Stand Sapwood Basal Area: An Indicator of Tree Vigor

by

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The control that stand leaf area exerts upon individual tree vigor was tested on a heterogeneous, even-aged Douglas-fir stand infected by Phellinus root rot.

Tree vigor, or growth efficiency ratio, was defined as the current basal area growth per unit of leaf area. Because of a close relationship between sapwood basal area and tree leaf area, stand sapwood basal area was used as an indirect estimator of competing leaf area to test changes in growth efficiency at different levels of canopy density. It was hypothesized that the growth efficiency ratio would decrease linearly with increasing stand sapwood basal area.

No significant relationships were found between stand sapwood basal area and either average or individual tree growth efficiency. However, the maximum efficiency of trees recorded under specified levels of canopy competition (upper boundary) did follow the expected

trends. No significant difference was found between the slope of the upper boundary linear equation and the slope of the line which described the relationship between mean growth efficiency and stand sapwood basal area in another homogeneous stand in the Oregon Coastal Range.

Several explanations for the lack of the general hypothesized relationship were offered, including the need for more refined measurement methods of sapwood basal area and basal area growth. Still, the most reasonable explanation is that root rot continued to reduce growth below the potential and induced unmeasured continuous changes in the canopy which affected individual tree response differently.

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AN INDICATOR OF TREE VIGOR

A recent publication (Waring et al. 1980) has suggested that trees in a stand supported varying amounts of canopy but produced an equal quantity of wood per unit of leaf surface.

Carbon assimilates are assumed to be more readily available for production of stemwood only after other essential needs are satisfied, such as respiratory demands (Mooney 1972, Ledig and Botkin 1974), new root (Harris et al. 1977, Fogel and Hunt 1979) and shoot growth (Gordon and Larson 1968, Rangnekar and Forward 1973, Larsson and Bengtson 1980), and the formation of reproductive tissues (Eis et al. 1965). This has implied that the proportion of carbon allocated to stemwood production indicates tree vigor (Waring et al. 1980, Waring 1980). Tree vigor is expected to be reduced in proportion to competition from canopy development of other trees, within a given environment (Boyer 1968, Waring et al. 1978, Waring 1980).

Growth in basal area reflects stemwood production (Waring 1979, Waring and Pitman 1980, Waring et al. 1980, Waring 1980), and sapwood basal area (SA) is proportional to, and reflects leaf area (Dixon 1971, Grier and Waring 1974, Waring et al. 1977, Snell and Brown 1978, Whitehead 1978). Growth efficiency is expressed as a ratio of the last year of basal area growth to sapwood basal area (BA_1/SA) (Waring et al. 1980, Waring 1980). Stated differently, BA_1/SA indicates the efficiency by which a unit of leaf area produces stemwood, and thus, for a specified environment, BA_1/SA indicates tree vigor. Current annual stemwood production of the stand is the product of stand average BA_1/SA and the total stand sapwood basal area (SSA). When annual growth is compared at different levels of competition, growth efficiency of individual trees, or the average for all trees in the stand, may serve as a measure of competitive stress, whereas stand sapwood basal area represents a measure of shading by the entire canopy.

In a 35-year-old stand of Douglas-fir (Pseudotsuga menziesii (Mirb.) Franco) thinned to different density levels, mean BA_1/SA decreased linearly with increasing leaf area (Newman 1979). This study tested the hypothesis that trees in a stand randomly thinned by Phellinus (Poria) weirii (Murr.) Gilbertson; root rot, would illustrate a similar pattern. The disease appeared to cause a reduction in growth and death of some individuals, resulting in a heterogeneous canopy (Thies 1980, in preparation). Thus, the stand, composed of trees growing under variable competition, provided the opportunity to test whether the above hypothesized relationship remained valid for a single stand that was naturally thinned.

As part of a root rot study, Thies (1980) conducted a destructive analysis of a 0.55 ha, 40-year-old natural stand of Douglas-fir. The analysis provided data on the precise location, current annual increment in basal area (BA_1) and SA for each of 160 trees in the stand. BA_1 and SA were determined at 1.3 m above ground. These data permitted a test of the proposed relationship between growth efficiency, and canopy competition as estimated by SSA, on 200 m^2 plots.

STUDY AREA

The stand was located in the Oregon Coast Range (46° Lat. 123° Long.) on Olympic silt loam at an elevation of 400 m with a slope of 15 percent. The climate is generally mild; estimated mean precipitation is 145 cm annually with less than 10 percent falling during the growing season (Franklin and Dyrness 1973). Mean tree height at 100 years was 52 m. At harvest, mean stocking was 280 trees ha^{-1} and mean basal area was 30 m^2 ha^{-1} (Waring et al. 1980).

