AN ABSTRACT OF THE DISSERTATION OF

Zachary A. Gelbaum for the degree of Doctor of Philosophy in Mathematics presented on April 17, 2013.

Title: Some Results in Probability from the Functional Analytic Viewpoint

Abstract approved: ______________________________

Harold R. Parks

This dissertation presents some results from various areas of probability theory, the unifying theme being the use of functional analytic intuition and techniques. We first give a result regarding the existence of certain stochastic integral representations for Banach space valued Gaussian random variables. Next we give a spectral geometric construction of Gaussian random fields over various manifolds that generalize classical fractional Brownian motion. Lastly we present a result describing the limiting distribution for the largest eigenvalue of a product of two random matrices from the β-Laguerre ensemble.

Some Results in Probability from the Functional Analytic Viewpoint

by

Zachary A. Gelbaum

A DISSERTATION

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Presented April 17, 2013
Commencement June 2013

Doctor of Philosophy dissertation of Zachary A. Gelbaum presented on April 17, 2013.

APPROVED:

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Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Zachary A. Gelbaum, Author

ACKNOWLEDGEMENTS

I would like to first thank my advisor, Hal Parks, for his constant encouragement, generosity, and friendship during the period in which this work was completed. I owe him almost completely for the opportunity to devote myself to mathematics and work that was meaningful to me, and will be forever in his debt.

There are also a number of faculty, both at Oregon State and elsewhere that deserve thanks. Ralph Showalter had the largest impact on me mathematically; aside from my own predispositions, it is from him that I learned how to take a functional analytic viewpoint in mathematics and that this perspective pays dividends when taken. My other analysis teacher was David Finch, whose patience and time I also appreciated greatly. I should also thank Mina Ossiander for introducing me to the study of Gaussian processes. Christine Escher, Bill Bogley, Yevgeniy Kovchegov, Bob Burton, Bob Higdon, and Kirk Lancaster also deserve thanks for their encouragement and time.

I would very likely not have continued in mathematics had it not been for my undergraduate analysis teacher, Nets Katz. He encouraged me to keep going when I was most discouraged, and taught me that "what matters is the mathematics." I'm deeply thankful for his continued time and encouragement.

Lastly, but of course not the least, I must thank my family for their constant support. In particular I thank my wife, Mica, for her patience, friendship, and understanding while I completed this work.

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In loving memory of my grandfather, B.R. Gelbaum.

Some Results in Probability from the Functional Analytic Viewpoint

Chapter 1: Introduction

This dissertation is a collection of three papers I wrote while at Oregon State. As the title is vague, a few words are in order as to what I mean by "the functional analytic viewpoint." In the words of one of my teachers, functional analysis is what one gets when linear algebra is combined with point set topology. Thus the term is extremely broad. In general, functional analysis deals with linear operators on linear spaces, usually infinite dimensional, but I allow linear algebra to be subsumed in this subject as well. About such operators there is a deep and powerful theory, and to look at a question in mathematics from a functional analytic point of view means attempting to frame the problem, or parts of it, in terms of some linear operator(s), thus bringing the above theory to bear. Probably the most well known examples of this strategy come from partial differential equations, and below I will describe how it is applied to the problems I consider in the following chapters.

Brownian motion, B_t, $t \in [0, 1]$, is the fundamental example of a stochastic process. With probability one the paths of B_t are continuous, and one can show that Brownian motion determines a probability measure on the Banach space $C[0, 1]$. This is one way in which one is led to study measures on Banach spaces, and in the case of Gaussian measures there is a rich functional analytic theory. One the most important features of Brownian motion to emerge is that one can define an integral with respect to it, and in attempting to extend this theory of integration to other Gaussian processes many authors relied on the existence of certain integral representations for the process in question, i.e., the existence of an integral kernel, say on $[0, 1] \times [0, 1]$, which when integrated against Brownian motion yields a Gaussian process. Thus the question arises: When does such an integral kernel exist? This question can be answered in terms of certain linear operators between various Hilbert spaces related to the process in question, and this is the subject of Chapter 2. It turns out that under very general conditions such an integral kernel exists, and one corollary to the proof of that fact is that, again in very general situations, a given Gaussian process is determined by a certain unbounded, self-adjoint operator on a Hilbert space.

The most important class of Gaussian processes, which contains Brownian motion, is the class of fractional Brownian motions. In light of the above, one can wonder what are the corresponding unbounded operators. It turns out that they are powers of the Laplacian. In Chapter 3 I address a question that has attracted steady attention since the mid twentieth century, namely how to extend Brownian motion, and more generally fractional Brownian motion, from a Gaussian process indexed by $\mathbb{R}$ to a process indexed by a manifold, now called a random field. It is a fact of life that it is often easier to recognize a differential operator and its various extensions than to recognize the corresponding inverses, and this is one reason that focusing on the corresponding unbounded operator for fractional Brownian motion yields an approach to the above question that succeeds rather spectacularly compared to previous approaches. Again we see how focusing on the functional analytic aspects of the problem, the associated linear operators, bears fruit.

In the above two paragraphs we have described the use of linear operators to study certain stochastic processes, i.e., random functions. However, if one considers functional analysis and probability together, in particular if one has followed the usual analytical training whereby one passes from the study of functions, to spaces of functions, and then linear operators between them, it doesn't take long to arrive at the naive notion of a random linear operator as a possible extension of the theory of random functions. The fundamental example of a linear operator is a linear transformation on Euclidean space, that is, a matrix. Thus one could wonder if there is a theory of random matrices, and indeed there is rich, vast, and growing theory of such random operators. Much of the theory of random matrices is concerned with the behavior of the eigenvalues as the dimensions of the matrix in question approaches infinity.

In Chapter 4 we investigate this question for a product of random matrices of a certain type, and in fact this is the first such study of its kind for any class, or ensemble, of random matrices. The method is essentially to realize the random matrices in question as discrete approximations to a certain random differential operator, and many of the arguments follow the pattern of the classical numerical analysis of such deterministic differential operators, in particular the tools of functional analysis. The added ingredient is the random nature of the operators in question, however to the reader familiar with such tools, e.g., coercivity bounds, the general pattern of proof will be clear.

The chapters are ordered chronologically, Chapter 2 having been written first, and Chapter 4 most recently. What remained clear to me during the writing of each of these papers, at least when it was clear at all what was happening, was the functional analytic picture of linear operators between linear spaces, and it is my hope that this introduction will aid the reader in seeing this picture throughout the work.

Chapter 2: White Noise Representation of Gaussian Random Fields

2.1 Introduction

Much of literature regarding the representation of Gaussian fields as integrals against white noise has focused on processes indexed by $\mathbb{R}$, in particular canonical representations (most recently see [26] and references therein) and Volterra processes (e.g. [3, 6]). An example of the use of such integral representations is the construction of a stochastic calculus for Gaussian processes admitting a white noise representation with a Volterra kernel (e.g. [3, 36]).

In this paper we study white noise representations for Gaussian random variables in Banach spaces, focusing in particular on Gaussian random fields indexed by a measure space. We show that the existence of a representation as an integral against a white noise on a Hilbert space H is equivalent to the existence of a version of the field whose sample paths lie almost surely in H. For example a consequence of our results is that a centered Gaussian process Y_t indexed by $[0, 1]$ admits a representation

$$Y_t \stackrel{d}{=} \int_0^1 h(t, z)\, dW(z)$$

for some $h \in L^2([0, 1] \times [0, 1], d\nu \times d\nu)$, ν a measure on $[0, 1]$ and W the white noise on $L^2([0, 1], d\nu)$ if and only if there is a version of Y_t whose sample paths belong almost surely to $L^2([0, 1], d\nu)$.

The stochastic integral for Volterra processes developed in [36] depends on the existence of a white noise integral representation for the integrator. If there exists an integral representation for a given Gaussian field then the method in [36] can be extended to define a stochastic integral with respect to this field. We describe this extension for Gaussian random fields indexed by a measure space whose sample paths are almost surely square integrable.

Accepted for publication in *Communications on Stochastic Analysis*, 2013.

Section 2.2 contains preliminaries we will need from Malliavin Calculus and the theory of Gaussian measures over Banach spaces. In section 2.3, Theorem 2.3.1 gives our abstract representation theorem and Corollary 2.3.2 specializes to Gaussian random fields indexed by a measure space. Section 2.4 contains the extension of results in [36].

2.2 Preliminaries

2.2.1 Malliavin Calculus

We collect here only those parts of the theory that we will explicitly use, see [48].

Definition 2.2.1. Suppose we have a Hilbert space H. Then there exists a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a map $W : H \to L^2(\Omega, \mathbb{P})$ satisfying the following:

1. $W(h)$ is a centered Gaussian random variable with $E[W(h)^2] = \|h\|_H$
2. $E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_H$

This process is unique up to distribution and is called the *Isonormal* or *White Noise Process* on H.

The classical example is $H = L^2[0,1]$ and $W(h)$ is the Wiener-Ito integral of $h \in L^2$.

Let $\mathcal{S}$ denote the set of random variables of the form

$$F = f(W(h_1), ..., W(h_n))$$

for some $f \in C^\infty(\mathbb{R}^n)$ such that f and all its derivatives have at most polynomial growth at infinity. For $F \in \mathcal{S}$ we define the derivative as

$$DF = \sum_1^n \partial_j f(W(h_1), ..., W(h_n))h_j.$$

We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm induced by the inner product

$$\langle F, G \rangle_D = E[FG] + E[\langle DF, DG \rangle_H].$$

We also define a directional derivative for $h \in H$ as

$$D_h F = \langle DF, h \rangle_H.$$

D is then a closed operator from $L^2(\Omega)$ to $L^2(\Omega, H)$ and $\text{dom}(D) = \mathbb{D}^{1,2}$. Further, $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$. Thus we can speak of the adjoint of D as an operator from $L^2(\Omega, H)$ to $L^2(\Omega)$. This operator is called the divergence operator and denoted by δ. Next, $\text{dom}(\delta)$ is the set of all $u \in L^2(\Omega, H)$ such that there exists a constant c (depending on u) with

$$|\mathbb{E}[\langle DF, u\rangle_H]| \leq c\|F\|$$

for all $F \in \mathbb{D}^{1,2}$. For $u \in \text{dom}(\delta)$, $\delta(u)$ is characterized by

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_H]$$

for all $F \in \mathbb{D}^{1,2}$.

For examples and descriptions of the domain of δ see [48], section 1.3.1.

When we want to specify the isonormal process defining the divergence we write δ^W. We will also use the following notations interchangeably

$$\delta^W(u)\,, \int u\, dW.$$

2.2.2 Gaussian Measures on Banach Spaces

Here we collect the necessary facts regarding Gaussian measures on Banach spaces and related notions that we will use in what follows. For proofs and further details see e.g. [11, 41]. All Banach spaces are assumed real and separable throughout.

Definition 2.2.2. Let B be a Banach space. A probability measure μ on the Borel sigma field $\mathcal{B}$ of B is called *Gaussian* if for every $l \in B^*$ the random variable $l(x) : (B, \mathcal{B}, \mu) \to \mathbb{R}$ is Gaussian. The *mean* of μ is defined as

$$m(\mu) = \int_B x\, d\mu(x).$$

The measure μ is called *centered* if $m(\mu) = 0$. The *(topological) support* of μ in B, denoted B_0, is defined as the smallest closed subspace of B with μ-measure equal to 1.

Suppose we have a probability space $(\Omega, \mathcal{F}, P)$ and a measurable map $X : \Omega \to B$, i.e. X is a B-valued random variable. Then we say μ is the *distribution* of X if $P(X^{-1}(A)) = \mu(A)$ for any Borel set $A \subset B$. Such an X always exists, for we can let X be the identity map on B as B is a probability space with measure μ.

The mean of a Gaussian measure is always an element of B, and thus it suffices to consider only centered Gaussian measures as we can then acquire any Gaussian measure via a simple translation of a centered one. For the remainder of the paper all measures considered are centered.

Definition 2.2.3. The covariance of a Gaussian measure is the bilinear form $C_\mu : B^* \times B^* \to \mathbb{R}$ given by

$$C_\mu(k,l) = \mathbb{E}\left[k(X)l(X)\right] = \int_B k(x)l(x)\,d\mu(x).$$

Any Gaussian measure is completely determined by its covariance: if for two Gaussian measures μ, ν on B we have $C_\mu = C_\nu$ on $B^* \times B^*$ then $\mu = \nu$.

If H is a Hilbert space then

$$C_\mu(f,g) = \mathbb{E}[\langle X,f\rangle\langle X,g\rangle] = \int_B \langle x,f\rangle\langle x,g\rangle\,d\mu(x)$$

defines a continuous, positive, symmetric bilinear form on $H \times H$ and thus determines a positive symmetric operator K_μ on H. K_μ is of trace class and is injective if and only if $\mu(H) = 1$. Conversely, any positive trace class operator on H uniquely determines a Guassian measure on H [19]. Whenever we consider a Gaussian measure μ over a Hilbert space H we can after restriction to a closed subspace assume $\mu(H) = 1$ and do so throughout.

We will denote by H_μ the Reproducing Kernel Hilbert Space (RKHS) associated to a Gaussian measure μ on B . There are various equivalent constructions of the RKHS. We follow [55] and refer the interested reader there for complete details.

For any fixed $l \in B^*$, $C_\mu(l,\cdot) \in B$ (this is a non trivial result in the theory). Consider the linear span of these functions,

$$S = span\{C_\mu(l,\cdot)\ :\ l \in B^*\}.$$

Define an inner product on S as follows: if $\phi(\cdot) = \sum_1^n a_i C_\mu(l_i,\cdot)$ and $\psi(\cdot) = \sum_1^m b_j C_\mu(k_j,\cdot)$ then

$$<\phi,\psi>_{H_\mu} \equiv \sum_1^n \sum_1^m a_i b_j C_\mu(l_i,k_j).$$

H_μ is defined to be the closure of S under the associated norm $\|\cdot\|_{H_\mu}$. This norm is stronger than $\|\cdot\|_B$, H_μ is a dense subset of B_0 and H_μ has the reproducing property with reproducing kernel $C_\mu(l,k)$:

$$\langle \phi(\cdot), C_\mu(l,\cdot)\rangle_{H_\mu} = \phi(l) \qquad \forall\, l \in B^*,\, \phi \in H_\mu.$$

Remark 2.2.1. Often one begins with a collection of random variables indexed by some set, $\{Y_t\}_{t\in T}$. For example suppose (T,ν) is a finite measure space. Then setting $K(s,t) = \mathbb{E}[Y_sY_t]$ and supposing that application of Fubini-Tonelli is justified we have for $f,g \in L^2(T)$

$$\mathbb{E}[\langle Y,f\rangle\langle Y,g\rangle] = \int_T\int_T \mathbb{E}[Y_s,Y_t]f(s)g(t)\,d\nu d\nu = \langle K(s,t)(f),g\rangle$$

where we denote $\int_T K(s,t)f(s)d\nu(s)$ by $K(s,t)(f)$. If one verifies that this last operator is positive symmetric and trace class then the above collection $\{Y_t\}_{t\in T}$ determines a measure μ on $L^2(T)$ and the above construction goes through with $C_\mu(f,g) = \langle K(s,t)(f),g\rangle$ and the end result is the same with H_μ a space of functions over T.

Define H_X to be the closed linear span of $\{X(l)\}_{l\in B^*}$ in $L^2(\Omega,\mathbb{P})$ with inner product $\langle X(l),X(l')\rangle_{H_X} = C_\mu(l,l')$ (again for simplicity assume X is nondegenerate). From the reproducing property we can define a mapping R_X from H_μ to H_X given initially on S by

$$R_X(\sum_1^n c_kC_\mu(l_k,\cdot)) = \sum_1^k c_kX(l)$$

and extending to an isometry. This isometry defines the isonormal process on H_μ.

In the case that H is a Hilbert space and μ a Gaussian measure on H with covariance operator K it is known that $H_\mu = \sqrt{K}(H)$ with inner product $\langle\sqrt{K}(x),\sqrt{K}(y)\rangle_{H_\mu} = \langle x,y\rangle_H$.

It was shown in [40] that given a Banach space B there exists a Hilbert space H such that B is continuously embedded as a dense subset of H. Any Gaussian measure μ on B uniquely extends to a Gaussian measure μ_H on H. The converse question of whether a given Gaussian measure on H restricts to a Gaussian measure on B is far more delicate. There are some known conditions e.g. [23]. The particular case when X is a metric space and $B = C(X)$ has been the subject of extensive research [42]. Let us note here however that either $\mu(B) = 0$ or $\mu(B) = 1$ (an extension of the classical zero-one law, see [11]).

From now on we will not distinguish between a measure μ on B and its unique extension to H when it is clear which space we are considering.

2.3 White Noise Representation

2.3.1 The General Case

The setting is the following: B is a Banach space densely embedded in some Hilbert space H (possibly with $B = H$), where H is identified with its dual, $H = H^*$. (A Hilbert space equal to its dual in this way is called a Pivot Space, see [4]).

The classical definition of canonical representation has no immediate analogue for fields not indexed by $\mathbb{R}$, but the notion of strong representation does. Let $L : H_\mu \to H$ be unitary. Then $W_X(h) = R_X(L^*(h))$ defines an isonormal process on H and $\sigma(\{W_X(h)\}_{h\in H}) = \sigma(H_X) = \sigma(\{X(l)\}_{l\in B^*})$ where the last inequality follows from the density of H in B^*.

We now state our representation theorem.

Theorem 2.3.1. *Let B be a Banach space, μ a Gaussian measure on B, and C_μ the covariance of μ on $B^* \times B^*$. Then μ is the distribution of a random variable in B given as a white noise integral of the form*

$$X(l) = \int h(l)\, dW. \tag{3.1}$$

for some $h : B^* \to H$ and a Hilbert space H, where $h|_H$ is a Hilbert-Schmidt operator on H. Moreover, the representation is strong in the following sense: $\sigma(\{W_X(h)\}_{h\in H}) = \sigma(\{X(l)\}_{l\in B^*})$.

proof. Let $B \subset H = H^*$ as above. Let W_X be the isonormal process constructed above and $C_\mu(l, k)$ the covariance of μ. Let L be a unitary map from H_μ to H and define the function $k_L(l) : B^* \to H$ by

$$k_L(l) \equiv L(C_\mu(l, \cdot)).$$

Consider the Gaussian random variable determined by

$$Y(l) \equiv \int k_L(l)\, dW_X.$$

We have

$$\mathbb{E}[Y(l_1), Y(l_2)] = \langle k_L(l_1), k_L(l_2)\rangle_H = \langle C_\mu(l_1, \cdot), C_\mu(l_2, \cdot)\rangle_{H_\mu} = C_\mu(l_1, l_2)$$

so that μ is the distribution of $Y(l)$ and

$$X(l) \stackrel{d}{=} \int k_L(l)\, dW_X.$$

It is clear that k_L is linear and if $C_\mu(h_1, h_2) = \langle K(h_1), h_2\rangle_H,\ h_1, h_2 \in H$, then from above

$$k_L^* k_L = K.$$

Because K is trace class this implies that k_L is Hilbert-Schmidt on H.

From the preceding discussion we have $\sigma(\{W_X(h)\}_{h\in H}) = \sigma(\{X(l)\}_{l\in B^*})$. □

Remark 2.3.1. While the statement of the above theorem is more general than is needed for most applications, this generality serves to emphasize that having a "factorable" covariance and thus an integral representation are basic properties of all Banach space valued Gaussian random variables.

Remark 2.3.2. The kernel $h(l)$ is unique up to unitary equivalence on H, that is if $L' = UL$ for some unitary U on H L as above, then

$$\int h_{L'}(l)\, dW \stackrel{d}{=} \int U\left(h_L(l)\right)\, dW \stackrel{d}{=} \int h_L(l)\, dW.$$

Remark 2.3.3. In the proof above,

$$\langle k_L(l_1), k_L(l_2)\rangle_H = C_\mu(l_1, l_2) \tag{3.2}$$

is essentially the "canonical factorization" of the covariance operator given in [56], although in a slightly different form.

Remark 2.3.4. In the language of stochastic partial differential equations, what we have shown is that every Gaussian random variable in a Hilbert space H is the solution to the operator equation

$$L(X) = W$$

for some closed unbounded operator L on H with inverse given by a Hilbert-Schmidt operator on H.

2.3.2 Gaussian Random Fields

The proof of Theorem 2.3.1 has the following corollary for Gaussian random fields:

Corollary 2.3.2. *Let X be a Hausdorff space, ν a positive Radon measure on the Borel sets of X and $H = L^2(X, d\nu)$. If $\{B_x\}$ is a collection of centered Gaussian random variables indexed by X, then $\{B_x\}$ has a version with sample paths belonging almost surely H if and only if*

$$B_x \stackrel{d}{=} \int h(x,\cdot)\, dW \tag{3.3}$$

for some $h : X \to H$ such that the operator $K(f) \equiv \int_X h(x,z)f(z)\, d\nu(z)$ is Hilbert-Schmidt. In this case (3.2) takes the form

$$\mathbb{E}[B_x B_y] = \int_X h(x,z)h(y,z)\, d\nu(z).$$

In other words, the field B_x determines a Gaussian measure on $L^2(X, d\nu)$ if and only if B_x admits an integral representation (3.3).

