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Title: PERFORMANCE OF SOME TESTS FOR BIVARIATE INDEPENDENCE, INTERCHANGEABILITY, AND EQUALITY OF DISTRIBUTIONS OF RATIOS

Abstract approved: Redacted for Privacy
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The purpose of this thesis is to derive and compare asymptotic performances of certain tests of independence, interchangeability and equality of the distributions of ratios Y/X for positive variates X and Y with joint c.d.f. H(x,y) of particular forms. Three bivariate families, Morgenstern, Plackett, and Moran, are considered which permit marginal c.d.f.'s F(x) and G(y) of specified forms.

For testing independence of X and Y, asymptotic locally optimal parametric tests are derived for bivariate gamma and Weibull distributions for each of three families. Locally most powerful nonparametric tests are also derived for each of the three bivariate families, and asymptotic relative efficiency comparisons are made among the various tests of independence.

Nonparametric tests for interchangeability of X and Y, based on the Wilcoxon, symmetric squared ranks, and Savage statistics are considered. Asymptotic relative efficiency comparisons of the nonparametric tests are made for scale alternatives in the Morgenstern

and Moran bivariate gamma distributions and bivariate lognormal distributions. Wilcoxon, symmetric squared ranks, and normal score tests are considered for testing equality of two distributions of ratios Y/X. Asymptotic relative efficiency comparisons of the non-parametric tests are made for scale alternatives in some Morgenstern bivariate gamma distributions.

The techniques and results of this study may have application in the analysis of data from rain-making experiments, reliability studies and medical research.

Performance of Some Tests for Bivariate Independence,
Interchangeability, and Equality of
Distributions of Ratios

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PERFORMANCE OF SOME TESTS FOR BIVARIATE INDEPENDENCE, INTERCHANGEABILITY, AND EQUALITY OF DISTRIBUTIONS OF RATIOS

I. INTRODUCTION

Statistical analyses for bivariate, or more generally multivariate, continuous data have not been developed very completely except for multivariate normal models. Transformations, such as logarithmic and square root, are frequently applied to non-normal data so that normal-theory analyses may be justified. One difficulty commonly encountered is that the transformations required to obtain exact marginal normality may depend on unknown parameters. Moreover, even if the transformed variables have marginal normal distributions, the joint distributions need not be multivariate normal. In analyses of non-normal data, asymptotic distribution theory is often applied to statistics on which the inference procedures are based. In certain small sample size cases the asymptotic theory approximations may be quite poor; for example, normal approximations to the distribution of sample means can be quite inaccurate when the sampling is from skewed distributions.

This thesis is concerned with some inference problems for bivariate (BV) distributions which permit marginal distributions of specified forms. Since BV distributions are not uniquely determined from the marginal distributions, several different procedures for constructing BV distributions with specified marginals have been proposed. We consider the Morgenstern [24], Plackett [24], and Moran [19] forms which are described in Chapter II. Each of these three forms has the convenient property that the BV cumulative distribution function (c.d.f.) $H(x,y)$ depends on x and y only through the marginal c.d.f.'s $F(x)$, $G(y)$ and one unknown parameter θ ; that is, $H(x,y) = Q[F(x) , G(y) ; \theta]$ for some function Q . This property may be used in developing certain inference procedures which are distribution free relative to the marginal c.d.f.'s F and G . We are particularly interested in BV distributions for positive random variables X and Y with skewed marginal distributions, including the gamma, Weibull and lognormal families.

This study was motivated, in part, by inference problems arising from rain-making experiments. Cloud seeding with silver iodide smoke is used to increase the ice crystals in the clouds in order to increase the rainfall. Measurable precipitation data, $X > 0$, have been found [23] to be fit quite well by gamma distributions with shape parameter less than unity. The X and Y measurements may represent, respectively, precipitations resulting from unseeded and seeded clouds. If these X and Y measurements are taken near the same location at approximately the same time, X and Y will in general be dependent random variables. In addition to rain-making experiments, the BV models and inference procedures studied in this thesis should have application in other areas of research, including reliability and medical studies.

In Chapter II, the BV distribution are described and compared. Conditional distributions of $Y|x$ are also studied for the Morgenstern and Plackett BV distributions. In Chapter III, the problem of testing independence, $H_o : H(x,y) = F(x)\ G(y)$, in the Morgenstern, Plackett, and Moran distributions is considered. Asymptotic optimal parametric $C(\alpha)$ tests and locally most powerful rank tests are derived and compared. Asymptotic relative efficiency, ARE, comparisons of the tests for independence are also made in Chapter III.

In Chapter IV, some nonparametric tests for interchangeability, $H_o : H(x,y) = H(y,x)$, are studied. Sen [25] developed conditional permutation tests for H_o based on linear rank statistics. We consider the Wilcoxon (W), sum of squared rank (SSR), and Savage (S) statistics. Also using Sen's general asymptotic theory results, ARE comparisons of the three rank tests are made for gamma scale alternatives, $G_\alpha(y) = F_\alpha(x/\theta)$ for some $\theta \neq 1$ and common shape parameter α , in the Morgenstern and Moran BV families.

In Chapter V, the Wilcoxon statistic, the symmetric squared rank statistic [17] and normal score statistic are used for comparing ratios Y/X from two bivariate samples. The applications of these tests in two different designs used in rain-making experiments are discussed. ARE's among the Wilcoxon, the symmetric squared rank test and normal score test are also evaluated for the Morgenstern BV gamma distribution scale alternatives with common shape parameter $\alpha = 1,2$ and 3 .

II BIVARIATE DISTRIBUTIONS

Consider positive random variables X and Y with joint probability density function (p.d.f.) $h(x,y)$ and marginal p.d.f.'s $f(x)$ and $g(y)$. As indicated earlier, we let $H(x,y)$, $F(x)$ and $G(y)$ denote the c.d.f.'s corresponding to p.d.f.'s $h(x,y)$, $f(x)$ and $g(y)$. The following five types of BV distributions have been considered by others.

(1) Morgenstern [5,24]:

$$h(x,y) = f(x)g(y)\left[1+\gamma(2F(x)-1)(2G(y)-1)\right] \qquad (2.1)$$

$$H(x,y) = F(x)G(y)\left[1+\gamma(F(x)-1)(G(y)-1)\right] \qquad (2.2)$$

where

$$-1 \leq \gamma \leq 1 \ .$$

(2) Plackett[24,26]:

$$h(x,y) = \frac{\psi fg\left\{(\psi-1)(F+G-2FG)+1\right\}}{\left\{[1+(\psi-1)(F+G)]^2-4\psi(\psi-1)FG\right\}^{3/2}} \qquad (2.3)$$

$$H(x,y) = \frac{S-\left(S^2-4\psi(\psi-1)FG\right)^{1/2}}{2(\psi-1)} \qquad (2.4)$$

where

$$0 \leq \psi < \infty \ , \qquad \text{and} \qquad S = 1+[F(x)+G(y)]\,(\psi-1) \ .$$

(3) Moran [19]:

$$h(x,y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} e^{-\frac{1}{2(1-\rho^2)}(A^2-2\rho AB+B^2)} \frac{dA}{dx}\frac{dB}{dy} \quad (2.5)$$

where

$A = \Phi^{-1}(F(x))$

$B = \Phi^{-1}(G(y))$

Φ^{-1} is the inverse function of unit normal c.d.f. .

(4) Gumbel's [7] BV exponential distributions:

$$h(x,y) = ((1+\delta x)(1+\delta y)-1)e^{-x-y-\delta xy} \quad (2.6)$$

$$H(x,y) = 1-e^{-x}-e^{-y}+e^{-x-y-\delta xy} \quad , \quad (2.7)$$

where

$0<\delta<1$, $f(x) = e^{-x}$, and $g(y) = e^{-y}$ for $0\leq x<\infty, 0\leq y<\infty$.

(5) Lancaster [11]:

$$h(x,y) = f(x)g(y)(1+\sum_0^\infty \rho_i x^{(i)} y^{(i)}) \quad (2.8)$$

where $x^{(i)}$, $y^{(i)}$ are certain sets of orthonormal functions and ρ_i are the "canonical correlations".

Distribution types 1, 2, 3, and 5 can be used for any random variables X and Y which have absolutely continuous c.d.f.'s F and G. We see that use of relations

$$F(x) = \Phi(\frac{\ln x - \mu_x}{\sigma_x})$$

and (2.9)

$$G(y) = \Phi(\frac{\ln y - \mu_y}{\sigma_y})$$

in equation (2.5) gives the classical BV lognormal distributions. For positive parameters $\alpha_1, \alpha_2, \beta_1$, and β_2, transformations $x = (\frac{U}{\beta_1})^{\alpha_1}$ and $y = (\frac{V}{\beta_2})^{\alpha_2}$ can be used in Gumbel's BV exponential distributions to give BV Weibull distributions. Marshall and Olkin [16] and Downton [3] have also considered different BV exponential and Weibull distributions which are useful in reliability studies.

It is instructive to investigate the conditional distributions of X, given Y=y, for BV distributions with gamma or exponential marginals. Attention is restricted to only the Morgentern, Plackett, and Moran BV distributions because they are the underlying distributions used in the next two chapters.

2.1 Morgenstern's BV Distribution

The BV distribution, equation (2.1), has been extended to a more general form by Farlie [5],

$$h(x,y) = f(x)g(y)\left\{1+\gamma \frac{\partial[F\cdot A(F)]}{\partial F} \cdot \frac{\partial[G\cdot B(G)]}{\partial G}\right\}, \qquad (2.10)$$

$$H(x,y) = F(x)G(y)[1+\gamma A(F(x))\, B(G(y))] \quad ,$$

where

$A(F(x)) \longrightarrow 0$ as $F(x) \longrightarrow 1$, $B(G(y)) \longrightarrow 0$ as $G(y) \rightarrow 1$,

and $A(F(x))$ and $B(G(y))$ are bounded and have bounded first order derivatives in their arguments $F(x)$ and $G(y)$.

The regression curve of X on y, as derived in [5], is given by

$$E(X|y) = \int x \cdot \frac{h(x,y)}{g(y)} \, dx$$
$$= \mu_x + K \cdot \gamma \cdot \frac{\partial[G \cdot B(G)]}{\partial G} , \tag{2.11}$$

where

$$K = \int x \frac{\partial[F \cdot A(F)]}{\partial F} \, dx .$$

Gumbel [7] studied the conditional distributions for the Morgenstern type BV distributions with exponential marginals in addition to the BV exponential distribution of equation (2.6). The following analysis of the Morgenstern BV gamma distributions generalizes Gumbel's work since an exponential density is a special case of a gamma density with shape parameters equal to unity.

Densities

$$f(x;\alpha_1,\beta_1) = \frac{1}{\Gamma(\alpha_1)} \cdot \frac{1}{\beta_1} e^{-\frac{x}{\beta_1}} \cdot \left(\frac{x}{\beta_1}\right)^{\alpha_1 - 1} \tag{2.12}$$

where

$$0 < \alpha_1, \ 0 < \beta_1, \ 0 < x < \infty$$

and

$$g(y;\alpha_2,\beta_2) = \frac{1}{\Gamma(\alpha_2)} \frac{1}{\beta_2} \left(\frac{y}{\beta_2}\right)^{\alpha_2-1} e^{-\frac{y}{\beta_2}} , \tag{2.13}$$

where

$$0 < \alpha_2,\ 0 < \beta_2,\ 0 < y < \infty$$

are used in equation (2.1) for the Morgenstern BV distribution. For integer α, c.d.f.'s corresponding to equations (2.12) and (2.13) can be written as

$$F(x;\alpha_1,\beta_1) = 1 - \sum_{i=0}^{\alpha_1-1} \frac{\left(\frac{x}{\beta_1}\right)^i e^{-\frac{x}{\beta_1}}}{i!} , \tag{2.14}$$

and

$$G(y;\alpha_2,\beta_2) = 1 - \sum_{i=0}^{\alpha_2-1} \frac{\left(\frac{y}{\beta_2}\right)^i e^{-\frac{y}{\beta_2}}}{i!} . \tag{2.15}$$

The conditional p.d.f. of X, given y, is

$$f(x|y) = f(x;\alpha_1,\beta_1)[1+\gamma|2F(x;\alpha_1,\beta_1)-1|\,|2G(y;\alpha_2,\beta_2)-1|] , \tag{2.16}$$

The boundary p.d.f. of X given y is

$$f(x|0) = f(x;\alpha_1,\beta_1)\left|1+ \gamma \left[2 \sum_{i=0}^{\alpha_1-1} \frac{\left(\frac{x}{\beta_1}\right)^i e^{-\frac{x}{\beta_1}}}{i!} - 1\right]\right| \tag{2.17}$$

where

$$\alpha_1 = 1, 2, 3, \ldots \qquad 0 < \beta_1 \quad .$$

2.1.1. The Regression Curves

Using equation (2.16), the nonlinear regression curve of X on y can be expressed in the form

$$E(X|y) = \int x f(x|y) dx$$

$$= \mu_x [1 + \gamma \{2G(y;\alpha_2,\beta_2)-1\}] - \gamma \beta_1 \{2G(y;\alpha_2,\beta_2)-1\} \cdot K(\alpha_1) \quad , \tag{2.18}$$

where

$$\mu_x = \alpha_1 \beta_1 \quad \text{is mean of X}$$

and

$$K(\alpha_1) = \sum_{i=0}^{\alpha_1-1} \frac{\Gamma(\alpha_1+i+1)}{\Gamma(\alpha_1)\Gamma(i+1)} \cdot \frac{1}{2^{\alpha_1+i}} \quad . \tag{2.19}$$

The symmetry relation

$$1/2 \left[E_{-\gamma}(X|y) + E_{\gamma}(X|y) \right] = \mu_x = \alpha_1 \beta_1$$

is easily found from equation (2.18).

For the case where $\alpha_1 = 1$, equation (2.18) reduces to

$$E_{\gamma}(X|y)$$

$$= \beta_1 \left[1 + \gamma \{2G(y;\alpha_2,\beta_2)-1\} - \frac{\gamma}{2} \{2G(y;\alpha_2,\beta_2)-1\} \right] \quad . \tag{2.20}$$

Graphs of equation (2.20) for $\gamma = -1$, 0, and 1, and $\alpha_2 = 1/2$, 1, and 2 are given in Figure 1. When $\alpha_1 = \alpha_2 = 1$, Figure 1 corresponds to Gumbel's Graph 2 in [7,p.705].

2.1.2. The Conditional Variance

Using equation (2.16), we obtain

$$E(X^2 | y)$$

$$= \beta_1^2 \left\{ \alpha_1^2+\alpha_1+\gamma[2G(y;\alpha_2,\beta_2)-1] \left[\alpha_1^2+\alpha_1 - \sum_{i=0}^{\alpha_1-1} \frac{\Gamma(\alpha_1+i+2)}{\Gamma(\alpha_1)\Gamma(i+1)} \frac{1}{2^{\alpha_1+i+1}}\right] \right\} \tag{2.21}$$

Moreover,

$$1/2\ [E_{-\gamma}(x^2 | y)+E_{\gamma}(x^2 | y)] = \alpha_1^2\beta_1^2 + \alpha_1\beta_1^2 = EX^2 \quad .$$

Thus, from equations (2.20 and (2.21), we have the conditional variance

$$\sigma^2_{X|y}$$

$$= \beta_1^2 \left\{ \alpha_1+\gamma[2G(y;\alpha_2\beta_2)-1] \left[-\alpha_1^2+\alpha_1 - \sum_{i=1}^{\alpha_1-1} \frac{\Gamma(\alpha_1+i+2)}{\Gamma(\alpha_1)\Gamma(i+1)} \frac{1}{2^{\alpha_1+i+1}}\right.\right. \tag{2.22}$$

$$\left. + \alpha_1 \sum_{i=0}^{\alpha_1+1} \frac{\Gamma(\alpha_1+i+1)}{\Gamma(\alpha_1)\Gamma(i+1)} \frac{1}{2^{\alpha_1+i+1}} \right] - \gamma^2[2G(y;\alpha_2\beta_2)-1]^2$$

$$\left.\left[\alpha_1^2-\alpha_1 \sum_{i=1}^{\alpha_1-1} \frac{\Gamma(\alpha_1+i+1)}{\Gamma(\alpha_1)\Gamma(i+1)} \frac{1}{2^{\alpha_1+i-1}} + \left(\sum_{i=0}^{\alpha_1-1} \frac{\Gamma(\alpha_1+i+1)}{\Gamma(\alpha_1)\Gamma(i+1)} \frac{1}{2^{\alpha_1+i}} \right)^2\right] \right\} .$$

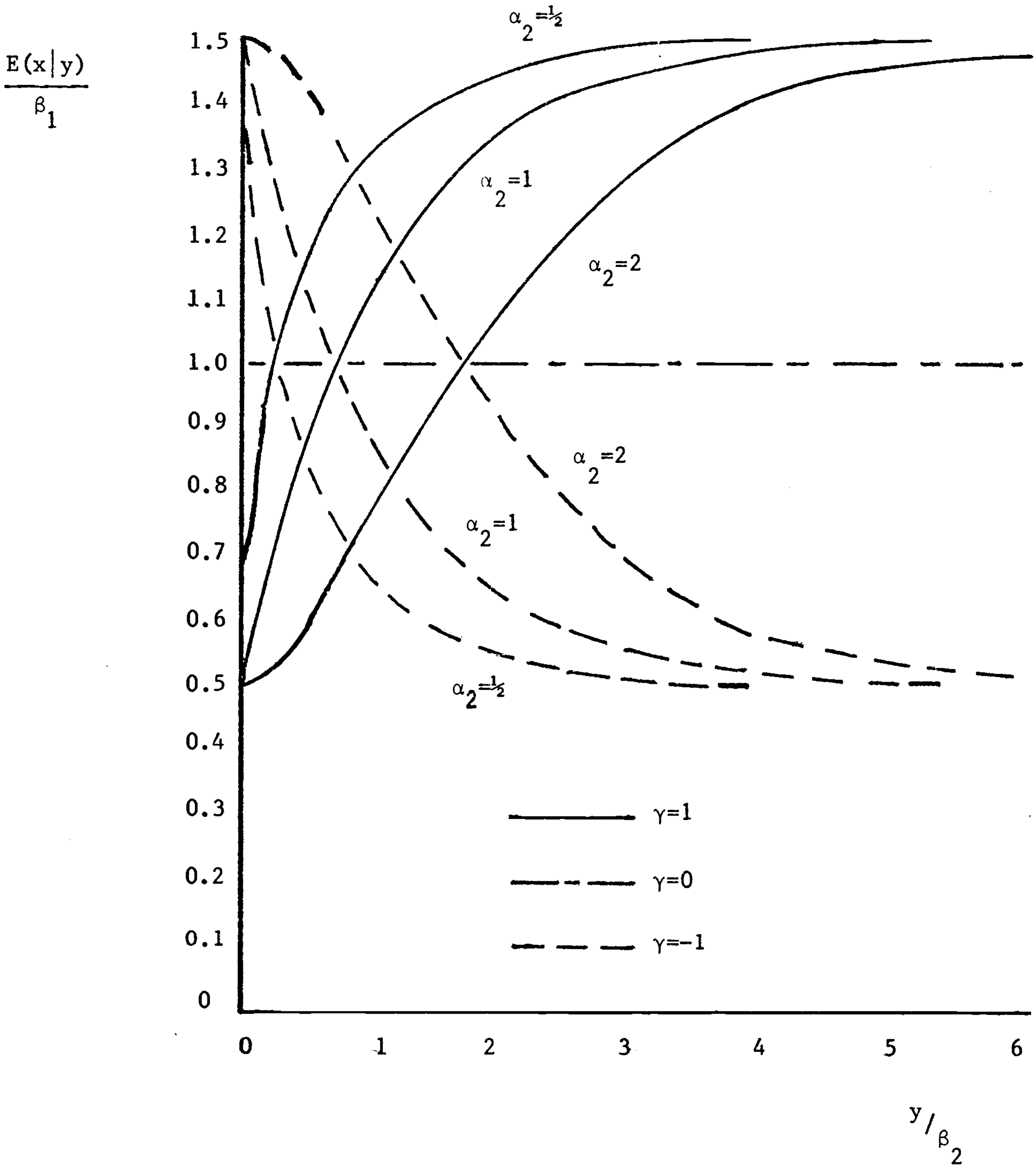

Figure 1. Regression curves for Morgenstern BV gamma distributions with X gamma $(\alpha_1=1,\beta_1)$ and Y gamma (α_2,β_2).

