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Abstract approved Signature redacted for privacy.
(R. B. Guenther)

It is proved that: if the coefficients of a second order parabolic equation in an infinite space-time cylinder, $D \times (-\infty, \infty)$; the nonhomogeneous term; and the mixed data on the boundary of $D \times (-\infty, \infty)$, are all periodic in t with period T, then there exists a solution in $\bar{D} \times (-\infty, \infty)$ which also is periodic in t with period T. This result is also extended (by the use of the Schauder fixed point theorem) to the case with nonlinear right hand side and the case with nonlinear data on the boundary. In the linear case, uniqueness is established. In the nonlinear case, uniqueness is obtained only if stronger conditions are imposed on the "non-homogeneous terms".

Periodic Solutions of Parabolic Partial Differential Equations

by

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PERIODIC SOLUTIONS OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

I. INTRODUCTION

The determination of periodic solutions to a parabolic partial differential equation began with Fourier's work on heat flow [4] (see also Carslaw [2]). Fourier treated the problem of finding bounded solutions $u(x, t)$ of

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad -\infty < t < \infty,$$

satisfying $u(0, t) = A \cos \omega t$, and applied his results to the study of temperature variations in the earth's surface. Lord Kelvin [10] took up the problem in 1861; he used periodic solutions of the heat equation in order to estimate the thermal conductivity of the soil. More specifically, taking the surface of the earth to be the plane $x = 0$ and assuming that the surface has the periodic temperature

$$(*) \qquad u(0, t) = \mu_0 + \sum_{j=1}^{\infty} \mu_j \cos (j\omega t - \epsilon_j),$$

the temperature at depth x was found theoretically by solving the equation

$$u_t = ku_{xx}, \qquad 0 < x < \infty, \qquad -\infty < t < \infty,$$

subject to the boundary condition (*), to be

$$u(x,t) = \mu_0 + \sum_{j=1}^{\infty} \mu_j \cos(j\omega t - \epsilon_j + kx\sqrt{j})$$

where $k = (\frac{\omega}{2\eta})^{1/2}$ and η is the thermal conductivity of soil. Experimental results showed that the temperatures u_1, u_2 at depths x_1, x_2 were represented as

$$u_1(t) = \mu_0' + \sum_{j=1}^{\infty} \mu_j' \cos(j\omega t - \epsilon_j')$$

and

$$u_2(t) = \mu_0'' + \sum_{j=1}^{\infty} \mu_j'' \cos(j\omega t - \epsilon_j'').$$

Comparing the theoretical and observed results one has

$$\mu_j' = \mu_j e^{-kx_1\sqrt{j}}, \quad \mu_j'' = \mu_j e^{-kx_2\sqrt{j}}, \quad \mu_0 = \mu_0' = \mu_0''$$

$$\epsilon_j' = kx_1\sqrt{j} + \epsilon_j, \qquad \epsilon_j'' = kx_2\sqrt{j} + \epsilon_j$$

and hence $\frac{\ell n\, \mu_j' - \ell n\, \mu_j''}{x_2 - x_1} = \frac{\epsilon_j'' - \epsilon_j'}{x_2 - x_1} = k\sqrt{j}$. From this it is seen that either the amplitude or the phase of any harmonic can be used to compute k and hence the thermal conductivity η.

Another application is that of determining the conductivity of a metal [1]. Suppose that the metal is in the form of a cylindrical rod. One end of the rod is kept at constant temperature while the other end is subjected to periodic variations in temperature. By comparing the theoretical and the observed temperatures at various points on the rod, one can obtain estimates of the metal's conductivity. This procedure, which is originally due to Ångström [1], is similar to that used by Lord Kelvin [4].

More recently [11] periodic solutions of the heat equation have been used to study thermal stresses in cylindrical walls of steam and combustion engines. Further applications of periodic solutions of the heat equation, as well as an extensive bibliography, can be found in Carslaw's book [2].

In the above examples it is necessary that there exist periodic, or nearly periodic, solutions to some parabolic equation. The general problem of obtaining existence theorems for periodic solutions of parabolic boundary value problems seems to have started in 1939 with a paper by Karimov [6]. In a series of articles from 1939-1946 (see [6, 7, 8, 9]), Karimov solved the boundary value problem

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - \frac{\partial u}{\partial t} = \phi(x,t) + \mu f(u) \qquad 0 < x < 1$$

$$0 < t < 1$$

$$u(0,t) = u(1,t) = 0 \qquad 0 \leq t \leq 1,$$

$$u(x,0) = u(x,1) \qquad 0 \leq x \leq 1 .$$

Karimov assumed that (i) $\phi(x,t) = \phi(x,t+1)$, (ii) $\phi(0,t) = \phi(1,t) = 0$, (iii) $\frac{\partial\phi(x,t)}{\partial x}$ exists and is bounded, (iv) f satisfies a Lipschitz condition, (v) $f(0) = 0$, and (vi) $p(x)$ is continuously differentiable and satisfies $p(x) > 0$ on $0 \leq x \leq 1$. μ is a real parameter. (The fact that $u(x,0) = u(x,1)$ insures that $u(x,t)$ can be extended periodically into $0 \leq x \leq 1$, $-\infty < t < \infty$.) Karimov was able to demonstrate existence of a solution to this generally nonlinear problem, but was able to ensure uniqueness only by imposing the additional restriction that the parameter μ be "sufficiently small".

The next significant existence theorem for parabolic equations appeared in 1952 in a paper by G. Prodi [15]. Here Prodi showed that, under suitable conditions, the nonlinear boundary value problem

$$u_t = u_{xx} + f(x,t,u,u_x), \qquad 0 < x < \ell, \quad -\infty < t < \infty,$$

$$u(0,t) = u(\ell,t) = 0 \qquad -\infty < t < \infty$$

has at least one solution $u = u(x,t)$ which is periodic in t. In addition to being assumed periodic in t, the function f must also satisfy the growth condition: $f(x,t,z,\omega)/\omega^2 \to 0$, $|\omega| \to \infty$ for $|z| \leq M$, where $M > 0$ is an arbitrary, finite constant. Prodi obtains this result by using the method of Leray and Schauder (see [14] or also the textbook of A. Friedman [5]). Although Prodi's paper is a significant extension of Karimov's existence theory, he does not examine uniqueness.

Two years later [16] in 1954 Prodi showed that the more general problem

$$u_t = u_{xx} + f(x,t,u,u_x), \qquad 0 < x < \ell, \quad -\infty < t < \infty$$

$$u(0,t) = \phi_1(t, u(0,t))$$

$$u(\ell,t) = \phi_2(t, u(\ell,t)) \qquad -\infty < t < \infty$$

has at least one periodic solution. Prodi again used the Leray-Schauder method to get this result and again did not obtain uniqueness.

In 1961 Šmulev [18] extended Prodi's [16] existence theory to the problem

$$u_t = a(x,t,u)u_{xx} + f(x,t,u,u_x), \qquad 0 < x < \ell, \quad -\infty < t < \infty$$

$$u_t(0,t) = \phi_1(t, u(0,t), u_x(0,t)), \quad -\infty < t < \infty,$$

$$u_t(\ell,t) = \phi_2(t, u(\ell,t), u_x(\ell,t)), \quad -\infty < t < \infty,$$

where f, ϕ_1, and ϕ_2 satisfy certain smoothness, growth and periodicity conditions. This was the first paper that allowed the principal part of the equation to be nonlinear. Šmulev did not, however, take up the question of uniqueness.

