AN ABSTRACT OF THE THESIS OF

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Title: LEAST SQUARES ADJUSTMENT COMPUTER PROGRAMS FOR HORIZONTAL AND VERTICAL POSITIONS

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The application of the least squares adjustment techniques to the area of surveying has been considered and two computer programs called HCONTRL and VCONTRL are developed for horizontal and vertical control respectively. The principle of least squares is discussed in chapter II. The observation and condition equations are compared and the observation equations technique is applied in the programs.

The general forms of the observation equations for distance and angle (or direction) are derived in chapter III. The solution and the precision of the solution from the least squares adjustment technique are also discussed. A brief discussions on future systems; the satellite positioning, the inertial positioning, and the non-classical method, are included in chapter III.

In chapter IV, the development and the procedures to be used in using the computer program HCONTRL are discussed. The least squares application to the various techniques of the horizontal position control; traverse, intersection, resection, triangulation, trilateration, and combined networks, are described in chapter V.

The theory of vertical position control is discussed in chapter VI where techniques of direct leveling and trig leveling are described. This chapter also includes a brief description on; gravimetric leveling, barometric leveling, hydrostatic leveling, tacheometric leveling, satellite altimetry, and steric leveling. The development and procedures of the computer program VCONTRL, to adjust the direct leveling and the trig leveling by the least squares technique are discussed in chapter VII. The application of the least squares to the direct leveling and the trig leveling is given in chapter VIII. In chapter IX, practicality of the least squares adjustment is considered.

The summary and conclusions are in chapter X. Appendix I consists of the numerical examples of the least squares adjustment of the various techniques in the horizontal position control by the computer program HCONTRL. The numerical examples of direct leveling and trig leveling by the computer program VCONTRL are given in Appendix II. A numerical example of the gravimetric leveling computation to find the geopotential numbers is also presented in Appendix II. Appendix III contains the complete listings of the computer programs HCONTRL and VCONTRL.

Least Squares Adjustment Computer Programs For Horizontal and Vertical Positions

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Ramesh Lal Shrestha

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Redacted for Privacy

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LEAST SQUARES ADJUSTMENT COMPUTER PROGRAMS FOR HORIZONTAL AND VERTICAL POSITIONS

I. INTRODUCTION

The purpose of an adjustment is to have a series of observed quantities consistent within themselves and with geometrical or other data conditions. Once consistency has been achieved, computations involving the adjusted values will give unique results. Since observed values are the best evidence for determining the true value of the desired quantities, the best adjustment is that which can achieve consistency with as little disturbance to the observation as possible.

A good adjustment technique should give due consideration and order to all relevant factors and permit the simultaneous interaction of these factors in the derivation of the adjusted values. The least squares adjustment technique satisfies all of these criterions and requirements, because it produces a best set of consistent values - the most probable values (MPV) or the best estimates - by simultaneous consideration of all factors, while at the same time it causes the least affect on the observations themselves.

In some adjustment techniques, factors affecting the observations are taken arbitrarily and the adjustment is carried out step by step with the previous results affecting successive results. For example, there are a number of techniques for adjusting a traverse. Some are: arbitrary, transit, compass (or Bowditch) and Crandal techniques. Whatever technique is applied, the final positions depend on the adjusted angles (or directions). They are usually corrected first by arbitrarily applying a constant correction to each angle. Any error not corrected for at this stage, is carried forward when the distance is adjusted by making sum of the latitudes and the departures equal to zero.

In this paper, the principle of the least squares has been described. Amongst the various computational techniques of the least squares adjustment cited in chapter II, the observation equations technique has been presented in detail, and applied, to obtain the most probable values of the horizontal and the vertical position controls in the classical survey adjustment.

The theory of the horizontal position control has been described in chapter III where the distance and the angle have been considered as the observations. The classical techniques of traverse, intersection, resection, triangulation, trilateration, and combined networks in the horizontal position control have been discussed in this chapter. A brief description on the precision from the least squares adjustment has also been considered. The future systems in this chapter gives a short description of the satellite and inertial positioning which have been practiced frequently in the recent years.

The non-classical method is just meant to show that the three dimensional representation and the adjustment of the positions, might be the ultimate goal in surveying. The theory of the vertical position control has been discussed separately from the horizontal control in chapter VI, where the techniques of leveling, trig leveling, and gravimetric leveling have been discussed. A very brief description on barometric leveling, hydrostatic leveling, stadia leveling, satellite altimetry, and steric leveling are also given in chapter VI.

For each technique of the horizontal position control, a numerical example has been chosen and adjusted by the program HCONTRL and the respective numerical adjustment is given in Appendix I. The numerical examples of the leveling and the trig leveling utilizing VCONTRL are given in the Appendix II where a computational procedure to obtain the geopotential numbers in gravimetric leveling are also given. The complete listings of the HCONTRL and VCONTRL are provided in the Appendix III.

II. THE LEAST SQUARES PRINCIPLE

The principle of least squares dates back to 1806 when Legendre proposed a method to obtain the most probable value (MPV) of a quantity. He postulated that, given a set of equally reliable measured values of a quantity, the most probable value, or the best estimate, of a quantity would be the one which makes the sum of the squares of the residuals a minimum. A residual is defined as

$$v = \hat{\ell} - \ell \tag{2.01}$$

where ℓ is the observed (measured) value and $\hat{\ell}$ is the MPV of the quantity observed.

When there are more observations than necessary to uniquely determine the MPV's of the desired quantities, it is said that there exists redundancy in the observations. In such circumstances, the observations no longer give a unique solution of the MPV's, but they give the inconsistent infinite solutions for the MPV's. On the other hand, if the principle of least squares is applied to the redundant observations, one unique and consistent set of best estimates or MPV's will be obtained. This principle provides the best estimate $\hat{\ell}$, assuming that no systematic errors are involved in the observations. Consequently, any variation of the observed quantity from the MPV produced by the redundant observations should be small. True value of the observed quantities for the large sets of normally observed data should be the MPV's. If n is the number observations and n_0 is the minimum number of variables required to determine estimates uniquely, then the redundancy r or degrees of freedom as expressed in statistics is

$$r = n - n_o \tag{2.02}$$

If a set of residuals is given by v_1, v_2,..., etc., where a residual is defined by the equation (2.01), the residuals can be represented in the matrix form by

$$\underline{v} = \begin{vmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{vmatrix} \tag{2.03}$$

where a slash below the letter v symbolizes the matrix, and this notation to represent a matrix will be used in this paper. A transpose of the matrix $\underline{v}$ is defined as $\underline{v}^t$, such that

$$\underline{v}^t = \begin{vmatrix} v_1 & v_2 \ldots\ldots v_n \end{vmatrix} \tag{2.04}$$

Then, the sum of the squares of the residuals is

$$v_1^2 + v_2^2 + \ldots\ldots + v_n^2$$

which, in the matrix form, is given by

$$\underline{v}^t \ \underline{v}$$

For a set of equally reliable measured values, according to Legendre's postulation, the function defined by the equation (2.04), should be a minimum for the MPV of the measured quantity. This is known as the least squares principle which can be stated as; the sum of the squares of the residuals of equally reliable measured values is a minimum. If the measured values are not equally reliable, the relative reliabilities of the measurements must be considered. The different reliabilities may be due to different factors involved in the measurements. For example, one angle of a triangle might have been measured more number of times than the other, the two angles might have been measured by two different transits or theodolites, etc. In such circumstances, measurements are adjusted by weighting them with the corresponding weights. Usually, the weight is taken as a function of the standard deviation of the measurement. If the standard deviation of the i^{th} observation is σ_i, the weight w_i, of that observation is usually taken as:

$$w_i = \frac{1}{\sigma_i^2} \tag{2.05}$$

where σ_i^2 is known as the variance of the observation. Therefore, for n measurements of observations, (e.g. n angles in a traverse), there will be n residuals given by the equation (2.03), and n number of weights, given by

$$\underline{w} = \begin{vmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{vmatrix} \tag{2.06}$$

where w_1, w_2, ..., w_n are the corresponding weights of the measurements. Here, it has been assumed that the one measurement does not affect the other. It is then said that the measurements are independent from each other, and there is no correlation between any two measurements. In practice, this principle of no correlation is used, but in theory, it may not be true. The column weight matrix $\underline{w}$ of equation (2.06) may, in theory, contain other elements. In such circumstances, the column matrix of the equation (2.06) is replaced by a square matrix $\underline{W}$. The properties of the weight matrix will be discussed later in the section Weights. It is, however, important to note that $\underline{W}$ is a square matrix of order n by n and is of the form:

$$\underline{W} = \begin{vmatrix} w_1 & & \text{other} \\ & w_2 & \text{elements} \\ & & \cdot \\ \text{other} & & \cdot \\ \text{elements} & & & w_n \end{vmatrix} \tag{2.07}$$

The principle of the least squares can then be shown [26];

$$\phi = \underline{v}^t \ \underline{W} \ \underline{v} \ \rightarrow \ \text{minimum} \tag{2.08}$$

where ϕ is a function of $\underline{v}$ and $\underline{W}$.

The criterion expresed by the equation (2.08) is the most general case. Other special cases can be derived from it by considering the special structure of the weight matrix $\underline{W}$, and they are:

(a) $\underline{W}$ being a diagonal matrix, i.e. all the off diagonal elements are zero, which implies that there is no correlation between two of the measurements. If this matrix, called the diagonal matrix, is represented by $\underline{W}^*$, then the principle of least squares becomes:

$$\phi = \underline{v}^t \, \underline{W}^* \, \underline{v} \rightarrow \text{minimum} \tag{2.09}$$

which is equivalent to

$$\phi = \sum_{i=1}^{n} (w_i \, v_i^2) \rightarrow \text{minimum} \tag{2.10}$$

where w_i is the i^{th} diagonal element of $\underline{W}^*$ and v_i is the residual associated with the corresponding i^{th} observation.

(b) $\underline{W}$ being an identity matrix, i.e. all the diagonal elements are one and all the off diagonal elements are zeroes. This implies that there is no correlation between any two of the observations and all the observations have the same and equal weight one. Then the weight matrix $\underline{W}$ is represented by $\underline{I}$. This gives the principle of the least squares as:

$$\phi = \underline{v}^t \, \underline{I} \, \underline{v} \rightarrow \text{minimum} \tag{2.11}$$

which is equivalent to

$$\phi = \sum_{i=1}^{n} (v_i^2) \rightarrow \text{minimum} \tag{2.12}$$

where v_i is the i^{th} residual.

It follows from the least squares principle that the arithmetic mean ($\hat{\ell}$) of a series of equally reliable observations is the MPV. This can be mathematically derived. For the simplest case, where $\underline{W}$ is an identity matrix, the least squares principle is given by the expression (2.12), i.e.

$$\phi = \sum_{i=1}^{n} (v_i^2) = \sum_{i=1}^{n} (\hat{\ell} - \ell_i)^2 \rightarrow \text{minimum} \qquad (2.13)$$

This equation to be minimized, therefore,

$$\frac{\partial \phi}{\partial \hat{\ell}} = 0$$

Taking the partial derivative of the equation (2.13) gives

$$2\,n\,\hat{\ell} - 2 \sum_{i=1}^{n} (\ell_i) = 0$$

Hence, $\hat{\ell} = \frac{1}{n} \sum_{i=1}^{n} (\ell_i)$, which is the arithmetic mean, noting that

$\frac{\partial^2 \phi}{\partial^2 \hat{\ell}} = 2n > 0$ which indicates the minimum.

A. Weights

The elements of the weight matrix must be known in advance to adjust the data by the least squares technique as seen from the equation (2.08). The reason for using the weight matrix $\underline{W}$ is that the measurements of different reliabilities are often made in practice, and it is necessary to find the MPV's. For example, in triangulation, some angles may have been measured more precisely than the others.

If both angles and distances are included in the observations, as in traverse, the relative reliabilities must be taken into account, if the MPV's of the observations are desired. The reliabilities of the different measurements are given by the corresponding precisions. The precision is the degree of closeness of the measurements to its mean and it is given by the standard deviation σ_ℓ. The weight of a measurement is given by the equation (2.05). On page 65 of [26], the weight matrix $\underline{W}$ is defined as

$$\underline{W} = \underline{Q}_{\underline{\ell\ell}}^{-1} = \underline{Q}^{-1} \tag{2.14}$$

where $\underline{Q}_{\underline{\ell\ell}}$ is called the cofactor matrix of the observations, and for convenience, it is written as $\underline{Q}$, which is given by, page 21,[26];

$$\underline{Q} = \frac{1}{\sigma_o^2} \begin{vmatrix} \sigma_{\ell_1}^2 & \sigma_{\ell_1\ell_2} & \cdot & \cdot & \cdot & \sigma_{\ell_1\ell_n} \\ \sigma_{\ell_2\ell_1} & \sigma_{\ell_2}^2 & \cdot & \cdot & \cdot & \sigma_{\ell_2\ell_n} \\ \vdots & \vdots & & & & \vdots \\ \sigma_{\ell_n\ell_1} & \sigma_{\ell_n\ell_2} & \cdot & \cdot & \cdot & \sigma_{\ell_n}^2 \end{vmatrix} \tag{2.15}$$

The off diagonal elements of this matrix are called the co-variances of and σ_o^2 is an arbitrary constant with the arbitrary dimension, and it is known as the reference variance; and $+\sqrt{\sigma_o^2}$ is known as the standard error of the unit weight. In the section 11.3 of [26], it is shown that the best estimate (MPV), $\hat{\sigma}_o^2$, of the reference variance σ_o^2, is given by

$$\hat{\sigma}_o^2 = \frac{\underline{v}^t \, \underline{W} \, \underline{v}}{r} \tag{2.16}$$

Therefore, if the variances and the covariances in the equation (2.15) are known, the weight matrix $\underline{W}$, needed for the least squares adjustment given by the equation (2.08), can be determined from the equation (2.14). These elements may be determined from the measurement data.

Let the m observational sets of data be

$$\begin{matrix} x_{11}, & x_{12}, & \cdots, & x_{1n} \\ x_{21}, & x_{22}, & \cdots, & x_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_{m1}, & x_{m2}, & \cdots, & x_{mn} \end{matrix} \tag{2.17}$$

where n is the number of measurements for each set. The set of measurements may be the measurements of an angle measured n times and the second set for a distance. The estimated values of the variance (σ_ℓ^2) and covariance ($\sigma_{m-1,m}$) of the expression (2.15) are respectively given by s_m^2 and $s_{m-1,m}$ such that

$$\bar{x}_m = \frac{1}{n} \sum_{i=1}^{n} (x_{mi}) \tag{2.18}$$

$$s_m^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_{mi} - \bar{x}_m)^2 \tag{2.19}$$

$$s_{m-1,m} = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_{m-1,i} - \bar{x}_{m-1}) (x_{mi} - \bar{x}_m) \tag{2.20}$$

For detail see chapter 3, [26] .

In classical survey, there are basically two quantities observed. They are angle (or direction) and distance. Each set of measurements, taken at different times and environmental conditions produces different variances, but observations of similar quantities, angle or distance, are assumed to have the same variances. It is usually also assumed that the observations are uncorrelated, producing zero covariances in the weight matrix $\underline{W}$. Consequently, a diagonal cofactor matrix of the expression (2.15) will be left, with the same variance (or precision) for all the angular measurements and the variances for all the distance observations. However, if different variances for different measurements are available or computable, they should be used accordingly. Obviously, the capability of measuring a quantity greatly depends on the instrument used. Instrument manufacturer provides the precision of the instrument which in general, can be used as the estimation of the variance to be utilized in the weight matrix, if there is no other way to find the variances and covariances of the cofactor matrix.

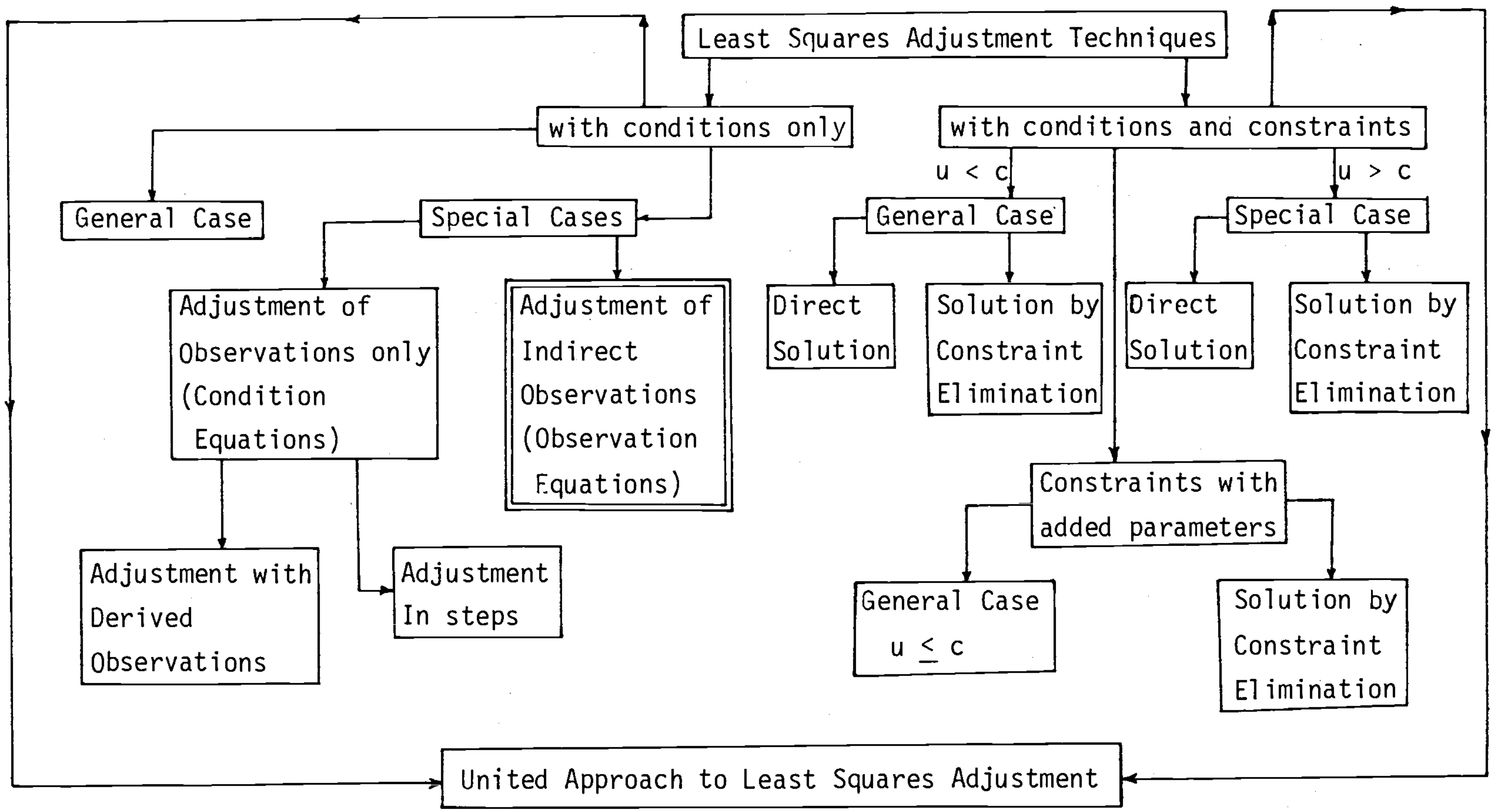
B. Problem Analysis

There are several different computational techniques in least squares adjustment which yield a unique result. It should be emphasized that whatever technique is used, the final answers are always the same. Table 1 gives the different techniques cited in [26]. It is the "Adjustment of Indirect Observation" that this paper will follow for all computations. The technique will be referred to as "observation equations technique" in the remainder of this paper. Observation equations technique means Indirect method.

Among the different techniques tabulated in Table 1, general case technique under conditions only is used if the adjustment consists of both the observations and the parameters (or unknowns). Adjustment of observations only technique is used when the adjustment includes observations only. This technique is also referred to as Direct method. Since this technique involves adjusting condition equations, it will be referred to as "condition equations" in this paper. This technique is often used for simple geometrical figures, such as; a plane triangle, a small level net, etc. Observation and condition equations techniques are simpler compared to the general case. Therefore, they are used more frequently in the adjustments. The least squares adjustment with conditions and constraints is used when the part or all of the parameters in the adjustment must be constrained corresponding to some other constraints in the data. For example, the elevations of points on a lake shore may be constrained to have the same elevation in the adjustment procedures. Details of all the techniques in Table 1 are given in [26].

In the observation equations technique, one observation formulates one equation in the adjustment process. Hence, there are as many observation equations as there are observations themselves. In this case, the parameters are the unknowns. In the case of condition equations, there will be no parameters, but only the residuals. The detail adjustment procedure by the observation equations will be demonstrated later in this paper.

Table 1. Techniques of the Least Squares Adjustment



Whatever technique is applied, the least squares adjustments are performed with linear functions. This is because the computation by the least squares technique is possible only when the function is linear. Therefore, some means of linearization, in the case of non-linear function, must be used. Taylor's series expansion is often used for this purpose, where only the zero and the first order terms are retained, and all other higher order terms are neglected. Details of linearization procedures are given in Appendix B of [26] and in [32].

In survey adjustment, a simple technique is sometimes used to linearize the non-linear function when the parameters are co-ordinates of any form, such as: rectangular, geodetic, or three dimensional. The linear form is obtained by partially differentiating the non-linear function with respect to the unknown parameters. This technique is often called the Variation of Co-ordinates technique and will be used in this paper where applicable.

C. Observation Equations vs Condition Equations

An important advantage of the observation equations technique is that there are as many observation equations as there are the observations themselves. In the condition equations technique, it is essential to correctly determine the necessary and sufficient conditions (c), for a unique solution, such that $c = r + u$, where u is the number of parameters (unknowns), and r is the redundancy or the degrees of freedom. It is often difficult to determine r, which is given by, $r = n - n_0$, where n_0 is the minimum number of variables required to uniquely determine the MPV's, and it is difficult to determine n_0 as it depends on the geometrical conditions of the observations. Consequently, it is sometimes difficult to determine c.

In the observation equations technique, there is not as good a check as there is in the condition equations technique, because the right hand side of a condition equation is an indication of the mis-closure by which the measurements fail to satisfy the condition. Any large number, here indicates possible mistake or mistakes in the measurements. No such check is available in the observation equations technique.

In both of the techniques, the solution of the desired quantities is obtained by solving the equations in the form of $\underline{B}^t \underline{W} \underline{B} \underline{x} = \underline{f}$, called the normal equations, where $(\underline{B}^t \underline{W} \underline{B})$ is symmetric and definite. the derivation of this equation for the observation equations technique will be given later in the paper. In the above normal equations, everything is known except $\underline{x}$, the solution of which are obtained by inverting the matrix $(\underline{B}^t \underline{W} \underline{B})$ and post-multiplying it by $\underline{f}$. Thus, the smaller the normal equations matrix, the smaller amount of storage space is required for a given technique of solution. In the observation equations technique, the dimension of the normal equations matrix will be the number of unknown parameters u, whereas the condition equations technique will have the dimension of the independent conditions, i.e., c. Usually, c is less than u.

Current computers can provide a reasonable amount of storage space for most of the project. In the past, however, because of the lack of computer capabilities, the condition equations technique was most widely used. Moreover, it is much easier and simpler to write a computer program in the form of observation equations than in the form of condition equations. Hence, computer programs utilizing observation equations have been developed for the adjustments in this paper.

III. THEORY OF THE HORIZONTAL POSITION CONTROL

The purpose of horizontal position control is to establish the horizontal positions, rectangular X and Y co-ordinates, of points surveyed. This can be done in a number of ways. Whichever technique is employed, the basic principle is that the horizontal positions are the result of measurements of directions and/or distances. The precision of the observed directions or angles and distances can be used to determine the theoretical accuracy of the horizontal positions.

In adjusting horizontal positions by the observation equation technique, one equation for each observation is written. This technique is often referred to as the variation of co-ordinates method.

Observation equations for a distance and an angle will now be developed.

A. Distance Equation

For a measured distance such as that between points i and j, Figure 1, let the most probable value (MPV) of the distance be $\hat{d}_i$ and the measured value of the distance be d_i. Then if v_i is the residual of the measurement, rewriting equation (2.01), so the observation equation becomes:

$$v_i = \hat{d}_i - d_i \tag{3.01}$$

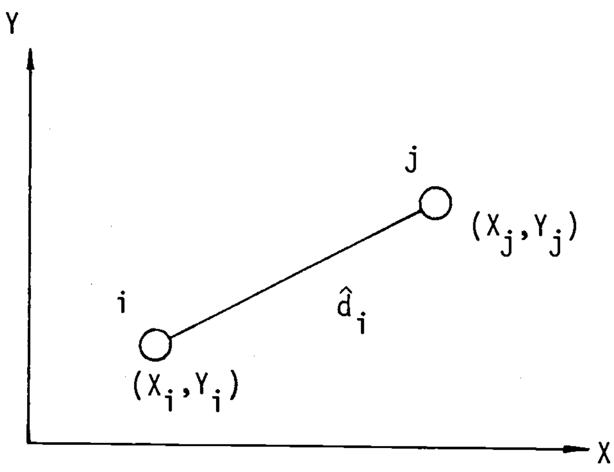


Figure 1. Sketch for Distance Observation Equation.

where $$\hat{d}_i = \{(X_j - X_i)^2 + (Y_j - Y_i)^2\}^{\frac{1}{2}} \tag{3.02}$$

and X_i, Y_i, X_j and Y_j are the MPVs of the co-ordinates.

Equation (3.02) is non-linear. In order to obtain a linear observation equation, let X_i^o, Y_i^o, X_j^o and Y_j^o be the approximate co-ordinates so that

$$\begin{aligned} X_i &= X_i^o + \delta X_i^o \;; \qquad & X_j &= X_j^o + \delta X_j^o \\ Y_i &= Y_i^o + \delta Y_i^o \;; \qquad & Y_j &= Y_j^o + \delta Y_j^o \end{aligned} \tag{3.03}$$

and $$\hat{d}_i = D_i + \delta D_i \tag{3.04}$$

where $$D_i = \{(X_j^o - X_i^o)^2 + (Y_j^o - Y_i^o)^2\}^{\frac{1}{2}} \tag{3.05}$$

Partially differentiating (3.05) gives

$$2D_i \delta D_i = 2(X_j^o - X_i^o)(\delta X_j^o - \delta X_i^o) + 2(Y_j^o - Y_i^o)(\delta Y_j^o - \delta Y_i^o)$$

or

$$\delta D_i = \frac{X_j - X_i^o}{D_i} (\delta X_j^o - \delta X_i^o) + \frac{Y_j^o - Y_i^o}{D_i} (\delta Y_j^o - \delta Y_i^o) \tag{3.06}$$

or

$$\delta D_i = \left(\frac{X_j^o - X_i^o}{D_i}\right) \delta X_j^o + \left(\frac{X_i^o - X_j^o}{D_i}\right) \delta X_i^o + \left(\frac{Y_j^o - Y_i^o}{D_i}\right) \delta Y_j^o + \left(\frac{Y_i^o - Y_j^o}{D_i}\right) \delta Y_i^o \quad (3.07)$$

Let

$$P_i = \frac{X_i^o - X_j^o}{D_i}, \qquad R_i = \frac{X_j^o - X_i^o}{D_i}$$
$$Q_i = \frac{Y_i^o - Y_j^o}{D_i} \text{ and } S_i = \frac{Y_j^o - Y_i^o}{D_i} \quad (3.08)$$

Substituting from (3.08) to (3.07) and rearranging terms, equation (3.07) becomes

$$\delta D_i = P_i \delta X_i^o + Q_i \delta Y_i^o + R_i \delta X_j^o + S_i \delta Y_j^o \quad (3.09)$$

Recalling equation (3.01),

$$v_i = \hat{d}_i - d_i$$

Now substituting $\hat{d}_i = D_i + \delta D_i$ from equation (3.04) gives

$$v_i = D_i + \delta D_i - d_i$$
$$= D_i - d_i + \delta D_i$$

Substituting for δD_i from equation (3.09) gives

$$v_i = D_i - d_i + P_i \delta X_i^o + Q_i \delta Y_i^o + R_i \delta X_j^o + S_i \delta Y_j^o \quad (3.10)$$

Substituting "O" for the observed value d_i, and "C" for the computed value D_i in equation (3.10), the general observation equation for a distance becomes

$$P_i \delta X_i^o + Q_i \delta Y_i^o + R_i \delta X_j^o + S_i \delta Y_j^o = (O - C) + v_i \quad (3.11)$$

where δX_i^o, δY_i^o, δX_j^o, and δY_j^o are the unknown quantities. The unknown quantities here, are the correction to be added to the approximate co-ordinates.

B. Direction or Angle Equation

The observation equation for a direction will be derived first. The difference between two directions will give the one observation equation for each angle. The observation equation for a direction from i to j. Figure 2a, will be

$$v_{ij} = \hat{\alpha}_{ij} - \alpha_{ij} \tag{3.12}$$

where $\hat{\alpha}_{ij}$ is the MPV of the direction, α_{ij} is the observed direction, and v_{ij} is the residual. Then it follows that

$$\tan\hat{\alpha}_{ij} = \frac{X_j - X_i}{Y_j - Y_i} \tag{3.13}$$

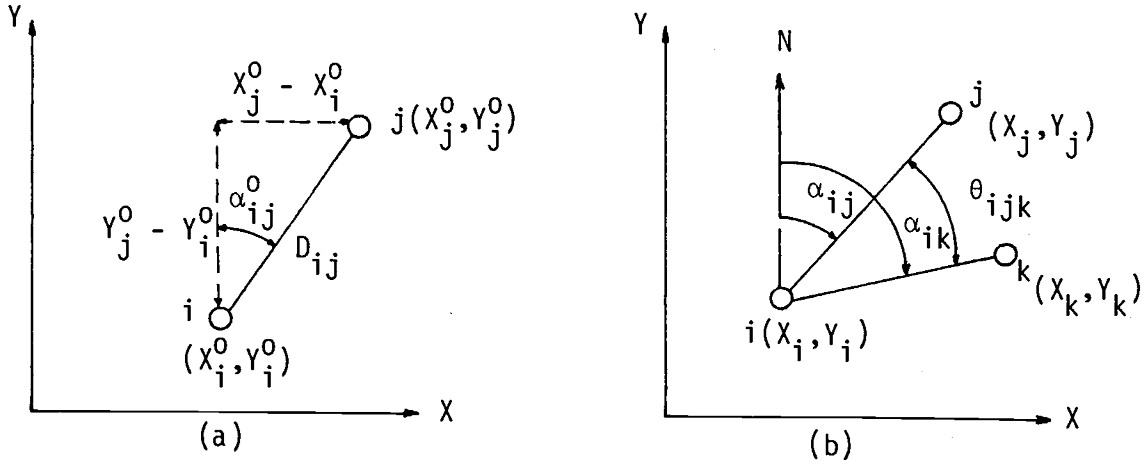


Figure 2. Sketches for Direction and Angle Observation Equations.