2.1.3. Correlations of X and Y

The correlation ratio $\eta(X|y)$, as defined in [7 , equation (1.9)], reduces in our case to equation (2.23).

$$\eta(X|y) = \left\{ \frac{1}{\sigma_y^2} \int [E(X)-E(X|y)]^2 g(y;\alpha_2\beta_2)dy \right\}^{1/2}$$

$$= \left\{ \frac{\gamma^2}{3\alpha_2\beta_2^2} [\alpha_1\beta_1 - \beta_1 K(\alpha_1)]^2 \right\}^{1/2} = \frac{\gamma\beta_1}{\sqrt{3}\,\sigma_y} [\alpha_1 - K(\alpha_1)] \quad . \tag{2.23}$$

As in [7], we now show that $\eta(X|y)$ is a multiple of the coefficient of correlation ρ. By and using definition (2.19), we find

$$E(XY) = \beta_1\beta_2 \left\{ \alpha_1\alpha_2 + \gamma [\alpha_1-K(\alpha_1)] [\alpha_2-K(\alpha_2)] \right\} \quad ,$$

which gives

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_x \quad \sigma_y}$$

$$= \frac{\gamma[\alpha_1 - K(\alpha_1)] [\alpha_2 - K(\alpha_2)]}{\sqrt{\alpha_1} \quad \sqrt{\alpha_2}} \tag{2.24}$$

The inequality $|\gamma| \leq 1$ implies

$$|\rho| \leq \frac{1}{\sqrt{\alpha_1\alpha_2}} [\alpha_1-K(\alpha_1)] [\alpha_2-K(\alpha_2)] \quad .$$

Using equations (2.23) and (2.24) yields

$$\eta(x|y) = \rho \cdot \frac{\sqrt{\alpha_1} \cdot \beta_1}{\sqrt{3}\,[\alpha_2 - K(\alpha_2)]} \tag{2.25}$$

2.1.4. Conditional Median of X Given y

From equation (2.1), the conditional c.d.f. of X, given y, is found:

$$F(x|y) = F(x) + \gamma F(x)\,[1-F(x)]\,[1-2G(y)] \quad . \tag{2.26}$$

The conditional median $M_{x|y}$ is defined by

$$F(M_{x|y}|y) = 1/2 \quad . \tag{2.27}$$

Solving equation (2.27), we obtain

$$F(M_{x|y}) = \frac{-1-\gamma(1-2G) \pm \sqrt{[1+\gamma(1-2G)]^2 + 2(2G-1)}}{2(2G-1)} = Q_\gamma(G) \tag{2.28}$$

and

$$M_{x|y} = F^{-1}(Q_\gamma(G)) \quad . \tag{2.29}$$

For the case where $\alpha_1=1$, equation (2.29) reduces to

$$M_{x|y} = -\beta_1 \ln[\gamma\, Q_\gamma(G)] \quad . \tag{2.30}$$

Notice that $F(x|y)$ and consequently $M_{x|y}$, depends on x and y only through F and G. Graphs of equations (2.28) are given in Figure 2 for γ = -1, 0, 1. Using expression (2.14) for F with $\alpha_1=1$ and a table of chi-square distributions with degrees of freedom 1, 2, and 4 for evaluation of G, graphs of equation (2.30) are given in

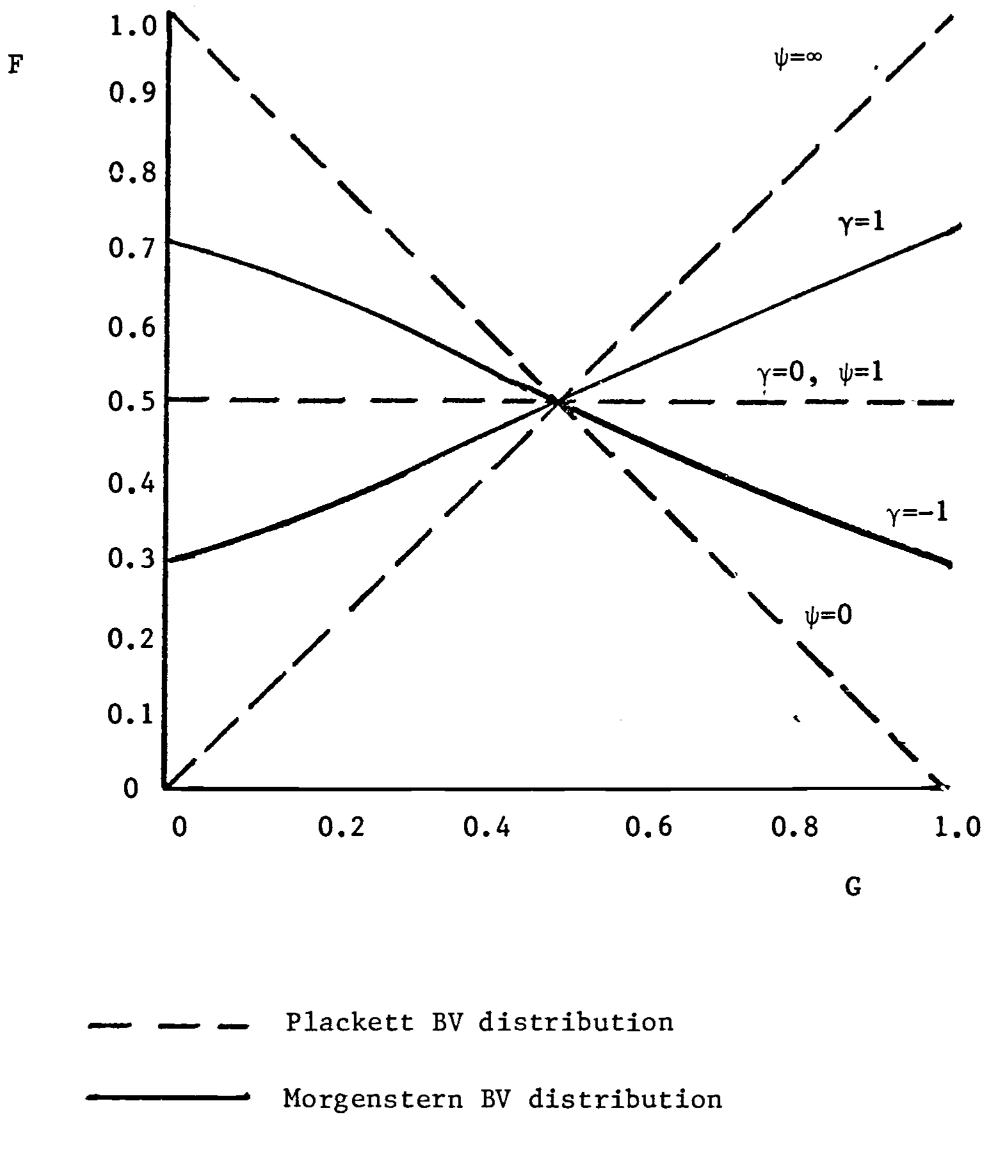

Figure 2. Regression curves of F on G for Plackett and Morgenstern BV distributions.

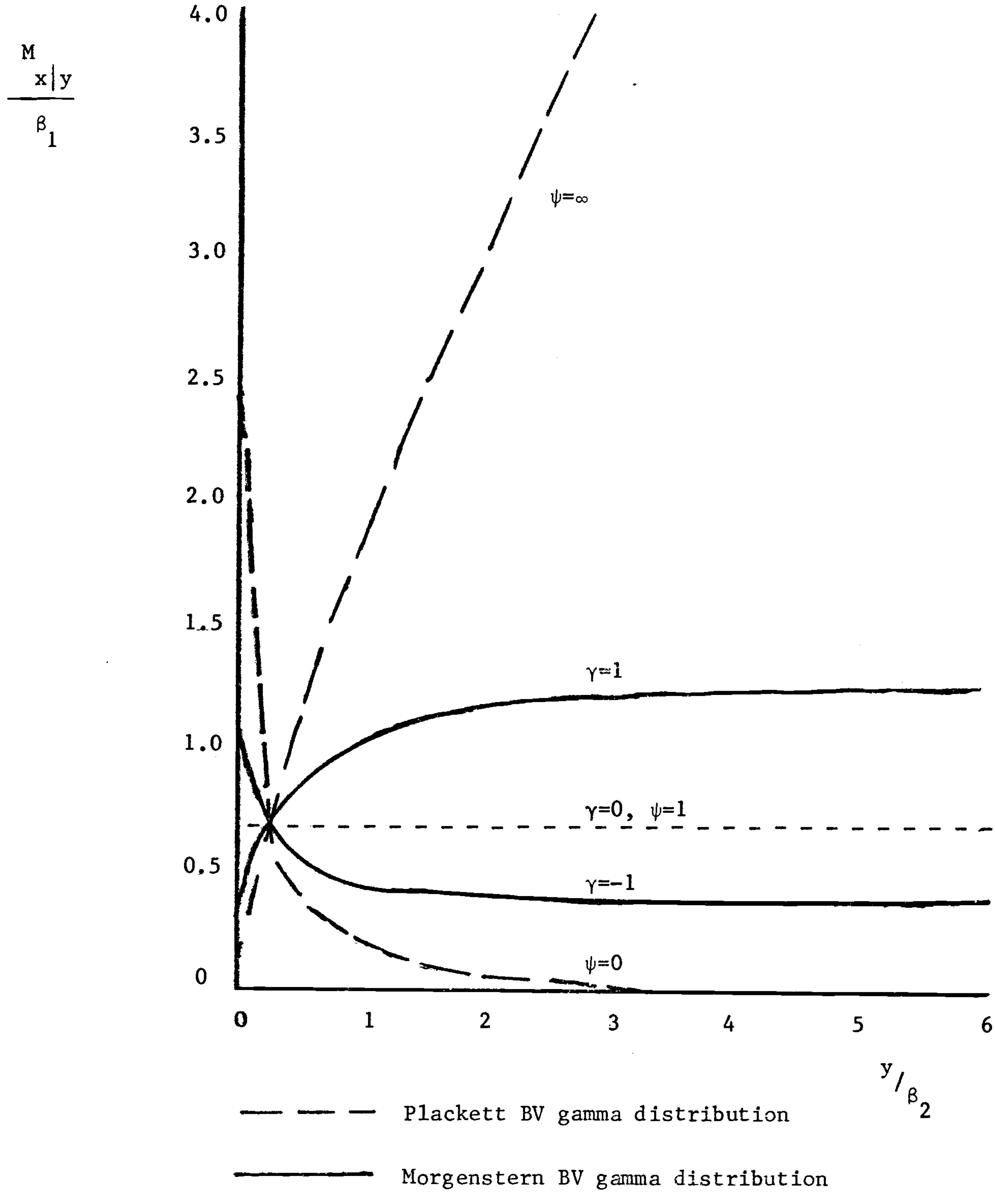

Figure 3. Median regression curves for two BV gamma distributions with X gamma (α_1=1, β_1) and y gamma (α_2=0.5, β_2).

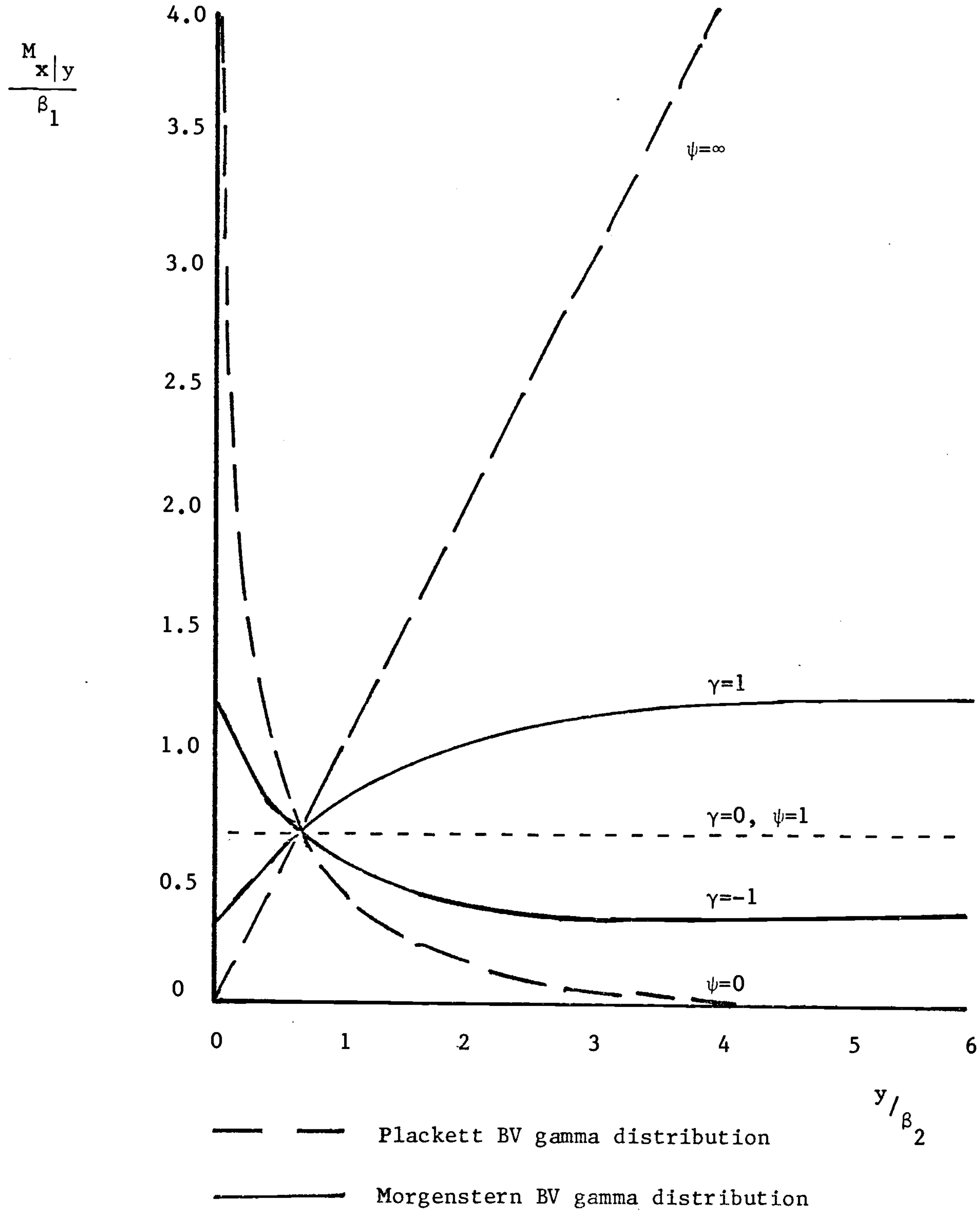

Figure 4. Median regression curves for two BV gamma distributions with x gamma (α_1=1,β_1) and y gamma (α_2=1,β_2).

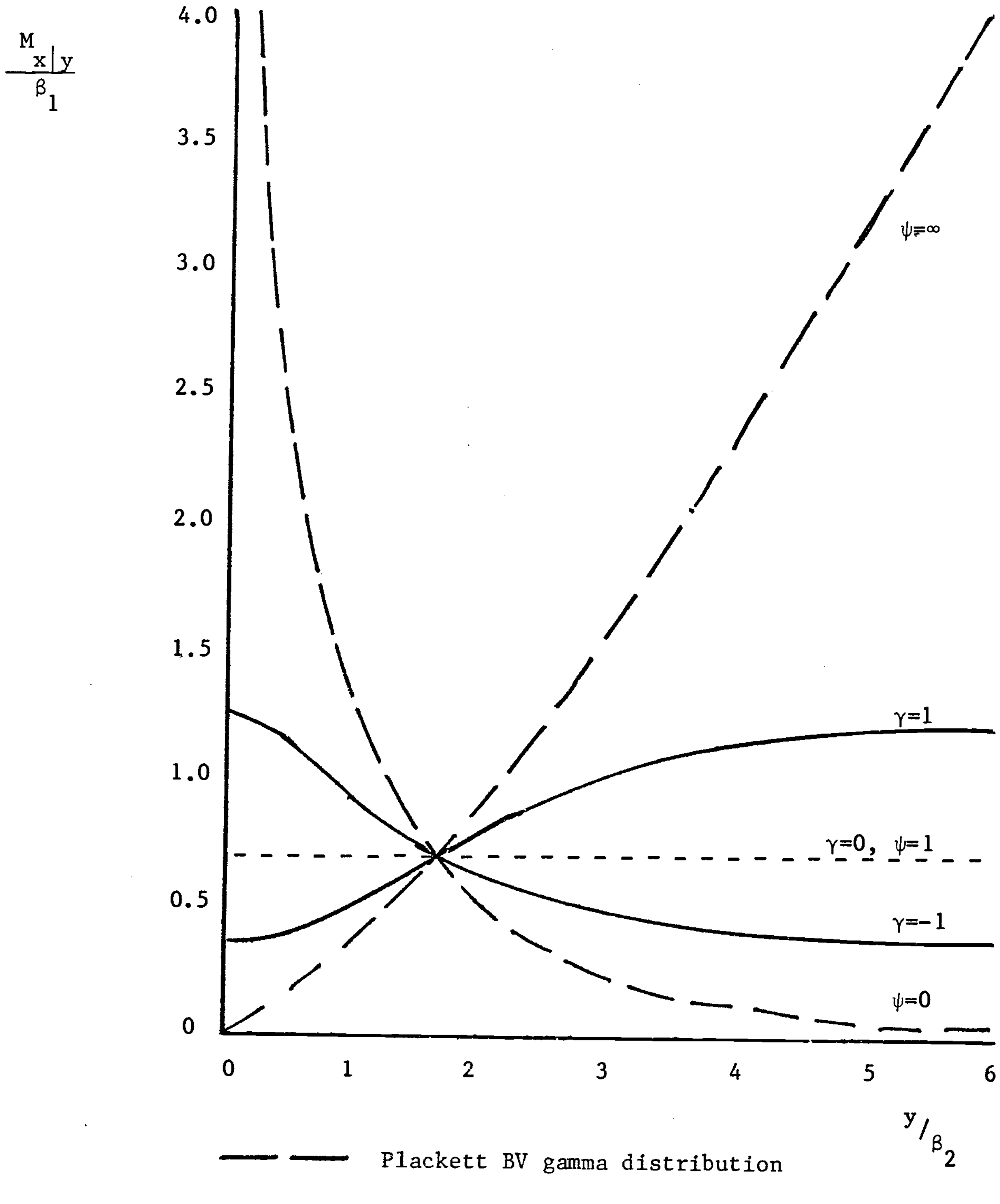

Figure 5. Median regression curves for two BV gamma distributions with x gamma (α_1=1,β_1) and y gamma (α_2=2,β_2).

Figures 3, 4, and 5 for the case $\alpha_2 = 1/2$, 1, and 2 respectively.