In this same paper Šmulev considered, for the first time, periodic parabolic problems in several space variables. He treated the parabolic boundary value problem

$$u_t = \sum_{i,j=1}^{n} a_{ij}(x,t,u)u_{x_i}u_{x_j} + \sum_{i=1}^{n} b_i(x,t,u)u_{x_i} + a(x,t,u)$$

$$(x,t) \in D \times (-\infty, \infty),$$

$$u(x,t) = 0 \qquad (x,t) \in \partial D \times (-\infty,\infty),$$

where D is a suitably smooth, bounded domain in n dimensional Euclidean space, ∂D is its boundary, and $x = (x_1, x_2, \cdots x_n)$ is a point in n dimensional Euclidean space. In this problem Šmulev needs to assume a_{ij}, b_i, a are all periodic functions in

t with the same period T, and that the functions a_{ij} satisfy the usual parabolicity condition.

Later in the same year, Šmulev [19] considered the parabolic partial differential equation

$$(1) \qquad \frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u + f(x,t),$$

$$(x,t) \in D \times (-\infty, \infty)$$

with the boundary condition

$$(2) \qquad u(x,t) = \psi(x,t) \qquad (x,t) \in \partial D \times (-\infty, \infty),$$

where the functions f and ψ are assumed to be periodic[1/] functions in t with the period T. Assuming the functions a_{ij}, b_i, c, f, ψ, and the boundary, ∂D, to be sufficiently smooth, he shows the existence of a unique, periodic solution to the problem (1), (2). In the same paper, Šmulev gives an existence and uniqueness proof for solutions to the equation (1) subject to the Neumann boundary condition

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1/ There is an oversight in Šmulev's paper. The assumption that ψ be periodic in t is omitted, but is needed in the proof of the theorem.

$$(3) \qquad \frac{\partial u(x,t)}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (-\infty,\infty),$$

where $\frac{\partial u}{\partial \nu}$ is a directional derivative in the direction of the interior conormal to ∂D. More specifically, if $\mu(x) = (\mu_1(x), \cdots, \mu_n(x))$ is the unit interior normal to ∂D at the point x, then the vector $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$, where the components are $\nu_i(x) = \sum_{j=1}^{n} a_{ij}(x)u_j(x)$, $i = 1, \cdots, n$, lies in the direction of the interior conormal. To obtain his existence and uniqueness results for the problem (1), (3), Šmulev had to assume, in addition to certain periodicity and smoothness requirements, that $\beta(x,t) \leq b_0 < 0$, b_0 a constant. The existence proofs in this paper make essential use of Fourier's method of separation of variables.

In his most recent paper [20], Šmulev proves that if the parabolic boundary value problem

$$(4) \qquad \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u - u_t = f(x,t) \qquad (x,t) \in D \times (-\infty,\infty)$$

$$(5) \qquad u(x,t) = \psi(x,t) \qquad (x,t) \in \partial D \times (-\infty,\infty),$$

has periodic data in t, in addition to other conditions, then the problem has a unique solution, period in t. In the linear case just described, Šmulev was able to obtain both a uniqueness and an existence theorem; however, when he took up the nonlinear problem

$$u_t = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x,t,u)\frac{\partial u}{\partial x_i}\right] + a(x,t,u,u_x)$$

$$(x,t) \epsilon D \times (-\infty, \infty)$$

$$u(x,t) = \psi(x,t) \qquad (x,t) \epsilon \partial D \times (-\infty, \infty),$$

he was able to extend only the existence theory for periodic solutions, but not the uniqueness theory.

More recently [13] in 1966, Kusano showed existence of periodic solutions for systems of quasilinear equations. He considered the following two boundary value problems (6), (7) and (8), (9):

(6) $$\sum_{i,j=1}^{n} a_{ij}^{k}(x,t)u_{x_i x_j}^{k} - u_t^k = a^k(x,t,u,\Delta u^k)$$

$$(x,t) \epsilon D \times (-\infty, \infty)$$

(7) $$u^k(x,t) = \psi^k(x,t) \qquad (x,t) \epsilon \partial D \times (-\infty, \infty),$$

$$(k = 1, 2, \cdots, n)$$

$$(8) \qquad \sum_{i,j=1}^{n} a_{ij}(x,t,u)u^k_{x_i x_j} - u^k_t = a^k(x,t,u,\Delta u)$$

$$(x,t) \in D \times (-\infty, \infty)$$

$$(9) \qquad u^k(x,t) = \psi^k(x,t) \qquad (x,t) \in \partial D \times (-\infty, \infty),$$

$$(k = 1, 2, \cdots, n)$$

where

$$u = (u^1, u^2, \cdots, u^n), \quad \Delta u^k = (u^k_{x_1}, u^k_{x_2}, \cdots, u^k_{x_n})$$

and

$$\Delta u = (\Delta u^1, \Delta u^2, \cdots, \Delta u^n) .$$

In problem (6), (7), Kusano made use of the Schauder fixed point theorem and in problem (8), (9) he used the Leray-Schauder method to obtain the existence of periodic solutions. In neither problem does he discuss uniqueness questions.

It is the purpose of this thesis to extend the current literature by showing that, under suitable conditions, the four following parabolic boundary value problems I-IV have periodic solutions. In all these problems the operator L is defined by

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u - u_t$$

and the functions a_{ij} satisfy the usual parabolicity conditions, namely see definition 2.4 of the next section.

Problem I

$$Lu = f(x,t) \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (-\infty, \infty)$$

Problem II

$$Lu = f(x,t) \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t,u) \qquad (x,t) \in \partial D \times (-\infty, \infty)$$

Problem III

$$Lu = f(x,t,u) \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (-\infty, \infty)$$

Problem IV

$$Lu = f(x,t,u) \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t,u) \qquad (x,t) \in \partial D \times (-\infty, \infty) .$$

Problem I generalizes Šmulev's [19] result for the Neumann problem by allowing the coefficients of the operator to depend on t. We also obtain uniqueness for problem I. The proof of the existence of a periodic solution to problem I follows along the same lines as Šmulev's ingenious proof for the corresponding Dirichlet problem.

Problem II is an extension of problem I, since in problem II the function g depends on the dependent variable u, as well as on x and t. In problem II uniqueness is obtained, but only under stronger conditions on the function g. The proof of the existence of a periodic solution to problem II depends on the concept of a sequence of families of functions converging *in an equi-convergent manner,* which will be introduced below. Using this concept, one can show that a certain class of functions is compact and hence one is able to apply the Schauder fixed point theorem [3] (see also [17]).

Problem III is also an extension of problem I, since the function f depends on x, t and u. For this problem we also obtain a uniqueness theorem but again under fairly strong conditions on the function f. The existence proof for this problem also uses the Schauder fixed point theorem, but in this case one must obtain *a priori* bounds on the Hölder constants of the solution.

Problem IV extends problems I, II, III. The existence of a periodic solution to this problem is proved along the same lines as in problems II and III. In problem IV we can also obtain a uniqueness theorem if we assume strong enough conditions on the functions f and g.

II. PRELIMINARIES

We now introduce the notation and terminology to be used throughout the rest of this dissertation. Further, for the convenience of the reader, we shall state several theorems which will be of fundamental importance in obtaining our main results.

Let E_n denote real, Euclidean n-space and D a bounded domain in E_n. Let points of E_n be denoted by x, ξ etc., with their respective coordinates $(x_1, x_2, \cdots, x_n)$, $(\xi_1, \xi_2, \cdots, \xi_n)$. Denote by $\overline{D}$ the closure of D, by ∂D the boundary of D, and let $|x| = (\sum_{k=1}^{n} x_k^2)^{1/2}$. An open interval in E_1 will be denoted by (a, b), closed by $[a, b]$, etc.