Note that the direction α_{ij} is the azimuth of the line from i to j. This transcendental function is not linear and can be linearized by taking the approximate co-ordinates as X_i^o, Y_i^o, X_j^o, and Y_j^o such that

$$X_i = X_i^o + \delta X_i^o, \qquad X_j = X_j^o + \delta X_j^o$$
$$Y_i = Y_i^o + \delta Y_i^o, \qquad Y_j = Y_j^o + \delta Y_j^o \tag{3.14}$$

If α_{ij}^o is the approximate direction then

$$\tan\alpha_{ij}^o = \frac{X_j^o - X_i^o}{Y_j^o - Y_i^o} \tag{3.15}$$

and

$$\hat{\alpha}_{ij} = \alpha_{ij}^o + \delta\alpha_{ij}^o \tag{3.16}$$

where δX_i^o, δY_i^o, δX_j^o, and δY_j^o are the corrections to be added to the approximate co-ordinates and $\delta\alpha_{ij}^o$ is the correction to be added to the approximate direction. X_i, Y_i, X_j, and Y_j are the MPV's of the co-ordinates.

Partial differentiation of equation (3.15) gives

$$\sec^2\alpha_{ij}^o\,\delta\alpha_{ij}^o = \frac{(Y_j^o - Y_i^o)\,(\delta X_j^o - \delta X_i^o) - (X_j^o - X_i^o)\,(\delta Y_j^o - \delta Y_i^o)}{(Y_j^o - Y_i^o)^2}$$

or

$$\delta\alpha_{ij}^o = \frac{\cos^2\alpha_{ij}^o}{(Y_j^o - Y_i^o)^2}\ \{(Y_j^o - Y_i^o)(\delta X_j^o - \delta X_i^o) - (X_j^o - X_i^o)(\delta Y_j^o - \delta Y_i^o)\} \tag{3.17}$$

From Figure 2a,

$$\cos\alpha_{ij}^o = \frac{Y_j^o - Y_i^o}{D_{ij}}$$

Hence,

$$\frac{\cos^2\alpha_{ij}^o}{(Y_j^o - Y_i^o)^2} = \frac{1}{D_{ij}^2} \tag{3.18}$$

Where

$$D_{ij}^2 = (X_j^o - X_i^o)^2 + (Y_j^o - Y_i^o)^2 \tag{3.19}$$

Substituting (3.18) into equation (3.17)

$$\delta\alpha^{o}_{ij} = \frac{1}{D^2_{ij}} \{(Y^o_j - Y^o_i)(\delta X^o_j - \delta X^o_i) - (X^o_j - X^o_i)(\delta Y^o_j - \delta Y^o_i)\}$$

or

$$\delta\alpha^{o}_{ij} = \frac{Y^o_j - Y^o_i}{D^2_{ij}}\delta X^o_j + \frac{Y^o_i - Y^o_j}{D^2_{ij}}\delta X^o_i + \frac{X^o_i - X^o_j}{D^2_{ij}}\delta Y^o_j + \frac{X^o_j - X^o_i}{D^2_{ij}}\delta Y^o_i \quad (3.20)$$

Let $P_{ij} = \frac{Y^o_i - Y^o_j}{D^2_{ij}}$, $Q_{ij} = \frac{X^o_j - X^o_i}{D^2_{ij}}$

$R_{ij} = \frac{Y^o_j - Y^o_i}{D^2_{ij}}$, $S_{ij} = \frac{X^o_i - X^o_j}{D^2_{ij}}$ (3.21)

Substituting (3.21) into (3.20) gives

$$\delta\alpha^{o}_{ij} = P_{ij}\delta X^o_i + Q_{ij}\delta Y^o_i + R_{ij}\delta X^o_j + S_{ij}\delta Y^o_j \quad (3.22)$$

Substituting for $\hat{\alpha}_{ij}$ from equation (3.16) into equation (3.12) gives

$$v_{ij} = \alpha^o_{ij} + \delta\alpha^o_{ij} - \alpha_{ij}$$

$$v_{ij} = \delta\alpha^{o}_{ij} + (\alpha^o_{ij} - \alpha_{ij}) \quad (3.23)$$

Substituting for $\delta\alpha^o_{ij}$ from equation (3.22) into equation (3.23) gives

$$v_{ij} = P_{ij}\delta X^o_i + Q_{ij}\delta Y^o_i + R_{ij}\delta X^o_j + S_{ij}\delta Y^o_j + (\alpha^o_{ij} - \alpha_{ij})$$

Recognizing α_{ij} as observed value "O", and α^o_{ij} as computed value "C", then the general observation equation for a direction is given by

$$P_{ij}\delta X^o_i + Q_{ij}\delta Y^o_i + R_{ij}\delta X^o_j + S_{ij}\delta Y^o_j = (O - C)_{ij} + v_{ij} \quad (3.24)$$

Similarily, for the direction from i to k in Figure 2b, the observation equation for the line i - k is given by

$$P_{ik}\delta X_i^o + Q_{ik}\delta Y_i^o + R_{ik}\delta X_k^o + S_{ik}\delta Y_k^o = (0 - C)_{ik} + v_{ik} \qquad (3.25)$$

where

$$P_{ik} = \frac{Y_i^o - Y_k^o}{D_{ik}^2}, \qquad Q_{ik} = \frac{X_k^o - X_i^o}{D_{ik}^2}$$
$$R_{ik} = \frac{Y_k^o - Y_i^o}{D_{ik}^2}, \qquad S_{ik} = \frac{X_i^o - X_k^o}{D_{ik}^2} \qquad (3.26)$$

Subtracting equation (3.24) from equation (3.25) gives a general observation equation for an angle $_{ijk}$, Figure 2b, as

$$(P_{ik} - P_{ij})\delta X_i^o + (Q_{ik} - Q_{ij})\delta Y_i^o - R_{ij}\delta X_j^o - S_{ij}\delta Y_j^o + R_{ik}\delta X_k^o +$$
$$S_{ik}\delta Y_k^o = (0 - C)_{ik} - (0 - C)_{ij} + v_{ik} - v_{ij}$$

Let $0 - C = (0 - C)_{ik} - (0 - C)_{ij}$, and $v_{ijk} = v_{ik} - v_{ij}$

Then the final general equation for an angle θ_{ijk} becomes

$$(P_{ik} - P_{ij})\delta X_i^o + (Q_{ik} - Q_{ij})\delta Y_i^o - R_{ij}\delta X_j^o - S_{ij}\delta Y_j^o + R_{ik}\delta X_k^o +$$
$$S_{ik}\delta Y_k^o = (0 - C) + v_{ijk} \qquad (3.27)$$

It is very important to realize here that (0 - C) in equation (3.27) is the difference between the observed and the computed angle. The computed angle is obtained from the approximate co-ordinates assigned to the three points under consideration. The observed angle, of course, is the angle observed and comes directly from the field work.

C. Least Squares Solution

As stated earlier, the observation equations technique which is also known as the adjustment of indirect observations, will be adopted in this paper where one observation equation for each observation is written. The observations in our case could be direction(s), distance(s), angle(s), or the combination of any of these items. For each observation, one of the respective observation equations of the form given by equations (3.11), (3.24) or (3.27) can be written which yields n observation equations on u unknowns (n x u). These observation equations can be represented by the following equations.

$$
\begin{array}{ccccccccc}
b_{11}x_1 & + & b_{12}x_2 & + \ldots\ldots + & b_{1u}x_u & = & f_1 & + & v_1 \\
b_{21}x_1 & + & b_{22}x_2 & + \ldots\ldots + & b_{2u}x_u & = & f_2 & + & v_2 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
b_{n1}x_1 & + & b_{n2}x_2 & + \ldots\ldots + & b_{nu}x_u & = & f_n & + & v_n
\end{array}
\tag{3.28}
$$

Each of these equations has the corresponding weight w_1, w_2, ..., w_n. In the equations above, the x's are the unknown parameters which represent δX_i^o, δY_j^o, ..., etc. in the equations of distance (3.11), direction (3.24), or angle (3.27). Similarily, b_{11}, b_{12}, ..., etc. represent P_i, Q_{ij}, ..., etc. whereas f_1, f_2,..., etc. represent (O - C)'s, and v_1, v_2, ..., etc. represent v_i, v_{ij}, ..., etc. in the equations (3.11), (3.24), and (3.27).

In the matrix notation, equation (3.28) is given by

$$\underline{B}\ \underline{\Delta} = \underline{f} + \underline{v} \tag{3.29a}$$

where a slash below a letter represents a matrix. Therefore:

$$\underline{B} = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1u} \\ b_{21} & b_{22} & \cdots & b_{2u} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nu} \end{vmatrix}, \quad \underline{f} = \begin{vmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_n \end{vmatrix}, \quad \underline{v} = \begin{vmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{vmatrix}, \text{ and } \underline{\Delta} = \begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_u \end{vmatrix} \tag{3.29b}$$

where $\underline{B}$ will be called the coefficient matrix, $\underline{f}$ the right hand side matrix, $\underline{\Delta}$ the parameters, and $\underline{v}$ the residuals respectively in this paper.

For the least squares solution (shown in chapter II), the most probable values of the x's in the observation equations will give a set of v's such that

$$\phi = \underline{v}^t\ \underline{W}\ \underline{v} \rightarrow \text{minimum} \tag{3.30}$$

where properties of $\underline{W}$ have been discussed in chapter II. From the equation (3.29a),

$$\underline{v} = \underline{B}\ \underline{\Delta} - \underline{f}$$

Substituting for $\underline{v}$ in equation (3.30),

$$\phi = (\underline{B}\ \underline{\Delta} - \underline{f})^t\ \underline{W}\ (\underline{B}\ \underline{\Delta} - \underline{f})$$

The transposition factor "t" in the above equation can be taken inside the bracket giving:

$$\phi = \{(\underline{B}\ \underline{\Delta})^t - \underline{f}^t\}\ \underline{W}\ (\underline{B}\ \underline{\Delta} - \underline{f})$$

or

$$\phi = (\underline{\Delta}^t\ \underline{B}^t - \underline{f}^t)\ \underline{W}\ (\underline{B}\ \underline{\Delta} - \underline{f}) \tag{3.31}$$

Note that $(\underline{B}\ \underline{\Delta})^t = \underline{\Delta}^t\ \underline{B}^t$. For this proof, see page 437 of [26].

Matrix multiplication is accomplished in a manner similar to ordinary multiplication except that the multiplication must be done in order. Thus, equation (3.31) gives

$$\phi = (\underline{\Delta}^t\ \underline{B}^t\ \underline{W} - \underline{f}^t\ \underline{W})\ (\underline{B}\ \underline{\Delta} - \underline{f})$$

$$\phi = \underline{\Delta}^t\ \underline{B}^t\ \underline{W}\ \underline{B}\ \underline{\Delta} - \underline{f}^t\ \underline{W}\ \underline{B}\ \underline{\Delta} - \underline{\Delta}^t\ \underline{B}^t\ \underline{W}\ \underline{f} + \underline{f}^t\ \underline{W}\ \underline{f} \tag{3.32}$$

To determine the least squares solution of the unknown parameters, $\underline{\Delta}$, the condition is

$$\frac{\partial\phi}{\partial\underline{\Delta}} = 0 \tag{3.33}$$

Matrix differentiation, however, is not the same as the ordinary differentiation. The procedures for differentiating matrix functions are given in page 457 [26]. Only the final derivatives will be given here. The differentiation of equation (3.32) with respect to $\partial\underline{\Delta}$ is then,

$$\frac{\partial\phi}{\partial\underline{\Delta}} = \frac{\partial}{\partial\underline{\Delta}}(\underline{\Delta}^t\underline{B}^t\ \underline{W}\ \underline{B}\ \underline{\Delta}) - \frac{\partial}{\partial\underline{\Delta}}(\underline{f}^t\ \underline{W}\ \underline{B}\ \underline{\Delta}) - \frac{\partial}{\partial\underline{\Delta}}(\underline{\Delta}^t\ \underline{B}^t\ \underline{W}\ \underline{f}) + \frac{\partial}{\partial\underline{\Delta}}(\underline{f}^t\ \underline{W}\ \underline{f}) \tag{3.34}$$

Matrix differentiation will yield [26]:

$$\frac{\partial}{\partial\underline{\Delta}}(\underline{\Delta}^t\ \underline{B}^t\ \underline{W}\ \underline{B}\ \underline{\Delta}) = 2\ \underline{\Delta}^t\ \underline{B}^t\ \underline{W}\ \underline{B}$$

$$\frac{\partial}{\partial\underline{\Delta}}(\underline{f}^t\ \underline{W}\ \underline{B}\ \underline{\Delta}) = \frac{\partial}{\partial\underline{\Delta}}(\underline{\Delta}^t\ \underline{B}^t\ \underline{W}\ \underline{f}) = \underline{f}^t\ \underline{W}\ \underline{B} \tag{3.35}$$

and $\frac{\partial}{\partial \underline{\Delta}} = (\underline{f}^t \; \underline{W} \; \underline{f}) = 0$

since $\underline{f}^t \; \underline{W} \; \underline{f}$ is a constant.

Substituting into equation (3.34),

$$\frac{\partial \phi}{\partial \underline{\Delta}} = 2 \; \underline{\Delta}^t \; \underline{B}^t \; \underline{W} \; \underline{B} - \underline{f}^t \; \underline{W} \; \underline{B} - \underline{f}^t \; \underline{W} \; \underline{B} + 0$$

$$= 2 \; \underline{\Delta}^t \; \underline{B}^t \; \underline{W} \; \underline{B} - 2 \; \underline{f}^t \; \underline{W} \; \underline{B}$$

Imposing the least squares condition from equation (3.33) of

$$\frac{\partial \phi}{\partial \underline{\Delta}} = 0 \quad \text{gives:}$$

$$2 \; \underline{\Delta}^t \; \underline{B}^t \; \underline{W} \; \underline{B} - 2 \; \underline{f}^t \; \underline{W} \; \underline{B} = 0$$

or, $\underline{\Delta}^t \; \underline{B}^t \; \underline{W} \; \underline{B} = \underline{f}^t \; \underline{W} \; \underline{B}$

Transposing both sides gives

$$(\underline{\Delta}^t \; \underline{B}^t \; \underline{W} \; \underline{B})^t = (\underline{f}^t \; \underline{W} \; \underline{B})^t$$

$$\underline{B}^t \; \underline{W} \; \underline{B} \; \underline{\Delta} = \underline{B}^t \; \underline{W} \; \underline{f} \qquad (3.36)$$

Again, for this derivation see page 437 [26].

Equation (3.36) is the normal equation where matrix $\underline{B}^t \; \underline{W} \; \underline{B}$ is: symmetric and square, of the order u, and with the number of unknown parameters $\underline{\Delta}$. Post multiplying both sides of equation (3.36) by $(\underline{B}^t \; \underline{W} \; \underline{B})^{-1}$ gives

$$(\underline{B}^t \; \underline{W} \; \underline{B})^{-1} \; (\underline{B}^t \; \underline{W} \; \underline{B}) \; \underline{\Delta} = (\underline{B}^t \; \underline{W} \; \underline{B})^{-1} \; (\underline{B}^t \; \underline{W} \; \underline{f})$$

But $(\underline{B}^t \; \underline{W} \; \underline{B})^{-1} \; (\underline{B}^t \; \underline{W} \; \underline{B}) = \underline{I}$, Identity matrix

and since $\underline{I} \; \underline{\Delta} = \underline{\Delta}$

then $\underline{\Delta} = (\underline{B}^t \; \underline{W} \; \underline{B})^{-1} \; (\underline{B}^t \; \underline{W} \; \underline{f}) \qquad (3.37)$

This is the least squares solution for the unknown parameters $\underline{\Delta}$. All the terms at the right hand side of this equation are known. The unknown parameters $\underline{\Delta}$, coefficient matrix $\underline{B}$, the weight matrix $\underline{W}$, and right hand side matrix $\underline{f}$ are given by the equations in the expression (3.29b). Usually, the matrices $\underline{B}^t\ \underline{W}\ \underline{B}$ and $\underline{B}^t\ \underline{W}\ \underline{f}$ are respectively represented by $\underline{N}$ and $\underline{t}$. Then, the equation (3.37) becomes

$$\underline{\Delta} = \underline{N}^{-1}\ \underline{t} \tag{3.38}$$

Matrix $\underline{N}$ is called the normal matrix.

D. Precision from the Least Squares

The adjustments made by the least squares techniques give the MPV's of the measured quantities and in any adjustment it is important to know the precisions of the adjusted quantities. For instance, we might like to know the precisions of the adjusted co-ordinates of the stations in a traverse. This precision can be obtained by the technique of propagation described in chapter IV of [26].

Recalling equation (3.37)

$$\underline{\Delta} = (\underline{B}^t\ \underline{W}\ \underline{B})^{-1}\ (\underline{B}^t\ \underline{W}\ \underline{f})$$

This gives the adjusted values (MPV's) of the unknown parameters $\underline{\Delta}$, and it is desired to find the precisions of these parameters. In equation (3.29b), $\underline{\Delta}$ is given by

$$\underline{\Delta} = \begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_u \end{vmatrix} \tag{3.39}$$

where $x_1, x_2, \ldots, x_u$ are defined as the δX^o's and δY^o's, which are the corrections to be added to the approximate co-ordinates X^o's and Y^o's. Therefore, the parameters in this case, are the unknown co-ordinates. These parameters can be other quantities, such as the unknown elevations in direct leveling or trig leveling. Whatever the parameters may be, the least squares adjustment can provide their precisions.

Under the section Weights in chapter II, it has been shown from [26] that the cofactor matrix $Q_{\ell\ell}$ ($\underline{Q}$) for the <u>observations</u> $\underline{\ell}$ is given by:

$$\underline{Q_{\ell\ell}} = \frac{1}{\sigma_o^2} \begin{vmatrix} \sigma_{\ell_1}^2 & \sigma_{\ell_1\ell_2} & \cdots & \sigma_{\ell_1\ell_n} \\ \sigma_{\ell_2\ell_1} & \sigma_{\ell_2}^2 & \cdots & \sigma_{\ell_2\ell_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \sigma_{\ell_n\ell_1} & \sigma_{\ell_n\ell_2} & \cdots & \sigma_{\ell_n}^2 \end{vmatrix} \tag{3.40}$$

where σ_o^2 is a constant and $\underline{\ell}$ is given by:

$$\underline{\ell} = \begin{vmatrix} \ell_1 \\ \ell_2 \\ \cdot \\ \cdot \\ \cdot \\ \ell_n \end{vmatrix} \tag{3.41}$$

Comparing equations (3.41) and (3.39), by analogy, $\underline{\ell}$ can be replaced by $\underline{\Delta}$ and ℓ's by x's. Substituting $\underline{\Delta}$ for $\underline{\ell}$ and x's for ℓ's in the equation (3.40) gives:

$$\underline{Q}_{\underline{\Delta\Delta}} = \frac{1}{\sigma_o^2} \begin{vmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_u} \\ \sigma_{x_2x_1} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_u} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{x_ux_1} & \sigma_{x_ux_2} & \cdots & \sigma_{x_u}^2 \end{vmatrix} \quad (3.42)$$

(u x u)

Since there are u number of parameters, the order of the matrix given by the equation (3.42) will be u by u. The expected value of σ_o^2 is given by $\hat{\sigma}_o^2$ which is shown in chapter XI of [26] as

$$\hat{\sigma}_o^2 = \frac{\underline{v}^t \; \underline{W} \; \underline{v}}{r} \quad (3.43)$$

It is shown in [26], that

$$\underline{Q}_{\underline{\Delta\Delta}} = (\underline{B}^t \; \underline{W} \; \underline{B})^{-1} \quad (3.44)$$

In the process of the least squares adjustment, the term $(\underline{B}^t \; \underline{W} \; \underline{B})^{-1}$ has already been computed when the solution of the parameters $\underline{\Delta}$ was obtained from the equation (3.37). It can be shown that the order of the matrix $(\underline{B}^t \; \underline{W} \; \underline{B})^{-1}$ is u by u. Substituting for $\underline{Q}_{\ell\ell}$ from the equation (3.45) into the equation (3.42) and rearranging gives:

$$\begin{vmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_u} \\ \sigma_{x_2x_1} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_u} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{x_ux_1} & \sigma_{x_ux_1} & \cdots & \sigma_{x_u}^2 \end{vmatrix} = \sigma_o^2 \; (\underline{B}^t \; \underline{W} \; \underline{B})^{-1} \quad (3.45)$$

(u x u) (u x u)

Note that σ_o^2 is a known constant computed from the equation (3.43). Therefore, the elements of the right hand side matrix of the equation (3.45) are all known. Suppose, that $\sigma_o^2(\underline{B}^t \; \underline{W} \; \underline{B})^{-1}$ is represented by:

$$\sigma_o^2 \; (\underline{B}^t \; \underline{W} \; \underline{B})^{-1} = \begin{vmatrix} n11 & n12 & \cdots & n1u \\ n21 & n22 & \cdots & n2u \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ nu1 & nu2 & & nuu \end{vmatrix} \qquad (3.46)$$

$$(u \times u)$$

where n's are the known numbers. Comparing this equation (3.46) to the equation (3.45):

$$\underset{(u \times u)}{\begin{vmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_4} & \cdots & \sigma_{x_1 x_u} \\ \sigma_{x_2 x_1} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2 x_u} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \sigma_{x_u x_1} & \sigma_{x_u x_2} & \cdots & \sigma_{x_u}^2 \end{vmatrix}} = \underset{(u \times u)}{\begin{vmatrix} n11 & n12 & \cdots & n1u \\ n21 & n22 & \cdots & n2u \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ nu1 & nu2 & \cdots & nuu \end{vmatrix}} \qquad (3.47)$$

Note that all the elements of the left hand side matrix can now be computed. For example

$$\sigma_{x_1}^2 = n11, \quad \sigma_{x_2 x_u} = n2u, \; \ldots, \text{ etc.}$$

Therefore, $\sigma_{x_i} = \pm \sqrt{nii}$, $i = 1, 2, 3, \ldots, u.$ (3.48)

where σ_{x_i} gives the precision of the adjusted parameter x_i.

In our case of horizontal position control, the parameter for the j^{th} station are δX_j^o and δY_j^o, and they are given in the equations (3.03) and (3.14) as:

$$X_j = X_j^o + \delta X_j^o, \qquad Y_j = Y_j^o + \delta Y_j^o \tag{3.49}$$

It has been defined earlier that x's represent the parameters δX's and δY's. Suppose, $\delta X_j^o = x_1$ and $\delta Y_j^o = x_2$

Then,

$$X_j = X_j^o + x_1 \quad \text{and} \quad Y_j = Y_j^o + x_2 \tag{3.50}$$

where (X_j, Y_j) and (X_j^o, Y_j^o) are the adjusted and approximate co-ordinates of the j^{th} station. Assuming (X_j^o, Y_j^o) are constants, it can be shown [26] that:

$$\sigma_{X_j}^2 = \sigma_{x_1}^2, \qquad \text{and } \sigma_{Y_j}^2 = \sigma_{x_2}^2$$

or

$$\sigma_{X_j} = \pm\sqrt{\sigma_{x_1}^2}, \qquad \text{and } \sigma_{Y_j} = \pm\sqrt{\sigma_{x_2}^2} \tag{3.51}$$

where σ_{X_j} and σ_{Y_j} are the precisions of the adjusted co-ordinates X_j and Y_j respectively, and these are the quantities we wish to determine. From the equation (3.47),

$$\sigma_{x_1}^2 = n11, \quad \text{and} \quad \sigma_{x_2}^2 = n22$$

Therefore,

$$\sigma_{X_j} = \pm\sqrt{n11}, \text{ and } \sigma_{Y_j} = \pm\sqrt{n22} \tag{3.52}$$

Thus, for all the adjusted co-ordinates, their corresponding precisions can be computed. This technique has been used in HCONTRL to compute the precisions (printed as the standard errors in the computer output) of the adjusted co-ordinates, X and Y. The precision of the j^{th} position will be computed by:

$$\pm \sqrt{\sigma_{X_j}^2 + \sigma_{Y_j}^2} = \pm \sqrt{n11 + n22} \qquad (3.53)$$

In the case of the vertical position control, since the adjusted parameters are elevations, (one quantity as opposed to two quantities, X and Y, in the horizontal control); the precision for the adjusted elevation of the j^{th} elevation station will be directly given by the equation (3.48). For the example, the third adjusted elevation, the precision will be given by:

$$\sigma_{x_3} = \pm \sqrt{n33} \qquad (3.54)$$

This technique has been utilized in the program VCONTRL.

E. Future Systems

So far we have discussed the traditional or classical techniques of the horizontal control positioning. Current technological development has grown beyond the imagination of a few decades ago. New techniques of precise horizontal and vertical positioning are rapidly developing; some of which are already in use. In this section, a brief description of these techniques will be presented. These techniques are:

1. Satellite Positioning
2. Inertial Positioning
3. Non-Classical Technique

Satellite Positioning

Since the introduction of the first artificial satellite in 1958, the development in satellite positioning has leaped ahead, especially, since the development of the high speed computer. The first operational satellite, for the position fixing on the surface of the earth, was introduced in January 1964 as a U.S. Navy Navigation Satellite System (NAVSAT). It was developed by the Applied Physics Laboratory (APL) of the John Hopkins University and is known as Transit. The Transit became commercial available in July 1969.

Currently, there are five operational Transit satellites in orbit. These satellites are in circular polar orbits, about 1,075 kilometers high, circling the earth at about every 107 minutes. Whenever a satellite passes above the horizon, there is an opportunity to obtain a position fix.

The satellite continuously transmits very stable frequencies at a constant interval of two minutes. A similar frequency may also be generated at the observer's position. By comparing the transmitted frequencies from two positions of a satellite to the frequency of the observer, it is possible to compute the difference between the two distances (distances between the observer and the two satellite positions), called the range rate (O), Figure 3. The satellite, also transmits a message in terms of orbital parameters [27] as a function

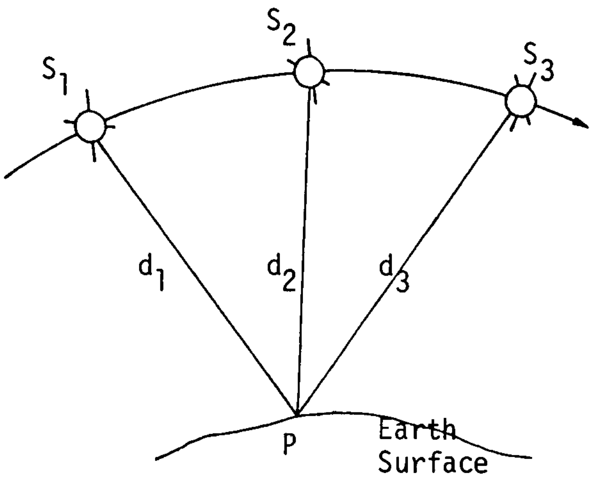


Figure 3. Position Fix from Artificial Satellites.

of time, from which its position can be computed [27]. From such two positions, and the approximate co-ordinates of the observer P, Figure 3, the range rate (C), can be computed. Since the range rate is the function of the unknown co-ordinates of the observer and the fixed co-ordinates of the satellites at the specific times, it can be differentiated to form a linear observation equation, where the parameters will be the correction quantities to be algebraically added to the assumed approximate co-ordinates of the observer. The observation equation can be represented by the similar type of the equation as given by the equation (3.11), where the right hand side is given by (O-C) for the range rate. Therefore, for each observation of frequencies comparison, one observervation equation can be formulated. Typically, twenty to forty such comparisons are possible from one pass of the satellite at a point on the surface of the earth. Hence, twenty to forty observation equations per pass can be formed. The redundant observations, can then be adjusted by the least squares technique to obtain the MPV's of the co-ordinates of the observer. All the computations and the adjustments are done in the digital computer which gives the position of the observer at any desired moment, almost instantaneously [31].

Inertial Positioning

In inertial positioning, the horizontal position of an observer is obtained from the measurements of the velocity and displacement of the observer from a starting point by sensing the acceleration acting on the observer or the vehicle in which the inertial positioning equipment is transported. This technique is fundamentally different from the others because it depends only on the measurements made on the vehicle itself.

The inertial positioning system consists of three components; (a) Gyroscope, (b) Accelerometers, and (c) A Computer. A gyroscope is an instrument which spins about its axis. The property of the gyroscope is such that its spin axis always points towards a fixed direction in space unless a torque is applied. When the torque is applied, the resultant change in direction is then the measure of the applied torque. This change in direction of the spin axis of the gyroscope provides the reading in terms of the rate of change of direction with respect to the original spin axis at the instrument. This information is then integrated by the computer with respect to time and this gives the direction.

An accelerometer, on the other hand, gives distance travelled from the starting point. It is a device, a sensor, for measuring the acceleration of the vehicle. Therefore, integration of the accelerometer output by the computer gives velocity which, further integrated provides the distance travelled. This information combined with the integrated gyroscope output, will provide the change in position of the observer relative to the initial known point, from which the position of the observer can be computed.

The position of the observer, thus obtained by the inertial system, is then the function of the direction and the distance, which can be differentiated and linearized to obtain the observation equation. For the redundant data, the observations can then be adjusted by the usual least squares technique to obtain the desired parameters.

Non-Classical Technique

So far we have discussed and dealt with the position of a point in two dimensions only; that is X and Y co-ordinates only. Besides these co-ordinates, there is a third dimensional co-ordinate, which is the elevation or altitude of the point. This co-ordinate, the vertical control position, is presented in this paper in chapter VI, and it is represented by Z. The classical technique has been to deal with the vertical and horizontal controls separately. Therefore, traditionally, horizontal and vertical positions controls have been adjusted independently.

A new aspect of non-classical technique combines both the horizontal and vertical positions control techniques, into one technique. This technique considers the measurements or the observations, and the adjustment in three dimensions, instead of the conventional two dimensions. The unified three dimensional approach in surveying is a very advanced mathematical approach and is accomplished by the relevant mathematical tools such as vector and tensor calculus. The concepts of the three dimensional adjustment are very complicated. None the less, it is possible to get the unified three dimensional solution. The concepts and reality of a position in three dimensions, as well as the adjustment techniques, are illustrated in [20]. As the standards of the surveying sciences, especially geodesy, get higher, it seems in the near future that the use of three dimensional aspect of surveying may be close at hand, because, it is desirable to think and get the position of a point in three dimensions X, Y, and Z.

IV. DEVELOPMENT OF THE COMPUTER PROGRAM FOR THE HORIZONTAL POSITION CONTROL

A computer program called HCONTRL has been developed to adjust the horizontal position control by the least squares technique. The development procedures and running of this program will be discussed here. A similar program called VCONTROL for vertical position control has also been developed and is discussed in chapter VII.

A. Program "HCONTROL"

This is a computer program to compute the horizontal positions by the various classical techniques such as: traverse, intersection, resection, triangulation, trilateration, and combined networks utilizing the least squares technique of adjustment when the redundant measurements are taken. The program is capable of adjusting the measurements of distance(s), or angle(s), or both. It is written in Fortran IV language.

