a > 0$, then there are no strictly increasing convergent sequences in S.

Proof: Since $a > 0$, the maximum length of the representations of x in A, for any $x \in S$, is finite. Hence, the proof is by induction over the maximum of the sum of the coefficients of linear combinations of elements of A. Let S_p denote the set of elements generated from A by linear combinations with coefficient sum not exceeding p.

When $p = 1$, we have $S_1 = A$. Any strictly increasing sequence in S_1 must begin with an element in $\{a_n\}$, say a_j. However, $\{a_n\}$ is strictly decreasing and hence exactly $j - 1$ elements of A are greater than a_j. Therefore, there exist no strictly increasing convergent sequences in $S_1 = A$.

For our induction step, we assume that there are no strictly increasing convergent sequences in S_k, for some $k (k \geq 1)$.

Let $x \in R^+$. We must show that there are no strictly increasing sequences in S_{k+1} which converge to x. From an induction hypothesis, we know that there are no strictly increasing sequences in S_k which converge to x, none converging to $x - a$, and none converging to $x - a_i$ for each positive integer i. Therefore, there exist positive numbers ε_x, ε_a, and ε_i for each positive integer i such that the open intervals $(x - \varepsilon_x, x)$,

$(x - a - \varepsilon_a, x - a)$, and $(x - a_i - \varepsilon_i, x - a_i)$ for each positive integer i contain no elements of S_k.

Since a_n converges to a, we can find an integer $J > 0$ such that $a_i - a < \frac{1}{2}\varepsilon_a$ for all $i > J$. Let

$$(2) \qquad \varepsilon = \min \{\varepsilon_x, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_J, \frac{1}{2}\varepsilon_a\}.$$

We proceed to show that there are no elements of S_{k+1} in the open interval $(x - \varepsilon, x)$.

By (2), we have $\varepsilon \leq \varepsilon_x$. Therefore, there are no elements of S_k in $(x - \varepsilon, x)$, since there are no elements of S_k in $(x - \varepsilon_x, x)$. Hence, in order for an element of S_{k+1} to be in $(x - \varepsilon, x)$, there must be an element of S_k in at least one of the intervals $(x - a_i - \varepsilon, x - a_i)$ because all elements in S_{k+1} not also in S_k are of the form $y + a_i$ where $y \in S_k$ and $a_i \in A$.

There are no elements of S_k in $(x - a_i - \varepsilon, x - a_i)$ for $i (1 \leq i \leq J)$, because by (2), $\varepsilon \leq \varepsilon_i$ for all $i (1 \leq i \leq J)$. If $i > J$, then $a_i - a < \frac{1}{2}\varepsilon_a$. Therefore, $x + a_i - a < x + \frac{1}{2}\varepsilon_a$, and so $x - a - \varepsilon_a < x - a_i - \frac{1}{2}\varepsilon_a$. Since $\varepsilon \leq \frac{1}{2}\varepsilon_a$ by (2), we have $x - a - \varepsilon_a < x - a_i - \varepsilon$. Hence, $(x - a_i - \varepsilon, x - a_i) \subset (x - a - \varepsilon_a, x - a)$ which contains no elements of S_k. Therefore, there are no elements of S_k in $(x - a_i - \varepsilon, x - a_i)$ for $i > J$.

We have shown that there are no elements of S_{k+1} in $(x - \varepsilon, x)$. Hence, there are no strictly increasing sequences in

S_{k+1} which converge to x. But x is arbitrary, so the proof is complete.

This proof is valid only when $\{a_n\}$ is bounded away from zero; that is, when $a > 0$. The reason may not be immediately clear, but suppose $\lim_{n \to \infty} a_n = 0$. Then for any positive integer k, there are no strictly increasing sequences in S_k converging to x. However, there may be strictly increasing sequences in S converging to x. We obtain such a case by taking $A = \{\frac{1}{n}\}$ for all $n (n \geq 1)$, and letting S be the set generated by A. The induction proof remains valid, however, as long as $\{a_n\}$ is bounded away from zero, because then given any $x \in R^+$ there exists an integer $K > 0$ such that for all $k \geq K$, we have S and S_k identical in the interval $(0, x]$.

Theorem 3.11. Let $\{a_n\}$ be a strictly decreasing (convergent) sequence of positive real numbers, where $\lim_{n \to \infty} a_n = a$. Let A denote the set of elements in $\{a_n\}$ and let S be the set generated by A. If $a > 0$, then S is an s-set.

Proof: Axioms s-1 and s-2 are immediate. To verify Axiom s-3, we must show that $L(x)$ is finite for each $x \in S$. Assume that there is an $x_o \in S$ such that $L(x_o)$ is not finite. By Theorem 3.2, we know that x_o is not finitely represented in A.

Moreover, since $\{a_n\}$ is strictly decreasing and $\lim_{n \to \infty} a_n = a > 0$, we have $0 < a < a_i$ for all $i (i \geq 1)$. Let h be the greatest integer part of $\frac{x_o}{a}$. Then each representation of x_o in A has length h or less.

Let $\{a_n^*\}$ be the set of all elements of A, listed in strictly decreasing order, which occur in at least one representation of x_o. Then $\{a_n^*\}$ contains an infinite number of elements. If not, x_o would be finitely represented in A, since each representation has length h or less and there would be only finitely many elements of A available.

Consider the sequence $\{x_o - a_n^*\}$. Each $x_o - a_n^* \in S$ because each a_n^* is in at least one representation of x_o. The sequence $\{x_o - a_n^*\}$ is strictly increasing, because $\{a_n^*\}$ is strictly decreasing. We have $\lim_{n \to \infty} (x_o - a_n^*) = x_o - a$, since $\{a_n^*\}$ is an infinite subsequence of the convergent $\{a_n\}$ and hence converges to the same limit. But now we have a strictly increasing convergent sequence in S, which contradicts Lemma 3.10. Therefore, $L(x)$ is finite for each $x \in S$ and Axiom s-3 holds. Hence, S is an s-set and the proof is complete.

Theorem 3.11 can be generalized to the following theorems.

<u>Theorem 3.12.</u> Given a finite number of strictly decreasing (convergent) sequences of positive real numbers, let A be the

set of elements which occur in at least one of the sequences. Let S be the set generated by A. If each of the given sequences has a positive limit point and S is non-empty, then S is an s-set.

Theorem 3.13. Given a countably infinite collection of strictly decreasing (convergent) sequences of positive real numbers, let A be the set of elements which occur in at least one of the sequences. Let S be the set generated by A. Then S is an s-set if the following conditions hold:

(i) for each $x \in R^+$, the interval $(0, x]$ contains elements from only a finite number of the sequences,

(ii) each of the given sequences has a positive limit point.

The proofs of Theorems 3.12 and 3.13 parallel that of Theorem 3.11. We omit the details of these proofs but indicate how they may be constructed from the proof of Theorem 3.11. In each case we first show that there is no strictly increasing convergent sequence in S, which uses a proof paralleling that of Lemma 3.10. Then we show that S is an s-set. The basic difference is that at each step we must work with a finite number of sequences instead of just one. We only need to work with a finite number of sequences, even in the proof of Theorem 3.13, because given any $x \in R^+$, all but a finite number of the sequences occur completely to the right of x on the real line. In Theorem 3.13 this follows immediately from condition (i). In fact, if we do not have condition (i), the set S is not always an s-set. For example, if the first elements of some of the sequences form the strictly increasing sequence $\{1 - \frac{1}{n}\}$ and if one of the strictly decreasing sequences is $\{1 + \frac{1}{n}\}$, an s-set is not generated, as we shall see in Theorem 3.17.

We can generalize Theorems 3.12 and 3.13 to the following theorem.

<u>Theorem 3.14.</u> Given the conditions of either Theorem 3.12 or Theorem 3.13, let a be the minimum of the limit points of the given sequences. Let B be a finite set of positive real numbers in the open interval $(0, a)$. Let S be the set generated by $A \cup B$. Then S is an s-set.

Proof: Order the elements in B in decreasing order. Call the last element (the smallest) in this finite sequence b. We can extend this strictly decreasing finite sequence of positive reals to a strictly decreasing infinite sequence of positive real numbers which converges to $\frac{b}{2} > 0$ by inserting appropriate elements from the interval $(\frac{b}{2}, b)$ into the sequence. Call these newly inserted elements B'. Let S' be the set generated by $A \cup B \cup B'$. The conditions of Theorem 3.12 or Theorem 3.13 still hold.

Therefore, S' is an s-set. Hence, by Theorem 2.1, the set S is an s-set, because S is a closed subset of S'.

Suppose we have a set S generated by the elements of a strictly decreasing sequence $\{a_n\}$ of real numbers and that $\lim_{n \to \infty} a_n = 0$. Then sometimes S is an s-set and sometimes it is not as we see from the following two theorems.

<u>Theorem 3.15.</u> If A is the set of elements in the sequence $\{\frac{1}{n}\}$ and S is the set generated by A, then S is not an s-set.