We begin with the following

Definition 2.1. A real-valued function f, defined on a compact subset, $\overline{D}$, of E_n is called Hölder continuous (with exponent α, $0 < \alpha < 1$) if there exists a positive constant M such that

$$|f(x) - f(y)| \leq M|x-y|^{\alpha}$$

for all x, y in $\overline{D}$. If $\alpha = 1$, f is called Lipschitz continuous.

More generally, if X is any real metric space, we shall denote by $C(X)$ the normed, linear space of continuous real-valued functions f, defined on X for which the norm

$$\|f\| = \sup_{x \in X} |f(x)|,$$

is finite.

We denote by $C^p(\overline{D} \times (-\infty, \infty))$ the normed, linear space of all continuous real-valued functions f, defined for $(x, t) \in \overline{D} \times (-\infty, \infty)$, that are periodic in t with period T. The norm is the uniform norm

$$\|f\| = \sup_{\overline{D} \times (-\infty, \infty)} |f(x, t)|.$$

(This will sometimes be abbreviated to C^p when the context makes the meaning clear.)

We denote by $C^p_\alpha(\overline{D} \times (-\infty, \infty))$ the normed, linear space of all C^p functions which are also Hölder continuous in the x variables (exponent α), uniformly over $-\infty < t < \infty$. (This space will sometimes be referred to as C^p_α.)

Finally we denote by $C_{1+\lambda}(\overline{D})$ the class of real-valued functions defined on $\overline{D}$ whose derivatives are Hölder continuous with exponent λ.

Definition 2. 2. The second order linear differential operator L is defined by

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u - u_t \; .$$

Definition 2.3. L is said to be parabolic on $\overline{D} \times (-\infty, \infty)$ if the matrix $(a_{ij}(x,t))$ is symmetric and positive definite for all $(x,t) \in \overline{D} \times (-\infty, \infty)$.

Definition 2.4. L is said to be uniformly parabolic on $\overline{D} \times (-\infty, \infty)$ if there exist positive constants a, b such that for all $\xi \in E_n$

$$a \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i \xi_j \le b \sum_{i=1}^{n} \xi_i^2 \; .$$

The boundary ∂D is of class $C_{1+\lambda}$ if locally ∂D can be represented by $x_i = h(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ for some i where h belongs locally to $C_{1+\lambda}(\overline{D})$.

Definition 2.5. We say that ∂D satisfies the strong inside sphere property at the point $x^{(0)} \in \partial D$ if there exists a closed ball B with center $x^{(1)} \in D$ satisfying

(i) $B \subset \overline{D}$

(ii) $B \cap \partial D = \{x^{(0)}\}$.

Definition 2.6. Let ∂D have the strong inside sphere property at the point $x^{(0)} \epsilon \partial D$ and let $\mu(x^{(0)}, t_0) = (\mu_1(x^{(0)}, t_0), \cdots \mu_n(x^{(0)}, t_0), 0)$ denote the unit interior normal to $\partial D \times (-\infty, \infty)$ at $(x^{(0)}, t_0)$. We define the inward conormal direction to $\partial D \times (-\infty, \infty)$ at the point $(x^{(0)}, t_0)$ by the vector $\nu = (\nu_1, \cdots, \nu_n, 0)$ where

$$\nu_i = \sum_{j=1}^{n} a_{ij}(x^{(0)}, t_0)\mu_j(x^{(0)}, t_0) .$$

Having defined the conormal direction we now define the conormal derivative.

Definition 2.7. Let $x^{(0)} \epsilon \partial D$ and denote $\nu(x^{(0)}, t)$ the inward conormal direction at $(x^{(0)}, t)$. The conormal derivative, $\frac{\partial u(x,t)}{\partial \nu(x^{(0)}, t)}$, of $u(x,t)$, $x \epsilon D$, in the direction of the inward conormal at $(x^{(0)}, t)$ is defined by

$$\frac{\partial u(x,t)}{\partial \nu(x^{(0)}, t)} = \sum_{i,j=1}^{n} a_{ij}(x^{(0)}, t)\mu_j(x^{(0)}, t) \frac{\partial u(x,t)}{\partial x_i} .$$

We shall now define the conormal derivative at a point on the boundary ∂D. Before doing this, however, we need to introduce the idea of a finite closed cone.

Definition 2.8. A finite closed cone K in E_n with vertex at the origin is any set of points in E_n satisfying the implications

(i) $x \in K \Rightarrow \alpha x \in K$ for $0 \leq \alpha \leq M < \infty$, M fixed

(ii) $x, y \in K \Rightarrow x + y \in K$

(iii) $x \in K \Rightarrow -x \notin K$.

By a finite closed cone K in E_n with vertex $x^{(0)}$ is meant a set in E_n which is congruent under translation with a finite closed cone whose vertex is at the origin.

We can now define the conormal derivative at a point on the boundary.

Definition 2.9. The conormal derivative of $u(x,t)$ for $x \in \partial D$ is defined by

$$\frac{\partial u}{\partial \nu} \equiv \frac{\partial u(x,t)}{\partial \nu(x,t)} = \lim_{\substack{y \to x \\ y \in K}} \frac{\partial u(y,t)}{\partial \nu(x,t)} ,$$

where K is any finite closed cone with vertex x and satisfying $K \sim \{x\} \subset D$.

Definition 2.10. We say that $u = u(x,t)$ is a solution to the initial-boundary value problem

(i) $Lu = f(x,t)$ $\qquad (x,t) \in D \times (0,T]$

(ii) $u(x,0) = \psi(x)$ $\qquad x \in \overline{D}$

(iii) $\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t)$ $\qquad (x,t) \in \partial D\ (0,T]$

if:

(1) u is continuous on $\overline{D} \times [0,T]$

(2) $u_{x_i}, u_{x_i x_j}$, $i,j = 1,\cdots,n$, are continuous on $D \times (0,T]$

(3) u_t is continuous on $D \times (0,T]$

(4) u satisfies the above conditions (i), (ii), (iii).

For the convenience of the reader we shall state two theorems, the proofs of which may be found on pages 144, 147 of Friedman's text [5], and which we shall make use of in the sequel.

Theorem 2. 1

The boundary value problem

$$Lu = f(x,t) \qquad (x,t) \in D \times (0,T]$$

$$u(x,0) = \psi(x) \qquad x \in \overline{D}$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (0,T]$$

has a unique solution $u = u(x,t)$ provided the following conditions are satisfied:

(i) L is uniformly parabolic on $\overline{D} \times [0, T]$

(ii) the coefficients of L are continuous and satisfy the following Hölder conditions on $\overline{D} \times [0, T]$:

$$|a_{ij}(x,t) - a_{ij}(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

$$|b_i(x,t) - b_i(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

$$|c(x,t) - c(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

(iii) ∂D is of class $C_{1+\lambda}$

(iv) f is Hölder continuous in x (exponent α) and is uniformly continuous on $\overline{D} \times (0, T]$

(v) g, β are continuous on $\partial D \times [0, T]$

(vi) ψ is continuous in $\overline{D}$ and vanishes in some neighborhood of ∂D.

The second theorem taken from Friedman's book contains an a priori estimate on the solution of the above boundary value problem. We shall state a slightly modified version, which will be sufficient for our purposes.

Theorem 2.2

Let $u = u(x,t)$ be a solution of the boundary value problem stated in the previous theorem, and assume that L is uniformly parabolic in $\overline{D} \times [0,T]$ with continuous coefficients. Assume also that ∂D is of class $C_{1+\lambda}$ and that

(i) $c(x,t) \leq 0$ on $\overline{D} \times [0,T]$

(ii) $\beta(x,t) \leq b_0 < 0$ on $\partial D \times [0,T]$.