2.2 Plackett's BV Distribution

Let ψ be an arbitrary positive number. Plackett constructed the joint c.d.f. H(x,y) (equation 2.4) as the root of the quadratic equation

$$\psi = \frac{H(x,y)\cdot [1-F(x) - G(y) + H(x,y)]}{[F(x)-H(x,y)]\ [G(y) - H(x,y)]} \tag{2.31}$$

Mardia [14] showed that only one of the two roots of equation (2.31) satisfies Frechet inequality

$$\mathrm{Max}(0,\ F(x) + G(y)-1) \le H(x,y) \le \mathrm{Min}(F(x),G(y)) \quad . \tag{2.32}$$

When $H(x,y) = \mathrm{Max}(0,F(x)+G(y)-1)$, the entire BV distribution lies on the line $F(x)+G(y)=1$. From equation (2.31), we see that this is true when $\psi=0$. When $H(x,y) = \mathrm{Min}(F(x),G(y))$, the entire BV distribution lies on the $F(x)=G(y)$, corresponding to $\psi=\infty$ in equation (2.31).

Notice that equation (2.31) can be written in terms of probabilities as

$$\psi = \frac{P(X<x\ ,\ Y<y)\ P(X>x\ ,\ Y>y)}{P(X<x\ ,\ Y>y)\ P(X>x\ ,\ Y<y)} \tag{2.33}$$

The Plackett distribution has been called [14] a contingency type BV distribution. Plackett [24] considered ψ as a measure of association in a four-fold contingency table. $\psi = 0$ implies the independence of X and Y.

2.2.1. Conditional Median of X Given y

Plackett determined the conditional c.d.f. of X, given Y=y,

$$F(x|y) = \frac{\partial H(x,y)}{\partial G(y)}$$

$$= \frac{F(x)\psi - H(x,y)(\psi-1)}{1 + [F(x) + G(y) - 2H(x,y)]\,(\psi-1)} \quad . \tag{2.34}$$

The conditional mean of X, given Y=y, is usually not of simple form; therefore Plackett considers the conditional median of X, given Y=y,

$$M_{x|y} = F^{-1}\left(\frac{1}{\psi+1}(1 + (\psi-1)\,G(y))\right) \quad , \tag{2.35}$$

or correspondingly,

$$F(M_{x|y}) = \frac{1}{\psi+1}(1 + (\psi-1)\,G(y)) = W_{\psi}(G) \quad . \tag{2.36}$$

Equation (2.36) is graphed in Figure 2 for ψ=0, 1, and ∞.

Using equations (2.14), (2.15), and (2.35) for the gamma marginals with $\alpha_1 = 1$, we obtain

$$M_{x|y} = -\beta \ln\left|1 - \frac{1}{\psi+1}\left[1+(\psi-1)\left(1-\sum_{i=0}^{\alpha_2-1}\frac{\left(\frac{y}{\beta_2}\right)^i e^{-\frac{y}{\beta_2}}}{i!}\right)\right]\right| \tag{2.37}$$

Equation (2.37) is graphed for ψ=0, 1, and ∞ in Figures 3, 4, and 5 for α_2 = 1/2, 1, and 2, respectively. For α_2 = 1/2, the evaluation of G(y) is found by using the table of chi-square distribution with 1 degree of freedom.

2.2.2. The Correlation Coefficient

According to Mardia [14], the correlation coefficient of X and Y based on contingency type BV distribution can be expressed in the form

$$\rho_{a,b,c,d}(\psi) = \frac{1}{\sigma_1\sigma_2}\int_a^b\int_c^d (H-FG)\,dx\,dy \tag{2.38}$$

where $\sigma_1^2 = \mathrm{Var}(X)$ and $\sigma_2^2 = \mathrm{Var}(Y)$.

In [14, p.239] $\rho_{a,b,c,d}(\psi)$ is shown to have a limit $\rho(\psi)$ as $a,c \longrightarrow -\infty$ and $b,d \longrightarrow \infty$ under the following conditions:

(1) $\lim_{x\to\pm\infty} x\left[F(x|y) - F(x)\right] = 0$,

(2) $\lim_{y\to\pm\infty} y\,[H-FG] = 0$,

(3) $\int_{-\infty}^{\infty}\left|F(x|y)-F(x)\right|dx < I(y)$

where I(y) is an integrable function of Y. Mardia [14] and Plackett [24] further establishes the relations

$$\begin{array}{lll} 0 < \rho(\psi) \le 1 & \text{if} & 1 < \psi \le \infty \;, \\ \rho(\psi) = 0 & \text{if} & \psi = 1 \;, \\ -1 \le \rho(\psi) < 0 & \text{if} & 0 \le \psi < 1 \;. \end{array}$$

Moreover, the correlation coefficient $\rho_{F,G}(\psi)$ of random variables F(X) and G(Y) is given

$$\rho_{F,G}(\psi) = (\psi^2-1-2\psi\ln\psi) \;/\; (\psi-1)^2 \tag{2.39}$$

2.3. Moran's BV Distribution

Motivated by analysis of rainfall data, Moran [19] constructed BV distributions equation (2.5) with gamma marginals. Moran [19] and Mardia [13] showed that equation (2.5) satisfies the Frechet inequality

$$\mathrm{Max}(F(x) + G(y) - 1, 0) \leq H(x,y) \leq \mathrm{Min}(F(x), G(y)) \quad .$$

Moran also discussed an iterative procedure for obtaining the maximum likelihood estimators for the gamma parameters α_1, α_2, β_1, β_2, and the parameter ρ. Moreover a test for equal scale parameters based on maximum likelihood estimators is discussed for rain-making experiments.

These BV gamma distributions are constructed as follows: Let (W,Z) follow a standard unit BV normal distribution with correlation coefficient ρ,

$$\phi(w,z) = \frac{1}{2\pi(1-\rho^2)^{1/2}} e^{-\frac{1}{2(1-\rho^2)}(w^2 - 2\rho wz + z^2)}$$

with marginal c.d.f's $\qquad \Phi(w) = \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt \quad ,$

and

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt \quad .$$

Let X and Y each have p.d.f.'s given by equations (2.12) and (2.13). Equating,

$$\Phi(w) = F(x) \qquad \text{and} \qquad \Phi(z) = G(y)$$

gives

$$w = \Phi^{-1}(F(x)) \quad \text{and} \quad z = \Phi^{-1}(G(y)) \quad ,$$

and the resulting p.d.f.

$$h(x,y) = \phi(\Phi^{-1}(F(x)),\ \Phi^{-1}(G(y)))\frac{dw}{dx}\ \frac{dz}{dy}\ dxdy$$

$$= \frac{1}{(1-\rho^2)^{\frac{1}{2}}}\ e^{-\frac{1}{2(1-\rho^2)}[\rho^2\Phi^{-1^2}(F(x))-2\rho\Phi^{-1}(F(x))^{-1}(G(y))+\Phi^{-1^2}(G(y))]}$$

$$\cdot \frac{1}{\Gamma(\alpha_1)} \cdot \frac{1}{\Gamma(\alpha_2)}\ e^{-(\frac{x}{\beta_1}+\frac{y}{\beta_2})} \cdot (\frac{x}{\beta_1})^{\alpha_1-1}(\frac{y}{\beta_2})^{\alpha_2-1} \cdot \frac{1}{\beta_1} \cdot \frac{1}{\beta_2}\ dxdy \quad .$$

Among the Morgenstern, Plackett and Moran BV distributions, the first gives the simplest form. However the Morgenstern distribution does not attain the Frechet inequality [7, 14, 24], whereas the other two do. For these three BV distributions, H(x,y) and $\frac{h(x,y)}{f(x)g(y)}$ depend on X and Y only through F(x) and G(y). The special forms are utilized for the nonparametric consideration in the following two chapters. For the Morgenstern and Plackett distributions, H(x,y) have explicit forms in F(x) and G(y); however, two variable numerical intergration is needed to evaluate H(x,y) for the Moran distribution.

III TESTS FOR INDEPENDENCE

For testing the null hypothesis of independence of random variables X and Y, asymptotically optimal $C(\alpha)$ tests and locally most powerful rank tests (l.m.p.r.t.'s) are considered for the Morgenstern, Plackett and Moran BV distributions. The general class of $C(\alpha)$ tests was developed by Neyman [21], who used locally root-n consistent estimators for nuisance parameters. We compare the forms of these two kinds of test statistics and their asymptotic relative efficiency. ARE comparisons are also made with other tests for independence.

3.1. Locally Asymptotically Optimal $C(\alpha)$ Tests

First, the general development of optimal $C(\alpha)$ tests is summarized.

Suppose a sample $(X_1, X_2, \ldots, X_n)$ is taken from a population with p.d.f. $f(x;\xi,\underline{\theta})$ where $\underline{\theta}=(\theta_1,\ldots,\theta_r)$. The observations X_i may be vector values. For testing the null hypothesis $H_o:\xi=\xi_o$, against the alternatives $H_1:\xi>\xi_o$ (or $\xi<\xi_o$) or $H_2:\xi\neq\xi_o$ with unknown nuisance parameters $\underline{\theta}$, Neyman [21] considered tests based on the statistics

$$Z_n(\underset{\sim}{X};\xi_o,\hat{\underline{\theta}}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{\phi_\xi(x_j;\xi_o,\hat{\underline{\theta}}) - \sum_{i=1}^{r} a_i^o \phi_{\theta_i}(x_j;\xi_o,\hat{\underline{\theta}})}{\sigma_{\underline{\theta}}}, \tag{3.1}$$

where

$$\phi_\xi(x_j;\xi_o,\underline{\hat{\theta}}) = \left.\frac{\partial \ln f(x_j;\xi,\underline{\theta})}{\partial \xi}\right|_{\substack{\xi=\xi_o \\ \underline{\theta}=\underline{\hat{\theta}}}} \tag{3.2}$$

$$\phi_{\theta_i}(x_j;\xi_o,\underline{\hat{\theta}}) = \left.\frac{\partial \ln f(x_j;\xi,\underline{\theta})}{\partial \theta_i}\right|_{\substack{\xi=\xi_o \\ \underline{\theta}=\underline{\hat{\theta}}}} , \tag{3.3}$$

and the a_i^o's are chosen so as to minimize the variance of $\phi_\xi(x_j;\xi_o,\underline{\hat{\theta}}) - \sum_{i=1}^{r} a_i^o \phi_{\theta_i}(x_j;\xi_o,\underline{\hat{\theta}})$ under H_o. The resulting minimum variance is denoted by $\sigma_{\underline{\theta}}^2$. The estimators $\underline{\hat{\theta}} = (\hat{\theta}_1,\ldots,\hat{\theta}_r)$ are assumed to be either locally root-n consistent or root-n consistent for $\underline{\theta}$. Neyman [21] defines an estimator $\hat{\theta}_{jn}$ to be locally root-n consistent for θ_j if there exists a number $A_j \neq 0$, such that as $n \to \infty$, the product $\left|\hat{\theta}_{jn} - \theta_j - A_j(\xi-\xi_o)\right|\sqrt{n}$ remains bounded in probability for all ξ and θ. If $\left|\hat{\theta}_{jn} - \theta_j\right|\sqrt{n}$ remains bounded in probability as $n \to \infty$, $\hat{\theta}_{jn}$ is said to be root-n consistent for θ_j. The p.d.f. $f(x;\xi,\underline{\theta})$ is also assumed to satisfy Cramér-type regularity conditions [21,p.215].

The statistic of equation (3.1) has also been shown to have a limiting unit normal distribution under H_o. The optimal symmetric $C(\alpha)$ tests based on equation (3.1) for two-sided alternatives $H_2:\xi \neq \xi_o$ has critical regions $S_2(\alpha) = \left\{ Z_n : \left| Z_n \right| > z_{\alpha/2} \right\}$, where $z_{\alpha/2}$

is such that

$$\int_{z_{\alpha/2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \alpha/2 \quad .$$

For optimal one-tailed $C(\alpha)$ tests of $H_o: \xi = \xi_o$ against $H_1: \xi > \xi_o$ the critical region is $S_1(\alpha) = \left\{ Z_n : Z_n \geq z_\alpha \right\}$.

Moran [18] indicates that Z_n is asymptotically equivalent to tests using the maximum likelihood estimator (MLE) $\tilde{\xi}$ of ξ , and to the likelihood ratio test based on the statistic

$$2\left\{\sum_{j=1}^{n} \ln f(x_j;\tilde{\xi},\underline{\tilde{\theta}}) - \sum_{j=1}^{n} \ln f(x_j;\xi_o,\underline{\tilde{\theta}}_o)\right\} \quad ,$$

where $\tilde{\xi}$ and $\underline{\tilde{\theta}}$ are MLE's and $\underline{\tilde{\theta}}_o$ are MLE's under the null hypothesis. An advantage of the $C(\alpha)$ test over the other two types of tests is that the $C(\alpha)$ tests are frequently easier to compute.

Now we consider optimal $C(\alpha)$ tests for independence. Suppose (X,Y) follows a certain BV distribution with p.d.f. $h(x,y;\xi,\underline{\theta})$ where ξ is a parameter indicating the association between X and Y. That is, $h(x,y;\xi,\underline{\theta}) = f(x;\underline{\theta})g(y;\underline{\theta})$ if, and only if, for some number $\xi_o, \xi=\xi_o$. For the Morgenstern, Plackett, and Moran distributions, the null hypothesis of independence is given by $\gamma=0, \psi=1$, and $\rho=0$, respectively, in equations (2.1), (2.3) and (2.5).

The $C(\alpha)$ test statistics of independence for the Morgenstern, Plackett, and Moran distributions will now be shown to be a function of ϕ_ξ only. This result will follow from the fact that ϕ_ξ and ϕ_{θ_i}, $i=1,\ldots,r$, are uncorrelated under the null hypothesis, which gives

$a_1^o = \ldots = a_r^o = 0$. By using the p.d.f.'s of equations (2.1), (2.3), and (2.5) in equations (3.2) and (3.3), it may be seen that for appropriate functions K(U), $0 \le U \le 1$, ϕ_ξ can be expressed in the form

$$\phi_\xi = K(F(x))K(G(y)) \quad , \tag{3.4}$$

and

$$\phi_{\theta_j} = \frac{\partial \ln f}{\partial \theta_j} + \frac{\partial \ln g}{\partial \theta_j} \qquad j=1,\ldots,r \ . \tag{3.5}$$

Under H_o: $\xi = \xi_o$, the relations

$$E\ \phi_\xi = E[K(F(X))K(G(Y))] = E[K(F(X))]\ E[K(G(Y))]$$
$$= [E(F(X))]^2 = 0$$

imply that

$$E[K(F(X))] = E[K(G(Y))] = 0 \quad . \tag{3.6}$$

Using relation (3.5) and the fact that $E\ \phi_{\theta_j}=0$, we have, for $j=1,\ldots,r$

$$\mathrm{cov}(\phi_\xi,\phi_{\theta_j}) = E(\phi_\xi \cdot \phi_{\theta_j})$$

$$= E\left[K(F(X))\cdot K(G(Y))\cdot\left[\frac{\partial \ln f(X;\xi_o,\underline{\theta})}{\partial\theta_j} + \frac{\partial \ln g(Y;\xi_o,\underline{\theta})}{\partial\theta_j}\right]\right]$$

$$= E\left[K(F(X))\cdot K(G(Y))\cdot\frac{\partial}{\partial\theta_j}\ln f(X;\xi_o,\underline{\theta})\right] + E\left[K(F(X))\cdot K(G(Y))\cdot\frac{\partial}{\partial\theta_j}\ln g(Y;\xi_o,\underline{\theta})\right]$$

$=0$

Hence, the asymptotic optimal $C(\alpha)$ test for independence, $H_o:\xi=\xi_o$, is based on

$$t_n = \frac{1}{\sqrt{n}} \frac{\sum_{i=1}^{n} \phi_\xi(x_i,y_i;\xi_o,\underline{\hat{\theta}})}{\sigma_{\phi_\xi}} , \qquad (3.7)$$

where $\underline{\hat{\theta}}$ are any locally root-n consistent or simply root-n consistent estimators of $\underline{\theta}$ and $\sigma^2_{\phi_\xi}$ is the variance of $\phi_\xi(x,y;\xi_o,\underline{\hat{\theta}})$.

Now consider estimation of the parameters $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ of both the gamma and Weibull distributions. For the case where there are no linear restrictions on the parameters, any estimators, $\hat{\alpha}_1(\underline{X}),\hat{\alpha}_2(\underline{Y}),\hat{\beta}_1(\underline{X}),\hat{\beta}_2(\underline{Y})$, which are functions of only one of the variates $\underline{X}$ or $\underline{Y}$, and are root-n consistent with respect to the corresponding marginal distributions, such as MLE's and moment estimators, will be root-n consistent with respect to the bivariate distributions. Under the assumption of common shape parameter, $\alpha_1=\alpha_2=\alpha$, any linear function $C_1\hat{\alpha}_1(\underline{X})+C_2\hat{\alpha}_2(\underline{Y})$, with positive coefficients C_1 and C_2 satisfying $C_1+C_2=1$, of root-n consistent estimators $\hat{\alpha}_1(\underline{X})$ and $\hat{\alpha}_2(\underline{Y})$ will be root-n consistent since

$$\sqrt{n}\left|C_1\hat{\alpha}_1(\underline{X})+C_2\hat{\alpha}_2(\underline{Y})-\alpha\right| \leq C_1\sqrt{n}\left|\hat{\alpha}_1(\underline{X})-\alpha\right|+C_2\sqrt{n}\left|\hat{\alpha}_2(\underline{Y})-\alpha\right| .$$

The optimal $C(\alpha)$ test statistics of equation (3.7) are easily evaluated for the Morgenstern, Plackett, and Moran BV distributions, and are given below. Using densities of equation (2.3) in equation (3.2) gives

$$K(U) = 2U-1 \qquad (3.8)$$

in equation (3.4) for both the Morgenstern and the Plackett alternatives.

Using the additional result

$$\sigma^2_{\phi_\gamma} = \sigma^2_{\phi_\psi} = 1/9$$

in equation (3.7) then gives

$$t_n = \frac{3}{\sqrt{n}} \sum_{i=1}^{n} [2F(x_i;\underline{\hat{\theta}})-1]\,[2G(y_i;\underline{\hat{\theta}})-1] \tag{3.9}$$

Similarly, for Moran alternative

$$K(U) = \Phi^{-1}(U) \quad , \tag{3.10}$$

and

$$\sigma^2_{\phi_\xi} = 1$$

yields

$$t_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Phi^{-1}[F(x_i;\underline{\hat{\theta}})]\,\Phi^{-1}[G(y_i;\underline{\hat{\theta}})] \quad . \tag{3.11}$$

3.2 Locally Most Powerful Rank Tests

In this section, we consider l.m.p.r.t.'s for the null hypothesis of independence in the Morgenstern, Plackett and Moran families of distributions.

The l.m.p.r.t.'s for independence are derived in general as follows. For a random sample of size n from any BV distribution with p.d.f. $h(x,y),(X_{(1)},Y_{(R_1)}),\ldots,(X_{(n)},Y_{(R_n)})$ is a sufficient reduction of data, where $X_{(1)} < X_{(2)}\cdots < X_{(n)}$ and $R_1,R_2,\ldots,R_n$ are ranks of the corresponding Y's among $Y_1,\ldots,Y_n$. The joint p.d.f. for this sufficient statistic $(X_{(1)},Y_{(R_1)}),\ldots,(X_{(n)},Y_{(R_n)})$ is

$$n!\prod_{i=1}^{n} h(x_{(i)},y_{(R_i)}) \quad .$$

The problem of testing H_0: $H(x,y)=F(x)G(y)$ against H_1:$H(x,y)\neq F(x)G(y)$ for some (x,y) in each of the Morgenstern, Plackett, and Moran distributions is invariant under the group of function transformations G with elements

$$\begin{aligned} g_{\phi_1,\phi_2} &= g_{\phi_1,\phi_2}\left[(X_{(1)},Y_{(R_1)}),\ldots,(X_{(n)},Y_{(R_n)})\right] \\ &= (\phi_1(X_{(1)}),\phi_2(Y_{(R_1)})),\ldots,(\phi_1(X_{(n)}),\phi_2(Y_{(R_n)})) \quad , \end{aligned} \tag{3.12}$$

where ϕ_1 and ϕ_2 are any continuous increasing functions from the real line onto the real line. The induced group of function transformations $\bar{G}$ on the parameter space has elements

$$\bar{g}_{\phi_1,\phi_2} = \bar{g}_{\phi_1,\phi_2}(F,G,\xi) = (F\phi_1^{-1},G\phi_2^{-1},\xi) \tag{3.13}$$

where ξ is the parameter associated with independence of X and Y. In particular, $\xi = \gamma, \psi$, and ρ respectively for Morgenstern, Plackett and Moran distributions.