First, note that $\{\frac{1}{n}\}$ is strictly decreasing and $\lim_{n \to \infty} \frac{1}{n} = 0$.

Proof: Consider any positive rational number r. We can write $r = \frac{p}{q}$ where p and q are positive integers. Now $\frac{1}{q} \in A$. Therefore, $r = p \cdot \frac{1}{q} \in S$. Hence, S is the set of all positive rational numbers. The value of $L(r)$ is not finite for any positive rational r, because $L(r)$ is the set of all positive rationals less than or equal r. Therefore, Axiom s-3 fails and S is not an s-set.

<u>Theorem 3.16.</u> If A is the set of elements in the sequence $\{t^n\}$ where $0 < t < 1$ and t is transcendental, and S is the set generated by A, then S is an s-set.

First, note that $\{t^n\}$ is strictly decreasing since $0 < t < 1$, and that $\lim_{n \to \infty} t^n = 0$.

Proof: Axioms s-1 and s-2 are immediate. To show that Axiom s-3 holds, consider any $x \in S$ and assume we have two distinct representations of x in A. Thus,

$$x = \sum_{k=1}^{m} b_k t^k ,$$

and

$$x = \sum_{k=1}^{m} c_k t^k ,$$

where the b_k and c_k are non-negative integers, $b_k \neq c_k$ for some $k \,(1 \leq k \leq m)$, and $m = \max \{ k \mid b_k + c_k > 0 \}$.

Then

$$0 = x - x = \sum_{k=1}^{m} b_k t^k - \sum_{k=1}^{m} c_k t^k = \sum_{k=1}^{m} (b_k - c_k)\, t^k ,$$

and so t satisfies a polynomial equation in one unknown and with integral coefficients not all zero. Hence, t is algebraic, a contradiction. Therefore, x has a unique representation in A. By Theorem 3. 2, we have $L(x)$ is finite. But x is an arbitrary element of S. Therefore Axiom s-3 is valid and S is an s-set.

3. 4. Some s-sets Generated by Other Subsets of the Positive Real Number

Suppose we have a set S generated by the elements of a strictly increasing and a strictly decreasing sequence of positive real numbers. Then sometimes S is an s-set and sometimes it is not as we see from the following two theorems.

Theorem 3.17. If A is the set of elements in at least one of the two sequences $\{1 + \frac{1}{n} \mid n = 1, 2, \ldots\}$ and $\{1 - \frac{1}{n} \mid n = 2, 3, \ldots\}$, and S is the set generated by A, then S is not an s-set.

Note that $\{1 - \frac{1}{n}\}$ is strictly increasing and converges to 1 while $\{1 + \frac{1}{n}\}$ is strictly decreasing and converges to 1.

Proof: We can write $2 = (1 + \frac{1}{n}) + (1 - \frac{1}{n})$ for each integer $n(n \geq 2)$. Therefore, since $1 + \frac{1}{n} \in A$ and $1 - \frac{1}{n} \in A$ for each $n(n \geq 2)$, the element 2 is not finitely represented in A. Hence, by Theorem 3.2, L(2) is infinite, Axiom s-3 fails, and S is not an s-set.

Theorem 3.18. If A is the set of elements in at least one of the two sequences $\{1 - t^n\}$ where $0 < t < 1$ and t is transcendental, and $\{1 + \frac{1}{n}\}$, and S is the set generated by A, then S is an s-set.

Note that $\{1 - t^n\}$ is strictly increasing and converges to 1 while $\{1 + \frac{1}{n}\}$ is strictly decreasing and converges to 1.

Proof: Axioms s-1 and s-2 are immediate. To show that Axiom s-3 holds, consider any $x \in S$ and assume we have two distinct representations of x in A. Thus,

$$x = \sum_{k=1}^{m} b_k (1 - t^k) + \sum_{k=1}^{m} g_k (1 + \frac{1}{k}),$$

and

$$x = \sum_{k=1}^{m} c_k (1 - t^k) + \sum_{k=1}^{m} h_k (1 + \frac{1}{k}),$$

where the $b_k, c_k, g_k,$ and h_k are non-negative integers, $b_k \neq c_k$ or $g_k \neq h_k$ for some $k\ (1 \leq k \leq m)$, and $m = \max \{ k \mid b_k + c_k + g_k + h_k > 0 \}$.

Then

$$0 = x - x = \sum_{k=1}^{m} (b_k - c_k)(1 - t^k) + \sum_{k=1}^{m} (g_k - h_k)(1 + \frac{1}{k})$$

$$= \sum_{k=1}^{m} (c_k - b_k) t^k + \sum_{k=1}^{m} [(g_k - h_k)(1 + \frac{1}{k}) + (b_k - c_k)],$$

and so t satisfies a polynomial equation in one unknown and with rational coefficients. Since t is transcendental, we must have $c_k = b_k$ for all $k\,(1 \leq k \leq m)$. Therefore, x uniquely determines the elements from the sequence $\{1 - t^n\}$ which occur in any representation of x in A.

Let

$$x' = \sum_{k=1}^{m} b_k (1 - t^k) .$$

Since x uniquely determines x', we know that x and $x - x'$ both have the same number of representations in A. Therefore, by Theorem 3. 2, we have $L(x)$ is finite iff $L(x - x')$ is finite. However, any representation of $x - x'$ contains only elements from the sequence $\{1 + \frac{1}{n}\}$, which is strictly decreasing and has a positive limit. Now the set S' generated by the elements of the sequence $\{1 + \frac{1}{n}\}$ is an s-set by Theorem 3. 11. Thus, $L(x - x')$ is finite. Therefore, $L(x)$ is finite and S is an s-set.

Theorems 3. 3 through 3. 18 provide us with several sufficient conditions for s-sets of positive real numbers. A corresponding set of theorems can be listed for negative real numbers.

Unsolved Problem. Give necessary and sufficient conditions for a subset of the positive (negative) real numbers to generate an s-set.

3. 5. Other Examples of s-sets

The most useful s-set in our study of density spaces is the set I of positive integers with the usual addition. If, in Theorem 2. 2, we set $S_\delta = I$ for each $\delta \in \Delta$, then

$$S = \Pi \{I \mid \delta \in \Delta\},$$

which Freedman [3, p. 28] denotes by I^n, is an s-set. We will use this example in Chapter VI.

We can construct an unlimited number of different s-sets using Theorems 2.1 through 2.3 and 3.3 through 3.14. All of Freedman's examples [3, p. 28-30] follow from Theorems 2.2, 3.3, and Corollary 3.4.

Unsolved Problem. Construct an s-set which is essentially different from those we can construct with present theorems.

3.6. Construction of Fundamental Families

Theorem 2.4 is very useful in the construction of fundamental families for s-sets. For example, if we are given the s-set I, we can construct a fundamental family on I by letting $B(x) = \{1, 2, \ldots, x\}$ for all $x \in I$. Here we obtain the density space $(I, \mathcal{K})$ which is the one Schnirelmann [9, 10] worked with. Furthermore, if we let $B(x) = \{x\}$ for all $x \in I$ we obtain the density space $(I, \mathcal{D})$. Here $\mathcal{K}$ and $\mathcal{D}$ are defined as in Section 2.2.

From this point on, whenever we define $B(x)$ in order to obtain a particular density space, it is to be understood that conditions (i), (ii), and (iii) of Theorem 2.4 are satisfied.

Theorem 2.7 states that the intersection of any non-empty class of fundamental families on an s-set S is itself a fundamental family. If we replace intersection by union, the theorem fails as we see in the following example on the s-set I.

Let $\mathcal{F}_{B_1}$ be defined by

$$B_1(x) = \begin{cases} \{1,3\} & \text{if } x = 3, \\ \{x\} & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}_{B_2}$ be defined by

$$B_2(x) = \begin{cases} \{2,3\} & \text{if } x = 3, \\ \{x\} & \text{otherwise.} \end{cases}$$

Let $\mathcal{F} = \mathcal{F}_{B_1} \cup \mathcal{F}_{B_2}$. Then $\{1,3\} \in \mathcal{F}_{B_1} \subseteq \mathcal{F}$ and $\{2,3\} \in \mathcal{F}_{B_2} \subseteq \mathcal{F}$, but $\{1,3\} \cap \{2,3\} = \{3\} \notin \mathcal{F}$. Therefore, $\mathcal{F}$ does not satisfy Axiom f-3 and is not a fundamental family on I.

Definition 3.4 [4, p. 189]. A lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound. A lattice is complete if any subset has a least upper bound and a greatest lower bound.

Freedman [3, p. 24] proved the following theorem.

Theorem 3.19. The class of all fundamental families on an s-set S forms a complete lattice with respect to the partial ordering by set inclusion.