Then the following estimate holds on $\overline{D} \times [0,T]$,

$$|u(x,t)| \leq K \,(\mathrm{SUP}\, |f| + \mathrm{SUP}\, |g| + \mathrm{SUP}\, |\psi|)$$

where the supremums are taken over their respective domains and K is a constant depending on L, β and D.

We now state two more well-known theorems which will be needed later.

Schauder Fixed Point Theorem (see e.g. [3] or [17])

If X is a normed linear space and S is a continuous mapping of K, where K is a closed convex subset of X, into a compact subset of K, then S has a fixed point in K.

Before stating the final theorem we now recall the concept of equicontinuity.

Definition 2.11. A family of real-valued functions $\{f_\alpha\}$, α belonging to some index set A, defined on a domain D in E_n, is called equicontinuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f_\alpha(x) - f_\alpha(y)| < \epsilon$ for all x, y such that $|x-y| < \delta$ and for all α in the index set A.

We can now state the final theorem.

Theorem (Arzela-Ascoli)

A necessary and sufficient condition that a family of real-valued continuous functions defined on a compact metric space X be compact in $C(X)$ is that the family be uniformly bounded and equicontinuous.

For a proof of this theorem see e.g. [12].

III. PERIODIC SOLUTIONS OF THE SECOND BOUNDARY VALUE PROBLEM

It is the purpose of this section to extend Šmulev's [19] theorem by proving, under suitable conditions on L, the existence and uniqueness of a periodic solution $u = u(x,t)$ to the second boundary value problem

(3.1) $\quad Lu = f(x,t) \qquad (x,t) \in D \times (-\infty, \infty)$

(3.2) $\quad \dfrac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (-\infty, \infty)$.

Uniqueness and periodicity theorems can be proved *a priori* without using all the conditions needed for existence. We shall accordingly state and prove these two results first.

Theorem 3.1 (Uniqueness)

Let $u = u(x,t)$ be a bounded solution of (3.1), (3.2) and assume that

(i) L is uniformly parabolic on $\overline{D} \times (-\infty, \infty)$,

(ii) $c(x,t) \leq 0 \qquad (x,t) \in \overline{D} \times (-\infty, \infty)$,

(iii) $\beta(x,t) \leq b_0 < 0$, b_0 a constant, $(x,t) \in \partial D \times (-\infty, \infty)$,

(iv) ∂D belongs to $C_{1+\lambda}$.

Then there exists at most one solution, $u = u(x,t)$, to the problem (3.1), (3.2).

Proof: Let u_1, u_2 be two solutions of (3.1), (3.2). The difference $u \equiv u_1 - u_2$ satisfies

$$Lu = 0 \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = 0 \qquad (x,t) \in \partial D \times (-\infty, \infty) .$$

Let $(\bar{x}, \bar{t})$ be an arbitrary point in $D \times (-\infty, \infty)$. If we can show that $u(\bar{x}, \bar{t}) = 0$, the theorem will be proved. For this purpose we let $v(x,t) = e^t u(x,t)$, and let $t^* < \bar{t}$ be arbitrary. Then the function $v(x,t)$ satisfies

$$Lv - v = 0 \qquad (x,t) \in D \times (t^*, \infty) ,$$

$$\frac{\partial v}{\partial \nu} + \beta(x,t)v(x,t) = 0 \qquad (x,t) \in \partial D \times (t^*, \infty) ,$$

$$v(x,t^*) = e^{t^*} u(x,t^*) \qquad x \in \bar{D} .$$

From Theorem 3.2 follows the estimate

$$|v(x,t)| \leq K e^{t^*} \sup_{x \in \bar{D}} |u(x,t^*)| .$$

The functions u_1 and u_2 are bounded by assumption, so u is bounded. Denote the bound on u by B. Then

$$|v(x,t)| \le K\,B\,e^{t^*}, \qquad (x,t) \in \overline{D} \times [t^*, \infty),$$

and in particular, at $(\bar{x}, \bar{t})$,

$$|v(\bar{x}, \bar{t})| \le K B e^{t^*}.$$

But $t^* < \bar{t}$ was arbitrary and hence, letting t^* tend to $-\infty$, we find that $v(\bar{x}, \bar{t}) = 0$, and consequently, by the definition of v, $u(\bar{x}, \bar{t}) = 0$, which completes the proof.

Theorem 3.2 (Periodicity)

If the problem (3.1), (3.2) possesses a unique solution, $u = u(x,t)$, and if the functions a_{ij}, b_i, c, f, β and g are periodic in t with period T, then the solution u is periodic in t with period T.

Proof: Let $v(x,t) \equiv u(x,t+T)$ and observe that

$$\begin{aligned} L(x,t)v(x,t) &= L(x,t+T)u(x,t+T) = f(x,t+T) \\ &= f(x,t) \qquad (x,t) \in D \times (-\infty, \infty) \end{aligned}$$

$$\begin{aligned} \frac{\partial v(x,t)}{\partial \nu} + \beta(x,t)v(x,t) &= \frac{\partial u(x,t+T)}{\partial \nu} + \beta(x,t+T)u(x,t+T) \\ &= g(x,t+T) \\ &= g(x,t) \qquad (x,t) \in \partial D \times (-\infty, \infty). \end{aligned}$$

Thus, both u and v are solutions to (3. 1), (3. 2). By the uniqueness assumption, $v(x,t) \equiv u(x,t)$. That is, $u(x,t) = u(x, t+T)$.

We now state and prove the main result of this section.

Theorem 3.3 (Existence)

The boundary value problem (3. 1), (3. 2), that is,

$$Lu = f(x,t) \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (-\infty, \infty)$$

has a unique, bounded solution $u = u(x,t)$, which is periodic in t with period T, provided the following conditions are satisfied:

(i) L is uniformly parabolic in $\overline{D} \times (-\infty, \infty)$

(ii) the coefficients of L are continuous and satisfy the following Hölder conditions on $\overline{D} \times (-\infty, \infty)$:

$$|a_{ij}(x,t) - a_{ij}(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

$$|b_i(x,t) - b_i(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

$$|c(x,t) - c(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

(iii) ∂D belongs to class $C_{1+\lambda}$

(iv) f satisfies the following Hölder condition on $\overline{D} \times (-\infty, \infty)$:

$$|f(x,t) - f(x^{(0)}, t)| \leq M\,|x - x^{(0)}|^{\alpha}$$

and is uniformly continuous in both variables x and t on $\overline{D} \times (-\infty, \infty)$

(v) β and g are continuous on $\partial D \times (-\infty, \infty)$

(vi) $c(x,t) \leq 0 \qquad (x,t) \in \overline{D} \times (-\infty, \infty)$

(vii) $\beta(x,t) \leq b_0 < 0$, b_0 a constant, $(x,t) \in \partial D \times (-\infty, \infty)$

(viii) the functions a_{ij}, b_i, c, f, β and g are periodic in t with period T.

<u>Proof:</u> Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of negative real numbers, which decrease strictly to minus infinity. By Theorem 2. 1 this determines a sequence of functions, denoted by $\{u^n\}$, defined by:

$$Lu^n = f(x,t) \qquad (x,t) \in D \times (t_n, t^*]$$

$$\frac{\partial u^n}{\partial \nu} + \beta(x,t)u^n(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (t_n, t^*]$$

$$u^n(x, t^*) = 0 \qquad x \in \overline{D},$$

where t^* is an arbitrary, positive number. Since t^* is arbitrary, we conclude that the above problem has a unique solution, u^n, defined on $\overline{D} \times [t_n, +\infty)$.