2.3.3 Some Consequences and Examples

In principle, all properties of a field are determined by its integral kernel. Without making an exhaustive justification of this statement we give some examples:

In Corollary 2.3.2 above, being the kernel of a Hilbert-Schmidt operator, $h \in L^2(X \times X, d\nu \times d\nu)$. This means that we can approximate h by smooth kernels (supposing these are available). If we assume $h(x,\cdot)$ is continuous as a map from X to H i.e.

$$\lim_{x \to y} \|h(x,\cdot) - h(y,\cdot)\|_H = 0$$

for each $y \in X$ and let $h_n \in C^\infty(X)$, $h_n \xrightarrow{L^2} h$ it follows that $\|h_n(x,\cdot) - h(x,\cdot)\|_H \to 0$ pointwise so that if

$$B_x^n = \int h_n(x,\cdot)\, dW$$

we have

$$\mathbb{E}[B_x^n B_y^n] \to \mathbb{E}[B_x B_y]$$

pointwise. This last condition is equivalent to

$$B^n \xrightarrow{d} B$$

and we can approximate in distribution any field over X with a continuous (as above) kernel by fields with smooth kernels.

The kernel of a field over $\mathbb{R}^d$ describes its local structure [27]: The limit in distribution of

$$\lim_{\substack{r_n \to 0 \\ c_n \to 0}} \frac{X(t + c_n x) - X(t)}{r_n}$$

is

$$\lim_{\substack{r_n \to 0 \\ c_n \to 0}} \int \frac{h(t + c_n x) - h(t)}{r_n}\, dW$$

where h is the integral kernel of X, and this last limit is determined by the limit in H of

$$\lim_{\substack{r_n \to 0 \\ c_n \to 0}} \frac{h(t + c_n x) - h(t)}{r_n}.$$

The representation theorem yields a simple proof of the known series expansion using the RKHS. The setting is the same as in Theorem 2.3.1.

Proposition 2.3.3. *Let $Y(l)$ be a centered Gaussian random variable in a Hilbert space H with integral kernel $h(l)$. Let $\{e_k\}_1^\infty$ be a basis for H. Then there exist i.i.d. standard normal random variables $\{\xi_k\}$ such that*

$$Y(l) = \sum_1^\infty \xi_k \Phi_k(l)$$

where $\Phi_k(l) = \langle h(l), e_k \rangle_H$ and the series converges in $L^2(\Omega)$ and a.s.

proof. For each l

$$h(l) \stackrel{H}{=} \sum_1^\infty \Phi_k(l) e_k.$$

We have

$$Y(l) \stackrel{L^2}{=} \int \sum_1^\infty \Phi_k(l) e_k \, dW \stackrel{L^2}{=} \sum_1^\infty \Phi_k(l) \xi_k$$

where $\{\xi_k\} = \left\{\int e_k dW\right\}$ are i.i.d. standard normal as $\int dW$ is unitary from H to $L^2(\Omega)$. As $\{\Phi_k(l)\} \in l^2(\mathbb{N})$ the series converges a.s. by the martingale convergence theorem. □

2.4 Stochastic Integration

Combined with Theorem 2.3.1 above, [36] furnishes a theory of stochastic integration for a large class of Gaussian fields. In particular, by Corollary 2.3.2, if $(X, d\nu)$ is a (positive) radon measure space and B_x a centered gaussian random field indexed by X with sample paths almost surely belonging to $L^2(X, d\nu)$ then using [36] we can define a stochastic integral with respect to B_x as follows:

Denote by μ the distribution of $\{B_x\}$ in $H = L^2(X, d\nu)$ and as above the RKHS of B_x by $H_\mu \subset H$. Let

$$B_x = \int h(x, \cdot)\, dW$$

and $L^*(f) = \int h(x, y) f(y) d\nu(y)$. Then $L^* : H \to H_\mu$ is an isometry and the map $v \mapsto R_B(L^*(v)) \equiv W(v) : H \to H_B$ (H_B is the closed linear span of $\{B_x\}$ as defined in Section 2.2.2) defines an isonormal process on H. Denote this particular process by W in what follows.

First note that as $H_\mu = L^*(H)$ and L is unitary, it follows immediately that $\mathbb{D}^{1,2}_{H_\mu} = L^*(\mathbb{D}^{1,2}_H)$ where we use the notation in [48, 36] and the subscript indicates the underlying Hilbert space.

The following proof from [36] carries over directly: For a smooth variable $F(h) = f(B(L^*(h_1), ..., B(L^*(h_n))$ we have

$$\begin{aligned}
\mathbb{E}\langle D^B(F), u\rangle_{H_\mu} =& \mathbb{E}\langle \sum_1^n f'(B(L^*(h_1), ..., B(L^*(h_n))L^*(h_k), u\rangle_{H_\mu} \\
&= \mathbb{E}\langle \sum f'(B(L^*(h_1), ..., B(L^*(h_n))h_k, L(u)\rangle_H \\
&= \mathbb{E}\langle \sum f'(W(h_1), ..., W(h_n))h_k, L(u)\rangle_H \\
&= \mathbb{E}\langle D^W(F), L(u)\rangle_H
\end{aligned}$$

which establishes

$$\text{dom}(\delta^B) = L^*(\text{dom}(\delta^W))$$

and

$$\int L^*(u)\,dB = \int u\,dW \qquad \forall\, u \in \mathrm{dom}(\delta^W)$$

The series approximation in [36] also extends directly to this setting.

Theorem 2.4.1. *If $\{\Phi_k\}$ is a basis of H_μ then there exists i.i.d. standard normal $\{\xi_k\}$ such that:*

1. *If $f \in H$ then*

$$\int L^*(f)\,dB = \sum_1^\infty \langle L^*(f), \Phi_k\rangle_{H_\mu} \xi_k \quad a.s.$$

2. *If $u \in \mathbb{D}^{1,2}_{H_\mu}$ then*

$$\int u\,dB = \sum_1^\infty (\langle u, \Phi_k\rangle_{H_\mu} - \langle D^B_{\Phi_k} u, \Phi_k\rangle_{H_\mu}) \quad a.s.$$

proof. The proof follows that in [36].

□

Remark 2.4.1. For our purposes the method of approximation via series expansions above seems most appropriate. However in [3] a Riemann sum approximation is given under certain regularity hypotheses on the integral kernel of the process, and this could be extended in various situations as well.

Remark 2.4.2. The availability of the kernel above suggests the method in [3] whereby conditions are imposed on the kernel in order to prove an Ito Formula as promising for extension to more general settings.

Chapter 3: Fractional Brownian fields over Manifolds

The fractional Brownian motions and their stationary counterparts are the basic examples of Gaussian random fields over $\mathbb{R}$ and it is natural to ask what are the corresponding examples when $\mathbb{R}$ is replaced by a manifold. The first to do so was Paul Lévy (see [44]), who extended the standard Brownian motion on $\mathbb{R}$ to the standard Brownian field over $\mathbb{R}^d$, now called Lévy's Brownian motion. Lévy then extended this field to the sphere $\mathbb{S}^d$. Since then there have been a number of studies aimed at extending both the Brownian motion and the fractional Brownian motion to other manifolds. This is a natural step in the theory of Gaussian fields in general as one would like to understand how the structure of the index set determines the kinds of fields that can be defined over it. The geometric and topological structure of Riemannian manifolds make them a convenient and interesting setting for such a study. When one extends the fractional Brownian motions from $\mathbb{R}$ to $\mathbb{R}^d$ the resulting fields are called *Euclidean fractional Brownian fields* (some authors prefer *Lévy fractional Brownian motions*) and our purpose in this article is to construct fields over Riemannian manifolds that generalize the Euclidean fractional Brownian fields.

Much of the interest in the fractional Brownian fields (fBf's) over $\mathbb{R}^d$ stems from their distributional invariance and scaling properties. In particular, if $\alpha \in (0,1)$ denotes the Hurst index and the corresponding field is denoted by fBf^α, the increments of the fBf^α are invariant under rotation and translation and the distribution of the fBf^α scales by a power c^α when $\mathbb{R}^d$ is dilated by $c > 0$. Any extension of the fBf's should possess these properties and also reflect the geometry of the index set in question.

As mentioned above the first attempt to extend Lévy's Brownian motion, $fBf^{\frac{1}{2}}$, from $\mathbb{R}^d$ to a manifold was by Lévy himself in [44]. There he constructed a field over $\mathbb{S}^d$ with covariance given by

$$d(x,o) + d(y,o) - d(x,y),$$

$d(x,y)$ being the geodesic distance between x and y and o being a fixed origin point on the sphere. Further progress in this direction was made in the work of Molchan (see e.g. [47]) and Gangolli (see [30]) where the authors dealt with extensions of Lévy's Brownian motion to other manifolds including the sphere.

Accepted for publication in *Transactions of the American Mathematical Society*, 2013.

Most recently Istas in [37] studied fields over certain Riemannian manifolds with covariance given by

$$\frac{1}{2}\left(d(x,o)^{2\alpha}+d(y,o)^{2\alpha}-d(x,y)^{2\alpha}\right) \tag{3.1}$$

where $d(x,y)$ is the metric of the manifold and o is a chosen point. In particular Istas showed there that (1.1) defines a Gaussian field over compact rank one symmetric spaces and hyperbolic space $\mathbb{H}^d$ if and only if $\alpha \in (0,1/2]$.

A common feature of the above approaches is that they begin by looking for covariances of the form $f(x,o)+f(y,o)-f(x,y)$ for some symmetric function f; the idea being that, over $\mathbb{R}^d$, $o=0$ and $f(x,y)=\|x-y\|_{\mathbb{R}^d}$. The issue then is to prove that the function so defined does, in fact, define a covariance, i.e., one must establish positive definiteness. A necessary and sufficient condition for positive definiteness is that f be of *negative type*, for example one can take the above approach on metric spaces (X,d) with metric of negative type (e.g. [38, 34]). In general if $d(x,y)$ is the metric of a Riemannian manifold, establishing that $d(x,y)^{2\alpha}$ is of negative type for some $\alpha \in (0,1)$ is non-trivial and indeed, as in [37], it has been shown $d(x,y)^{2\alpha}$ can fail to be of negative type. Moreover, in all the above work this approach necessitates symmetry assumptions on the underlying manifold.

In the present article we take an essentially different approach inspired by the work of Benassi, Jaffard, and Roux (see [8] and more recently [9]). In particular we extend a characterization of the fBf^α in terms of the Laplacian on $\mathbb{R}^d$ to the Riemannian setting via the Laplace-Beltrami operator and the associated heat kernel. Using this approach we are able to extend the fBf^α to a variety of both compact and non-compact manifolds without any assumptions regarding symmetry of the manifolds and for the full range of $\alpha \in (0,1)$ (see Theorems 3.2.1-3.2.4 below).

Broadly speaking, in order to build a Gaussian random field over a manifold (or any index set) there are two things we must do: Determine a covariance function and prove that this covariance determines a probability measure on a suitable space of functions, e.g., some space of continuous functions. If we build our covariance correctly the resulting field will have the properties we would like, and we will be able to use some theorems from probability to show that we get a good probabilistic model, that is, a well defined random element of an appropriate function space.

This article is structured as follows: in Section 3.1 we cover some preliminaries regarding Gaussian random fields and analysis on manifolds, in particular the heat kernel of a Riemannian manifold. In Section 3.2.1 we describe the motivation behind our approach and define our candidate covariance functions before we study conditions which ensure these covariances exist for a given manifold in section 3.2.2. Section 3.2.3 deals with probability measures determined by our fields on a space of continuous functions and in 3.2.4 we establish the appropriate distributional invariance properties. In Section 3.3 we construct stationary counterparts to the fields of Section 3.2 and establish the corresponding distributional and sample path properties. Section 3.4 contains some open questions concerning geometry and probability encountered in the course of the article and in section 3.5 we collect some necessary results concerning sample path regularity of Gaussian fields over manifolds.

3.1 Preliminaries

3.1.1 Gaussian Random Fields

Given a complete probability space $(\Omega, \mathcal{F}, P)$ and some index set I we call a collection of random variables on Ω, $\{X_i(\omega)\}_{i\in I}$, a *Gaussian random field (GRF) over* I if for any finite subset $\{i_k\}_1^n \subset I$ the random vector $(X_{i_k})_1^n$ has a joint normal distribution. Then for each $\omega \in \Omega$, $X_i(\omega)$ defines a real valued function on I called a *sample path* of the field $\{X_i\}$. We let $\mathbb{E}$ denote the expectation operator,

$$\mathbb{E}[X_i] \equiv \int_\Omega X_i(\omega)\, dP(\omega) \qquad i \in I$$

and we call

$$\mathbb{E}[(X_s - \mathbb{E}[X_s])(X_t - \mathbb{E}[X_t])] = \mathbb{E}[X_s X_t] - \mathbb{E}[X_s]\mathbb{E}[X_t] \qquad s,t \in I$$

the *covariance* of $\{X_i\}$. The covariance of a GRF over I defines a symmetric positive definite function on $I \times I$.

We say two GRF's are equal in *finite dimensional distribution* or simply *in distribution*, denoted $\stackrel{d}{=}$, if their covariances are equal. We also say two GRF's defined on the same probability space are *versions* of each other if $P(X_i = Y_i) = 1$ for all $i \in I$. The salient analytical feature of GRF's is that for any set I the collection of all GRF's over I is in one to one correspondence up to equality in distribution with the set of all symmetric, positive definite functions on $I \times I$. In other words a GRF is uniquely determined in distribution by its covariance and every symmetric positive definite function K on $I \times I$ is the covariance of a GRF over I, that is, there exists some complete probability space $(\Omega, \mathcal{F}, P)$ and a *GRF* $\{X_i(\omega)\}_I$ where for each $i \in I$ X_i is a random variable on Ω.

We call a GRF *centered* if $\mathbb{E}[X_i] = 0 \ \forall i \in I$ and in this case its covariance is given by $\mathbb{E}[X_t X_s]$, $s, t \in I$. Throughout this article we will only consider centered GRF's.

3.1.1.1 The Euclidean Fractional Brownian Fields

The standard Brownian motion B_t over $[0, \infty)$ is the centered GRF with covariance

$$\mathbb{E}[B_s B_t] = \frac{|s| + |t| - |t - s|}{2}.$$

From this one generalizes to obtain the fractional Brownian motion fBm^α for $\alpha \in (0, 1)$:

$$\mathbb{E}[fBm_s^\alpha fBm_t^\alpha] = \frac{|s|^{2\alpha} + |t|^{2\alpha} - |t - s|^{2\alpha}}{2}.$$

We then have $B_t = fBm^{\frac{1}{2}}$.

One then further generalizes to $\mathbb{R}^d$, obtaining the fBf^α as the centered GRF over $\mathbb{R}^d$ with covariance

$$\mathbb{E}[fBf_x^\alpha fBf_y^\alpha] = \|x\|_{\mathbb{R}^d}^{2\alpha} + \|y\|_{\mathbb{R}^d}^{2\alpha} - \|x - y\|_{\mathbb{R}^d}^{2\alpha}$$

(note that some authors include the constant factor 1/2). We remark here that throughout the article we will make a slight abuse of notation and use $\mathbb{R}^d$ to refer both to the usual vector space and to Euclidean space as a manifold, though we doubt this will cause much confusion as the context will make clear what is meant.

One easily sees that the fBf^α is *self similar* of order α, i.e., if fBf_c^α denotes the field rescaled field $\{fBf_{cx}^\alpha\}_{x \in \mathbb{R}^d}$ then

$$fBf_c^\alpha \stackrel{d}{=} c^\alpha fBf^\alpha \qquad \forall\, c > 0,$$

and that it has *stationary (or homogeneous) increments*:

$$\mathbb{E}[|fBf_x^\alpha - fBf_y^\alpha|^2] = \|x - y\|^{2\alpha} = \|\iota(x) - \iota(y)\|^{2\alpha} = \mathbb{E}[|fBf_{\iota(x)}^\alpha - fBf_{\iota(y)}^\alpha|^2]$$

for any isometry ι on $\mathbb{R}^d$. Moreover it is known that there exists a version X_x of the fBf^α such that with probability one the sample paths $X_x(\omega)$ are Hölder continuous of any order

$\gamma < \alpha$ and fail to be Hölder continuous of any order $\gamma > \alpha$ at every point in $\mathbb{R}^d$ (see [1]).

3.1.1.2 White Noise

The treatment here follows [39]. Given a probability space $(\Omega, \mathcal{F}, P)$ we call a complete subspace G of $L^2(\Omega, \mathcal{F}, P)$ a *Gaussian Hilbert space* if every element of G is a centered Gaussian random variable. Note that the inner product H inherits from $L^2(\Omega, \mathcal{F}, P)$ is then

$$\langle X, Y \rangle_G = \mathbb{E}[XY].$$

Given any (real) Hilbert space H there exists a Gaussian Hilbert space G and a unitary map $W : H \to G$ called the *isonormal process* or *white noise process* on H (one can also consider complex white noises). If, as is the case below, $H = L^2(M, \mathcal{S}, d\mu)$ for some measure space $(M, \mathcal{S}, d\mu)$ then if $B = \{A \in \mathcal{S} : \mu(A) < \infty\}$ the map from $B \to G$ given by

$$W(A) \equiv W(\chi_A)$$

determines a Gaussian random measure on M. The properties of such measures will not be important for us here, but we mention them to motivate the notation for $W : H \to G$, given by

$$W(f) = \int_M f(z)\, dW(z),$$

which we refer to as a *white noise integral* (this is also commonly called a *stochastic integral*). Starting from a random measure one can construct the integral $\int_M dW$ in close analogy with classical measure theory. All that will be important for us is the property

$$\langle f, g \rangle_H = \mathbb{E}\left[\int_M f\, dW \cdot \int_M g\, dW\right].$$

Now suppose we have a function $h(x, z) : M \to L^2(M, d\mu)$, $x \mapsto h(x, z) \in L^2(M, d\mu(z))$. We can then define a centered GRF Y_x over X by

$$Y_x \overset{d}{=} \int_M h(x, z)\, dW(z).$$

The covariance of Y_x is then given by

$$\mathbb{E}[Y_x Y_y] = \langle h(x,z), h(y,z)\rangle_{L^2} = \int_M h(x,z)h(y,z)\, d\mu(z).$$

Note that the last expression on the right is in fact positive definite and symmetric. In this case we call h the *integral kernel* of Y.

3.1.2 Analysis on Manifolds

In what follows we assume throughout that all Riemannian manifolds are complete and of dimension d, with $2 \leq d < \infty$. For a manifold M let Δ denote the Laplace-Beltrami operator, or simply the Laplacian for short, on M. In any local coordinate system the action of Δ on $C^\infty(M)$ is given by

$$\Delta = \frac{1}{\sqrt{g}} \sum \partial_j \left(g^{ij} \sqrt{g}\, \partial_i \right)$$

where (g_{ij}) is the matrix of the Riemannian metric in these coordinates, $(g^{ij}) = (g_{ij})^{-1}$, and $\sqrt{g} = (det(g_{ij}))^{\frac{1}{2}}$. Because M is complete, Δ is essentially self adjoint (see e.g. [54]) and so we may consider from now on the unique minimal self-adjoint extension of Δ, which we shall write as Δ also. Moreover the spectrum of Δ is contained in $(-\infty, 0]$ (see e.g. [54]). By the spectral theorem we can define the heat semigroup

$$e^{t\Delta} = \int_0^\infty e^{-t\lambda}\, dE_\lambda$$

where dE_λ is the spectral measure of $-\Delta$. The action of $e^{t\Delta}$ on $L^2(M, dV_g)$, where dV_g denotes the measure derived from the metric g, is given by a kernel $H_t(x,y)$:

$$e^{t\Delta}(f)(x) = \int_M H_t(x,y) f(y)\, dV_g(y).$$

$H_t(x,y)$ is called the *heat kernel* of M. It is known that H_t is strictly positive, symmetric, and contained in $C^\infty(M \times M \times (0,\infty))$. Moreover we have the semigroup property

$$\int_M H_t(x,z) H_s(z,y)\, dV_g(z) = H_{t+s}(x,y).$$

As a consequence H_t is positive definite for each $t > 0$. As its name suggests, $H_t(x,y)$ is a fundamental solution to the heat equation on $M \times (0,\infty)$:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \Delta_x\right) H_t(x,y) = 0 \\ \lim\limits_{t\downarrow 0} \displaystyle\int_M H_t(x,y) f(x)\, dx = f(y) \quad \forall f \in C_0(M). \end{cases}$$

There are various constructions of the heat kernel, that given in [18] being most suited to our purposes. In particular if we let

$$\mathcal{E}_t(x,y) \equiv \frac{e^{-\frac{d(x,y)^2}{4t}}}{\sqrt{(4\pi t)^d}}$$

then there is an open neighborhood of the diagonal $U \subset M \times M$ such that on U

$$\frac{H_t(x,y)}{\mathcal{E}_t(x,y)} = \Phi(t,x,y) \tag{3.2}$$

where $\Phi(t,x,y)$ is symmetric in x and y, $\Phi \in C^k([0,T] \times U)$ $\forall\, T > 0$ where k can be chosen arbitrarily large (see [15] and [10]), and

$$\lim_{t\to 0,\, x\to y} \Phi(t,x,y) = 1.$$

In other words, for x and y close $H_t \sim \mathcal{E}_t$ as $t \to 0$. Thus on any manifold heat diffusion behaves locally for small times as in Euclidean space.

If M is compact then we also have the following eigenfunction expansion of H_t:

$$H_t(x,y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x)\phi_k(y) \tag{3.3}$$

where $0 = \lambda_0 < \lambda_1 \leq ... \leq \lambda_k \uparrow \infty$ and $\{\phi_k\}$ are the spectrum and orthonormalized L^2 eigenfunctions of $-\Delta$ respectively and where (3.3) converges absolutely and uniformly for each $t > 0$ (see [15]).