The HCONTRL employs the observation equations technique of adjustment. The measurements must be either angle(s), or distances(s), or both. For each measurement, HCONTRL formulates one observation equation given by the equation (3.27) for the angle, and by the equation (3.11) for the distance. This is accomplished by computing the coefficients of the unknown parameters and assigning them into a matrix form. For example, for a distance measurement between the two points i and j, Figure 1, the observation equation is of the form given by the equation (3.11). The computer computes the coefficients P_i, Q_i, R_i, and S_i; given by the equation (3.08), of the unknown parameters, δX_i^o, δY_i^o, δX_j^o and δY_j^o, respectively. These coefficients are computed from the given approximate co-ordinates (X_i^o, Y_i^o) and (X_j^o, Y_j^o) of the points i and j respectively. The HCONTRL then assigns these computed coefficients as the elements of the coefficient matrix $\underline{B}$. This is illustrated by the following example.

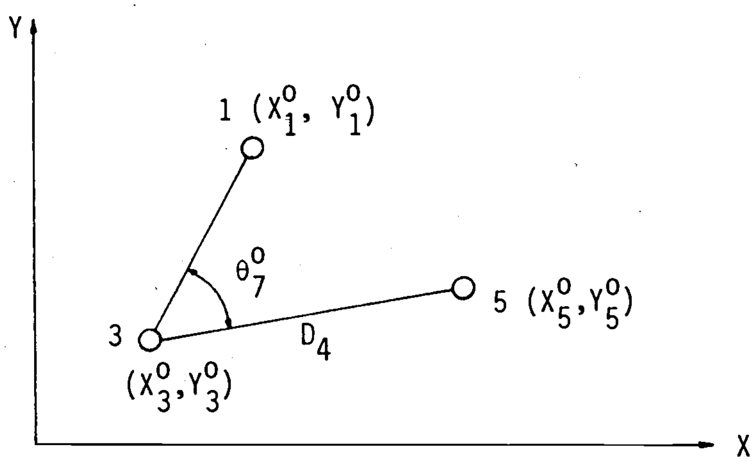


Figure 4. Sample Coefficients Computation.

In Figure 4, 1, 3, and 5 are three unknown points whose X and Y co-ordinates are required, and X_1^o, Y_1^o,..., etc. are their corresponding approximate co-ordinates. If the distance D_4, between the two points (unknown stations) 3 and 5 is inputed into the computer as the fourth observation, HCONTRL computes the approximate distance D_4^o from the equation (3.19) and assigns as C, which is then subtracted from D_4, and puts the result in the matrix form as

$$f(k) = D_k - D_k^o \tag{4.01}$$

where k = 4 (for the fourth input observation). The coefficients of the unknown parameters will be computed from:

$$P_i = \frac{X_3^o - X_5^o}{D_4^o}, \qquad Q_i = \frac{Y_3^o - Y_5^o}{D_4^o}$$
$$R_i = \frac{X_5^o - X_3^o}{D_4^o} \qquad S_i = \frac{Y_5^o - Y_3^o}{D_4^o} \tag{4.02}$$

These coefficients will be assigned as the elements of the matrix $\underline{B}$ according to the stations numbers 3 and 5, and the order of the input of the observation. These will be given by;

$$b(k,2i-1) = P_i, \qquad b(k,2j-1) = R_i$$
$$b(k,2i) = Q_i, \qquad b(k,2j) = S_i \tag{4.03}$$

where k = 4 (for the fourth input), i = 3, and j = 5. If there is additional unknown station(s), the computer will make the coefficients of all the other unknown parameters zero. Together with the distance, the standard deviation for the measured distance is also inputed. The program computes the weight from the standard deviation, and assigns it in the matrix form by

$$w(k) = \frac{1}{\sigma^2(k)} \tag{4.04}$$

where k = 4.

This completes the formation of one observation equation for a distance. The process is repeated for all the measured distance and the elements of the matrix $\underline{B}$, $\underline{f}$ and $\underline{w}$ are filled.

When the input is an angle, the observation equation is given by the equation (3.27). The procedures of forming the observation equation for an angle is almost the same as in the formation of distance observation equation. The difference is that there are six unknown parameters (two each, for i, j, and k unknown stations) as opposed to four parameters in the distance observation equation. The six coefficients for the six parameters are obtained from the equation (3.21) and (3.26) by computing P_{ij}, Q_{ij}, R_{ij}, S_{ij}, P_{ik}, Q_{ik}, R_{ik}, and S_{ik}. The element of the right hand side of the equation (3.27), (O - C), is the difference between the observed and computed angles. The computed angle is obtained from the approximate co-ordinates of the three unknown stations.

The procedure for assigning the computed values to the matrix $\underline{B}$ is illustrated from the following example. If the angle between the stations 1, 3, and 5, Figure 4, is inputed as the seventh observation, the elements of the matrix $\underline{B}$ $\underline{f}$, and $\underline{w}$ will be given by;

$$
\begin{array}{llll}
b(k,2i-1) = (P_{ik} - P_{ij}), & & b(k,2j-1) = -R_{ij}, & \\
b(k,2i) = (Q_{ik} - Q_{ij}), & & b(k,2j) = -S_{ij} & (4.05) \\
b(k,2m-1) = R_{ik} & , & b(k,2m) = S_{ik} &
\end{array}
$$

where k = 7 (for the seventh input), i = 1, j = 3, and m = 5. From the approximate co-ordinates of the stations 1, 3, and 5, approximate angle θ_7^o will be computed by the program. If the input angle is represented by θ_7, and the input standard deviation for the angle by σ_7, the element of $\underline{f}$ and $\underline{w}$ will be given by

$$f(k) = \theta_7 - \theta_7^0 , \qquad w(k) = \frac{1}{\sigma_7^2} \tag{4.06}$$

where k is the order number of the input angle, i.e. 7, in our case. All other coefficients of the other unknown station or stations (if any) will be assigned zero for this observation equation. For all the input angles, this procedure is repeated and the corresponding elements of $\underline{B}$, $\underline{f}$, and $\underline{w}$ are computed and assigned.

For any fixed input azimuth, the observation equation for it is formed, almost exactly the same way as the angle observation equation. The coefficients of the unknown parameters are computed from the equation (3.21). For a fixed distance, the procedure for the formation of the observation equation is exactly the same as in the measured distance observation equation. The difference between the fixed and the observed observation equations is that, in the fixed observation equation, the elements of the matrix $\underline{f}$ will be made zero by the computer. For all the fixed azimuths and distances, computation of the coefficients is repeated and the formation of matrix $\underline{B}$ is completed. The formation of the weight matrix $\underline{w}$ is completed according to the equation (4.06). Thus, the matrices $\underline{B}$, $\underline{f}$, and $\underline{w}$ are formed.

In the process of the adjustment into the computer, HCONTRL converts the column weight matrix $\underline{w}$ into $\underline{W}$, and the least squares adjustment is accomplished by solving the equation (3.37), which is

$$\underline{\Delta} = (\underline{B}^t \; \underline{W} \; \underline{B})^{-1} \; (\underline{B}^t \; \underline{W} \; \underline{f}) \tag{4.07}$$

The adjustment gives $\underline{\Delta}$, which is given by

$$\underline{\Delta} = \begin{vmatrix} \delta X_1^o \\ \delta Y_1^o \\ \cdot \\ \cdot \\ \delta X_u^o \\ \delta Y_u^o \end{vmatrix} \tag{4.08}$$

The equation (4.08) is the same equation as the equation (3.29b), written in terms of the actual parameters. The adjusted co-ordinates or MPV's are then obtained from

$$X_i = X_i^o + \delta X_i^o \,, \qquad Y_i = Y_i^o + \delta Y_i^o \tag{4.09}$$

where $i = 1, 2, \ldots, u$.

HCONTRL, then, compares each element (absolute value) of $\underline{\Delta}$ with the number 0.001. If it is greater than 0.001, the co-ordinates given by the equation (4.09) are taken as the new approximate co-ordinates and the procedures repeat themselves to obtain the new set of $\underline{\Delta}$, and the new adjusted co-ordinates of all the unknown stations, until all of the elements (absolute values) of $\underline{\Delta}$ are, separately, not greater than 0.001. When this condition is met, the computer computes the residuals for all the observations from which the estimate of the reference variance $\hat{\sigma}_0^2$ is computed from the equation (2.16). The precisions (standard errors) of the adjusted co-ordinates and the position, are then computed according to the discussion given in the section, Precision from Least Squares, chapter II. All the necessary informations are printed out. Detail listings are given in Appendix I. A complete listing of the HCONTRL is provided in Appendix III. A flow chart for HCONTRL is shown in Figure 6.

The general compilation and running of the program HCONTRL is shown in Figure 5. The following conditions must be satisfied to run the program.

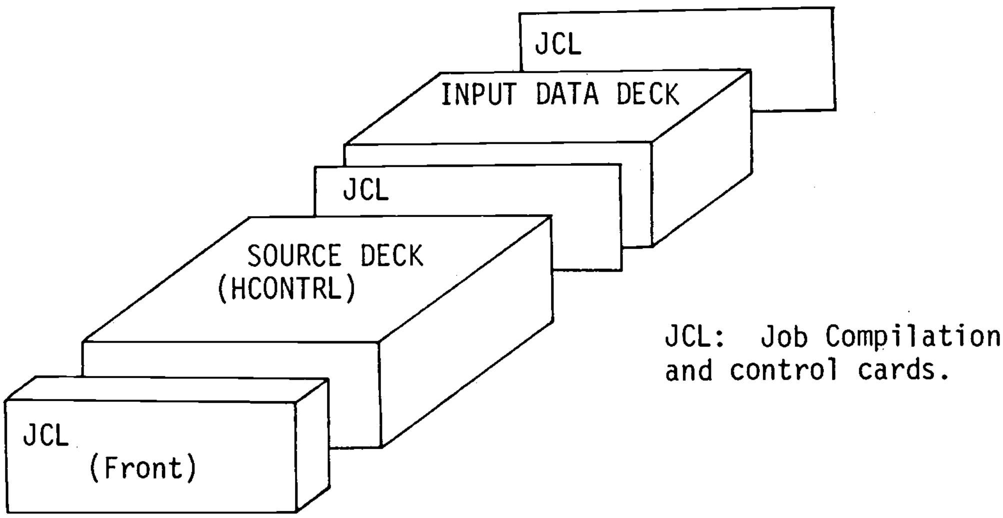


Figure 5. Deck Set Up for HCONTRL

1. The maximum number of observations must not exceed 104.
2. The total number of fixed and unknown stations must not exceed 10 and 40 respectively.
3. The maximum number of fixed azimuths and distances must not exceed four each.

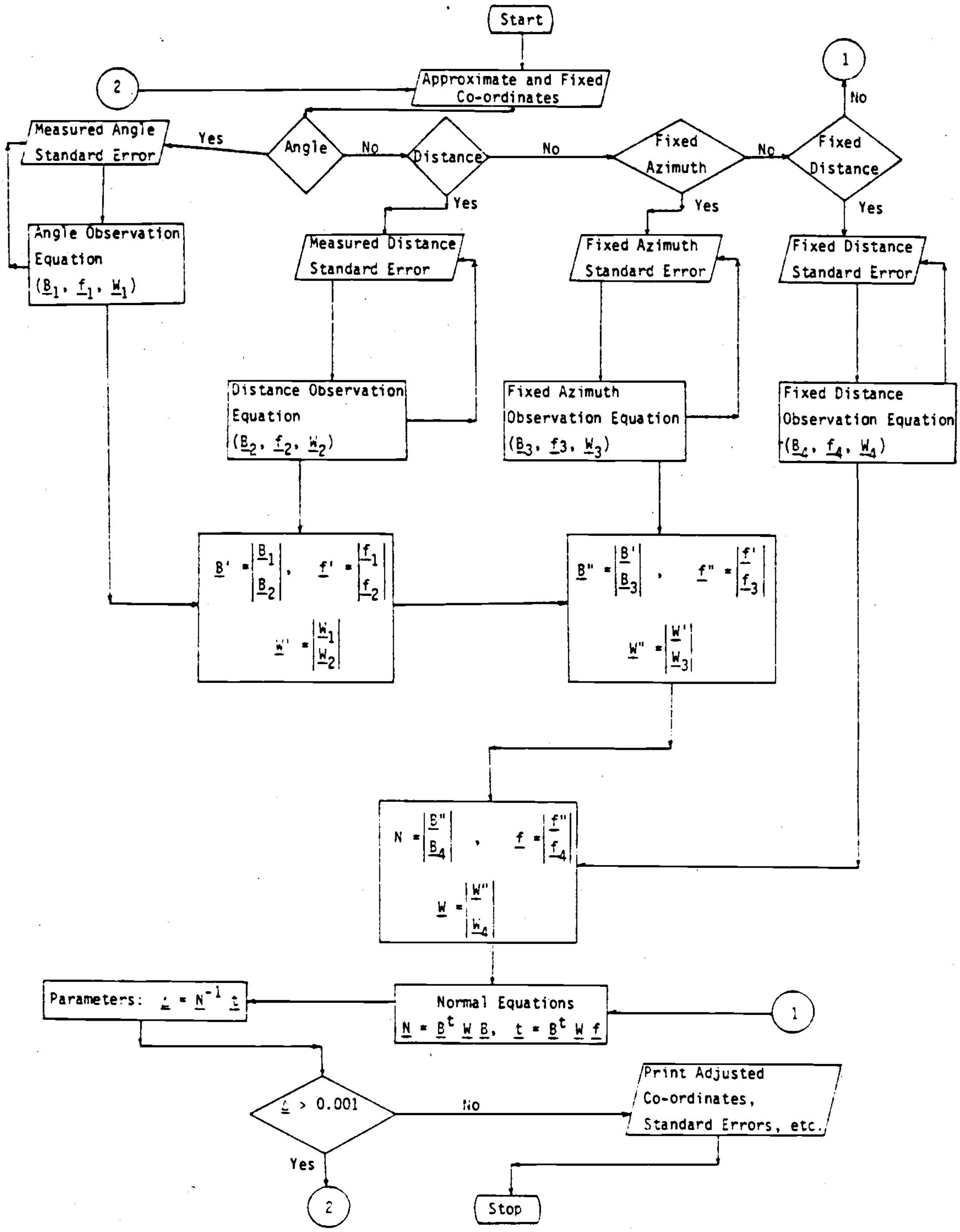


Figure 6. Flow Chart for HCONTRL.

4. At least one, fixed azimuth, distance, and fixed control stations (X and Y co-ordinates) to control the scale and orientation of the net must be given. The scale and orientation may also be fixed by providing two fixed control stations.

5. The approximate co-ordinates of all the unknown station(s) must be pre-determined. These co-ordinates can be, computed approximately, scaled from any available map or any other possible means.

Once the above criterions are satisfied, all that is needed in the adjustment procedure to run the program, are the observed angles, and/or the distances and their respective standard errors to account for the weighting of the observations. These standard errors may be obtained according to the discussion in the section Weights, chapter II.

To run the program, number the unknown station(s) first as, 1, 2, ..., m (m is number, no other character is allowed), and then the fixed stations as m+1, m+2, ..., etc. Any station number must not exceed two digits. Thus the maximum number, a station can have is 99. Set up the free form input data in the following order:

<u>Step or Rule</u>	<u>Input Quantity</u>
1.	Title of the project, up to 75 characters (letters, numerals, or both).
2.	Number of fixed station(s), number of unknown station(s) number of observed angle(s) (zero, if none), number

Step or Rule	Input Quantity
	of observed distance(s) (zero, if none), number of fixed azimuth(s) (zero, if none), number of fixed distance(s) (zero, if none).
3.	Approximate co-ordinates as X_i and Y_i; i = 1, 2, ..., m: where i denotes the i^{th} unknown station.
4.	Fixed co-ordinates as X_j and Y_j; j = m+1, m+2, ..., etc., where j denotes the j^{th} fixed station.
5.	Observed angle in degrees, minutes, and seconds (or any decimal part of second), the standard error of the angle in seconds (or any decimal part of second), the station number I, J, and K such that I is the station number at angle observation point; J and K are such that they, (in that order), appear in the clockwise manner when viewed through I towards the the measured angle. For example, in Figure 7, if the measured value of the angle 3 is, 69°23'16.2",

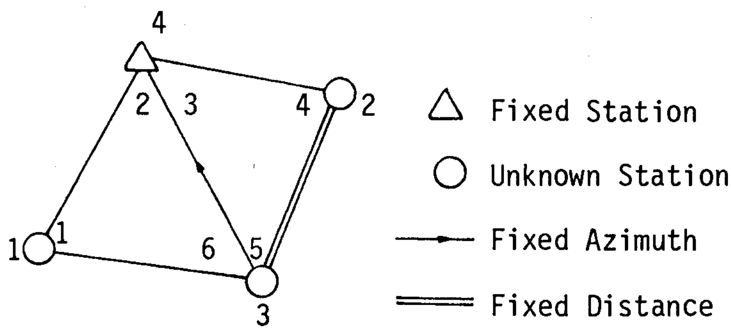


Figure 7. Sample Sketch for HCONTRL Input.

with the standard error of 5.0", then the input for this angle would be

Step or Rule	Input Quantity
	69 23 16.2 (angle) 5.0 (standard error) 4 2 3 (station numbers (I,J,K))
	when viewed through the angle observation point 4 (I), towards the measured angle 3 (69^o 23' 16.2") 2 (J) and 3 (K), in that order, appear in the clockwise manner relative to 4.
	(If no angle observation, go to step 7).
6.	Repeat step 5 for all the observed angles.
7.	The measured distance (in the same unit as the approximate and fixed co-ordinates or vice versa), its standard deviation (in the same unit as the measured distance), station numbers of the two ends (I,J or J,I). For example, if the measured distance between the stations 3 and 1 is 6,254.53 m. with the standard error of 5 cm, the input for the distance will be
	6254.53 0.05 3 1 (or 1 3)
	(If no distance observation, go to step 9)
8.	Repeat step 7 for all the measured distances.
9.	The fixed azimuth from station I to the station J in degrees, minutes, and seconds (any decimal part of second), its standard error in seconds, station numbers I,J (in that order). For example, in Figure 7, the fixed azimuth is 349^o 55' 48.6" from the station 3 to the station 4, if the standard error is 0.1", the input will be

Step or Rule	Input Quantity
	359 55 48.6 0.1 3 4
	(If no fixed azimuth, go to step 11)
10.	Repeat step 9 for all the fixed azimuths.
11.	Repeat step 7, replacing the measured distance by the fixed distance for all the fixed distances.

The classical techniques of horizontal position control such as: traverse, intersection, resection, triangulation, trilateration, etc. have been adjusted using the program HCONTRL. All the numerical examples for the various techniques are given in Appendix I. For each of these techniques, corresponding input data set up is also given in this Appendix. A complete listing of the computer program HCONTRL is given in Appendix III.

V. APPLICATION OF LEAST SQUARES SOLUTION FOR THE HORIZONTAL POSITION CONTROL TECHNIQUES

Generally, classical techniques of horizontal position control include:

1. Traverse
2. Intersection
3. Resection
4. Triangulation
5. Trilateration
6. Combined Networks

A. Traverse

This is the most widely used technique in general practice. A continuous set of directions or angles together with distances are measured. The least squares adjustment is accomplished by simultaneously adjusting two different types of measurements. The adjustment procedure to be followed here is the one which adjusts angle and distance.

For each angle and distance, observation equations like equations (3.27) and (3.11) respectively are written. Utilizing these equations, the computer program HCONTRL forms the observation equations for all the measured angles and distances. The program adjusts all of these equations to give the MPV's, that is, co-ordinates of the unknown stations, by the procedures described in chapter IV. To run the program for the traverse adjustment, the following quantities are needed: the measured angles, the measured distances, standard errors of the measured angles and the distances, the approximate co-ordinates of all the unknown stations, and the fixed control co-ordinates. The scale and the orientation of the traverse must be fixed. The procedures for fixing the scale and the orientation are discussed in chapter IV.

The output of the computer program includes: the adjusted co-ordinates of the unknown stations, the standard errors of the adjusted co-ordinates X_i and Y_i, and of position, of each adjusted station. The standard errors are computed according to the discussion presented in the section Precision from Least Squares, chapter II. The output also includes the adjusted angles and the distances. A listing of HCONTRL is provided in Appendix III. Details of the input procedures and running the program is given in chapter IV.

A traverse of Figure 8, has been adjusted by the program. The data, for checking purpose, are taken from [17]. Following are the fixed and measured quantities.

Fixed Data

	Co-Ordinates		
Station	X(m)	Y(m)	Azimuth from North
4	163,208.49	104,375.29	$\alpha_1 = 48^{\circ}\ 27'\ 30.0''$
5	165,074.49	105,227.47	$\alpha_2 = 67^{\circ}\ 48'\ 48.0''$

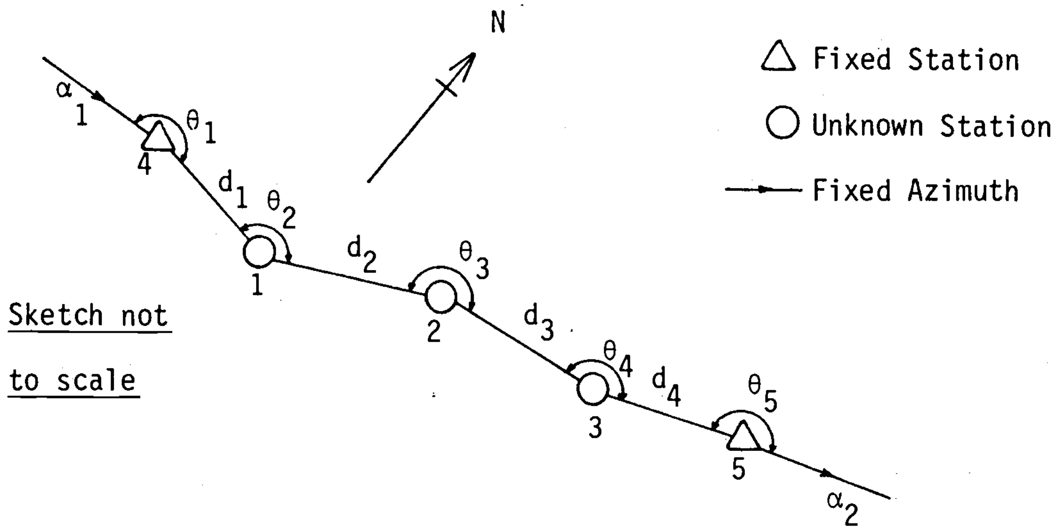


Figure 8. Traverse Sketch.

Measured Data

(a) Angles	(b) Horizontal Distance (m)
$\theta_1 = 203^{\circ}\ 41'\ 28''$	$d_1 = 703.28$
$\theta_2 = 162\ \ 37\ \ 21$	$d_2 = 473.29$
$\theta_3 = 193\ \ 18\ \ 06$	$d_3 = 687.48$
$\theta_4 = 170\ \ 08\ \ 49$	$d_4 = 202.31$
$\theta_5 = 189\ \ 35\ \ 52$	

The standard errors of a measured angle and a distance are arbitrarily chosen as $\pm 3''$ and $\pm$ 0.001 m. respectively, and for all measured angles and the distance, these values for the standard errors are assumed. The data fed into the computer program HCONTRL gives the following adjusted co-ordinates of stations 1, 2, and 3 with their corresponding standard errors. The detail output is given in Appendix I.

	Adjusted Co-ordinates		Standard Errors of Adj. Co-ord.		
Station	X(m)	Y(m)	X(m)	Y(m)	Position (m)
1	163,877.98	104,590.94	± 0.088	± 0.068	± 0.111
2	164,264.64	104,864.04	± 0.098	± 0.079	± 0.126
3	164,902.44	105,120.86	± 0.079	± 0.052	± 0.095

The last digit of the numbers on the above table are rounded digits and this rounding off rule will be followed throughout this chapter.

B. Intersection

Intersection is a simple technique to obtain the horizontal position of one point by measuring horizontal angles only from the fixed points to the point to be intersected. This technique is very convenient when the position of an inaccessible point is needed. At least two horizontal angles from the two fixed control points must be measured to obtain a unique horizontal position. Any more measurements will give the redundant observations, which can be adjusted by the least squares technique.

An observation equation of the form given by the equation (3.27) can be written for each angle measured. All of these observations equations can be represented in the matrix form given by the equation (3.29a). Note here that the elements of the matrices $\underline{B}$, $\underline{W}$, and $\underline{f}$, will be different from the corresponding matrices obtained in traverse adjustment. Elements of the coefficient matrix $\underline{B}$ depends on the approximate co-ordinates of the unknown point(s). Elements of the weight matrix W depends on the standard errors of the observation and the elements of the right hand side matrix f are the functions of the observed and computed quantities, angles in the case of intersection. Since, each adjustment has different fixed and observed data, all these matrices will then be different for different techniques of the horizontal position control.

For a numerical example, angles θ_1 through θ_8 in Figure 9, have been measured from the five fixed control stations (points),

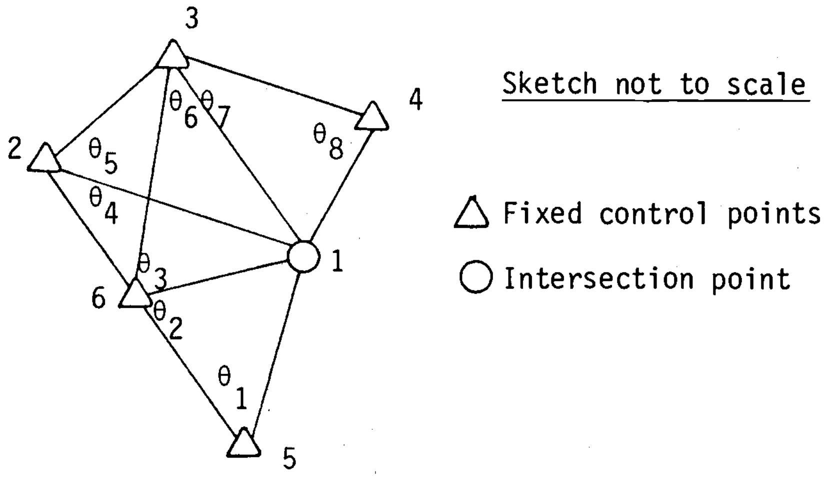


Figure 9. Intersection Sketch.

station 2 through 6, to the station 1 to be intersected. Fixed and measured data are given on the next page.

Assuming the standard error of all the measured angles to be the same, equal to ± 0.1", and inputing the data into the computer program HCONTRL, the following results extracted from the output at Appendix I, are obtained. Detail input procedures for running the program are given in chapter IV.

Fixed Data*

Station	X(M)	Y(m)
2	345,780.67	150,394.05
3	350,044.25	150,752.70
4	356,442.71	148,778.96
5	351,240.22	138,628.80
6	347,490.50	145,480.79

(Co-ordinates)

Measured Data*

Angle	°	'	"
θ_1	32	14	18.8
θ_2	53	18	32.5
θ_3	72	09	20.7
θ_4	27	35	52.1
θ_5	48	01	23.9
θ_6	40	59	38.9
θ_7	57	42	28.2
θ_8	56	00	48.8

	Adjusted Co-Ordinates		Standard Errors of Adj. Co-ordinates		
Station	X(m)	Y(m)	X(m)	Y(m)	Position (m)
1	351,629.08	144,899.05	± 0.044	± 0.036	± 0.057

C. Resection

Resection is the opposite of intersection. In resection, horizontal angles from the point whose horizontal position is required, to the fixed control stations are measured. A minimum of two horizontal angles are necessary. Any more observations gives the redundancy. As stated earlier, each observed angle formulates one observation equation given by the equation (3.27). All the observation equations are represented

*These and the rest of the data in this chapter, unless otherwise stated, are from [2].

by the matrix equation (3.29a) whose least squares solution is given by the equation (3.37). The adjustment procedure is the same as described in intersection or traverse. The same computer program HCONTRL is used for the adjustment.

For a numerical example, six horizontal angles θ_1 through θ_6, in Figure 10, are measured from the resection.

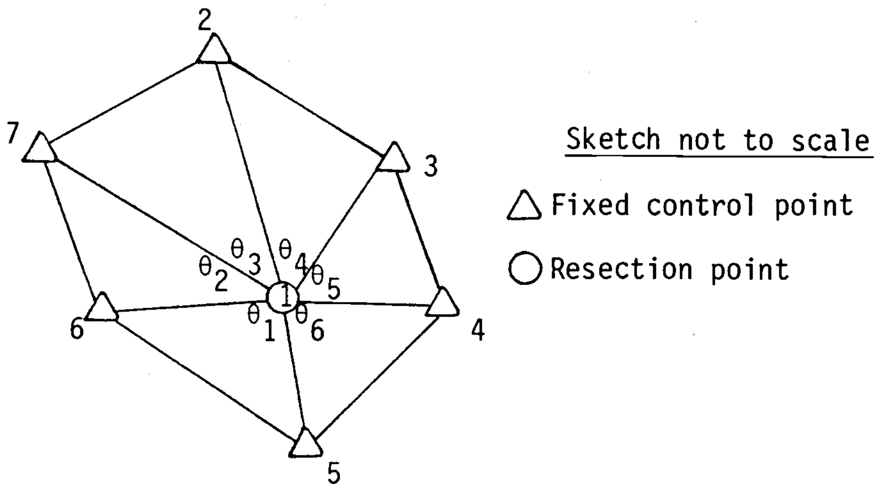


Figure 10. Resection Sketch.

Fixed and measured data are given below:

Fixed Data			Measured Data			
	Control Co-ordinates		Horizontal Angles			
Station	X(m)	Y(m)	Angle	°	'	"
2	350,044.25	150,752.70	θ_1	94	27	06.5
3	356,442.71	148,778.96	θ_2	35	12	51.4
4	356,788.67	144,328.27	θ_3	31	38	05.6
5	351,240.22	138,628.80	θ_4	66	16	48.9
6	347,490.50	145,480.79	θ_5	45	10	57.2
7	345,780.67	150,394.05	θ_6	87	14	09.4

Standard error for each measured angle is assumed ± 0.1". These data are adjusted using HCONTRL to obtain the following results. For the input procedure, see chapter IV. The complete listing of the computer output is given in Appendix I.

	Adjusted Co-Ordinates		Standard Errors of Adj. Co-Ordinates		
Station	X(m)	Y(m)	X(m)	Y(m)	Position (m)
1	351,629.12	144,899.09	± 0.016	± 0.012	± 0.020

D. Triangulation

In a traditional and classical technique of triangulation, co-ordinates of one station together with one fixed distance and azimuth are given. All the horizontal angles in the network are measured. From these items, the horizontal position(s) of all the point(s) are computed. Such a net is shown in Figure 11. The measured quantities are the angles, θ_1 through θ_{14}. The adjustment procedure is exactly the same as described in the previous three techniques. The fixed and measured data are given on the next page.