Given an s-set S and two fundamental families $\mathfrak{F}_{B_1}$ and $\mathfrak{F}_{B_2}$ on S defined by $B_1(x)$ and $B_2(x)$ respectively, we have seen that $\mathfrak{F}_{B_1} \cup \mathfrak{F}_{B_2}$ is not always a fundamental family. However, Theorem 3.19 assures the existence of a fundamental family $\mathfrak{F}_*$ such that $\mathfrak{F}_{B_1} \cup \mathfrak{F}_{B_2} \subseteq \mathfrak{F}_* \subseteq \mathfrak{F}$ for all fundamental families $\mathfrak{F}$ with the property that $\mathfrak{F}_{B_1} \cup \mathfrak{F}_{B_2} \subseteq \mathfrak{F}$.

Theorem 3.20. Let $\mathfrak{F}_{B_1}$ and $\mathfrak{F}_{B_2}$ be two fundamental families, on an s-set S, defined by $B_1(x)$ and $B_2(x)$ respectively and let $\mathfrak{F}_* = \ell\mu b\ \{\mathfrak{F}_{B_1}, \mathfrak{F}_{B_2}\}$. Then $\mathfrak{F}_*$ is defined by $B(x)$ where $B(x) = B_1(x) \cap B_2(x)$ for all $x \in S$.

Proof: First, we show that $B(x)$ satisfies conditions (i), (ii), and (iii) of Theorem 2.4. Since $x \in B_1(x)$ and $x \in B_2(x)$ for all $x \in S$, we have $x \in B(x) = B_1(x) \cap B_2(x)$. Therefore, condition (i) is satisfied. Now $B(x) = B_1(x) \cap B_2(x) \subseteq B_1(x) \subseteq L(x)$ for all $x \in S$. Hence, condition (ii) is satisfied. Furthermore, if $y \in B(x) = B_1(x) \cap B_2(x)$, we have $y \in B_1(x)$ and $y \in B_2(x)$. Thus, $B_1(y) \subseteq B_1(x)$ and $B_2(y) \subseteq B_2(x)$. Therefore, $B(y) = B_1(y) \cap B_2(y) \subseteq B_1(x) \cap B_2(x) = B(x)$ and condition (iii) is satisfied. Therefore, the family $\mathfrak{F}_B$, defined by $B(x) = B_1(x) \cap B_2(x)$, is a fundamental family on S.

Now $B(x) \subseteq B_1(x)$ and $B(x) \subseteq B_2(x)$. Thus, $\mathfrak{F}_{B_1} \subseteq \mathfrak{F}_B$ and $\mathfrak{F}_{B_2} \subseteq \mathfrak{F}_B$, by the way $\mathfrak{F}_{B_1}$, $\mathfrak{F}_{B_2}$, and $\mathfrak{F}_B$ are defined, and so $\mathfrak{F}_{B_1} \cup \mathfrak{F}_{B_2} \subseteq \mathfrak{F}_B$. We also have $B_1(x) \in \mathfrak{F}_{B_1} \subseteq \mathfrak{F}_*$ and $B_2(x) \in \mathfrak{F}_{B_2} \subseteq \mathfrak{F}_*$, by the way $\mathfrak{F}_{B_1}$ and $\mathfrak{F}_{B_2}$ are defined. Therefore, $B(x) = B_1(x) \cap B_2(x) \in \mathfrak{F}_*$ for each $x \in S$, by Axiom f-3 for fundamental families. However, $B(x) \in \mathfrak{F}_*$ for all $x \in S$ implies $\mathfrak{F}_B \subseteq \mathfrak{F}_*$. Therefore, $\mathfrak{F}_{B_1} \cup \mathfrak{F}_{B_2} \subseteq \mathfrak{F}_B \subseteq \mathfrak{F}_*$ and so $\mathfrak{F}_B = \mathfrak{F}_*$ and the proof is complete.

We conclude the chapter by showing that if $B_1(x)$ and $B_2(x)$ determine fundamental families, then $B_1(x) \cup B_2(x)$ does not always define a fundamental family. Let the s-set be I and let

$$B_1(x) = \begin{cases} \{2, 4\} & \text{if } x = 4, \\ \{1, 3\} & \text{if } x = 3, \\ \{x\} & \text{otherwise.} \end{cases}$$

and

$$B_2(x) = \begin{cases} \{3, 4\} & \text{if } x = 4, \\ \{x\} & \text{otherwise.} \end{cases}$$

Then

$$B_1(x) \cup B_2(x) = \begin{cases} \{2, 3, 4\} & \text{if } x = 4, \\ \{1, 3\} & \text{if } x = 3, \\ \{x\} & \text{otherwise.} \end{cases}$$

However, $B_1(x) \cup B_2(x)$ does not define a fundamental family on I because condition (iii) of Theorem 2.4 fails. To see this note that $3 \in B_1(4) \cup B_2(4)$, but $B_1(3) \cup B_2(3) \not\subseteq B_1(4) \cup B_2(4)$.

CHAPTER IV

DISCRETE FUNDAMENTAL FAMILIES

In this chapter we study discrete fundamental families, and find restrictions needed for various density properties to hold. We also state some problems which have come up in this study and remain unanswered.

4.1. Discrete Fundamental Families in General

Freedman [3, p. 101] defines a discrete fundamental family of order n as follows:

Definition 4.1. A fundamental family $\mathfrak{F}$ on an s-set S is discrete of order n if $\mathfrak{F}$ satisfies the following two conditions:

(i) $\mathfrak{F}$ is separated,

(ii) for each $x \in S$, we have $S([x]) \leq n$ with equality holding for some x.

Recall that $S([x])$ is the number of elements in $S \cap [x]$, which is the same as the number of elements in $S \cap B(x)$. Since $\mathfrak{F}$ is separated, we know by Theorem 2.12 that K-density and C-density are identical. Therefore, whenever we are working with discrete fundamental families we can say that the set A has density α, knowing that α is both the K-density and the C-density of A.

Freedman [3, p. 102-103] proves the following theorem:

Theorem 4.1. Let $(S, \mathfrak{F})$ be a density space where $\mathfrak{F}$ is discrete of order 1 or 2. Let $A, B,$ and $C = A + B$ be subsets of S with densities α, β, and γ respectively. Then $\gamma \geq \min\{1, \alpha + \beta\}$.

The following example shows that the preceding theorem fails for discrete fundamental families of order n for each $n (n \geq 3)$. In fact, Theorem 2.8, which is true for all density spaces, gives the strongest result. Let the s-set be I and let

$$B(x) = \begin{cases} \{1, 3\} & \text{if } x = 3, \\ \{1, 3, 4\} & \text{if } x = 4, \\ \{2, 5\} & \text{if } x = 5, \\ \{2, 5, 6\} & \text{if } x = 6 \\ \{11, 12, \ldots, 10+n\} & \text{if } x = 10+n \ (n \geq 3), \\ \{x\} & \text{otherwise.} \end{cases}$$

It can be verified in a straightforward manner that $B(x)$ satisfies conditions (i), (ii), and (iii) of Theorem 2.4 and hence $\mathfrak{F}_B$ is a fundamental family on I. Likewise it can be verified that $\mathfrak{F}_B$ is separated. Also $I([10+n]) = n$ and $I([x]) \leq n$ for all $x \in I$. Therefore, by Definition 4.1, $\mathfrak{F}_B$ is discrete of order n. Now let $A = B = \{1, 2, 7, 8, 9, \ldots\}$. Then $C = \{1, 2, 3, 4, 7, 8, 9, \ldots\}$. Since $q(A, [x])$, $q(B, [x])$, and $q(C, [x])$ are all minimized at $x = 6$, we have $\alpha = \beta = \gamma = \frac{1}{3}$. Hence, Theorem 2.8 gives the strongest valid result.

In the rest of this chapter we place various restrictions on discrete fundamental families so that we can prove results which are stronger than that given by Theorem 2.8. For some of these results it may be possible, of course, to weaken our additional restrictions somewhat without changing the conclusions. A conjecture of such a result is included later in the form of an unsolved problem.

4.2. Purely Discrete Fundamental Families

Definition 4.2. A fundamental family $\mathfrak{F}$ on an s-set S is *purely discrete* of order n if $\mathfrak{F}$ satisfies the following two conditions:

(i) $\mathfrak{F}$ is discrete of order n,

(ii) if $y \in [x]$ and $y \neq x$, then $[y] = \{y\}$.

Definition 4.2 restricts $\mathfrak{F}$ severely. However, we prove the following theorem which is useful later.

Theorem 4.2. Let $(S, \mathfrak{F})$ be a density space where $\mathfrak{F}$ is purely discrete of order n. Then $\gamma \geq \min\{1, \alpha + \beta\}$.

Proof: Let A be any subset of S, and let α be the density of A. From condition (ii) of Definition 4.2, we conclude that for any $x \in S$, either

(a) $[x] = \{x\}$,

or (b) $[x] = \{x_1, \ldots, x_{i-1}, x\}$ where $[x_j] = \{x_j\}$ for all $j (1 \leq j \leq i-1)$.