Following [15] we define a *regular domain* to be an open, connected, relatively compact subset D of a complete Riemannian manifold such that $\partial D \neq \emptyset$ is smooth. In what follows when we refer to the Laplacian of a regular domain we mean the Dirichlet Laplacian with corresponding the heat kernel (see [15], Chapter 7). As in the compact case we have an eigenfunction expansion (3.3), the only difference being that $\lambda_0 > 0$. If (M, g) is a regular domain in a manifold (N, g) then, as noted in [16], (3.2) holds in this setting as well.

Now suppose M is complete and non-compact, $\{D_k\}_1^\infty$ is any increasing exhaustion of M by regular domains, and $H_t^k(x, y)$ denotes the Dirichlet heat kernel of D_k. Then if we extend each H^k to be zero outside $\overline{D} \times \overline{D}$, $\{H_t^k(x, y)\}_1^\infty$ forms a pointwise increasing sequence on $M \times M \times (0, \infty)$. It was shown in [24] that

$$\lim_{k\to\infty} H_t^k(x, y) = H_t(x, y)$$

where $H_t(x, y)$ is the heat kernel defined above.

3.2 The Riesz Fields

3.2.1 Motivation and Definition

As mentioned in the introduction, our first task is to write down a candidate covariance for our fields. We could write down all the properties we want our field to have and see if this determines a covariance, however even on $\mathbb{R}^d$ this is non-trivial and as we shall see below, on a general manifold the properties of the Euclidean fractional Brownian fields described above do not uniquely determine a GRF. The other strategy is to find a characterization of the Euclidean fields that suggests a generalization to manifolds and then verify that this ansatz does indeed yield a probability measure on a nice function space with the properties we want. This is the strategy we will follow, and so our first task is to find a suitable characterization of the Euclidean field fBf^α.

In [8] the authors begin by defining a symbol class of pseudodifferential operators over $\mathbb{R}^d$. From such an operator A they define a Gaussian random field with covariance given by the integral kernel of A^{-1}. The authors are then able to derive all the important properties of this field from properties of the symbol of the operator A. This approach to constructing and studying GRF's is a natural extension of the classical spectral theory of Gaussian processes on $\mathbb{R}$ and demonstrates of the power of the spectral point of view.

The basic heuristic can be described as follows: Beginning with an unbounded operator A on some L^2 space, define and study the GRF determined by the integral kernel of A^{-1}. So in attempting to extend the fBf^α to a Riemannian manifold, we should first seek an operator A that determines the fBf^α in the manner above.

Our starting point is the well known (e.g. [8] or [59]) spectral representation of the fBf^α,

$$fBf_x^\alpha \stackrel{d}{=} C_{d,\alpha} \int_{\mathbb{R}^d} \frac{e^{i\langle x,\xi\rangle} - 1}{\|\xi\|^{\frac{d}{2}+\alpha}} \, d\widehat{W}(\xi), \tag{3.4}$$

where $\widehat{W}$ is a complex white noise on $L^2(\mathbb{R}^d, dx)$, dx being Lebesgue measure, and $C_{d,\alpha}$ is a constant. Examining (3.4) we see that, formally (for example, for f such that $\hat{f} = \|\xi\|^{\frac{d}{2}+\alpha}\hat{g}$ for some $g \in C_c^\infty$) and up to a constant,

$$\int_{\mathbb{R}^d} \frac{e^{i\langle x,\xi\rangle} - 1}{\|\xi\|^{\frac{d}{2}+\alpha}} \hat{f}(\xi) \, d\xi = (-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}(f)(x) - (-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}(f)(0).$$

Thus if we denote this last operator above by A then the fBf^α is the unique (in distribution) GRF with covariance given by the Schwarz kernel of the operator A^*A,

$$\mathbb{E}[fBf_x^\alpha fBf_y^\alpha] = C \int_{\mathbb{R}^d} \frac{e^{i\langle x-y,\xi\rangle} - e^{i\langle x,\xi\rangle} - e^{i\langle y,\xi\rangle} + 1}{\|\xi\|^{d+2\alpha}} \, d\xi.$$

We now have a characterization that extends immediately to manifolds: Simply replace the Laplacian on $\mathbb{R}^d$ by the Laplace-Beltrami operator of the manifold in question and determine the kernel of the operator A^*A. Following [54] we arrive at the following definitions:

Definition 3.2.1. For a complete Riemannian manifold M with heat kernel $H_t(x,y)$ define the *Riesz field* R^α to be the GRF with covariance given by

$$\mathbb{E}[R_x^\alpha R_y^\alpha] \equiv \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} \left(H_t(x,y) - H_t(x,o) - H_t(y,o) + H_t(o,o)\right) \, dt \tag{3.5}$$

where $o \in M$ is a fixed "origin" and the *stationary (or homogeneous) Riesz field* hR^α is the GRF with covariance

$$\mathbb{E}[hR_x^\alpha hR_y^\alpha] \equiv \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} H_t(x,y) \, dt. \tag{3.6}$$

Because $H_t(x, y)$ is positive definite for each $t > 0$ and

$$\begin{aligned} &H_t(x,y) - H_t(x,o) - H_t(y,o) + H_t(o,o) \\ &\quad = \int_M \left(H_{t/2}(x,z) - H_{t/2}(o,z)\right)\left(H_{t/2}(y,z) - H_{t/2}(o,z)\right) dV_g(z), \end{aligned}$$

each of these expressions is symmetric and positive definite, and thus when the integrals exist each determines a GRF over M. Of course the convergence of the above integrals is by no means obvious and our first task in Section 3.2.2 will be to determine manifolds for which they do converge.

Remark 3.2.1. We will see shortly that if either (3.5) or (3.6) exist for some $\alpha_0 \in (0,1)$ then it also exists for any $\alpha \in (0, \alpha_0)$. We say R^α (resp. hR^α) *exists* for all $\alpha \in (0, b)$ if (3.5) (resp. (3.6)) is finite for all $\alpha \in (0, b)$, $b \leq 1$, and all $x, y \in M$.

It turns out (Proposition 3.2.5) that the Riesz field (3.5) extends the fBf^α and that they agree up to a constant in distribution over $\mathbb{R}^d$. However we will also see that the stationary Riesz field has some claim to be an extension of the fBf^α, for example over negatively curved manifolds, even though it does not exist on $\mathbb{R}^d$.

Now let W denote the white noise over $L^2(M, dV_g)$. We will show that when they exist the Riesz fields admit the following integral representations:

$$R_x^\alpha \stackrel{d}{=} \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_M \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} \left(H_t(x,z) - H_t(o,z)\right) \, dt \, dW(z) \tag{3.7}$$

and

$$hR_x^\alpha \stackrel{d}{=} \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_M \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_t(x,z) \, dt \, dW(z). \tag{3.8}$$

The issue is whether or not the functions appearing in the above are in fact square integrable for each $x \in M$. Let us consider this in detail, first for hR^α:

Letting

$$h_{hR}(x,z) = \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_t(x,z) \, dt$$

we have

$$
\begin{aligned}
&\langle h_{hR}(x,z), h_{hR}(y,z)\rangle_{L^2} \\
&\qquad = \int_M \left(\frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)} \int_0^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1} H_t(x,z)\,dt \right) \\
&\qquad\qquad \times \left(\frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)} \int_0^\infty s^{\frac{d}{4}+\frac{\alpha}{2}-1} H_t(y,z)\,ds \right) dV_g(z) \\
&\qquad = \int_M \int_0^\infty \int_0^\infty \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)^2} t^{\frac{d}{4}+\frac{\alpha}{2}-1} s^{\frac{d}{4}+\frac{\alpha}{2}-1} H_t(x,z) H_s(y,z)\,dt\,ds\,dV_g(z) \\
&\qquad = \int_0^\infty \int_0^\infty \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)^2} t^{\frac{d}{4}+\frac{\alpha}{2}-1} s^{\frac{d}{4}+\frac{\alpha}{2}-1} \int_M H_t(x,z) H_s(y,z)\,dV_g(z)\,dt\,ds \\
&\qquad = \int_0^\infty \int_0^\infty \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)^2} t^{\frac{d}{4}+\frac{\alpha}{2}-1} s^{\frac{d}{4}+\frac{\alpha}{2}-1} H_{t+s}(x,y)\,dt\,ds \\
&\qquad = \int_0^\infty \int_s^\infty \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)^2} (t-s)^{\frac{d}{4}+\frac{\alpha}{2}-1} s^{\frac{d}{4}+\frac{\alpha}{2}-1} H_t(x,y)\,dt\,ds \\
&\qquad = \int_0^\infty \int_0^t \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)^2} (t-s)^{\frac{d}{4}+\frac{\alpha}{2}-1} s^{\frac{d}{4}+\frac{\alpha}{2}-1}\,ds H_t(x,y)\,dt
\end{aligned}
$$

by the positivity of $H_t(x,y)$ and the semigroup property.

Next note that if $g(s) = \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)} s^{\frac{d}{4}+\frac{\alpha}{2}-1}$ then

$$
\int_0^t \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)^2} (t-s)^{\frac{d}{4}+\frac{\alpha}{2}-1} s^{\frac{d}{4}+\frac{\alpha}{2}-1}\,ds = g * g(t)
$$

where $*$ denotes the finite convolution $f * g(t) \equiv \int_0^t f(t-s) g(s)\,ds$. If $\mathcal{L}$ denotes the Laplace transform we have the well known property $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$. Applying this to $g * g$ above we have

$$
\mathcal{L}(g * g)(s) = (\mathcal{L}(g))^2(s) = \left(s^{-\left(\frac{d}{4}+\frac{\alpha}{2}\right)}\right)^2 = s^{-\left(\frac{d}{2}+\alpha\right)}.
$$

Then inverting $\mathcal{L}$ we obtain

$$\frac{1}{\Gamma(\frac{d}{2}+\alpha)}t^{\frac{d}{2}+\alpha-1} = \mathcal{L}^{-1}\left(s^{-(\frac{d}{2}+\alpha)}\right) = \int_0^t \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)^2}(t-s)^{\frac{d}{4}+\frac{\alpha}{2}-1}s^{\frac{d}{4}+\frac{\alpha}{2}-1}\,ds.$$

Substituting this into the integral defining $\langle h_{hR}(x,z), h_{hR}(y,z)\rangle_{L^2}$ above yields

$$\frac{1}{\Gamma(\frac{d}{2}+\alpha)}\int_0^\infty t^{\frac{d}{2}+\alpha-1}H_t(x,y)\,dt.$$

Thus whenever hR^α exists it is given by (3.8).

Turning now to (3.5), let $h_R(x,z) = \dfrac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)}\displaystyle\int_0^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1}\left(H_t(x,z)-H_t(o,z)\right)\,dt$. Then

$$\begin{aligned}
&\|h_R(x,z)\|_{L^2}^2 \\
&\quad\le \int_M\int_0^\infty\int_0^\infty s^{\frac{d}{4}+\frac{\alpha}{2}-1}t^{\frac{d}{4}+\frac{\alpha}{2}-1}|H_t(x,z)-H_t(o,z)||H_s(x,z)-H_s(o,z)|\,ds\,dt\,dV_g(z)\\
&\quad= \int_0^\infty\int_0^\infty s^{\frac{d}{4}+\frac{\alpha}{2}-1}t^{\frac{d}{4}+\frac{\alpha}{2}-1}\int_M |H_t(x,z)-H_t(o,z)||H_s(x,z)-H_s(o,z)|\,dV_g(z)\,ds\,dt\\
&\quad\le \int_0^\infty\int_0^\infty s^{\frac{d}{4}+\frac{\alpha}{2}-1}t^{\frac{d}{4}+\frac{\alpha}{2}-1}\|H_t(x,\cdot)-H_t(o,\cdot)\|_2\|H_s(x,\cdot)-H_s(o,\cdot)\|_2\,ds\,dt\\
&\quad= \left(\int_0^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1}\|H_t(x,\cdot)-H_t(o,\cdot)\|_2\,dt\right)^2\\
&\quad= \left(\int_0^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1}\sqrt{H_t(x,x)-2H_t(x,o)+H_t(o,o)}\,dt\right)^2.
\end{aligned}$$

Recall that if M is any Riemannian manifold then from (3.2) for any $x,y\in M$ we have that $H_t(x,y) = O(t^{-\frac{d}{2}})$ as $t\to 0$. So then

$$\int_0^1 t^{\frac{d}{4}+\frac{\alpha}{2}-1}\sqrt{H_t(x,x)-2H_t(x,o)+H_t(o,o)}\,dt < \infty$$

and

$$\int_0^1 t^{\frac{d}{2}+\alpha-1}\left(H_t(x,x)-2H_t(x,o)+H_t(o,o)\right)\,dt < \infty$$

for all $\alpha\in(0,1)$.

Next notice that if $\alpha + \epsilon < b$

$$\begin{aligned}\int_1^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1}\sqrt{H_t(x,x)-2H_t(x,o)+H_t(o,o)}\,dt \\ &= \int_1^\infty t^{\frac{d}{4}+\frac{\alpha}{2}+\epsilon-(1+\epsilon)}\sqrt{H_t(x,x)-2H_t(x,o)+H_t(o,o)}\,dt \\ &\leq \left(\int_1^\infty t^{-(1+\epsilon)}\,dt\right)^{\frac{1}{2}}\left(\int_1^\infty t^{\frac{d}{2}+\alpha+\epsilon-1}\left(H_t(x,x)-2H_t(x,o)+H_t(o,o)\right)\,dt\right)^{\frac{1}{2}}\end{aligned}$$

by Cauchy-Schwarz. Thus if R^α exists for all $\alpha \in (0,b)$ we may interchange the order of integration as with hR^α to obtain

$$\begin{aligned}\langle h_R(x,z), h_R(y,z)\rangle_{L^2} \\ &= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)}\int_0^\infty t^{\frac{d}{2}+\alpha-1}\left(H_t(x,y)-H_t(x,o)-H_t(y,o)+H_t(o,o)\right)\,dt \\ &= \mathbb{E}[R_x^\alpha R_y^\alpha]\end{aligned}$$

for all such α.

In either case of (3.5) or (3.6) we see that the integrands are continuous on $(0,\infty)$ so by (3.2) convergence depends only on the behavior of the integrand at infinity. Thus the existence of both R_x^α and hR_x^α will depend on the large-time asymptotics of $H_t(x,y)$. These depend on the manifold in question and we will treat distinct cases below.

3.2.2 Existence

3.2.2.1 The Compact Case

We have the following:

Theorem 3.2.1. *If M is a compact Riemannian manifold, then the Riesz field of order α exists over M for any $\alpha \in (0,1)$ and the stationary Riesz field does not exist over M for any $\alpha \in (0,1)$.*

proof. Recall (3.3):

$$H_t(x,y) = \sum_{k=0}^\infty e^{-\lambda_k t}\phi_k(x)\phi_k(y).$$

We have

$$H_t(x,x) - 2H_t(o,x) + H_t(o,o) = \sum_{k=1}^{\infty} e^{-\lambda_k t}|\phi_k(x) - \phi_k(o)|^2 = O(e^{-\lambda_1 t}) \qquad \forall\, x \in M$$

and $\lambda_1 > 0$. Then (3.5) is clearly finite for any $x \in M$ and all $\alpha \in (0,1)$.

To see that hR_x^α does not exist on M notice that $\lim_{t\to 0} H_t(x,y) = \mathrm{Vol}(M)^{-1} \neq 0$ $\forall\, x, y \in M$.

□

Theorem 3.2.2. *If M is a regular domain then hR^α, and thus by linearity R^α, exists for any $\alpha \in (0,1)$.*

proof. As above let

$$H_t(x,y) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\phi_k(x)\phi_k(y).$$

Then $\lambda_0 > 0$ and $H_t(x,y) = O(e^{-\lambda_0 t})$ for each $x, y \in M$.

□

We note here that in either case above we may integrate term by term using the eigenfunction expansions of H_t to obtain a series expression for the covariance of R^α and hR^α as follows: For R^α and M compact we have

$$\begin{aligned}
\mathbb{E}[R_x^\alpha R_y^\alpha] &= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} H_t(x,y) - H_t(x,o) - H_t(y,o) + H_t(o,o)\, dt \\
&= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} \sum_{k=0}^{\infty} e^{-\lambda_k t}(\phi_k(x) - \phi_k(o))(\phi_k(y) - \phi_k(o))\, dt \\
&= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} \sum_{k=1}^{\infty} e^{-\lambda_k t}(\phi_k(x) - \phi_k(o))(\phi_k(y) - \phi_k(o))\, dt \\
&\leq \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \left(\int_0^\infty t^{\frac{d}{2}+\alpha-1} \sum_{k=1}^{\infty} e^{-\lambda_k t}|\phi_k(x) - \phi_k(o)|^2\, dt\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^\infty t^{\frac{d}{2}+\alpha-1} \sum_{k=1}^{\infty} e^{-\lambda_k t}|\phi_k(y) - \phi_k(o)|^2\, dt\right)^{\frac{1}{2}} \\
&= \left(\mathbb{E}[|R_x^\alpha|^2]\mathbb{E}[|R_y^\alpha|^2]\right)^{\frac{1}{2}},
\end{aligned}$$

which we know from above to be finite.

Then by dominated convergence we may integrate term by term to obtain

$$\begin{aligned}\mathbb{E}[R_x^\alpha R_y^\alpha] &= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{d}{2}+\alpha\right)}{\lambda_k^{\frac{d}{2}+\alpha}}(\phi_k(x)-\phi_k(o))(\phi_k(y)-\phi_k(o)) \\ &= \sum_{k=1}^{\infty}(\lambda_k)^{-\left(\frac{d}{2}+\alpha\right)}(\phi_k(x)-\phi_k(o))(\phi_k(y)-\phi_k(o)).\end{aligned}$$

In particular

$$R_x^\alpha \stackrel{d}{=} \sum_{k=1}^{\infty}(\lambda_k)^{-\left(\frac{d}{4}+\frac{\alpha}{2}\right)}(\phi_k(x)-\phi_k(o))\xi_k$$

where $\{\xi_k\}$ is an i.i.d. collection of standard normal random variables, the series converging in $L^2(M)$ almost surely.

The same equality holds for M a regular domain if we number the spectrum as $\{\lambda_k\}_1^\infty$. Similar arguments show that for M a regular domain

$$\mathbb{E}[hR_x^\alpha hR_y^\alpha] = \sum_{k=1}^{\infty}(\lambda_k)^{-\left(\frac{d}{2}+\alpha\right)}\phi_k(x)\phi_k(y)$$

and

$$hR_x^\alpha \stackrel{d}{=} \sum_{k=1}^{\infty}(\lambda_k)^{-\left(\frac{d}{4}+\frac{\alpha}{2}\right)}\phi_k(x)\xi_k.$$

Example 3.2.1. Let $M=\mathbb{S}^2$. Then in terms of the spherical harmonics $\{Y_{km}\}$ we have

$$H_t(x,y) = \sum_{k=0}^{\infty}e^{-k(k+1)t}\sum_{m=-k}^{k}Y_{km}(x)Y_{km}(y).$$

Applying the harmonic addition formula we have

$$H_t(x,y) = \sum_{k=0}^{\infty}e^{-k(k+1)t}\frac{2k+1}{4\pi}P_k(\cos\theta_{xy})$$

where P_k is the k-th Legendre Polynomial and $\langle x,y\rangle = \cos\theta_{xy}$. Fixing an origin point

$o \in \mathbb{S}^2$ we then have

$$\mathbb{E}[R_x^\alpha R_y^\alpha] = \sum_{k=1}^{\infty} (k(k+1))^{-(\frac{d}{2}+\alpha)} \frac{2k+1}{4\pi} \left(P_k(\cos\theta_{xy}) - P_k(\cos\theta_{xo}) - P_k(\cos\theta_{yo}) + P_k(1)\right).$$

Example 3.2.2. Let $M = \mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ and J_k the Bessel function of the first kind of order k, $k = 0, 1, 2...$ Then if $\lambda_k^1 < \lambda_k^2 < ...$ are the positive zeroes of J_k, using polar coordinates on $\mathbb{D}$ we have

$$\mathbb{E}\left[hR^\alpha_{(r,\theta)} hR^\alpha_{(R,\phi)}\right] = \frac{\sqrt{2}}{\pi} \sum_{k,l} \frac{(\lambda_k^l)^{-(d+2\alpha)}}{|J_{k+1}(\lambda_k^l)|} J_k(\lambda_k^l r) J_k(\lambda_k^l R) \left(\cos(k(\theta - \phi)) + \sin(k(\theta + \phi))\right).$$

3.2.2.2 The Non-Compact Case

For the case of M non-compact, first let us show by example that we cannot establish existence in general.

Example 3.2.3. Let $M = \mathbb{S}^1 \times \mathbb{R}$. Then we have

$$H_t^M((\theta, x), (\phi, y)) = H_t^{\mathbb{S}}(\theta, \phi) H_t^{\mathbb{R}}(x, y)$$

where H^M is the heat kernel of M, $H^{\mathbb{S}}$ is the heat kernel of $\mathbb{S}^1$, and $H^{\mathbb{R}}$ is the usual heat kernel on $\mathbb{R}$ (see [33], Theorem 9.11).