A maximal invariant statistic for the group of transformation is $(R_1,\ldots,R_n)$ and the probability function for $(R_1,\ldots,R_n)$ is given by

$$P_\xi(r_1,\ldots,r_n) = n!\int_S \prod_{i=1}^{n} h(x_i,y_i)dx_i dy_i \quad , \qquad (3.14)$$

where $S = \left\{(X_1,Y_{r_1}),\ldots,(X_n,Y_{r_n});\ X_1< ,\ldots, < X_n\right.$ and r_i is the rank of the Y-observation corresponding to X_i among the Y-sample$\left.\right\}$. The l.m.p.r.t. of H_o: $\xi = \xi_o$ against H_1:$\xi>\xi_o$ rejects H_o for

$$T = \frac{\partial}{\partial\xi} \ln P(r_1,\ldots,r_n)\Big|_{\xi=\xi_o} > K \quad . \qquad (3.15)$$

The tests which reject H_o: $\xi = \xi_o$ for

$$|T| > K \qquad (3.16)$$

may be used for the two-sided alternatives H_2 : $\xi \neq \xi_o$.

Using the relation

$$\prod_{i=1}^{n} h_\xi(x_i,y_i) = e^{\sum_{i=1}^{n} \ln h_\xi(x_i,y_i)}$$

and (3.1) for the Morgenstern, Plackett, and Moran distributions, we obtain

$$\frac{\partial}{\partial \xi} \prod_{i=1}^{n} h_\xi(x_i,y_i)\Big|_{\xi=\xi_o} = \sum_{i=1}^{n} \frac{\partial}{\partial \xi} \ln h_\xi(x_i,y_i) \prod_{i=1}^{n} h_\xi(x_i,y_i)\Big|_{\xi=\xi_o} \tag{3.17}$$

$$= \left[\sum_{i=1}^{n} \phi_\xi(x_i,y_i)\right] \prod_{i=1}^{n} f(x_i)g(y_i)$$

From equations (3.14) and (3.15), we then evaluate the l.m.p.r.t.

$$T = \frac{\partial}{\partial \xi} \ln P \Big|_{\xi=\xi_o} \propto \int_S n! \frac{\partial}{\partial \xi} \prod_{i=1}^{n} h_\xi(x_i,y_i) \prod_{i=1}^{n} dx_i dy_i \Big|_{\xi=\xi_o}$$

$$= n! \int_S \left[\sum_{i=1}^{n} \phi_\xi(x_i,y_i)\right] \prod_{i=1}^{n} f(x_i)g(y_i) \prod_{i=1}^{n} dx_i dy_i \tag{3.18}$$

$$= \sum_{i=1}^{n} E_{\xi_o} \phi_\xi (X_{(i)}, Y_{(r_i)}) = T^* \quad .$$

For the Morgenstern and Plackett distributions, using equations (3.4) and (3.8) in equation (3.18), we obtain

$$T^* = \sum_{i=1}^{n} E[2F(X_{(i)})-1] \cdot E[2G(Y_{(r_i)})-1]$$

$$= \sum_{i=1}^{n} [2\frac{i}{n+1} - 1]\,[2\,\frac{r_i}{n+1} - 1] \qquad (3.19)$$

$$= 1/3\,\frac{n(n-1)}{n+1}\,[\frac{12}{n^3-n} \sum_{i=1}^{n} (i - \frac{n+1}{2})(r_i - \frac{n+1}{2})] = 1/3\,\frac{n(n-1)}{n+1} \cdot \gamma_s$$

where $\gamma_s = \frac{12}{n^3-n} \sum_{i=1}^{n} (i - \frac{n+1}{2})(r_i - \frac{n+1}{2})$ is Spearman's rank correlation coefficient. Hence, the test which rejects H_o for $\gamma_s \geq K\,\frac{3(n+1)}{n(n-1)}$ is equivalent to rejection of H_o for $T^* \geq K$. Therefore, the one-tailed test based on γ_s is a l.m.p.r.t. for both the Morgenstern and Plackett distributions. Kendall [9,p.76] shows the asymptotic normality of γ_s and has the table of probability function of γ_s. Farlie [5] shows that the tests based on γ_s are asymptotically efficient with respect to tests using the MLE $\hat{\gamma}$ for the correlation index γ for the Morgenstern distributions.

For the Moran distributions, using equations (3.4) and (3.10), equation (3.18) gives the Fisher-Yates rank correlation coefficient

$$T^*/\sum_{i=1}^{n}[EZ_{(i)}]^2 = \sum_{i=1}^{n} E\left[\Phi^{-1}[F(X_{(i)})]\right] E\left[\Phi^{-1}[F(Y_{(r_i)})]\right] \Big/ \sum_{i=1}^{n}[E(Z_{(i)}]^2$$

$$= \sum_{i=1}^{n} EZ_{(i)}EZ_{(r_i)} \Big/ \sum_{i=1}^{n} [EZ_{(i)}]^2 \qquad (3.20)$$

where $Z_{(k)}$ is the kth order statistic from the unit normal distribution. The l.m.p.r.t.'s for Moran alternatives can also be obtained from the previous work of Hajěk and Šidák [8,p.112] who showed that l.m.p.r.t.'s of independence against the BV normal alternatives $\rho > 0$ can be based on the Fisher-Yates rank correlation coefficient. Since rank tests are distributed independently of F(x) and G(y), any l.m.p.r.t. of independence against BV normal alternatives will also be a l.m.p.r.t. against Moran alternatives. Hence the Fisher-Yates rank correlation coefficient gives a l.m.p.r.t. of independence against the Moran distribution with $\rho > 0$.

Using the approximate scores $\Phi^{-1}(i/(n+1))$ in place of $EZ_{(i)}$ in equation (3.20) gives the Van der Waerden correlation coefficient

$$\sum_{i=1}^{n} \Phi^{-1}\left(\frac{i}{n+1}\right) \Phi^{-1}\left(\frac{r_i}{n+1}\right) \Big/ \sum_{i=1}^{n} \left[\Phi^{-1}\left(\frac{i}{n+1}\right)\right]^2 \quad . \tag{3.21}$$

Hajěk and Šidák [8,p.112] have showed the asymptotic equivalence of test statistics (3.20) and (3.21).

It is interesting to note a relation between the optimal $C(\alpha)$ tests of independence, equations (3.9) and (3.11), and the corresponding l.m.p.r.t.'s. Replacing $F(X_i;\hat{\underline{\theta}})$ and $G(Y_i;\hat{\underline{\theta}})$ by $F(X_i)=i/(n+1)$ and $G(Y_i)=r_i/(n+1)$, respectively, in equations (3.9) and (3.11) yields statistics which are proportional to the corresponding Spearman and Van der Waerden correlation coefficients given in equations (3.19) and (3.21).

In the next section we will show that l.m.p.r.t.'s are asymptotically efficient with respect to the corresponding $C(\alpha)$ tests. Unless F and G have simple forms, the rank tests may be much easier to compute than the corresponding $C(\alpha)$ tests.

3.3. Asymptotic Relative Efficiency of Nonparametric Tests with Respect to Optimal $C(\alpha)$ Tests

In this section we compute the ARE of tests based on the Spearman correlation coefficient and the Fisher-Yates (normal scores) correlation coefficient with respect to the asymptotically optimal $C(\alpha)$ tests of equation (3.7) for the three distributions under consideration.

The general form for the efficacy of $C(\alpha)$ tests is derived as follows. Letting μ_ξ denote the asymptotic mean of the statistic t_n in equation (3.7), we find

$$\mu_\xi = \frac{1}{\sqrt{n}} \frac{\sum_{i=1}^{n} E_\xi \phi_\xi (X_i, Y_i; \xi_o, \theta)}{\sigma_{\phi_\xi}}$$

$$= \sqrt{n} \frac{E_\xi \phi_\xi (X, Y; \xi_o, \underline{\theta})}{\sigma_{\phi_\xi}} \tag{3.22}$$

$$= \frac{\sqrt{n}\, E_\xi [K(F(X))K(G(Y))]}{EK^2(F(X))}$$

where $\sigma^2_{\phi_\xi} = E\phi^2_\xi - (E\phi_\xi)^2$. Using equation (3.4), we obtain

$$\sigma^2_{\phi_\xi} = EK^2F(X)\cdot EG^2(Y) = [EK^2F(X)]^2 \quad .$$

Differentiation of equation (3.22) then gives

$$\left.\frac{\partial \mu_\xi}{\partial \xi}\right|_{\xi=\xi_o} = \frac{\sqrt{n}\, E_{\xi_o} K^2(F(X)) \cdot E_{\xi_o} K^2(G(Y))}{E_{\xi_o} K^2(F(X))}$$

$$= \sqrt{n}\, E_{\xi_o} K^2(G(Y)) = \sqrt{n}\, E_{\xi_o} K^2(F(X)) .$$

Using the result that t_n has asymptotic unit variance under the null hypothesis, we have for the efficacy

$$\lim_{n\to\infty} e\,(t_n) = \lim_{n\to\infty} \frac{\left.\left(\frac{\partial E_\xi(t_n)}{\partial \xi}\right)^2\right|_{\xi=\xi_o}}{n \operatorname{Var}_{\xi_o}(t_n)}$$

$$= \left.\left(\frac{\partial \mu_\xi}{\partial \xi}\right)^2\right|_{\xi=\xi_o} \tag{3.23}$$

$$= [E\, K^2(F(X))]^2 \quad .$$

For the Morgenstern and Plackett distributions, the efficacy of the optimal $C(\alpha)$ tests is, from equations (3.8) and (3.23),

$$\lim_{n\to\infty} e(t_n) = [EK^2(F)]^2$$

$$= [E(2F-1)^2]^2 = 1/9 \quad . \tag{3.24}$$

For the Moran distributions, using equations (3.10) and (3.23), we obtain

$$\lim_{n \to \infty} e(t_n) = [E(\Phi^{-1}(F))^2]^2$$
$$= 1 \quad . \tag{3.25}$$

The efficacy of Spearman rank correlation coefficient γ_s for the Morgenstern and Plackett distributions is calculated as follows. The term γ_s in equation (3.19) can be written as

$$\gamma_s = \frac{4n^3}{(n+1)^2} \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n} \cdot \frac{r_i}{n}\right)\right] - n$$
$$= \frac{4n^3}{(n+1)^2} S_n - n \quad , \tag{3.26}$$

where

$$S_n = \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{r_i}{n} \quad .$$

Computing the efficacy of S_n is sufficient to compute γ_s. In terms of the empirical distribution functions

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x)$$

$$G_n(y) = \frac{1}{n} \sum_{i=1}^{n} I(Y_i \leq y)$$

$$H_n(x,y) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \le x,\ Y_i \le y) \qquad ,$$

where I(A) is the indicator function of the set A.

S_n may be written as

$$S_n = \frac{1}{n} \sum_{i=1}^{n} F_n(x_{(i)}) G_n(y_{(r_i)})$$

$$= \iint F_n(x) G_n(y) dH_n(x,y) \qquad .$$

The asymptotic mean of S_n is given by

$$\begin{aligned} \mu_\xi &= \iint F(x)G(y)dH_\xi(x,y) \\ &= \iint F(x)G(y)h_\xi(x,y)dxdy \qquad . \end{aligned} \tag{3.27}$$

Using equation (3.4), we find

$$\left.\frac{\partial \mu_\xi}{\partial \xi}\right|_{\xi=\xi_o} = \iint F(x)G(y)\frac{\partial h_\xi(x,y)}{\partial \xi} \frac{1}{h_\xi(x,y)} \cdot h_\xi(x,y)dxdy \Big|_{\xi=\xi_o}$$

$$= \iint F(x)G(y)(2F(x)-1)(2G(y)-1)dF(x)dG(y)$$

$$= 1/36$$

The exact variance of S_n under the null hypothesis, as given in [8,p.114], is

$$\mathrm{Var}(S_n) = \frac{(n+1)^2(n-1)}{144\ n^4} \tag{3.28}$$

Hence, the efficacy of γ_s and S_n is

$$\lim_{n\to\infty} e(\gamma_s) = \lim_{n\to\infty} e(S_n) = \lim_{n\to\infty} \frac{\left[\frac{\partial}{\partial\xi}\mu_\xi\right]^2\Big|_{\xi=\xi_o}}{n \operatorname{Var}(S_n)}$$

$$= \lim_{n\to\infty} \frac{\left[\frac{1}{36}\right]^2}{\frac{n(n+1)^2(n-1)}{144\, n^4}} = 1/9 \quad . \tag{3.29}$$

Evaluation of the efficacy of γ_s for the Moran distributions is similar to that of the Morgenstern and Plackett distributions. Using equation (3.27), we obtain

$$\lim_{n\to\infty} \frac{\partial\mu_\rho}{\partial\rho}\Big|_{\rho=o} = \iint F(x)G(y)\Phi^{-1}(F(x))\Phi^{-1}(G(y))dF(x)dG(y)$$

$$= [\int F(x)\Phi^{-1}(F(x))dF(x)]^2$$

$$= [\int \Phi(w)w d\Phi(w)]^2 \tag{3.30}$$

$$= (1/2 \times 0.56419)^2 = .079524 \quad .$$

The last equality follows from the fact that $2\int w\Phi(w)d\Phi(w)$ is the expectation of the minimum of two independent unit normal random variables. From equations (3.30) and (3.28), we obtain

$$\lim_{n\to\infty} e(\gamma_s) = \lim_{n\to\infty} e(S_n)$$

$$= \frac{(1/2 \times 0.56419)^4}{1/144} \tag{3.31}$$

$$= 9(0.56419)^4 = .91169 \quad .$$

We will now calculate the efficacy of the statistic

$$T'_n = \frac{1}{n}\sum_{i=1}^{n}\Phi^{-1}\left(\frac{i}{n+1}\right)\Phi^{-1}\left(\frac{r_i}{n+1}\right) \tag{3.32}$$

which is $\frac{1}{n}$ times the numerator of the Van der Waerden correlation coefficient (expression 3.21) and is asymptotically equivalent to the Fisher-Yates correlation coefficient equation (3.20). The statistic T'_n has asymptotic mean

$$\mu' = \iint \Phi^{-1}(F(x))\Phi^{-1}(G(y))h_\xi(x,y)dxdy$$

Then for the Morgenstern and Plackett distributions, we have

$$\begin{aligned}
\left.\frac{\partial \mu_\xi}{\partial \xi}\right|_{\xi=\xi_o} &= \left.\iint \Phi^{-1}(F(x))\,\Phi^{-1}(F(y))\frac{\partial h_\xi(x,y)}{\partial \xi}\,\frac{1}{h_\xi(x,y)}\,h_\xi(x,y)dxdy\right|_{\xi=\xi_o} \\
&= \iint \Phi^{-1}(F(x))\Phi^{-1}(G(y))\,\phi(x,y)f(x)g(y)dxdy \\
&= \iint \Phi^{-1}(F(x))\Phi^{-1}(G(y))[2F(x)-1][2G(y)-1]dF(x)dG(y) \\
&= [\int \Phi^{-1}(F(x))(2F(x)-1)dF(x)]^2 \\
&= [\int w[2\Phi(w)-1]d\Phi(w)]^2 \\
&= 4[\int w\Phi(w)d\Phi(w)]^2
\end{aligned} \tag{3.33}$$

,

and

$$\mathrm{Var}(T'_n) = \frac{1}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n}[\Phi^{-1}(\frac{i}{n+1})]^2\right]^2 \tag{3.34}$$

Using equations (3.33) and (3.34) we obtain the efficacy

$$\lim_{n\to\infty} e(T_n') = \lim_{n\to\infty} \frac{\left|4[\int w\Phi(w)\,d\Phi(w)]^2\right|^2}{\frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\Phi^{-1}\left(\frac{i}{n+1}\right)\right)^2\right]^2}$$

$$= \frac{16[\int w\Phi(w)\,d\Phi(w)]^2}{[\int[\Phi^{-1}(F(x))]^2 dF(x)]^2} \tag{3.35}$$

$$= 16[\int w\Phi(w)\,d\Phi(w)]^4 = 0.1012999 \quad .$$

For Moran distributions, using equation (3.10) we can reduce equation (3.33) to

$$\lim_{n\to\infty} \left.\frac{\partial\mu_\rho'}{\partial\rho}\right|_{\rho=0} = \iint \Phi^{-1}(F)\Phi^{-1}(G)\Phi^{-1}(F)\Phi^{-1}(G)\,dF\,dG$$

$$= [EZ^2]^2 = 1 \quad . \tag{3.36}$$

Then equations (3.34) and (3.36) give the efficacy

$$\lim_{n\to\infty} e(T_n') = \frac{1}{[\int \Phi^{-1^2}(F)\,dF]^2} = 1 \quad . \tag{3.37}$$

The ARE's of the nonparametric tests relative to the $C(\alpha)$ tests are computed by taking the appropriate ratios of efficacies.

TABLE 1. THE ARE'S OF NONPARAMETRIC TESTS FOR INDEPENDENCE BASED ON THE SPEARMAN AND FISHER-YATES (VAN DER WAERDEN) STATISTICS RELATIVE TO THE ASYMPTOTIC OPTIMAL $C(\alpha)$ TESTS

Test statistics \ BV distributions	Morgenstern and Plackett	Moran
Spearman correlation coefficient	1	.91169
Fisher-Yates (Van der Waerden) correlation coefficient	.91169	1

3.4. ARE's of Two Other Tests of Independence Relative to the $C(\alpha)$ Test for the Plackett Distribution

Plackett [24] considered the consistent estimator

$$\psi_n^+ = \frac{ad}{bc} \qquad ,$$

where a,b,c and d are defined as the frequencies of pairs (X_i,Y_i) respectively in the quadrants $(X \leq h,\ Y \leq k)$, $(X \leq h,\ Y > k)$, $(X > h,\ Y \leq k)$ and $(X > h,\ Y > k)$.

Mardia [14] considered the consistent estimator ψ_n^* which is the solution of the equation

$$\bar{r} = \rho_U(\psi),$$

where

$$\rho_U(\psi) = \text{Corr }(F(x),G(y)) = (\psi^2-1-2\psi\ln\psi)/(\psi-1)^2 \qquad ,$$

and $\bar{r}$ is the Pearson product moment correlation of (F_i,G_i) =

$(F(X_i,\hat{\underline{\theta}}),G(Y_i,\hat{\underline{\theta}}))$, $i = 1,\ldots,n$.

Mardia also evaluated the efficacies

$$\lim_{n\to\infty} e(\psi_n^*) = 1/9$$

and

$$\lim_{n\to\infty} e(\psi_n^+) = 1/16.$$

Hence from equation (3.24), the ARE's of ψ_n^+ and ψ_n^* relative to asymptotically optimal $C(\alpha)$ tests are respectively 9/16 to 1.

IV. COMPARISON OF NONPARAMETRIC TESTS FOR BIVARIATE INTERCHANGEABILITY

Given a random sample $(X_1,Y_1),\ldots,(X_n,Y_n)$ from a BV absolutely continuous c.d.f. $H(x,y)$, we consider rank tests for the null hypothesis of interchangeability, H_0: $H(x,y) = H(y,x)$. Three linear rank test statistics, Wilcoxon (W_N), sum of squared ranks (SSR_N), and Savage (S_N), are described in section 4.1. In section 4.2, ARE comparisons of the three types of tests are made for Morgenstern and Moran BV alternatives with marginal distributions satisfying $G(x) = F(x/\theta)$ for some $\theta \neq 1$. Both gamma and lognormal marginal distributions are used.