MAPPING METHOD

Each tree on the study area 10 cm dbh or larger was numbered, its dbh recorded and its location, established through an alidade survey, plotted in a grid map. Trees were depicted on the map as small circles drawn proportional to tree diameters (Thies and Hoopes 1979) (figure 1).

ANALYSIS OF DATA

Several analysis were performed to test the relationship between BA_1/SA and SSA:

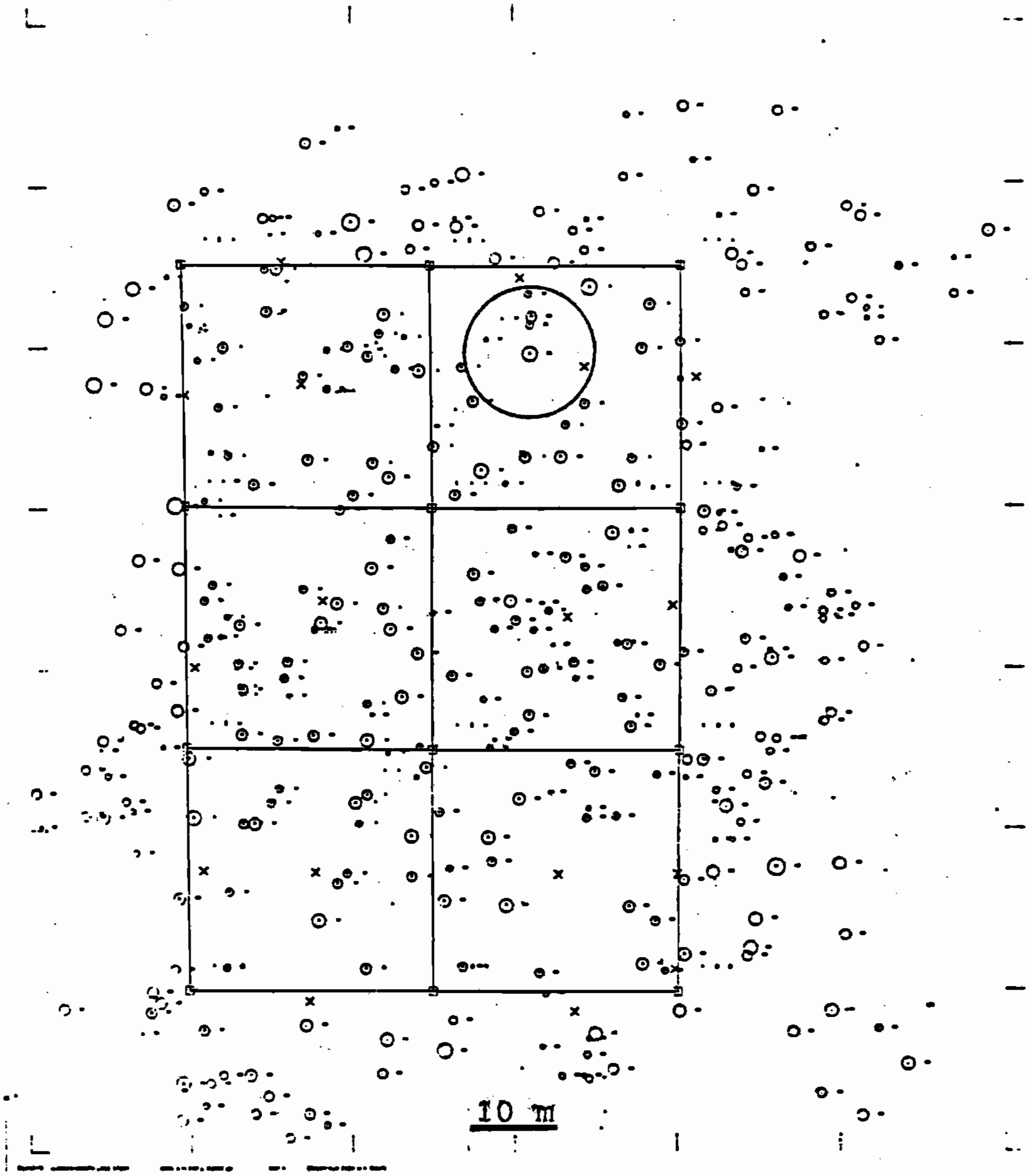

Figure 1. A map of the study area composed of six blocks. Trees are depicted as small circles drawn proportionally to tree diameters. Trees were numbered (not discernable at this scale) clockwise from north around reference points (x). The large circle in the upper right square represents a 200 m^2 plot.