We then have that

$$\begin{aligned} H_t^M((\theta,x),(\theta,x)) - 2H_t^M((\theta,x),(\phi,y) + H_t^M((\phi,y),(\phi,y)) &\sim \frac{1}{\pi} \frac{1 - e^{\frac{-|x-y|^2}{4t}}}{\sqrt{(4\pi t)}} \\ &= O(t^{\frac{3}{2}}) \quad \text{as } t \to \infty \end{aligned}$$

for any $(\theta, x), (\phi, y) \in M$. So $\mathbb{E}[|R_p^\alpha|^2] = \infty \; \forall p \in M$ and $\alpha \geq 1/2$ and thus R^α does not exist over M for this range of α. Using $\mathbb{S}^2$ instead of $\mathbb{S}^1$ in the above we have that R^α fails to exist for all $\alpha \in (0, 1)$.

Example 3.2.3 notwithstanding, for certain manifolds such that $\text{Vol}(M) < \infty$ we have a situation similar to the compact case:

Theorem 3.2.3. *Suppose M is non-compact with $Ric(M) \geq -\kappa^2$, $\kappa \in \mathbb{R}$, and $Vol(M) < \infty$. Let $\overline{\lambda}(M) = \inf_{\Omega \subset M} \{\lambda_1 : \sigma(\Omega) = \{\lambda_k\}_{k=0}^{\infty}\}$ where the infimum is taken over* *regular domains $\Omega \subset M$ and $\sigma(\Omega)$ denotes the Dirichlet spectrum of Ω. Then if $\overline{\lambda}(M) > 0$ R^α exists over M for any $\alpha \in (0,1)$ and hR^α does not.*

proof. That hR^α does not exist follows from the fact that on such M

$$\lim_{t\to\infty} H_t(x,y) = \frac{1}{\mathrm{Vol}(M)} \neq 0 \qquad \forall\, x,y \in M.$$

For R^α, under the hypothesis of the theorem it was shown in [43] that

$$H_t(x,y) - \frac{1}{\mathrm{Vol}(M)} = O\left(e^{-\frac{\overline{\lambda}(M)}{2}t}\right)$$

and so (3.5) converges $\forall\ \alpha \in (0,1)$.

□

We now turn to our main existence theorem for the Riesz fields over non-compact manifolds followed by some examples. Below we use the following notation:

$$D_p(r) \equiv \{x \in M : d(x,p) < r\}$$

and

$$V_p(r) \equiv \mathrm{Vol}\left(D_p(r)\right) = \int_{D_p(r)} dV_g.$$

We write $H_t = \widetilde{O}(t^{-\frac{\nu}{2}})$ if there exist two distinct points $x_k \in M,\ k = 1,2$, and constants $C_k > 0$ such that

$$H_t(x_k,x_k) \leq C_k t^{-\frac{\nu}{2}} \qquad \forall\, t \geq 1.$$

In that case using Theorem 1.1 of [32] we know that for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that for all $t \geq 1$ and all $x,y \in M$

$$H_t(x,y) \leq C_\delta t^{-\frac{\nu}{2}} e^{-\frac{d(x,y)^2}{(4+\delta)t}}.$$

Theorem 3.2.4. *Let M be non-compact.*

(1) Suppose $Ric(M) \geq 0$. Then hR^α does not exist for any $\alpha \in (0,1)$. If

$$H_t = \widetilde{O}\left(t^{-\left(\frac{d}{2}-\beta\right)}\right)$$

and

$$\overline{\lim_{r\to\infty}} \frac{V_x(r)}{r^{d-2\beta}} < \infty \qquad \forall\, x \in M$$

for some $\beta \in [0,1)$ *then* R^α *exists over* M *for any* $\alpha \in (0, 1-\beta)$.

(2) Suppose that

$$H_t = \widetilde{O}\left(t^{-\left(\frac{d}{2}+\beta\right)}\right)$$

for some $\beta > 0$. *Then* hR^α *(and thus* R^α *also) exists for any* $\alpha \in (0, \min\{\beta, 1\})$.

proof. (1): To begin we note that our hypothesis $H_t = \widetilde{O}(t^{-(d/2-\beta)})$ implies the following gradient bound for H_t (see [20]): For all $x, y \in M$ and $t \geq 1$

$$|\nabla_x H_t(x,y)| \leq C'_\delta t^{-\left(\frac{d}{2}-\beta+\frac{1}{2}\right)} e^{-\frac{d(x,y)^2}{(4+\delta)t}} \tag{3.9}$$

for some constant $C'_\delta > 0$.

Recall that by Cauchy-Schwarz in order for for (3.5) to converge it is sufficient to show that

$$\int_1^\infty t^{\frac{d}{2}+\alpha-1} \left(H_t(x,x) - 2H_t(x,o) + H_t(o,o)\right)\, dt < \infty$$

for the specified range of α. Moreover, by first restricting to a compact subset $K \subset M$ we may assume positive injectivity radius, i.e., $\exists\, r > 0$ such that $d(x,y) < r$ implies that x, y belong to some normal neighborhood. By repeated use of the triangle inequality we see that existence for all such x, y implies existence on all of K, and since K was arbitrary, on all of M.

To that end let $D = D_p(r)$ be a normal neighborhood containing x and o. We first apply the mean value theorem:

$$\begin{aligned}\int_1^\infty & t^{\frac{d}{2}+\alpha-1} \left(H_t(x,x) - 2H_t(x,o) + H_t(o,o)\right)\, dt \\ &= \int_1^\infty t^{\frac{d}{2}+\alpha-1} \int_M |H_t(x,z) - H_t(o,z)|^2\, dV_g(z)\, dt \\ &\leq d(x,o)^2 \int_1^\infty t^{\frac{d}{2}+\alpha-1} \int_M |\nabla_x H_t(\xi_z, z)|^2\, dV_g(z)\, dt\end{aligned}$$

for some ξ_z lying on some curve (parametrized to have unit velocity) contained in D_p and joining x and o. We now apply (3.9),

$$\int_1^\infty t^{\frac{d}{2}+\alpha-1} \int_M |\nabla_x H_t(\xi_z, z)|^2 \, dV_g(z)\, dt \\ \leq C \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \int_M e^{-\frac{2d(\xi_z,z)^2}{(4+\delta)t}} \, dV_g(z)\, dt.$$

We have

$$\begin{aligned} &\int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \int_M e^{-\frac{2d(\xi_z,z)^2}{(4+\delta)t}} \, dV_g(z)\, dt \\ &= \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \int_D e^{-\frac{2d(\xi_z,z)^2}{(4+\delta)t}} \, dV_g(z)\, dt \\ &\quad + \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \int_{M\setminus D} e^{-\frac{2d(\xi_z,z)^2}{(4+\delta)t}} \, dV_g(z)\, dt \\ &\leq \mathrm{Vol}(D) \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \, dt \\ &\quad + \int_{M\setminus D} \int_0^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} e^{-\frac{2d(\xi_z,z)^2}{(4+\delta)t}} \, dt\, dV_g(z). \end{aligned}$$

By hypothesis $\int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2}\, dt < \infty$ so we only need to show

$$\int_{M\setminus D} \int_0^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} e^{-\frac{2d(\xi_z,z)^2}{(4+\delta)t}} \, dt\, dV_g(z) < \infty.$$

We have

$$\begin{aligned} &\int_{M\setminus D} \int_0^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} e^{-\frac{2d(\xi_z,z)^2}{(4+\delta)t}} \, dt\, dV_g(z) \\ &= \left(\frac{4+\delta}{2}\right)^{\frac{d}{2}-\alpha-2\beta+1} \Gamma\left(\frac{d}{2}-\alpha-2\beta+1\right) \int_{M\setminus D} d(\xi_z, z)^{-d+2\alpha+4\beta-2} \, dV_g(z). \end{aligned}$$

Recall $D = D_p(r)$ and let

$$A_k = D_p(r+k)\setminus D_p(r+k-1) \qquad k = 1, 2, 3...$$

By monotone convergence

$$\begin{aligned}
\int_{M\setminus D} d(\xi_z, z)^{-d+2\alpha+4\beta-2}\, dV_g(z) &= \sum_{k=1}^{\infty} \int_{A_k} d(\xi_z, z)^{-d+2\alpha+4\beta-2}\, dV_g(z) \\
&\leq \sum_{k=1}^{\infty} \frac{\mathrm{Vol}(A_k)}{(r+k-1)^{d-2\alpha-4\beta+2}} \\
&= \sum_{k=1}^{\infty} \frac{V_p(r+k) - V_p(r+k-1)}{(r+k-1)^{d-2\alpha-4\beta+2}}.
\end{aligned}$$

Because $Ric(M) \geq 0$ we have (see [21] or [17])

$$V_p(cr) \leq c^d V_p(r) \qquad \forall\, r > 0,\ c \geq 1.$$

Thus

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{V_p(r+k) - V_p(r+k-1)}{(r+k-1)^{d-2\alpha-4\beta+2}} &\leq \sum_{k=1}^{\infty} \frac{V_p(r+k-1)\left(\frac{(r+k)^d-(r+k-1)^d}{(r+k-1)^d}\right)}{(r+k-1)^{d-2\alpha-4\beta+2}} \\
&\leq C \sum_{k=1}^{\infty} \frac{(r+k-1)^{d-2\beta}\left(\frac{(r+k)^d-(r+k-1)^d}{(r+k-1)^d}\right)}{(r+k-1)^{d-2\alpha-4\beta+2}} \\
&= C \sum_{k=1}^{\infty} \frac{(r+k)^d - (r+k-1)^d}{(r+k-1)^{d-2\alpha-2\beta+2}}
\end{aligned}$$

The convergence of this last sum is equivalent to that of

$$\sum_{k=1}^{\infty} \frac{k^{d-1}}{k^{d-2\alpha-2\beta+2}} = \sum_{k=1}^{\infty} k^{2\alpha+2\beta-3}.$$

By hypothesis $\alpha < 1 - \beta$, which implies

$$\sum_{k=1}^{\infty} k^{2\alpha+2\beta-3} < \sum_{k=1}^{\infty} k^{-(1+\epsilon)} < \infty$$

for some $\epsilon > 0$.

To see that hR^α does not exist on M for any α, we note that (see e.g. [21])

$$Ric(M) \geq 0 \Rightarrow H_t(x,y) \geq (4\pi t)^{-\frac{d}{2}} e^{-\frac{d(x,y)^2}{4t}}$$

for all $x, y \in M$ and $t > 0$. Thus

$$\int_0^\infty t^{\frac{d}{2}+\alpha-1} H_t(x,y)\, dt = \infty$$

for all $x, y \in M$ and any $\alpha \in (0,1)$.

To prove (2), simply write

$$\int_1^\infty t^{\frac{d}{2}+\alpha-1} H_t(x,y)\, dt \leq C \int_1^\infty t^{\alpha-\beta-1}\, dt < \infty.$$

□

We are now in a position to show that, over $\mathbb{R}^d$, R^α agrees up to a constant with the fBf^α in distribution. We could do this abstractly using arguments along the lines of Section 3.2.1, however we can also make a simple explicit calculation. Note that $\mathbb{R}^d$ satisfies the first hypothesis of Theorem 3.2.4 with $\beta = 0$. Thus R^α exists there and if we choose $o = 0$ has covariance

$$\mathbb{E}[R_x^\alpha R_y^\alpha] = \frac{1}{\Gamma(\frac{d}{2}+\alpha)} \int_0^\infty t^{\frac{d}{2}+\alpha-1}(H_t(0,0) - H_t(x,0) - H_t(y,0) + H_t(x,y))\, dt.$$

Proposition 3.2.5. *If* $M = \mathbb{R}^d$ *then* $H_t(x,y) = \frac{1}{\sqrt{(4\pi t)^d}} e^{\frac{-\|x-y\|^2}{4t}}$ *and for all* $x, y \in \mathbb{R}^d$ *and for* $\alpha \in (0,1)$

$$\mathbb{E}[R_x^\alpha R_y^\alpha] = C_\alpha \left(\|x\|^{2\alpha} + \|y\|^{2\alpha} - \|x-y\|^{2\alpha}\right)$$

where C_α *is the positive constant given by*

$$C_\alpha = \frac{-\Gamma(-\alpha)}{4^{\frac{d}{2}+\alpha}(\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}+\alpha)}.$$

proof. First note that if either $x = 0$ or $y = 0$ the result is trivial; thus we assume otherwise.

The integral defining $\mathbb{E}[R_x^\alpha R_y^\alpha]$ reduces to

$$\frac{1}{\sqrt{(4\pi)^d}}\int_0^\infty t^{\alpha-1}\left(1-e^{\frac{-\|x\|^2}{4t}}-e^{\frac{-\|y\|^2}{4t}}+e^{\frac{-\|x-y\|^2}{4t}}\right)dt,$$

which we recognize as a Mellin transform. Let $F_a(t)=\chi_{[a,\infty)}(t)-e^{\frac{-\|x\|^2}{4t}}-e^{\frac{-\|y\|^2}{4t}}+e^{\frac{-\|x-y\|^2}{4t}}$ with $a>0$. Then $F_a(t)=O(t^{-1})$ as $t\to\infty$ and $F_a(t)=o(t^N)$ as $t\to 0\ \forall\ N>0$. Thus

$$\int_0^\infty t^{s-1}F_a(t)\,dt$$

converges absolutely for all $s\in\mathbb{C}$ with $\Re(s)<1$ and defines an analytic function there.

On the other hand for $-1<\Re(s)<0$ we have by direct calculation that

$$\int_0^\infty t^{s-1}F_a(t)\,dt=\frac{a^s}{s}+\frac{-\|x\|^{2s}-\|y\|^{2s}+\|x-y\|^{2s}}{4^s}\Gamma(-s).$$

By analytic continuation this last equality holds for $0<\Re(s)<1$ as well. For such s we have by dominated convergence

$$\int_0^1 t^{s-1}F_0(t)\,dt=\lim_{a\to 0}\int_0^1 t^{s-1}F_a(t)\,dt.$$

Now for $a<1$

$$\int_1^\infty t^{s-1}F_a(t)\,dt=\int_1^\infty t^{s-1}F_0(t)\,dt$$

and so, noting $F_0(t)\geq 0$, we have using dominated convergence

$$\begin{aligned}\int_0^\infty t^{s-1}F_0(t)\,dt&=\int_0^1 t^{s-1}F_0(t)\,dt+\int_1^\infty t^{s-1}F_0(t)\,dt\\&=\left(\lim_{a\to 0^+}\int_0^1 t^{s-1}F_a(t)\,dt\right)+\int_1^\infty t^{s-1}F_0(t)\,dt\\&=\lim_{a\to 0^+}\left(\int_0^1 t^{s-1}F_a(t)\,dt+\int_1^\infty t^{s-1}F_0(t)\,dt\right)\\&=\lim_{a\to 0^+}\int_0^\infty t^{s-1}F_a(t)\,dt\\&=\frac{-\|x\|^{2s}-\|y\|^{2s}+\|x-y\|^{2s}}{4^s}\Gamma(-s)\end{aligned}$$

□

Example 3.2.4. Suppose M is non-compact with $Ric(M) \geq 0$ and

$$\lim_{R\to\infty} \frac{V_p(R)}{R^d} = \theta \in (0,1)$$

for some $p \in M$ (cf. the Bishop-Gromov comparison theorem). Then R^α exists over M for any $\alpha \in (0,1)$ and hR^α does not. Indeed, in [45] it is shown that $H_t(x,y) = O(t^{-\frac{d}{2}})$ for every $x, y \in M$. Theorem 3.2.4 applies once we note that for all $p \in M$

$$Ric(M) \geq 0 \Rightarrow V_p(R) \leq \omega_d R^d \qquad \forall\, R \geq 0,$$

ω_d being the volume of the unit ball in $\mathbb{R}^d$.

Example 3.2.5. If M is simply connected with all sectional curvatures $K \leq k$ for some $k < 0$ and $Ric(M) \geq -\kappa^2 > -\infty$ then hR^α exists over M for any $\alpha > 0$. For example this holds if $M = \mathbb{H}^d$, d-dimensional hyperbolic space. This follows from [46] in which it is shown that $\sigma(-\Delta)) \subset [(d-1)^2|k|/4, \infty)$, which in turn implies the following upper bound on H_t (see [22]):

$$H_t(x,y) \leq Ce^{\frac{(d-1)^2kt}{4}} \qquad \forall\, t \geq 1$$

for some $C > 0$ and all $x, y \in M$. Theorem 3.2.4 then applies.

In particular for $M = \mathbb{H}^2$, letting $\rho = d(x,y)$ we have the well known formula

$$H_t(x,y) = \frac{\sqrt{2}}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{1}{4}t} \int_\rho^\infty \frac{se^{-\frac{s^2}{4t}}}{\cosh(s) - \cosh(\rho)}\, ds.$$

Then

$$\mathbb{E}[hR_x^\alpha hR_y^\alpha] = \frac{\sqrt{2}}{(4\pi)^{\frac{3}{2}}\Gamma(1+\alpha)} \int_0^\infty \int_\rho^\infty t^{\alpha-\frac{3}{2}} \frac{se^{-\frac{1+s^2}{4t}}}{\cosh(s) - \cosh(\rho)}\, ds\, dt.$$

Remark 3.2.2. On negatively curved manifolds, hR^α can also be viewed as an extension of the fBf^α in the following way: In Section 3.2.1 we saw how the covariance of the fBf^α is the integral kernel of the operator A^*A where

$$A(f) = (-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}(f)(x) - (-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}(f)(0),$$

which can be seen as a correction to $(-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}$ when this operator does not have an integral kernel. However on manifolds with spectrum as in Example 3.2.5, $(-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}$ does have an integral kernel and no correction is needed. So if we view the fBf^{α} as the GRF with covariance that is the integral kernel of the minimal correction to $(-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}$ that yields an integral operator, then on such manifolds as above we obtain the hR^{α}.

3.2.3 Hölder Regularity

Having done the analytical work to build our covariances and check when they exist, we now turn to verifying that these covariances do in fact define random fields with the desired properties. The first of those properties is in some ways the most fundamental: Do the corresponding GRF's define probability measures on nice function spaces? In order to answer this we need some extensions of criteria for continuity of GRF's indexed by Euclidean space to the manifold case, that statements and proofs of which we postpone until Section 3.5 below. What we shall see is that if M is compact or a compact subset of a regular domain or non-compact manifold over which the Riesz fields exist, then with probability one they have continuous sample paths and thus they determine probability measures on $C(M)$ in the usual way (cf. [39], Example 8.27).

If M is any Riemannian manifold or regular domain with heat kernel $H_t(x,y)$ then the maximum principle implies

$$H_t(x,y) \leq H_t(x,x) \qquad \forall\, x,y \in M$$

with equality if and only if $y = x$. We then have that

$$H_t(x,x) - 2H_t(x,y) + H_t(y,y) > 0 \qquad \forall\, y \neq x.$$

In particular $\mathbb{E}[|R^{\alpha}_x - R^{\alpha}_y|^2]$ and $\mathbb{E}[|hR^{\alpha}_x - hR^{\alpha}_y|^2]$ both define metrics on M when they exist.

Note also that

$$\mathbb{E}[|R^{\alpha}_x - R^{\alpha}_y|^2] = \mathbb{E}[|hR^{\alpha}_x - hR^{\alpha}_y|^2]$$

when both exist. In particular in the proof below we will not distinguish these two metrics as the context of the Theorem will make clear which is being discussed.

We are now in a position to prove the following:

Theorem 3.2.6. *Let M be a compact Riemannian manifold, a regular domain, or non-*

compact under the hypothesis of Theorem 3.2.4. We then have the following:

1. *If M is compact then there exists a version, $\tilde{R}^\alpha$, of R^α such that with probability 1 the sample paths of $\tilde{R}^\alpha$ are uniformly Hölder continuous of any order $\gamma < \alpha$ on M, and there exists a dense subset of M such that with probability 1 the sample paths of $\tilde{R}^\alpha$ fail to be Hölder continuous at these points for any $\gamma > \alpha$.*

2. *If M is a regular domain or non-compact under the hypothesis of Theorem 3.2.4, then for any compact set $K \subset M$ there exists a version, $\tilde{R}^\alpha$, of R^α such that with probability 1 the sample paths of $\tilde{R}^\alpha$ are uniformly Hölder continuous of any order $\gamma < \alpha$ on K, and there exists a dense subset of K such that with probability 1 the sample paths of $\tilde{R}^\alpha$ fail to be Hölder continuous at these points for any $\gamma > \alpha$.*

proof. In order to apply Theorem 3.5.4 below we need to compare the metric $\mathbb{E}[|R^\alpha_x - R^\alpha_y|^2]$ (resp. $\mathbb{E}[|hR^\alpha_x - hR^\alpha_y|^2]$) on (M, g) with the metric $d(x, y)$ derived from g, in particular we need to study the boundedness of

$$\frac{\mathbb{E}[|R^\alpha_x - R^\alpha_y|^2]}{(d(x,y))^{2\gamma}} \tag{3.10}$$

for $d(x, y)$ small and $\gamma \in (0, 1)$. What we will show is that this ratio is unbounded if $\gamma > \alpha$ and approaches zero if $\gamma < \alpha$.

Our approach to controlling (3.10) will be to split the integral defining $\mathbb{E}[|R^\alpha_x - R^\alpha_y|^2]$ into two parts:

$$\begin{aligned}\int_0^\infty & t^{\frac{d}{2}+\alpha-1}(H_t(x,x) - 2H_t(x,y) + H_t(y,y))\,dt \\ &= \int_0^1 t^{\frac{d}{2}+\alpha-1}(H_t(x,x) - 2H_t(x,y) + H_t(y,y))\,dt \qquad (3.11)\\ &\quad + \int_1^\infty t^{\frac{d}{2}+\alpha-1}(H_t(x,x) - 2H_t(x,y) + H_t(y,y))\,dt. \qquad (3.12)\end{aligned}$$

We start with (3.11). Recall that in any case around any point p there is a closed disk D_p such that (3.2) holds with $\Phi \in C^k(\overline{D_p} \times \overline{D_p} \times [0, T])$ where we can choose $k > 2$ and $T > 0$.