4.1. Description of the Test Statistics

Let $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(N)}$, with $N = 2n$, represent the order statistics of the combined sample of X's and Y's. Define $C_{N,1},\ldots,C_{N,N}$ by

$$C_{N,i} = \begin{cases} 1 & \text{if } Z_{(i)} \text{ is an X-observation} \\ 0 & \text{if } Z_{(i)} \text{ is a Y-observation.} \end{cases}$$

Sen [25] studied linear rank statistics

$$T_N = \frac{1}{n} \sum_{i=1}^{N} E_{N,i} C_{N,i} \quad , \tag{4.1}$$

where the coefficients $E_{N,i} = J_N(i/(N+1))$, $i = 1,\ldots,N$, are specified.

The coefficients for the three tests under consideration are defined by

$$W_N : J_N(i/(N+1)) = J(i/(N+1)) = i/(N+1) \quad , \qquad (4.2)$$

$$SSR_N : J_N(i/(N+1)) = J(i/(N+1)) = (i/(N+1))^2 \quad , \qquad (4.3)$$

$$S_N : J_N(i/(N+1)) = \sum_{j=1}^{i} 1/(N-j+1)$$

$$\doteq -\ln(1-i/(N+1))-1 \quad = J(i/(N+1)). \qquad (4.4)$$

Sen considered permutation tests based on T_N. A summary of Sen's development of the permutation tests is included here for completion. Define the 2xn matrix A_N as

$$A_N = \begin{bmatrix} X_1 X_2 \ldots X_n \\ Y_1 Y_2 \ldots Y_n \end{bmatrix} \qquad (4.5)$$

and define rank matrix R_N as

$$R_N = \begin{bmatrix} R_{X_1} R_{X_2} \ldots R_{X_n} \\ \\ R_{Y_1} R_{Y_2} \ldots R_{Y_n} \end{bmatrix} \qquad (4.6)$$

where R_{X_i} and R_{Y_i} are the ranks of X_i and Y_i in the combined sample $(Z_1, \ldots, Z_N)$.

Also, define the set $S(R_N)$ as

$$S(R_N) = \left\{ \begin{array}{l} R_N^*: R_N^* \text{ can be obtained from } R_N \text{ by permutations} \\ \text{of two rank elements within each column.} \end{array} \right\}$$

There are 2^n possible rank matrices in $S(R_N)$. $S(R_N)$ is called the permutation set of R_N. Under H_o: $H(x,y) = H(y,x)$.

$$P(R_N = R_N^* \mid S(R_N)) = \frac{1}{2^n} \quad \text{for all } R_N^* \in S(R_N) \quad . \tag{4.7}$$

Thus, the conditional distribution of T_N given $R_N = R_N^*$, under H_o can be obtained from equation (4.7). It should be pointed out that the unconditional distribution of R_N, and consequently of T_N, does depend on $H(x,y)$ under H_o. Hence, an unconditional test based on T_N would not be distribution free.

Calculation of the level of the conditional Wilcoxon test is now illustrated by the following example. Consider five BV observations given in the data matrix form of equation (4.5)

$$A_{10} = \begin{bmatrix} 3 & 10 & 7 & 13 & 6 \\ 4 & 9 & 5 & 11 & 2 \end{bmatrix} \quad .$$

Using the ordered combined sample

$$(z_{(1)}, z_{(2)}, z_{(3)}, z_{(4)}, z_{(5)}, z_{(6)}, z_{(7)}, z_{(8)}, z_{(9)}, z_{(10)})$$

$$= (y_5, x_1, y_1, y_3, x_5, x_3, y_2, x_2, y_4, x_4)$$

$$= (2,3,4,5,6,7,9,10,11,13)$$

the rank matrix R_{10} is evaluated.

$$R_{10} = \begin{bmatrix} 2 & 8 & 6 & 10 & 5 \\ 3 & 7 & 4 & 9 & 1 \end{bmatrix} \quad .$$

Using equation (4.2) in (4.1), we obtain

$$T_{10}^{o} = \frac{1}{5}\,\frac{2+8+6+10+5}{11} = \frac{31}{55} .$$

The following three R_{10}^{*} elements of $S(R_{10})$ also have T_{10} greater than or equal to T_{10}^{o}:

$$\begin{bmatrix} 3 & 8 & 6 & 10 & 5 \\ 2 & 7 & 4 & 9 & 1 \end{bmatrix} , \begin{bmatrix} 3 & 7 & 6 & 10 & 5 \\ 2 & 8 & 4 & 9 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 8 & 6 & 9 & 5 \\ 2 & 7 & 4 & 10 & 1 \end{bmatrix} .$$

The other 28 R_{10}^{*} elements of $S(R_{10})$ have T_{10} less than T_{10}^{o}.

Therefore,

$$P(T_{10} \geq T_{10}^{o} \,|\, S(R_{10})\ ) = 4/32 = .125.$$

The conditional significance level of the test statistic T_{10}^{o} is then equal to $2P(T_{10} \geq T_{10}^{o}\ ) = .25$.

Under suitable conditions on the coefficients $E_{N,i}$, Sen [25, Th.5.1] establishes the asymptotic normality of the conditional distributions of the test statistics. The three statistics under consideration satisfy Sen's conditions. Now, for the example, we compare the normal approximation with the exact tail probability for $T_{10} \geq T_{10}^{o}$.

The conditional expection and conditional variance of T_N are

$$E(T_N | S(R_N)) = \frac{1}{N} \sum_{i=1}^{N} E_{N,i} \tag{4.8}$$

$$\text{var}\ (T_N \mid S(R_N)) = \frac{1}{N^2} \sum_{k=1}^{n} (E_{N,R_{1.k}} - E_{N,R_{2.k}})^2 \tag{4.9}$$

where $R_{j.k}$ denotes the rank of the jth element of kth pair, j = 1,2 .

In our example,

$$E(T_{10} \mid S(R_{10})) = 1/2$$

$$\text{var}\ (T_{10} \mid S(R_{10})) = \frac{23}{12100}$$

and

$$z^o = \frac{T_{10}^o - E(T_{10} \mid S(R_{10}))}{[\text{Var}(T_{10} \mid S(R_{10}))]^{\frac{1}{2}}} = \frac{7}{\sqrt{23}} = 1.49 \ .$$

For the normal approximation to $P(T_{10} \geq T_{10}^o \mid S(R_{10}))$, we have

$$P(Z \geq z^o) = .068 \qquad ,$$

where $Z \sim N\ (\ 0,\ 1\ )$. Using the correction for continuity improves the approximation to $P\ (Z \geq 1.08) = .100$

4.2. ARE's for Morgenstern and Moran BV Gamma Distributions

The ARE's of the Wilcoxon, sum of squared ranks, and Savage tests are now evaluated for the Morgenstern and Moran BV gamma distributions with common shape parameter $\alpha = 1/2,1,2,3,\ldots,16$ and the BV lognormal distribution.

The general form for the asymptotic efficacy of conditional tests for interchangeability based on linear rank order statistics with the alternatives satisfying $G(y) = F(y \mid \theta)$ with $\theta \neq 1$ was found by Sen [25]:

$$\lim_{N \to \infty} e(T_N) = \lim_{N\to\infty} \frac{\left[\dfrac{\partial E_\theta(T_N)}{\partial \theta}\right]^2_{\theta=1}}{N \operatorname{Var}_{\theta=1}(T_N)}$$

$$= \frac{1/4\ C^2(F,J)}{1/2\ A^2(J)[1-B(H,J)]} \tag{4.10}$$

where

$$A^2(J) = \int_0^1 J^2(U)dU - \left[\int_0^1 J(U)dU\right]^2 \tag{4.11}$$

$$B(H,J) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))J(F(y))dH(x,y) - \left[\int_0^1 J(U)dU\right]^2}{A^2(J)} \tag{4.12}$$

$$C(F,J) = \int_{-\infty}^{\infty} xJ'(F(x))f(x)dF(x) \tag{4.13}$$

and

$$J(U) = \lim_{N\to\infty} J_N(U) \quad \text{for} \quad 0 < U < 1 \quad .$$

Concerning the constants A, B, and C, notice that A is independent of distributions, B depends on the BV distribution form and is equal to zero for the case of independence, whereas C depends on the BV distribution only through the marginals. The quantities in Equations (4.10), (4.11), (4.12) and (4.13) are evaluated below.

First we consider the case of the Morgenstern BV gamma distributions with common shape parameter α.

For the Wilcoxon test statistic (W_N) with $J(U) = U$, $0 < U < 1$,

$$A^2(J) = 1/12 \tag{4.14}$$

Evaluation of B(H,J) requires the integration:

$$\begin{aligned}&\iint J(F(x))J(F(y))dH(x,y)\\ &= \iint F(x)F(y)f(x)f(g)[1+\gamma(2F(x)-1)(2F(y)-1)]dxdy \qquad (4.15)\\ &= 1/4 + \gamma/36\end{aligned}$$

For C(F,J), we find

$$C(F,J) = \int xf^2(x)dx = \frac{\Gamma(2\alpha)}{2^{2\alpha}[\Gamma(\alpha)]^2} \quad . \tag{4.16}$$

Hence,

$$\lim_{N \to \infty} e(W_N) = 1/2 \cdot \frac{3}{1-\frac{\gamma}{3}} \left\{ \frac{\Gamma(2\alpha)}{2^{2\alpha}[\Gamma(\alpha)]^2} \right\}^2 \tag{4.17}$$

For the sum of squared ranks test statistic (SSR_N) with $J(U) = U^2$, $0 < U < 1$, we have

$$A^2(J) = 4/45 \tag{4.18}$$

$$\begin{aligned}&\iint J(F(x))J(F(y))dH(x,y)\\ &= \iint F^2(x)F^2(y)f(x)f(g)[1+\gamma(2F(x)-1)(2F(y)-1)]dxdy \qquad (4.19)\\ &= 1/9 + \gamma/36\end{aligned}$$

Using equations (4.19) and (4.18) in (4.12), we have

$$B(H,J) = 5\gamma/16 \; . \tag{4.20}$$

The term

$$C(F,J) = \int_0^\infty 2xF(x)f^2(x)dx$$

reduces to

$$C(F,J) = \int_0^\infty 2x\left(1-\sum_{i=o}^{\alpha-1}\frac{x^i e^{-x}}{i!}\right)\left[\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}\right]^2 dx$$

$$= 2\cdot\left[\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2\cdot 2^{2\alpha}} - \sum_{i=0}^{\alpha-1}\frac{\Gamma(i+2\alpha)}{i![\Gamma(\alpha)]^2\cdot 3^{i+2\alpha}}\right] \text{ for integer } \alpha \tag{4.21}$$

and to

$$C(F,J) = \int_0^\infty\int_0^x 2x\,\frac{y^{-\frac{1}{2}}e^{-y}}{\sqrt{\pi}}\cdot\frac{x^{-1}e^{-2x}}{\pi}\,dydx$$

$$= 2\int_0^\infty\int_0^x \frac{y^{-\frac{1}{2}}e^{-y}e^{-2x}}{\pi^{3/2}}dydx = \int_0^\infty\int_y^\infty 2e^{-2x}dx\,\frac{y^{-\frac{1}{2}}e^{-y}}{\pi^{3/2}}\,dy \tag{4.22}$$

$$= \int_0^\infty \frac{e^{-3y}y^{-\frac{1}{2}}}{\pi^{3/2}}\,dy = \frac{\sqrt{3}}{3\pi} \qquad \text{for } \alpha = 1/2 \; .$$

Using equations (4.18), (4.20), (4.21) and (4.22) in (4.10), we have

$$\lim_{N \to \infty} e(SSR_N) = \begin{cases} \dfrac{2\left[\dfrac{\Gamma(2\alpha)}{\Gamma(\alpha)^2 \cdot 2^{2\alpha}} - \displaystyle\sum_{i=o}^{\alpha-1} \dfrac{\Gamma(i+2\alpha)}{\Gamma(\alpha)\cdot i!\cdot 3^{i+2\alpha}}\right]^2}{4/45 \;-\; \gamma/36} & \text{for integer } \alpha. \qquad (4.23) \\[2em] \dfrac{1}{6\pi^2(4/45 - 1/36\gamma)} & \text{for } \alpha = 1/2. \qquad (4.24) \end{cases}$$

For Savage test statistics (S_N) with $J(U) = -\ln(1-U)-1$, we have

$$A^2(J) = 1 \tag{4.25}$$

and

$$\iint J(F(x))J(F(y))dH(x,y)$$

$$= \int_o^1\!\!\int_o^1 J(U)J(V)[1+\gamma(2U-1)(2V-1)]dUdV \tag{4.26}$$

$$= \gamma/4$$

where $U = F(x)$ and $V = F(y)$.

Using equations (4.26) and (4.25) in equation (4.12), we obtain

$$B(H,J) = \gamma/4. \tag{4.27}$$

The term

$$C(F,J) = \int_o^\infty \frac{x}{1-F(x)} \cdot f^2(x)dx \tag{4.28}$$

reduces to

$$
C(F,J) = \begin{cases} \displaystyle\int_0^\infty \frac{x^{2\alpha-1}\, e^{-x}}{[\Gamma(\alpha)]^2 \sum_{i=o}^{2\alpha-1} \frac{x^i}{i!}}\, dx & \text{for integer } \alpha\,, \quad (4.29) \\[2em] \displaystyle\int_0^\infty \frac{xe^{-2x}}{[1-F_\alpha(x)]\pi}\, dx & \text{for } \alpha = 1/2\,, \quad (4.30) \end{cases}
$$

where expressions (4.29) and (4.30) are approximated by 32 point quadrature numerical integration $\int_0^\infty k(x)e^{-x}dx = \sum_{i=1}^{32} k(x_i)A_i$.

Tables of x_i and A_i given in [9,p.352] were used.

Using equations (4.25), (4.27), (4.29) and (4.30) in equation (4.10), we have

$$
\lim_{N\to\infty} e(S_N) = \begin{cases} \dfrac{\left[\displaystyle\int_0^\infty \frac{x^{2\alpha-1}}{(\Gamma(\alpha))^2 \sum_{i=0}^{\alpha-1} \frac{x^i}{i!}} \cdot e^{-x}dx\right]^2}{2\,(\,1 - \gamma/4\,)} & \text{for integer } \alpha\,, \quad (4.31) \\[3em] \dfrac{\left[\displaystyle\int_0^\infty \frac{1}{1-F_\alpha(x)}\, \frac{xe^{-2x}}{\pi}\, dx\right]^2}{2\,(\,1 - \gamma/4\,)} & \text{for } \alpha = 1/2\,. \quad (4.32) \end{cases}
$$

The ARE's for the Wilcoxon, Savage and SSR tests are tabulated in Tables II-IV for the Morgenstern BV gamma distributions with common shape parameter $\alpha = 1/2,1,2,3,\ldots,16$ and parameter $\gamma = -1(.1)1$. For the case of independence ($\gamma = 0$), our results of the ARE of W_N relative to SSR_N for all integer α's, and the ARE of W_N relative to S_N for $\alpha = 1$ agree to four decimal places with those of Duran and Mielke [4]. Also for $\gamma = 0$, the ARE of SSR_N relative to S_N agrees with that of Mielke [4] to three decimal places. In figure 6, values of α and γ for which each of the three tests is most efficient are indicated. The Savage test is seen to be best among the three tests for negative γ. For positive γ, either the Wilcoxon or the sum of squared ranks test is preferred to the Savage test depending on the combination of α and γ.

For computing ARE's for Moran BV gamma distributions with common shape parameter α, we only need to evaluate $B(H,J)$ since $A(J)$ and $C(F,J)$ do not depend on the form of the BV distribution. We use the following transformations

$$F(x) = \Phi(w)$$

$$F(y) = \Phi(z)$$

and

$$H(x,y) = \Phi(w,z) \quad ,$$

where (W,Z) are BV unit normal random variables with correlation coefficient ρ.

For the Wilcoxon statistic W_N then,

$$\begin{aligned} &\iint J(F(x))J(F(y))dH(x,y) \\ &= \iint J(\Phi(w))J(\Phi(z))d\Phi(w,z) \qquad (4.33) \\ &= \quad P(U < W, V < Z) \quad , \end{aligned}$$

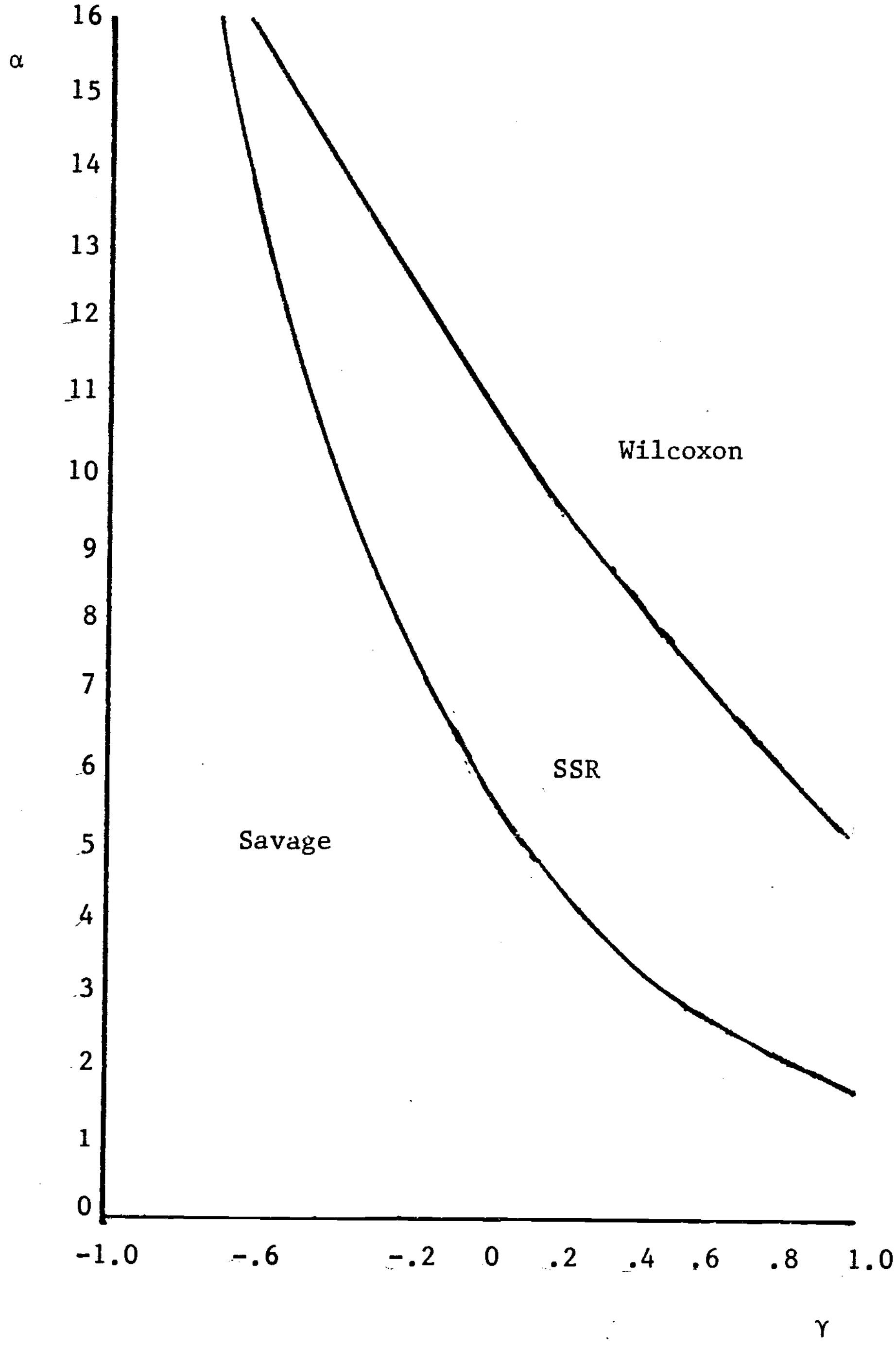

Figure 6. Regions of the most efficient tests for the Morgenstern BV distribution