In Analysis I, 20 circular plots were randomly positioned on the map. Mean BA_1/SA for all trees on the plot and SSA was determined for each plot. Mean BA_1/SA was then compared with its respective SSA.

Analysis II included the same procedures used in Analysis I, but mean BA_1/SA of only dominant and codominant trees of each plot were compared to the SSA of the plot.

In Analysis III, 30 dominant and codominant trees were randomly selected and their BA_1/SA compared with competition from the total canopy in the surrounding 200 m^2 areas (competing SSA).

In Analysis IV, procedures were similar to those in Analysis III but were confined to 19 codominant trees.

In Analysis V, procedures were similar to those in Analysis III but were restricted to eight suppressed individuals.

In Analysis VI, procedures were similar to those in Analysis III but included all trees in the study area with the necessary measurements (a total of 104).

All relationships were evaluated through linear regression technique (Zar 1974).

A boundary analysis was performed on all trees included in Analysis VI. Absolute measurement error was calculated for the upper data points (appendix) with regard to growth efficiency. A boundary equation of the line that fitted through most of the error bars was estimated. In addition, upper boundary equation were developed through linear and polynomial regression techniques.

Data from a study by Newman (1979) were recalculated in terms of mean BA_1/SA and SSA. The relationship between the two variables were evaluated through linear regression analysis. Finally, the resulting regression line was compared with the linear expression for the upper boundary of this study. The two regression lines were tested as to whether they were identical and then, whether the slopes of the lines were equal (Neter and Wasserman 1974).

RESULTS

Comparison of mean BA_1/SA of a plot for all trees (Analysis I) and for dominant and codominant trees only (Analysis II) with the total leaf area of the plot (SSA) did not reveal significant relationships. Inspection of the randomly located plots indicated considerable heterogeneity (even on 200 m^2 plots) which prevented the individuals of any plot from being subjected to similar competition for light. Uneven competition within plots decreased the sensitivity of mean BA_1/SA as a measure of competitive stress. Consequently, Analysis III-VI were conducted to accomodate the heterogeneous conditions.

Analysis III-VI compared individual trees' BA_1/SA with competing SSA, and presumably provided a more precise measure of competitive stress at different levels of competition in a heterogeneous stand. BA_1/SA of a single center tree was thought to be sensitive to small changes in surrounding leaf area because, unlike mean BA_1/SA, that of a single tree does not average out the variations in BA_1/SA of trees subjected to different levels of competition within a plot. Yet Analysis III-VI did not reveal significant trends in the growth efficiency of the center tree with respect to different levles of competing stand leaf area.

Through the boundary analysis, a clear trend appeared between the maximum observed BA_1/SA of the center tree and stand leaf area (or SSA) (figure 2). A single datum point above the apparent boundary (not shown in fig. 2) represented a tree with less than six percent of its basal area as functioning sapwood. A comparison with other trees, displaying ratios of sapwood to basal area of 24 to 56 percent, suggested that this particular tree was perhaps in the process of dying.

Calculations of absolute measurement error for the upper boundary trees produced error bars of varying size (figure 3). The equation of the line that best fitted through most of the bars was:

$$BA_1/SA=0.1586-4.3*10^{-3}(SSA) \qquad (1.)$$

The boundary equation was derived from linear regression (figure 3):

$$BA_1/SA=0.1618-4.8*10^{-3}(SSA) \qquad r^2=0.95 \qquad (2.)$$

The polynomial regression analysis provided the best fit for the upper boundary but was not significantly different from the linear