As a consequence we have, denoting the integral (3.10) by I_1 and $d(x, y)$ by ρ,

$$I_1 = (4\pi)^{-\frac{d}{2}} \int_0^1 t^{\alpha-1}(\Phi(t,x,x) + \Phi(t,y,y) - 2\Phi(t,x,y)e^{-\frac{\rho^2}{4t}})\,dt. \tag{3.13}$$

Because $\Phi \in C^k(\overline{D_p} \times \overline{D_p} \times [0,T])$ with $k > 2$ and is symmetric, by Lemma 3.5.1,

$$\Phi(t,x,x) + \Phi(t,y,y) - 2\Phi(t,x,y) = O(\rho^2) \qquad \text{as } \rho \to 0$$

uniformly for $t \in [0,1]$. Thus we have

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}(\Phi(t,x,x) + \Phi(t,y,y) - 2\Phi(t,x,y)e^{-\frac{\rho^2}{4t}})\,dt \\
&\quad = 2\int_0^1 t^{\alpha-1}\Phi(t,x,y)(1 - e^{-\frac{\rho^2}{4t}})\,dt \\
&\qquad + \int_0^1 t^{\alpha-1}(\Phi(t,x,x) + \Phi(t,y,y) - 2\Phi(t,x,y))\,dt \\
&\quad = 2\int_0^1 t^{\alpha-1}\Phi(t,x,y)(1 - e^{-\frac{\rho^2}{4t}})\,dt + O(\rho^2)
\end{aligned}$$

Because

$$\overline{\lim_{x\to y}} \int_0^1 t^{\alpha-1}\Phi(t,x,y)(1 - e^{-\frac{\rho^2}{4t}})\,dt = \overline{\lim_{x\to y}} \rho^{2\alpha} \int_0^{\rho^{-2}} t^{\alpha-1}\Phi(\rho^2 t,x,y)(1 - e^{-\frac{1}{4t}})\,dt$$

and

$$\overline{\lim_{x\to y}} \int_0^{\rho^{-2}} t^{\alpha-1}\Phi(\rho^2 t,x,y)(1 - e^{-\frac{1}{4t}})\,dt < \infty,$$

$$I_1 = O(\rho^{2\alpha}) = O(d(x,y)^{2\alpha}) \qquad \text{as } d(x,y) \to 0 \tag{3.14}$$

for $x, y \in D_p$.

For (3.12), which we denote I_2, we first deal with the case of M compact. Using (3.3) we have for $t \geq 1$

$$\begin{aligned}
H_t(x,x) - 2H_t(x,y) + H_t(y,y) &= \sum_{k=0}^{\infty} e^{-\lambda_k t}|\phi_k(x) - \phi_k(y)|^2 \\
&= \sum_{k=1}^{\infty} e^{-\lambda_k t}|\phi_k(x) - \phi_k(y)|^2 \\
&\leq d(x,y)^2 \sum_{k=1}^{\infty} e^{-\lambda_k t}\|\nabla\phi_k\|_\infty.
\end{aligned}$$

Now we apply the following bound on $\|\nabla\phi_k\|_\infty$ (see [52]):

$$\|\nabla\phi_k\|_\infty \le C_M \lambda_k^{\frac{d+1}{4}}$$

where C_M is a constant depending only on M. We then have

$$H_t(x,x) - 2H_t(x,y) + H_t(y,y) \le C_M d(x,y)^2 \sum_{k=1}^{\infty} e^{-\lambda_k t}\lambda_k^{\frac{d+1}{4}} = C_M d(x,y)^2 O\left(e^{-\lambda_1 t}\right),$$

which yields

$$I_2 \le C_M d(x,y)^2 \int_1^\infty t^{\frac{d}{2}+\alpha-1} O\left(e^{-\lambda_1 t}\right)\, dt = C d(x,y)^2 \tag{3.15}$$

as $\lambda_1 > 0$.

If M is a regular domain then a similar argument using the corresponding bound (see [58])

$$\|\nabla\phi_k\|_\infty \le C_M \lambda_k^{\frac{d+1}{4}}$$

for the Dirichlet eigenfunctions on M we obtain (3.15) in this case as well. Thus for either M compact or a regular domain

$$I_2 = O\left(d(x,y)^2\right) \qquad \text{as } d(x,y) \to 0.$$

Turning now to the case of M non-compact, first suppose the first hypothesis of Theorem 3.2.4 is in force. As in that proof we have, for x, y contained in a sufficiently small geodesic disc,

$$\begin{aligned}
&\int_1^\infty t^{\frac{d}{2}+\alpha-1}\left(H_t(x,x) - 2H_t(x,y) + H_t(y,y)\right)\, dt \\
&\qquad = \int_1^\infty t^{\frac{d}{2}+\alpha-1} \int_M |H_t(x,z) - H_t(y,z)|^2\, dV_g(z)\, dt \\
&\qquad \le d(x,y)^2 \int_1^\infty t^{\frac{d}{2}+\alpha-1} \int_M |\nabla_x H_t(\xi_z, z)|^2\, dV_g(z)\, dt,
\end{aligned}$$

which was shown to be finite.

Next suppose the second hypothesis holds. For this case we will use a Schauder estimate and Lemma A.1: We choose a geodesic disc D_p and let L be Δ in geodesic normal coordinates on D_p, $\mathbf{D} = \exp^{-1}(D_p)$, $P = \partial_t - L$ on $C^\infty(\mathbf{D} \times (0,1))$, and $u(x', y', t) \in$

$C^\infty(\mathbf{D}\times\mathbf{D}\times(0,1))$ be $H_t(x,y)$ in our chosen coordinates. For any $T>0$ we then have

$$Pu(x',y',t+T)=\partial_t u(x',y',t+T)-L_{x'}u(x',y',t+T)=0$$

for each for all $x',y',t\in\mathbf{D}\times\mathbf{D}\times(0,1/2)$. In other words, u satisfies $Pu(x',y',t)=0$ on $\mathbf{D}\times(T,T+1/2)$ for each $y'\in\mathbf{D}$ and $T>0$.

Because L is uniformly elliptic on $\mathbf{D}$ and its coefficients are all C^∞ (and independent of T, t), using the Schauder estimate (Theorem 5 p.64 in [29] and choosing $\alpha=1$) we obtain for each closed disk $\mathbf{D}_r$ contained in $\mathbf{D}$ a constant $K_r>0$ such that

$$\sup_{(x',t)\in\mathbf{D}_r\times(0,1/2)}\left|\frac{\partial^2 u}{\partial x_i' x_j'}(x',y',t+T)\right|\le K_r\sup_{(x',t)\in\mathbf{D}_r\times(0,1/2)}|u(x',y',t+T)|$$

for each i,j and $y'\in D_r$. We then have

$$\sup_{(x',y',t)\in\mathbf{D}_r\times\mathbf{D}_r\times(0,1/2)}\left|\frac{\partial^2 u}{\partial x_i' x_j'}(x',y',t+T)\right|\le K_r\sup_{(x',y',t)\in\mathbf{D}_r\times\mathbf{D}_r\times(0,1/2)}|u(x',y',t+T)|.$$

We note that K_r is independent of T and by our hypothesis that $H_t=\widetilde{O}\left(t^{-\left(\frac{d}{2}+\beta\right)}\right)$, $\sup_{(x,y)\in D_p\times D_p} H_t(x,y)\le Ct^{-(\frac{d}{2}+\beta)}$, $\beta>0$. Thus, returning to $D_r=exp\,(\mathbf{D}_r)$, for all $T>1$

$$\sup_{(x,y,t)\in D_r\times D_r\times(0,1/2)}\left|\frac{\partial^2 H}{\partial x_i x_j}(x,y,t+T)\right|\le CK_rT^{-(\frac{d}{2}+\beta)}.$$

Then applying Lemma 3.5.1 and assuming without loss of generality we have chosen our disc D_p such that the above estimates hold, we have

$$\begin{aligned}&\int_1^\infty t^{\frac{d}{2}+\alpha-1}\left(H_t(x,x)-2H_t(x,y)+H_t(y,y)\right)\,dt\\&\quad\le Cd(x,y)^2\int_1^\infty t^{\frac{d}{2}+\alpha-1}\sup_{\overline{D}_p\times\overline{D}_p}\left|\sum_{i,j=1}^d\frac{\partial^2 H}{\partial x_i\partial x_j}(t,x,y)\right|dt\\&\quad\le Cd(x,y)^2\int_1^\infty t^{\frac{d}{2}+\alpha-1}(t-1/2)^{-(\frac{d}{2}+\beta)}\,dt\end{aligned}$$

for some $C > 0$. By hypothesis $\beta > 0$, so $\int_1^\infty t^{\frac{d}{2}+\alpha-1}(t-1/2)^{-(\frac{d}{2}+\beta)}\,dt < \infty$. Lastly recall that when hR^α exists for $\alpha \in (0,b)$ for some $b > 0$ then R^α does as well. Moreover in that case

$$\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2] = \mathbb{E}[|hR_x^\alpha - hR_y^\alpha|^2],$$

so in the second case of Theorem 3.2.4 the arguments above apply to R^α as well.

Thus in each case from the preceeding discussion we know that for each $p \in M$ there exists a closed disc D_p centered at p such that for all $\gamma \leq \alpha$

$$\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2] \leq C_p(d(x,y)^{2\gamma})$$

for some constant $C_p > 0$ and all $x, y \in D_p$ and that such a condition fails for any $\gamma > \alpha$ in light of (3.14). Then if M is compact or K is a compact subset of M, there exists a constant $C > 0$ such that for all $\gamma \leq \alpha$

$$\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2] \leq Cd(x,y)^{2\gamma}$$

for all $x, y \in M$ (resp. $x, y \in K$). Then by Theorem 3.5.4 there is a version of R^α that is almost surely uniformly Hölder continuous over M (resp. K) of order γ for any $\gamma < \alpha$. Moreover from the discussion following Theorem 3.5.4 there is a dense subset of M (resp. K) on which R^α fails to satisfy any Hölder condition of order γ for any $\gamma > \alpha$ with probability 1. By the remarks preceding the Theorem the same holds for hR^α, when it exists. □

Remark 3.2.3. From the proof above we see that

$$\lim_{x\to y} \frac{\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2]}{d(x,y)^{2\alpha}} = \lim_{x\to y} \int_0^{d(x,y)^{-2}} t^{\alpha-1}\Phi(d(x,y)^2 t, x, y)(1 - e^{-\frac{1}{4t}})\,dt$$

and thus the exact comparison between the Riemannian metric of M and the metric induced by R^α depends on the local geometry of M, in particular on the comparison with the Euclidean heat kernel contained in $\Phi(t,x,y)$.

Remark 3.2.4. It would be desirable in the case of regular domains to extend continuity to the closure of M. However the local Euclidean approximation of the heat kernel is not uniform near the boundary of M and so some other method of proof seems necessary. On the other hand it is easy to show that for any sequence $(x_1^k, \ldots, x_n^k)$ that approaches the boundary of M, $P(\|(hR^\alpha_{x_1^k},\ldots,hR^\alpha_{x_n^k})\|>\epsilon)\xrightarrow{k}0$ for any $\epsilon>0$. This combined with the existence of a continuous version as close as we like to the boundary seems sufficient for most applications, at least from the point of view of simulation.

3.2.4 Distributional Scaling and Invariance

3.2.4.1 Stationarity

Definition 3.2.2. Let (M,g) be a complete Riemannian manifold and $I(M)$ the group of isometries of (M,g). If Y_x is a centered GRF over (M,g) we say that Y_x is *stationary (or homogeneous)* if

$$\mathbb{E}[Y_{\iota(x)}Y_{\iota(y)}]=\mathbb{E}[Y_xY_y]$$

for any $\iota\in I(M)$ and all $x,y\in M$. We say Y_x has *stationary (or homogeneous) increments* if

$$\mathbb{E}[|Y_{\iota(x)}-Y_{\iota(y)}|^2]=\mathbb{E}[|Y_x-Y_y|^2]$$

for any $\iota\in I(M)$ and all $x,y\in M$.

Because for any manifold (M,g) we have $H_t(\iota(x),\iota(y))=H_t(x,y)$ for any $\iota\in I(M)$ (see [33], Theorem 9.12) it is clear from the definitions,

$$\mathbb{E}[R^\alpha_xR^\alpha_y]=\frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)}\int_0^\infty t^{\frac{d}{2}+\alpha-1}\left(H_t(x,y)-H_t(x,o)-H_t(y,o)+H_t(o,o)\right)dt$$

and

$$\mathbb{E}[hR^\alpha_xhR^\alpha_y]=\frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)}\int_0^\infty t^{\frac{d}{2}+\alpha-1}H_t(x,y)\,dt,$$

that when they exist, R^α and hR^α have stationary increments and are stationary respectively.

3.2.4.2 Self-Similarity

Turning to self-similarity, let us first recall how this property is defined for random fields on Euclidean space: If Y_x is a random field over $\mathbb{R}^d$, then Y_x is *self-similar* of order $\alpha>0$ if $c^\alpha Y_{\frac{1}{c}x}\overset{d}{=}Y_x$. The Euclidean fractional Brownian field fBf^α is self similar of order α, and we want to extend this property to manifolds. To do this we must define an operation that extends the scaling operation on $\mathbb{R}^d$, $x\mapsto cx$. This operation scales the distance between any two points by $c > 0$:

$$\|x - y\| \mapsto \|cx - cy\| = c\|x - y\|,$$

or written another way,

$$d(x, y) \mapsto cd(x, y).$$

Viewing $\mathbb{R}^d$ as a manifold, we see this is equivalent to scaling the Riemannian metric $(g_{ij}) = (\delta_{ij})$ of $\mathbb{R}^d$ by c^2,

$$\sum_{i,j=1}^{d} x_i x_j g_{ij} = \sum_{i=1}^{d} x_i^2 = \|x\|^2 \mapsto c^2\|x\|^2 = \sum_{i,j=1}^{d} x_i x_j c^2 g_{ij}.$$

Thus a natural definition of scaling for a manifold M is to simply scale the metric as above. Indeed, if M is an embedded submanifold of $\mathbb{R}^d$ with induced metric g_M, then scaling the ambient space $\mathbb{R}^d$ results in the induced scaling on M

$$g_M \mapsto c^2 g_M.$$

Of course, we'd like a definition of scaling that is intrinsic to the manifold in question, i.e., independent of any ambient Euclidean space, but that also agrees with the scaling induced by scaling any ambient space. If we take the above operation as the definition of scaling for a general manifold M we achieve this goal.

We are thus ready to prove that the Riesz fields are self-similar.

Proposition 3.2.7. *Let (M, g) be a complete Riemannian manifold or regular domain. Both the Riesz field R^α and the stationary Riesz field hR^α over (M, g) are self-similar of order α (if they exist on M) in the sense that if $\bar{R}^\alpha$ and $h\bar{R}^\alpha$ are the Riesz fields over $(M, c^2 g)$ then*

$$c^\alpha R_x^\alpha \stackrel{d}{=} \bar{R}_x^\alpha$$

and

$$c^\alpha hR_x^\alpha \stackrel{d}{=} h\bar{R}_x^\alpha$$

for any $c > 0$.

proof. First we note from the coordinate expression for Δ, if we denote by Δ_g the Laplacian of (M, g) and $H_t^g(x, y)$ the corresponding heat kernel, we have $\Delta_{c^2 g} = \dfrac{1}{c^2}\Delta_g$. But then because $L^2(M, dV_g) = L^2(M, dV_{c^2 g})$ we can write

$$\begin{aligned}\int_M c^d H_t^{c^2 g}(x,y) f(y)\, dV_g(y) &= \int_M H_t^{c^2 g}(x,y) f(y)\, dV_{c^2 g}(y)\\ &= e^{-t\Delta_{c^2 g}}(f)\\ &= e^{-\frac{t}{c^2}\Delta_g}(f)\\ &= \int_M H^g_{\frac{t}{c^2}}(x,y) f(y)\, dV_g(y)\end{aligned}$$

for any $f \in L^2(M, dV_g)$. Thus by symmetry

$$\frac{1}{c^d} H^g_{\frac{t}{c^2}}(x,y) = H_t^{c^2 g}(x,y) \qquad \forall\, x, y \in M.$$

We then have

$$\begin{aligned}c^{2\alpha}\mathbb{E}[R_x^\alpha R_y^\alpha] &= \frac{c^{2\alpha}}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} \left(H_t^g(x,y) - H_t^g(o,x) - H_t^g(o,y) + H_t^g(o,o)\right) dt\\ &= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} \frac{1}{c^d} \left(H^g_{\frac{t}{c^2}}(x,y) - H^g_{\frac{t}{c^2}}(o,x) - H^g_{\frac{t}{c^2}}(o,y) + H^g_{\frac{t}{c^2}}(o,o)\right) dt\\ &= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} \left(H_t^{c^2 g}(x,y) - H_t^{c^2 g}(o,x) - H_t^{c^2 g}(o,y) + H_t^{c^2 g}(o,o)\right) dt\\ &= \mathbb{E}[\bar{R}_x^\alpha \bar{R}_y^\alpha]\end{aligned}$$

and similarly for hR^α.

□

Remark 3.2.5. Here we see that hR^α exhibits essentially non-Euclidean phenomena; on $\mathbb{R}^d$ there cannot exist a GRF that is both stationary and self similar (see e.g. [7]). We will return to the questions this raises in Section 5.

3.2.4.3 Uniqueness

We now come to a natural question: Are the Riesz fields the only fields with stationary increments that are also self-similar? In other words, does requiring stationarity and self-similarity as above uniquely determine a GRF over a given manifold M? To answer this we examine an example, $M = \mathbb{S}^1$, which we normalize to have total volume 2π. Using the expansion of section 3.2.2.1 we have

$$R^{\alpha}(x) \stackrel{d}{=} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{\sqrt{2\pi}}|k|^{-\frac{1}{2}-\alpha}(e^{ikx}-1)\xi_k.$$

In [37] the author constructs a GRF, denoted R_{α}, with the following covariance

$$\frac{1}{2}(d(x,0)^{2\alpha}+d(y,0)^{2\alpha}-d(x,y)^{2\alpha}).$$

In particular it is shown that

$$R_{\alpha}(x) \stackrel{d}{=} \sum_{k\in\mathbb{Z}\setminus\{0\}} d_k(e^{ikx}-1)\xi_k$$

where

$$d_k = \frac{\sqrt{-\int_0^{|k|\pi} u^{2\alpha}\cos(u)du}}{\sqrt{2\pi}|k|^{\frac{1}{2}+\alpha}}.$$

Note however that for $\alpha = \frac{1}{2}$,

$$d_k = \begin{cases} 0 & k \text{ even}, \\ (\sqrt{\pi}|k|)^{-1} & k \text{ odd} \end{cases}.$$

Thus

$$R_{\frac{1}{2}}(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{\sqrt{\pi}}|2k+1|^{-1}(e^{i(2k+1)x}-1)\xi_k$$

and

$$\sqrt{2}R^{\frac{1}{2}}(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{\sqrt{\pi}}|k|^{-1}(e^{ikx}-1)\xi_k.$$

We then find that

$$\mathbb{E}[|\sqrt{2}R^{\frac{1}{2}}(x)|^2] - \mathbb{E}[|R_{\frac{1}{2}}(x)|^2] = \sum_{k=-\infty}^{\infty} \frac{1}{\pi}|2k|^{-2}|e^{i2kx}-1|^2,$$

which is not identically zero. As their variances are not identical, these two fields are not equal in distribution. However it is easy to see that both fields have stationary increments and are self-similar of order 1/2.

Thus even in the simple case of $\mathbb{S}^1$ we do not have uniqueness, and so in general the Riesz fields are not the only GRF's that are self-similar with stationary increments over a given manifold M. It then remains an open question to determine the general form of the covariance of a GRF with stationary increments that is also self-similar over a given manifold other than $\mathbb{R}^d$.

3.3 The Bessel Field

We now turn to constructing stationary counterparts to R^α by analogy with the Brownian motion and Ornstein-Uhlenbeck processes on $\mathbb{R}$. We define the *Bessel Field* of order $\alpha \in (0,1)$ by

$$B_x^\alpha \stackrel{d}{=} \frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)} \int_M \int_0^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1} e^{-t} H_t(x,z)\, dt\, dW(z), \tag{3.16}$$

which extends the Ornstien-Uhlenbeck fields with covariance given (up to a constant) by

$$\int_{\mathbb{R}^d} \frac{e^{i\langle x,y\rangle}}{(1+|\xi|^2)^{\frac{d}{2}+\alpha}}\, d\xi.$$

These fields are altogether more well behaved than the Riesz fields, which is not surprising in light of the analogy with the Riesz and Bessel potentials.

Theorem 3.3.1. *The Bessel field exists over any complete Riemannian manifold or regular domain M for all $\alpha \in (0,1)$.*

proof. Proceeding as for hR^α, for each $x, y \in M$

$$\begin{aligned}
\mathbb{E}[B_x^\alpha B_y^\alpha] &= \left(\frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)}\right)^2 \int_M \int_0^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1} e^{-t} H_t(x,z)\, dt \int_0^\infty s^{\frac{d}{4}+\frac{\alpha}{2}-1} e^{-s} H_s(y,z)\, ds\, dV_g(z) \\
&= \left(\frac{1}{\Gamma\left(\frac{d}{4}+\frac{\alpha}{2}\right)}\right)^2 \int_0^\infty \int_0^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1} s^{\frac{d}{4}+\frac{\alpha}{2}-1} e^{-(t+s)} H_{t+s}(x,y)\, dt\, ds \\
&= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} e^{-t} H_t(x,y)\, dt
\end{aligned} \tag{3.17}$$

From the fact that the heat kernel always satisfies $\overline{\lim_{t\to\infty}} H_t(x,y) < \infty$ for any x and y, we see that (3.17) converges everywhere on $M \times M$.