TABLE II. ASYMPTOTIC RELATIVE EFFICIENCY OF THE WILCOXON TEST RELATIVE TO THE SAVAGE TEST OF H_o: $H(x,y)=H(y,x)$ AGAINST ALTERNATIVES $G(y)=F(y/\theta), \theta \neq 1$, FOR THE MORGENSTERN BV DISTRIBUTION

$\alpha \backslash \gamma$	-1	-.9	-.8	-.7	-.6	-.5	-.4	-.3	-.2	-.1	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
1	.703	.707	.711	.715	.719	.723	.728	.733	.738	.744	.750	.757	.763	.771	.779	.788	.797	.807	.818	.830	.844
2	.803	.807	.811	.816	.821	.826	.831	.837	.843	.850	.857	.864	.872	.880	.889	.899	.910	.922	.934	.948	.964
3	.851	.855	.860	.865	.870	.875	.881	.887	.894	.901	.908	.916	.924	.933	.943	.953	.965	.977	.990	1.005	1.021
4	.881	.885	.890	.895	.900	.906	.912	.918	.925	.932	.940	.948	.956	.966	.976	.987	.998	1.011	1.025	1.040	1.057
5	.902	.906	.911	.916	.922	.928	.934	.940	.947	.954	.962	.970	.979	.989	.999	1.010	1.022	1.035	1.050	1.065	1.082
6	.917	.922	.927	.932	.938	.944	.950	.956	.963	.971	.979	.987	.996	1.006	1.016	1.027	1.040	1.053	1.068	1.083	1.101
7	.930	.935	.940	.945	.950	.956	.963	.969	.976	.984	.992	1.000	1.009	1.019	1.030	1.041	1.054	1.067	1.082	1.098	1.116
8	.940	.945	.950	.955	.961	.967	.973	.980	.987	.994	1.002	1.011	1.020	1.030	1.041	1.053	1.065	1.079	1.094	1.110	1.128
9	.948	.953	.958	.964	.969	.975	.982	.988	.996	1.003	1.011	1.020	1.029	1.039	1.050	1.062	1.075	1.088	1.103	1.120	1.038
10	.955	.960	.965	.971	.977	.983	.989	.996	1.003	1.011	1.019	1.028	1.037	1.047	1.058	1.070	1.083	1.097	1.112	1.128	1.146
11	.962	.966	.972	.977	.983	.989	.995	1.002	1.010	1.017	1.026	1.034	1.044	1.054	1.065	1.077	1.090	1.104	1.119	1.136	1.154
12	.967	.972	.977	.983	.988	.995	1.001	1.008	1.015	1.023	1.031	1.040	1.050	1.060	1.071	1.083	1.096	1.110	1.125	1.142	1.160
13	.972	.977	.982	.988	.993	1.000	1.006	1.013	1.020	1.028	1.037	1.046	1.055	1.065	1.076	1.088	1.101	1.115	1.131	1.148	1.166
14	.976	.981	.986	.992	.998	1.004	1.011	1.018	1.025	1.033	1.041	1.050	1.060	1.070	1.081	1.093	1.106	1.120	1.136	1.153	1.171
15	.980	.985	.990	.996	1.002	1.008	1.015	1.022	1.029	1.037	1.045	1.054	1.064	1.074	1.086	1.098	1.111	1.125	1.140	1.157	1.176
16	.984	.989	.994	.999	1.005	1.012	1.018	1.025	1.033	1.041	1.049	1.058	1.068	1.078	1.089	1.102	1.115	1.129	1.144	1.161	1.180

TABLE III. ASYMPTOTIC RELATIVE EFFICIENCY OF THE SUM OF SQUARED RANKS TEST RELATIVE TO THE SAVAGE TEST OF H_o: $H(x,y)=H(y,x)$ AGAINST ALTERNATIVES $G(y)=F(y/\theta)$, $\neq 1$, FOR THE MORGENSTERN BV DISTRIBUTION

$\alpha \backslash \gamma$	-1	-.9	-.8	-.7	-.6	-.5	-.4	-.3	-.2	-.1	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
1	.827	.830	.833	.837	.841	.845	.849	.853	.858	.863	.868	.874	.880	.886	.893	.900	.908	.917	.926	.936	.947
2	.889	.892	.896	.900	.904	.908	.913	.917	.923	.928	.934	.939	.946	.953	.960	.968	.977	.986	.996	1.007	1.018
3	.917	.920	.924	.928	.932	.937	.941	.946	.951	.957	.963	.969	.976	.983	.990	.988	1.007	1.017	1.027	1.038	1.050
4	.934	.937	.941	.945	.949	.954	.958	.963	.969	.974	.980	.987	.993	1.001	1.008	1.017	1.025	1.035	1.046	1.057	1.070
5	.945	.949	.953	.957	.961	.965	.970	.975	.981	.986	.992	.999	1.006	1.013	1.021	1.029	1.038	1.048	1.058	1.070	1.082
6	.954	.957	.961	.965	.970	.974	.979	.984	.990	.995	1.001	1.008	1.015	1.022	1.030	1.038	1.047	1.057	1.068	1.080	1.092
7	.960	.964	.968	.972	.976	.981	.986	.991	.996	1.002	1.008	1.015	1.022	1.029	1.037	1.045	1.055	1.065	1.075	1.087	1.100
8	.966	.969	.973	.977	.982	.986	.991	.996	1.002	1.008	1.014	1.020	1.027	1.035	1.043	1.051	1.061	1.071	1.081	1.093	1.106
9	.970	.974	.978	.982	.986	.991	.996	1.001	1.006	1.012	1.018	1.025	1.032	1.039	1.048	1.056	1.065	1.075	1.086	1.098	1.111
10	.974	.978	.982	.986	.990	.995	1.000	1.005	1.010	1.016	1.022	1.029	1.036	1.044	1.052	1.060	1.070	1.080	1.091	1.102	1.115
11	.977	.981	.985	.989	.993	.998	1.003	1.008	1.014	1.020	1.026	1.032	1.039	1.047	1.055	1.064	1.073	1.083	1.094	1.106	1.119
12	.980	.984	.988	.992	.996	1.001	1.006	1.011	1.017	1.023	1.029	1.035	1.043	1.050	1.058	1.067	1.076	1.086	1.097	1.109	1.122
13	.982	.986	.990	.994	.999	1.004	1.008	1.014	1.019	1.025	1.031	1.038	1.045	1.052	1.061	1.070	1.079	1.089	1.100	1.112	1.125
14	.985	.988	.992	.997	1.001	1.006	1.011	1.016	1.022	1.028	1.034	1.040	1.048	1.055	1.063	1.072	1.081	1.092	1.103	1.115	1.128
15	.987	.990	.994	.999	1.003	1.008	1.013	1.018	1.024	1.030	1.036	1.043	1.050	1.057	1.065	1.074	1.084	1.094	1.105	1.117	1.130
16	.988	.992	.996	1.001	1.005	1.010	1.015	1.020	1.026	1.032	1.038	1.045	1.052	1.059	1.067	1.076	1.086	1.096	1.107	1.119	1.132

TABLE IV. ASYMPTOTIC RELATIVE EFFICIENCY OF THE WILCOXON TEST RELATIVE TO THE SUM OF SQUARED RANKS TEST OF H_o: $H(x,y)=H(y,x)$ AGAINST ALTERNATIVES $G(y)=F(y/\theta), \theta \neq 1$, FOR THE MORGENSTERN BV DISTRIBUTION

$\alpha \backslash \gamma$	-1	-.9	-.8	-.7	-.6	-.5	-.4	-.3	-.2	-.1	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
½	.788	.789	.790	.791	.792	.793	.794	.796	.797	.798	.800	.802	.804	.806	.808	.810	.813	.815	.818	.821	.825
1	.851	.852	.853	.854	.855	.856	.858	.859	.861	.862	.864	.866	.868	.870	.872	.875	.878	.880	.884	.887	.891
2	.903	.904	.906	.907	.908	.909	.911	.912	.914	.916	.918	.920	.922	.924	.926	.929	.932	.935	.938	.942	.946
3	.928	.929	.931	.932	.933	.935	.936	.938	.939	.941	.943	.945	.947	.950	.952	.955	.958	.961	.964	.968	.973
4	.944	.945	.946	.947	.949	.950	.952	.953	.955	.957	.959	.961	.963	.965	.968	.971	.974	.977	.980	.984	.989
5	.954	.955	.957	.958	.959	.961	.962	.964	.966	.967	.969	.971	.974	.976	.979	.982	.985	.988	.991	.995	1.000
6	.962	.963	.965	.966	.967	.969	.970	.972	.974	.975	.977	.980	.982	.984	.987	.990	.993	.996	1.000	1.004	1.008
7	.968	.970	.971	.972	.974	.975	.977	.978	.980	.982	.984	.986	.988	.991	.993	.996	.999	1.002	1.006	1.010	1.014
8	.973	.975	.976	.977	.979	.980	.982	.983	.985	.987	.989	.991	.993	.996	.998	1.001	1.004	1.008	1.011	1.015	1.019
9	.978	.979	.980	.982	.983	.984	.986	.987	.989	.991	.993	.995	.998	1.000	1.003	1.006	1.009	1.012	1.016	1.020	1.024
10	.981	.982	.984	.985	.986	.988	.989	.991	.993	.995	.997	.999	1.001	1.004	1.006	1.009	1.012	1.016	1.019	1.023	1.028
11	.984	.985	.987	.988	.989	.991	.993	.994	.996	.998	1.000	1.002	1.004	1.007	1.009	1.012	1.016	1.019	1.023	1.027	1.031
12	.987	.988	.989	.991	.992	.994	.995	.997	.999	1.001	1.003	1.005	1.007	1.010	1.012	1.015	1.018	1.022	1.025	1.029	1.034
13	.989	.991	.992	.993	.995	.996	.998	.999	1.001	1.003	1.005	1.007	1.010	1.012	1.015	1.018	1.021	1.024	1.028	1.032	1.036
14	.991	.993	.994	.995	.997	.998	1.000	1.001	1.003	1.005	1.007	1.009	1.012	1.014	1.017	1.020	1.023	1.026	1.030	1.034	1.039
15	.993	.995	.996	.997	.999	1.000	1.002	1.003	1.005	1.007	1.009	1.011	1.014	1.016	1.019	1.022	1.025	1.028	1.032	1.036	1.041
16	.995	.996	.998	.999	1.000	1.002	1.003	1.005	1.007	1.009	1.011	1.013	1.015	1.018	1.021	1.024	1.027	1.030	1.034	1.038	1.043

where U and V are independent unit normal random variables which are also independent of W and Z. Let

$$S_1 = (W-U)/\sqrt{2}$$

and

$$S_2 = (Z-V)/\sqrt{2} \quad ,$$

with second order moments

$$\mathrm{var}(S_1) = 1/2(\mathrm{var}(W) + \mathrm{var}(U)) = 1 \quad ,$$

$$\mathrm{var}(S_2) = 1/2(\mathrm{var}(Z) + \mathrm{var}(V)) = 1 \quad ,$$

and

$$\mathrm{cov}(S_1,S_2) = 1/2\ \mathrm{cov}(W,Z) = 1/2\ \rho \quad .$$

Equation (4.33) then is equal to

$$P(0 < S_1, 0 < S_2) = 1/4 + 1/(2\pi)\ \sin^{-1}(\rho/2) \quad . \tag{4.34}$$

The equality of equation (4.34) was established in [2,p.290]. Therefore, from equations (4.12), (4.14), (4.33) and (4.34) we have

$$B(H,J) = (6/\pi)\ \sin^{-1}(\rho/2) \quad . \tag{4.35}$$

Using equations (4.14), (4.35) and (4.16) in equation (4.10) gives

$$\lim_{N \to \infty} e(W_N) = \begin{cases} \dfrac{\left[\dfrac{\Gamma(2\alpha)}{\Gamma(\alpha)^2 \cdot 2^{2\alpha}} \right]^2}{2[1/12 - 1/(2\pi)\sin^{-1}(\rho/2)]} & \text{for integer } \alpha , \qquad (4.36) \\ \\ \dfrac{3}{2\pi^2 \, [1 - 6/\pi \sin^{-1}(\rho/2)]} & \text{for } \alpha = 1/2 \quad . \qquad (4.37) \end{cases}$$

For the sum of squared ranks test statistic (SSR_N),

$$\begin{aligned} &\iint J(F(x))J(F(y))dH(x,y) \\ &= \iint J(\Phi(w))J(\Phi(z))d\Phi(w,z) \\ &= \iint \Phi^2(w)\Phi^2(z)d\Phi(w,z) \\ &= \iint P(U_1 < w, U_2 < w,\ V_1 < z, V_2 < z)d\Phi(w,z) \end{aligned} \qquad (4.38)$$

where U_1, U_2, V_1 and V_2 are independent unit normal random variables which are also independent of W and Z. Let

$$S_1 = (W-U_1)/\sqrt{2},\ S_2 = (W-U_2)/\sqrt{2},\ S_3 = (z-V_1)/\sqrt{2},\ \text{and}\ S_4 = (Z-V_2)/\sqrt{2},$$

so that (S_1, S_2, S_3, S_4) has a quadrivariate normal distribution with zero mean vector and covariance matrix.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{\rho}{2} & \frac{\rho}{2} \\ \frac{1}{2} & 1 & \frac{\rho}{2} & \frac{\rho}{2} \\ \frac{\rho}{2} & \frac{\rho}{2} & 1 & \frac{1}{2} \\ \frac{\rho}{2} & \frac{\rho}{2} & \frac{1}{2} & 1 \end{bmatrix} \qquad (4.39)$$

Then equation (4.38) may be written as

$$P(\ 0 < S_1,\ 0 < S_2,\ 0 < S_3,\ 0 < S_4\ )$$

$$= 1/16 + \left[\sin^{-1}(1/2) + 2\sin^{-1}(\tfrac{\rho}{2})\right]/4\pi + \left[[\sin^{-1}(1/2)]^{-2} - 2[\sin^{-1}(\tfrac{\rho}{2})]^2\right]/4\pi^2$$

$$+ \left[2L_{i_2}[f, \cos^{-1}(\tfrac{\rho}{2})] + 1/2 L_{i_2}[-f^2] - L_{i_2}[f^2, \cos^{-1}(1/2)]\right]/\pi^2$$

$$= \quad L(\rho) \tag{4.40}$$

Equation (4.40) was derived in [1,p.153] with $L_{i_2}(x)$ and $L_{i_2}(r,\theta)$ representing logarithm functions given in [11].

Hence, from equations (4.12), (4.18), (4.38) and (4.40) we have

$$B(H,J) = \frac{L(\rho) - 1/9}{4/45} \tag{4.41}$$

Using equations (4.18), (4.41), (4.21) and (4.22) in equation (4.10) gives

$$\lim_{N \to \infty} e(SSR_N) = \begin{cases} \dfrac{\left\{2\left[\dfrac{\Gamma(2\alpha)}{\Gamma(\alpha)^2 \cdot 2^{2\alpha}} - \displaystyle\sum_{i=0}^{\alpha-1} \dfrac{\Gamma(i+2\alpha)}{\Gamma(\alpha)^2 \cdot i! \cdot 3^{i+2\alpha}}\right]\right\}^2}{2\,[1/5 - L(\rho)]} & \text{for integer } \alpha\ , \qquad (4.42) \\[2ex] \dfrac{1}{6\,\pi^2\,[1/5 - L(\rho)]} & \text{for } \alpha = 1/2\ . \qquad (4.43) \end{cases}$$

For Savage test statistics (S_N) we have

$$B(H,J) = \iint J(\Phi(w))J(\Phi(z))d\Phi(w,z)$$

$$= \iint \ln(1-\Phi(w))\ln(1-\Phi(z))\phi(w,z)dwdz - 1 \tag{4.44}$$

$$= \int \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \ln(1-\Phi(z))e^{-\frac{(1-\rho^2)z^2}{2(1-\rho^2)}} \left[\int \ln(1-\Phi(w))e^{-\frac{(w-\rho z)^2}{2(1-\rho^2)}}dw\right]dz - 1$$

Using the transformation

$$S = \frac{W - \rho Z}{\sqrt{2(1-\rho^2)}}, \quad T = \frac{Z}{\sqrt{2}}$$

and letting

$$h_1(s,t) = \ln[1-\Phi(\sqrt{2(1-\rho^2)}\cdot s + \sqrt{2}\rho t)],$$

$$h_2(t) = \ln[1 - \Phi(\sqrt{2}\, t)],$$

equation (4.44) may be written as

$$B(H,J) = \frac{1}{\pi}\int h_2(t)\int h_1(s,t)e^{-s^2}ds\, e^{-t^2}dt - 1 \tag{4.45}$$

The double integral in equation (4.45) is approximated by

$$\sum_{i=1}^{20} h_2(t_i)\left[\sum_{j=1}^{20} h_1(s_i,t_i)A_j\right]A_i$$

where the $t_i = s_i$ and A_i were taken from [9,p.343-346].

Using equations (4.25), (4.44), (4.29) and (4.30) in equation (4.10), we obtain

$$\lim_{N\to\infty} e(W_N) = \begin{cases} \dfrac{1/2\left[\displaystyle\int \frac{x^{2\alpha-1}}{[\Gamma(\alpha)]^2 \sum_{i-0}^{\alpha-1} \frac{x^i}{i!}} e^{-x} dx\right]^2}{2-\displaystyle\iint \ln[1-\Phi(w)]\ln[1-\Phi(z)] \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} e^{-\frac{w^2-2\rho wz+z^2}{2(1-\rho^2)}} dwdz} & \text{for integer } \alpha \,, \quad (4.46) \\[2em] \dfrac{1/2\left[\displaystyle\int_o^\infty \frac{1}{1-F(x)} \frac{xe^{-2X}}{\pi} dx\right]^2}{2-\displaystyle\iint \ln[1-\Phi(w)]\ln[1-\Phi(z)] \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} e^{-\frac{w^2-2\rho wz+z^2}{2(1-\rho^2)}} dwdz} & \text{for } \alpha = 1/2 \,. \quad (4.47) \end{cases}$$

The ARE's for the Wilcoxon, Savage, and SSR tests are tabulated in Tables V, VI and VII for the Moran BV gamma distributions with common shape parameter α = 1/2, 1, 2, 3,...,16. The correlation coefficients ρ's are chosen so that $\sin^{-1}(\rho/2) = -30^o \ (+5^o) \ 30^o$ (see equation 4.40). When $\rho = 0$, the results agree with those in Tables II - IV. Figure 7, similar to Figure 6, graphs in α - ρ space those regions in which each of the three tests is most efficient.