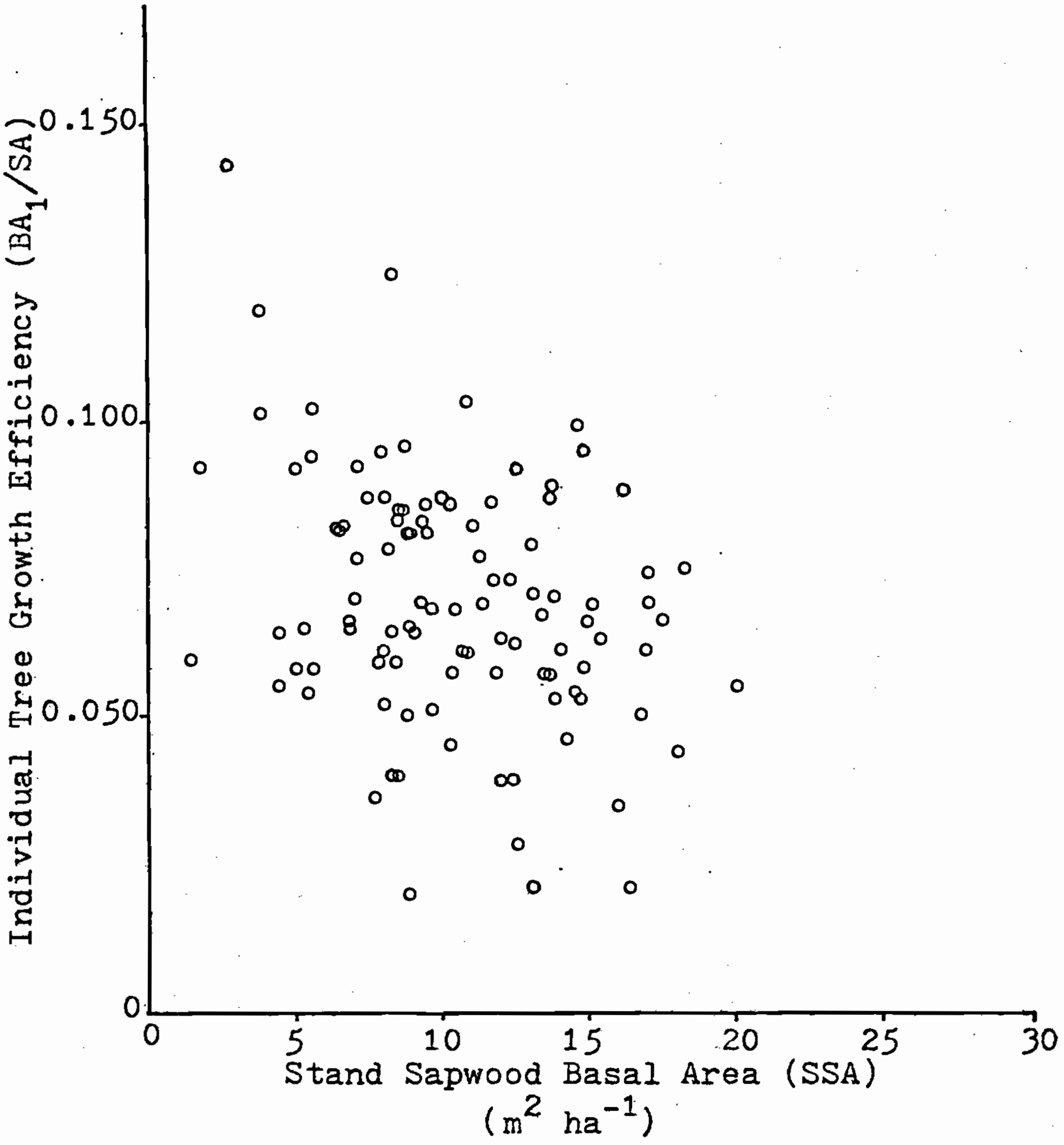

Figure 2. The relationship between individual tree growth efficiency (BA_1/SA) and stand sapwood basal area (SSA) in the mapped Douglas-fir stand. A single datum point (SSA=15.5, BA_1/SA= 0.127) was omitted from the figure based on the observation that the tree was dying as indicated by its abnormally small proportion of sapwood to total basal area (<6%).