□

Clearly B_x^α is stationary and we can see that it does not possess the scaling properties of the Riesz fields. Turning to sample path regularity we have the following result.

Theorem 3.3.2. *The Bessel field B^α has a version with sample paths almost surely uniformly Hölder continuous of order γ for any $\gamma < \alpha$ and almost surely failing to satisfy a Hölder condition of order γ for any $\gamma > \alpha$ on a dense subset of M.*

proof. Split the integral

$$\begin{aligned}\mathbb{E}[|B_x^\alpha - B_y^\alpha|^2] &= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha 2-1} e^{-t} \left(H_t(x,x) - 2H_t(x,y) + H_t(y,y)\right) \, dt \\ &= \frac{1}{\Gamma\left(\frac{d}{2}+\alpha\right)} (I_1 + I_2)\end{aligned}$$

where

$$I_1 = \int_0^1 t^{\frac{d}{2}+\alpha 2-1} e^{-t} \left(H_t(x,x) - 2H_t(x,y) + H_t(y,y)\right) \, dt$$

and

$$I_2 = \int_1^\infty t^{\frac{d}{2}+\alpha 2-1} e^{-t} \left(H_t(x,x) - 2H_t(x,y) + H_t(y,y)\right) \, dt$$

and argue as in Theorem 3.2.6.

□

3.4 Conclusion and Further Work

3.4.1 Existence and Uniqueness

Using a spectral theoretic approach we have constructed analogues of the fractional Brownian fields over arbitrary compact manifolds and a wide class of non-compact manifolds. There are still many questions remaining. For example in light of the non-uniqueness result in Section 3.2.4.3, one could ask how many different such fields there are over any given manifold. One could also attempt to determine the general form the covariance of such objects must take.

We also saw in Example 3.2.3 that R^α does not exist on $\mathbb{S}^1 \times \mathbb{R}$ (with the product metric) for $\alpha > 1/2$. This raises the following question: Does there exist any Gaussian field over $\mathbb{S}^1 \times \mathbb{R}$ with stationary increments that is also self similar of order α for some $\alpha \in (1/2, 1)$? More generally, are there geometric conditions that ensure a given manifold can have such a field defined over it?

We conjecture that it is possible to construct such fields over any manifold M in the following way: Somewhat informally, the Riesz fields are solutions to the stochastic equation

$$(-\Delta)^{\frac{d}{4}+\frac{\alpha}{2}} X = W,$$

where W is Gaussian white noise over M and Δ is the Laplacian of M with certain "boundary conditions," i.e., with domain restricted to include only functions f such that $f(o) = 0$ for some fixed point $o \in M$. As we saw, for example in the case of compact manifolds, this restriction of the domain led to the existence of a continuous integral kernel for the corresponding inverse and it seems plausible that in general we could always obtain such a kernel through restricting the domain of Δ by determining a sufficient number of derivatives of $f \in \mathrm{dom}(\Delta)$ at the point o. Of course finding an explicit expression for such a kernel may be very difficult in general.

3.4.2 Restriction to Submanifolds

There is one aspect of this theory we did not touch upon, that being the behavior of our fields when restricted to geodesics and more general submanifolds. One thing we can say is that for a given manifold M, following the discussion of self-similarity and dilation in Section 3.2.4, the Riesz fields over M when restricted to an embedded submanifold N determine self-similar fields over N. Also, being embedded, the isometry group of N determines a (possibly trivial) subgroup of the general isometry group of the M. However, the resulting restricted field may be stationary or have stationary increments (for example, consider the fBf^α over $\mathbb{R}^d$ restricted to $\mathbb{S}^{d-1}$). Moreover, as we already saw, stationarity and self-similarity alone do not uniquely determine a GRF in general, and so we cannot say that R^α over M when restricted to a submanifold N agrees with R^α over N.

While we have avoided symmetry hypothesis in our treatment, when dealing with invariance properties involving isometry groups one is naturally led towards general harmonic analysis and it would be interesting to study GRF's over manifolds from this point of view.

For example, one could consider GRF's that are only stationary with respect to a subgroup of the entire isometry group, analogous to GRF's over $\mathbb{R}^d$ that are only rotationally invariant (so called *isotropic* random fields).

One property of the Euclidean fractional Brownian fields (or more generally any GRF that is self-similar with translation invariant increments) is that when restricted to lines through the origin they agree with the usual fractional Brownian motion, up to a constant. One could then ask if this holds more generally. For example one could require that a field over M when restricted to infinite geodesics became a fractional Brownian motion. This would require a subgroup of the isometry group of M that restricted to translation of the given geodesic. Of course, in general geodesics may be closed or infinite. Again, one could study such questions from a general harmonic analytic point of view.

3.4.3 Hyperbolic GRF's

We also mentioned above that the existence of hR^α raises interesting questions regarding negatively curved manifolds and what we could loosely call hyperbolic Gaussian random fields. For example, although the proof of existence of hR^α over $\mathbb{H}^d$ uses properties of the heat kernel, one can ask if there are more geometric or topological conditions one can put on a manifold M to ensure the existence of some self-similar and stationary GRF. Conversely one can ask what are the implications of such a field existing over M. Is hR^α the only such field or are there others?

The above is only a first attempt to state some questions at the intersection of geometry and probability that, at least on the face of it, seem novel and interesting; doubtless there are others. The study of random fields over manifolds, although its history is not short, seems to the author to still be wide open. It is our hope that the work here and the questions raised above will be of interest to both researchers in geometry or geometric analysis and probabilists and lead to further interaction between the two.

3.5 Auxiliary Results

First we record the following Lemma involving Taylor approximation.

Lemma 3.5.1. *Let M be complete and suppose $f \in C^\infty(M \times M)$ is symmetric. Around any point $p \in M$ there exists a closed geodesic disk D_p centered at p and a constant $C_p > 0$*

such that

$$|f(x,x) - 2f(x,y) + f(y,y)| \leq C_p d(x,y)^2 \sup_{D_p \times D_p} \left| \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} \right|$$

for all $x, y \in D_p$.

proof. Let $F \in C^2(\mathbb{R}^d)$ and recall Taylor's Theorem: for each $p \in \mathbb{R}^d$ and all $x \in \mathbb{R}^d$

$$\begin{aligned} F(x) = F(p) &+ \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(p)(x_i - p_i) \\ &+ \sum_{i,j=1}^{d} (x_i - p_i)(x_j - p_j) \frac{2}{1+\delta_{ij}} \int_0^1 (1-t) \frac{\partial^2 F}{\partial x_i \partial x_j}(p + t(x-p)) dt. \end{aligned}$$

Now let $f \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $f(x,y) = f(y,x)$. Fix $x, y \in \mathbb{R}^d$. Then letting $p = (x,y)$, from the symmetry of f we have

$$\begin{aligned} f(x,x) - &2f(x,y) + f(y,y) \\ &= \sum_{i,j=1}^{d} (x_i - y_i)(x_j - y_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(x + t(y-x), x) dt \\ &\quad + \sum_{i,j=1}^{d} (x_i - y_i)(x_j - y_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(y + t(x-y), y) dt \\ &= \int_0^1 \sum_{i,j=1}^{d} (x_i - y_i)(x_j - y_j)(1-t) \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x + t(y-x), x) \right. \\ &\quad \left. + \frac{\partial^2 f}{\partial x_i \partial x_j}(y + t(x-y), y) \right) dt \\ &= c \sum_{i,j=1}^{d} (x_i - y_i)(x_j - y_j) \left(\frac{\partial^2 f}{\partial x_i x_j}(x + \theta_1, x) + \frac{\partial^2 f}{\partial x_i x_j}(y + \theta_2, y) \right) \end{aligned}$$

for some constant $c > 0$ and $\theta_k \in \mathbb{R}^d$ with $\|\theta_k\|_{\mathbb{R}^d} < \|x - y\|_{\mathbb{R}^d}$. In particular for x, y in a closed disk D_ϵ of radius $\epsilon > 0$ we have

$$|f(x,x) - 2f(x,y) + f(y,y)| \leq C_1 \|x - y\|_{\mathbb{R}^d}^2 \sup_{D_\epsilon \times D_\epsilon} \left| \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} \right|$$

for some $C_1 > 0$.

Now suppose $f \in C^\infty(M \times M)$ is symmetric and let D_p be a geodesic disk centered at $p \in M$. Then the above implies

$$|f(x,x) - 2f(x,y) + f(y,y)| \leq C_2 d(x,y)^2 \sup_{\overline{D}_p \times \overline{D}_p} \left| \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} \right| \tag{3.18}$$

for all $x, y \in D_p$.

□

3.5.1 Continuity of Gaussian random fields

Here we provide analogues of results given for Gaussian fields over $\mathbb{R}^d$ in the setting of manifolds. These proofs are simple modifications of the originals and we include them for convenience. The first result is an analytical lemma, given for hypercubes in $\mathbb{R}^d$. We will replace the cubes with metric disks and $\mathbb{R}^d$ by a d-dimensional manifold M. Let p be even and continuous on $[-1,1]$, $p(|x|)$ monotone increasing, and satisfy $\lim_{x \to 0} p(x) = 0$.

Lemma 3.5.2. *(Manifold version of Lemma 1 in [31]): Let $f \in C(I_0)$ where $I_0 \subset M$ is compact, has non-empty interior, and has no isolated points. Suppose that*

$$\int_D \int_D \exp\left(\frac{f(x) - f(y)}{p(\operatorname{diam}(D))}\right)^2 dx\, dy \leq B$$

for all closed metric disks $D \subset I_0$. Then for some $C > 0$

$$|f(x) - f(y)| \leq 8 \int_0^{d(x,y)} \sqrt{\log(BCu^{-2d})}\, dp(u)$$

for all $x, y \in I_0$.

proof. Fix $x, y \in I_0$. Then choose a sequence of disks $D_k = \{z \in M : d(z,x) < r_k\}$ such that $D_k \subset I_0$, $2r_1 \leq d(x,y)$, $r_k \to 0$, and if $d_k = 2r_k$ we have

$$p(d_k) = \frac{1}{2} p(d_{k-1}).$$

Let $f_{D_k} = \mathrm{Vol}(D_k)^{-1} \int_{D_k} f\, dV$. We apply Jensen's inequality to obtain

$$\begin{aligned}
&\exp\left(\frac{f_{D_k} - f_{D_{k-1}}}{p(d_{k-1})}\right)^2 \\
&\quad \leq [\mathrm{Vol}(D_k)\mathrm{Vol}(D_{k-1})]^{-1} \int_{D_k}\int_{D_{k-1}} exp\left(\frac{f(x)-f(y)}{p(d_{k-1})}\right)^2 dV(x)\, dV(y) \\
&\quad \leq B[\mathrm{Vol}(D_k)\mathrm{Vol}(D_{k-1})]^{-1}.
\end{aligned}$$

We then have

$$|f_{D_k} - f_{D_{k-1}}| \leq p(d_{k-1})\sqrt{\log(B[\mathrm{Vol}(D_k)\mathrm{Vol}(D_{k-1})]^{-1})} \tag{3.19}$$

By the definition of D_k we have

$$p(d_{k-1}) = 4[p(d_k) - p(d_{k+1})].$$

Then because

$$\mathrm{Vol}(D_k) = O\left((d_k)^d\right) \qquad \text{as } k \to \infty,$$

$\exists\ C > 0$ such that

$$\mathrm{Vol}(D_k) \geq C(d_k)^d$$

so that $d_{k+1} \leq u \leq d_k \Rightarrow u^{-2d} \leq C[\mathrm{Vol}(D_k)\mathrm{Vol}(D_{k-1})]^{-1}$. Then we can write (3.19) as

$$|f_{D_k} - f_{D_{k-1}}| \leq 4\int_{d_{k+1}}^{d_k} \sqrt{\log(BCu^{-2d})}\, dp(u).$$

Summing these and using continuity of f we get

$$|f(x) - f_{D_1}| = \overline{\lim_{k\to\infty}}\, |f_{D_k} - f_{D_1}| \leq 4\int_0^{d_2} \sqrt{\log(BCu^{-2d})}\, dp(u).$$

Now $d_2 < d(x,y)$ so if we need to we can replace B by a larger bound to ensure the integrand is defined, and after doing so we have

$$|f(x) - f_{D_1}| \leq 4\int_0^{d(x,y)} \sqrt{\log(BCu^{-2d})}\, dp(u).$$

The argument is symmetric in x and y, so an application of the triangle inequality yields the conclusion.

□

Suppose now we are given a (centered) Gaussian random field X_x over (M, g) and consider its restriction to a compact set I_0 as above. Suppose further that the function $K(x, y) = \mathbb{E}[X_x X_y]$ is continuous on $I_0 \times I_0$. Then $K(x, y)$ determines a positive trace class integral operator on $L^2(I_0, dV_g)$ and by Mercer's theorem we have

$$K(x, y) = \sum_{k=0}^{\infty} \lambda_k \phi_k(x) \phi_k(y)$$

uniformly on $I_0 \times I_0$, where λ_k and ϕ_k are the eigenvalues and eigenfunctions of K respectively.

Let

$$p(u) = \sup\{\sqrt{\mathbb{E}[|X_x - X_y|^2]} : d(x, y) \leq |u|\}$$

and

$$X_x^n = \sum_{k=0}^{n} \sqrt{\lambda_k} \phi_k(x) \theta_k$$

where the θ_k are independent standard normal random variables.

We then have the following adaptation of Garsia's theorem to the manifold setting:

Theorem 3.5.3. *(Manifold version of Theorem 1 in [31]): Suppose that for $x, y \in I_0$ as above*

$$\int_0^{diam(I_0) \wedge 1} \sqrt{-\log(u)}\, dp(u) < \infty.$$

Then with probability 1

$$|X_x^m - X_y^m| \leq \frac{1}{8} \int_0^{d(x,y)} \sqrt{\log(BCu^{-2d})}\, dp(u)$$

where $C > 0$ and

$$\sup_m \int_{I_0} \int_{I_0} exp \frac{1}{4} \left(\frac{X_x^m - X_y^m}{p(d(x, y))} \right)^2 dV(x)\, dV(y) \leq B < \infty$$

almost surely. In particular the partial sums X_x^m are almost-surely equicontinuous and *uniformly convergent on* I_0.

proof. Let

$$P_n = \exp\frac{1}{8}\left(\frac{X^n_x - X^n_y}{p(d(x,y))}\right)^2 = P_{n-1}\exp\frac{1}{8}\left(\frac{(Y^n(x,y))^2 - 2Y^n(x,y)(X^{n-1}_x - X^{n-1}(y))}{p(d(x,y))}\right)^2$$

where $Y^k(x,y) = \sqrt{\lambda_k}(\phi_k(x) - \phi_k(y))\theta_k$. Then by independence of the θ_k and Jensen's inequality for conditional expectation

$$\begin{aligned}
&\mathbb{E}[P_{n+1}\,|P_n,...,P_1] \\
&\quad= P_n\left(\mathbb{E}\left[\exp\frac{1}{8}\left(\frac{X^{n+1}_x - X^{n+1}_y}{p(d(x,y))}\right)^2\middle| P_n,...,P_1\right]\right) \\
&\quad\geq P_n\exp\frac{1}{8}\left(\mathbb{E}\left[\left(\frac{(Y^{n+1}(x,y))^2 - 2Y^{n+1}(x,y)(X^{n-1}_x - X^{n-1}(y))}{p(d(x,y))}\right)\middle| P_n,...,P_1\right]\right)^2 \\
&\quad= P_n\exp\frac{1}{8}\left(\mathbb{E}\left[\left(\frac{(Y^{n+1}(x,y))^2}{p(d(x,y))}\right)\middle| P_n,...,P_1\right]\right)^2 \\
&\quad\geq P_n \qquad a.s.
\end{aligned}$$

Thus $\{P_n\}$ is a submartingale. Next note that $\mathbb{E}[P_n^2] \leq \sqrt{2}$, as

$$\frac{X^n_x - X^n_y}{p(d(x,y))}$$

is centered, Gaussian, and has variance less than or equal to one. Then applying the classical submartingale inequalities we have

$$\mathbb{E}[\max_{m\leq n} P_m^2] \leq 4\mathbb{E}[P_n^2] \leq 4\sqrt{2}.$$

Applying the Fubini-Tonelli theorem we then have

$$\mathbb{E}\left(\int_{I_0}\int_{I_0}\max_{m\leq n}\exp\frac{1}{4}\left(\frac{X^n_x - X^n_y}{p(d(x,y))}\right)^2 dV(x)\,dV(y)\right) \leq 4\sqrt{2}\,(V(I_0))^2\,.$$

Letting n tend to infinity and applying monotone converge yields

$$\mathbb{E}[B] \leq 4\sqrt{2}\,(V(I_0))^2 < \infty.$$

We then have that almost surely

$$\int_{I_0}\int_{I_0} \exp\frac{1}{4}\left(\frac{X_x^n - X_y^n}{p(d(x,y))}\right)^2 dV(x)\,dV(y) \le B < \infty \qquad \forall\, n$$

so that Lemma (3.5.2) applies.

Lastly note that from

$$\mathbb{E}\left[\sum_{k=0}^{\infty}\lambda_k\theta_k^2\right] = \sum_{k=0}^{\infty}\lambda_k = \int_{I_0} K(x,x)\,dV(x) < \infty$$

we obtain with probability one

$$\sum_{k=0}^{\infty}\lambda_k\theta_k^2 < \infty,$$

which together with the conclusion of Lemma (3.5.2) implies the almost sure uniform convergence of $\{X_x^n\}$ on I_0.

□

As remarked in [31] this result gives a sufficient condition for the existence of an almost surely continuous version of X_x. The next result establishes Hölder continuity.

Theorem 3.5.4. *(Manifold version of Thm 8.3.2 in [1]): Let the field X over $I_0 \subset M$ be as above and let $\gamma = \sup\{\beta : \mathbb{E}[|X_x - X_y|^2] = o(d(x,y)^{2\beta})$ uniformly on $I_0\}$. Then there exists a version of X with sample paths that are almost surely uniformly Hölder continuous over I_0 of any order $\beta < \gamma$.*

proof. Let $\rho = d(x,y)$. First note that, with $p(u)$ as above, we have for any $L > 0$

$$\int_L^{\infty} p(e^{-x^2})\,dx \le c_\epsilon \int_L^{\infty} e^{-(\gamma-\epsilon)x^2}\,dx < \infty$$

for any $0 < \epsilon < \gamma$ and some constant c_ϵ. But this is equivalent to

$$\int_0^{\mathrm{diam}(I_0)\wedge 1} \sqrt{-\log(u)}\,dp(u) < \infty.$$

Thus by the previous result we have a version (which we also denote by X) for which

$$|X_x - X_y| \le Bp(\rho) + C\int_0^{\rho}\sqrt{-\log(u)}\,dp(u) \qquad a.s.$$

for some constant $C > 0$ and some positive random variable B which is almost surely finite.

Now for any $0 < \epsilon < \gamma$ we have some constant $C_\epsilon > 0$ such that $p(\rho) < C_\epsilon \rho^{\gamma-\epsilon}$, and similarly $\int_0^\rho \sqrt{-\log(u)}\, dp(u) < C'_\epsilon \rho^{\gamma-\epsilon}$ for some $C'_\epsilon > 0$. Thus, with probability 1, for each $\epsilon > 0$ there is an almost surely finite positive random variable A_ϵ such that

$$|X_x - X_y| \leq A_\epsilon d(x,y)^{\gamma-\epsilon} \qquad \forall\, x, y \in I_0.$$

□

Note that we can also show under the hypotheses of the theorem that in any disk of positive radius in I_0 the sample paths of X fail to be uniformly Hölder of any order greater than γ. Indeed,

$$\frac{X_x - X_y}{d(x,y)^{\gamma+\epsilon}}$$

is a centered Gaussian random variable with variance $O(d(x,y)^{-\frac{\epsilon}{2}})$ and thus becomes almost surely unbounded as $x \to y$. For example we can pick any countable dense subset of I_0 and modify X on a set of measure zero to obtain the failure of Hölder continuity at each point in the set. Any stronger converse statement will require more refined tools, i.e., local times, which we will not attempt to develop here.

Remark 3.5.1. We mention here that the results in [50], of which the author became aware after submission of the present article, may be an alternative to the results above for establishing sample path continuity in Theorem 3.6.

Chapter 4: On the Largest eigenvalue of products from the β-Laguerre ensemble

4.1 Introduction

The limiting spectral behavior of products of random matrices has been the subject of a number of studies in random matrix theory and various results on the limiting spectral distribution of such products are by now known (e.g. [49, 14, 12]). In general the spectra of such products will be complex, but in the event it is real, e.g., that of the product of two Hermitian matrices where one is non-negative definite (see for example [5, 53, 13]), it makes sense to speak of the largest eigenvalue. There are strong limit laws known for these largest eigenvalues, but so far there are no results regarding the distribution of the fluctuations around the strong limit. The purpose of this paper is to investigate this limiting distribution in the setting of the β-Laguerre ensembles.