4.3 ARE's for the BV Lognormal Distribution

For the case of the BV lognormal distributions, we need only to compute the constant C(F,J) in equation (4.13) since the evaluation of $A^2(J)$ and B(H,J) are the same as those obtained for the Moran BV

TABLE V. ASYMPTOTIC RELATIVE EFFICIENCY OF THE WILCOXON TEST RELATIVE TO THE SAVAGE TEST OF H_o: $H(x,y)=H(y,x)$ AGAINST ALTERNATIVES $G(y)=F(y/\theta),\theta\neq 1$, FOR THE MORAN BV DISTRIBUTION

α \ ρ	-.8452	-.6840	-.5176	-.3472	-.1744	0	.1744	.3472	.5176	.6840	.8452
½	.3638	.4247	.4767	.5245	.5714	.6223	.6855	.7779	.9407	1.3003	2.5190
1	.4384	.5118	.5746	.6321	.6888	.7500	.8261	.9377	1.1338	1.5673	3.0366
2	.5007	.5845	.6561	.7218	.7865	.8565	.9434	1.0708	1.2947	1.7898	3.4677
3	.5306	.6195	.6954	.7651	.8336	.9078	.9999	1.1349	1.3723	1.8970	3.6755
4	.5492	.6412	.7198	.7919	.8628	.9396	1.0349	1.1747	1.4203	1.9635	3.8041
5	.5622	.6564	.7368	.8106	.8833	.9618	1.0594	1.2024	1.4540	2.0099	3.8942
6	.5720	.6678	.7496	.8247	.8986	.9785	1.0778	1.2233	1.4792	2.0448	3.9618
7	.5796	.6767	.7597	.8358	.9107	.9917	1.0922	1.2398	1.4991	2.0723	4.0151
8	.5859	.6841	.7679	.8448	.9205	1.0024	1.1040	1.2532	1.5153	2.0947	4.0584
9	.5912	.6902	.7748	.8524	.9287	1.0113	1.1139	1.2644	1.5288	2.1134	4.0947
10	.5956	.6954	.7806	.8588	.9357	1.0190	1.1223	1.2739	1.5404	2.1293	4.1256
11	.5995	.6999	.7857	.8644	.9418	1.0256	1.1296	1.2822	1.5504	2.1432	4.1523
12	.6029	.7038	.7901	.8693	.9471	1.0314	1.1360	1.2894	1.5591	2.1553	4.1758
13	.6059	.7074	.7940	.8736	.9518	1.0365	1.1416	1.2959	1.5669	2.1660	4.1966
14	.6086	.7105	.7976	.8775	.9561	1.0411	1.1467	1.3016	1.5738	2.1756	4.2152
15	.6110	.7133	.8007	.8810	.9559	1.0453	1.1513	1.3068	1.5801	2.1843	4.2320
16	.6132	.7159	.8036	.8841	.9633	1.0490	1.1554	1.3115	1.5858	2.1922	4.2473

TABLE VI. ASYMPTOTIC RELATIVE EFFICIENCY OF THE SUM OF SQUARED RANKS TEST RELATIVE TO THE SAVAGE TEST OF H_0: $H(x,y)=H(y,x)$ AGAINST ALTERNATIVES $G(y)=F(y/\theta), \theta \neq 1$, FOR THE MORAN BV DISTRIBUTION

α \ ρ	-.8452	-.6840	-.5176	-.3472	-.1744	0	.1744	.3472	.5176	.6840	.8452
½	.4852	.5626	.6237	.6751	.7241	.7778	.8473	.9510	1.1361	1.5487	2.9531
1	.5416	.6277	.6958	.7535	.8077	.8681	.9460	1.0613	1.2673	1.7273	3.2922
2	.5823	.6751	.7482	.8102	.8686	.9335	1.0172	1.1412	1.3628	1.8574	3.5400
3	.6006	.6962	.7716	.8356	.8958	.9627	1.0491	1.1770	1.4055	1.9155	3.6510
4	.6115	.7089	.7857	.8509	.9121	.9802	1.0682	1.1985	1.4311	1.9504	3.7174
5	.6190	.7176	.7953	.8613	.9233	.9922	1.0813	1.2131	1.4486	1.9743	3.7630
6	.6246	.7240	.8025	.8691	.9315	1.0011	1.0910	1.2240	1.4616	1.9921	3.7968
7	.6289	.7290	.8081	.8751	.9380	1.0081	1.0986	1.2325	1.4718	2.0059	3.8232
8	.6324	.7331	.8125	.8800	.9432	1.0137	1.1047	1.2394	1.4800	2.0171	3.8445
9	.6354	.7365	.8163	.8840	.9476	1.0184	1.1098	1.2451	1.4868	2.0264	3.8622
10	.6378	.7394	.8195	.8874	.9513	1.0223	1.1141	1.2500	1.4926	2.0343	3.8773
11	.6400	.7418	.8222	.8904	.9544	1.0258	1.1179	1.2541	1.4976	2.0411	3.8902
12	.6418	.7440	.8246	.8930	.9572	1.0287	1.1211	1.2578	1.5019	2.0470	3.9015
13	.6435	.7459	.8267	.8953	.9597	1.0314	1.1240	1.2610	1.5058	2.0522	3.9115
14	.6449	.7476	.8286	.8973	.9619	1.0337	1.1265	1.2639	1.5092	2.0569	3.9204
15	.6463	.7491	.8303	.8992	.9638	1.0358	1.1288	1.2665	1.5123	2.0611	3.9284
16	.6474	.7505	.8318	.9008	.9656	1.0378	1.1309	1.2688	1.5151	2.0649	3.9357

TABLE VII. ASYMPTOTIC RELATIVE EFFICIENCY OF THE SUM OF SQUARED RANKS TEST RELATIVE TO THE WILCOXON TEXT OF H_0: $H(x,y)=H(y,x)$ against alternatives $G(y)=F(y/\theta)$, $\theta \neq 1$, FOR THE MORAN BV DISTRIBUTION

α \ ρ	-.8452	-.6840	-.5176	-.3472	-.1744	0	.1744	.3472	.5176	.6840	.8452
½	1.3340	1.3246	1.3082	1.2870	1.2672	1.2498	1.2361	1.2225	1.2078	1.1910	1.1723
1	1.2354	1.2265	1.2110	1.1921	1.1728	1.1574	1.1452	1.1319	1.1178	1.1021	1.0842
2	1.1632	1.1549	1.1403	1.1225	1.1043	1.0898	1.0783	1.0658	1.0525	1.0377	1.0209
3	1.1319	1.1238	1.1096	1.0922	1.0745	1.0605	1.0493	1.0371	1.0242	1.0098	.9933
4	1.1135	1.1055	1.0916	1.0745	1.0571	1.0432	1.0322	1.0202	1.0075	.9934	.9772
5	1.1011	1.0932	1.0794	1.0625	1.0453	1.0316	1.0207	1.0089	.9963	.9823	.9663
6	1.0920	1.0842	1.0705	1.0538	1.0367	1.0231	1.0123	1.0006	.9881	.9742	.9584
7	1.0850	1.0773	1.0637	1.0470	1.0300	1.0166	1.0058	.9942	.9818	.9680	.9522
8	1.0794	1.0717	1.0582	1.0416	1.0247	1.0113	1.0006	.9890	.9767	.9630	.9473
9	1.0748	1.0671	1.0536	1.0371	1.0203	1.0070	.9963	.9848	.9725	.9588	.9432
10	1.0709	1.0632	1.0498	1.0333	1.0166	1.0033	.9927	.9812	.9690	.9554	.9398
11	1.0675	1.0599	1.0465	1.0301	1.0134	1.0002	.9896	.9781	.9569	.9524	.9369
12	1.0646	1.0570	1.0436	1.0273	1.0107	.9974	.9869	.9754	.9633	.9498	.9343
13	1.0620	1.0545	1.0411	1.0248	1.0082	.9950	.9845	.9731	.9610	.9475	.9321
14	1.0598	1.0522	1.0389	1.0226	1.0061	.9929	.9824	.9710	.9589	.9454	.9301
15	1.0577	1.0502	1.0369	1.0206	1.0041	.9910	.9805	.9691	.9571	.9436	.9283
16	1.0559	1.0483	1.0351	1.0189	1.0024	.9892	.9788	.9674	.9554	.9420	.9266

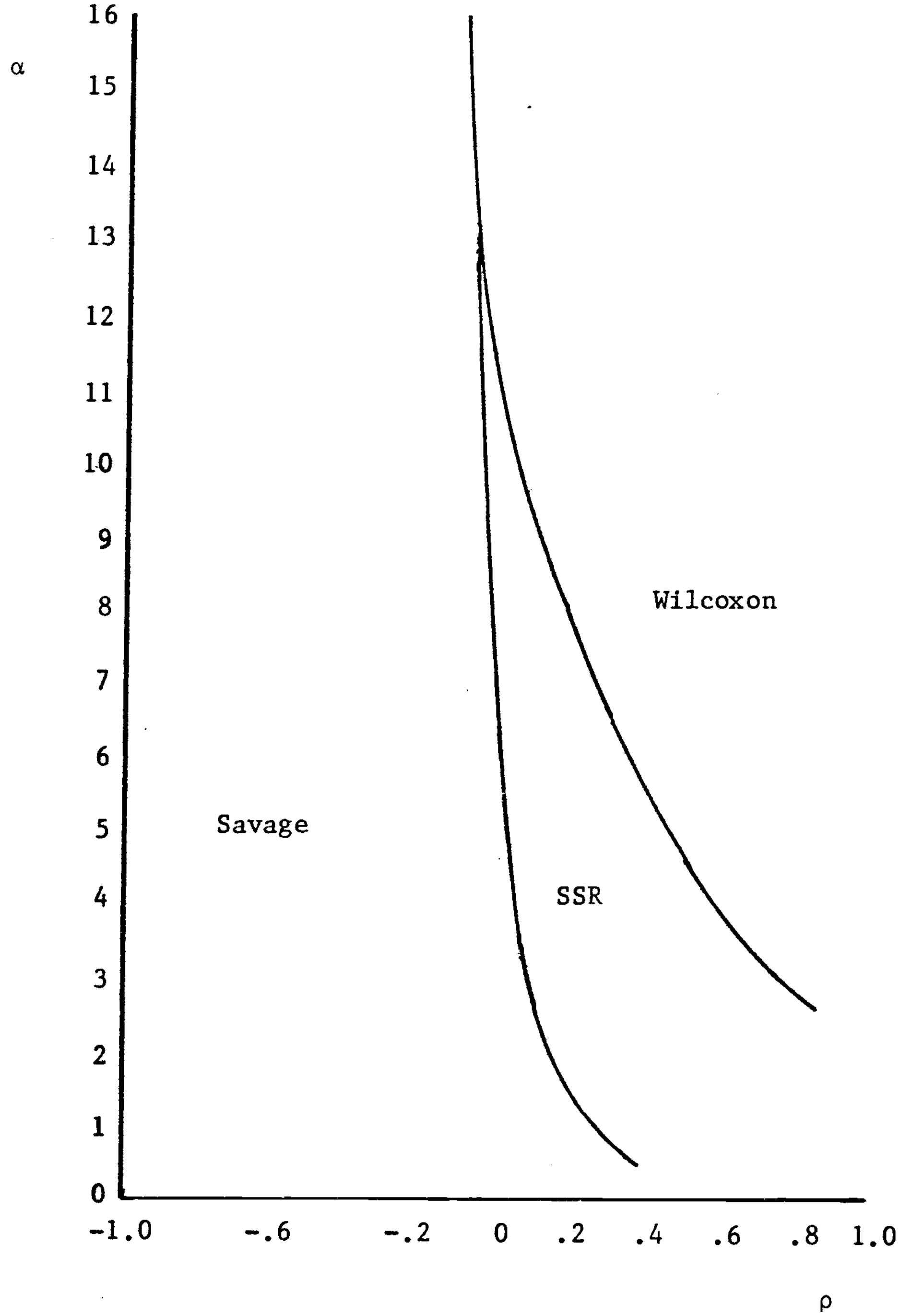

Figure 7. Regions for the most efficient tests for the Moran BV distribution.

distributions.

For the Wilcoxon test statistic (W_N)

$$C(F,J) = \int x f^2(x)dx$$

$$= \int x\left[\frac{1}{x\sqrt{2\pi}} e^{-1/2(\ln x)^2}\right]^2 dx$$

$$= \frac{1}{2\pi}\int \frac{1}{x} e^{-(\ln x)^2} dx \qquad ,$$

which, by the transformation y = lnx, reduces to

$$C(F,J) = \frac{1}{2\pi}\int e^{-y^2} dy = \frac{1}{2\sqrt{\pi}} \quad . \tag{4.48}$$

Using equations (4.14), (4.35) and (4.48) in equation (4.10), we have

$$\lim_{N\to\infty} e(W_N) = \frac{1}{4\pi\ [\ 1/6 - 1/\pi \sin^{-1}(\frac{\rho}{2})\ ]} \quad . \tag{4.49}$$

For sum of squared ranks test (SSR_N)

$$C(F,J) = \int 2xF(x)f^2(x)dx$$

$$= \int 2x\Phi(\ln x)\ \frac{1}{x^2\cdot 2\pi} e^{-(\ln x)^2} dx$$

which by the transformation y = lnx reduces to

$$= \int 2\ \ \Phi(y)\ \frac{1}{2\pi} e^{-y^2} dy$$

$$= \frac{1}{2\sqrt{\pi}} \tag{4.50}$$

Using equations (4.18), (4.41) and (4.50) in equation (4.10), we have

$$\lim_{N \to \infty} e(SSR_N) = \frac{1}{8\pi(1/5 - L(\rho))} \tag{4.51}$$

Similarly, for Savage test statistic (S_N)

$$C(F,J) = \int x \frac{1}{1-F(x)} f^2(x)dx$$

$$= \int \frac{1}{x \cdot 2\pi} e^{-(\ln x)^2} \frac{1}{1-\Phi(\ln x)} dx \tag{4.52}$$

$$= \frac{1}{2\pi} \int \frac{1}{1-\Phi(y)} e^{-y^2} dy$$

equation (4.52) is approximated by 20 point quadrature numerical integration, $\int k(x)e^{-x^2}dx \doteq \sum_{i=1}^{20} k(x_i)A_i$. Tables of x_i, A_i given in [9,p.343-346] are used.

Using equations (4.26), (4.44) and (4.52) in equation (4.10), we obtain

$$\lim_{N \to \infty} e(S_N) = \frac{\left[\int \frac{1}{1-\Phi(y)} e^{-y^2} dy \right]^2}{8\pi^2 \left[2 - \iint \ln(1-\Phi(w))\ln(1-\Phi(z)) \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} e^{-\frac{1}{2(1-\rho^2)}(w^2 - 2\rho wz + z^2)} dwdx \right]} \tag{4.53}$$

The ARE's for the Wilcoxon, Savage, and SSR tests are tabulated in Table VIII for the BV lognormal distributions. The entry of the ARE of SSR_N relative to W_N for $\rho = 0$ agrees with that of [4,p.343] to three decimal places.

TABLE VIII. ASSYMPTOTIC EFFICIENCY COMPARISONS OF H_o:$H(x,y)=H(y,x)$ AGAINST $G(y)=F(y/\theta)$, $\theta \neq 1$, FOR THE BV LOGNORMAL DISTRIBUTION

ρ	-.8452	-.6840	-.5176	-.3472	-.1744	0.	.1744	.3472	.5176	.6840	.8452
W_N relative to S_N	.6844	.7990	.8968	.9867	1.0750	1.1705	1.2895	1.4637	1.7695	2.4473	4.7400
SSR_N relative to S_N	.6846	.7936	.8799	.9525	1.0217	1.0973	1.1956	1.3419	1.6028	2.1851	4.1664
SSR_N relative to W_N	1.0005	.9935	.9812	.9653	.9504	.9374	.9272	.9168	.9158	.8932	.8790

Note: W_N, SSR_N and S_N represent statistics of the Wilcoxon test, sum of squared ranks test and the Savage test, respectively.

V. COMPARISON OF TWO-SAMPLE NONPARAMETRIC TESTS FOR RATIOS OF BIVARIATE OBSERVATIONS

Given two independent BV samples $(X_{11},Y_{11}),\ldots,(X_{1m},Y_{1m})$ and $(X_{21},Y_{21}),\ldots,(X_{2n},Y_{2n})$, let

$$Z_{1j} = Y_{1j} / X_{1j} \ , \ j=1,\ldots,m \text{ and } Z_{2j} = Y_{2j}/X_{2j}, \ j=1,\ldots,n \ , \quad (5.1)$$

and

$$N = m + n \ . \quad (5.2)$$

Let $F_1(z)$ and $F_2(z)$ represent, respectively, c.d.f.'s for the Z_1 and Z_2 variables. We are interested in nonparametric tests for H_o: $F_1(z)=F_2(z)$ against H_1: $F_1(z)\neq F_2(z)$. This is a standard two sample problem with respect to the Z_1 and Z_2 variables.

The power functions of nonparametric tests of H_o: $F_1(z)=F_2(z)$ against H_1:$F_1(z)\neq F_2(z)$ will depend on the forms of the c.d.f.'s $F_1(z)$ and $F_2(z)$. These c.d.f.'s depend on the joint c.d.f. $H(x,y)$ of the X and Y variables. ARE comparisons among the Wilcoxon, normal score, and symmetric squared rank tests are made below in section 5.2 for Morgenstern BV gamma scale alternatives for X and Y

$$G(x) = F(x/\lambda), \ \lambda\neq 1 \ ,$$

where F and G have common shape parameter α .

5.1 Rain-making experiments

In rain-making experiments, it is commonly assumed that the seeding effect is multiplicative [23] with the precipitation following gamma distributions [4,19]. Let X and Y denote nonseeded and seeded precipitations respectively. $G(y)=F(y/\theta)$ where $\theta \neq 1$ indicates a seeding effect. Two designs for rain-making experiments will now be discussed.

5.1.1. Design I

This design, introduced by Neymen and Scott [22], includes one target area and s control areas. The cloud seeding is conducted over the target area. The control areas are chosen such that their precipitation is highly correlated with that of target area. Moreover, the control areas are somewhat isolated from the target area so that their precipitation is not affected by the cloud seeding. When a storm is approaching, if it is seedable as determined by a meteorologist, a random scheme is used to determine whether or not to seed the target area. For instance, a fair coin may be tossed and if it comes up heads, the target area will be seeded. Precipitation measurements are then taken over both the target and the control areas. Comparison between the precipitation measurements of seeded and nonseeded "seedable" storms is made in order to judge the effect of seeding.

Neyman and Scott formulate the problem of testing for a seeding effect in the following way: Let Y and $\underline{X} = (X_1,\ldots,X_s)$ be the precipitations in the target and s control areas. The distribution of Y depends on the value of $\underline{X}$ and some parameters $\vartheta(\theta)$ where $\theta\neq 1$ denotes the seeding effect. The conditional p.d.f. of Y given $\underline{X}=\underline{x}$ can be written as $g(y|\underline{x},\vartheta(\theta))$. The p.d.f. of $\underline{X}$, $f(\underline{x})$, is rather arbitrary. If the nonseeded conditional expected precipitation in the target is $\eta(\underline{x})$, then the seeded conditional expectation is assumed to be $\eta(\underline{x})\theta$. $\theta=1$ implies no seeding effect. Since the distributions of Y and $\underline{X}$ are found to be far from normal, Neyman and Scott use a square root transformation on Y, i.e. $T=\sqrt{Y}$, and assume the conditional p.d.f. of T for given $\underline{X}=\underline{x}$ is normal with constant variance,

$$p(t\,|\,\underline{x},\vartheta(\theta))=\frac{1}{\sigma(\theta)\sqrt{2\pi}}\exp\left[-(t-a_o(\theta)-\sum_{i=1}^{s}a_i(\theta)x_i)^2/2\sigma(\theta)^2\right]$$

where $\vartheta(\theta)=(\sigma(\theta),\ a_o(\theta),\ i=0,1,2,\ldots,s)$.

Both $p(t|\underline{x}\ \vartheta(\theta))$ and $f(\underline{x})$ should satisfy Cramér conditions [21]. For testing the null hypothesis $\theta=1$, Neyman and Scott construct an optimal $C(\alpha)$ test.