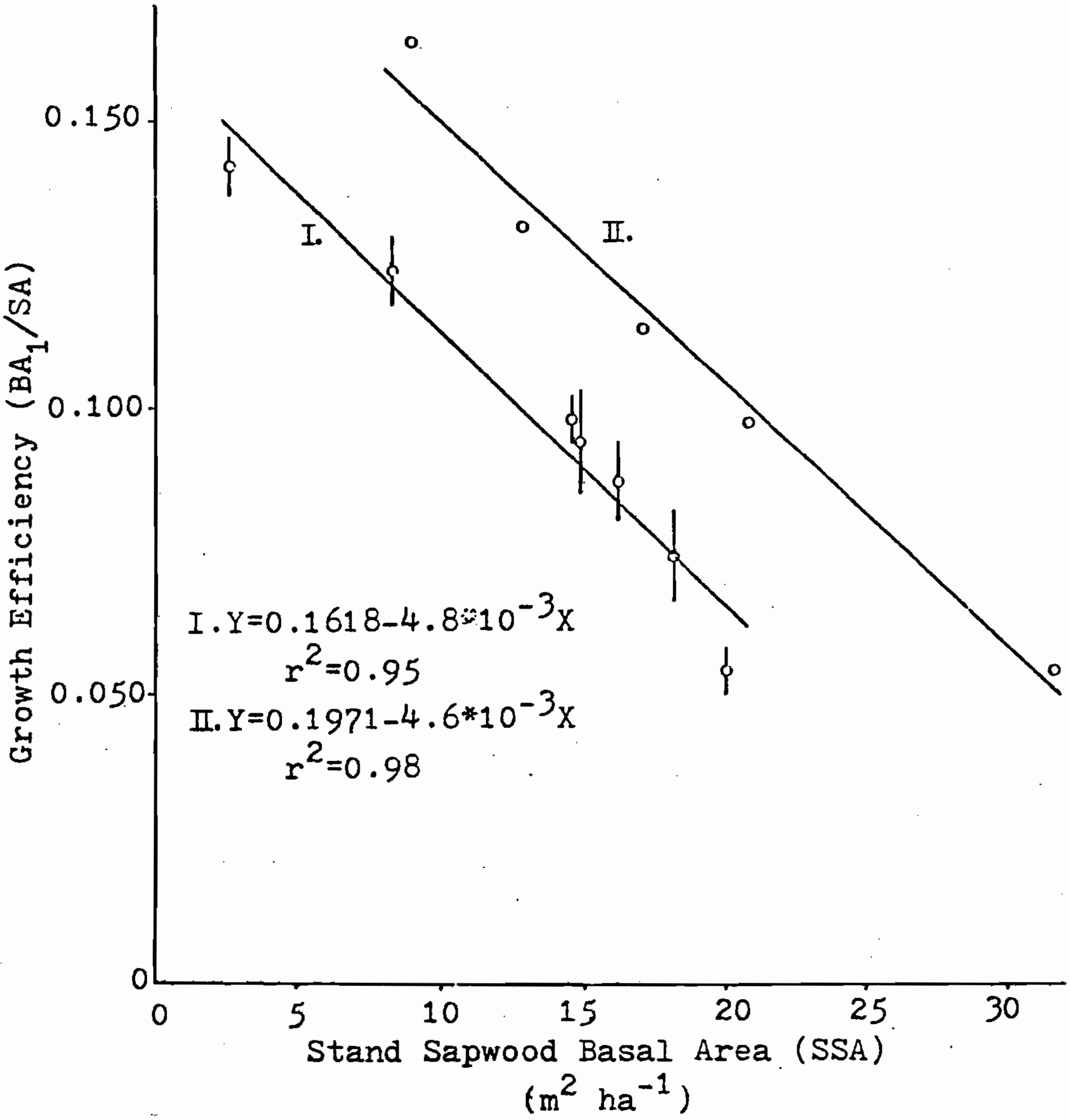

Figure 3. Maximum and average growth efficiencies of Douglas-fir trees decrease as stand sapwood basal area (SSA) increases. I. Upper boundary or maximum potential BA_1/SA values (see fig. 2), ± absolute measurement errors for individual trees at different levels of SSA in the study area. II. Values are means of 30 trees (after Newman 1979).

regression line:

$$BA_1/SA=0.1428-2.0*10^{-4}(SSA)^2 \qquad r^2=0.99 \qquad (3.)$$

The linear equation which described the relationship between mean BA_1/SA and SSA for the data from Newman (1979) was (figure 3):

$$\bar{X}\ BA_1/SA=0.1971-4.6*10^{-3}(SSA) \qquad r^2=0.98 \qquad (4.)$$

Comparison of this line (eqn. 4) with the linear expression for the upper boundary (eqn. 2) indicated that the intercepts were significantly different at the .05 level, but that the slopes were similar.

DISCUSSION

Results of all the regression analysis indicated no apparent relationships between growth efficiencies of either individual trees or plot means and canopy leaf area. Because such relationships were found in managed stands where spacing was controlled (Jackson 1977, Newman 1979) (figure 3) an explanation was sought for their absence in this natural stand.

Other reported studies were conducted in homogeneous, thinned stands and plantations (Jackson 1977, Newman 1979) while this study represented a heterogeneous stand. A heterogeneous forest canopy apparently requires more sensitive characterization of tree vigor and competing canopy than is needed for homogeneous stands. The mean of only two or three radii measurements of a disk or of increment borings for estimating growth efficiency, and the assumption that all the surrounding leaf area competes with the center tree may be incorrect for a complex heterogeneous stand.

A study by Waring and Pitman (personal communication)[1] indicated that up to an eight-fold difference in growth efficiency may exist on a single tree experiencing shading on one side and none on the other. To increase the sensitivity of leaf area for predicting growth efficiency of a center tree, one might core the center tree in the direction of its competitors and towards openings as well. The growth efficiencies derived from a center tree could then be compared with the competing leaf area from their respective side of the tree. If a mean plot growth efficiency is sought, the growth efficiency of the trees in the plot should be based on measurements taken from the side of the tree facing the plot center to reduce the influence of trees outside the plot boundary.