The β-Laguerre ensemble generalizes the classical Laguerre ensemble by allowing β to vary over the positive reals in

$$c^{\beta}_{n,\kappa} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \prod_{k=1}^{n} \lambda_k^{\frac{\beta}{2}(\kappa-n+1)-1} e^{-\frac{\beta}{2}\lambda_k}, \tag{4.1}$$

where without loss of generality $\kappa \geq n$ and $c^{\beta}_{n,\kappa}$ is a normalizing constant (see e.g. [28]). The above densities first arose in the study of certain quantum systems and orthogonal polynomials (see [28] and references therein), however there were initially no known random matrices with these eigenvalue densities. Then in [25] the authors constructed families of tridiagonal random matrices whos eigenvalue densities agreed with the above, and in [51] the limiting distribution of the largest eigenvalues was determined, thus generalizing the classical Tracy-Widom laws for $\beta = 1, 2, 4$ to a family of distributions indexed by $\beta > 0$, denoted TW_{β}.

In a first approach to the general problem of finding the limiting distribution of the largest eigenvalue of a product of random matrices, we are free to choose which matrix

Submitted for publication, January 2013.

ensemble to work with and the β-ensembles along the methods employed in [51] are particularly amenable to such a study (the reader may note that throughout this paper we make the slight abuse of language in referring to both the above density and the corresponding family of random matrices as the β-Laguerre ensemble). Our results are as follows:

Theorem 4.1.1. *Let X_n^p and X_n^q be two independent elements of the β-Laguerre ensemble, with $\kappa = p$ and q respectively. Assume that $n \leq p \leq q$ and that $p = O(n) = q$. Then if $\lambda_{n,0}$ is the largest eigenvalue of $X_n^p X_n^q$,*

$$\frac{\lambda_{n,0} - \mu_n}{\sigma_n} \xrightarrow{d} TW_{\beta_0},$$

where TW_{β_0} denotes the Tracy-Widom Law with parameter β_0 and

$$\beta_0 = \lim_{n\to\infty} C_n \beta, \quad \mu_n = (\sqrt{n} + \sqrt{p})^2(\sqrt{n} + \sqrt{q})^2, \quad \sigma_n = c_n \frac{(\sqrt{n} + \sqrt{p})^{\frac{4}{3}}(\sqrt{n} + \sqrt{q})^{\frac{4}{3}}}{(\sqrt{np})^{\frac{1}{3}}(\sqrt{nq})^{\frac{1}{3}}},$$

the constants C_n and c_n being defined by (4.8) and (4.9) in section 2.4 below.

We have written the scaling terms to ease comparison to the case of a single matrix (e.g. [51], Theorem 1.4), where by the hypothesis $p = O(q)$ we have $c_n \to c \in \mathbb{R}$. It is worth noting that if both matrices are identically distributed, i.e. $p = q$, then $C_n = 2$, so even in the i.i.d. case the parameter of the limiting Tracy-Widom law is different than that of the factors.

In [51] the authors show how elements of the β-Laguerre ensemble can be realized as finite difference approximations to a stochastic differential operator on $[0, \infty)$. Just as in the usual finite difference schemes, e.g., for the Laplacian on $[0, \infty)$, the lowest k eigenvalues and eigenvectors converge to those of the limiting operator. This characterization of the limiting distributions is robust and we make full use of the results and techniques in [51] below, in particular Section 5 in that paper. We note here that although we assume in Theroem 4.1.1 that $n \leq p \leq q$, this is only to simplify the proof; one can relabel parameters without altering the arguments in any essential way.

In the next section we outline the setup from [51] and then proceed to the proof of Theorem 4.1.1. We end with some remarks and further questions in section 4.3.

4.2 Proof of Theorem 4.1.1

4.2.1 Tridiagonal elements of the β-Laguerre ensemble

Here we briefly describe the tridiagonal matrix ensemble that realizes (4.1); for proofs and further discussion see [25] and [28]. Let χ_α denote the random variable with density

$$\chi_\alpha \sim \frac{2}{\Gamma\left(\frac{\alpha}{2}\right)} x^{\alpha-1} e^{-x^2},$$

said to be a chi random variable with parameter α. Let B_n^κ, $\kappa \geq n$ be the following matrix:

$$B_n^\kappa = \begin{bmatrix} \tilde{\chi}_{\beta\kappa} & & & \\ \chi_{\beta(n-1)} & \tilde{\chi}_{\beta(k-1)} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \tilde{\chi}_{\beta(\kappa-n+1)} \end{bmatrix},$$

where $\tilde{\chi}_\alpha$ and χ_α denote independent chi random variables. Then the eigenvalues of

$$X_n^\kappa \equiv \left(B_n^\kappa\right)^* B_n^\kappa$$

have the density (4.1). Note that X_n^κ has

$$\tilde{\chi}^2_{\beta(\kappa-j+1)} + \chi^2_{\beta(n-j)}$$

along the main diagonal, $j = 1, \ldots, n$, and

$$\tilde{\chi}_{\beta(\kappa-j)} \chi_{\beta(n-j)}$$

above and below the main diagonal.

4.2.2 Notation and Setup from [51]

Unless specified otherwise, for vectors $v, u \in \mathbb{R}^n$, $\langle v, u \rangle$ denotes the Euclidean inner product and $\|v\| = \sqrt{\langle v, v \rangle}$.

Fix $\beta > 0$ and let X_n^i, $i = p, q$, be as above. Define

$$H_n^p \equiv \frac{\mu_{n,p} - X_n^p}{\sigma_{n,p}}, \qquad H_n^q \equiv \frac{\mu_{n,q} - X_n^q}{\sigma_{n,q}},$$

$$m_{n,i} = \left(\frac{\sqrt{ni}}{\sqrt{n} + \sqrt{i}}\right)^{\frac{2}{3}} = n^{\frac{1}{3}} \left(\frac{\sqrt{\frac{i}{n}}}{1 + \sqrt{\frac{i}{n}}}\right)^{\frac{2}{3}},$$

$$\mu_{n,i} = (\sqrt{n} + \sqrt{i})^2, \qquad \sigma_{n,i} = \frac{(\sqrt{n} + \sqrt{i})^{\frac{4}{3}}}{(\sqrt{ni})^{\frac{1}{3}}}.$$

Note here that the X_n^i, and hence the H_n^i, are independent, a fact we will use repeatedly below.

Let L^* be the following subspace of L^2,

$$L^* = \{f \in L^2[0, \infty) : \, f(0) = 0, \, \|f\|_*^2 < \infty\}$$

where

$$\|f\|_*^2 = \int_0^\infty (f')^2 + xf^2 + f^2 dx.$$

Let B be standard Brownian motion on $[0, \infty)$ and for $f \in L^*$ define

$$H_\beta(f) = -\frac{d^2}{dx^2} f + xf + \frac{2}{\sqrt{\beta}} B' f$$

where $B'f$ is the distribution given by

$$\frac{d}{dt} \int_0^t f \, dB$$

and where we denote the action of $H_\beta f$ on a test function $\phi \in C_c^\infty$ by

$$(\phi, H_\beta f).$$

Thus if ϕ is a test function,

$$(B'f, \phi) = -(f'B, \phi) - (fB, \phi').$$

In [51] it is shown that $(g, H_\beta f)$ defines a continuous bilinear form on L^* and if Λ denotes the smallest eigenvalue of H_β, given by

$$\Lambda = \inf\{(f, H_\beta f), : \ f \in L^*, \ \|f\|_{L^2} = 1\}, \tag{4.2}$$

then $-\Lambda$ is distributed as TW_β, that is, $-\Lambda \sim TW_\beta$.

Next let $L^*_{n,i}$ be the subspace of $L^2[0, \infty)$ consisting of step functions of the following form:

$$f = \sum_{k=1}^{n} c_k \chi_{[\frac{k-1}{m_{n,i}}, \frac{k}{m_{n,i}}]}.$$

Let P_n be the projection from L^2 onto this subspace. Then $L^*_{n,i}$ is isometric to $\mathbb{R}^n$ with the inner product

$$m_{n,i}^{-1}\langle v, u\rangle = m_{n,i}^{-1}\sum_{k=1}^{n} v_k u_k,$$

$$\langle f, g\rangle_{L^2} = \sum_{k=1}^{n} c_k d_k m_{n,i}^{-1} = m_{n,i}^{-1}\langle f, g\rangle_{\mathbb{R}^n}.$$

We let T_n denote the shift operator

$$(T_n v)_k = v_{k+1},$$

that is, the operator given by the $n \times n$ matrix with 1's above the main diagonal and zero's elsewhere. Then define the difference operator

$$\Delta_n^i v_k = m_{n,i}(v_k - v_{k-1}) = m_{n,i}(I - T_n^*)v_k,$$

i.e., for $\phi \in C_c^\infty$ $\Delta_n^i \Delta_n^{i*} P_n \phi \to \phi''$ in L^2, and note $\|T_n\| = 1$. Additionally, for two vectors $u, v \in \mathbb{R}^n$ we denote by $u_\times v$ the vector

$$(u_1 v_1, \ldots, u_n v_n).$$

Now H_n^i takes the following form:

$$H_n^i v = -\Delta_n^i \Delta_n^{i*} v + \left(\Delta_n^i y_{n,1}^i\right)_\times v + \frac{1}{2}\left(\Delta_n^i y_{n,2}^i\right)_\times T_n v + \frac{1}{2} T_n^* \left(\Delta_n^i y_{n,2}^i\right)_\times v,$$

$$\Delta_n^i y_{n,j}^i = \eta_{n,j}^i + \Delta_n^i w_{n,j}^i,$$

$$(\eta_{n,1}^i)_k = \frac{m_{n,i}^2}{\sqrt{ni}}(n + i - \beta^{-1}\mathbb{E}[\tilde{\chi}^2_{\beta(i-k+1)} + \chi^2_{\beta(n-k)}]) = \frac{m_{n,i}^2}{\sqrt{ni}}(2k-1),$$

$$(\eta_{n,2}^i)_k = \frac{m_{n,i}^2}{\sqrt{ni}}2(\sqrt{ni} - \beta^{-1}\mathbb{E}[\chi_{\beta(n-k)}\tilde{\chi}_{\beta(i-k)}]),$$

$$(w_{n,1}^i)_k = \frac{m_{n,i}}{\sqrt{ni}}\sum_{j=1}^{k}\left(n + i - \beta^{-1}(\chi^2_{n-j} + \tilde{\chi}^2_{i-j+1})\right) - m_{n,i}^{-1}(\eta_{n,1}^i)_k,$$

$$(w_{n,2}^i)_k = \frac{m_{n,i}}{\sqrt{ni}}2\sum_{j=1}^{k}\left(\sqrt{ni} - \beta^{-1}\chi_{\beta(n-j)}\tilde{\chi}_{\beta(i-j)}\right) - m_{n,i}^{-1}(\eta_{n,2}^i)_k.$$

We now collect some bounds we will need in the proof below in Section 4.2.3. In [51] it is shown that for each i and any subsequence $H_{n_m}^i$ there exists a further subsequence and a probability space such that the statements below hold almost surely; from now on we will assume we are working with such a subsequence.

First we have that for any $\epsilon > 0$ there is a $c_\epsilon^i > 0$ such that

$$|\Delta_n^i w_{n,j,k}^i| \leq m_{n,i}\sqrt{\epsilon\tilde{\eta}_{n,k}^i + c_\epsilon^i} \tag{4.3}$$

where

$$\tilde{\eta}_{n,k}^i = \frac{k}{m_{n,i}}.$$

Next we have the following two bounds

$$\eta_{n,j,k}^i \leq 2m_{n,i}^2, \qquad c_1^\eta\tilde{\eta}^i \leq \eta_{n,1,k}^i + \eta_{n,2,k}^i \leq c_2^\eta\tilde{\eta}^i \tag{4.4}$$

for some $c_i^\eta > 0$. Finally (cf. Section 6 in [51]), there exist independent Brownian motions B^i and processes $y_j^i(x)$ such that

$$y_{n,j}^i(x) \equiv (y_{n,j}^i)_{\lfloor xm_{n,q}\rfloor}\mathbf{1}_{xm_{n,q}\in[0,n]} \to y_j^i(x) \tag{4.5}$$

and

$$y_{n,1}^i(x) + y_{n,2}^i(x) \to \frac{2}{\sqrt{\beta}}B^i + \frac{x^2}{2}$$

in the Skorokhod topology on $D[0,\infty)$.

4.2.3 Outline of the proof

Let H_β^i denote the operator H_β above with B^i in place of B. In [51] the authors show, for each subsequence restricted to a further subsequence such that the above bounds hold a.s., that the smallest eigenvalue and corresponding eigenvector of H_n^i converge to that of H_β^i using three Lemmas, numbered $5.6-5.8$, the content of which is as follows: Lemma 5.6 states that there are positive constants c_k^i independent of n such that for all $v \in \mathbb{R}^n$

$$c_1^i \|v\|_{i,n*}^2 - c_2^i m_{n,i}^{-1} \|v\|_2^2 \le m_{n,i}^{-1} \langle H_n^i v, v\rangle_{\mathbb{R}^n} \le c_3^i \|v\|_{n,i*}^2$$

where

$$\|v\|_{i,n*}^2 = m_{n,i}^{-1}(\|\Delta_n^i v\|_{\mathbb{R}^n}^2 + \|(\bar{\eta}_n^i)_\times^{\frac{1}{2}} v\|_{\mathbb{R}^n}^2 + \|v\|_{\mathbb{R}^n}^2).$$

This is a coercivity bound used to control the eigenvectors as $n \to \infty$. Lemma 5.7 establishes convergence in the sense of distributions, i.e., if $f_n \in L_{n,i}^*$ is such that $f_n \to f$ and $\Delta_n^i f_n \to f'$ weakly in L^2 then for any $\phi \in C_c^\infty$

$$\langle \phi, H_n^i f_n\rangle_{L^2} \to (\phi, H_\beta^i f).$$

Lastly Lemma 5.8 ensures that the eigenvectors of H_n^i contain a subsequence converging to those of H^i: If $f_n \in L_{n,i}^*$, $\|f\|_{n,i*}^2 \le c < \infty$, and $\|f\|_{L^2}^2 = 1$ then there exists a subsequence f_{n_k} such that $f_{n_k} \to_{L^2} f \in L^*$ and $\langle \phi, H_{n_k}^i f_{n_k}\rangle_{L^2} \to (\phi, H_\beta^i f)$ for all $\phi \in C_c^\infty$.

We want to study the smallest eigenvalue of

$$\begin{aligned} H_n = \frac{\mu_{n,p}\mu_{n,q} I - X_n^p X_n^q}{\sigma_{n,p}^2 \sigma_{n,q}^2} &= \frac{\mu_{n,p} I - X_n^p}{\sigma_{n,p}} \frac{X_n^q}{\sigma_{n,q}^2 \sigma_{n,p}} + \frac{\mu_{n,p}}{\sigma_{n,p}^2 \sigma_{n,q}} \frac{\mu_{n,q} I - X_n^q}{\sigma_{n,q}} \\ &= \frac{\mu_{n,q}}{\sigma_{n,q}^2 \sigma_{n,p}} H_n^p (I - \frac{\sigma_{n,q}}{\mu_{n,q}} H_n^q) + \frac{\mu_{n,p}}{\sigma_{n,p}^2 \sigma_{n,q}} H_n^q \\ &= a_n \bar{H}_n^p + b_n \bar{H}_n^q - \frac{m_{n,p}^2 m_{n,q}^2}{m_n^4 \sigma_{n,p}\sigma_{n,q}} \bar{H}_n^p \bar{H}_n^q, \end{aligned}$$

where

$$\bar{H}_n^i = \frac{m_n^2}{m_{n,i}^2} H_n^i, \qquad m_n = \left(\frac{\left(\frac{\mu_q}{\sigma_{n,q}^2 \sigma_{n,p}} m_{n,p}^2 + \frac{\mu_p}{\sigma_{n,p}^2 \sigma_{n,q}} m_{n,q}^2 \right) m_{n,p} m_{n,q}}{\frac{\mu_q}{\sigma_{n,q}^2 \sigma_{n,p}} m_{n,q} + \frac{\mu_p}{\sigma_{n,p}^2 \sigma_{n,q}} m_{n,p}} \right)^{\frac{1}{3}}$$

and

$$a_n = \frac{m_{n,p}^2 \mu_{n,q}}{m_n^2 \sigma_{n,q}^2 \sigma_{n,p}}, \qquad b_n = \frac{m_{n,q}^2 \mu_{n,p}}{m_n^2 \sigma_{n,p}^2 \sigma_{n,q}}.$$

This choice of m_n ensures the proper scaling for the convergence we need below.

In the next section we determine the limiting operator of H_n in the sense above. The product term $\bar{H}_n^p \bar{H}_n^q$ prevents us from directly applying Theorem 5.1 in [51], so instead we will follow the proof of that Theorem, stating and proving Lemmas analogous to those above.

4.2.4 Convergence

To begin we first establish analogous almost sure bounds to those above. We have

$$\bar{H}_n^i v = -\Delta_n \Delta_n^* v + \left(\Delta_n \bar{y}_{n,1}^i\right)_\times v + \frac{1}{2}\left(\Delta_n \bar{y}_{n,2}^i\right)_\times T_n v + \frac{1}{2} T_n^* \left(\Delta_n \bar{y}_{n,2}^i\right)_\times v$$

where

$$\Delta_n = m_n(I - T_n^*),$$

$$\Delta_n \bar{y}_{n,j}^i = \bar{\eta}_{n,j}^i + \Delta_n \bar{w}_{n,j}^i,$$

$$\bar{\eta}_{n,j}^i = \frac{m_n^2}{m_{n,i}^2} \eta_{n,j}^i, \qquad \bar{w}_{n,j}^i = \frac{m_n}{m_{n,i}} w_{n,j}^i,$$

i.e.,

$$(\bar{y}_{n,j}^i)_k = \frac{1}{m_n} \sum_{i=1}^{k} (\bar{\eta}_{n,j}^i)_k + (\bar{w}_{n,j}^i)_k = \frac{m_n}{m_{n,i}} (y_{n,j}^i)_k.$$

Noting that by hypothesis

$$m_n = O(m_{n,p}) = O(m_{n,q}) = O(n^{1/3}),$$

it follows easily from (4.3) and (4.4) that we can reduce to subsequences as above such that

$$|(\Delta_n \bar{w}_{n,j}^i)_k| \leq m_n \sqrt{\epsilon \tilde{\eta}_{n,k} + c_\epsilon}, \tag{4.6}$$

$$\bar{\eta}_{n,j,k}^i \leq 2m_n^2, \qquad c_1^\eta \tilde{\eta} \leq \bar{\eta}_{n,1,k}^i + \bar{\eta}_{n,2,k}^i \leq c_2^\eta \tilde{\eta}, \tag{4.7}$$

and the processes defined by

$$\bar{y}^i_{n,j}(x) \equiv (\bar{y}^i_{n,j})_{\lfloor xm_n \rfloor} \mathbf{1} xm_n \in [0,n]$$

are convergent in the Skorokhod topology on $D[0,\infty)$, where $\tilde{\eta}_{n,k} = k/m_n$ and where we have used the same notation for the constants as in (4.4), though they may be different here. With the bounds (4.6)–(4.7) in hand, the proofs of Lemmas 5.6–5.8 in [51] apply to $\bar{H}^i_n$ without change, a fact we will use below.

If we now let $\bar{y}_{n,j} = \dfrac{a_n}{c_n}\bar{y}^p_{n,j} + \dfrac{b_n}{c_n}\bar{y}^q_{n,j}$ where

$$c_n = a_n + b_n = \frac{(\sqrt{np}+\sqrt{nq})^2\left((\sqrt{n}+\sqrt{q})^2\sqrt{np}+(\sqrt{n}+\sqrt{p})^2\sqrt{nq}\right)}{(\sqrt{n}+\sqrt{p})^4(\sqrt{n}+\sqrt{q})^4}, \tag{4.8}$$

then by our choice of m_n and using the independence of the y^i, it follows from [51], Section 6, that there is a Brownian motion B_x such that

$$\bar{y}_{n,1}(x) + \bar{y}_{n,2}(x) \to \frac{x^2}{2} + \frac{2}{\sqrt{C\beta}} B_x,$$

$$C = \lim_{n\to\infty}\left(\frac{m_n^3}{m_{n,p}^3}\frac{a_n^2}{c_n^2} + \frac{m_n^3}{m_{n,q}^3}\frac{b_n^2}{c_n^2}\right)^{-1} = 1 + \lim_{n\to\infty}\frac{p(\sqrt{n}+\sqrt{p})^2 + q(\sqrt{n}+\sqrt{q})^2}{\sqrt{pq}\left((\sqrt{n}+\sqrt{p})^2+(\sqrt{n}+\sqrt{q})^2\right)}, \tag{4.9}$$

in law with respect to the Skorokhod topology on $D[0,\infty)$. As already noted, we can reduce to a further subsequence such that this convergence holds almost surely on some probability space. We now have a candidate limiting operator:

$$H_n \to c\left(-\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{C\beta}}B'_x\right) = cH_{\beta_0}, \qquad \beta_0 = C\beta,\ c = \lim c_n,$$

the idea being that $c_n^{-1}\left(a_n\bar{H}^p_n + b_n\bar{H}^q_n\right) \to H_{\beta_0}$ and the product term $\bar{H}^p_n\bar{H}^q_n$ vanishes in the limit.

In the following lemma we let L^*_n be the analogue of the discrete spaces already defined above for our new scaling term, e.g., L^*_n is the space of step functions of the form

$$f = \sum_{k=1}^{n} c_k \chi_{[\frac{k-1}{m_n},\frac{k}{m_n}]}$$

and P_n denotes the projection from L^2 onto this space.