In case (a), we have $\alpha = 0$ if $x \notin A$. In case (b), we have $\alpha = 0$ if $x_j \notin A$ for some j. Thus, if $\alpha \neq 0$, we must have $q(A, [x]) = 1$ or $q(A, [x]) = \frac{i-1}{i}$. Therefore, the only densities α, β, and γ can take on are $0, 1$, or $\frac{i-1}{i}$ for all $i (2 \leq i \leq n)$. If either α or β is 0, then by Theorem 2.8, we have $\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha+\beta\}$. If either α or β is 1, then by Theorem 2.9, we have $\gamma = 1 = \min\{1, \alpha+\beta\}$. If $\alpha = \frac{a-1}{a}$ for some $a(2 \leq a \leq n)$ and $\beta = \frac{b-1}{b}$ for some $b(2 \leq b \leq n)$, then

$$\alpha + \beta = \frac{a-1}{a} + \frac{b-1}{b} \geq \frac{1}{2} + \frac{1}{2} = 1.$$

Thus, by Theorem 2.9, we have $\gamma = 1 = \min\{1, \alpha+\beta\}$.

4.3. Singularly Discrete Fundamental Families

Definition 4.3. A fundamental family $\mathcal{F}$ on an s-set S is singularly discrete of order n if $\mathcal{F}$ satisfies the following two conditions:

(i) $\mathcal{F}$ is discrete of order n.

(ii) $S([x]) = i$ for at most one $x \in S$ where $i = 2, 3, \ldots, n$.

Theorem 4.3. Let $(S, \mathfrak{F})$ be a density space where $\mathfrak{F}$ is singularly discrete of order 3. Then $\gamma \geq \min\{1, \alpha+\beta\}$.

Proof: Consider the following cases:

Case I: Let $\mathfrak{F}_B$ be defined by

$$B(x) = \begin{cases} \{x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{x\} & \text{otherwise,} \end{cases}$$

where $x_1 \prec x_2 \prec x_3$. Then $\mathfrak{F}_B$ is purely discrete of order 3 and, by Theorem 4.2, we have $\gamma \geq \min\{1, \alpha+\beta\}$.

Case II: Let $\mathfrak{F}_B$ be defined by

$$B(x) = \begin{cases} \{x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{x_4, x_5\} & \text{if } x = x_5, \\ \{x\} & \text{otherwise,} \end{cases}$$

where $x_1 \prec x_2 \prec x_3$ and $x_4 \prec x_5$. Then $\mathfrak{F}_B$ is purely discrete of order 3 and, by Theorem 4.2, we have $\gamma \geq \min\{1, \alpha+\beta\}$.

Case III: Let $\mathfrak{F}_B$ be defined by

$$B(x) = \begin{cases} \{x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{x_1, x_2\} & \text{if } x = x_2, \\ \{x\} & \text{otherwise.} \end{cases}$$

where $x_1 \prec x_2 \prec x_3$. The only densities possible for α or β are $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.8, we have $\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha+\beta\}$. If

$\alpha + \beta \geq 1$, then by Theorem 2.9, we have $\gamma = 1 = \min\{1, \alpha+\beta\}$. The only cases left to consider are $\alpha = \beta = \frac{1}{3}$ and $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$ (or $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$).

If $\alpha = \beta = \frac{1}{3}$, then $x_2 \notin A, B$ and $x_3 \notin A, B$ and all other elements of S are in both A and B, hence in C. Since $x_1 \prec x_2 \prec x_3$, we have $x_2 - x_1 \in S$. Also $x_2 - x_1 \prec x_2$ because $x_2 - (x_2 - x_1) = x_1 \in S$. Therefore, $x_2 - x_1 \prec x_2 \prec x_3$, and so $x_2 - x_1 \neq x_2$ and $x_2 - x_1 \neq x_3$ since "$\prec$" is transitive. Thus, $x_2 - x_1 \in A$ because x_2 and x_3 are the only two elements of S not in A. However, $x_2 - x_1 \in A$ and $x_1 \in B$ imply $(x_2 - x_1) + x_1 = x_2 \in C$. Therefore, $\gamma \geq \frac{2}{3} = \min\{1, \alpha+\beta\}$.

If $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, then $x_2 \notin A, B$; $x_3 \notin A$; $x_3 \in B$; and all other elements of S are in both A and B, hence in C. Again since $x_1 \prec x_2 \prec x_3$, we have $x_2 - x_1 \in S$, and again $x_2 - x_1 \neq x_2$ and $x_2 - x_1 \neq x_3$. Therefore, $x_2 - x_1 \in A$. Now $x_1 \in B$, so $(x_2 - x_1) + x_1 = x_2 \in C$. Since $x_3 \in B \subseteq C$, all elements of S are in C and so $\gamma = 1 > \frac{5}{6} = \min\{1, \alpha+\beta\}$.

We do not have the case where $\mathfrak{F}_B$ is defined by

$$B(x) = \begin{cases} \{x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{x_1, x_4\} & \text{if } x = x_4, \\ \{x\} & \text{otherwise,} \end{cases}$$

where $x_1 \prec x_2 \prec x_3$ and $x_1 \prec x_4$ because here $\mathfrak{F}_B$ is not separated and hence not discrete. Therefore the proof is complete.

Indications are that we can prove similar theorems for singularly discrete fundamental families of order 4, 5, 6, and so on. However, each successive proof seems to require a greater number of cases and we have found no generalization of the proof for $n = 3$.

Unsolved Problem. Prove (or disprove) that if $(S, \mathfrak{F})$ is a density space where $\mathfrak{F}$ is singularly discrete of order n, then $\gamma \geq \min\{1, \alpha + \beta\}$.

Before going on, we should show that $\gamma \geq \min\{1, \alpha + \beta\}$ is the strongest valid result for a density space $(S, \mathfrak{F})$ where $\mathfrak{F}$ is singularly discrete of order n. Consider the following example where the s-set is I:

Let $\mathfrak{F}_B$ be defined by

$$B(x) = \begin{cases} \{i \mid 1 \leq i \leq x\} & \text{if } 1 \leq x \leq n, \\ \{x\} & \text{otherwise,} \end{cases}$$

where $n \geq 2$. Then $(I, \mathfrak{F}_B)$ is a density space where $\mathfrak{F}_B$ is singularly discrete of order n. Let $A = B = \{1, n+1, n+2, \ldots\}$. Then $C = \{1, 2, n+1, n+2, \ldots\}$. Therefore $\alpha = \beta = \frac{1}{n}$ and $\gamma = \frac{2}{n}$, and so $\gamma = \alpha + \beta$.

4.4. Nested Singularly Discrete Fundamental Families

Definition 4.4. A fundamental family $\mathfrak{F}_B$ on an s-set S is nested singularly discrete of order n if $\mathfrak{F}_B$ is defined by

$$B(x) = \begin{cases} \{x_i \mid 1 \leq i \leq m\} & \text{if } x = x_m \ (1 \leq m \leq n), \\ \{x\} & \text{otherwise,} \end{cases}$$

and $x_1 \prec x_2 \prec \cdots \prec x_n$.

Case III of Theorem 4. 3, which is the most difficult case to prove, concerns nested singularly discrete fundamental families of order 3.

Theorem 4. 4. Let $(S, \mathcal{F})$ be a density space where $\mathcal{F}$ is nested singularly discrete of order 4. Then $\gamma \geq \min\{1, \alpha + \beta\}$.

The inequality $\gamma \geq \min\{1, \alpha + \beta\}$ is the strongest possible result according to the example at the end of Section 4. 3. Before proving Theorem 4. 4, we prove two lemmas.

Lemma A. Let $(S, \mathcal{F}_B)$ be a density space where $\mathcal{F}_B$ is the nested singularly discrete fundamental family of order 4 defined by

$$B(x) = \begin{cases} \{x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{x_1, x_2\} & \text{if } x = x_2, \\ \{x\} & \text{otherwise,} \end{cases}$$

where $x_1 \prec x_2 \prec x_3 \prec x_4$. Let A and B be subsets of S with densities α and β respectively and let C = A + B. If $\alpha > 0$ and $x_1 \in B$, then $x_2 \in C$.

Proof: Since $x_1 \prec x_2 \prec x_3 \prec x_4$, we have $x_2 - x_1 \in S$.

Now $x_2 - x_1 \prec x_2$ because $x_2 - (x_2 - x_1) = x_1 \in S$. Hence, $x_2 - x_1 \prec x_2 \prec x_3 \prec x_4$ and since "$\prec$" is transitive we have $x_2 - x_1 \neq x_2$, $x_2 - x_1 \neq x_3$, and $x_2 - x_1 \neq x_4$. Since $\alpha > 0$, only x_2, x_3, or x_4 can be missing from A. Therefore, $x_2 - x_1 \in A$. Now $x_1 \in B$ so $(x_2 - x_1) + x_1 = x_2 \in C$.

