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The purpose of this dissertation is to formulate, analyze and numerically solve the dynamic harvesting problem for uneven-aged stands. The problem is to find the optimal numbers of trees to remove from diameter classes over a finite time horizon and is formulated as a discrete-time optimal-control problem with bounded control variables and free terminal point. A solution algorithm called the method of steepest descent is described and demonstrated with a whole-stand/diameter-class simulator for Wisconsin hardwood stands. Optimal management regimes that maximize present net worth (PNW) from harvests taken on a 5-year cutting cycle during a 150-year time horizon are developed for three stumpage value functions.

Harvest regimes derived with the gradient method contradicted optimal steady-state management regimes determined with static analysis. Pontryagin's maximum principle is used to establish optimality conditons for the dynamic and static optimization problems. Comparison

of these conditions shows that for a stand with any initial diameter distribution: (1) the optimal transition regime does not converge to the steady state that maximizes land expectation value defined by the Faustmann equation; (2) the PNW of the optimal transition and steady-state regime is greater than the PNW of the statically determined regime; and (3) the optimal steady-state regime is invariant. These results invalidate the use of investment-efficient diameter distributions.

The gradient method produces stationary solutions for harvest control variables in the first period and beyond; however, because of large discount factors in distant periods, the algorithm fails to provide stationary solutions for long-term management within reasonable execution times. To avoid this problem, a restart procedure that takes advantage of the stability of the first-period solution and the sequential nature of the problem is developed.

An economic model for harvesting forest stands is presented and used to contrast the two major timber harvesting systems: even-aged and uneven-aged management. In contrast to even-aged management, the value of uneven-aged stand harvesting cannot be separated into independent components for stand value and land value. Thus, conclusions about the most profitable harvesting system depend on the joint productivity of the land and existing timber. This result is demonstrated by developing optimal management regimes for Arizona ponderosa pine (*Pinus ponderosa* Laws.) stands.

Optimal Timber Harvesting in Uneven-aged Forest Stands: a Discrete-Time Optimal-Control Approach

by

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PREFACE

This dissertation is composed of three manuscripts. The first two (chapters 2 and 3) have been submitted to FOREST SCIENCE, and the third (chapter 4) has been submitted to LAND ECONOMICS. I am grateful for the helpful comments from J. Douglas Brodie and Darius M. Adams, professors of forest management at Oregon State University and the University of Washington, respectively, who are co-authors on the first manuscript. Dr. Brodie assisted in interpreting the results and reviewed earlier drafts of the manuscript. Dr. Adams provided the original formulation of the dynamic uneven-aged management problem and reviewed earlier drafts of the manuscript. Of course, the remaining errors are mine.

OPTIMAL TIMBER HARVESTING IN UNEVEN-AGED FOREST STANDS: A DISCRETE-TIME OPTIMAL-CONTROL APPROACH

INTRODUCTION

A fundamental problem in forest management is the determination of the best sequence of stocking levels to maintain in a stand to meet yield oriented objectives. Thus, stand-level management has been identified as an important class of forestry problems to which operations research techniques can be fruitfully applied (Hann and Bare 1979, Hann and Brodie 1980). Brodie and Haight (1985) reviewed the application of dynamic programming to the determination of profit-maximizing harvest policies for even-aged stands. In contrast, the application of any operations research techniques to uneven-aged stand management is still in its infancy.

One distinguishing feature of uneven-aged stand management is the characterization of stand structure by the numbers of trees in prespecified diameter classes. The harvesting problem consists of determining the optimal sequence of residual stand diameter distributions and selection harvests.

Management guides for uneven-aged stands have most often been constructed by subjective extrapolation from limited field experiments. The difficulties of field experimentation, however, generally preclude an exhaustive examination of a broad range of harvest policies. Thus, with this limited approach, it is impossible to be certain that the best sequence of stocking levels has been tested.

With recent advances in forest growth modeling it is now possible to simulate the movement of trees between diameter classes and the mortality of trees in each diameter class. Simulators such as those developed by Ek (1974) for Wisconsin hardwood stands and by Hann (1980) for Arizona ponderosa pine stands can accurately predict the effects of a wide range of harvest policies on stand structure, growth and yield. To date, uneven-aged stand optimization studies have focused on the problem of determining maximum equilibrium rent policies. The purpose of this research is to develop techniques for determining the transient and equilibrium harvest policies that maximize the present net worth (PNW) of harvests.

This dissertation is composed of three papers, each comprising a separate chapter. In chapter 1, I formulate the uneven-aged stand harvesting problem and trace the developement of this formulation through the forestry literature. A gradient-based nonlinear programming technique called _the method of steepest descent_ is presented in the context of the harvesting problem. The method is demonstrated with a simulator for Wisconsin hardwood stands.

Numerical solutions for uneven-aged stand harvesting provided two results that required further study. First, equilibrium management regimes develeped with static optimization were not optimal when used as starting conditions in dynamic formulations. Second, the gradient method did not provide stationary solutions for distant-period control variables within reasonable execution times. These problems are taken up in the second paper.

In chapter 2, I compare the problem formulations for dynamic and static harvest optimization. I use Pontryagin's maximum principle to develop marginal produciton rules that must be satisfied by equilibrium harvesting regimes for each problem. A comparison of the marginal production rules highlights the conditions under which the equilibria will differ. These results place limits on the use of statically determined equilibrium regimes as goals in uneven-aged management. Further, the results point out that the valuation of the uneven-aged harvesting system cannot be made on the basis of equilibrium management alone. The value of uneven-aged management must include the values of both transition and equilibrium management regimes.