Nothing is known as to convergence of $\{u^n\}$ on $\overline{D} \times (-\infty, \infty)$. However, given any $a < 0$ we shall produce a sub-sequence of $\{u^n\}$ which converges uniformly on $\overline{D} \times (a, \infty)$, namely any sub-sequence $\{u^n\}_{n_0}^{\infty}$ for which $t_{n_0} < a$. (We shall henceforth call sets of the form $\overline{D} \times (a, \infty)$ left bounded subsets of $\overline{D} \times (-\infty, \infty)$.) To show that such a sub-sequence $\{u^n\}_{n_0}^{\infty}$ — which we re-name $\{u^n\}$ — converges uniformly in the sense of Cauchy on $\overline{D} \times (a, \infty)$, we first use Theorem 2.2 to obtain the following estimate on $\overline{D} \times [t_n, \infty)$:

$$|u^n(x,t)| \leq K\,(\mathrm{SUP}\,|f| + \mathrm{SUP}\,|g|) \equiv C_0 < \infty$$

the supremum of $|f|$ being taken over $\overline{D} \times (-\infty, \infty)$ and that of $|g|$ over $\partial D \times (-\infty, \infty)$. The constant K depends only on L, β and D. Next we pick integers $q > p > 0$ and observe that the difference $u^{p,q} \equiv u^q - u^p$ satisfies

$$Lu^{p,q} = 0 \qquad (x,t) \in D \times (t_p, \infty)$$

$$\frac{\partial u^{p,q}}{\partial \nu} + \beta(x,t)u^{p,q}(x,t) = 0 \qquad (x,t) \in \partial D \times (t_p, \infty)$$

$$\begin{aligned} u^{p,q}(x,t_p) &= u^q(x,t_p) - u^p(x,t_p) \\ &= u^q(x,t_p) \qquad x \in \overline{D}\,. \end{aligned}$$

Making the transformation $v^{p,q} = e^t u^{p,q}$, we find that $v^{p,q}$ satisfies

$$Lv^{p,q} - v^{p,q} = 0 \qquad (x,t) \epsilon D \times (t_p, \infty)$$

$$\frac{\partial v^{p,q}}{\partial \nu} + \beta(x,t)v^{p,q}(x,t) = 0 \qquad (x,t) \epsilon \partial D \times (t_p, \infty)$$

$$v^{p,q}(x,t) = e^{t_p} u^{p,q}(x, t_p)$$

$$= e^{t_p} u^q(x, t_p) \qquad x \epsilon \overline{D}.$$

Again applying Theorem 2.2, we estimate $v^{p,q}$ on $\overline{D} \times [t_p, \infty)$ by

$$|v^{p,q}(x,t)| \leq K e^{t_p} \sup_{x \epsilon \overline{D}} | u^q(x, t_p)|.$$

But we have already seen that the elements of the sequence $\{u^n\}$ are bounded by C_0, so that $|v^{p,q}(x,t)| \leq K e^{t_p} C_0$ on $\overline{D} \times [t_p, \infty)$, where K and C_0 depend only on L, β, f, g and D. Substituting $u^{p,q}(x,t) = e^{-t} v^{p,q}(x,t)$, we have

$$|u^{p,q}(x,t)| \leq K C_0 e^{t_p - t}$$

for $(x,t) \epsilon \overline{D} \times [t_p, \infty)$. From this it follows that on every left bounded subset of $\overline{D} \times (-\infty, \infty)$ the sequence $\{u^n\}$ is a uniformly Cauchy sequence of continuous functions.

Defining $u(x,t)$ to be the pointwise limit of $\{u^n\}$, we now show that $u(x,t)$ satisfies (3.1), (3.2). To this end suppose that $v(x,t)$ is a function that satisfies

$$Lv = f(x,t) \qquad (x,t) \in D \times (t^*, \infty)$$

$$\frac{\partial v}{\partial \nu} + \beta(x,t)v(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (t^*, \infty)$$

$$v(x, t^*) = u(x, t^*) \qquad x \in \overline{D}$$

where t^* is an arbitrary negative number. Choose n sufficiently large so that $t_n < t^*$. Letting $w^n = v - u^n$, we find that

$$Lw^n = 0 \qquad (x,t) \in D \times (t^*, \infty)$$

$$\frac{\partial w^n}{\partial \nu} + \beta(x,t)w^n(x,t) = 0 \qquad (x,t) \in \partial D \times (t^*, \infty)$$

$$w^n(x, t^*) = u(x, t^*) - u^n(x, t^*) \qquad x \in \overline{D}\ .$$

Applying Theorem 2. 2 again, we have

$$|v(x,t) - u^n(x,t)| \leq K \sup_{x \in \overline{D}} |u(x,t^*) - u^n(x,t^*)|$$

for $(x,t) \in \overline{D} \times [t^*, \infty)$ and $K = K(L, \beta, D)$. From this estimate, we see that the sequence $\{u^n\}$ converges to v uniformly on $\overline{D} \times [t^*, \infty)$ and since $\{u^n\}$ also converges to u in the same region, we conclude that $u = v$. That is, u satisfies

$$Lu = f(x,t) \qquad (x,t) \in D \times (t^*, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (t^*, \infty).$$

Since t^* was an arbitrary negative number, we conclude that u satisfies (3.1), (3.2). Finally the uniqueness and periodicity assertions follow from Theorems 3.1 and 3.2, respectively.

IV. PERIODIC SOLUTIONS OF THE NONLINEAR SECOND BOUNDARY VALUE PROBLEM

The purpose of this section is to extend the results in the previous section to the case where the right hand side of the boundary condition $\frac{\partial u}{\partial \nu} + \beta u = g$ depends on x, t and u. We also consider the case where the right hand side of the operator equation $Lu = f$ depends on x, t and u and finally, we take up the case where both f and g depend on x, t and u.

Before stating and proving the major results, we first give the following definition and prove a lemma.

Definition 4.1. If the pair (X, d) is a metric space, where X is a linear space and d is a metric defined on X, we say that a sequence of families of real-valued functions $F_n = \{f_\alpha^n\}$, α belonging to some index set A, each f_α^n defined on X, converges to the family $\{f_\alpha\}$ in an equiconvergent manner as $n \to \infty$ if, for each $\epsilon > 0$, there exists an integer N, independent of α, such that $n \geq N$ implies

$$|f_\alpha^n(x) - f_\alpha(x)| < \epsilon ;$$

for all $x \in X$ and all $\alpha \in A$.

Lemma 4.1

If for each $n = 1, 2, \cdots$, $F_n = \{f_\alpha^n\}$ is a family of equicontinuous real-valued functions defined on a compact metric space (X, d) and if $\{f_\alpha^n\}$ converges to $\{f_\alpha\}$ in an equiconvergent manner as $n \to \infty$, then the family $\{f_\alpha\}$ is equicontinuous.

Proof: Since each family F_n is equicontinuous, we know that for every $\epsilon > 0$ there exists a $\delta_n > 0$, independent of α, such that $d(x, y) < \delta_n$ implies $|f_\alpha^n(x) - f_\alpha^n(y)| < \frac{\epsilon}{3}$ for all $\alpha \epsilon A$, by use of the definition. Also since $\{f_\alpha^n\}$ converges to $\{f_\alpha\}$ in an equiconvergent manner we know that if $\epsilon > 0$ then there exists an integer N independent of α such that for all $x \epsilon X$, $\alpha \epsilon A$,

$$|f_\alpha^n(x) - f_\alpha(x)| < \frac{\epsilon}{3}$$

for all $n \geq N$. Now pick $\epsilon > 0$. We conclude that there exists a $\delta_n > 0$, independent of $\alpha \epsilon A$, such that if $d(x, y) < \delta_n$, then for all $\alpha \epsilon A$ we have

$$|f_\alpha(x) - f_\alpha(y)| \leq |f_\alpha(x) - f_\alpha^n(x)| + |f_\alpha^n(x) - f_\alpha^n(y)| + |f_\alpha^n(y) - f_\alpha(y)|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon .$$

This proves the lemma.