We formulate the problem of testing for a seeding effect as follows: Let the random variables U_A and U_B denote unseeded rainfalls in the control area A and target area B respectively. To simplify the problem, only one control area is used . Let $W=U_B\ /\ U_A$. Let there be m occasions when area B is seeded and n occasions when area B is not seeded. Assuming a multiplicative effect for seeding, let $(X_{1i}\ ,\ Y_{1i})=(U_{Ai},\theta U_{Bi})$, $i=1,\ldots,m$, represent observed rainfalls for the m occasions where area B is seeded, and $(X_{2j},\ Y_{2j})=(U_{Aj},\ U_{Bj})$, $j=1,\ldots,n$ for the n occasions when area B is not seeded. Then ratios of the X and Y variables may be expressed in terms of the W variables

$$Z_{1i}=Y_{1i}\ /\ X_{1i}=\theta U_{Bi}\ /\ U_{Ai}=\theta W_i,\ i=1,\ldots,m \tag{5.3}$$

$$Z_{2j}=Y_{2j}\ /\ X_{2j}=U_{Bj}\ /\ U_{Aj}=W_j,\ j=1,\ldots,n\ . \tag{5.4}$$

The c.d.f.'s $F_1(z)$ and $F_2(z)$ for the Z_1 and Z_2 variables, respectively, are then scale alternatives,

$$F_1(z) = F_2(z/\theta) \quad . \tag{5.5}$$

5.1.2. Design II.

This is so-called cross-over design [20]. There are two areas which are so close to each other that there is a high correlation between their unseeded precipitations, yet the areas are far enough apart that seeding in one area does not influence the precipitation in the other area. When a storm approaches, we randomly (e.g. flip a fair coin) select one of the two areas as the target area and the other as the control. Moran [20] has shown that this design is better than the previous one provided that contamination between the two areas is of little consequence. We will also show this is true in section 5.2. Moran's statistical theory is based on the assumption that the logarithmic transformation of precipitation follows a normal distribution. Mielke [17] introduced a symmetric squared rank test to test the seeding effect for this design. He prefers the symmetric squared rank test over the Wilcoxon test because the symmetric squared rank test gives greater weight to large precipitation ratios Y/X. We now formulate Design II similar to the formulation of Design I.

Let U_A and U_B denote the unseeded rainfalls in area A and area B respectively, and $W = U_B / U_A$. Let there be m occasions when area A is seeded and area B is not seeded. $N = n + m$. $\lim_{N \to \infty} \frac{m}{N} = p$, $0 < p < 1$. Again, a multiplicative effect for seeding is assumed. Let $(X_{1i}, Y_{1i}) = (U_{Ai}, \theta U_{Bi})$, $i=1,\ldots,m$, represent observed rainfalls for the m occasions when area B is seeded, and $(X_{2j}, Y_{2j}) = (\theta U_{Aj}, U_{Bj})$, $j=1,\ldots,n$, for the n occasions when area A is seeded. The ratios of the X and Y variables may be expressed in terms of the W variables

$$Z_{1i} = Y_{1i}/X_{1i} = \theta U_{Bi}/U_{Ai} = \theta W_i, \quad i=1,\ldots,m \quad , \tag{5.6}$$

$$Z_{2j} = Y_{2j}/X_{2j} = U_{Bj}/\theta U_{Aj} = \frac{1}{\theta} W_j, \quad j=1,\ldots,n \quad . \tag{5.7}$$

The c.d.f.'s $F_1(z)$ and $F_2(z)$ for the Z_1 and Z_2 variables, respectively, then are scale alternatives,

$$F_1(z) = F_2(z/\theta^2) \quad . \tag{5.8}$$

5.2. ARE's for Morgenstern BV gamma alternatives

For Morgenstern alternatives, we compute the efficiency for three nonparametric tests for H_o: $F_1(z) = F_2(z)$ against H_1: $F_1(z) = F_2(z/\lambda)$. Notice that this formulation holds for both designs, since $\lambda = \theta$ for Design I and $\lambda = \theta^2$ for Design II.

Recall that the efficacy of the rank statistic $T_N = \frac{1}{m} \sum_{i=1}^{N} J(\frac{i}{N+1}) C_{N,i}$ where $C_{N,i} = \begin{cases} 1 \text{ if it is } Z_1\text{-observation} \\ 0 \text{ of it is } Z_2\text{-observation} \end{cases}$.

with weight function $J(U)$ is

$$\lim_{N \to \infty} e(T_N) = \frac{\left[\frac{\partial}{\partial \lambda} E_\lambda(T_N) \Big| \lambda=1 \right]^2}{1/2 \left[\int_o^1 J^2(U)dU - [\int_o^1 J(U)dU]^2 \right]} \tag{5.9}$$

Notice that equation (5.9) is a special case of equation (4.10).

Evaluation of equation (5.9) requires the p.d.f. of Z where $Z=Y/X$. In the rain-making experiments, $Z=\lambda W$ for $\lambda=1, \theta$ and θ^2. Therefore it is sufficient to derive the p.d.f. $f(w)$ and c.d.f. $F(w)$ for W. Using the Morgenstern p.d.f. equation (2.1) possessing common gamma marginal distributions with integer shape parameter α, we find

$$f(w) = (1+\gamma)\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\frac{w^{\alpha-1}}{(1+w)^{2\alpha}} + \gamma\left[4\sum_{i=0}^{\alpha-1}\sum_{j=0}^{\alpha-1}\frac{\Gamma(2\alpha+i+j)}{[\Gamma(\alpha)]^2 i!\,j!}\frac{w^{\alpha+i-1}}{[2(1+w)]^{2\alpha+i+j}}\right.$$

$$\left. - 2\sum_{i=0}^{\alpha-1}\frac{\Gamma(2\alpha+i)}{[\Gamma(\alpha)]^2 i!}\frac{w^{\alpha-1}}{(2+w)^{2\alpha+i}} + \frac{w^{\alpha+i-1}}{(2w+1)^{2\alpha+1}}\right] \qquad (5.10)$$

and

$$F(w) = (1+\gamma)\left[1 - \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\alpha-1}\frac{\Gamma(2\alpha-k-1)}{\Gamma(\alpha-k)}\frac{w^{\alpha-k-1}}{(1+w)^{2\alpha-k-1}}\right]$$

$$+\gamma\left[4\sum_{i=0}^{\alpha-1}\sum_{j=0}^{\alpha-1}\frac{\Gamma(\alpha+i)\Gamma(\alpha+j)}{[\Gamma(\alpha)]^2 i!\,j!\,2^{2\alpha+i+j}} - \sum_{k=0}^{\alpha+i+1}\frac{\Gamma(\alpha+i)\Gamma(2\alpha+i+k+1)}{\Gamma(i+\alpha-k)}\frac{w^{\alpha+i-1-k}}{(1+w)^{2\alpha+i+j-k-1}}\right.$$

$$(5.11)$$

$$- 2\sum_{j=0}^{\alpha-1}\frac{1}{[\Gamma(\alpha)]^2\cdot j!}\left[\frac{\Gamma(\alpha+j)}{2^{\alpha+j}} - \sum_{k=0}^{\alpha-1}\frac{\Gamma(2\alpha+j-k-1)}{\Gamma(\alpha-k)}\frac{w^{\alpha-1-k}}{(2+w)^{2\alpha+j-k-1}}\right]$$

$$\left. -2\sum_{i=0}^{\alpha-1}\frac{1}{\Gamma(\alpha)\cdot i!}\left[\frac{\Gamma(\alpha+i)\Gamma(\alpha)}{2^{\alpha+i}} - \sum_{k=0}^{\alpha+i-1}\frac{\Gamma(\alpha+i)\Gamma(2\alpha+i-k-1)}{\Gamma(\alpha+i-k)\cdot 2^{k+1}}\cdot\frac{w^{\alpha+i-1-k}}{(2w+1)^{2\alpha+i-k-1}}\right]\right].$$

for $|\gamma| \leq 1$, $0 < w < \infty$, $\alpha = 1,2,\ldots$.

From equations (5.1) and (5.2) for Design I, the asymptotic mean of T_N is given by

$$E_\theta(T_N) = \int J[\frac{m}{N} F(w/\theta) + (1- \frac{m}{N}) F(w)]dF(w) \quad ,$$

with

$$\left.\frac{\partial E_\theta(T_N)}{\partial \theta}\right|_{\theta=1} \doteq - \frac{m}{N}\int w\ J'(F(w))f^2(w)dw \quad . \tag{5.12}$$

Assuming m/N -> p in probability, 0 < p < 1, we have

$$\lim_{N\to\infty} \left.\frac{\partial}{\partial\theta} E_\theta(T_N)\right|_{\theta=1} = -p\int w\ J'(F(w))f^2(w)dw \quad . \tag{5.13}$$

Similarly for Design II, we have

$$E_\theta(T_N) \doteq \int J[\frac{m}{N} F(w/\theta) + (1- \frac{m}{N}) F(\theta w)]\theta f(\theta w)dw \tag{5.14}$$

with

$$\left.\frac{\partial E_\theta(T_N)}{\partial\theta}\right|_{\theta=1}$$

$$\doteq \int \left[w\ J'(F(w))[- \frac{m}{N} f(w)+(1- \frac{m}{N})f(w)]f(w)+J(F(w))[f'(w)+f(w)]\right]dw$$

$$= (1-2 \frac{m}{N})\int w\ J'(F(w))f^2(w)dw-\int w\ J'(F(w))f^2(w)dw \tag{5.15}$$

$$= -2 \frac{m}{N}\int wJ'(F(w))f^2(w)dw \quad ,$$

and

$$\lim_{N \to \infty} \frac{\partial E_\theta(T_N)}{\partial \theta} \bigg|_{\theta=1} = -2p \int wJ'(F(w)) f^2(w)dw \quad . \tag{5.16}$$

Using equation (5.13) in equation (5.9), we have the efficacy of any rank test (T_N) for Design I as

$$\lim_{N \to \infty} e_1\ (T_N) = \frac{p^2\ [\int J'(F(w)) f^2(w)dw]^2}{1/2 \left\{ \int J^2(U)dU - [\int J(U)dU]^2 \right\}} \quad . \tag{5.17}$$

Similarly, using equation (5.16) in equation (5.9), we get the efficacy of any rank test (T_N) for Design II as

$$\lim_{N \to \infty} e_2\ (T_N) = \frac{4p^2 [\int J'(F(w)) f^2(w)dw]^2}{1/2 \left\{ \int J^2(U)dU - [\int J(U)dU]^2 \right\}} \quad . \tag{5.18}$$

Since the ARE of test T_1 with respect to test T_2 is defined as the ratio of their efficacies, i.e.

$$e\ (T_1, T_2) = \frac{\lim_{N \to \infty} e\ (T_1)}{\lim_{N \to \infty} e\ (T_2)} \quad , \tag{5.19}$$

from equations (5.17), (5.18) and (5.19), it is easily seen that the ARE of T_1 with respect to T_2 for Design I is identical to that of Design II.

Also, for any particular rank test (T_N), Design II is more powerful than Design I, since equation (5.18) is greater than equation (5.17) for a given test.

The weighting functions for the Van de Waerden (asymptotically equivalent to normal score) and symmetric squared rank tests are

$$J(U) = \Phi^{-1}(U) \qquad 0 < U < 1 \quad , \tag{5.20}$$

and

$$J(U) = \begin{cases} -(U-\tfrac{1}{2})^2 & \text{for } 0 < U < 1/2 \\ (U-\tfrac{1}{2})^2 & \text{for } 1/2 \leq U \leq 1 \quad . \end{cases} \tag{5.21}$$

Notice that the J function in equation (5.21) is symmetric about U = 1/2. The weighting function for the Wilcoxon test is given in equation (4.2).

Using Simpson's rule, we compute the ARE's for $\alpha = 1,2,3$, which are presented in Tables IX, X and XI. The entries for $\gamma = 0$ agree to at least two decimal places with those obtained by direct integration. Among the three tests, the Wilcoxon test is the best for small α and/or positive correlation. The symmetric squared rank test is best for $\gamma = -1$ and the combination of $\gamma = -.8$ and $\alpha = 2,3$. The normal score test is the best for large α. Figure 8 indicates the values of α and γ for which each of the three tests is most efficient.

For the case $\gamma = 0$ and $\alpha = 1$, ln(w) has a logistic distribution. In this case the Wilcoxon test is known to be asymptotically efficient. Similarly for $\gamma = 0$ and $\alpha \to \infty$, $\ln(w)/\sqrt{\alpha}$ is asymptotically normally distributed. Consequently, the normal score test would be nearly asymptotically efficient for $\gamma = 0$ and large α.

TABLE IX. ASYMPTOTIC RELATIVE EFFICIENCIES OF THE SYMMETRIC SQUARED RANK TEST WITH RESPECT TO WILCOXON TEST FOR EQUALITY OF RATIOS FROM MORGENSTERN BV GAMMA DISTRIBUTIONS

α \ γ	-1.0	-0.8	-0.6	-0.4	-0.2	-0.0	0.2	0.4	0.6	0.8	1.0
1	1.019	.997	.978	.961	.947	.936	.929	.924	.922	.923	.927
2	1.065	1.041	1.020	1.002	.988	.978	.969	.964	.963	.965	.980
3	1.082	1.057	1.036	1.018	1.003	.992	.984	.980	.978	.980	.986

TABLE X. ASYMPTOTIC RELATIVE EFFICIENCIES OF THE SYMMETRIC SQUARED RANK TEST WITH RESPECT TO NORMAL SCORE RANK TEST FOR EQUALITY OF RATIOS FROM MORGENSTERN BV GAMMA DISTRIBUTIONS

α \ γ	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
1	1.057	1.043	1.031	1.020	1.012	1.006	1.002	.999	.998	.997	.998
2	1.013	1.004	.996	.991	.987	.985	.984	.985	.987	.989	.990
3	1.009	1.001	.994	.989	.986	.984	.984	.984	.986	.988	.989

TABLE XI. ASYMPTOTIC RELATIVE EFFICIENCIES OF THE WILCOXON TEST WITH RESPECT TO NORMAL SCORE RANK TEST FOR EQUALITY OF RATIOS FOR MORGENSTERN BV GAMMA DISTRIBUTIONS

α \ γ	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
1	1.038	1.046	1.054	1.062	1.069	1.074	1.079	1.081	1.082	1.081	1.077
2	.951	.964	.977	.988	.999	1.008	1.016	1.021	1.024	1.025	1.022
3	.933	.947	.960	.972	.983	.992	1.000	1.005	1.008	1.007	1.003

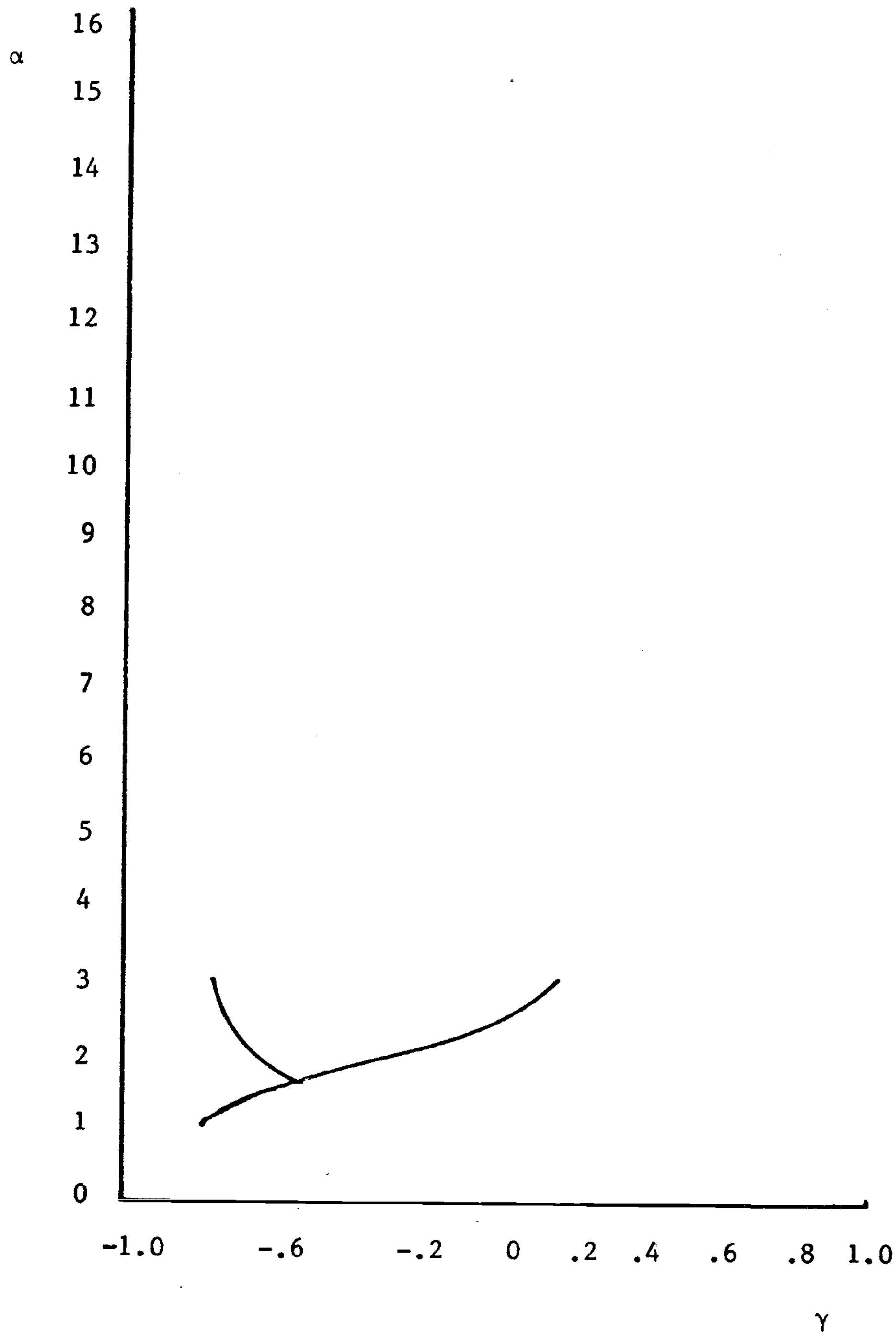

Figure 8. Regions of the most efficient tests for the equality of ratios of Morgenstern gamma BV observations for the common shape parameter α - 1, 2 and 3.

VI. SUMMARY AND CONCLUSIONS

In the previous chapters, three BV statistical inference problems have been discussed. We have shown that optimal $C(\alpha)$ tests and l.m.p.r.t.'s are asymptotically equivalent for the tests of independence when any of the three BV distributions, Morgenstern, Plackett and Moran, are used as alternatives. The l.m.p.r.t.'s are usually easier to compute than $C(\alpha)$ tests. As to the tests for interchangeability of gamma bivariates from either BV Morgenstern or Moran distributions, comparisons of three nonparametric tests show that the Wilcoxon test is the best for positive correlations and large shape parameters α , the Savage test should be used for negative correlations and sum of squared rank test is good for some combinations of shape parameter and correlations. Finally, for the tests of equivalence of ratios from two BV Morgenstern gamma samples, three nonparametric tests, Wilcoxon, normal score and symmetric squared rank test, are considered for two designs of rain-making experiments. Among them, the Wilcoxon test is the best for small shape parameters α and/or large correlation; normal score test should be used for large α; and symmetric squared rank test is good for correlation index $\gamma=-1$ and combinations of $\gamma=-.8$ and $\alpha=2$ and 3. Since the distributions of rainfall data of two close areas have small shape parameters and their rainfall is highly correlated, the Wilcoxon test seems to be the best of the three tests when BV Morgenstern gamma distribution is used as the underlying distribution.

In this paper, three types of BV distributions and five nonparametric tests are used. Other types of BV distributions and other nonparametric tests could be considered. The techniques and results of this paper could be generalized to handle multivariate distributions so as to broaden the range of applications. An empirical test of the results of this paper could be performed to see whether the models fit real-world data.

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