Lemma 4.2.1. *Let $f_n \in L_n^*$ be such that $f_n \to f$ and $\Delta_n f_n \to f'$ weakly in L^2. Then for all $\phi \in C_c^\infty$*

$$\langle \phi, H_n f_n \rangle_{L^2} = \langle P_n \phi, H_n f_n \rangle_{L^2} \to (\phi, cH_{\beta_0} f).$$

proof. The bounds (4.6)–(4.7) can be extended additively to $a_n \bar{H}_n^p + b_n \bar{H}_n^q$ and the proof of Lemma 5.7 in [51] goes through without change to show that under the hypotheses above

$$\langle \phi, (a_n \bar{H}_n^p + b_n \bar{H}_n^q) f_n \rangle_{L^2} \to (\phi, cH_{\beta_0} f). \tag{4.10}$$

Next,

$$\frac{m_{n,p}^2 m_{n,q}^2}{m_n^4 \sigma_{n,p} \sigma_{n,q}} = O\left(m_n^{-2}\right),$$

so the proof of Lemma 4.2.1 reduces to showing

$$m_n^{-2} \langle \phi, \bar{H}_n^p \bar{H}_n^q f_n \rangle_{L^2} = m_n^{-2} \langle \bar{H}_n^p P_n \phi, -\Delta_n \Delta_n^* f_n \rangle_{L^2} + m_n^{-2} \langle \bar{H}_n^p P_n \phi, \bar{H}_n^q f_n + \Delta_n \Delta_n^* f_n \rangle_{L^2} \to 0.$$

First note that for $g \in L^2$, $T_n g \to g$ in L^2 and likewise for T_n^*. Then

$$\langle g, T_n f_n \rangle_{L^2} = \langle T_n^* g, f_n \rangle_{L^2} \to \langle g, f \rangle_{L^2}$$

so $T_n f_n \to f$ weakly and likewise for $T_n^* f_n$. Similarly $T_n T_n^* f_n \to f$ weakly. Thus

$$(T_n^* - I)(I - T_n) f_n \to 0$$

weakly. Next observe that

$$\langle g, \Delta_n (T_n^* - I)(I - T_n) f_n \rangle = \langle g, (I - T_n^*)(T_n - I) \Delta_n^* f_n \rangle = \langle (T_n^* - I)(I - T_n) g, \Delta_n^* f_n \rangle$$

and $(T_n^* - I)(I - T_n) g \to 0$ in L^2. We also have $\Delta_n^* f_n \to -f'$ weakly. Thus

$$\Delta_n (T_n^* - I)(I - T_n) f_n \to 0$$

weakly as well and Lemma 5.7 in [51] now implies

$$m_n^{-2} \langle \bar{H}_n^p P_n \phi, -\Delta_n \Delta_n^* f_n \rangle_{L^2} = \langle \phi, \bar{H}_n^p (T_n^* - I)(I - T_n) f_n \rangle_{L^2} \to 0.$$

For the terms

$$\langle m_n^{-2}\bar{H}_n^p P_n\phi, \bar{H}_n^q f_n + \Delta_n\Delta_n^* f_n\rangle_{L^2}$$

we note that from the proof of Lemma 5.7 in [51] we have the following: If $g_n \in L_n^*$ is such that g_n is bounded uniformly independent of n, g_n and $\Delta_n g_n$ both have supports that are contained in a finite interval I for all n, and both are convergent in L^2 with

$$g_n \overset{L^2}{\to} g \quad \text{and} \quad \Delta_n g_n \overset{L^2}{\to} g',$$

then

$$\langle g_n, \bar{H}_n^q f_n + \Delta_n\Delta_n^* f_n\rangle_{L^2} \to (g, \bar{H}^q f + f'')$$

for all f_n as above. Thus if we show that $g_n = m_n^{-2}\bar{H}_n^p P_n\phi$ satisfies the above hypothesis and $g_n \to 0$ the proof will be complete.

The existence of I comes from $\phi \in C_c^\infty$ and uniform boundedness follows easily from (4.6) and (4.7) together with the compact support and uniform boundedness of $P_n\phi$.

To control

$$\Delta_n m_n^{-2}\bar{H}_n^p P_n\phi$$

we first consider $\Delta_n(-m_n^{-2}\Delta_n\Delta_n^* P_n\phi) = (I - T_n^*)(T_n^* - I)\Delta_n^* P_n\phi$. By the arguments above this converges to 0 in L^2. For the potential term

$$\begin{aligned} &\Delta_n m_n^{-2}\left(\left(\Delta_n\bar{y}_{n,1}^p\right)_\times P_n\phi + \frac{1}{2}\left(\Delta_n\bar{y}_{n,2}^p\right)_\times T_n P_n\phi + \frac{1}{2}T_n^*\left(\Delta_n\bar{y}_{n,2}^p\right)_\times P_n\phi\right) \quad (4.11)\\ &= (I - T_n^*)\left(\left((I - T_n^*)\bar{y}_{n,1}^p\right)_\times P_n\phi + \frac{1}{2}\left((I - T_n^*)\bar{y}_{n,2}^p\right)_\times T_n P_n\phi\right.\\ &\quad \left. + \frac{1}{2}T_n^*\left((I - T_n^*)\bar{y}_{n,2}^p\right)_\times P_n\phi\right), \end{aligned}$$

we note that $\bar{y}_{n,j}^p(x)$ are locally bounded and convergent a.e. This combined with the compact support of $P_n\phi$ implies the $\bar{y}_{n,j}^p(x)$ converge locally in L^2, and by the arguments above regarding T_n we find that the above converges to 0 in L^2. That $m_n^{-2}\bar{H}_n^p P_n\phi \overset{L^2}{\to} 0$ follows similarly.

□

Lemma 4.2.2. *Define the following norm on* $\mathbb{R}^n$*:*

$$\|v\|_{*n}^2 = m_n^{-1}(\|\Delta_n v\|_{\mathbb{R}^n}^2 + \|(\tilde{\eta}_n)_\times^{\frac{1}{2}} v\|_{\mathbb{R}^n}^2 + \|v\|_{\mathbb{R}^n}^2).$$

Then we have constants $C_k > 0$ *and* $N > 0$ *such that for all* $n > N$

$$C_1\|v\|_{n*}^2 - C_2 m_n^{-\frac{1}{2}}\|v\|_{\mathbb{R}^n}\sqrt{\|v\|_{n*}^2} - C_3 m_n^{-1}\|v\|_{\mathbb{R}^n}^2 \leq \langle H_n v, v\rangle_{L^2}. \tag{4.12}$$

proof. We have by definition

$$\begin{aligned}\bar{H}_n^i v = -\Delta_n\Delta_n^* v &+ \left(\left(\bar{\eta}_{n,1}^i\right)_\times v + \frac{1}{2}\left(\bar{\eta}_{n,2}^i\right)_\times T_n v + \frac{1}{2}T_n^*\left(\bar{\eta}_{n,2}^i\right)_\times v\right)\\ &+ \left(\left(\Delta_n \bar{w}_{n,1}^i\right)_\times v + \frac{1}{2}\left(\Delta_n \bar{w}_{n,2}^i\right)_\times T_n v + \frac{1}{2}T_n^*\left(\Delta_n \bar{w}_{n,2}^i\right)_\times v\right)\\ &= A^i v + B^i v + C^i v.\end{aligned}$$

So letting

$$d_n = \frac{m_{n,p}^2 m_{n,q}^2}{m_n^4 \sigma_{n,p}\sigma_{n,q}},$$

we have

$$\begin{aligned}&a_n\langle \bar{H}_n^p v, v\rangle + b_n\langle \bar{H}_n^q v, v\rangle - \frac{m_{n,p}^2 m_{n,q}^2}{m_n^4\sigma_{n,p}\sigma_{n,q}}\langle \bar{H}_n^p v, \bar{H}_n^q v\rangle\\ &= a_n(\langle (A^p + B^p)(I - d_n a_n^{-1}(A^q + B^q))v, v\rangle) + b_n\langle (A^q + B^q)v, v\rangle \qquad (4.13)\\ &\quad + d_n\left(\langle C^q v, (A^p + B^p)v\rangle + \langle C^p v, (A^q + B^q)v\rangle + \langle C^q v, C^p v\rangle\right) \qquad (4.14)\\ &\quad + a_n\langle C^p v, v\rangle + b_n\langle C^q v, v\rangle.\end{aligned}$$

We first bound (4.14) and then (4.13). We have from (4.6)

$$m_n^{-1}\|\Delta_n^i \bar{w}_{n,j,k} v_k\| \leq \|\sqrt{\epsilon\tilde{\eta}_{n,k} + c_\epsilon} v_k\|. \tag{4.15}$$

Then for $m_n^{-2}\langle C^q v, C^p v\rangle$ we have

$$\begin{aligned}
m_n^{-1}\|C^i v\| &\leq \|\sqrt{\epsilon\tilde{\eta}_{n,k}+c_\epsilon}v_k\| + \frac{1}{2}\|\Delta_n^i \bar{w}_{n,2,k}^i T_n v_k\| + \frac{1}{2}\|T_n^*\Delta_n^i \bar{w}_{n,2,k}^i v_k\| \\
&= \|\sqrt{\epsilon\tilde{\eta}_{n,k}+c_\epsilon}v_k\| + \frac{1}{2}\|\Delta_n^i \bar{w}_{n,2,k}^i v_{k+1}\| + \frac{1}{2}\|T_n^*\Delta_n^i \bar{w}_{n,2,k}^i v_k\| \\
&\leq \|\sqrt{\epsilon\tilde{\eta}_{n,k}+c_\epsilon}v_k\| + \frac{1}{2}\|\sqrt{\epsilon\tilde{\eta}_{n,k}+c_\epsilon}v_{k+1}\| + \frac{1}{2}\|T_n^*\Delta_n^i \bar{w}_{n,2,k}^i v_k\| \\
&\leq \|\sqrt{\epsilon\tilde{\eta}_{n,k}+c_\epsilon}v_k\| + \frac{1}{2}\|\sqrt{\epsilon\tilde{\eta}_{n,k+1}+c_\epsilon}v_{k+1}\| + \frac{1}{2}\|\sqrt{\epsilon\tilde{\eta}_{n,k-1}+c_\epsilon}v_{k-1}\| \\
&\leq 2\|\sqrt{\epsilon\tilde{\eta}_{n,k}+c_\epsilon}v_k\|
\end{aligned}$$

and so

$$\begin{aligned}
|m_n^{-2}\langle C^q v, C^p v\rangle| \leq 4\|\sqrt{\epsilon\tilde{\eta}_{n,k}+c_\epsilon}v_k\|^2 &= 4\epsilon\|\sqrt{\tilde{\eta}_{n,k}}v_k\|^2 + c_\epsilon\|v\|^2 \\
&\leq 4\epsilon m_n\|v\|_{*n}^2 + c_\epsilon\|v\|_{\mathbb{R}^n}^2. \qquad (4.16)
\end{aligned}$$

For the $\langle A, C\rangle$ terms,

$$m_n^{-1}\|A^i v\| = c^i\|(I - T_n^*)\Delta_n^* v\| \leq 2c^i\|\Delta_n v\|$$

for constants $c^i > 0$, so we have

$$m_n^{-1}\|A^i v\| \leq c_A\|\Delta_n v\|$$

for some $c_A > 0$. Thus

$$\begin{aligned}
m_n^{-2}|\langle C^q v, A^p v\rangle| &\leq 2\|\sqrt{\epsilon\tilde{\eta}_{n,k}+c_\epsilon}v_k\|c_A\|\Delta_n v\| \\
&\leq 2c_A(\sqrt{\epsilon m_n\|v\|_{n*}^2} + \sqrt{c_\epsilon}\|v\|)\sqrt{m_n\|v\|_{n*}^2} \\
&= 2c_A\left(\sqrt{\epsilon}m_n\|v\|_{n*}^2 + \sqrt{c_\epsilon}\|v\|_{\mathbb{R}^n}\sqrt{m_n\|v\|_{n*}^2}\right), \qquad (4.17)
\end{aligned}$$

and similarly for $m_n^{-2}\langle A^q v, C^p v\rangle$.

For the $\langle B, C\rangle$ terms note that

$$m_n^{-2}|(\Delta_n w_{n,j}^i)_k| \leq m_n^{-1}\sqrt{\epsilon\tilde{\eta}+c_\epsilon} \leq \sqrt{\epsilon}\sqrt{m_n^{-2}\tilde{\eta}} + m_{m,q}^{-1}\sqrt{c_\epsilon} \leq c_1\sqrt{\epsilon} + m_{m,q}^{-1}\sqrt{c_\epsilon}.$$

By Cauchy-Schwarz and (4.7) we have

$$\begin{aligned} |m_n^{-2}\langle C^q v, B^p v\rangle| &\leq c_2(c_1\sqrt{\epsilon} + m_n^{-1}\sqrt{c_\epsilon})\sum(\tilde{\eta}_n)_k v_k^2 \\ &\leq c_3(c_1\sqrt{\epsilon} + m_n^{-1}\sqrt{c_\epsilon})m_n\|v\|_{n*}^2 \end{aligned} \tag{4.18}$$

and likewise for $m_n^{-2}\langle C^p v, B^q v\rangle$.

For the remaining noise terms, we have from the proof of Lemma 5.6 in [51] that

$$\langle C^i v, v\rangle \geq -c_4\sqrt{\epsilon} m_n\|v\|_{n*}^2 - c_5(\epsilon)\|v\|_{\mathbb{R}^n}^2.$$

For (4.13), first we note that from the same Lemma in [51] using (4.7) we have

$$\langle (A^p + B^p)v, v\rangle \geq 0.$$

After some algebra we find

$$\frac{d_n}{a_n} = \frac{\sqrt{\frac{q}{n}}}{(1+\sqrt{\frac{q}{n}})^2} m_n^{-2} \leq \frac{1}{4} m_n^{-2}.$$

By definition,

$$\begin{aligned} m_n^{-2}\langle (A^p + B^p)v, v\rangle &= \sum (m_n^{-2}(\bar{\eta}_{n,1}^p)_k - 2)v_k^2 + m_n^{-2}(\bar{\eta}_{n,2}^p)_k + 2)v_k v_{k+1} \\ &\leq \sum (m_n^{-2}(\bar{\eta}_{n,2}^p)_k + 2)v_k v_{k+1} \\ &\leq 4\|v\|^2 \end{aligned}$$

using (4.7) and Cauchy-Schwarz. Thus

$$d_n a_n^{-1}\langle (A^p + B^p)v, v\rangle \leq \|v\|^2$$

and so

$$I - d_n a_n^{-1}(A^p + B^p)$$

is Hermitian with spectrum contained in $[0,1]$. Thus

$$T \equiv (A^p + B^p)(I - d_n a_n^{-1}(A^p + B^p)),$$

being the product of two Hermitian, nonnegative matrices has only real, nonnegative eigenvalues (though it need not be normal). Then using standard results (see e.g. [35], Chapter 1 and [57]) on the numerical range of T,

$$\{\langle Tv, v\rangle : \|v\| = 1\},$$

we see that $\langle Tv, v\rangle \geq -\|v\|^2$. Thus

$$\langle (A^p + B^p)(I - d_n a_n^{-1}(A^p + B^p))v, v\rangle \geq -\|v\|^2. \tag{4.19}$$

Lastly, from [51], Lemma 5.6, we know

$$\langle (A^q + B^q)v, v\rangle \geq c_6 m_n \|v\|_{n*}^2 - c_7\|v\|^2.$$

Noting that a_n, b_n, and d_n are convergent, we now have constants $c_8, c_9, c_{10}(\epsilon), c_{11}(\epsilon), c_{12}(\epsilon) > 0$ such that

$$\begin{aligned} a_n\langle \bar{H}_n^p v, v\rangle + b_n\langle \bar{H}_n^q v, v\rangle - d_n\langle \bar{H}_n^p v, \bar{H}_n^q v\rangle \qquad\qquad (4.20)\\ \geq (c_8 - c_9 O(\epsilon) - c_{10}(\epsilon)m_n^{-1})m_n\|v\|_{n*}^2 - c_{11}(\epsilon)\|v\|\sqrt{m_n\|v\|_{n*}^2} - c_{12}(\epsilon)\|v\|^2. \end{aligned}$$

Taking ϵ small and then n large establishes the lemma. □

Lemma 4.2.3. *Suppose $f_n \in L_n^*$ with $\|f_n\|_{*n}^2 \leq c < \infty$ and $\|f_n\|_{L^2} = 1$. Then there exists $f \in L^*$ and a subsequence f_{n_k} such that $f_{n_k} \xrightarrow{L^2} f$ and for all $\phi \in C_c^\infty$ we have*

$$\langle \phi, H_{n_k} f_{n_k}\rangle_{L^2} \to (\phi, cH_{\beta_0} f).$$

proof. The proof is that same as that of Lemma 5.8 in [51] and we omit it. □

Let $\bar{\lambda}_{n,0}$ and $v_{n,0}$ be the smallest eigenvalue and corresponding eigenvector of H_n such that $\|v_{n,0}\|_{L^2}^2 = m_n^{-1}\|v_{n,0}\|_{\mathbb{R}^n}^2 = 1$, and let Λ_0 and f_0 be the same for H_{β_0}. To show that $\bar{\lambda}_{n,0} \to c\Lambda_0$ we can proceed exactly as in [51], repeating the arguments for completeness.

Suppose $\liminf \bar{\lambda}_{n,0} < \infty$. Lemma 4.2.2 shows that $\bar{\lambda}_{n,0}$ is uniformly bounded below so there exists a subsequence such that $\bar{\lambda}_{n_k,0} \to \liminf \bar{\lambda}_{n,0}$. Lemma 4.2.2 now implies that $\|v_{n_k,0}\|_{n*}^2$ are uniformly bounded, Lemma 4.2.3 then implies that a further subsequence converges to some $f \in L^*$ as in Lemma 4.2.1, and so Lemma 4.2.1 implies that for this further subsequence

$$\langle P_n\phi, H_{n_k} v_{n_k,0}\rangle_{L^2} \to (\phi, cH_{\beta_0} f).$$

Then it follows that

$$\frac{(\phi, cH_{\beta_0} f)}{\langle f, f\rangle_{L^2}} = \liminf \bar{\lambda}_{n,0} \frac{\langle \phi, f\rangle_{L^2}}{\langle f, f\rangle_{L^2}}$$

for all $\phi \in C_c^\infty$. Thus

$$\liminf \bar{\lambda}_{n,0} \geq c\Lambda_0.$$

To see $\limsup \bar{\lambda}_{n,0} \leq c\Lambda_0$, let $f^\epsilon \in C_c^\infty$ be such that $\|f^\epsilon - f_0\|_*^2 < \epsilon$. Then by the minmax principle and Lemma 4.2.1,

$$\begin{aligned} \limsup \bar{\lambda}_{n,0} &\leq \limsup_{n\to\infty} \frac{\langle P_n f^\epsilon, H_n P_n f^\epsilon\rangle_{L^2}}{\langle P_n f^\epsilon, P_n f^\epsilon\rangle_{L^2}} \\ &= \frac{(f^\epsilon, cH_{\beta_0} f^\epsilon)}{\langle f^\epsilon, f^\epsilon\rangle_{L^2}}. \end{aligned} \tag{4.21}$$

Letting $\epsilon \to 0$ we have

$$\limsup \bar{\lambda}_{n,0} \leq \frac{(f_0, cH_{\beta_0} f_0)}{\langle f_0, f_0\rangle_{L^2}} = c\Lambda_0.$$

Noting that by definition

$$-\bar{\lambda}_{n,0} = c_n \frac{\lambda_{n,0} - \mu_n}{\sigma_n},$$

what we have then is that for every subsequence of $\{\lambda_{n,0}\}$ there exists a probability space and a further subsequence along which

$$\frac{\lambda_{n,0} - \mu_n}{\sigma_n} \to -\Lambda_0$$

almost surely. Recalling that $-\Lambda_0 \sim TW_{\beta_0}$, Theorem 4.1.1 obtains.

4.3 Some remarks

The reader may note that contrary to the approach in the classical case, the framework in terms of a limiting operator allows us to avoid determining the eigenvalue densities for finite n, which, depending on one's point of view can be either an advantage or disadvantage to the approach.

Although Theorem 4.1.1 does not tell us about the largest eigenvalue of the product of two independent Wishart matrices, it does suggest some interesting questions regarding the classical ensembles. For example, in [13] the authors determine the limiting empirical spectral distribution for a product of independent Wisharts, the limit depending on the ratio of the two parameters in the product. The authors there conjecture that the limiting distribution of the largest eigenvalue of such a product is a Tracy-Widom law. One can then ask the following: If the limit does indeed follow a Tracy-Widom law TW_β, what is β, and does it depend on the parameters in a way similar to that in Theorem 4.1.1? Much is still unknown about the full family of TW_β distributions and it would be of interest to see them arise for $\beta \neq 1, 2, 4$ in the context of the classical ensembles.

Chapter 5: Conclusion

The above papers are examples of how one can solve problems in probability using functional analytic tools and intuition. Here we have only seen functional analytical ideas applied to probability, but in fact there is much work where the direction is reversed and probabilistic tools are used to solve problems in functional analysis. One particular area that seems to me to offer a wealth of possibilities in both directions is the general theory of random operators. Of course that is just one example of a still largely undeveloped subject where probability and functional analysis interact, and it is my hope that the work here helps to demonstrate the rich possibilities when these two fields are considered together.

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