<u>Lemma B.</u> Let $(S, \mathfrak{F}_B)$ be a density space where $\mathfrak{F}_B$ is the nested singularly discrete fundamental family of order 4 defined by

$$B(x) = \begin{cases} \{x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{x_1, x_2\} & \text{if } x = x_2, \\ \{x\} & \text{otherwise,} \end{cases}$$

where $x_1 \prec x_2 \prec x_3 \prec x_4$. Let A and B be subsets of S with densities α and β respectively and let $C = A + B$. If $\alpha > 0$, $x_1 \in B$, and $x_2 \in B$, then $x_3 \in C$.

Proof: Since $x_1 \prec x_2 \prec x_3 \prec x_4$, we have $x_3 - x_2 \in S$, $x_3 - x_1 \in S$, and $x_2 - x_1 \in S$. Now $x_3 - x_2 \prec x_3 \prec x_4$ because $x_3 - (x_3 - x_2) = x_2 \in S$, and so $x_3 - x_2 \neq x_3$ and $x_3 - x_2 \neq x_4$. Likewise, $x_3 - x_1 \prec x_3 \prec x_4$, and so $x_3 - x_1 \neq x_3$ and $x_3 - x_1 \neq x_4$. But $x_3 - x_2 \prec x_3 - x_1$, because $(x_3 - x_1) - (x_3 - x_2) = x_2 - x_1 \in S$, and so $x_3 - x_2 \neq x_3 - x_1$. Therefore, either $x_3 - x_2 \neq x_2$ or $x_3 - x_1 \neq x_2$. Since $\alpha > 0$, only $x_2, x_3,$ or x_4 can be missing from A. Therefore, either $x_3 - x_2 \in A$ or $x_3 - x_1 \in A$. But $x_1 \in B$ and $x_2 \in B$, so in either case, $x_3 \in C$.

Proof of Theorem 4.4: By an appropriate selection of notation, $\mathfrak{F}$ is defined by

$$B(x) = \begin{cases} \{x_1, x_2, x_3, x_4\} & \text{if } x = x_4, \\ \{x_1, x_2, x_3\} & \text{if } x = x_3, \\ \{x_1, x_2\} & \text{if } x = x_2, \\ \{x\} & \text{otherwise,} \end{cases}$$

where $x_1 \prec x_2 \prec x_3 \prec x_4$. The only densities possible for α and β are $0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and 1. If either $\alpha = 0$ or $\beta = 0$, then by Theorem 2.8, we have $\gamma \geq \max\{\alpha, \beta\} = \min\{1, \alpha + \beta\}$. If $\alpha + \beta \geq 1$, then by Theorem 2.9, we have $\gamma = 1 = \min\{1, \alpha + \beta\}$. We may select our notation so that $\alpha \leq \beta$. Then the cases which remain are (i) $\alpha = \beta = \frac{1}{4}$, (ii) $\alpha = \beta = \frac{1}{3}$, (iii) $\alpha = \frac{1}{4}$, $\beta = \frac{1}{3}$, (iv) $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, (v) $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, (vi) $\alpha = \frac{1}{4}$, $\beta = \frac{2}{3}$.

Case (i): If $\alpha = \beta = \frac{1}{4}$, then $x_2 \notin A, B$; $x_3 \notin A, B$; $x_4 \notin A, B$; and all other elements of S are in both A and B, hence in C. In particular, $x_1 \in B$. Therefore, by Lemma A, we have $x_2 \in C$, and so $\gamma \geq \frac{1}{2} = \min\{1, \alpha + \beta\}$.

Case (ii): If $\alpha = \beta = \frac{1}{3}$, then $x_2 \notin A, B$; $x_3 \notin A, B$; and all other elements of S are in both A and B, hence in C.

Again since $x_1 \in B$, we have $x_2 \in C$ by Lemma A. Hence, $\gamma \geq \frac{2}{3} = \min\{1, \alpha+\beta\}$.

Case (iii): If $\alpha = \frac{1}{4}$, $\beta = \frac{1}{3}$, then $x_2 \notin A, B$; $x_3 \notin A, B$; $x_4 \notin A$; $x_4 \in B$; all other elements of S are in both A and B, hence in C. Since $x_1 \in B$, we have $x_2 \in C$ by Lemma A. Since $x_4 \in B \subseteq C$, we have $\gamma = \frac{2}{3} > \frac{7}{12} = \min\{1, \alpha+\beta\}$.

Case (iv): If $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, then there are two possibilities, one of which is $x_2 \notin A, B$; $x_3 \notin A$; $x_3 \in B$; $x_4 \in A$; x_4 may or may not be in B; and all other elements of S are in A and B, hence in C. Since $x_1 \in B$, we have $x_2 \in C$ by Lemma A. Since $x_3 \in B \subseteq C$ and $x_4 \in A \subseteq C$, then all elements of S are in C and so $\gamma = 1 > \frac{5}{6} = \min\{1, \alpha+\beta\}$. The other possibility is $x_2 \notin A$; $x_2 \in B$; $x_3 \notin A, B$; $x_4 \in A$; $x_4 \notin B$; and all other elements are in A and B, hence in C. Since $x_1 \in B$ and $x_2 \in B$ we have $x_3 \in C$ by Lemma B. Since $x_2 \in B \subseteq C$ and $x_4 \in A \subseteq C$, then all all elements of S are in C and so $\gamma = 1 > \frac{5}{6} = \min\{1, \alpha+\beta\}$.

Case (v): If $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, then there are two possibilities, one of which is $x_2 \notin A, B$; $x_3 \notin A$; $x_3 \in B$; $x_4 \notin A$; x_4 may or may not be in B; and all other elements of S are in both A and B, hence in C. Since $x_1 \in B$, we have $x_2 \in C$ by Lemma A. Since $x_3 \in B \subseteq C$, we have $\gamma \geq \frac{3}{4} = \min\{1, \alpha+\beta\}$. The other possibility is $x_2 \notin A$; $x_2 \in B$; $x_3 \notin A, B$; $x_4 \notin A, B$;

and all other elements of S are in both A and B, hence in C. Since $x_1 \in B$ and $x_2 \in B$, we have $x_3 \in C$ by Lemma B. Since $x_2 \in B \subseteq C$, we have $\gamma \geq \frac{3}{4} = \min\{1, \alpha + \beta\}$.

Case (vi): If $\alpha = \frac{1}{4}$, $\beta = \frac{2}{3}$, then $x_2 \notin A$; $x_2 \in B$; $x_3 \notin A, B$; $x_4 \notin A$; $x_4 \in B$; and all other elements of S are in both A and B, hence in C. Since $x_1 \in B$ and $x_2 \in B$, we have $x_3 \in C$ by Lemma B. Since $x_2 \in B \subseteq C$ and $x_4 \in B \subseteq C$, then all elements of S are in C and so $\gamma = 1 > \frac{11}{12} = \min\{1, \alpha+\beta\}$.

The proof of Theorem 4.4 is complete.

Now we examine the density space $(I, \mathfrak{F})$ where $\mathfrak{F}$ is nested singularly discrete of order n. Even here, we have been unable to prove that $\gamma \geq \min\{1, \alpha + \beta\}$. However, we prove below both the Landau-Schnirelmann inequality, $\gamma \geq \alpha + \beta - \alpha\beta$, and the Schur inequality, $\gamma \geq \frac{\beta}{(1 - \alpha)}$ when $\alpha + \beta < 1$. The proofs depend on the nested property of $\mathfrak{F}$ and are based on proofs given by Landau [7] and Schur [11] for the density space $(I, \mathcal{K})$.

<u>Definition 4.5.</u> Let $(S, \mathfrak{F}_B)$ be a density space and let $x \in S$. Then x is called a <u>singleton</u> if $B(x) = \{x\}$.

The following definition for $X(x)$ is used in the remainder of this chapter and is not to be confused with the $B(x)$ defined in Theorem 2.4.

Definition 4.6. Let X be a subset of I. Then $X(x)$ denotes the number of integers in X which do not exceed x.

The following theorem is useful in proving Theorem 4.6 and 4.7.