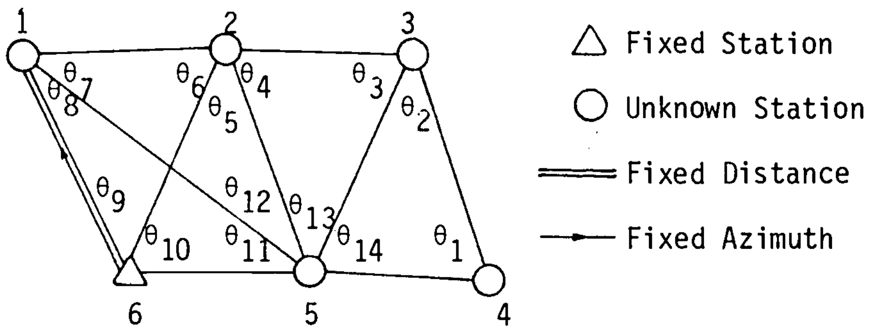


Figure 11. Triangulation Sketch.

Fixed Data

Station	Co-ordinates X(m)	Y(m)
6	347,490.50	145,480.79

Line	Distance (m)
6 - 1	5,202.273

Azimuth from North

6 - 1 340^{0} 48' 43.0"

Measured Data

Angle	o	'	"	Angle	o	'	"
θ_1	79	14	33.5	θ_8	27	35	52.1
θ_2	55	34	32.3	θ_9	45	02	04.0
θ_3	56	00	48.8	θ_{10}	72	09	20.7
θ_4	57	42	28.2	θ_{11}	35	12	51.4
θ_5	40	59	38.9	θ_{12}	31	38	05.6
θ_6	59	20	44.4	θ_{13}	66	16	48.9
θ_7	48	01	23.9	θ_{14}	45	10	57.2

The standard error for each of the measured angles and the fixed azimuth are assumed equal and is taken as $\pm$ 1.0". The standard error of the fixed distance is taken as $\pm$ 0.05 m. The computer program HCONTRL is used to adjust these data by the least square technique and the following adjusted co-ordinates and standard errors are obtained. The complete listing of the output is given in Appendix I.

Station	Adjusted Co-Ordinates X(m)	Y(m)	Standard Errors of Adj. Co-ordinates X(m)	Y(m)	Position (m)
1	345,780.00	150,394.00	± 0.091	± 0.151	± 0.177
2	350,043.67	150,753.20	± 0.127	± 0.177	± 0.218
3	356,442.61	148,780.20	± 0.312	± 0.226	± 0.385
4	356,789.15	144,324.41	± 0.334	± 0.209	± 0.394
5	351,629.29	144,899.56	± 0.154	± 0.092	± 0.179

E. Trilateration

In the trilateration technique only the distances are measured and reduced to horizontal distances. As the Electronic Distance Measuring (EDM) equipment became more and more precise, the feasibility of the trilateration technique compared to the triangulation technique for the horizontal control positioning became a reality. In general practice in the past, this technique was not used due to the expensive EDM equipment. But this is no longer the case. Precise EDM equipment is now available at a reasonable price for the general use. Because of the tremendous saving of time, this technique is now economically practicle.

Once the scale and the orientation of a network is fixed, any kind of net can be adjusted by the trilateration technique. To adjust the net by the least squares technique using the computer program HCONTRL, horizontal distances must be obtained.

For a numerical example, the same net as in resection, Figure 10, is taken. Instead of measuring angles at station 1, the horizontal distances from six control stations to the station 1, were recorded, and they are:

Line	Distance (m)
2 - 1	6,064.34
3 - 1	6,182.65
4 - 1	5,191.05
5 - 1	6,282.32
6 - 1	4,179.31
7 - 1	8,024.87

The control co-ordinates of the fixed stations 2 through 7 are the same as given in Resection, page 58.

Each measured horizontal distance produces one observation equation of the form given by the equation (3.27). For the seven measured distances, there will be seven observation equations, but only two unknown parameters, that is X and Y co-ordinates of the station 1. These redundant observations have been adjusted by the least squares technique using the program HCONTRL, by utilizing the equation(3.37) to obtain the following adjusted co-ordinates of the station 1 and the corresponding standard errors. The standard errors are computed according to the procedure described in the section Precision from Least Squares, chapter II.

	Adjusted Co-Ordinates		Standard Errors of Adj. Co-ord.		
Station	X(m)	Y(m)	X(m)	Y(m)	Position (m)
1	351,629.08	144,899.07	± 0.022	± 0.024	± 0.032

In the adjustment, the standard errors for all the measured distances are assumed to be the same and taken as ± 0.01 m. Input and the run procedure for the program HCONTRL are given in chapter IV. The complete listing of the output is given in Appendix I.

F. Combined Networks

The most general technique in the horizontal position control is the technique of combined networks. This technique is the combination of all the techniques previously described. Once the orientation and scale of the net are fixed, the combination of the horizontal distance(s) and horizontal angle(s) are measured. Then the basic net produced by the combination(s) of figures of triangle(s), quadrilateral(s), central point figure(s), or any other figure(s), by the measurements of distance(s) or angle(s), or both, are adjusted by the simultaneous adjustments of the distance(s) and angle(s). In practice, the scale and orientation of the net are usually controlled and checked by fixing distance(s) and azimuth(s) at chosen places in the net.

Regardless of the shape and size of the triangulated figures, the basic observation equations are angle(s) and distance(s). The adjustment procedure involves formulating one observation equation for one measurement which is represented by equation (3.11) for distance and equation (3.27) for the angle. All the observation equations, put together, can be represented in the matrix form given by the equation (3.29a). Then, the adjustment is carried out as usual; obtaining the unknown parameters by the equation(3.37) using the computer program HCONTRL.

For a numerical example, 21 horizontal angles and 6 distances in Figure 12, are measured with one fixed azimuth and one fixed distance. the fixed azimuth from the north and the fixed distance are between station 7 and station 3. Also fixed in the net, are the co-ordinates of stations 6 and 7. The measured angles are labeled θ_1 through θ_{21}. Following are the fixed and measured data:

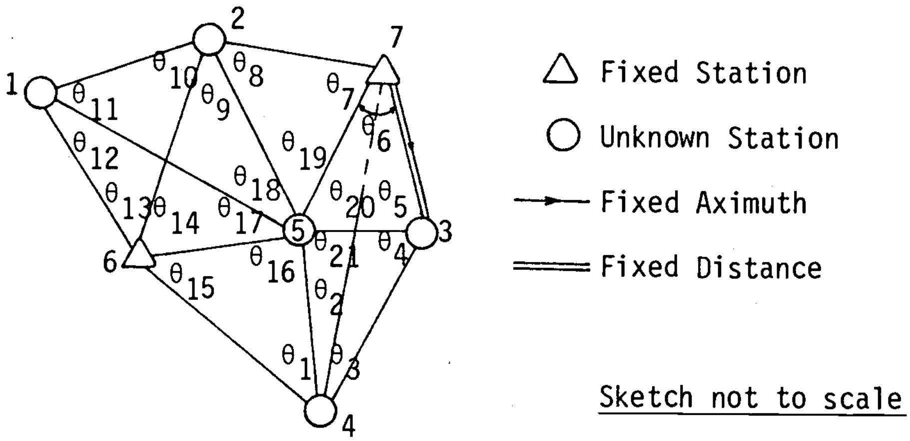


Figure 12. Sketch for Combined Networks

Fixed Data

	Co-Ordinates		Line	Distance (m)
Station	X(m)	Y(m)	7 - 3	4,464.116
				Azimuth
6	347,490.50	145,480.79		
7	356,442.71	148,778.96	7 - 3	175° 33' 18.85"

Measured Data

Angle	°	'	"	Angle	°	'	"	Angle	°	'	"
θ_1	32	14	18.8	θ_8	57	42	28.2	θ_{15}	53	18	32.5
θ_2	23	35	17.6	θ_9	40	59	38.9	θ_{16}	94	27	06.5
θ_3	17	05	36.2	θ_{10}	59	20	44.4	θ_{17}	35	12	51.4
θ_4	52	04	57.1	θ_{11}	48	01	23.9	θ_{18}	31	38	05.6
θ_5	79	14	33.5	θ_{12}	27	35	52.1	θ_{19}	66	16	48.9
θ_6	55	34	32.3	θ_{13}	45	02	04.0	θ_{20}	45	10	57.2
θ_7	56	00	48.8	θ_{14}	72	09	20.7	θ_{21}	87	14	09.4

Line	Measured Distance (m)
4 - 5	6,282.32
3 - 5	5,191.05
7 - 5	6,182.65
2 - 5	6,064.34
1 - 5	8,024,87
6 - 5	4,179.31

The standard errors for each of the measured angle and the distance are arbitrarily taken as ± 1.0" and ± 0.01 m. respectively, whereas the standard errors of the fixed azimuth and the distance are taken as ± 0.1" and ± 0.005 m. respectively. These data are adjusted by the least squares technique using HCONTRL and the following adjusted co-ordinates together with the corresponding standard errors of the adjusted co-ordinates are obtained. To input and run the program, see chapter IV. The standard errors are computed according to the technique described in chapter II.

	Adjusted Co-ordinates		Standard Errors of Adj. Co-ord.		
Station	X(m)	Y(m)	X(m)	Y(m)	Position (m)
1	345,780.70	150,394.03	± 0.043	± 0.043	± 0.061
2	350,044.25	150,752.64	± 0.043	± 0.029	± 0.052
3	356,788.64	144,328.21	± 0.005	± 0.012	± 0.013
4	351,240.17	138,628.77	± 0.045	± 0.031	± 0.055
5	351,629.08	144,899.04	± 0.014	± 0.020	± 0.024

The complete listing of the computer output is given in Appendix I.

VI. THEORY OF THE VERTICAL POSITION CONTROL

The vertical is the line which a plumb line takes, Figure 13, when it is hung up freely under the affect of the gravitational attraction of the earth. A surface, everywhere at right angle to the vertical, is called the level surface. A reference surface, called

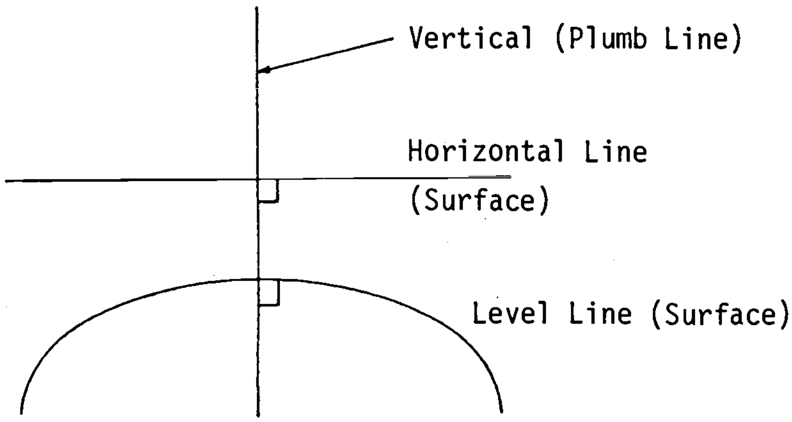


Figure 13. Reference Surface.

datum, is any level surface, chosen according to the conveniency. The datum that coincides with the mean sea level surface is known as the mean sea level surface (MSL) datum. The mean sea level surface is determined from the observations on the tide gauges over a period of years. The MSL surface is not a regular simple figure and computation on its surface is difficult. To ease the computational work, the MSL surface is approximated by an oblate spheroid, which is the figure obtained by revolving an ellipse about its minor axis. In surveying, the linear distance, up or down the vertical; from the datum to a point is known as the elevation, or the altitude, or the height of the point above or below the datum.

There are several techniques for determining elevation. The classical or the most frequently used are the direct leveling and the trigonometric leveling. For conveniency, direct leveling, in this paper, will be called leveling, and the trigonometric leveling as trig leveling. Each of these techniques will be discussed. Also discussed is the gravimetric leveling, which is applicable only to the highest order of leveling work. A brief discussion will also be presented on the non-classical techniques of the vertical position control, which includes: barometric leveling, hydrostatic leveling, stadia (tacheometric) leveling, satellite altimetry, and steric leveling.

A. Direct Leveling

Direct leveling employs the use of a spirit or automatic level which establishes a near horizontal line of sight, Figure 14. The elevation difference between two points are determined by taking the difference between two consecutively sighted horizontal line of sights to two uniformly marked vertical rods held on the two points. If R_A and R_B are the two rod readings at the points A and B respectively, Figure 14, when the level is at position (1), the elevation difference ΔH_{AB}, between the points A and B is given by;

$$\Delta H_{AB} = R_A - R_B \tag{6.01}$$

The process is continued to obtain the successive elevation differences between the two concucative points until the point X, whose

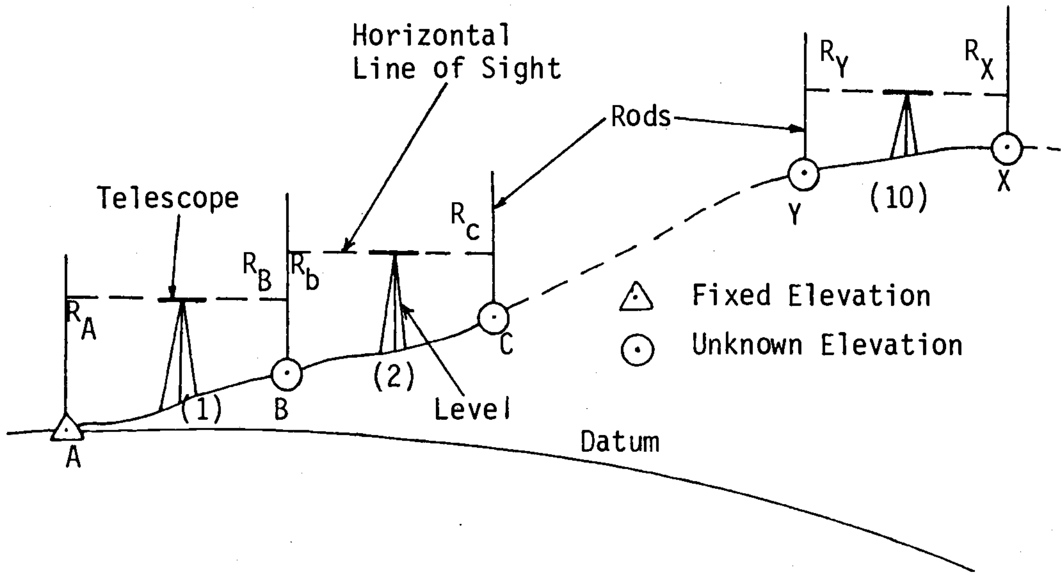


Figure 14. Direct Leveling

elevation is desired, is reached. All these elevation differences are algebraically added to obtain the elevation difference between the two points. In Figure 14, if A is the fixed elevation station, the elevation of the point X can then be obtained from;

$$\text{Elevation X} = \text{Elevation A} + (R_A - R_B) + (R_b - R_c) + \ldots$$
$$\ldots\ldots + (R_Y - R_X) \tag{6.02}$$

where R_b, R_c, R_X, and R_Y are the rod readings shown in the Figure 14 at different points.

B. Trig (Trigonometric) Leveling

Trig leveling is the technique used to determine the elevation difference between two points by measuring the vertical angle from an instrument over one point to a rod over the second point. To compute the elevation difference, either the horizontal or the slant distance must be known.

In Figure 15, the elevation difference, QR, between the two points P and Q is given by;

$$QR = D\tan\theta \qquad (6.03)$$

where D is the horizontal distance and θ is the verticle angle between the two points; θ is determined by a transit or a theodolite. If the slant distance S has been measured instead of the horizontal distance, the same elevation difference is given by;

$$QR = S\sin\theta \qquad (6.04)$$

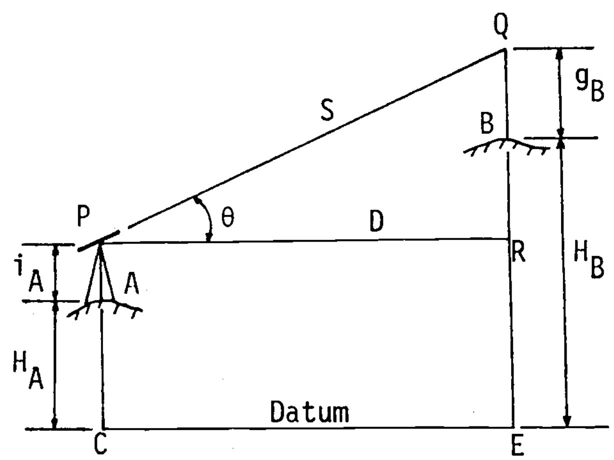


Figure 15. Trig Leveling.

In Figure 15, H_A and H_B are the elevations of the points A and B respectively, which in most surveying work, are considered linear elevations; i_A and g_B are the height of the instrument (HI) and height of the target (HT) respectively. All these quantities are related by the expression

$$H_A + i_A + D\tan\theta = H_B + g_B$$

which gives

$$H_B - H_A = D\tan\theta + i_A - g_B \tag{6.05}$$

If the slant distance S, is measured, then;

$$H_B - H_A = S\sin\theta + i_A - g_B \tag{6.06}$$

These are the basic elevation difference expressions between the two points as determined by the trig leveling technique. These equations are true for all values provided the following conventions are followed:

(i) Height above a station is positive and below it, is negative.

(ii) $H_B - H_A$ is defined as the elevation difference between A and B; from A to B.

(iii) The angle θ is positive for the elevation angle, and negative for the depression angle.

From the equations (6.05) and (6.06), the elevation difference between the two points can be computed if the vertical angle and, either the horizontal or the slant distance are measured. However, the measured vertical angle θ, must be corrected for the effects of the atmospheric refraction and the curvature of the earth, if the greater precision in the elevation difference is desired from the trig leveling.

Atmospheric Refraction

In Figure 16a, θ is the vertical angle at A between the horizontal at A and the chord AB. Due to the atmospheric refraction, the measured vertical angle is α. Therefore, a correction β, must be applied to to obtain θ, such that

$$\theta = \alpha - \beta \qquad (6.07)$$

In Figure 16a,

R = mean radius of curvature of the earth.

λ = angle between the two R's at A and B.

$\gamma = \lambda/2$

A' and B' are the projection of A and B respectively along the verticals to the datum.

If the slant path is assumed to be circular and if σ is the radius of the path, from Figure 16b,

$$\text{arc } AB = 2\sigma\beta \qquad (6.08)$$

From Figure 16a, $$\text{arc } A'B' = R\lambda \qquad (6.09)$$

and since $A'B' \doteq AB$

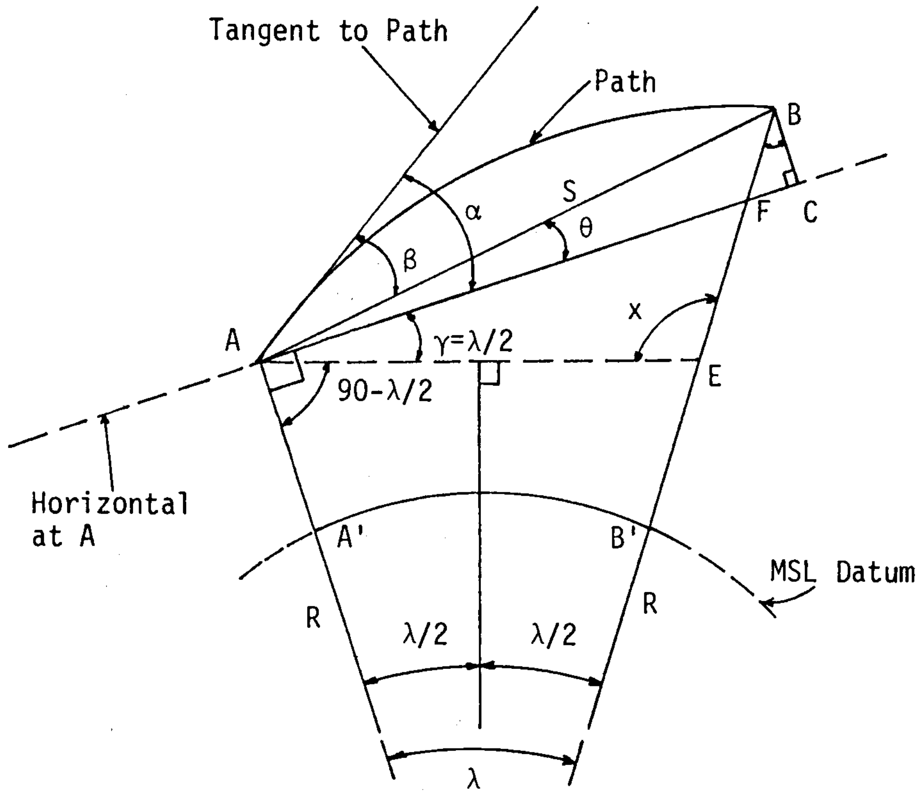
$$\text{arc } A'B' \doteq \text{arc } AB = R\lambda$$

Substituting the arc AB in the equation (6.08)

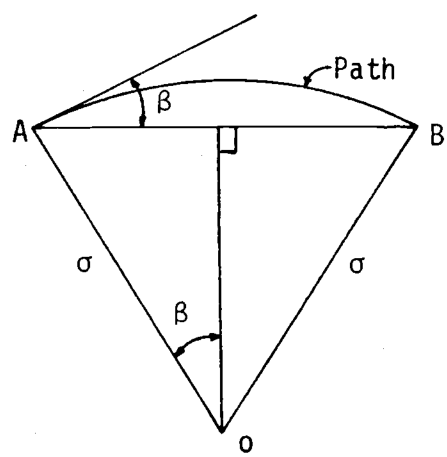
$$R\lambda \doteq 2\sigma\beta$$

Therefore, $$\beta \doteq \frac{R\lambda}{2\sigma} \qquad (6.10)$$

From Figure 16a, $$R\lambda \doteq S$$



(a)



(b)

Figure 16. Refraction and Curvature of Earth.

Therefore, substituting for $R\lambda$ in the equation (6.10) gives

$$\beta \doteq \frac{S}{2\sigma} \tag{6.11}$$

where R is defined as [18];

$$R = (\rho\nu)^{1/2} \tag{6.12}$$

such that

$$\rho = \frac{a(1 - e^2)}{(1 - e^2\sin^2\phi)^{3/2}} \tag{6.13}$$

$$\nu = \frac{a}{(1 - e^2\sin^2\phi)^{1/2}} \tag{6.14}$$

where ϕ is the latitude of the point A.

a is the semi-major radius of the spheroid used.

e is the eccentricity of the spheroid.

For a spheroid, a and e are fixed quantities. In this paper, the Clarke spheroid of 1866 has been taken for the computations and they are:

a = 6,378,206.4 meters, and e = 0.0067 686 579 79291

It is also defined on page 97 [17] that

$$\kappa = \frac{R}{2\sigma} \tag{6.15}$$

where κ is called the coefficient of refraction. Substituting for $\sigma = \frac{R}{2\kappa}$ from equation (6.15) into the equation (6.11) gives

$$\beta \doteq \frac{\kappa S}{R} \text{ radians} \tag{6.16}$$

For visible light, it is shown in [17] that $\sigma \doteq 43 \times 10^6$ meters. Taking the mean radius of curvature $R = 6.36 \times 10^6$ m, the value for the value for the coefficient of refraction κ is given by

$$\kappa = \frac{R}{2\sigma} \doteq 0.07 \tag{6.17}$$

For the case of microwave, which may be used in some EDM measurements, it is shown in [17] that the value of κ varies between 0.1 and 1 with the value of 0.125 taken as the standard. These values of κ, 0.07 for the visible light waves, and 0.125 for the microwaves, are used later in this paper in the computer program VCONTRL.

Thus, from the equation(6.16), the value of β can be computed, if S and R are known. This value of β is then subtracted to obtain the vertical angle corrected for the atmospheric refraction, and is given by the equation (6.07).

Curvature of the Earth

In the derivation of the equations (6.05) and (6.06), the datum CE, Figure 15, has been assumed as straight, but it is actually curved and shown as A'B' in Figure 16a. The elevation difference obtained from (6.05) or (6.06), gives the elevation difference above a flat earth. The effect of the curvature of the earth must be corrected if the better elevation difference is desired. This effect is shown as γ in Figure 16a.

At point E, $x = 90 + \lambda/2$ (6.18)

Applying the sine rule to the parts of the triangle ABE;

$$\frac{BE}{\sin(\theta + \lambda/2)} = \frac{S}{\sin x}$$

Substituting for x from the equation (6.18)

$$\frac{BE}{\sin(\theta + \lambda/2)} = \frac{S}{\sin(90 + \lambda/2)}$$

or,

$$BE = S \cdot \frac{\sin(\theta + \lambda/2)}{\cos(\lambda/2)}$$

Therefore, $$BE = S \cdot \sin\,(\theta + \lambda/2) \cdot \sec(\lambda/2) \qquad (6.19)$$

Since $\lambda/2$ is very small for usual work, $\sec(\lambda/2)$ can be considered as unity. Then the effect of the curvature of the earth is accounted by adding Υ to θ, where

$$\Upsilon = \lambda/2$$

Substituting for $\lambda = (\text{arc}A'B')/R$ from the equation (6.09)

$$\gamma \doteq \frac{\text{arc } A'B'}{2R}$$

From Figure 16a, assuming arc $A'B' \doteq S$, therefore,

$$\Upsilon \doteq \frac{S}{2R} \qquad (6.20)$$

The refraction and the curvature corrections are often combined together so that the corrected vertical angle θ is given by

$$\theta = \alpha - \beta + \gamma \qquad (6.21)$$

where α is the measured vertical angle. Substituting for β and γ from the equations (6.16) and (6.20) respectively, gives

$$\theta = \alpha - \frac{\kappa S}{R} + \frac{S}{2R}$$

or, $$\theta = \alpha + \left(\frac{S}{2R} - \frac{\kappa S}{R}\right)$$

Therefore, $$\theta = \alpha + \frac{S}{2R}(1 - 2\kappa) \qquad (6.22)$$

In this paper, this equation has been used to compute the corrected vertical angle, and thus the elevation difference from the equations (6.05) or (6.06). The effects of the refraction and the curvature for the various distances are tabulated in Table 2, by computing the value of $\frac{S}{2R}(1 - 2\kappa)$. The value of R is taken as 6,378,206.4 m which is the semi-major axis of the Clarke spheroid of 1866.

Table 2. Effects of Refraction and Curvature.

Distance	Coefficient of Refraction (κ)	
meters	0.07	0.125
100	00' 01.4"	00' 01.2"
500	00 07.0	00 06.1
1,000	00 13.9	00 12.1
2,000	00 27.8	00 24.3
5,000	01 09.5	01 00.6
10,000	02 19.1	02 01.3
50,000	11 35.3	10 06.4
100,000	23 10.6	20 12.7

C. Gravimetric Leveling

It should be realized that the gravimetric leveling is usually only applied to the first order leveling where the misclosure of a leveling loop, according to the National Geodetic Survey (NGS) specifications [34], must be within ± 3mm $\sqrt{K}$; where K is the single direction path distance of the level line in kilometers. Elevation of a point is the amount that the point is above or below the mean sea level (MSL) surface. Mathematically, the MSL surface is a surface of equal gravity potential and it is conventionally known as the Geoid.

Due to the: rotation of the earth around the sun as well as about its own rotational axis; the transcendental attractions of the sun, the moon, and other planets on the earth; differing densities of the earth at various depths underneath the surface; the Earth's gravitation produces a very complicated Geoid. The Geoid is not spherical in shape with a longer radius at the equator than at the pole. Consequently, gravity at the pole is greater than at the equator. The equal gravity potential surface or the equipotential surface, is closely approximated by a spheroid, sometimes referred to the best fitting ellipsoid. An ellipse revolved around the minor axis produces a mathematical figure called an oblate ellipsoid. Fitting an oblate ellipsoid or spheroid to the earth eases the computational work involved in the horizontal and the vertical control positions.

In leveling, elevation of a point can be a linear distance of the point above or below the geoid measured along the vertical. This elevation or height is called an Orthometric height and is the value normally quoted in the literature.

Due to the different gravity values from the equator to the pole, the equipotential surfaces are not parallel, but they are closer together at the pole than at the equator, Figure 17. It should be

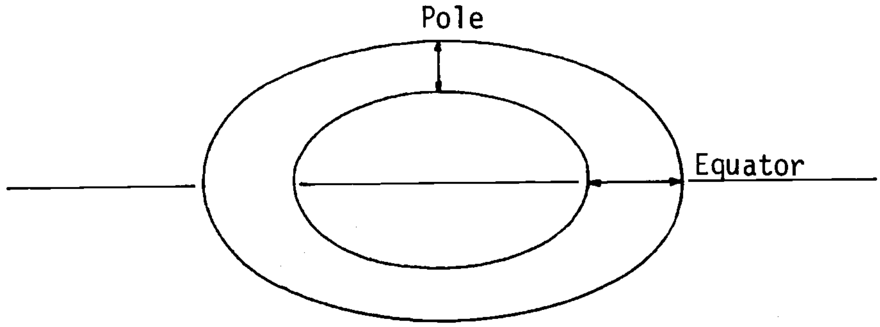


Figure 17. Equipotential Gravity Surfaces.

realized that nonparallelism of the equipotential or the level surfaces is significant only in the north-south direction due to the varying gravity along this direction. Therefore, leveling in the north-south direction over a long distance will be in error in otherwise errorless leveling. This is clearly seen in Figure 18., A and B are two points at sea level while a and b are vertically above them

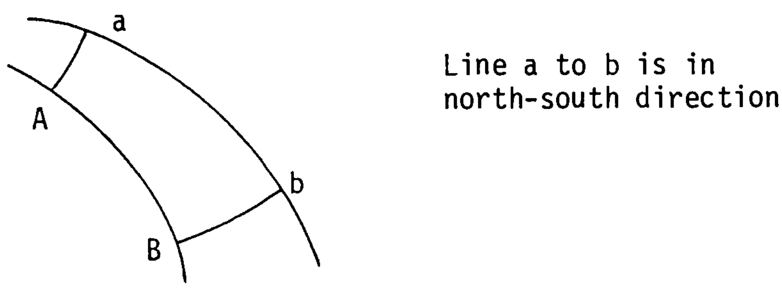


Figure 18. Non-parallelism of Equipotential Gravity Surfaces.

at another equipotential surface. Note that the linear distances Aa and Bb are not straight lines, but curved. This is due to the fact that these vertical lines have to be perpendicular to all the infinite equipotential surfaces between A and a (B and b). It is clearly seen Aa is not equal to Bb. Thus the determination of an elevation of a point may be dependent upon the leveling route taken.

From the previous vertical control techniques described, only the route dependent elevations are obtained. To make them route independent, what is known as geopotential numbers are used. A geopotential number is the product of the observed gravity, in Kilogals, at the point and elevation of the point in Meters. Geopotential numbers (GPU) are measured in geopotential units (gpu) such that 1 gal = 1 cm/sec^2 and 1 gpu = 1000 gal meters. Additional information may be found in [14] and [11]. In the most simple form GPU is given by

$$\text{GPU} = gh \tag{6.23}$$

where g is the observed gravity at the point in kgals and h is the elevation of the point in meters.