Estimating competing leaf area indirectly through sapwood area could also be modified for use in heterogeneous stands. The actually competing leaf area, rather than the total leaf area in the plot, may be calculated according to the vertical distribution of the foliage within the plot in reference to the crown of the center tree. Assuming

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[1] Waring, R. H., and G. B. Pitman. 1980. School of Forestry, Oregon State University, Corvallis. Personal communication.

leaf area within the canopy is distributed predictably (Stephens 1969, Kinerson et al. 1974, Schreder and Swank 1974, Gary 1978, Whitehead 1978), combining leaf area distribution with tree height and length of the live crown would permit an estimate of the percentage of the leaf area in the canopy which is actually competing with a selected tree. Thus, the competing leaf area with a particular tree in a plot would be expressed as a fraction of the total sapwood area, representative of the canopy in direct competition with that tree.

Even with more precise measurements, the hypothesized relationship might still not develop in a stand with obvious root rot infection. Waring (unpublished)[2] conducted a study in another healthy heterogeneous stand. His analysis illustrated a good linear correlation between leaf area and mean growth efficiency (fig. 4) although not as strong as those reported in homogeneous stands (Jackson 1977, Newman 1979). This correlation, which might be improved with refinements previously suggested, raised the question as to whether trees in my stand were actually in dynamic equilibrium. The large variation observed (fig. 2) suggested that more precision in measuring competition and growth efficiency would still be insufficient to establish a general relationship similar to that illustrated in figures 3 and 4.

Approximately 20 percent of the stand was infected with root rot. Root rot infected trees had shown an average reduction in volume

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[2] Waring, R. H. 1979. Data from field experiment by the Silviculture Institute. Oregon State University, Corvallis. (Unpublished).