Theorem 4.5. Let $(I, \mathfrak{F})$ be a density space where $\mathfrak{F}$ is nested singularly discrete of order n. Let A be any subset of I and α be the density of A. Let T be the set of singletons in I which are greater than 1. If $T \cup \{1\} \subseteq A$, then

$$\alpha = g\ell b \left\{ \frac{A(x) - T(x)}{x - T(x)} \,\middle|\, x \in I \right\} .$$

Proof: Since $\mathfrak{F}$ is discrete, we have

$$\alpha = g\ell b \{ q(A, [x]) \mid x \in I \} .$$

Now the set of all singletons, $T \cup \{1\}$, is contained in A, and so $q(A, [x]) = 1$ for all $x \in T \cup \{1\}$. Therefore,

$$\alpha = g\ell b \{ q(A, [x]) \mid x \in I,\ x \notin T \cup \{1\} \}.$$

Since $\mathfrak{F}$ is nested singularly discrete of order n, then using the notation of Definition 4.4, we have

(1) $$\{ x \mid x \in I,\ x \notin T \cup \{1\} \} = \{ x_i \mid 2 \leq i \leq n \} .$$

Thus,

(2) $$\alpha = \min \{ q(A, [x_i]) \mid 2 \leq i \leq n \} .$$

Hence, for each $i\ (2 \leq i \leq n)$, we have $[x_i] = \{x_1, x_2, \ldots x_i\}$. Since $T(x_i)$ is the number of singletons from 2 through x_i, then

$$T(x_i) = (x_i - 1) - (i - 1)$$

$$= x_i - i$$

(3) $$= x_i - I([x_i])$$

whether $x_1 = 1$ or $x_1 > 1$. Let R be the number of non-singletons in A which do not exceed x_i $(2 \leq i \leq n)$. Since all singletons are in A, we have

(4) $$A(x_i) - (T(x_i) + 1) = R.$$

However, since the non-singletons in A which do not exceed x_i form the set $\{x_j \mid 2 \leq j \leq i\}$, then

$$A([x_i]) = A(\{x_1\} \cup \{x_2, \ldots, x_i\})$$

$$= A(\{x_1\}) + A(x_2, \ldots, x_i)$$

(5) $$= 1 + R$$

Combining (4) and (5), we obtain

(6) $$A([x_i]) = A(x_i) - T(x_i).$$

Equations (3) and (6) yield

$$q(A, [x_i]) = \frac{A([x_i])}{I([x_i])} = \frac{A(x_i) - T(x_i)}{x_i - T(x_i)}$$

where $2 \leq i \leq n$, and so by (2), we have

$$\alpha = \min \left\{ \frac{A(x_i) - T(x_i)}{x_i - T(x_i)} \,\middle|\, 2 \leq i \leq n \right\}$$

and by (1),

$$(7) \qquad \alpha = g\ell b \left\{ \frac{A(x) - T(x)}{x - T(x)} \,\middle|\, x \in I,\ x \notin T \cup \{1\} \right\}.$$

If $x \in T \cup \{1\}$, and there exists a y such that $y < x$, $y \notin T \cup \{1\}$, and $u \in T \cup \{1\}$ whenever $y < u < x$, then

$$(8) \qquad \frac{A(x) - T(x)}{x - T(x)} = \frac{A(x-1) - T(x-1)}{x-1 - T(x-1)} = \ldots = \frac{A(y) - T(y)}{y - T(y)}.$$

If no such y exists then $A(x) = x$, $T(x) = x - 1$, and so

$$(9) \qquad \frac{A(x) - T(x)}{x - T(x)} = 1.$$

Therefore, by (7), (8), and (9), we have

$$\alpha = g\ell b \left\{ \frac{A(x) - T(x)}{x - T(x)} \,\middle|\, x \in I \right\},$$

and the proof is complete.

Theorem 4.6. Let $(I, \mathcal{F})$ be a density space where $\mathcal{F}$ is nested singularly discrete of order n. Then $\gamma \geq \alpha + \beta - \alpha\beta$.

Proof: Recall that $X(x)$ denotes the number of integers in $X \cap I$ which do not exceed x. Let A and B be subsets of I, and α, β, γ be the densities of $A, B, C = A + B$ respectively. If $A = I$, then $\alpha = \gamma = 1$ and the theorem follows. Therefore, we assume that at least one positive integer m is not in A. We construct integers a_i and b_i where

$$0 \le a_1 < b_1 < a_2 < b_2 < \ldots < a_{k-1} < b_{k-1} < a_k < m$$

as follows. Let $a_1 + 1$ be the least positive integer missing from A. Let $b_1 + 1$ be the least integer greater than $a_1 + 1$ which is in A. In general, let $a_i + 1$ be the least integer greater than $b_{i-1} + 1$ which is not in A and let $b_i + 1$ be the least integer greater than $a_i + 1$ which is in A. This process terminates when we reach $a_k < m$ and find that either b_k does not exist or $b_k \ge m$.

We can assume that all singletons are in both A and B, hence in C. Otherwise, $\alpha = 0$ or $\beta = 0$, and the theorem is immediate. Let T be the set of all singletons in I which are greater than 1, as in Theorem 4.5.

We have

$$C(m) \ge A(m) + B(b_1 - a_1) + \ldots + B(b_{k-1} - a_{k-1}) + B(m - a_k) ,$$

and so

$$(10)\quad \frac{C(m)-T(m)}{m-T(m)} \geq \frac{A(m)-T(m)}{m-T(m)} + \frac{B(b_1-a_1)+\ldots+B(b_{k-1}-a_{k-1})+B(m-a_k)}{m-T(m)}.$$

By Theorem 4. 5, we have

$$\alpha \leq \frac{A(m)-T(m)}{m-T(m)} .$$

Hence, there exists a non-negative constant α_m such that

$$(11)\qquad \alpha + \alpha_m = \frac{A(m)-T(m)}{m-T(m)} .$$

Also by Theorem 4. 5, we have

$$(12)\qquad \beta \leq \frac{B(m-a_k)-T(m-a_k)}{m-a_k-T(m-a_k)} \leq \frac{B(m-a_k)}{m-a_k} ,$$

and

$$(13)\quad \beta \leq \frac{B(b_i-a_i)-T(b_i-a_i)}{b_i-a_i-T(b_i-a_i)} \leq \frac{B(b_i-a_i)}{b_i-a_i}, \quad i=1,2,\ldots,k-1.$$

Combining (10), (11), (12), and (13), we obtain

$$(14)\quad \frac{C(m)-T(m)}{m-T(m)} \geq \alpha+\alpha_m + \frac{\beta}{m-T(m)} \{(b_1-a_1)+\ldots(b_{k-1}-a_{k-1})+(m-a_k)\} .$$

But $A(m) = a_k - (b_1 - a_1) - \ldots - (b_{k-1}-a_{k-1})$, so (14) becomes

$$\frac{C(m)-T(m)}{m-T(m)} \geq a+a_m + \frac{\beta}{m-T(m)}\{m-A(m)\}$$

$$= a+a_m + \frac{\beta}{m-T(m)}\{m-T(m)+T(m)-A(m)\}$$

$$= a+a_m+\beta-\beta\left\{\frac{A(m)-T(m)}{m-T(m)}\right\}$$

$$\geq a+a_m+\beta-\beta\{a+a_m\}$$

$$= a+\beta-a\beta+a_m\{1-\beta\}$$

$$\geq a+\beta-a\beta$$

because a_m is non-negative and $\beta \leq 1$. Therefore,

(15) $$\frac{C(m)-T(m)}{m-T(m)} \geq a+\beta-a\beta$$

for all $m \notin A$.

If $m \in A$ and all integers less than m are in A, we have

(16) $$\frac{C(m)-T(m)}{m-T(m)} = 1 \geq a+\beta-a\beta.$$

Otherwise, we find the largest integer m' less than m and in A. Then

(17) $$\frac{C(m)-T(m)}{m-T(m)} \geq \frac{C(m')-T(m')}{m'-T(m')} \geq a+\beta-a\beta.$$

From (15), (16), and (17) we can conclude that

$$\frac{C(x) - T(x)}{x - T(x)} \geq \alpha + \beta - \alpha\beta$$

for all $x \in I$. Therefore,

$$g\ell b\ \{\frac{C(x) - T(x)}{x - T(x)}\ \Big|\ x \in I\} \geq \alpha+\beta-\alpha\beta\ ,$$

and so by Theorem 4. 5, we have $\gamma \geq \alpha+\beta-\alpha\beta$ and the proof is complete.

<u>Theorem 4. 7.</u> Let $(I, \mathfrak{F})$ be a density space where $\mathfrak{F}$ is nested singularly discrete of order n. If $\alpha+\beta < 1$, then $\gamma \geq \beta / (1 - \alpha)$.