A numerical example of a GPU computation is given on page 140. Table 3 gives the GPU of the five stations extracted from Table 5, Appendix II.

Since 1 gpu = 1 kgals-meter, and the average gravity value is 980 gals which is equal to 0.98 kgals, gpu values will always be less than the orthometric height (elevation).

Table 3. Surface Gravities, Station Elevations, and Geopotential Numbers

Station	Surface Gravity (gals)	Elevation (meter)	Geopotential Number (gpu) kgals-meter
Corvallis OSU-PC	980.573 149	77.142	75.643
U-54	980.573 253	76.787	75.295
τβπ	980.575 467	71.337	69.951
Corvallis OSU-KL	980.573 844	73.336	71.911
College	980.573 302	73.219	70.816

D. Non-Classical Techniques

It is just meant to show in this section that there exists other techniques of vertical position control. Only a brief discussion will be presented here. The non-classical technique of the vertical position control include:

1. Barometric Leveling
2. Hydrostatic Leveling
3. Stadia (Tacheometric) Leveling
4. Satellite Altimetry
5. Steric Leveling

Barometric Leveling

The atmospheric pressure (P), at a point is a function of the elevation of the point (H), gravity (g), and the density of air (ρ) [2]. That is

$$P = f(g, \rho, H) \tag{6.24}$$

If g and ρ at the two points are constant, then it is shown on page 388,[2] that

$$\frac{H_1}{H_2} = k \log_e \left(\frac{P_1}{P_2}\right) \tag{6.25}$$

where H_1, H_2 and P_1, P_2 are the corresponding elevations and pressures respectively, and k is a constant. Thus, if one elevation is known, other elevations can be computed from the known pressures at the two points. The pressure is measured by the instrument known as a Barometer [13].

Hydrostatic Leveling

If a U-shaped tube is partially filled with water, the elevations of the two water level surfaces at the two limbs will be equal due to the balance of water set by the force of gravity. This idea is used in hydrostatic leveling by laying and filling water into long flexible pipes, from a known elevation point. Pipes as long as 10 km. have been used to determine the elevations by this technique [2]. The technique is useful where the elevation differences are very small and has been much used in Holland.

Stadia (Tacheometric) Leveling

The stadia or tacheometric leveling is based on the measurements of the vertical angle and the horizontal distance. Though this technique looks like the trig leveling, the later is more precise than the previous. For detail information on stadia leveling, see [2] and [13].

A transit or a theodolite provides the top and bottom cross wires besides the central cross wire in the raticule. The transit or the theodolite is sighted at a near vertical rod, and the rod readings at the top and the bottom cross wires, S_T and S_B respectively are recorded, Figure 19. If α is the vertical angle, the vertical distance

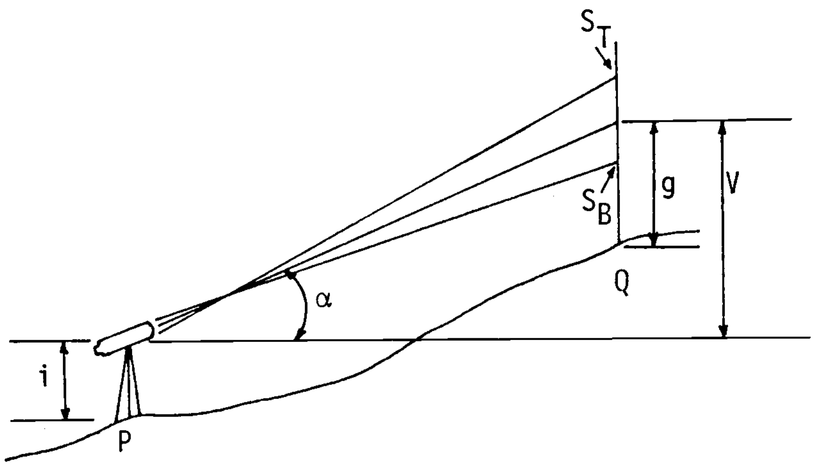


Figure 19. Stadia Leveling.

V, as shown in the figure, can be mathematically shown as [13];

$$V = C(S_T - S_B)\ (\tfrac{1}{2}\sin 2\alpha) \tag{6.26}$$

where C is an instrument constant, and usually made equal to 100. In Figure 19;

i = Height of the instrument (HI)

g = Rod reading

H_P = Elevation of the point P

H_Q = Elevation of the point Q.

Then it can be shown [13] that;

$$H_Q - H_P = i + V - g \tag{6.27}$$

If the elevation of one point is known, the elevation of the other point can then be computed. This equation very much looks like the equation (6.05). The difference is the computation of V. The adjustment by least squares technique can then be applied in the similar manner as described in the trig leveling adjustment.

Satellite Altimetry

The determination of the X and Y co-ordinates from the artificial satellites have been briefly described in the section Future Systems in chapter III. During the horizontal position (X,Y) fix, as described in chapter III, from the satellites, the Z co-ordinate or the elevation, is also computed applying the adjustment of least squares technique described in the chapter. For more information, see [27].

Steric Leveling

The technique of the steric leveling is applied by the oceanographers to find the elevations of points on the ocean surface. It is meant to indicate here that the technique of the steric leveling exists. It is out of the scope of this paper to describe the procedures of the steric leveling. The technique involves the measurements of: density, pressure, salinity of sea water, temperature, etc. at different points on the ocean surface. For more information, see [19].

VII. DEVELOPMENT OF THE COMPUTER PROGRAM FOR THE VERTICAL POSITION CONTROL

A computer program called VCONTRL has been developed for adjusting the classically obtained vertical control positions by the least squares technique. The classical obtained vertical control means either the direct leveling or the trig leveling. The program is written in Fortran IV language.

A. Program VCONTRL

This program employs the observation equation technique of the least squares adjustment, where one observation equation, given by the equations(6.01) for the direct leveling, or the equation (6.05) or (6.06) for trig leveling, for each measurement is written. This is accomplished by the computer by assigning the coefficients of the parameters as the elements, $b(i,j)$; $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, u$, of the coefficient matrix $\underline{B}$. The parameters $\underline{\Delta}$, are the elevations of the unknown stations, and can be represented by

$$\underline{\Delta} = \begin{vmatrix} H_1 \\ H_2 \\ \cdot \\ \cdot \\ H_u \end{vmatrix} \qquad (7.01)$$

where H_m; m = 1, 2, ..., u, is the elevation of the m^{th} unknown elevation station. The formation of the elements of the matrix $\underline{B}$ is illustrated as following. The elements of $\underline{B}$, are represented according to the numbers of the elevation stations and the order of the elevation difference, put into the computer as the observation. For example, if the elevation difference between the two elevation stations I and J is ΔH_{IJ}, then

$$H_J - H_I = \Delta H_{IJ} \tag{7.02}$$

where I and J must be numerals. If this observation is put into the computer as the i^{th} observation, then the VCONTRL assigns the coefficients of the parameters H_I and H_J in

$$b(i,I) = -1, \; b(i,J) = 1 \tag{7.03}$$

The right hand side number ΔH_{IJ}, will be represented by the element of the right hand side matrix $\underline{f}$ as

$$f(i) = \Delta H_{IJ} \tag{7.04}$$

If there are any additional stations besides I and J, the coefficients of all the other parameters will be assigned to zero. Note that all the elements of the matrix $\underline{B}$ will be either 1, -1, or 0. For a numerical example, Figure 20, if the elevation difference of +16.201 m occuring between the stations 3 and 1 is put into the computer as the fourth observation, then

$$i = 4,\; I = 3,\; J = 1, \text{ and } \Delta H_{IJ} = +16.201 \text{ m.}$$

which gives the elements of $\underline{B}$ and $\underline{f}$ as

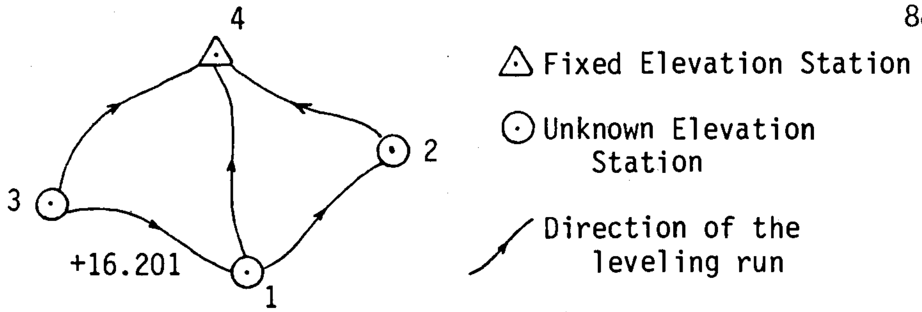


Figure 20. Sample Level Net.

b(4,3) = -1, b(4,1) = 1, and f(4) = 16.201;

Also: b(4,2) = b(4,4) = 0

When the observation is put into the computer, its standard error is also inputed into the computer at the same time. VCONTRL converts this to the element of the column weight matrix $\underline{w}$. If, for the above example, the standard error of the fourth observation is 10 mm, then the fourth element of $\underline{w}$ will be given by

$$w(4) = 1/(0.010)^2$$

Note in the above equation that input units of the elevation difference and the standard error are, and must be the same. For each input observation, the process is repeated and the matrices $\underline{B}$, $\underline{w}$, and $\underline{f}$ are formed. The least squares adjustment is then performed according to the equation (3.37) which yields the parameter $\underline{\Delta}$, given by the equation (7.01). Adjusted, elevation differences, standard errors, and all the relevant quantities are then printed out. A listing of such an adjustment is given in Appendix II.

The adjustment procedures for the trig leveling is almost the same as in the direct leveling technique. The difference here is that the elements of the right hand side matrix $\underline{f}$ are computed either from the equation (6.05), or the equation (6.06), which are respectively given by

$$H_B - H_A = D\tan\theta + i_A - g_B$$

and

$$H_B - H_A = S\sin\theta + i_A - g_B$$

where θ is given by the equation (6.22). The computation of θ requires that ϕ, a, e, R, κ, etc be specified.

There is, however, one difference between the two adjustment procedures, and that is in the computation of the elements of the weight matrix $\underline{w}$. Since the trig leveling is a function of two observed quantities, vertical angle and the distance in the same observation equation, given by the equation (6.05) or (6.06); as opposed to two observation equations for the horizontal position, given by the equations (3.11) and (3.27), for the distance and the angle respectively, the resultant standard error for an observation of the trig leveling will be a function of the standard errors of the vertical angle and the distance. The technique for computing the weight of trig leveling is described in the section trig leveling, chapter VIII. Standard errors of the vertical angle and the distance are converted by VCONTRL by the equation (8.06) to obtain the weight of the trig leveling observation.

The general deck set up and the flow chart for VCONTRL are given in Figures 21 and 22 respectively.

To run the program VCONTRL the following conditions must be satisfied.

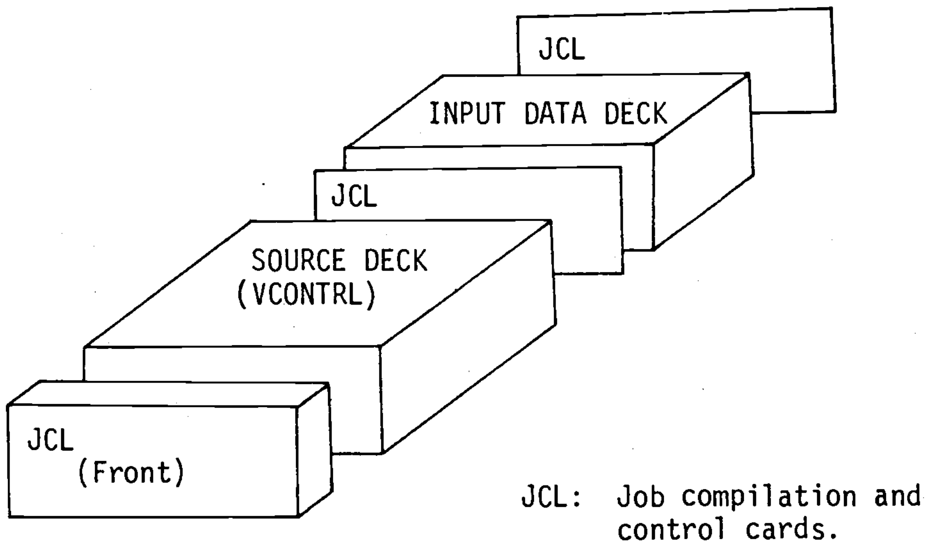


Figure 21. Deck Set Up for VCONTRL

1. The maximum number of observations must not exceed 100.
2. The maximum number of fixed elevation stations must not exceed 10.
3. A maximum number of 90 unknown elevation stations can be adjusted.
4. At least one fixed elevation must be given.

To run the program, number the unknown elevation stations first as 1, 2, ..., u. Then number the fixed elevation station(s) as u+1, u+2, ..., M; where u and M must be numerals. No character is is allowed. Any station number must not exceed two digits. Then, set up the free form input data in the following order.

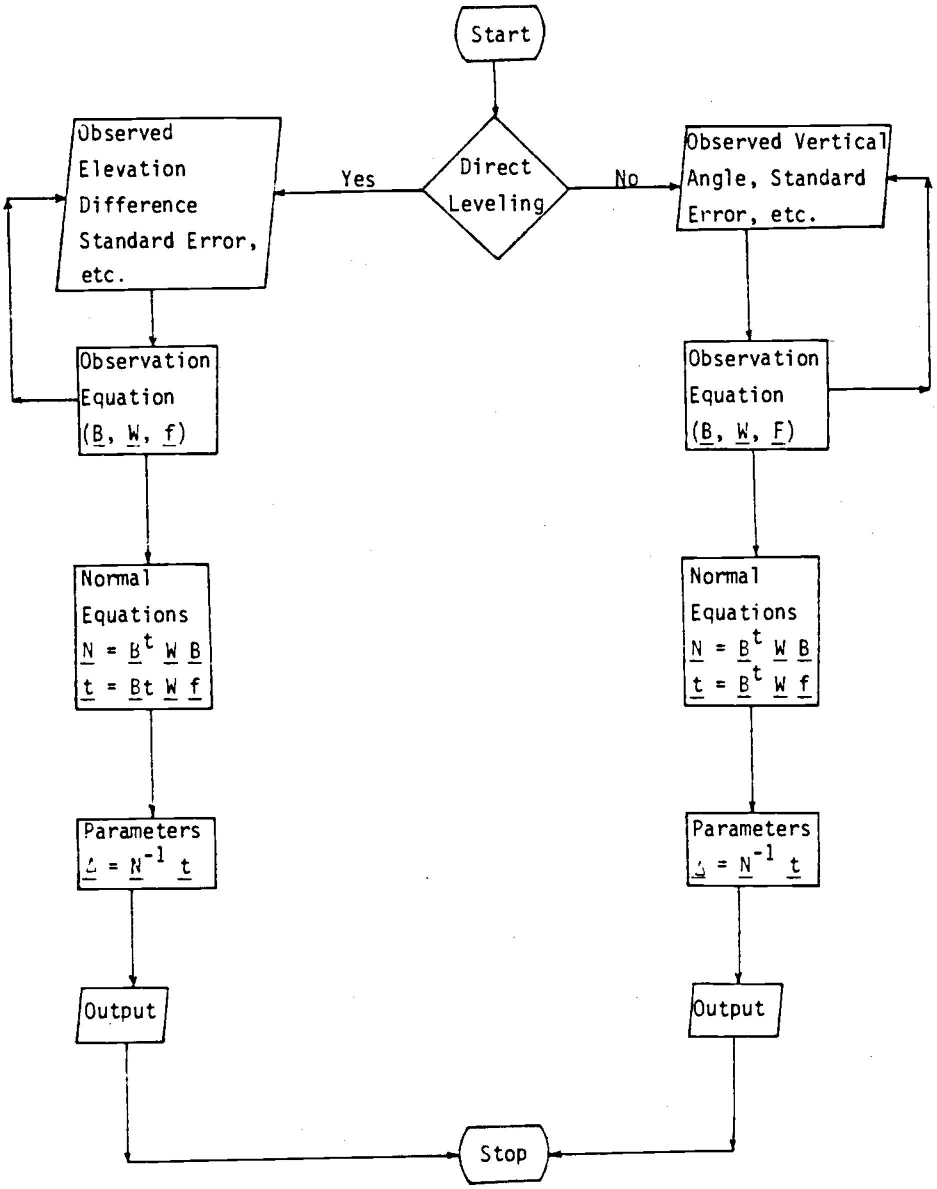


Figure 22. Flow Chart for VCONTRL.

Step/Rule	Input Quantity
1.	Technique identifier, 1 for direct leveling, 0 for trig leveling.
2.	Project title, up to 75 characters (letters, numerals, or both).
	(If 1 in step 1, input the following; otherwise go to step 8).
3.	Number of observed direct level line(s), number of fixed elevation station(s), and number of unknown elevation station(s).
4.	Fixed elevation(s).
5.	Repeat step 4 for all the fixed elevation(s) in order of the fixed elevation station numbers.
6.	The observed elevation difference from elevation station number I to the elevation station number J (a negative value is entered as minus), standard error of the observed elevation difference in the same unit as the elevation difference, station number I, and station number J. For example, in Figure 20, for the observed elevation difference of +16.201 m from the station 3 to the station 1, with the standard error of ± 0.010 m (± 10 mm), the input will be 16.201 0.01 3 1

Step/Rule	Input Quantity
7.	Repeat step 6 for all the observed elevation difference(s). (If 1 in step 1, input ends here).
8.	Approximate average latitude of the project area in degrees, minutes and seconds (or any decimal part of the second); e.g. latitude of 45° 40' 33" will be inputed as 45 40 33 (or 33.00).
9.	Number of observed trig leveling line(s), number of unknown elevation station(s), number of fixed elevation station(s); 1 if the input measured distance(s) is/are slant distance(s), 0 if horizontal, 1 if the distance(s) is/are measured with the visible light wave system EDM equipment, 0 for the microwave.
10.	Unit identifier, 1 for feet, 0 for meters.
11.	Fixed elevation in the same unit as the measured distance(s).
12.	Repeat step 11 for all the fixed elevation(s) in the order of the numbered fixed elevation station(s).
13.	The observed vertical angle in seconds (or any decimal part of second) as positive upward and negative downward, its standard error in seconds (any decimal part), measured distance, standard error of the measured distance in the same unit as

Step/Rule	Input Quantity
	the measured distance, height of the instrument, height of the target (HI and HT must be in the same unit as the measured distance), instrument station number, and target station number.
	Example: In Figure 20, if the instrument is at station 2, a vertical angle of -2^{o} 18' 10.0" is measured by sighting instrument to the target at station 3, the height of the instrument HI = 1.72 m, the target height HT = 4.85 m, the measured horizontal distance = 6,286.342 m, the standard error of the observed vertical angle and distance are $\pm$ 10.0" and ± 0.001 m (± 1 mm), then the input for this observation would be
	-8290 (-8290.0) 10 6286.342 0.001 1.72 4.85 2 3
14.	Repeat step 13 for all the measured vertical angle.

Numerical example of the vertical position control least squares adjustment by the direct and trig leveling are given in Appendix II.

VIII. APPLICATION OF LEAST SQUARES SOLUTION FOR THE VERTICAL POSITION CONTROL TECHNIQUES

The least squares solution for the vertical position control is obtained in the same way as described in the horizontal position control by the equation (3.37). The difference here is that the observation equation is already linear, given by the equations (6.01), and (6.05) or (6.06) for the direct and trig leveling respectively. Therefore, it is relatively simpler to form the observation equation in the vertical position control in comparison to the horizontal position control.

A. Direct Leveling

For each observation of the elevation difference between the two elevation stations, one observation equation of the form given by the equation (6.01) can be written. For all the observations, the observation equations can then be represented in the matrix form given by the equation (3.29a). Rewriting this equation

$$\underline{B}\,\underline{\Delta} = \underline{f} + \underline{v} \tag{8.01}$$

where the parameters $\underline{\Delta}$, here, are the unknown elevations of the elevation stations given by the equation (7.01), $\underline{B}$ is the coefficient matrix, $\underline{f}$ is the column right hand side matrix. It is important to realize that the elements of $\underline{B}$ will be different from those obtained in the horizontal position control. The elements of $\underline{B}$ in the vertical position control will be either 1, -1, or 0 as discussed in chapter VII, and the elements of $\underline{f}$ will be the observed elevation differences (ΔH_{IJ} from the equation (7.02) for the direct leveling, and the value of the right hand side of the equation (6.05) or (6.06) for the trig leveling). For the least squares adjustment of the equation (8.01), given by the equation (3.37), the weight matrix $\underline{W}$ is needed. In direct leveling, the elements of the weight matrix are usually taken as the reciprocal of the distance between the corresponding elevation stations of observation. This implies, according to the discussion in the section Weights, chapter II, that the standard error of the observation is proportional to the square root of the distance between the two stations. In other words, the precision of the elevation difference is proportional to $\sqrt{L}$, where L is the length of the level line. However, the standard error should be determined by the actual observation, if possible. One way to find this quantity is, by taking the repetitive measurements of one elevation difference between the two points and finding the standard deviation given by the equation (2.19). This may, then be taken as the representative standard deviation (error) for all the observations. Another way of finding the precision is, by utilizing the precision of the instrument provided by the instrument manufacturer.

A level net of Figure 23, has been adjusted by the least squares technique by VCONTRL as a numerical example. The observed elevation differences and the other data are tabulated below.

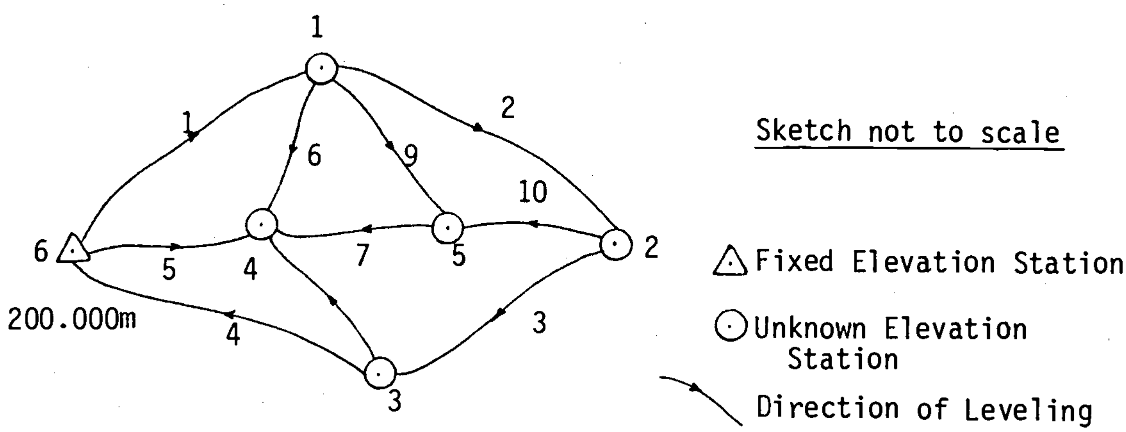


Figure 23. Level Net.

Level From	Line To	Observed Elevation Difference* (m)	Length of Line (L) (km)	Standard Error $\sqrt{L}$
6	1	+16.298	1.3	1.1402
1	2	-17.700	1.9	1.3784
2	3	- 0.687	2.3	1.5166
3	6	+ 2.086	2.7	1.6432
6	4	+23.615	0.3	0.5477
1	4	+ 7.304	0.9	0.9487
5	4	+14.162	1.2	1.0954
3	4	+25.709	1.6	1.2649
1	5	- 6.855	1.1	1.0488
2	5	+10.863	0.5	0.7071

*For computational purposes, these data are taken from [17].

These data have been adjusted by the computer program VCONTRL and the following adjusted elevations and their corresponding standard errors are obtained. The standard errors of the adjusted elevations are computed according to the discussion in the section Precision from Least Squares, chapter II.

Station	Adjusted Elevation (m)	Standard Error (m)
1	216.304	±0.0044
2	198.594	±0.0056
3	197.908	±0.0057
4	223.614	±0.0030
5	209.454	±0.0051

The complete listing of the computer is given in Appendix II.

B. Trig (Trigonometric) Leveling

The adjustment procedure is the same as in the direct leveling. One observation equation for each measured vertical angle is written. For the trig leveling, either the horizontal or the slant distance must be known. The observation equation is given, either by the equation (6.05) or the equation (6.06), depending whether the measured distance is the horizontal or the slant respectively. Note that in both equations, θ is given by the equation (6.22).

Setting up the observation equation in the trig leveling in the matrix form is the same as in the direct leveling. The solution, by the least squares technique, obtained by the equation (3.37), requires the weight matrix $\underline{W}$. Various methods for determining the elements of the weight matrix are discussed in the direct leveling, as well as in chapter II. However, since the observed elevation difference in trig leveling is the function of two measurements, the vertical angle and the distance, a separate technique for obtaining the elements of the weight matrix will be used. This technique is used because the trig leveling observation equation consists both of the measurements in the same observation equation. Rewriting equation (6.05)

$$H_B - H_A = D\tan\theta + i_A - g_B$$

Representing the elevation from the point A to the point B by

$$\Delta H_{AB} = H_B - H_A$$

$$\Delta H_{AB} = D\tan\theta + i_A - g_B \tag{8.02}$$

If the variances of the observed elevation difference, the measured horizontal distance, and the measured vertical angle are $\sigma^2_{\Delta H}$, σ^2_D, and σ^2_θ respectively; then, from the technique of the propagation of the variances, chapter IV, [26] it can be shown that

$$\sigma^2_{\Delta H} = \left(\frac{\partial(\Delta H_{AB})}{\partial D}\ \sigma_D\right)^2 + \left(\frac{\partial(\Delta H_{AB})}{\partial\theta}\ \sigma\theta\right)^2 \tag{8.03}$$

which gives

$$\sigma^2_{\Delta H} = \tan^2\theta\sigma^2_D + D^2\sec^4\theta\sigma^2_\theta \tag{8.04}$$

In this derivation, it has been assumed that there is no correlation between the distance (D) and the vertical angle (θ) measurements. It is further assumed that the both i_A and g_B are constants. Similarily, if the technique of the propagation is applied to the equation (6.06), it can be shown that

$$\sigma^2_{\Delta H} = \sin^2\theta\sigma^2_S + S^2\cos^2\theta\sigma^2_\theta \tag{8.05}$$

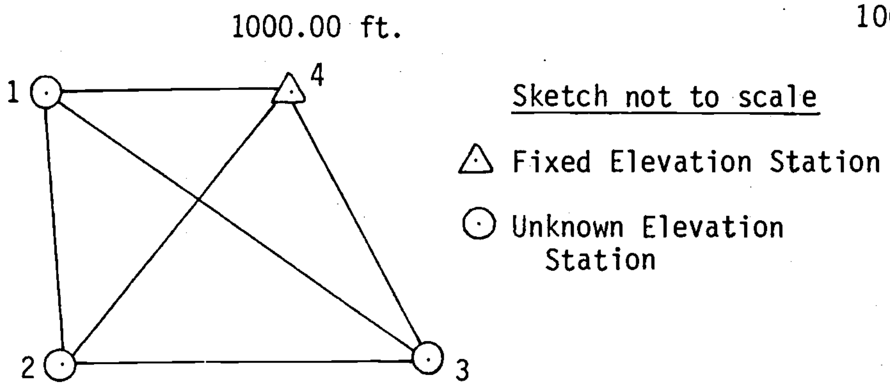


Figure 24. Trig Leveling Net.

Table 4. Extract from Field Book of Vertical Angle Observations**

At station 4	Peg	i = +4.96ft	g = 10.08 ft	Top
To 1	$-02^{o}38'00''$	Top pillar		
2	-02 17 22	Top skirt		
3	-01 39 24	Top pillar		
At station 1	Pillar	i = +0.68ft	g = 0	
To 4	$+02^{o}36'57''$	Top beacon		
2	-00 25 25	Top beacon		
3	+00 07 39	Top pillar		
At station 2	Peg	i = +4.80ft	10.42 Top beacon 9.24 Top skirt	
To 4	$+02^{o}16'27''$	Top beacon		
1	+00 23 48	Top pillar		
3	+00 48 46	Top pillar		
At station 3	Pillar	i = +0.83ft	g = 0	
To 4	$+01^{o}37'27''$	Top beacon		
1	-00 11 33	Top pillar		
2	-00 49 25	Top beacon		

**For computation checking purposes, these data are taken from [2].

where σ_S^2 is the variance of the measured slant distance S. Thus, from the given σ_θ and σ_D or σ_S, $\sigma_{\Delta H}$ can be determined from the equation (8.04), or (8.05). The determination of the quantities, σ_θ, σ_D, and σ_S have been discussed in the section Weights, chapter II. The computer program VCONTRL utilizes either the equation (8.04), or the equation (8.05) to determine the standard error fo the observed elevation difference by the trig leveling technique. The weight of an observation is then computed from

$$w = \frac{1}{\sigma_{\Delta H}^2} \tag{8.06}$$

Table 4 gives the data for the trig leveling net of Figure 24 for the least squares adjustment. The standard errors for all the vertical angles and the distance measurements are arbitrarily taken as ± 1" and ± 0.10 m respectively. The data are put into the computer for VCONTRL and the following adjusted elevations and the corresponding standard errors are obtained.

Station	Adjusted Elevation (ft)	Standard Error (ft)
1	363.851	±0.2006
2	234.586	±0.2154
3	437.433	±0.2291

The complete listing of the computer output is given in Appendix II.

IX. PRACTICALITY OF THE ADJUSTMENT

Since the introduction of the principle of the least squares by Legendre in 1806, the technique has been widely used in the various types of problems where there is redundancy in the observations. Since then, various computational techniques have been developed to ease the computation. In the past, it has not been an easy computation whatever the technique was used, especially in a project containing many observations. The introduction of high speed digital computers has removed most of the tedium of the computations. The continuous development in the digital computer capabilities, now and in the future, makes many kinds of complicated and difficult computational systems in mathematics routine work. Therefore, it looks like the adjustments of surveying in the future will be by the least squares technique, as more reliable and precise data are required with a precision estimate.

Currently, under U.S. Department of Commerce, the re-adjustment of the North American Geodetic Horizontal Control Network is in progress. When the project is completed in 1983, it will replace the present North American Datum of 1929 (NAD 29), and it will be called North American Datum of 1983 (NAD 83). The reason for obtaining NAD 83 is that NAD 29 is not precise enough for currently required precision in the horizontal and the vertical positions control computations and adjustments. The importance of redefining the North American Datum may be judged from the statement by [36]: "The precision required for today's scientific technology, plus a pressing need for accurate inventories of our Earth's resources, resulted in the creation of an international project to readjust the North American Geodetic Horizontal Control Network.... Many of these papers deal with computer software and have not been published previously. Computer specialists will find the information on utilizing computer capability to solve for as many as 400,000 unknowns fascinating and applicable to other fields..." The NAD 83 will provide, the best MPV's of the horizontal and the vertical positions from the most currently available data, adjusted by the least squares techniques; consequently, giving the most reliable values.