With the help of this lemma we can now state and prove the major theorems of this section.

Theorem 4.2 (Existence of a periodic solution to PROBLEM II)

The problem

(4.1) $Lu = f(x,t)$ $\qquad (x,t) \in D \times (-\infty, \infty)$

(4.2) $\dfrac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t,u)$ $\qquad (x,t) \in \partial D \times (-\infty,\infty)$

has at least one solution $u = u(x,t)$, periodic in t with period T provided the following conditions are satisfied:

(i) L is uniformly parabolic in $\overline{D} \times (-\infty, \infty)$

(ii) the coefficients of L are continuous and satisfy the following Hölder conditions on $\overline{D} \times (-\infty, \infty)$:

$$|a_{ij}(x,t) - a_{ij}(x^{(0)},t)| \leq M|x - x^{(0)}|^{\alpha},$$

$$|b_i(x,t) - b_i(x^{(0)},t)| \leq M|x - x^{(0)}|^{\alpha},$$

$$|c(x,t) - c(x^{(0)},t)| \leq M|x - x^{(0)}|^{\alpha}.$$

(iii) ∂D belongs to class $C_{1+\lambda}$

(iv) f satisfies the Hölder condition on $\overline{D} \times (-\infty,\infty)$,

$$|f(x,t) - f(x^{(0)},t)| \leq M|x - x^{(0)}|^{\alpha}$$

and is uniformly continuous in x and t on $\bar{D} \times (-\infty, \infty)$

(v) β is continuous on $\partial D \times (-\infty, \infty)$

(vi) $c(x,t) \leq 0$ on $\bar{D} \times (-\infty, \infty)$

(vii) $\beta(x,t) \leq b_0 < 0$, b_0 a constant, on $\partial D \times (-\infty, \infty)$

(viii) the functions a_{ij}, b_i, c, f, β and g are periodic in t with period T

(ix) $g = g(x,t,v)$ is continuous in $(x,t) \in \partial D \times (-\infty, \infty)$, $-\infty < v < \infty$, and Lipschitz continuous in v with Lipschitz constant "sufficiently small", the Lipschitz constant being independent of x and t.

Proof: If in our nonlinear problem (4. 1), (4. 2) we replace $g(x,t,u(x,t))$ by $g(x,t,v(x,t))$ where $v(x,t)$ is an arbitrary function in C^p, we return to a linear problem

$$(S)\quad \begin{cases} Lw = f(x,t) & (x,t) \in D \times (-\infty, \infty) \\ \dfrac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t,v(x,t)) & (x,t) \in \partial D \times (-\infty, \infty) \end{cases}$$

of the type considered in Theorem 3. 3, which consequently possesses a unique solution $w(x,t)$ in C^p. We may accordingly consider

(S) as defining a transformation S of C^p into itself. Now if $w = Sv$ can be shown to have a "fixed point" u in C^p (that is, $Su = u$), we shall have a periodic solution to (4.1), (4.2).

In order to apply Schauder's fixed point theorem, we next show that:

1. S is a continuous mapping from the normed linear space C^p into itself.

2. S maps a closed convex subset of C^p into a compact subset of itself.

Having shown these two properties of S, we shall be in a position to apply the Schauder fixed point theorem and conclude that S has a fixed point in C^p and hence the problem (4.1), (4.2) has a periodic solution. To show S is continuous let $\|v_n - v\| \to 0$ as $n \to \infty$. Now, if we write $w_n = v_n - v$ we have

$$Lw_n = 0$$

$$\frac{\partial w_n}{\partial \nu} + \beta(x,t)w_n(x,t) = g(x,t,v_n(x,t)) - g(x,t,v(x,t)).$$

By Theorem 2.2, we obtain the following estimate

$$|w_n(x,t)| \leq K \ \mathrm{SUP} \ |g(x,t,v_n(x,t)) - g(x,t,v(x,t))| ,$$

where the supremum is taken over $\overline{D} \times (-\infty, \infty)$ and the positive constant K depends on L, β and D. Using the fact that g is Lipschitz continuous in the last argument, we can write

$$|w_n(x,t)| \leq K M \, \text{SUP} \, |v_n(x,t) - v(x,t)|$$

$$= K M \, \|v_n - v\|$$

where M is the Lipschitz constant. Taking the supremum of the left hand side we have

$$\|u_n - u\| \leq K M \, \|v_n - v\|$$

and hence S is a continuous mapping.

To show that S maps every closed convex subset of C^p into a compact subset of itself, let $B(0,R) \equiv \{v \in C^p : \|v\| \leq R\}$, a closed convex set. For $v \in B(0,R)$, we have that

$$|u(x,t)| \leq K \, (\text{SUP} \, |f(x,t)| + \text{SUP} \, |g(x,t,v(x,t))|).$$

To find a bound on the above expression one observes that

$$|g(x,t,v(x,t))| \leq |g(x,t,0)| + |g(x,t,v(x,t)) - g(x,t,0)|$$

$$\leq |g(x,t,0)| + M|v(x,t)| ,$$

and by taking the supremum of each side of the inequality we conclude that

$$\text{SUP } |g(x,t,v(x,t))| \leq \text{SUP } |g(x,t,0)| + M\,\text{SUP } |v(x,t)|$$

$$\leq \text{SUP } |g| + MR .$$

Using this estimate, we bound $|u(x,t)|$ by

$$|u(x,t)| \leq K\,(\text{SUP } |f| + \text{SUP } |g| + MR)$$

$$= K\,(MR + \text{CONSTANT})$$

where $\text{CONSTANT} = \text{SUP } |f| + \text{SUP } |g| < \infty$. From this inequality we finally arrive at

$$\|u\| \leq K\,(MR + \text{CONSTANT}).$$

This means that if $KM < 1$ then it is possible to pick R sufficiently large so that

$$\|u\| \leq K\,(MR + \text{CONSTANT}) \leq R$$

and hence $u \in B(0,R)$. That is, if the Lipschitz constant M is sufficiently small, say $M < \frac{1}{K}$, then for R sufficiently large, S maps the closed convex set $B(0,R)$ into itself.

We now show that $S(B(0,R))$ is a compact subset of $C^p\,(\overline{D} \times (-\infty, \infty))$. It is sufficient to show that $S(B(0,R))$ is compact in $C\,(\overline{D} \times (-\infty, \infty))$ since $S(B(0,R))$ will then be compact in the closed subset, $C^p(\overline{D} \times (-\infty, \infty))$, of $C\,(\overline{D} \times (-\infty, \infty))$. To show $S(B(0,R))$ is compact in $C(\overline{D} \times (-\infty, \infty))$ let

$F_n = \{u_v^n\}_{n=1}^{\infty}$ be a sequence of families of functions defined by

$$Lu_v^n = f(x,t) \qquad (x,t) \in D \times (t_n, t^*]$$

$$\frac{\partial u_v^n}{\partial \nu} + \beta(x,t)u_v^n(x,t) = g(x,t,v) \qquad (x,t) \in \partial D \times (t_n, t^*]$$

where

(i) $\{t_n\}$ is a sequence of negative number decreasing strictly to $-\infty$.