Proof: Recall that $X(x)$ denotes the number of integers in the set $X \cap I$ which do not exceed x. Let A and B be subsets of I, and α, β, γ be the densities of $A, B, C = A + B$ respectively. Let T be the set of all singletons in I which are greater than 1, as in Theorem 4. 5. We can assume all singletons are in both A and B, hence in C. Otherwise, $\alpha = 0$ or $\beta = 0$, and the theorem is immediate. If $\gamma = 1$, we have $\gamma > \alpha + \beta = \alpha\gamma + \beta$, and so $\gamma(1 - \alpha) > \beta$ and the theorem follows. Hence, we can assume that $\gamma < 1$, and so, at least one integer is missing from C. Let $x_1, x_2, \ldots$ be the integers missing from

C. Let $x_o = 0$. First, we show that

(18) $$x_i - x_{i-1} - 1 \geq B(x_i) - B(x_{i-1}) + A(x_i - x_{i-1} - 1),$$

for any i $(i \geq 1)$. Now $B(x_i) - B(x_{i-1})$ is the number of integers in B which lie in the interval $(x_{i-1}, x_i]$. Assume there are p such integers $b_1, b_2, \ldots, b_p$. Since $x_i \notin C$, we know that $x_i - b_j \notin A$ for each $j(1 \leq j \leq p)$. We also know that $0 < x_i - b_j \leq x_i - x_{i-1} - 1$ for each $j(1 \leq j \leq p)$. Therefore, $A(x_i - x_{i-1} - 1) \leq x_i - x_{i-1} - 1 - p$. Hence,

$$x_i - x_{i-1} - 1 = p + x_i - x_{i-1} - 1 - p \geq B(x_i) - B(x_{i-1}) + A(x_i - x_{i-1} - 1).$$

For any h such that $x_h \notin C$, we can sum (18) from 1 to h, obtaining

(19) $$x_h - h \geq B(x_h) + \sum_{i-1}^{h} A(x_i - x_{i-1} - 1).$$

However, $C(x_h) = x_h - h$ and by Theorem 4. 5, we have

$$\alpha \leq \frac{A(m) - T(m)}{m - T(m)} \leq \frac{A(m)}{m}$$

for any positive integer m. Therefore, (19) becomes

$$C(x_h) \geq B(x_h) + \sum_{i=1}^{h} \alpha(x_i - x_{i-1} - 1)$$

$$= B(x_h) + \alpha(x_h - h)$$

$$= B(x_h) + \alpha C(x_h) .$$

Hence,

(20) $$(1 - \alpha) C(x_h) \geq B(x_h) .$$

Therefore, by (20) and Theorem 4. 5, we have

(21) $$\frac{(1-\alpha) C(x_h) - T(x_h)}{x_h - T(x_h)} \geq \frac{B(x_h) - T(x_h)}{x_h - T(x_h)} \geq \beta$$

But $0 < 1 - \alpha \leq 1$ so $(1 - \alpha) T(x_h) \leq T(x_h)$. Hence, (21) becomes

$$(1 - \alpha) \frac{C(x_h) - T(x_h)}{x_h - T(x_h)} \geq \beta ,$$

and so

(22) $$\frac{C(x_h) - T(x_h)}{x_h - T(x_h)} \geq \frac{\beta}{(1 - \alpha)} .$$

If $m \in A$ and all integers less than m are in A, we have

(23) $$\frac{C(m) - T(m)}{m - T(m)} = 1 > \frac{\beta}{(1 - \alpha)} .$$

Otherwise, we find the largest integer m' less than m and in A. Then

$$\text{(24)} \qquad \frac{C(m) - T(m)}{m - T(m)} \geq \frac{C(m') - T(m')}{m' - T(m')} \geq \frac{\beta}{(1 - \alpha)} .$$

From (22), (23), and (24) we can conclude that

$$\frac{C(x) - T(x)}{x - T(x)} \geq \frac{\beta}{(1 - \alpha)}$$

for all $x \in I$. Therefore,

$$g\ell b \left\{ \frac{C(x) - T(x)}{x - T(x)} \;\middle|\; x \in I \right\} \geq \frac{\beta}{(1 - \alpha)} ,$$

and so by Theorem 4.5, we have $\gamma \geq \beta/(1 - \alpha)$ and the proof is complete.

Unsolved Problem. Study discrete fundamental families of infinite order. Such a family is defined by replacing condition (ii) of Definition 4.1 by " the set $\{ S([x]) | x \in S \}$ is unbounded." The density space $(I, \mathcal{K})$ is an example of a discrete fundamental family of infinite order.

CHAPTER V

THE TRANSFORMATION PROPERTIES

In this chapter we study some relations between the two transformation properties T - 1 and T - 2, and the Landau-Schnirelmann and Schur inequalities.

5.1. The Landau-Schnirelmann Inequality and T-1

Let $(S, \mathfrak{F})$ be any density space. Theorem 2.14 tells us that the Landau-Schnirelmann inequality, $\gamma \geq \alpha+\beta-\alpha\beta$, holds if $\mathfrak{F}$ is T-1. In this section we show that $\gamma \geq \alpha+\beta-\alpha\beta$ does not imply that $\mathfrak{F}$ is T-1. In order to show this we prove the following theorem which is of independent interest.

Theorem 5.1. Let $(I, \mathfrak{F}_B)$ be a density space where $\mathfrak{F}_B$ is discrete of order n. Then $\mathfrak{F}_B$ is T-1 iff $\mathfrak{F}_B$ is defined by $B(x) = \{x\}$ for all $x \in I$; that is, iff $n = 1$.

Proof: Let $\mathfrak{F}_B$ be defined by $B(x) = \{x\}$ for all $x \in I$. Then $\mathfrak{F}_B = \mathfrak{D}$, and so $\mathfrak{F}_B$ is T-1 because all non-empty subsets of I are in $\mathfrak{F}_B = \mathfrak{D}$.

Now let $\mathfrak{F}_B$ be T-1 and assume that $\mathfrak{F}_B$ is not defined by $B(x) = \{x\}$ for all $x \in I$. Then there exists an integer

$k(k \geq 2)$ and an integer $i(1 \leq i \leq k-1)$ such that $B(x) = \{x\}$ for all $x(1 \leq x \leq k-1)$ and $\{i, k\} \subseteq B(k)$.

Suppose $B(k+1) = \{k+1\}$. Then, using the notation of Definition 2.17, if $x = 1$ and $F = \{1, k+1\}$, we have $D = \{k+1\}$ and $T_1[D] = \{k\} \notin \mathfrak{F}_B \cup \{\phi\}$, by the way $\mathfrak{F}_B$ is defined. Thus, $\mathfrak{F}_B$ is not T-1, a contradiction. Therefore, $B(k+1)$ is not a singleton.

As our induction step we assume that $B(k+j)$ is not a singleton for an arbitrary integer $j(j \geq 1)$.

Suppose $B(k+j+1) = \{k+j+1\}$. Then, if $x = 1$ and $F = \{1, k+j+1\}$, we have $D = \{k+j+1\}$ and $T_1[D] = \{k+j\} \notin \mathfrak{F}_B \cup \{\phi\}$, by the way $\mathfrak{F}_B$ is defined. Thus, $\mathfrak{F}_B$ is not T-1, a contradiction. Therefore, $B(k+j)$ is not a singleton for any $j(j \geq 1)$. Hence, only the first $k-1$ positive integers are singletons.

Now each $B(x)$ must contain at least one singleton, because the smallest integer in each $B(x)$ is a singleton. Therefore, one or more of the first $k-1$ integers must occur in infinitely many of the $B(x)$. Let y be any such integer. Then $y \in \bigcap_{i=1}^{\infty} B(x_i)$ for some strictly increasing sequence $\{x_i\}$. Since $\mathfrak{F}_B$ is a discrete fundamental family it must be separated. Hence, since $y \in B(x_i) \cap B(x_{i+1})$ and $x_i < x_{i+1}$, we must have $B(x_i) \subset B(x_{i+1})$. Therefore,

$$B(x_1) \subset B(x_2) \subset \ldots \subset B(x_i) \subset \ldots$$

which contradicts the fact that $\mathfrak{F}_B$ is discrete of order n. Therefore, $\mathfrak{F}_B$ is defined by $B(x) = \{x\}$ for all $x \in I$ and the proof is complete.

Let $(I, \mathfrak{F}_B)$ be any density space where $\mathfrak{F}_B$ is discrete of order 2. By Theorem 4.1, we have $\gamma \geq \min\{1, \alpha+\beta\} \geq \alpha+\beta-\alpha\beta$. By Theorem 5.1, we know that $\mathfrak{F}_B$ is not T-1. Therefore, $\gamma \geq \alpha+\beta-\alpha\beta$ does not assure us that $\mathfrak{F}_B$ is T-1.

<u>Unsolved Problem.</u> Replace the T-1 property in the hypothesis of Theorem 2.14 with a weaker condition which still implies the Landau-Schnirelmann inequality.

5.2. The Schur Inequality and T-2

Let $(S, \mathfrak{F})$ be any density space. Theorem 2.15 tells us that if $\mathfrak{F}$ is T-2, then $\gamma \geq \beta/(1-\alpha)$ whenever $\alpha+\beta < 1$. In this section we show that the Schur inequality does not imply T-2. Consider the following example on the s-set I.