The computer programs, HCONTRL and VCONTRL, developed in this paper are for the adjustments of the horizontal and the vertical positions control respectively by the least squares technique when the distances and angles are obtained as the observations. The HCONTRL is capable of adjusting any classical techniques of horizontal position control such as; traverse, intersection, resection, triangulation, trilateration independently or the combination of any of these techniques which is specified here by combined networks. To use the program, one must have access to a computer, whose capability, in terms of storage space, must be able to store a matrix of the order of 104 by 100. The program VCONTRL, on the other hand, is capable of adjusting either the direct leveling or the trig leveling network. In this case, the maximum storage space needed in the computer is a matrix of the order of 100 by 100. These types of storage requirement should be able to be handled on the current mini-computers.

X. SUMMARY AND CONCLUSIONS

All of the adjustment techniques utilized in the least squares are based on the principle that the sum of the squares of the residuals is a minimum. Consequently, no one technique is better than the other, and whichever technique is adopted, the final adjusted quantities and the corresponding precisions are the same.

The observation equations technique provides the direct determination of the precisions from the covariance matrix of the adjusted quantities of any survey measurement. It is relatively easier to write observation equations than the condition equations in computer language. Though the condition equations technique, generally, requires lesser computer storage space than the observation equations technique, the availability of the computer storage space is no longer a problem.

Two computer programs, HCONTRL and VCONTRL, have been developed to adjust the respective classical horizontal and vertical positions control survey measurement data, by utilizing the least squares adjustment. The inputs for both of the programs are fairly simple. The basic input quantities are angle, distance, or both, and the control co-ordinates, for HCONTRL, and elevation difference, or vertical angle for VCONTRL. For both of the programs, the estimates of the precisions for the corresponding observations must also be inputed into the computer. The input data comes directly from the field work; consequently, no intermediate adjustment is necessary.

Any measurement or the adjustment is of little use if the precisions of the measurements or the adjusted values (MPV's) from the adjustment, are not known. These quantities are provided by the least squares adjustment techniques; a factor which makes the adjustment by the least squares so important. From the continued availability and the advancement of the digital computer, it is most likely that the adjustment of surveying data by the least squares techniques will probably be adopted in the future for most of the survey computations.

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XII. APPENDICES

APPENDIX I

A numerical example for each of the horizontal position control described in chapter IV will be adjusted by the computer program HCONTRL. All the necessary input data and the complete listing of the output will be given. The conditions, the requirements, and the procedures for running the program are discussed in chpater IV.

A. Traverse

For Figure 8, the following input data are obtained.

```
TRAVERSE BY LEAST SQUARES
2   3   5   4   0   0
163877.00       104590.00
164264.00       104865.00
164902.00       105120.00
163208.49       104375.29
165074.49       105227.47
006   41   42   3.0   4   5   1
162   37   21   3.0   1   4   2
193   18   06   3.0   2   1   3
170   08   49   3.0   3   2   5
007   14   20   3.0   5   3   4
703.28   0.01   1   4
473.29   0.01   1   2
687.48   0.01   2   3
202.31   0.01   3   5
```

Note that the input angles given at the stations 4 and 5, in Figure 8, are different from the given measured angles. For example, at station 4, the input angle is 06^{o} 41' 42" whereas the measured angle is 203^{o} 41' 28" (page 55). The reason for being different is that HCONTRL requires three station numbers for each input angle, and the approximate or fixed co-ordinates of all the input station numbers must be given. In Figure 8, however, only two station numbers 4 and 1 are given. So, from the fixed co-ordinates of the stations 4 and 5, the fixed azimuth α_1, and the measured angle of θ_1 = 203^{o} 41' 28", the included angle Ω_1 = 06^{o} 41' 42" (Figure 25), is computed. Similarly,

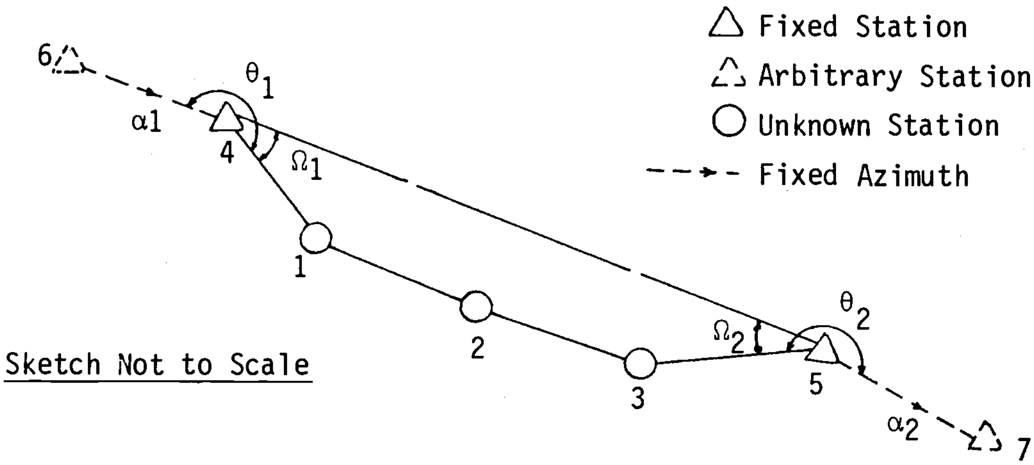

Figure 25. Sample Traverse Computation.

the input angle at station 5 has been computed. These input angles include the criterions of the fixed azimuths α_1 and α_2 in the traverse. Another way of inputing the angle 203^{o} 41' 42" as the observation, is by computing the co-ordinates of the station 6 (Figure 25) from the fixed azimuth α_1, and arbitrary distance d_1', and inputing the necessary data for the inputed angle of 203^{o} 41' 42". Whichever technique is used, the final answers from both will be the same.

The complete listing of the computer output by HCONTRL is given on the next page.

79/06/13. 11.02.01.S

```
TTTTTTT  RRRRRR    AAAAA   V     V  EEEEEEE  RRRRRR    SSSSS   EEEEEEE
   T     R     R  A     A  V     V  E        R     R  S     S  E
   T     R     R  A     A  V     V  E        R     R  S        E
   T     RRRRRR   A     A  V     V  EEEEEE   RRRRRR    SSSSS   EEEEEE
   T     R   R    AAAAAAA   V   V   E        R   R          S  E
   T     R    R   A     A    V V    E        R    R   S     S  E        ::
   T     R     R  A     A     V     EEEEEEE  R     R   SSSSS   EEEEEEE  ::
```

TRAVERSE BY LEAST SQUARES

APPROXIMATE CO-ORDINATES

STATION	X	Y
1	.1636770000E+06	.1045900000E+06
2	.1642640000E+06	.1048650000E+06
3	.1649020000E+06	.1051200000E+06

FIXED CO-ORDINATES

4	.1632084900E+06	.1043752900E+06
5	.1650744900E+06	.1052274700E+06

ITERATION NO. 1

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.979	.942
2	.642	-.962
3	.444	.860

ITERATION NO. 2

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.002	-.001
2	.000	.000
3	.000	-.002

ITERATION NO. 3

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	-.000	.000
2	-.000	-.000
3	-.000	-.000

COEFFICIENT MATRIX -B-

39.9116	-279.1306	0.0000	0.0000	0.0000	0.0000
-341.2859	635.0371	251.3744	-355.9005	0.0000	0.0000
251.3744	-355.9005	-363.4276	634.1858	112.0532	-278.2793
0.0000	0.0000	112.0532	-278.2793	-648.6490	1144.5388
0.0000	0.0000	0.0000	0.0000	536.7958	-866.2595
.9518	.3006	0.0000	0.0000	0.0000	0.0000
-.8106	-.5769	.8163	.5769	0.0000	0.0000
0.0000	0.0000	-.9276	-.3735	.9276	.3735
0.0000	0.0000	0.0000	0.0000	-.8500	-.5267

RIGHT HAND SIDE MATRIX -F-

```
13.915
 4.984
 4.517
-2.369
-3.047
 -.085
 -.090
 -.087
 -.090
```

STANDARD ERROR	RESIDUAL
.300E+01	-.139E+02
.300E+01	-.498E+01
.300E+01	-.452E+01
.300E+01	.237E+01
.300E+01	.305E+01
.100E-01	.854E-01
.100E-01	.902E-01
.100E-01	.873E-01
.100E-01	.899E-01

ADJUSTED ANGLES

ANGLE			DEG	MIN	SECS
4	5	1	6	41	28.30
1	4	2	162	37	16.02
2	1	3	193	13	1.48
3	2	5	170	8	51.37
5	3	4	7	14	22.83

LINE		ADJ.DISTANCE
1	4	703.365
1	2	473.380
2	3	687.557
3	5	202.400

STANDARD ERROR OF CO-ORDINATES

STATION	X	Y	POSITION
1	.883681E-01	.579080E-01	.111463E+00
2	.961208E-01	.787825E-01	.125835E+00
3	.794213E-01	.517326E-01	.947841E-01

ADJUSTED CO-ORDINATES

STATION	X	Y
1	.1638779804E+06	.1045909415E+06
2	.1642646419E+06	.1046640380E+06
3	.1649024444E+06	.1051208534E+06

B. Intersection

The input data corresponding to Figure 9, are:

```
INTERSECTION BY LEAST SQUARES
5   1   8   0   0   0
351625.00     144905.00
345780.67     150394.05
350044.25     150752.70
356442.71     148778.96
351240.22     138628.80
347490.50     145480.79
32   14   18.8   0.1   5   6   1
56   00   48.8   0.1   4   1   3
57   42   28.2   0.1   3   4   1
40   59   38.9   0.1   3   1   6
48   01   23.9   0.1   2   3   1
27   35   52.1   0.1   2   1   6
72   09   20.7   0.1   6   3   1
53   18   32.5   0.1   6   1   5
```

Complete listing of adjustment of these data by HCONTRL is given on the next page.

79/06/12. 11.06.29.S

INTER-
SECTION.

INTRESECTION BY LEAST SQUARES

APPROXIMATE CO-ORDINATES

STATION	X	Y
1	.3516250000E+06	.1449050000E+06

FIXED CO-ORDINATES

STATION	X	Y
2	.3457806700E+06	.1503940500E+06
3	.3500442500E+06	.1507527000E+06
4	.3564427100E+06	.1487789600E+06
5	.3512402200E+06	.1366288000E+06
6	.3474905000E+06	.1454807900E+06

ITERATION NO. 1

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	4.082	-5.949

ITERATION NO. 2

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.001	-.005

ITERATION NO. 3

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.000	-.000

COEFFICIENT MATRIX -B-

32.7698	-2.0323
20.9364	-25.9749
-32.8364	-8.8686
32.8364	8.8686
-17.6001	-18.7320
17.6001	18.7320
-6.8700	-48.8738
6.8700	48.8738

RIGHT HAND SIDE MATRIX -F-

1.084
1.151
1.916
-3.417
-2.044
4.923
.755
-.288

STANDARD ERROR	RESIDUAL
.100E+00	-.108E+01
.100E+00	-.115E+01
.100E+00	-.192E+01
.100E+00	.342E+01
.100E+00	.204E+01
.100E+00	-.492E+01
.100E+00	-.755E+00
.100E+00	.288E+00

ADJUSTED ANGLES

ANGLE			DEG	MIN	SECS
5	6	1	32	14	17.72
4	1	3	56	0	47.65
3	4	1	57	42	26.23
3	1	6	40	59	42.32
2	3	1	48	1	25.94
2	1	6	27	35	47.18
6	3	1	72	9	19.94
6	1	5	53	18	32.79

STANDARD ERROR OF CO-ORDINATES

STATION	X	Y	POSITION
1	.436068E-01	.363213E-01	.567520E-01

ADJUSTED CO-ORDINATES

STATION	X	Y
1	.3516290826E+06	.1448990462E+06

C. Resection

The input data corresponding to the Figure 10, are:

```
RESECTION BY LEAST SQUARE

6   1   6   0   0   0

351620.00     144905.00
350044.25     150752.70
356442.71     148778.96
356788.67     144328.27
351240.22     138628.80
347490.50     145480.79
345780.67     150394.05

94   27   06.5   0.1   1   5   6
35   12   51.4   0.1   1   6   7
31   38   05.6   0.1   1   7   2
66   16   48.9   0.1   1   2   3
45   10   57.2   0.1   1   3   4
87   14   09.4   0.1   1   4   5
```

The complete listing of the adjustment by HCONTRL is given on the next page.

79/07/13. 17.27.45.S

RESECTION..

RESECTION BY LEAST SQUARE

APPROXIMATE CO-ORDINATES

STATION	X	Y
1	.3516200000E+06	.1449050000E+06

FIXED CO-ORDINATES

2	.3500442500E+06	.1507527000E+06
3	.3564427100E+06	.1487789600E+06
4	.3567886700E+06	.1443282700E+06
5	.3512402200E+06	.1386288000E+06
6	.3474905000E+06	.1454807900E+06
7	.3457806700E+06	.1503940500E+06

ITERATION NO. 1

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	9.126	-5.914

ITERATION NO. 2

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	-.008	-.001

ITERATION NO. 3

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.000	-.000

COEFFICIENT MATRIX -B-

-39.6390	-46.8411
-10.7305	30.1414
-15.2306	9.8432
11.8939	34.8640
25.3059	13.5188
28.4002	-41.5263

RIGHT HAND SIDE MATRIX -F-

.242
-.128
-1.249
1.048
-.853
-.060

STANDARD ERROR	RESIDUAL
.100E+00	-.242E+00
.100E+00	.128E+00
.100E+00	.125E+01
.100E+00	-.105E+01
.100E+00	.853E+00
.100E+00	.596E-01

ADJUSTED ANGLES

ANGLE			DEG	MIN	SECS
1	5	6	94	27	6.25
1	6	7	35	12	51.53
1	7	2	31	38	6.85
1	2	3	66	16	47.85
1	3	4	45	10	58.05
1	4	5	87	14	9.46

STANDARD ERROR OF CO-ORDINATES

STATION	X	Y	POSITION
1	.160557E-01	.119578E-01	.200193E-01

ADJUSTED CO-ORDINATES

STATION	X	Y
1	.3516291181E+06	.1448990852E+06

D. Triangulation

The input data corresponding to the Figure 11, are:

```
ADJUSTMENT OF TRIANGULATION BY LEAST SQUARES

1   5   14   0   1   1

345780       150394
350044       150752
356442       148778
356788       144328
351629       144899
347490.50    145480.79

79   14   33.5   1   4   5   3
55   34   32.3   1   3   4   5
56   00   48.8   1   3   5   2
57   42   28.2   1   2   3   5
40   59   38.9   1   2   5   6
59   20   44.4   1   2   6   1
48   01   23.9   1   1   2   5
27   35   52.1   1   1   5   6
45   02   04.0   1   6   1   2
72   09   20.7   1   6   2   5
35   12   51.4   1   5   6   1
31   38   05.6   1   5   1   2
66   16   48.9   1   5   2   3
45   10   57.2   1   5   3   4
340  48   43.0   1   6   1
5202.273   0.05   6   1
```

The complete listing of the output by HCONTRL is given on the next page.

79/07/23. 13.17.02.S

TRIANGU-----
LATION..

ADJUSTMENT OF TRIANGULATION BY LEAST SQUARES

APPROXIMATE CO-ORDINATES

STATION	X	Y
1	.3457800000E+06	.1503940000E+06
2	.3500440000E+06	.1507520000E+06
3	.3564420000E+06	.1487780000E+06
4	.3567860000E+06	.1443280000E+06
5	.3516290000E+06	.1448990000E+06

FIXED CO-ORDINATES

6	.3474905000E+06	.1454807300E+06

ITERATION NO. 1

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	-.000	.000
2	-.330	1.195
3	.605	2.197
4	1.150	1.410
5	.295	.564

ITERATION NO. 2

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	-.000	.000
2	-.000	-.000
3	-.000	.000
4	-.000	.000
5	-.000	.000

COEFFICIENT MATRIX -B-

0.0000	0.0000	0.0000	0.0000	46.0642	3.5866
0.0000	0.0000	0.0000	0.0000	-25.1253	-29.5580
0.0000	0.0000	9.0760	29.4358	-30.0149	-3.4644
0.0000	0.0000	23.7523	-20.5433	9.0760	29.4358
0.0000	0.0000	-1.1380	-24.2385	0.0000	0.0000
-4.0409	48.0364	-27.6434	-32.6903	0.0000	0.0000
21.6440	-29.3027	-4.0469	48.0364	0.0000	0.0000
19.8403	-5.6980	0.0000	0.0000	0.0000	0.0000
-37.4434	-13.0357	31.6902	-15.3460	0.0000	0.0000
0.0000	0.0000	-31.6902	15.3460	0.0000	0.0000
17.5971	18.7336	0.0000	0.0000	0.0000	0.0000
-17.5971	-18.7336	32.8283	8.8925	0.0000	0.0000
0.0000	0.0000	-32.8283	-8.8925	20.9389	-25.9714
0.0000	0.0000	0.0000	0.0000	-20.9389	25.9714
37.4434	13.0357	0.0000	0.0000	0.0000	0.0000
-.3268	.9444	0.0000	0.0000	0.0000	0.0000

-41.7003	35.9061	-4.3639	-39.4927
46.0642	3.5865	-20.9389	25.9714
0.0000	0.0000	20.9389	-25.9714
0.0000	0.0000	-32.8283	-8.8925
0.0000	0.0000	32.8283	8.8925
0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	-17.5971	-18.7336
0.0000	0.0000	17.5971	18.7336
0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	-6.8634	-48.6731
0.0000	0.0000	-10.7337	30.1394
0.0000	0.0000	-15.2312	9.8411
0.0000	0.0000	11.8894	34.8639
-4.3639	-39.4927	25.3028	13.5213
0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000

RIGHT HAND SIDE MATRIX -F-

1.001
.997
1.971
1.968
-3.632
-.490
-1.729
4.510
2.109
1.507
.074
-1.349
1.961
1.002
0.000
0.000

STANDARD ERROR	RESIDUAL
.100E+01	-.100E+01
.100E+01	-.100E+01
.100E+01	-.197E+01
.100E+01	-.197E+01
.100E+01	.363E+01
.100E+01	.498E+00
.100E+01	.173E+01
.100E+01	-.451E+01
.100E+01	-.211E+01
.100E+01	-.150E+01
.100E+01	-.717E-01
.100E+01	.135E+01
.100E+01	-.197E+01
.100E+01	-.100E+01
.100E+01	-.626E-13
.500E-01	.530E-14

ADJUSTED ANGLES

ANGLE			DEG	MIN	SECS
4	5	3	79	14	32.50
3	4	5	55	34	31.30
3	5	2	56	0	46.83
2	3	5	57	42	26.23
2	5	6	40	59	42.53
2	6	1	59	20	44.89
1	2	5	48	1	25.63
1	5	6	27	35	47.59
6	1	2	45	2	1.89
6	2	5	72	9	19.19
5	6	1	35	12	51.33
5	1	2	31	36	6.95
5	2	3	66	16	46.94
5	3	4	45	10	56.20

STANDARD ERROR OF CO-ORDINATES

STATION	X	Y	POSITION
1	.914509E-01	.151466E+00	.176950E+00
2	.127367E+00	.176594E+00	.217733E+00
3	.311529E+00	.226322E+00	.385061E+00
4	.334179E+00	.209092E+00	.394202E+00
5	.153877E+00	.916465E-01	.179102E+00

ADJUSTED CO-ORDINATES

STATION	X	Y
1	.3457800000E+06	.1503940000E+06
2	.3500436696E+06	.1507531954E+06
3	.3564426050E+06	.1487801975E+06
4	.3567891500E+06	.1443294103E+06
5	.3516292946E+06	.1448995636E+06

E. Trilateration

The input data corresponding to the Figure 10 are:

```
ADJUSTMENT OF TRILATERATION BY LEAST SQUARES

6   1   0   6   0   0

351628.00      144897.00
350044.25      150752.70
356442.71      148778.96
356788.67      144328.27
351240.22      138628.80
347490.50      145480.79
345780.67      150394.05

6064.34   0.01   1   2
6182.65   0.01   1   3
5191.05   0.01   1   4
6282.32   0.01   1   5
4179.31   0.01   1   6
8024.87   0.01   1   7
```

The complete listing of the adjustment of these data by HCONTRL is given on the next page.

79/07/18. 13.08.47.S

```
TRILATE -----
RATION :
```

ADJUSTMENT OF TRILATERATION BY LEAST SQUARES

APPROXIMATE CO-ORDINATES

STATION	X	Y
1	.3516280000E+06	.1448970000E+06

FIXED CO-ORDINATES

STATION	X	Y
2	.3500442500E+06	.1507527000E+05
3	.3564427100E+06	.1487789500E+06
4	.3567986700E+06	.1443282700E+06
5	.3512402200E+06	.1396298000E+06
6	.3474905000E+06	.1454907900E+06
7	.3457806700E+05	.1503940500E+06

ITERATION NO. 1

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	1.084	2.074

ITERATION NO. 2

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	-.000	.000

COEFFICIENT MATRIX -B-

.2613	-.9652
-.7795	-.6275
-.9939	.1100
.0619	.9981
.9903	-.1392
.7288	-.6847

RIGHT HAND SIDE MATRIX -F-

```
-.035
 .052
-.014
-.000
 .043
-.011
```

STANDARD ERROR	RESIDUAL
.100E-01	.346E-01
.100E-01	-.521E-01
.100E-01	.143E-01
.100E-01	.196E-03
.100E-01	-.435E-01
.100E-01	.105E-01

LINE		ADJ.DISTANCE
1	2	6064.375
1	3	6182.598
1	4	5191.064
1	5	6282.320
1	6	4179.267
1	7	8024.881

STANDARD ERROR OF CO-ORDINATES

STATION	X	Y	POSITION
1	.221556E-01	.235432E-01	.323494E-01

ADJUSTED CO-ORDINATES

STATION	X	Y
1	.3516290837E+06	.1448990735E+06

F. Combined Networks

The input data for the Figure 12, are:

LEAST SQUARES ADJUSTMENT OF COMBINED NETWORKS

```
2    5    21    6    1    1

345780          150394
350044          150752
356788          144328
351240          138628
351629          144899
347490.50       145480.79
356442.71       148778.96

32   14   18.8   1   4   6   5
23   35   17.6   1   4   5   7
17   05   36.2   1   4   7   3
52   04   57.1   1   3   4   5
79   14   33.5   1   3   5   7
55   34   32.3   1   7   3   5
56   00   48.8   1   7   5   2
57   42   28.2   1   2   7   5
40   59   38.9   1   2   5   6
59   20   44.4   1   2   6   1
48   01   23.9   1   1   2   5
27   35   52.1   1   1   5   6
45   02   04.0   1   6   1   2
72   09   20.7   1   6   2   5
53   18   32.5   1   6   5   4
94   27   06.5   1   5   4   6
35   12   51.4   1   5   6   1
31   38   05.6   1   5   1   2
66   16   48.9   1   5   2   7
45   10   57.2   1   5   7   3
87   14   09.4   1   5   3   4

6282.32   0.01   4   5
5191.05   0.01   3   5
6182.65   0.01   7   5
6064.34   0.01   2   5
8024.87   0.01   1   5
4179.31   0.01   6   5

175   33   18.85   0.1   7   3

4464.116   0.005        7   3
```

The complete listing of the adjustment is given on the next page.

79/07/12. 13.50.51.S

COMBINED NETWORKS..

LEAST SQUARES ADJUSTMENT OF COMBINED NETWORKS

APPROXIMATE CO-ORDINATES

STATION	X	Y
1	.3457800000E+06	.1503940000E+06
2	.3500440000E+06	.1507520000E+06
3	.3567860000E+06	.1443280000E+06
4	.3512400000E+06	.1366280000E+06
5	.3516290000E+06	.1446990000E+06

FIXED CO-ORDINATES

6	.3474965000E+06	.1454307900E+06
7	.3564427100E+06	.1487789600E+06

ITERATION NO. 1

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.501	.141
2	.173	.792
3	.027	-.008
4	-.246	.936
5	-.147	.152

ITERATION NO. 2

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.011	-.006
2	.004	-.010
3	.026	-.006
4	.009	-.011
5	.010	-.009

ITERATION NO. 3

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.011	-.006
2	.004	-.009
3	.025	-.005
4	.009	-.010
5	.009	-.008

ITERATION NO. 4

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.010	-.005
2	.004	-.009
3	.024	-.003
4	.009	-.009
5	.009	-.008

ITERATION NO. 5

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.009	-.005
2	.004	-.008
3	.023	-.002
4	.010	-.009
5	.009	-.007

ITERATION NO. 6

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.009	-.005
2	.003	-.007
3	.022	-.001
4	.010	-.008
5	.008	-.006

ITERATION NO. 7

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.008	-.005
2	.003	-.007
3	.021	-.000
4	.010	-.007
5	.006	-.006

ITERATION NO. 8

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.008	-.004
2	.003	-.006
3	.020	.001
4	.010	-.007
5	.007	-.005

ITERATION NO. 9

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.007	-.004
2	.003	-.006
3	.013	.001
4	.010	-.006
5	.007	-.005

ITERATION NO.10

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.007	-.004
2	.003	-.005
3	.013	.002
4	.010	-.006
5	.007	-.005

. . . .

. . . .

ITERATION NO. 57

CO-ORDINATE VARIATIONS

STATION	DELTA X	DELTA Y
1	.000	-.000
2	.000	-.000
3	.001	.001
4	.001	-.000
5	.000	-.000

COEFFICIENT MATRIX -3-

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	18.5808	-18.0587
0.0000	0.0000	0.0000	0.0000	-22.9502	-21.4052
0.0000	0.0000	0.0000	0.0000	-41.6962	35.9134
0.0000	0.0000	0.0000	0.0000	46.0656	3.5804
0.0000	0.0000	9.0799	29.4359	0.0000	0.0000
0.0000	0.0000	23.7508	-20.5471	0.0000	0.0000
0.0000	0.0000	-1.1411	-24.2395	0.0000	0.0000
-4.0467	45.0368	-27.0469	-32.6879	0.0000	0.0000
21.6406	-29.3067	-4.0407	48.0358	0.0000	0.0000
13.3404	-5.7007	0.0000	0.0000	0.0000	0.0000
-37.4406	-13.0313	31.6895	-15.3508	0.0000	0.0000
0.0000	0.0000	-31.6895	15.3508	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
17.6001	18.7321	0.0000	0.0000	0.0000	0.0000
-17.6001	-18.7321	32.8307	8.8887	0.0000	0.0000
0.0000	0.0000	-32.8307	-8.8887	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	-4.3694	-39.4938
0.0000	0.0000	0.0000	0.0000	4.3694	39.4938
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	.9939	-.1100
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	-.2613	.9652	0.0000	0.0000
-.7268	.6547	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	-46.0656	-3.5804
0.0000	0.0000	0.0000	0.0000	.0775	-.9970

RIGHT HAND SIDE MATRIX -F-

				RIGHT HAND SIDE
-9.6041	14.7095	32.7696	-2.0326	.941
16.6763	6.2162	-32.7695	2.0326	-.737
-2.4875	9.8399	0.0000	0.0000	.166
18.5808	-18.0587	4.3694	39.4938	1.270
0.0000	0.0000	-4.3694	-39.4938	.793
0.0000	0.0000	-20.9364	25.9743	3.910
0.0000	0.0000	20.9364	-25.9743	2.549
0.0000	0.0000	-32.8307	-8.8887	.652
0.0000	0.0000	32.8307	8.8887	-4.757
0.0000	0.0000	0.0000	0.0000	-.799
0.0000	0.0000	-17.6001	-18.7321	-.820
0.0000	0.0000	17.6001	18.7321	4.272
0.0000	0.0000	0.0000	0.0000	1.747
0.0000	0.0000	-6.8701	-46.8738	1.390
-23.1655	-12.6770	6.8701	46.8733	-1.586
32.7696	-2.0326	-39.6397	-40.8412	-1.555
0.0000	0.0000	-10.7301	30.1417	.790
0.0000	0.0000	-15.2305	9.8433	-.824
0.0000	0.0000	11.6942	34.8635	2.659
0.0000	0.0000	25.3056	13.5190	-1.702
-32.7696	2.0326	28.4002	-41.5204	-.396
-.3619	-.9981	.0619	.9981	.001
0.0000	0.0000	-.9939	.1100	.010
0.0000	0.0000	-.7726	-.6276	.031
0.0000	0.0000	.2613	-.9652	-.013
0.0000	0.0000	.7268	-.6847	.005
0.0000	0.0000	.9903	-.1392	.038
0.0000	0.0000	0.0000	0.0000	0.000
0.0000	0.0000	0.0000	0.0000	0.000

STANDARD ERROR	RESIDUAL
.100E+01	-.941E+00
.100E+01	.741E+00
.100E+01	-.167E+00
.100E+01	-.129E+01
.100E+01	-.801E+00
.100E+01	-.387E+01
.100E+01	-.254E+01
.100E+01	-.669E+00
.100E+01	.477E+01
.100E+01	.793E+00
.100E+01	.618E+00
.100E+01	-.426E+01
.100E+01	-.175E+01
.100E+01	-.140E+01
.100E+01	.157E+01
.100E+01	.157E+01
.100E+01	-.794E+00
.100E+01	.619E+00
.100E+01	-.269E+01
.100E+01	.167E+01
.100E+01	.418E+00
.100E-01	-.976E-03
.100E-01	-.983E-02
.100E-01	-.309E-01
.100E-01	.131E-01
.100E-01	-.453E-02
.100E-01	-.379E-01
.100E+00	-.486E-01
.500E-02	-.833E-03

ADJUSTED ANGLES

ANGLE			DEG	MIN	SECS
4	6	5	32	14	17.66
4	5	7	23	35	18.34
4	7	3	17	5	36.03
3	4	5	52	4	55.63
3	5	7	79	14	32.71
7	3	5	55	34	20.39
7	5	2	56	0	40.25
2	7	5	57	42	27.54
2	5	6	40	59	43.66
2	6	1	59	20	45.20
1	2	5	48	1	24.72
1	5	6	27	35	47.83
6	1	2	45	2	2.25
6	2	5	72	9	19.31
6	5	4	53	15	34.09
5	4	6	94	27	6.06
5	6	1	35	12	50.61
5	1	2	31	38	6.42
5	2	7	66	16	46.21
5	7	3	45	10	58.90
5	3	4	67	14	9.00

LINE		ADJ.DISTANCE
4	5	6282.319
3	5	5191.040
7	5	6182.619
2	5	6064.353
1	5	6024.865
6	5	4179.272

STANDARD ERROR OF CO-ORDINATES

STATION	X	Y	POSITION
1	.431336E-01	.428346E-01	.607893E-01
2	.425915E-01	.292832E-01	.516870E-01
3	.529619E-02	.118447E-01	.129748E-01
4	.446555E-01	.315344E-01	.546208E-01
5	.140149E-01	.198062E-01	.242632E-01

ADJUSTED CO-ORDINATES

STATION	X	Y
1	.3457807015E+06	.1503940263E+06
2	.3500442537E+06	.1507526439E+06
3	.3567886433E+06	.1443232138E+06
4	.3512401657E+06	.1386287699E+06
5	.3516290841E+06	.1448990359E+06

APPENDIX II

A numerical example for each of the vertical position control techniques; the direct leveling and the trig leveling, described in chapter VIII, will be adjusted by the computer program VCONTRL. The conditions, the requirements, and the procedures for running the program are given in chapter VII. All the necessary input data and the corresponding output of the adjustment will be given here. A numerical example of the gravimetric leveling computation is also presented.