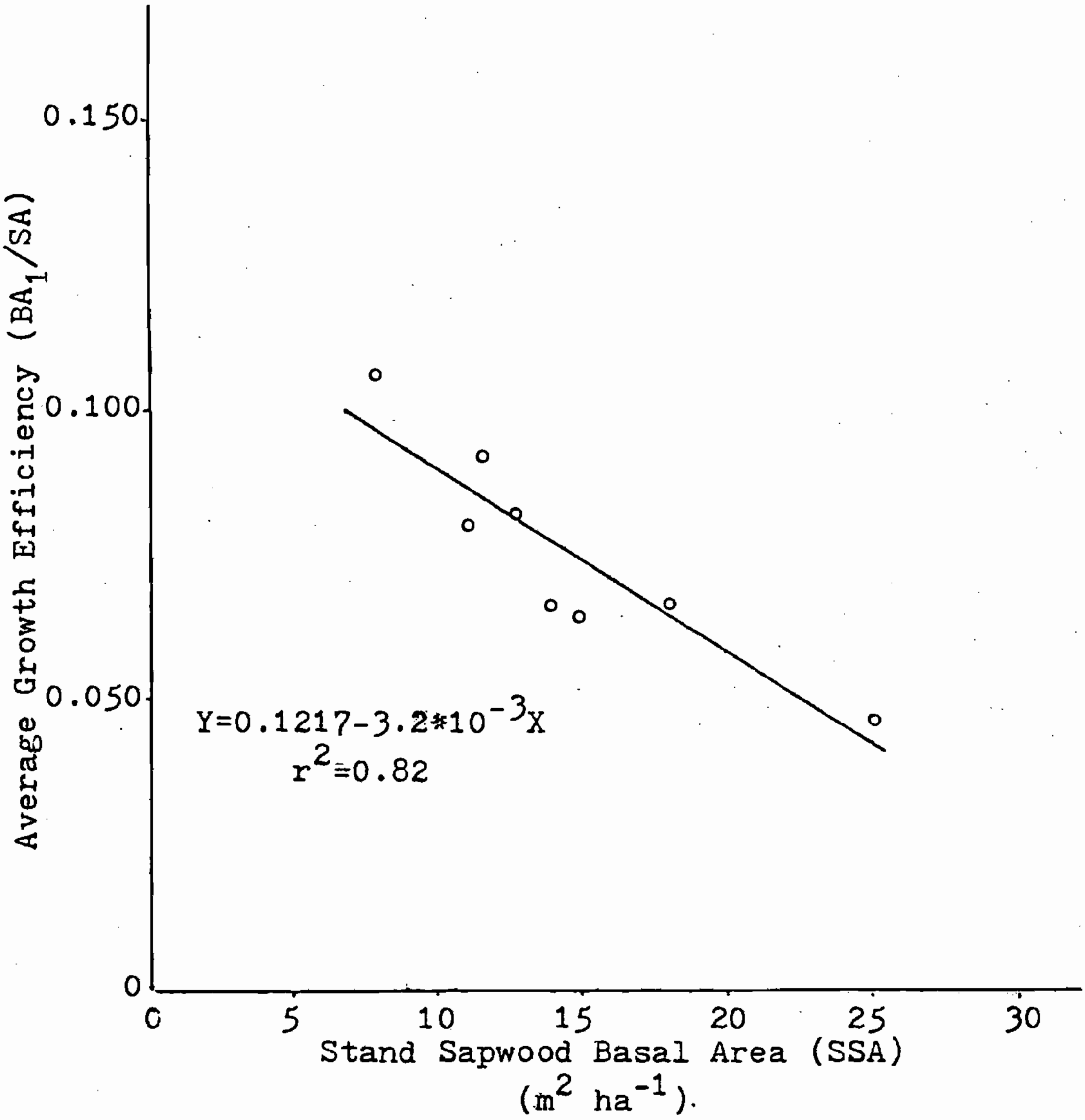

Figure 4. Average growth efficiency of Douglas-fir in Dunn Forest decreases as stand sapwood basal area increases. Values are the means of trees on 200 m^2 plots (after Waring, unpublished).

growth of more than 11 percent over the last ten years before harvest (Thies, 1980), which could have accounted for some of the variation in growth efficiency.

The clustered diseased trees exhibited a highly significant reduction in height growth, lost their position in the canopy and some subsequently died. A significantly lower competitive stress index (CSI as developed by Arney 1973) between pairs of diseased and healthy trees (Thies, 1980) might have been caused by reduced crown area and death of some diseased trees. Thus, the combined effects of the disease resulted in slowly reducing the competition with the healthy trees, affecting more and more trees as the stand evolves. Because of the instability caused by the disease, only some of the trees may have fully responded to their respective canopy competition. Those in balance with their competing canopy are, probably, represented as the trees occupying the upper boundary of the relationship between growth efficiency and competing leaf area. When a canopy is abruptly altered and a new equilibrium allowed to develop, significant responses by healthy trees may result within two years (Newman 1979). However, continuous canopy alteration and varying rate of response among individual trees could have prevented most of the healthy trees in this study from adjusting to the competition levels at the time of harvest.

In addition, some of the inhibited responses may have reflected impaired roots of apparently healthy trees. The infected roots could have reduced growth efficiency by decreasing the absorptive ability of the root system, and by changing the partitioning of the carbon assimilates among the plant parts.

The presence of the upper boundary suggested that, *ceteris paribus*, in a more stable stand growth efficiency is a function of leaf area and is constrained by it. Further support for this conclusion is derived from the similarity of the upper boundary slope to that described by Newman data (1979) (figure 3). The rate of decrease in BA_1/SA when leaf area increased could have, possibly, be both environmentally (Waring 1980) and genetically controlled. Higher maximum leaf area (Waring et al. 1978, Grier and Running, 1978, Gholtz 1979, Waring 1980) and higher BA_1/SA for each level of leaf area (Waring et al. 1980) is expected in a more favorable environment. The higher intercepts displayed by Newman's data suggested that the environmental conditions in his study area were, probably, less restrictive to tree growth than the environment in this area. Genetic differences between the two stands could also have partially accounted for the difference in the intercepts (Ledig 1974).

Future studies that may further test the competition-vigor hypothesis should assure that a stand has reached dynamic equilibrium prior to analysis, and that more refined measurement techniques be applied to heterogeneous (as compared to homogeneous) stands.

SUMMARY AND CONCLUSIONS

The efficiency of a stand in converting energy into biomass can be defined as growth per unit of leaf area. Tree vigor begins to decrease with competition for light and other resources. Vigor continues to decrease as the amount of competing foliage (or stand leaf area) increases.

Studies have shown that a direct and strong correlation exists between the leaf area of each tree and its cross-sectional area of conductive tissue at a given height. This relationship remained valid for stands that were thinned to various stocking densities (Whitehead 1978).

Average tree vigor (mean growth efficiency) varies largely, and is measured indirectly by growth per unit of sapwood. At a lower level of stocking, efficiency approaches a maximum. However, as canopy begins and continues to close, growth efficiency decreases. This relationship, which appeared to be linear (Newman 1979), was hypothesized to apply for a natural stand that had been thinned randomly by *Phellinus* root rot.

Although the results did not support the general hypothesized relationship, the maximum efficiency of trees recorded under specified levels of canopy competition did follow expected trends.