Let $\mathfrak{F}_B$ be defined by

$$B(x) = \begin{cases} \{2, 4\} & \text{if } x = 4, \\ \{1, 5\} & \text{if } x = 5, \\ \{x\} & \text{otherwise.} \end{cases}$$

If, using the notation of Definition 2.18, we let $x = 5$ and F be empty, then $D = \{1, 5\}$ and $T_2[D] = \{4\} \notin \mathfrak{F}_B \cup \{\phi\}$. Hence, $\mathfrak{F}_B$ is not T-2. However, the Schur inequality holds in $(I, \mathfrak{F}_B)$. To see this we note that if $\alpha + \beta < 1$, then by Theorem 4.1, we have

$$\gamma \geq \min\{1, \alpha + \beta\} = \alpha + \beta \geq \alpha\gamma + \beta,$$

since $\mathfrak{F}_B$ is discrete of order 2. Thus, $\gamma - \alpha\gamma \geq \beta$, and so $\gamma \geq \beta/(1 - \alpha)$.

Unsolved Problem. Replace the T-2 property in the hypothesis of Theorem 2.15 with a weaker condition which still implies the Schur inequality.

5.3. The Correspondence Between T-1 and T-2

Property T-2 does not imply property T-1 as we see by the following example on the s-set I.

Let $\mathfrak{F}_B$ be defined by

$$B(x) = \begin{cases} \{1, 2,\} & \text{if } x = 2, \\ \{x\} & \text{otherwise.} \end{cases}$$

Using the notation of Definition 2.18, the only choices allowed for D are $\{\phi\}$, $\{1, 2\}$, and $\{x\}$. Hence, the only possibilities for $T_2[D]$ are $\{\phi\}$ and $\{1\}$, which are both in $\mathfrak{F}_B \cup \{\phi\}$. Thus, $\mathfrak{F}_B$ is T-2. However, $\mathfrak{F}_B$ is discrete of order 2 and hence is not T-1 by Theorem 5.1.

It seems unlikely that T-1 could imply T-2 because then, by Theorem 2.15, T-1 would imply the Schur inequality. However, T-1 incorporates the characteristics of density spaces for which the Landau-Schnirelmann inequality hold, and the Landau-Schnirelmann and Schur inequalities are dissimilar.

Unsolved Problem. Prove that T-1 implies T-2 or find a density space which is T-1 but not T-2.

CHAPTER VI

DENSITY SPACES AND PROPERTIES

In this chapter we list several density spaces and summarize properties which they satisfy. We are particularly interested in the properties Freedman [3, p. 43-46] has defined and the inequalities made famous in the study of Schnirelmann density on the set of positive integers.

6.1. Examples and Properties

The properties we study are T-1, T-2, separation, the Schur inequality, the Landau-Schnirelmann inequality, and $\gamma \geq \min\{1, \alpha+\beta\}$. We consider both K-density and C-density. We examine the following density spaces which have been of particular interest in our work. These are only representatives of a much longer list which could be developed.

Example 1: $(S, \mathfrak{D})$. This is the density space $(S, \mathfrak{F})$, where $\mathfrak{F}$ is discrete of order 1.

Example 2: $(S, \mathfrak{F})$ where $\mathfrak{F}$ is discrete of order 2.

Example 3: $(S, \mathfrak{F})$ where $\mathfrak{F}$ is discrete of order $n (n > 2)$.

Example 4: $(S, \mathfrak{F})$ where $\mathfrak{F}$ is purely discrete of order n.

Example 5: $(S, \mathfrak{F})$ where $\mathfrak{F}$ is singularly discrete of order n.

Example 6: $(I, \mathcal{F})$ where $\mathcal{F}$ is nested singularly discrete of order n.

Example 7: $(S, \mathcal{K})$.

Example 8: $(I, \mathcal{K})$. This is the density space for which Schnirelmann density was first studied.

Example 9: $(I^n, \mathcal{K})$ where I^n is defined as in Section 3.5 and $n > 1$.

Example 10: $(I, \mathcal{F}_B)$ where $\mathcal{F}_B$ is defined by

$$B(x) = \begin{cases} \{2, 4, \ldots, x\} & \text{if } x \text{ is even,} \\ \{x\} & \text{if } x \text{ is odd.} \end{cases}$$

A clear way to summarize the results for these ten examples is with a chart like the one we have included at the end of this chapter. Each box in the chart represents one of the properties and one density example. A YES in the box means that the property under consideration is always true for that example, a NO means it is not always true, a UKN means that to our knowledge the truth of the result is unknown, and a CNJ means that the truth of the result is unknown but we conjecture a YES. Many boxes also contain letters which identify the person or persons who first verified the results. The results in boxes containing no letters are easily verified or are not attributed to any individual. The letter code for the chart is as follows:

S = Schnirelmann [9, 10]

L = Landau [7]

H = Schur [11]

M = Mann [8]

D = Dyson [2]

C = Cheo [1]

K = Kvarda [6]

F = Freedman [3]

A = The author

If a number occurs in a box, then a comment concerning the results in that box is found following that number in Section 6. 2.

6. 2. Comments

The number preceding each comment in this section identifies it with the box or boxes in the chart which contain that number.

1. These results follow from Theorem 4. 1.

2. The example following Theorem 4. 1 shows that all of the listed density inequalities fail to hold in general for Example 3.

3. These results follow from Theorem 4. 2.

4. These results hold for $n \leq 3$, by Theorems 4. 1 and 4. 3. We conjecture that they hold for all n.

5. These results follow from Theorem 4. 6.

6. These results follow from Theorem 4. 7.

7. These results hold for $n \leq 4$, by Theorems 4.1, 4.3, and 4.4. We conjecture that they hold for all n.

8. These results follow from Theorem 2.13.

9. This result follows from Theorems 2.13 and 2.14.

10. These results follow from Theorems 2.13 and 2.15.

11. Example 9 is a particular case of Example 7 for which this result is known to fail.

12. These results are easily verified.

Property \ Example	1	2	3	4	5	6	7	8	9	10
T-1	YES	NO	NO	NO	NO	NO	YES F-8	YES	YES	NO
T-2	YES	NO	NO	NO	NO	NO	YES F-8	YES	YES	YES
Separation	YES	YES	YES	YES	YES	YES	NO	YES	NO	YES
$\gamma \geq \alpha + \beta - \alpha\beta$	YES F-1	YES F-1	NO A-2	YES A-3	YES $n \leq 3$ CNJ A-4	YES A-5	YES F-9	YES L, S	YES K	YES 12
$\gamma_c \geq \alpha_c + \beta_c - \alpha_c\beta_c$	YES F-1	YES F-1	NO A-2	YES A-3	YES $n \leq 3$ CNJ A-4	YES A-5	UKN	YES L, S	UKN	YES 12
$\gamma \geq \beta/(1-\alpha)$ when $\alpha + \beta < 1$	YES F-1	YES F-1	NO A-2	YES A-3	YES $n \leq 3$ CNJ A-4	YES A-6	YES F-10	YES H	YES F-10	YES 12
$\gamma_c \geq \beta_c/(1-\alpha_c)$ when $\alpha_c + \beta_c < 1$	YES F-1	YES F-1	NO A-2	YES A-3	YES $n \leq 3$ CNJ A-4	YES A-6	UKN	YES H	UKN	YES 12
$\gamma \geq \min\{1, \alpha + \beta\}$	YES F-1	YES F-1	NO A-2	YES A-3	YES $n \leq 3$ CNJ A-4	YES $n \leq 4$ CNJ A-7	UKN	YES M, D	UKN	YES 12
$\gamma_c \geq \min\{1, \alpha_c + \beta_c\}$	YES F-1	YES F-1	NO A-2	YES A-3	YES $n \leq 3$ CNJ A-4	YES $n \leq 4$ CNJ A-7	NO 11	YES M, D	NO C	YES 12

BIBLIOGRAPHY

1. Cheo, Luther. On the density of sets of Gaussian integers. American Mathematical Monthly 58:618-620. 1951.

2. Dyson, F. A theorem on the densities of sets of integers. Journal of the London Mathematical Society 20:8-14. 1945.

3. Freedman, Allen R. A general theory of density in additive number theory. Ph. D. thesis. Corvallis, Oregon State University, 1965. 125 numb. leaves.

4. Jacobson, Nathan. Lectures in abstract algebra. Princeton, Van Nostrand, 1951. 217 p.

5. Kasch, Friedrich. Wesentliche Komponenten bei Gitterpunktmengen. Journal fur die reine und angewandte Mathematik 197:208-215. 1957.

6. Kvarda, Betty. On densities of sets of lattice points. Pacific Journal of Mathematics 13:611-615. 1963.

7. Landau, Edmund. Die Goldbachsche Vermutung und der Schnirelmannsche Satz. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, Fachgruppe II, 1930, p. 255-276.

8. Mann, Henry B. A proof of the fundamental theorem on the density of sums of sets of positive integers. Annals of Mathematics, Ser. 2, 43:523-527. 1942.

9. Schnirelmann, L. Über additive Eigenschaften von Zahlen. Annales d'Institut polytechnique, Novočerkask 14: 3-28. 1930.

10. ________________. Über additive Eigenschaften von Zahlen. Mathematische Annalen 107:649-690. 1933.

11. Schur, I. Über den Begriff den Dichte in der additiven Zahlentheorie. Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse, 1936, p. 269-297.