A. Direct Leveling

The input data corresponding to Figure 23, are:

```
1
ADJUSTMENT OF DIRECT LEVELING BY LEAST SQUARES
10      1      5
200
 16.298    1.1402   6   1
-17.700    1.3784   1   2
 -0.687    1.5166   2   3
  2.086    1.6432   3   6
 23.615    0.5477   6   4
  7.304    0.9487   1   4
 14.162    1.0954   5   4
 25.709    1.2649   3   4
 -6.855    1.0488   1   5
 10.863    0.7071   2   5
```

The complete listing of the adjustment output is given on the next page.

79/06/12. 23.17.28.3

LEVELING..

ADJUSTMENT OF DIRECT LEVELING BY LEAST SQUARES

COEFFICIENT MATRIX

1.	0.	0.	0.	0.
-1.	1.	0.	0.	0.
0.	-1.	1.	0.	0.
0.	0.	-1.	0.	0.
0.	0.	0.	1.	0.
-1.	0.	0.	1.	0.
0.	0.	0.	1.	-1.
0.	0.	-1.	1.	0.
-1.	0.	0.	0.	1.
0.	-1.	0.	0.	1.

RIGHT HAND SIDE MATRIX -F-

216.2980
-17.7000
-.6870
-197.9140
223.6150
7.3040
14.1620
25.7090
-6.8550
10.8630

LEV. LINE	OBS. ELEV. DIFF.	STAND. ERROR	ADJ.ELEV.DIFF.
6 - 1	16.298	1.1402	16.3045
1 - 2	-17.700	1.3784	-17.7104
2 - 3	-.687	1.5166	-.6861
3 - 6	2.086	1.6432	2.0920
6 - 4	23.615	.5477	23.6142
1 - 4	7.304	.9487	7.3096
5 - 4	14.162	1.0954	14.1600
3 - 4	25.709	1.2649	25.7061
1 - 5	-6.855	1.0488	-6.8504
2 - 5	10.863	.7071	10.8601

STATION	ADJ.ELEVATION	STAND.ERROR
1	216.3045	.00435
2	198.5941	.00558
3	197.9080	.00570
4	223.6142	.00302
5	209.4542	.00511
	FIXED ELEVATION	
6	200.000	

....

B. Trig Leveling

The input data corresponding to the Figure 24, are:

```
 0
 TRIG LEVELING BY LEAST SQUARE
 45   10   10
 12   3    1    0    1
 1
 1000
-9480  1.0  14037.6  0.1  4.96   0.00  4  1
-8242  1.0  19211.8  0.1  4.96   9.24  4  2
-5964  1.0  19896.3  0.1  4.96   0.00  4  3
 9417  1.0  14037.6  0.1  0.68  10.08  1  4
-1525  1.0  17064.9  0.1  0.68  10.42  1  2
 4459  1.0  26328.4  0.1  0.68   0.00  1  3
 8187  1.0  19211.8  0.1  4.80  10.08  2  4
 1428  1.0  17064.9  0.1  4.80   0.00  2  1
 2926  1.0  13706.8  0.1  4.80   0.00  2  3
 5847  1.0  19896.3  0.1  0.83  10.08  3  4
 -693  1.0  26328.4  0.1  0.83   0.00  3  1
-2965  1.0  13706.8  0.1  0.83  10.42  3  2
```

The complete listing of the output of the adjustment by VCONTRL is given on the next page.

79/06/12. 23.17.35.S

TRIG.:

LEVELING.:

TRIG LEVELING BY LEAST SQUARE

TRIG.LINE	OBS.VERT.ANGLE DEG MIN SEC	STAND.ERROR	OBS. DIST.	STAND.ERROR
4 - 1	-2 37 60.00	1.00	14037.60	.1000
4 - 2	-2 17 22.00	1.00	19211.80	.1000
4 - 3	-1 39 24.00	1.00	19896.30	.1000
1 - 4	2 36 57.00	1.00	14037.60	.1000
1 - 2	-0 25 25.00	1.00	17064.90	.1000
1 - 3	0 7 39.00	1.00	26328.40	.1000
2 - 4	2 15 27.00	1.00	19211.80	.1000
2 - 1	0 23 48.00	1.00	17064.90	.1000
2 - 3	0 45 46.00	1.00	13706.80	.1000
3 - 4	1 37 27.00	1.00	19896.30	.1000
3 - 1	-0 11 33.00	1.00	26328.40	.1000
3 - 2	-0 49 25.00	1.00	13706.80	.1000

COEFFICIENT MATRIX

```
 1.  0.  0.
 0.  1.  0.
 0.  0.  1.
-1.  0.  0.
-1.  1.  0.
-1.  0.  1.
 0. -1.  0.
 1. -1.  0.
 0. -1.  1.
 0.  0. -1.
 1.  0. -1.
 0.  1. -1.
```

RIGHT HAND SIDE MATRIX	WEIGHT MATRIX
363.3902	214.046
235.2358	114.703
437.6534	107.207
-364.0114	214.024
-129.9258	146.073
73.5125	61.375
-234.7340	114.688
128.9208	146.071
203.1143	226.251
-436.9563	107.202
-73.3834	61.375
-202.7737	226.262

TRIG.LINE	ADJ.ELEV.DIFF.
4 - 1	-636.1495
4 - 2	-765.4140
4 - 3	-562.5672
1 - 4	636.1495
1 - 2	-129.2645
1 - 3	73.5824
2 - 4	765.4140
2 - 1	129.2645
2 - 3	202.8469
3 - 4	562.5672
3 - 1	-73.5824
3 - 2	-202.8469

STATION	ADJ.ELEVATION	STAND.ERROR
1	363.8505	.20062
2	234.5860	.21540
3	437.4328	.22906
	FIXED ELEVATION	
4	1000.000	

....

C. Gravimetric Leveling

To demonstrate the theory of the geopotential numbers, field measurements with a LaCoste and Romberg Model G Geodetic Gravity Meter were taken on five stations located on the Oregon State University (OSU) Campus, Corvallis. These five stations are located in either line 15 of the National Geodetic Survey (NGS) Leveling Network or from the Oregon Gravity Base Station Network. Figure 26, shows the five stations and the gravity observation sequence.

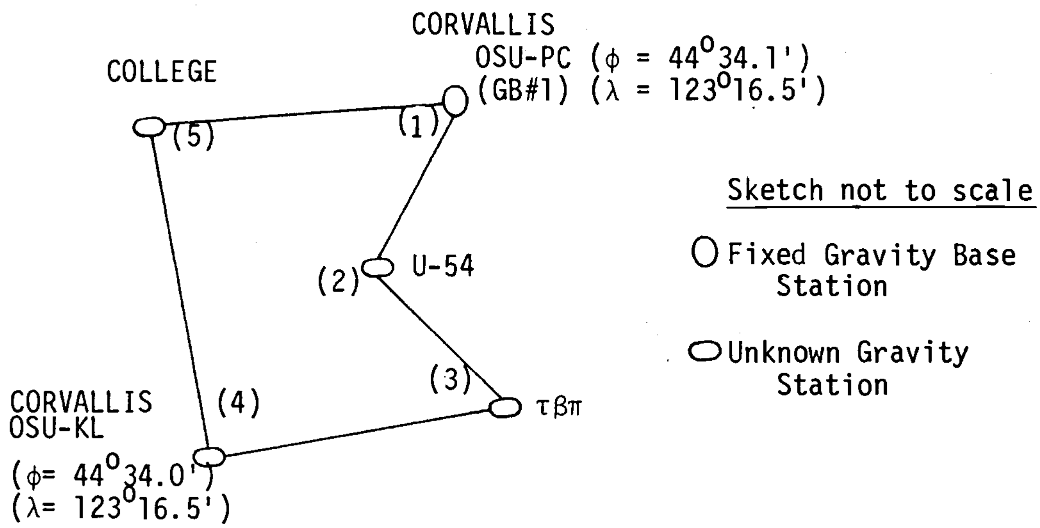

Figure 26. Level Net and Gravity Observation Sequence.

Computational Procedures

The detail computation procedures are given in [14]. Only a very brief description will be given here. The interest is in finding the actual gravity values, g's, at the five observation points. The mean readings from the gravity observations and the corresponding local times on two separate days are given in column (2) of Table 5. From the gravity meter calibration chart provided by the manufacturer (Table 7), corrected values (mgals) for the corresponding mean readings (counter reading in the calibration chart) are obtained and they are tabulated in column (5) of Table 5.

The next step is to apply the Earth Tide correction. For this, Earth Tide Correction Tables were obtained from the Department of Oceanography, Oregon State University computer. These are given in Tables 8 through 10. The earth tide correction for an instant of time is then interpolated from these tables and it is tabulated in column (6) of Table 5. The correction is algebraically added to the corrected value of column (5) to obtain the corrected reading in column (7). The instrument drift correction, in column (8), has been computed linearly for the difference of time from the corrected readings misclosure at the starting Gravity Base Station 1 (GB #1). This, algebraically added to the corrected reading, in column (7) gives the relative gravity in milligals (mgals) in column (9) of Table 5.

From the published data for the Oregon Gravity Network, the gravity value for the Corvallis OSU-PC (Gravity Base Station 1; GP #1 in Figure 26), [7], is 980,573.14 mgals. By comparing this value with the relative gravity at GP #1 in Table 5, the difference is

976,341.609 mgals.

This value must be added to each of the relative gravity values to obtain the surface gravity values, tabulated in column (2) of Table 6. Values in column (3) are computed similarily for the second day of

Table 5. Gravity Computation

Station	Mean Reading	Mean Local Time h m	GMT h m	Corrected Value (mgals)	Earth Tide (mgals)	Corrected Reading (mgals)	Drift Corr (mgals)	Relative Gravity (mgals)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
			Date: 04/20/1979, PST; GMT: 04/20,21/1979					
GB # 1 *	4024.132	15 36	23 36	4231.544	-0.013	4231.531	0.000	4231.531
U - 54	4024.236	15 48	23 48	4231.653	-0.009	4231.644	-0.005	4231.639
τ β π	4026.376	16 05	00 05	4233.903	-0.003	4233.874	-0.012	4233.862
GB # 4**	4024.804	16 15	00 15	4232.250	0.001	4232.251	-0.016	4232.235
College	4024.290	16 27	00 27	4231.710	0.005	4231.715	-0.020	4231.695
GB # 1	4024.135	16 41	00 41	4231.547	0.010	4231.557	-0.026	4231.531
			Date: 05/08/1979, PDT; GMT: 05/08/1979					
GB # 1	4024.008	15 26	22 26	4231.413	-0.022	4231.391	0.000	4231.391
U - 54	4024.126	15 36	22 36	4231.537	-0.027	4231.510	-0.002	4231.508
τ β π	4026.230	15 46	22 46	4233.750	-0.032	4233.718	-0.005	4233.713
GB # 4	4024.700	15 57	22 57	4232.141	-0.037	4232.104	-0.008	4232.096
College	4024.196	16 21	23 21	4231.611	-0.048	4231.563	-0.013	4231.550
GB # 1	4024.060	16 48	23 48	4231.468	-0.057	4231.411	-0.020	4231.391

* Corvallis OSU-PC (Gravity Base #1)

** Corvallis OSU-KL (Gravity Base #4)

observations. Values in columns (2) and (3) are averaged to obtain the mean surface gravity g, as explained earlier, in column (4). The elevations of U - 54, τβπ, and COLLEGE in column (6) are from the published values by the U.S. Department of Commerce in the Vertical Control Data, Sea Level Datum of 1929, and the elevations of the other two stations are taken from [14]. These elevations should be the values after the adjustment of the observation by the least squares adjustment, described in this paper. The elevations in column (5), are then multiplied by the corresponding mean surface gravity, in kgals, of column (4), to obtain the geopotential numbers (GPU) tabulated in column (6) of Table 6.

Table 6. Geopotential Numbers

Station	Surface Gravity 04/21/79 (mgals)	Surface Gravity 05/09/79 (mgals)	Mean Surface Gravity (mgals)	Elevation (m)	GPU (Kgal m)
(1)	(2)	(3)	(4)	(5)	(6)
GB # 1	980,573.14	980,573.14	980,573.140	77.142	75.643
U - 54	980,573.248	980,573.257	980,573.253	76.787	75.295
$\tau\ \beta\ \pi$	980,575.471	980,575.462	980,575.467	71.337	69.295
GB # 4	980,573.844	980,573.845	980,573.844	73.336	71.911
College	980,573.304	980,573.299	980,573.302	72.219	70.816

Table 7. Calibration Values.

Milligal Values for LaCoste & Romberg, Inc. Model G Gravity Meter No. 126

Counter Reading*	Value in Milligals	Factor for Interval
000	000.00	1.05230
100	105.23	1.05220
200	210.45	1.05210
300	315.66	1.05200
400	420.86	1.05190
500	526.05	1.05185
600	631.23	1.05175
700	736.41	1.05170
800	841.58	1.05160
900	946.74	1.05155
1000	1051.89	1.05150
1100	1157.04	1.05145
1200	1262.19	1.05140
1300	1367.33	1.05140
1400	1472.47	1.05140
1500	1577.62	1.05140
1600	1682.75	1.05140
1700	1787.89	1.05140
1800	1893.03	1.05140
1900	1998.17	1.05140
2000	2103.31	1.05140
2100	2208.45	1.05135
2200	2313.58	1.05135
2300	2418.72	1.05135
2400	2523.85	1.05135
2500	2628.99	1.05135
2600	2734.12	1.05135
2700	2839.26	1.05140
2800	2944.40	1.05140
2900	3049.54	1.05145
3000	3154.69	1.05145
3100	3259.83	1.05150
3200	3364.98	1.05150
3300	3470.13	1.05150
3400	3575.28	1.05150
3500	3680.43	1.05150

Counter Reading*	Value in Milligals	Factor for Interval
3500	3785.58	1.05150
3700	3890.73	1.05150
3800	3995.88	1.05145
3900	4101.02	1.05145
4000	4206.17	1.05145
4100	4311.31	1.05140
4200	4416.45	1.05135
4300	4521.59	1.05135
4400	4626.72	1.05130
4500	4731.85	1.05125
4600	4836.98	1.05120
4700	4942.10	1.05120
4800	5047.22	1.05105
4900	5152.33	1.05095
5000	5257.43	1.05090
5100	5362.52	1.05085
5200	5467.60	1.05075
5300	5572.68	1.05070
5400	5677.75	1.05060
5500	5782.81	1.05045
5600	5887.85	1.05025
5700	5992.88	1.05005
5800	6097.89	1.04985
5900	6202.87	1.04965
6000	6307.84	1.04950
6100	6412.79	1.04920
6200	6517.71	1.04895
6300	6622.60	1.04865
6400	6727.47	1.04840
6500	6832.31	1.04810
6600	6937.12	1.04780
6700	7041.90	1.04750
6800	7146.65	1.04720
6900	7215.37	1.04685
7000	7356.06	

Note: Right hand wheel on counter indicates approximately 0.1 milligal.

8-2-66

Table 8. Earth Tide Corrections - April 20, 1979.

EARTH TIDE GRAVITY CORRECTIONS TO BE ADDED ALGEBRAICALLY

Table for Location: 44 33.58 Lat. 123 17.68 Long., 240.0 Feet Elevation

On Day: 20 April 1979 From 0000 Hrs to 2400 Hours GMT

Time (GMT)	Tide (MGAL)	Slope (Mgal/Min)
0	0.036300	0.000000
30	0.045913	0.000275
100	0.054158	0.000216
130	0.060638	0.000146
200	0.065006	0.000066
230	0.066989	-0.000020
300	0.066403	-0.000108
330	0.063166	-0.000195
400	0.057307	-0.000278
430	0.048967	-0.000352
500	0.038398	-0.000415
530	0.025958	-0.000462
600	0.012095	-0.000492
630	-0.002668	-0.000503
700	-0.017760	-0.000494
730	-0.032587	-0.000466
800	-0.046563	-0.000419
830	-0.059141	-0.000357
900	-0.069842	-0.000282
930	-0.078289	-0.000198
1000	-0.084223	-0.000110
1030	-0.087521	-0.000023
1100	-0.088204	0.000059
1130	-0.086438	0.000131
1200	-0.082519	0.000189
1230	-0.076854	0.000231
1300	-0.069936	0.000254
1330	-0.062307	0.000260
1400	-0.054520	0.000247
1430	-0.047104	0.000219
1500	-0.040528	0.000179
1530	-0.035163	0.000130
1600	-0.031269	0.000076
1630	-0.028968	0.000024
1700	-0.028245	-0.000024
1730	-0.028950	-0.000062
1800	-0.030813	-0.000088
1830	-0.033466	-0.000100
1900	-0.036472	-0.000096
1930	-0.039356	-0.000076
2000	-0.041644	-0.000042
2030	-0.042896	0.000005
2100	-0.042737	0.000062
2130	-0.040885	0.000124
2200	-0.037171	0.000187
2230	-0.031555	0.000248
2300	-0.024126	0.000301
2330	-0.015106	0.000342
2400	-0.004833	0.000000

Table 9. Earth Tide Corrections - April 21, 1979.

EARTH TIDE GRAVITY CORRECTIONS TO BE ADDED ALGEBRAICALLY

Table for Location: 44 33.58 Lat., 123 17.68 Long., 240.0 Feet Elevation

On Day: 21 April 1979 From 0000 Hours to 2400 Hours GMT

Time (GMT)	Tide (MGAL)	Slope (Mgal/Min)
0	-0.004833	-0.000000
30	0.006248	0.000379
100	0.017618	0.000369
130	0.028701	0.000340
200	0.038903	0.000291
230	0.047641	0.000225
300	0.054379	0.000142
330	0.058653	0.000049
400	0.060109	-0.000053
430	0.058520	-0.000157
500	0.053809	-0.000258
530	0.046055	-0.000352
600	0.035502	-0.000432
630	0.022546	-0.000494
700	0.007723	-0.000535
730	-0.008313	-0.000551
800	-0.024833	-0.000541
830	-0.041061	-0.000505
900	-0.056225	-0.000446
930	-0.069601	-0.000365
1000	-0.080555	-0.000268
1030	-0.088589	-0.000159
1100	-0.093368	-0.000046
1130	-0.094745	0.000065
1200	-0.092776	0.000169
1230	-0.087711	0.000258
1300	-0.079985	0.000327
1330	-0.070187	0.000372
1400	-0.059020	0.000392
1430	-0.047257	0.000386
1500	-0.035684	0.000354
1530	-0.025054	0.000301
1600	-0.016028	0.000230
1630	-0.009137	0.000146
1700	-0.004746	0.000057
1730	-0.003026	-0.000031
1800	-0.003953	-0.000112
1830	-0.003707	-0.000180
1900	-0.012695	-0.000229
1930	-0.019574	-0.000257
2000	-0.027299	-0.000262
2030	-0.035163	-0.000243
2100	-0.042450	-0.000201
2130	-0.048483	-0.000140
2200	-0.052672	-0.000062
2230	-0.054551	0.000025
2300	-0.053811	0.000116
2330	-0.050320	0.000206
2400	-0.044136	0.000000

Table 10. Earth Tide Corrections - May 8, 1979

EARTH TIDE GRAVITY CORRECTIONS TO BE ADDED ALGEBRAICALLY

Table for Location: 44 33.58 Lat., 123 17.68 Long., 240.0 Feet Elevation

On Day: 8 May 1979 From 0000 Hours to 2400 hours GMT

Time (GMT)	Tide (MGAL)	Slope (Mgal/Min)
0	-0.054856	0.000000
30	-0.055841	0.000065
100	-0.053881	0.000156
130	-0.049201	0.000233
200	-0.042226	0.000289
230	-0.033544	0.000322
300	-0.023870	0.000329
330	-0.013994	0.000309
400	-0.004726	0.000263
430	0.003166	0.000195
500	0.009008	0.000109
530	0.012274	0.000011
600	0.012618	-0.000090
630	0.009905	-0.000189
700	0.004226	-0.000278
730	-0.004116	-0.000350
800	-0.014620	-0.000400
830	-0.026617	-0.000424
900	-0.039323	-0.000419
930	-0.051882	-0.000385
1000	-0.063429	-0.000324
1030	-0.073149	-0.000239
1100	-0.080323	-0.000135
1130	-0.084381	-0.000019
1200	-0.084940	0.000104
1230	-0.081827	0.000225
1300	-0.075091	0.000336
1330	-0.065003	0.000432
1400	-0.052041	0.000506
1430	-0.036860	0.000553
1500	-0.020255	0.000571
1530	-0.003118	0.000558
1600	0.013622	0.000514
1630	0.029047	0.000432
1700	0.042305	0.000345
1730	0.052667	0.000230
1800	0.059562	0.000102
1830	0.062617	-0.000031
1900	0.061684	-0.000161
1930	0.056840	-0.000218
2000	0.048393	-0.000384
2030	0.036863	-0.000464
2100	0.022947	-0.000515
2130	0.007483	-0.000536
2200	-0.008604	-0.000525
2230	-0.024367	-0.000484
2300	-0.038882	-0.000415
2330	-0.051324	-0.000323
2400	-0.061010	0.000000