(ii) t^* is an arbitrary positive number.

(iii) $v \in B(0,R)$.

We observe that from [5, p. 210] it follows that for each $n = 1, 2, \cdots$ $F_n \equiv \{u_v^n\}$ is an equicontinuous family of real-valued functions defined on $\overline{D} \times [t_n, t^*]$. Also for $q > p > 0$ we have by Theorem 2.2 the following estimate on $\overline{D} \times [t_p, t^*]$

$$\begin{aligned} |u_v^q(x,t) - u_v^p(x,t)| &\leq K e^{t_p - t} \text{SUP} |u_v^q(x, t_p)| \\ &\leq K^2 e^{t_p - t} (\text{SUP } |f| + \text{SUP } |g|) \\ &\leq K^2 e^{t_p - t} (MR + \text{CONSTANT}). \end{aligned}$$

This estimate tells us that for a fixed $v \in B(0,R)$ the sequence $\{u_v^n\}_{n=1}^{\infty}$ converges uniformly in the Cauchy sense on every <u>left</u>

bounded subset of $\overline{D} \times (-\infty, \infty)$ and hence converges pointwise to some function u_v. We can show by an argument very similar to the one given in the proof of Theorem 3.3, that u_v satisfies (4.1), (4.2). We also conclude from the above estimate that the sequence of families $\{u_v^n\}$ converges to $\{u_v\}$ in an equiconvergent manner, since the bound does not depend on v. Applying Lemma 4.1 we conclude that $\{u_v\}$ is an equicontinuous family on every left bounded subset of $\overline{D} \times (-\infty, \infty)$ and in particular on a set of the form $\overline{D} \times [t_1, t_2]$ where t_1, t_2 are numbers satisfying $-\infty < t_1 < t_2 < \infty$. Using the Arzela-Ascoli theorem, we conclude that $\{u_v\}$ is compact in $C(\overline{D} \times [t_1, t_2])$ and since t_1, t_2 were arbitrary, $\{u_v\}$ is compact in $C(\overline{D} \times (-\infty, \infty))$. From what we've observed before $\{u_v\}$ is compact in $C^p(\overline{D} \times (-\infty, \infty))$. The Schauder fixed point theorem now tells us that the mapping S has a fixed point $u \in C^p(\overline{D} \times (-\infty, \infty))$. That is, there is a function $u = u(x,t)$ periodic in t with period T satisfying (4.1), (4.2). This completes the proof.

We now state a slightly modified version of the above theorem. We first note the idea of a uniformly concave function. A function $\phi(v)$ is uniformly concave if

$$\phi\left[\frac{v_1+v_2}{2}\right] - \left[\frac{\phi(v_1) + \phi(v_2)}{2}\right] \geq \epsilon > 0 ,$$

for ϵ an arbitrary positive number.

Theorem 4.3 (Alternate existence theorem for PROBLEM II)

Let condition (ix) in Theorem 4.2 be replaced by:

(ix') $g = g(x,t,v)$ is continuous in $(x,t) \epsilon \partial D \times (-\infty, \infty)$, $-\infty < v < \infty$, and Lipschitz continuous in v, the Lipschitz constant being independent of x and t; furthermore $|g(x,t,v)| \leq c\phi(x,t,v)$ where the function ϕ is uniformly concave in u and c is a positive constant.

Then problem (4.1), (4.2) has at least one solution $u = u(x,t)$, periodic in t with period T.

Proof: The proof follows nearly the same lines as Theorem 4.2. The only difference between these two proofs occurs when showing that the mapping S maps C^p into itself. One can easily observe that the hypothesis $|g(x,t,v)| \leq c\phi(x,t,v)$ allows one to relax the condition that the Lipschitz constant of g be sufficiently small.

Remark: The function $\phi(v) = |v|^{\theta}$, $0 < \theta < 1$, is uniformly concave.

We now state and prove a uniqueness result for problem (4.1), (4.2).

Theorem 4.4 (Uniqueness theorem for PROBLEM II)

If $u = u(x,t)$ is a bounded solution of (4.1), (4.2) and if

(i) L is uniformly parabolic on $\overline{D} \times (-\infty, \infty)$

(ii) $c(x,t) \leq 0 \qquad (x,t) \in \overline{D} \times (-\infty, \infty)$

(iii) $\beta(x,t) \leq b_0 < 0 \qquad (x,t) \in \partial D \times (-\infty, \infty)$

(iv) ∂D belongs to class $C_{1+\lambda}$

(v) $g(x,t,v)$ is monotone increasing in v,

then the solution $u = u(x,t)$ is unique.

Proof: Let u_1, u_2 be two different solutions of (4.1), (4.2) and denote by w the difference $w = u_1 - u_2$. This difference satisfies

$$Lw = 0 \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial w}{\partial \nu} + \beta(x,t)w(x,t) = g(x,t,u_1) - g(x,t,u_2)$$

$$= \psi(x,t)w(x,t), \qquad (x,t) \in \partial D \times (-\infty, \infty)$$

where

$$\psi(x,t) = \begin{cases} \dfrac{g(x,t,u_1) - g(x,t,u_2)}{u_1(x,t) - u_2(x,t)}, & u_1(x,t) \neq u_2(x,t) \\ 0 & u_1(x,t) = u_2(x,t). \end{cases}$$

Since g is monotone increasing, we conclude that $\psi(x,t) \geq 0$.

We now write the boundary condition as

$$\frac{\partial w}{\partial \nu} + [\beta(x,t) - \psi(x,t)]\, w(x,t) = 0 \ .$$

By Theorem 3.1 we can conclude that the unique solution to the problem

$$Lw = 0$$

$$\frac{\partial w}{\partial \nu} + [\beta(x,t) - \psi(x,t)]\, w(x,t) = 0$$

is the solution $w \equiv 0$. Hence $u_1 \equiv u_2$.

We now state a second uniqueness theorem.

Theorem 4.5 (Alternate uniqueness theorem for PROBLEM II)

Let condition (iv) in Theorem 4.4 be replaced by:

(iv') The function $g(x,t,v)$ is differentiable with respect to v and the derivative g_v satisfies $|g_v(x,t,v)| \leq b_0$ (b_0 defined in condition (iii) of Theorem 4.4).

Then the solution $u = u(x,t)$ of (4.1), (4.2) is unique.

Proof: The proof follows the same lines as Theorem 4.4. The major step is carried out by a simple application of the mean value theorem.

We now take up the case where the right hand side of $Lu \equiv f$ depends on x, t and u.

Theorem 4.6 (Existence theorem for PROBLEM III)

The problem

(4.3) $$Lu = f(x, t, u(x, t)) \qquad (x, t) \in D \times (-\infty, \infty)$$

(4.4) $$\frac{\partial u}{\partial \nu} + \beta(x, t)u(x, t) = g(x, t) \qquad (x, t) \in \partial D \times (-\infty, \infty)$$

has at least one solution $u = u(x, t)$, periodic in t with period T, provided the following conditions are satisfied:

(i) L is uniformly parabolic on $\overline{D} \times (-\infty, \infty)$

(ii) the coefficients of L are continuous and satisfy the Hölder conditions on $\overline{D} \times (-\infty, \infty)$

$$|a_{ij}(x, t) - a_{ij}(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

$$|b_i(x, t) - b_i(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

$$|c(x, t) - c(x^{(0)}, t)| \leq M|x - x^{(0)}|^{\alpha}$$

(iii) ∂D belongs to class $C_{1+\lambda}$

(iv) $f(x,t,v)$ is uniformly continuous in x and t, and Lipschitz continuous in v with Lipschitz constant "sufficiently small"; furthermore the Lipschitz constant does not depend on x and t

(v) β, g are continuous on $\partial D \times (-\infty, \infty)$

(vi) $c(x,t) \leq 0 \qquad (x,t) \in \overline{D} \times (-\infty, \infty)$

(vii) $\beta(x,t) \leq b_0 < 0$, b_0 a constant, $(x,t) \in \partial D \times (-\infty, \infty)$

(viii) the functions a_{ij}, b_i, c, f, β and g are periodic in t with period T.