Possible reasons for the absence of a hypothesized linear relationship included: the need for more sensitive characterization of tree vigor and competing canopy when comparing tree growth in heterogeneous stands; recognition that root rot continues to reduce growth below the potential and induces undocumented continuous changes in canopy competition; and, the differential rate of response of individual trees to changes in competing leaf area.

Those trees exhibiting the highest growth efficiencies under a given level of competing leaf area showed a relationship similar to that reported on an experimentally thinned stand (Newman 1979). Although the slopes of the two lines were similar, the intercepts were significantly different, perhaps expressing differences in environmental conditions and genetic capabilities.

Recommendations were offered for achieving greater accuracy in determining leaf area and growth increment required for estimating competition and growth efficiency in heterogeneous stands.

I also caution that comparison of growth and competition be made in stands not continuously subjected to canopy alteration by a disease so that equilibrium can be established and potential growth realized.

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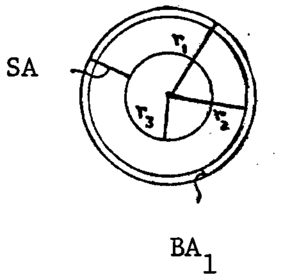
APPENDIX

APPENDIX: Absolute Error Analysis for the Growth Efficiency Ratio (BA_1/SA)

ratio BA_1/SA:

$$BA_1 = \pi r_1^2 - \pi r_2^2 = \pi(r_1^2 - r_2^2)$$
$$SA = \pi r_1^2 - \pi r_3^2 = \pi(r_1^2 - r_3^2)$$



$$BA_1/SA = \frac{\pi(r_1^2 - r_2^2)}{\pi(r_1^2 - r_3^2)} = \frac{r_1^2 - r_2^2}{r_1^2 - r_3^2} = Y$$

Assumption: Measurements include standard error that equals half of the smallest scale.

Note: Smallest scale in this study = 0.01 inch.

Standard measurement error = 0.005 inch.

Measurement Error with respect to (w.r.t.) three variables (r_1, r_2, r_3):

$$\delta Y \text{ (w.r.t.) } r_1 = \frac{(r_1^2 - r_3^2)\, 2r_1 - (r_1^2 - r_2^2) 2r_1}{(r_1^2 - r_3^2)^2}$$

$$= \frac{2r_1(r_2^2 - r_3^2)}{(r_1^2 - r_3^2)^2} \qquad (1.)$$

$$\delta Y \text{ (w.r.t.) } r_2 = \frac{-2r_2}{(r_1^2 - r_3^2)} \qquad (2.)$$

$$\delta Y \text{ (w.r.t.) } r_3 = \frac{(r_1^2 - r_3^2)0 - (r_1^2 - r_2^2)(-2r_3)}{(r_1^2 - r_3^2)^2}$$

$$= \frac{2r_3(r_1^2 - r_2^2)}{(r_1^2 - r_3^2)^2} \qquad (3.)$$

$$\delta Y = \delta r_1 \underbrace{\frac{2r_1(r_2^2 - r_3^2)}{(r_1^2 - r_3^2)^2}}_{A} - \delta r_2 \underbrace{\frac{2r_2}{(r_1^2 - r_3^2)}}_{B} + \delta r_3 \underbrace{\frac{2r_3(r_1^2 - r_2^2)}{(r_1^2 - r_3^2)^2}}_{C}$$

$$Y = \delta r_1 A - \delta r_2 B + \delta r_3 C$$

<u>Absolute Error:</u>

$$Y = \sqrt{A^2(\delta r_1)^2 + B^2(\delta r_2)^2 + C^2(\delta r_3)^2}$$

$$Y = \left[\frac{4r_1^2(r_2^2 - r_3^2)^2(\delta r_1)^2 + 4r_2^2(r_1^2 - r_3^2)^2(\delta r_2)^2 +}{(r_1^2 - r_3^2)^4}\right.$$

$$\left. + \underline{4r_3^2(r_1^2 - r_2^2)^2(\delta r_3)^2}\right]^{1/2}$$

Note: δr_1, δr_2 and δr_3 are equal to each other because all measurements were done with the same scale.

$$\delta r_1 = \delta r_2 = \delta r_3 = \delta r \qquad \delta r = 0.005 \text{ inch.}$$

$$\boxed{\delta Y = \frac{2(\delta r)}{(r_1^2 - r_3^2)^2} \left[r_1^2(r_2^2 - r_3^2) + r_2^2(r_1^2 - r_3^2)^2 + r_3^2(r_1^2 - r_2^2)^2\right]^{1/2}}$$