APPENDIX III

79/07/13. 17.26.49.S

```
H     H  CCCCC   OOOOO  N     N TTTTTTT RRRRRR  L
H     H C     C O     O NN    N    T    R     R L
H     H C       O     O N N   N    T    R     R L
HHHHHHH C       O     O N  N  N    T    RRRRRR  L
H     H C       O     O N   N N    T    R   R   L
H     H C     C O     O N    NN    T    R    R  L        ::
H     H  CCCCC   OOOOO  N     N    T    R     R LLLLLLL  ::
```

```
      PROGRAM HCONTRL(INPUT,OUTPUT,TAPE60=INPUT,TAPE61=OUTPUT)

      DIMENSION XO(50),YO(50),IDEG(50),MIN(50),SEC(50),
     1          SIGMA1(50),NUMI(50),NUMJ(50),NUMK(50),
     2          DIST(50),SIGMA2(50),LNUMI(50),LNUMJ(50),
     3          IAZD(4),IAZM(4),SECAZ(4),SIGMA3(4),
     4          INOI(4),INOJ(4),F(104),Q(104),
     5          A(104,80),AT(80,104),ATW(80,104),B(80,80),
     6          T(104),DELTA(80),DELX(40),DELY(40),DIG(80),
     7          AD(104),WV(104),TITLE(15),V(104),EST(100),
     8          FDIST(4),STE(4),NOI(4),NOJ(4)

C     COMPUTER LANGUAGE ---- FORTRAN IV

C     PROGRAMMED BY  RAMESH L. SHRESTHA

C     THIS PROGRAM ADJUSTS A NETWORK OF ANGLES, DISTANCES
C        OR AZIMUTHS ONLY, OR THE COMBINATION OF ANY TWO OF
C        ALL BY THE LEAST SQUARES TECHNIQUE USING OBSERVATION
C        EQUATIONS. CONSEQUENTLY, THIS PROGRAM IS CAPABLE OF
C        ADJUSTING TRAVERSE, INTERSECTION, RESECTION,
C        TRIANGULATION, TRILATERATION, OR COMBINED NETWORKS.

C     A MAXIMUM OF 104 OBSERVATION EQUATIONS CAN BE ADJUSTED,
C     SUCH THAT THE TOTAL NUMBER OF FIXED AND UNKNOWN POINTS
C     DOES NOT EXCEED 10 AND 40 RESPECTIVELY. THE MAXIMUM
C     NUMBER OF FIXED AZIMUTHS AND DISTANCES MUST NOT EXCEED
C     4 EACH.

C     MEASURED ANGLE(S),FIXED AZIMUTH(S),AND THEIR RESPECTIVE
C     STANDARD ERRORS CAN HAVE ANY DECIMAL PART OF **  SECOND **.

C        THE MEASURED ANGLES MUST BE IN DEGREES, MINUTES, AND
C        SECONDS. THE MEASURED DISTANCES CAN BE EITHER IN FEET,
C        OR METERS. THE FIXED AZIMUTH(S) MUST BE TAKEN FROM THE
C        NORTH AND MUST INPUT IN DEGREES, MINUTES, AND SECONDS.
C        THE FIXED DISTANCE(S) CAN BE EITHER FEET, OR METERS.
C        THE UNITS MUST BE CONSISTENT. FOR INSTANCE, IF THE
C        APPROXIMATE AND FIXED CO-ORDINATES ARE IN FEET,
C        THE DISTANCE(S) MUST ALSO BE IN FEET.

C              NFXD =NO. OF FIXED STATIONS
C              NTRAV=NO. OF UNKNOWN POINTS
C              NANGL=NO. OF OBSERVED ANGLES
C              NDIST=NO. OF OBSERVED DISTANCES
C              NAZ  =NO. OF FIXED AZIMUTHS
C              NFD  =NO. OF FIXED DISTANCES

      ITER=0
      PI=3.141592654
      ONE=PI/648000.
      CONV=1./SIN(ONE)

C     INPUT TITLE (UP TO 75 CHARACTERS,LETTERS,NUMERALS OR BOTH)

      READ(60,3) (TITLE(I),I=1,15)
    3 FORMAT(15A5)
      WRITE(61,4) (TITLE(I),I=1,15)
    4 FORMAT(*1*,T5,15A5//)

C     INPUT NUMBER OF FIXED STATIONS,NUMBER OF UNKNOWN POINTS,
C        NUMBER OF OBSERVED ANGLES, NUMBER OF OBSERVED DISTANCES,
C        NUMBER OF FIXED AZIMUTHS AND NUMBER OF FIXED DISTANCES.

      READ(60,*) NFXD,NTRAV,NANGL,NDIST,NAZ,NFD

      NSTA=NFXD+NTRAV
      NT=2*NTRAV
      NP=NANGL+NDIST
      NISUM=NP+NAZ
      NTOT=NP+NAZ+NFD

C     NUMBER THE UNKNOWN STATION(S) FIRST,AS 1,2,....,N
C     (N IS NUMBER,NO LETTER IS ALLOWED),AND FIXED STATION(S)
C     AS N+1,N+2,....,ETC. (ANY STATION NUMBER MUST NOT
C     EXCEED TWO DIGITS).
```

```
C      INPUT APPROX. CO-ORDINATES, X  AND  Y, OF THE
C      UNKNOWN STATION(S) ACCORDING TO THE ORDER OF THE NUMBERED
C      UNKNOWN STATION(S) AND THEN THE FIXED CO-ORDINATES,
C      X  AND  Y, ACCORDING TO THE ORDER OF THE OF THE
C      NUMBERED FIXED STATION(S).

       READ(60,*) (XC(I),YC(I),I=1,NSTA)

       WRITE(61,5)
     5 FORMAT(* *,T22,*APPROXIMATE CO-ORDINATES*//
     1          * *,T7,*STATION*,T24,*X*,T44,*Y*/)

       WRITE(61,10) (I,XC(I),YC(I),I=1,NTRAV)
    10 FORMAT(* *,T8,I3,T16,E16.10,T36,E16.10/)

       NCNT=NTRAV+1
       WRITE(61,15)
    15 FORMAT(*0*,T22,*FIXED CO-ORDINATES*/)
       WRITE(61,17) (I,XC(I),YC(I),I=NCNT,NSTA)
    17 FORMAT(* *,T8,I3,T16,E16.10,T36,E16.10/)

C    INPUT OBSERVED ANGLE IN DEGREES,MINUTES,AND SECONDS FOLLOWED
C      BY ITS STANDARD DEVIATION IN SECONDS AND STATION NUMBERS
C      I,J,K  WHERE I=NUMBER AT THE MEASURED ANGLE, J,K IS TAKEN
C      IN THE CLOCKWISE MANNER LOOKING THROUGH THE MEASURED
C      ANGLE FROM  I  TOWARDS  J  AND  K.
C    REPEAT FOR ALL ANGLES.

       IF(NANGL.EQ.0) GO TO 16
       READ(60,*)(IDEG(I),MIN(I),SEC(I),SIGMA1(I),NUMI(I),NUMJ(I),
     1            NUMK(I),I=1,NANGL)

C      INPUT MEASURED DISTANCE, ITS STANDARD DEVIATION
C      IN THE SAME UNIT AS THE DISTANCE FOLLOWED BY STATION
C      NUMBERS OF THE TWO ENDS (I,J OR J,I).
C    REPEAT FOR ALL DISTANCES.

    16 IF(NDIST.EQ.0) GO TO 19
       READ(60,*)(DIST(I),SIGMA2(I),LNUMI(I),LNUMJ(I),I=1,NDIST)

    19 IF(NAZ.EQ.0) GO TO 22

C    INPUT FIXED AZIMUTH IN DEGREES,MINUTES,AND SECONDS FOLLOWED BY
C    ITS STANDARD DEVIATION IN SECONDS AND STATION NUMBERS
C    OF ITS TWO ENDS I,J FOR THE FIXED AZIMUTH FROM  I  TO  J.
C    REPEAT FOR ALL FIXED AZIMUTHS.

       READ(60,*)(IAZD(I),IAZM(I),SECAZ(I),SIGMA3(I),INOI(I),
     2            INOJ(I),I=1,NAZ)

    22 IF(NFD.EQ.0) GO TO 20

C    INPUT FIXED DISTANCE, ITS STANDARD DEVIATION (IN THE SAME
C    UNIT AS THE DISTANCE) FOLLOWED BY STATION NUMBERS OF ITS
C    ENDS (I,J OR J,I).
C    REPEAT FOR ALL FIXED DISTANCES.

       READ(60,*)(FDIST(I),STE(I),NOI(I),NOJ(I),I=1,NFD)

C    FORMATION OF COEFFICIENT MATRIX FOR ANGLES.

    20 IF(NANGL.EQ.0) GO TO 31
       DO 30 M=1,NANGL
       I=NUMI(M)
       J=NUMJ(M)
       K=NUMK(M)

       X1=XC(J)-XC(I)
       Y1=YC(J)-YC(I)
       X2=XC(K)-XC(I)
       Y2=YC(K)-YC(I)

       CALL ANGLE(X1,Y1,X2,Y2,COMP,A12,A21)

       EST(M)=COMP
       ANG2=FLOAT(IDEG(M))*3600.+FLOAT(MIN(M))*60.+SEC(M)
```

```
      F(M)=ANG2-COMP
      DIST1=X1*X1+Y1*Y1
      DIST2=X2*X2+Y2*Y2
      Q(M)=1./(SIGMA1(M)*SIGMA1(M))
      A(M,2*I-1)=CONV*(Y1/DIST1-Y2/DIST2)
      A(M,2*I)=CONV*(X2/DIST2-X1/DIST1)
      A(M,2*J-1)=-CONV*(Y1/DIST1)
      A(M,2*J)=CONV*(X1/DIST1)
      A(M,2*K-1)=CONV*(Y2/DIST2)
      A(M,2*K)=-CONV*(X2/DIST2)
   30 CONTINUE
C     FORMATION OF COEFFICIENT MATRIX FOR DISTANCES.
   31 IF(NDIST.EQ.0) GO TO 41
      DO 40 II=1,NDIST
      I=LNUMI(II)
      J=LNUMJ(II)
      DX=XC(J)-XC(I)
      DY=YC(J)-YC(I)
      DSQR=DX*DX+DY*DY
      COMP=SQRT(DSQR)
      M=II+NANGL
      Q(M)=1./(SIGMA2(II)*SIGMA2(II))
      EST(M)=COMP
      F(M)=DIST(II)-COMP
      A(M,2*I-1)=-(DX/COMP)
      A(M,2*I)=-(DY/COMP)
      A(M,2*J-1)=DX/COMP
      A(M,2*J)=DY/COMP
   40 CONTINUE
   41 IF(NAZ.EQ.0) GO TO 60
C     FORMATION OF COEFFICIENT MATRIX FOR FIXED AZIMUTHS.
      DO 50 III=1,NAZ
      I=INOI(III)
      J=INOJ(III)
      C1=XC(J)-XC(I)
      C2=YC(J)-YC(I)
      C3=C1*C1+C2*C2
      M=III+NP
      Q(M)=1./(SIGMA3(III)*SIGMA3(III))
      A(M,2*I-1)=-CONV*(C2/C3)
      A(M,2*I)=CONV*(C1/C3)
      A(M,2*J-1)=CONV*(C2/C3)
      A(M,2*J)=-CONV*(C1/C3)
      F(M)=0.
   50 CONTINUE
   60 IF(NFD.EQ.0) GO TO 65
C     FORMATION OF COEFFICIENT MATRIX FOR FIXED DISTANCES.
      DO 61 LL=1,NFD
      I=NOI(LL)
      J=NOJ(LL)
      DDX=XC(J)-XC(I)
      DDY=YC(J)-YC(I)
      M=NISUM+LL
      Q(M)=1./(STE(LL)*STE(LL))
```

```
              F(M)=0.
              A(M,2*I-1)=-(DDX/FDIST(LL))
              A(M,2*I)=-(DDY/FDIST(LL))
              A(M,2*J-1)=DDX/FDIST(LL)
           61 A(M,2*J)=DDY/FDIST(LL)
        C   FORMATION OF NORMAL EQUATIONS.
           65 DO 70 I=1,NT
              DO 70 J=1,NTOT
              AT(I,J)=0.
           70 AT(I,J)=A(J,I)
              DO 80 I=1,NT
              DO 80 J=1,NTOT
           80 ATW(I,J)=AT(I,J)*Q(J)
              DO 90 I=1,NT
              DO 90 J=1,NT
              B(I,J)=0.
              DO 90 K=1,NTOT
           90 B(I,J)=B(I,J)+ATW(I,K)*A(K,J)
        C   INVERT NORMAL EQUATION MATRIX.
              DO 100 I=1,NT
              T(I)=0.
              DO 100 J=1,NTOT
          100 T(I)=T(I)+ATW(I,J)*F(J)
              DO 110 K=1,NT
              DO 120 J=1,NT
              IF(J-K) 140,120,140
          140 B(K,J)=B(K,J)/B(K,K)
          120 CONTINUE
              B(K,K)=1./B(K,K)
              DO 110 I=1,NT
              IF(I-K) 150,110,150
          150 DO 130 L=1,NT
              IF(L-K) 160,130,160
          160 B(I,L)=B(I,L)-B(I,K)*B(K,L)
          130 CONTINUE
              B(I,K)=-B(I,K)*B(K,K)
          110 CONTINUE
        C   COMPUTE PARAMETERS (UNKNOWNS).
              DO 170 I=1,NT
              DELTA(I)=0.
              DO 170 J=1,NT
          170 DELTA(I)=DELTA(I)+B(I,J)*T(J)
        C   PERFORM ITERATIONS  (IF NECESSARY).
              ITER=ITER+1
              IF(ITER.GT.10) GO TO 192
              WRITE(61,180) ITER
          180 FORMAT(≠0≠,T12,≠ITERATION NO.≠,I2//
             1         ≠ ≠,T20,≠CO-ORDINATE VARIATIONS≠//
             2         ≠ ≠,T16,≠STATION≠,T28,≠DELTA X≠,T38,≠DELTA Y≠/)
              DO 190 L=1,NTRAV
              L1=2*L-1
              L2=2*L
              WRITE(61,200) L,DELTA(L1),DELTA(L2)
          200 FORMAT(≠ ≠,T18,I2,T27,F8.3,T37,F8.3/)
          190 CONTINUE
          192 DO 205 I=1,NTRAV
              II=2*I-1
              JJ=2*I
              DELX(I)=XC(I)
              DELY(I)=YC(I)
              XC(I)=XC(I)+DELTA(II)
          205 YC(I)=YC(I)+DELTA(JJ)
```

```
            DO 210 I=1,NT
            DIG(I)=DELTA(I)
            AB=ABS(DELTA(I))
      210 IF(AB.GT.0.001) GO TO 20

            IF(ITER.LE.10) GO TO 218
            WRITE(61,212) ITER
      212 FORMAT(≠0≠,T21,≠.  .  .  .  .  .  .  .  .≠//
         1        ≠ ≠,T21,≠.  .  .  .  .  .  .  .  .≠///
         2        ≠ ≠,T12,≠ITERATION NO.≠,I3//
         3        ≠ ≠,T20,≠CO-ORDINATE VARIATIONS≠//
         4        ≠ ≠,T16,≠STATION≠,T28,≠DELTA X≠,T38,≠DELTA Y≠/)

            DO 213 L33=1,NTRAV
            L11=2*L33-1
            L22=2*L33
      213 WRITE(61,214)L33,DELTA(L11),DELTA(L22)
      214 FORMAT(≠ ≠,T16,I2,T27,F8.3,T37,F8.3/)

    C   COMPUTE RESIDUALS.

      216 DO 220 I=1,NTOT
            AD(I)=0.
            DO 220 J=1,NT
      220 AD(I)=AD(I)+A(I,J)*DIG(J)

            DO 230 I=1,NTOT
      230 V(I)=AD(I)-F(I)

            DO 240 I=1,NTOT
      240 WV(I)=Q(I)*V(I)

    C   COMPUTE POST PRIORI.

            VTWV=0.
            DO 250 I=1,NTOT
      250 VTWV=VTWV+V(I)*WV(I)

            PRIORI=VTWV/(NTOT-NT)

            WRITE(61,260)
      260 FORMAT(≠0≠,10X,≠COEFFICIENT MATRIX  -3-≠/)

            CON=(FLOAT(NT))/6.
            NOT=INT(CON)+1

            I1=0
            N11=0

            DO 265 II=1,NOT
            N1=NT-6-N11
            N11=NT-N1

            J1=II+I1
            N12=N11-J1+1
            IF(N1.LT.0) GO TO 266
            GO TO 269
      266 N12=6+N1
            N11=NT
            IF(N12.EQ.0) GO TO 283
      269 DO 270 I=1,NTOT
      270 WRITE(61,280) N12,(A(I,J),J=J1,N11)
      280 FORMAT(≠ ≠,= (F11.4))
            I1=5+I1
            WRITE(61,281)
      281 FORMAT(≠ ≠,T5,≠      ≠,//)
      265 CONTINUE

      283 WRITE(61,284)
      284 FORMAT(≠0≠,13X,≠RIGHT HAND SIDE MATRIX -F-≠/)
            WRITE(61,285) (F(I),I=1,NTOT)
      285 FORMAT(≠ ≠,15X,F13.3)

            WRITE(61,290)
      290 FORMAT(≠0≠,T15,≠STANDARD≠,T31,≠RESIDUAL≠/
         1        ≠ ≠,T16,≠ERROR≠/)
```

```
      DO 292 I=1,NTOT
      Q11=1./Q(I)
      Q1=SQRT(Q11)
  292 WRITE(61,300) Q1,V(I)
  300 FORMAT(≠ ≠,T14,E9.3,T30,E9.3/)
C     COMPUTE ADJUSTED ANGLES
      IF(NANGL.EQ.0) GO TO 304
      WRITE(61,301)
  301 FORMAT(≠0≠,T11,≠ADJUSTED ANGLES≠//
     1       ≠ ≠,T7,≠ANGLE≠,T16,≠DEG≠,T21,≠MIN≠,T27,≠SECS≠/)
      DO 302 M=1,NANGL
      SOL=EST(M)
      CALL DEGREE(SOL,IHAT,MHAT,SHAT)
      I=NJMI(M)
      J=NJMJ(M)
      K=NJMK(M)
  302 WRITE(61,303) I,J,K,IHAT,MHAT,SHAT
  303 FORMAT(≠ ≠,T4,3I3,T16,I3,T21,I3,T26,F5.2/)
  304 IF(NDIST.EQ.0) GO TO 308
      WRITE(61,305)
  305 FORMAT(≠0≠,T7,≠LINE≠,T14 ,≠ADJ.DISTANCE≠/)
      NO=NANGL+1
      DO 306 N=NO,NP
      NN=N-NANGL
      I=LNUMI(NN)
      J=LNUMJ(NN)
  306 WRITE(61,307) I,J,EST(N)
  307 FORMAT(≠ ≠,T5,2I3,T15,F10.3/)
C     COMPUTE VARIANCE/COVARIANCE MATRIX.
  308 DO 310 I=1,NT
      DO 310 J=1,NT
  310 B(I,J)=B(I,J)*PRIORI
      WRITE(61,320)
  320 FORMAT(≠0≠,T18,≠STANDARD ERROR OF CO-ORDINATES≠//
     1          ≠ ≠,T7,≠STATION≠,T24,≠X≠,T39,≠Y≠,T49,≠POSITION≠/)
C     COMPUTE STATDARD ERROR OF ADJUSTED CO-ORDINATES (X,Y)
      J=0
      DO 330 I=1,NTRAV
      J=J+1
      M=2*I-1
      N=2*I
      VAR=B(M,M)+B(N,N)
      SEP=SQRT(VAR)
      SEX=SQRT(B(M,M))
      SEY=SQRT(B(N,N))
      WRITE(61,340) J,SEX,SEY,SEP
  340 FORMAT(≠ ≠,T8,I3,T17,E13.6,T32,E13.6,T45,E13.6/)
  330 CONTINUE
      WRITE(61,350)
  350 FORMAT(≠0≠,T24,≠ADJUSTED CO-ORDINATES≠//
     1          ≠ ≠,T7,≠STATION≠,T24,≠X≠,T44,≠Y≠/)
      WRITE(61,360)(I,DELX(I),DELY(I),I=1,NTRAV)
  360 FORMAT(≠ ≠,T6,I3,T16,E16.10,T36,E16.10/)
      END
```

```
SUBROUTINE ANGLE        73/73    OPT=2                                   FTN 4.7+485

                SUBROUTINE ANGLE(A1,A2,A3,A4,Z,AZ1,AZ2)
          C   COMPUTE AZIMUTHS OF TWO LINES AND TAKE THE
          C     DIFFERENCE TO COMPUTE ANGLE.

                PI=3.141592654
                ONE=PI/648000.
                CONV=1./SIN(ONE)
                B1=ATAN(A1/A2)
                B2=ATAN(A3/A4)

                IF(A1.GE.0.0.AND.A2.GE.0.0) GO TO 100
                GO TO 210
            100 AZ1=B1
            210 IF(A1.GE.0.0.AND.A2.LT.0.0) GO TO 101
                GO TO 211
            101 AZ1=B1+PI
            211 IF(A1.LT.0.0.AND.A2.LT.0.0) GO TO 102
                GO TO 212
            102 AZ1=B1+PI
            212 IF(A1.LT.0.0.AND.A2.GE.0.0) GO TO 103
                GO TO 200
            103 AZ1=B1+PI*2.
            200 IF(A3.GE.0.0.AND.A4.GE.0.0) GO TO 301
                GO TO 310
            301 AZ2=B2
            310 IF(A3.GE.0.0.AND.A4.LT.0.0) GO TO 302
                GO TO 311
            302 AZ2=B2+PI
            311 IF(A3.LT.0.0.AND.A4.LT.0.0) GO TO 303
                GO TO 312
            303 AZ2=B2+PI
            312 IF(A3.LT.0.0.AND.A4.GE.0.0) GO TO 304
                GO TO 315
            304 AZ2=B2+PI*2.

            315 ACAT=AZ2-AZ1
                Z1=ACAT
                IF(ACAT.LT.0.0) Z1=ACAT+2.*PI
                Z=Z1*CONV
                RETURN
                END
```

```
SUBROUTINE DEGREE       73/73    OPT=2                                   FTN 4.7+485

                SUBROUTINE DEGREE(SECOND,IG,MG,SE1)
          C   CONVERSION OF ANGLE FROM SECONDS TO DEG,MIN,AND SEC

                G1=SECOND/3600.
                IG=INT(G1)
                G2=(G1-FLOAT(IG))*60.
                MG=INT(G2)
                SE1=(G2-FLOAT(MG))*60.
                RETURN
                END
```

79/07/18. 13.08.58

```
V     V   CCCCC    OOOOO   N      N  TTTTTTT  RRRRRR   L
V     V  C     C  O     O  NN     N     T     R     R  L
V     V  C        O     O  N N    N     T     R     R  L
V     V  C        O     O  N  N   N     T     RRRRRR   L
 V   V   C        O     O  N   N  N     T     R   R    L
  V V    C     C  O     O  N    NN      T     R    R   L       ..
   V      CCCCC    OOOOO   N      N     T     R     R  LLLLLLL ..
```

```
      PROGRAM VCONTRL(INPUT,OUTPUT,TAPE60=INPUT,TAPE61=OUTPUT)
      DIMENSION TITLE(15),ELEV(100),A(100,90),W(100),
     1          F(100),ATW(90,100),B(90,90),T(100),S(100),
     2          NOI(100),NOJ(100),ADIFF(100),SIG(100)
C     COMPUTER LANGUAGE ---- FORTRAN IV
C     PROGRAMMED BY ---- RAMESH L. SHRESTHA
C     VERTICAL CONTROL ADJUSTMENT, EITHER BY DIRECT LEVELING OR
C     TRIG LEVELING TECHNIQUE CAN BE PERFORMED BY THIS PROGRAM.
C     OBSERVATION EQUATION METHOD HAS BEEN EMPLOYED.
C     IN THE ADJUSTMENT BY TRIG LEVELING,ATMOSPHERIC REFRACTION
C     AND CURVATURE OF THE EARTH CORRECTIONS NEED MEAN RADIUS AND
C     ECCENTRICITY OF THE SPHEROID.CONSTANTS OF CLAKRE SPHEROID
C     OF 1866 ARE USED FOR THIS PURPOSE.
C     TOTAL NUMBER OF 100 OBSERVATION EQUATIONS CAN BE ADJUSTED.
C     THE MAXIMUM NUMBERS OF FIXED AND UNKNOWN ELEVATION
C     STATIONS MUST NOT EXCEED 10 AND 90 RESPECTIVELY.
C     THE APPROXIMATE LATITUDE OF THE PROJECT AREA MUST BE IN
C     DEGREES,MINUTES,AND SECONDS (OR ANY DECIMAL PART OF
C     SECONDS).THE MEASURED VERTICAL ANGLE MUST BE IN SECONDS.
C     ANY DECIMAL PART OF THE SECOND CAN BE HANDLED.
C     NUMBER THE UNKNOWN ELEVATION STATION(S) FIRST,AS 1,2,....,N,
C     (N IS NUMBER,NO LETTER IS ALLOWED.ANY STATION NUMBER MUST
C     NOT EXCEED TWO DIGITS),AND THEN THE FIXED ELEVATION
C     STATION(S) AS N+1,N+2,....,ETC.
C     AT LEAST ONE FIXED ELEVATION MUST BE GIVEN.
C     THE UNITS MUST BE CONSISTENT. IF THE APPROXIMATE AND FIXED
C     CO-ORDINATES ARE IN FEET,THEN THE DISTANCE(S) MUST ALSO
C     BE IN FEET.
      COEFF=0.125
      ESQ=0.006768657997291
      SEMIA=6378206.4*3.280833333
      PI=3.141592654
      PP=PI/(180.*3600.)
C     INPUT CONTROL IDENTIFIER. 1 FOR DIRECT LEVELING, 0 FOR
C     TRIG LEVELING.
      READ(60,*) ISELCT
      IF(ISELCT.EQ.1) GO TO 486
C     INPUT TITLE (UP TO 75 CHARACTERS,LETTERS,NUMERALS,OR BOTH)
      READ(60,10) (TITLE(I),I=1,15)
   10 FORMAT(15A5)
      WRITE(61,20) (TITLE(I),I=1,15)
   20 FORMAT(*1*,T5,15A5///)
C     INPUT APPROXIMATE AVERAGE LATITUDE OF THE PROJECT AREA
C     IN DEGREES,MINUTES,AND SECONDS.
      READ(60,*) IDEG,MIN,SEC
C     NOBS   = NO.OF OBSERVATIONS
C     NUN    = NO. OF UNKNOWN ELEVATIONS
C     NFIXED= NO.OF FIXED ELEVATIONS
C     ICHOSE=MEASURED DISTANCE IDENTIFIER (SLANT OR HORIZONTAL)
C     LIGHT =DISTANCE MEASURING EQUIPMENT IDENTIFIER (VISIBLE
C            LIGHT WAVE OR MICROWAVE SYSTEM)
C     INPUT NO. OF OBSERVATIONS, NO. OF UNKNOWN ELEVATIONS,
C     NO. OF FIXED ELEVATIONS, 1  OR  0  (1  IF THE INPUT MEASURED
C     DISTANCE(S) IS/ARE SLANT, 0 IF HORIZONTAL), 1  OR  0 (1 IF
C     THE DISTANCE(S) WERE MEASURED BY VISIBLE LIGHT WAVE EDM
C     EQUIPMENT SYSTEM, 0 IF MICROWAVE).
```

```
         READ(60,*) NOBS,NUN,NFIXED,ICHOSE,LIGHT
         IF(LIGHT.EQ.1) COEFF=0.07
C   INPUT UNIT IDENTIFIER, 1 FOR FEET 0 FOR METERS.
         READ(60,*) IUNIT
         IF(IUNIT.EQ.0) SEMIA=6378206.4
         PH1=3600.*FLOAT(IDEG)+60.*FLOAT(MIN)+SEC
         PHI=PP*PH1
         DENO=1.-ESQ*SIN(PHI)*SIN(PHI)
         AN=SEMIA*(1.-ESQ)
         RHO=AN/(DENO**1.5)
         ANU=SEMIA/SQRT(DENO)
         RAVE=SQRT(RHO*ANU)
         NB=NUN+1
         NTOT=NUN+NFIXED
C   INPUT FIXED ELEVATIONS
         READ(60,*) (ELEV(I),I=NB,NTOT)
         WRITE(61,25)
      25 FORMAT(≠0≠,T17,≠OBS.VERT.ANGLE≠/
       1        ≠ ≠,T5,≠TRIG.LINE≠,T18,≠DEG MIN SEC≠,
       2        ≠ ≠,T34,≠STAND.ERROR≠,T47,≠OBS. DIST.≠,
       3        ≠ ≠,T60,≠STAND.ERROR≠/)
C   INPUT VERTICAL ANGLE(POSITIVE UPWARD, NEGATIVE DOWNWARD)
C   IN SECONDS (OR IN ANY DECIMAL PART), ITS STANDARD ERROR
C   IN SECONDS (OR IN ANY DECIMAL PART),OBSERVED DISTANCE
C   (HORIZONTAL OR VERTICAL),ITS STANDARD ERROR (IN THE SAME
C   UNIT AS THE DISTANCE),INSTRUMENT HEIGHT,TARGET HEIGHT,
C   INSTRUMENT STA.NO.,TARGET STA.NO.
         DO 30 N=1,NOBS
         READ(60,*) SECS1,SIGA,SLANT,SIGS,AI,BT,I,J
         NOI(N)=I
         NOJ(N)=J
         ALFA=PP*SECS1
         DIV1=SECS1/3600.
         LDEG=INT(DIV1)
         IF(SECS1.LT.0.) DIV1=ABS(DIV1)
         DEG1=FLOAT(LDEG)
         REM=ABS(DEG1)
         REM1=(DIV1-REM)*60.
         LMIN=INT(REM1)
         SECO=(REM1-FLOAT(LMIN))*60.
         IF(SECS1.LT.0.0.AND.LDEG.EQ.0) GO TO 32
         WRITE(61,40)I,J,LDEG,LMIN,SECO,SIGA,SLANT,SIGS
      40 FORMAT(≠ ≠,T7,I2,≠ -≠,I2,T18,I3,T22,I2,T25,F5.2,
       1        T37,F5.2,T46,F10.2,T60,F7.4/)
         GO TO 43
      32 WRITE(61,41)I,J,LMIN,SECO,SIGA,SLANT,SIGS
      41 FORMAT(≠ ≠,T7,I2,≠ -≠,I2,T19,≠-0≠,T22,I2,T25,F5.2,
       1        T37,F5.2,T46,F10.2,T60,F7.4/)
      43 ALPHA=ALFA+((1.-2.*COEFF)*SLANT/(2.*RAVE))
         IF(ICHOSE.EQ.1) GO TO 42
         TANSQ=TAN(ALPHA)*TAN(ALPHA)
         SEC4=1./((COS(ALPHA))**4.)
         SS=SIGS*SIGS
         ST=(SIGA*SIGA)/(206265.*206265.)
         Q=TANSQ*SS+SLANT*SLANT*SEC4*ST
         GO TO 44
      42 S11=SIN(ALPHA)*SIN(ALPHA)
```

```
              S22=SIGS*SIGS
              C11=COS(ALPHA)*COS(ALPHA)
              C22=C11*SLANT*SLANT
              S44=(SIGA*SIGA)/(206265.*206265.)
              Q=S11*S22+C22*S44
           44 W(N)=1./Q

              A(N,I)=-1.
              A(N,J)=1.

              CONST1=AI-BT+SLANT*SIN(ALPHA)
              CONST2=AI-BT+SLANT*TAN(ALPHA)

              IF(ICHOSE.EQ.0) GO TO 38

              F(N)=CONST1

              IF(I.GT.NUN) F(N)=ELEV(I)+CONST1
              IF(J.GT.NUN) F(N)=-ELEV(J)+CONST1
              GO TO 30

           38 F(N)=CONST2

              IF(I.GT.NUN) F(N)=ELEV(I)+CONST2
              IF(J.GT.NUN) F(N)=-ELEV(J)+CONST2

           30 CONTINUE

              WRITE(61,50)
           50 FORMAT(≠0≠,T5,≠COEFFICIENT  MATRIX≠/)

              DO 60 I=1,NOBS

           60 WRITE(61,70) NUN,(A(I,J),J=1,NUN)
           70 FORMAT(≠ ≠,T5,=(F4.0))

              WRITE(61,80)
           80 FORMAT(≠0≠,T5,≠RIGHT HAND SIDE MATRIX≠,
             1        ≠ ≠,T20,≠WEIGHT MATRIX≠/)

              WRITE(61,90)(F(I),W(I),I=1,NOBS)
           90 FORMAT(≠ ≠,T13,F10.4,T34,F7.3)

              DO 100 I=1,NUN
              DO 100 J=1,NOBS
          100 ATW(I,J)=A(J,I)*W(J)

              DO 105 I=1,NUN
              DO 105 J=1,NUN
              B(I,J)=0.
              DO 105 K=1,NOBS
          105 B(I,J)=B(I,J)+ATW(I,K)*A(K,J)

              DO 110 K=1,NUN
              DO 120 J=1,NUN
              IF(J-K) 140,120,140
          140 B(K,J)=B(K,J)/B(K,K)
          120 CONTINUE
              B(K,K)=1./B(K,K)
              DO 110 I=1,NUN
              IF(I-K) 150,110,150
          150 DO 130 L=1,NUN
              IF(L-K) 160,130,160
          160 B(I,L)=B(I,L)-B(I,K)*B(K,L)
          130 CONTINUE
              B(I,K)=-B(I,K)*B(K,K)
          110 CONTINUE

              DO 170 I=1,NUN
              T(I)=0.
              DO 170 J=1,NOBS
          170 T(I)=T(I)+ATW(I,J)*F(J)

              DO 180 I=1,NUN
              ELEV(I)=0.
              DO 180 J=1,NUN
          180 ELEV(I)=ELEV(I)+B(I,J)*T(J)
```

```
      DO 185 I=1,NOBS
      ADIFF(I)=0.
      DO 185 J=1,NTOT
  185 ADIFF(I)=ADIFF(I)+A(I,J)*ELEV(J)

      WRITE(61,186)
  186 FORMAT(≠0≠,T5,≠TRIG.LINE≠,T30,≠ADJ.ELEV.DIFF.≠/)

      WRITE(61,187) (NOI(N),NOJ(N),ADIFF(N),N=1,NOBS)
  187 FORMAT(≠ ≠,T7,I2,≠ -≠,I2,T30,F10.4/)

      VTWV=0.
      DO 190 I=1,NOBS
      AE=0.
      DO 200 J=1,NUN
  200 AE=AE+A(I,J)*ELEV(J)
      V=AE-F(I)
      WV=W(I)*V
  190 VTWV=VTWV+V*WV

      PRIORI=VTWV/FLOAT(NOBS-NUN)

      DO 210 I=1,NUN
      DO 210 J=1,NUN
      B(I,J)=PRIORI*B(I,J)
  210 B(I,J)=SQRT(B(I,J))

      GO TO 700

C     INPUT TITLE (UP TO 75 CHARACTERS,LETTERS,NUMERALS,OR BOTH)

  406 READ(60,410) (TITLE(I),I=1,15)

  410 FORMAT(15A5)
      WRITE(61,420) (TITLE(I),I=1,15)
  420 FORMAT(≠1≠,T5,15A5//)

C     INPUT NUMBER OF LEVEL LINES, NUMBER OF FIXED ELEVATIONS
C     AND NUMBER OF UNKNOWN ELEVATION POINTS

      READ(60,*) NOBS,NFXD,NUN

      ISUM=NFXD+NUN
      IRIGIN=NUN+1

C     INPUT FIXED ELEVATIONS

      READ(60,*) (ELEV(I),I=IRIGIN,ISUM)

C     INPUT OBSERVED ELEVATION DIFFERENCE
C     (ELEV OF STA J - ELEV OF STA I), STANDARD ERROR OF THE
C     OBSERVED ELEVATION DIFFERENCE (IN THE SAME UNIT AS THE
C     ELEVATION DIFFERENCE), STATION NUMBERS I , J.

C     REPEAT FOR ALL OBSERVED ELEVATION DIFFERENCE

      DO 430 N=1,NOBS

      READ(60,*) F(N),SIG(N),I,J

      E(N)=F(N)

      NOI(N)=I
      NOJ(N)=J
      IF(I.GT.NUN) F(N)=F(N)+ELEV(I)
      IF(J.GT.NUN) F(N)=F(N)-ELEV(J)

      A(N,J)=1.
      A(N,I)=-1.

      SQR=SIG(N)*SIG(N)
  430 W(N)=1./SQR

      DO 450 I=1,NUN
```

```
      DO 450 J=1,NOBS
  450 ATW(I,J)=A(J,I)*W(J)

      DO 460 I=1,NUN
      DO 460 J=1,NUN
      B(I,J)=0.
      DO 460 K=1,NOBS
  460 B(I,J)=B(I,J)+ATW(I,K)*A(K,J)

      DO 470 I=1,NUN
      T(I)=0.
      DO 470 J=1,NOBS
  470 T(I)=T(I)+ATW(I,J)*F(J)

      DO 510 K=1,NUN
      DO 520 J=1,NUN
      IF(J-K) 540,520,540
  540 B(K,J)=B(K,J)/B(K,K)
  520 CONTINUE
      B(K,K)=1./B(K,K)
      DO 510 I=1,NUN
      IF(I-K) 550,510,550
  550 DO 530 L=1,NUN
      IF(L-K) 560,530,560
  560 B(I,L)=B(I,L)-B(I,K)*B(K,L)
  530 CONTINUE
      B(I,K)=-B(I,K)*B(K,K)
  510 CONTINUE

      DO 600 I=1,NUN
      ELEV(I)=0.
      DO 600 J=1,NUN
  600 ELEV(I)=ELEV(I)+B(I,J)*T(J)

      WRITE(61,645)
  645 FORMAT(≠0≠,T5,≠COEFFICIENT MATRIX≠/)

      DO 650 I=1,NOBS
  650 WRITE(61,660) NUN,(A(I,J),J=1,NUN)
  660 FORMAT(≠ ≠,=(F4.0))

      WRITE(61,670)
  670 FORMAT(≠0≠,T5,≠RIGHT HAND SIDE MATRIX -F-≠/)
      WRITE(61,680) (F(I),I=1,NOBS)
  680 FORMAT(≠ ≠,T7,F12.4)

      VTW=0.
      DO 681 I=1,NOBS
      AE=0.
      V=0.
      DO 682 J=1,NUN
  682 AE=AE+A(I,J)*ELEV(J)
      V=AE-F(I)
      WV=W(I)*V
  681 VTW=VTW+V*WV

      PRIORI=VTW/(NOBS-NUN)

      DO 683 I=1,NUN
      DO 683 J=1,NUN
      VAR1=PRIORI*B(I,J)
  683 B(I,J)=SQRT(VAR1)

      WRITE(61,684)
  684 FORMAT(≠0≠,T5,≠LEV. LINE≠,T16,≠OBS. ELEV. DIFF.≠,T35,
     1       ≠STAND. ERROR≠,T51,≠ADJ.ELEV.DIFF.≠/)

      DO 685 I=1,NOBS
      ADIFF(I)=0.
      DO 685 J=1,ISUM
  685 ADIFF(I)=ADIFF(I)+A(I,J)*ELEV(J)

      NB=IBIGIN
      NTOT=ISUM

      WRITE(61,686)(NOI(N),NOJ(N),F(N),SIG(N),ADIFF(N),N=1,NOBS)
  686 FORMAT(≠ ≠,T6,I2,≠ -≠,I3,T17,F10.3,T38,F6.4,
```

```
     1         ≠ ≠,T51,F10.4/)

 700 WRITE(61,720)
 720 FORMAT(≠0≠,T5,≠STATION≠,T15,≠ADJ.ELEVATION≠,
     1         ≠ ≠,T32,≠STAND.ERROR≠/)

     WRITE(61,730)(I,ELEV(I),Q(I,I),I=1,NUN)
 730 FORMAT(≠ ≠,T7,I2,T15,F10.4,T33,F8.5/)

     WRITE(61,740)
 740 FORMAT(≠ ≠,T15,≠FIXED ELEVATION≠/)

     WRITE(61,750)(I,ELEV(I),I=NB,NTOT)
 750 FORMAT(≠ ≠,T7,I2,T15,F9.3/)
     WRITE(61,840)
 840 FORMAT(≠0≠,T15,≠....................≠)

     END
```