<u>Proof:</u> Consider the mapping S from $C^p_\alpha(\overline{D} \times (-\infty, \infty))$ into $C^p_\alpha(\overline{D} \times (-\infty, \infty))$, where S, $Sv = u$, is defined by

$$Lu = f(x,t,v(x,t)) \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t) \qquad (x,t) \in \partial D \times (-\infty, \infty)\,.$$

To show $Sv \in C^p_\alpha(\overline{D} \times (-\infty, \infty))$, we first observe that since $f(x,t,v(x,t))$ is Hölder continuous in x, we can apply Theorem 2.1 to conclude that $Sv \in C^p(\overline{D} \times (-\infty, \infty))$. To show Sv is Hölder continuous in x uniformly on $-\infty < t < \infty$, that is, $Sv \in C^p_\alpha(\overline{D} \times (-\infty, \infty))$, it is sufficient to show that u_{x_i} is uniformly bounded on $\overline{D} \times (-\infty, \infty)$. From Friedman's text [5] we know that u_{x_i} is uniformly continuous on $D \times [0, T]$ and hence by the periodicity of u_{x_i} we conclude that u_{x_i} is uniformly bounded on $\overline{D} \times (-\infty, \infty)$ and hence Sv is Hölder continuous in x, uniformly on $-\infty < t < \infty$. We now show that:

1. S is a continuous map from the normed linear space C_α^p into itself.

2. S maps a closed convex subset of C_α^p into a compact subset of itself.

To show that S satisfies these two conditions we proceed along the same lines as the proof of Theorem 4.2. We then apply the Schauder fixed point theorem to obtain the existence of a periodic solution.

We now state a slightly modified version of the above theorem.

Theorem 4.7 (Alternate existence theorem for PROBLEM III)

Let condition (iv) in Theorem 4.6 be replaced by:

(iv') $f = f(x, t, v)$ is uniformly continuous in $(x, t) \in \overline{D} \times (-\infty, \infty)$, $-\infty < v < \infty$, and Lipschitz continuous in v, with Lipschitz constant independent of x and t; furthermore, f satisfies $|f(x,t,v)| \leq c\,\phi(x,t,v)$ where the function ϕ is uniformly concave in v and c is a positive constant.

Then problem (4.3), (4.4) has at least one solution $u = u(x,t)$, periodic in t with period T.

Proof: The proof follows nearly the same lines as Theorem 4.6. The only difference between these two proofs occurs when showing that the mapping S maps C^p_α into itself. One can easily observe that the hypothesis $|f(x,t,v)| \leq c\,\phi(x,t,v)$ allows one to relax the condition that the Lipschitz constant of f be sufficiently small.

We now state and prove a uniqueness theorem for problem (4.3), (4.4).

Theorem 4.8 (Uniqueness theorem for PROBLEM III)

If $u = u(x,t)$ is a bounded solution of problem (4.3), (4.4) and if the following conditions hold:

(i) L is uniformly parabolic on $\overline{D} \times (-\infty, \infty)$

(ii) $c(x,t) \leq 0 \qquad (x,t) \in \overline{D} \times (-\infty, \infty)$

(iii) $\beta(x,t) \leq b_0 < 0$, b_0 a constant, $(x,t) \in \partial D \times (-\infty, \infty)$

(iv) ∂D belongs to class $C_{1+\lambda}$

(v) $f(x,t,v)$ is monotone increasing in v,

then the solution $u = u(x,t)$ is unique.

Proof: Let u_1, u_2 be two solutions of (4. 5), (4. 6) and denote w the difference $w = u_1 - u_2$. The function w satisfies

$$Lw = f(x,t,u_1) - f(x,t,u_2) \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial w}{\partial \nu} + \beta(x,t)w(x,t) = 0 \qquad (x,t) \in \partial D \times (-\infty, \infty) .$$

We now write

$$Lw = f(x,t,u_1) - f(x,t,u_2) \equiv \psi(x,t)w(x,t)$$

where

$$\psi(x,t) = \begin{cases} \dfrac{f(x,t,u_1)-f(x,t,u_2)}{u_1(x,t)-u_2(x,t)}\, w(x,t), & u_1(x,t) \neq u_2(x,t) \\ 0 & u_1(x,t) = u_2(x,t) . \end{cases}$$

Now since g is monotone increasing, we conclude that $\psi(x,t) \geq 0$. We now write this operator equation as

$$Lw - \psi(x,t)w = 0$$

and by applying Theorem 3.1 we conclude that the problem

$$Lw - \psi(x,t)w = 0 \qquad (x,t) \in D \times (-\infty, \infty)$$

$$\frac{\partial w}{\partial \nu} + \beta(x,t)w = 0 \qquad (x,t) \in \partial D \times (-\infty, \infty)$$

has a unique solution. But the zero function is a solution and hence $w = 0$, that is, $u_1 = u_2$.

We now extend Theorems 4.2 and 4.6 by allowing both f and g to depend on x, t and u.

<u>Theorem 4.9</u> (Existence theorem for PROBLEM IV)

The problem

(4.5) $$Lu = f(x,t,u(x,t)) \qquad (x,t) \in D \times (-\infty, \infty)$$

(4.6) $$\frac{\partial u}{\partial \nu} + \beta(x,t)u(x,t) = g(x,t,u(x,t)) \qquad (x,t) \in \partial D \times (-\infty, \infty)$$

has at least one solution $u = u(x,t)$, periodic in t with period T, provided the following conditions are satisfied:

(i) L is uniformly parabolic in $\overline{D} \times (-\infty, \infty)$

(ii) the coefficients of L are continuous and satisfy the Hölder conditions on $\overline{D} \times (-\infty, \infty)$

$$|a_{ij}(x,t) - a_{ij}(x^{(0)},t)| \leq M\,|x - x^{(0)}|^{\alpha}$$

$$|b_i(x,t) - b_i(x^{(0)},t)| \leq M\,|x - x^{(0)}|^{\alpha}$$

$$|c(x,t) - c(x^{(0)},t)| \leq M\,|x - x^{(0)}|^{\alpha}$$

(iii) ∂D belongs to class $C_{1+\lambda}$

(iv) f is uniformly continuous in x and t, Lipschitz continuous in v with Lipschitz constant "sufficiently small", the Lipschitz constant being independent of x and t.

(v) β is continuous on $\partial D \times (-\infty, \infty)$

(vi) $c(x,t) \leq 0$ $\quad (x,t) \in \overline{D} \times (-\infty, \infty)$

(vii) $\beta(x,t) \leq b_0 < 0$, b_0 a constant, $(x,t) \in \partial D \times (-\infty, \infty)$

(viii) the functions a_{ij}, b_i, c, f, β and g are periodic in t with period T

(ix) $g(x,t,v)$ is continuous in x and t and is Lipschitz continuous in v, where the Lipschitz constant is "sufficiently small" and independent of x and t.

Proof: Consider the mapping S from $C^p_\alpha(\overline{D} \times (-\infty, \infty))$ into $C^p_\alpha(\overline{D} \times (-\infty, \infty))$ where S, $Sv = u$, is defined by