I also include in chapter 2 an improved solution algorithm that provides stable solutions for transition and equilibrium harvest policies. I demonstrate the algorithm with the Wisconsin hardwood stand simulator by computing optimal management regimes for comparison with a statically determined equilibrium regime and for analysis of the effect of increasing the cutting cycle length.

Optimal harvest regimes can be used to evaluate the performance of uneven-aged management relative to other uses of the land and timber. For example, the PNW of the best uneven-aged management regime for an existing stand can be compared with the timber liquidation value and the PNW of the best plantation management regime. This comparison is of prime importance when ranking the relative values of these two fundamental management systems.

The third chapter addresses the problem of how to value a forest stand when either even-aged or uneven-aged harvesting systems can be applied. A general economic model for stand managment is presented and used to compare the two harvesting systems. Results are demonstrated by developing optimal harvesting regimes for Arizona ponderosa pine stands.

OPTIMIZING THE SEQUENCE OF DIAMETER DISTRIBUTIONS AND SELECTION HARVESTS FOR UNEVEN-AGED STAND MANAGEMENT

by

Robert G. Haight, J. Douglas Brodie, and Darius M. Adams

ABSTRACT

The determination of an optimal sequence of diameter distributions and selection harvests for uneven-aged stand management is formulated as a discrete-time optimal-control problem with bounded control variables and free terminal point. An efficient programming technique utilizing gradients provides solutions that are stable and interpretable on the basis of economic principles. Methods and results are demonstrated using a whole-stand/diameter-class simulator developed for northern hardwood stands in Wisconsin. Examples in which the objective is present net worth maximization over a 150-year planning horizon with a 5-year cutting cycle suggest two types of optimal equilibrium stand structures: a downward-sloping diameter distribution if large value premiums are assigned to the largest diameter classes and a truncated diameter distribution if premiums for larger trees are gradual or absent. Transition strategies vary in length and harvest pattern depending on the stumpage value function used. It is emphasized that equilibrium management regimes developed with static analysis are not optimal when used as starting conditions in dynamic

formulations.

OPTIMIZING THE SEQUENCE OF DIAMETER DISTRIBUTIONS AND SELECTION HARVESTS FOR UNEVEN-AGED STAND MANAGEMENT

INTRODUCTION

The incorporation of even-aged stand growth and yield simulators into dynamic programming frameworks has improved the economic analysis of silvicultural investment decisions in stand-level even-aged management (Brodie and Kao 1979, Martin and Ek 1981, Riitters and others 1982). Uneven-aged stand simulators cannot be readily incorporated into dynamic programming frameworks because the large number of decision variables involved would require long computation time and massive computer storage. We introduce a more efficient optimization technique based on optimal control theory and demonstrate its use in finding the optimal sequence of diameter distributions and selection harvests for an uneven-aged stand.

The decision variables facing a forest manager interested in applying uneven-aged management at the stand level are (1) the cutting cycle length, and (2) the diameter distribution and species composition after each selection harvest during a given planning horizon. We use the optimization technique to solve the diameter distribution problem, and it can be expanded to analyze both cutting cycle length and species composition problems.

Hann and Bare (1979) have identified two separate but related issues in determining the optimal sequence of diameter distributions: (1) equilibrium diameter distribution, and (2) conversion strategy and conversion period length. Researchers have developed static-optimization techniques that focus on the first issue. Adams and Ek (1974) demonstrated a two-stage technique for determining the optimal equilibrium stand structure. In the first stage, they used a gradient projection method to determine the residual diameter distribution that maximized stand value growth for a given cutting cycle. In the second stage, having solved this nonlinear program for several alternative residual basal are levels, they chose the stand structure that satisfied a marginal value growth percent criterion (Duerr and Bond 1952). Adams (1976) subsequently distinguished between the use of value and basal area measures of growing stock as the appropriate constraints for determining investment-efficient diameter distributions. Martin (1982) derived investment-efficient diameter distributions using the Weibull distribution function. His problem formulation reduced the decision variables to the Weibull function parameters and the total number of trees. Solutions were obtained with a gradient projection method.

Adams and Ek (1974) addressed the second issue by formulating and solving a transition strategy problem with the constraint of achieving an investment-efficient diameter distribution after a specified number of periods. They used a gradient projection method, but the number of decision variables and constraints exceeded algorithm capacity for problems with more than three transition harvests.

The optimization technique presented here differs from these approaches because it considers the transition and equilibrium problems jointly. Interpretation of solutions generates new insights into the economics of uneven-aged management.

This paper is presented in four sections. In the first we formulate the problem of determining the optimal sequence of diameter distributions and selection harvests for an existing stand without constraints of period-to-period sustainability or specified equilibrium endpoint. We trace the development of this formulation through the forestry literature.

The second section shows how to solve this problem with a gradient-based non-linear programming technique called the method of steepest descent.

In the third section we demonstrate the solution algorithm with the use of a whole-stand/diameter-class simulator for mixed-species northern hardwood stands developed by Ek (1974) and modified by Adams and Ek (1974). We use the algorithm to determine stand-specific management regimes for three different stumpage value functions. Finally, we examine the sensitivity of solutions to changes in the termination criterion and starting conditions for the algorithm.

We conclude with a discussion of optimal equilibrium stand structures and methods for expanding the problem statement and solution algorithm to consider cutting cycle and species composition problems.

PROBLEM STATEMENT

The determination of the optimal sequence of diameter distributions and selection harvests for an uneven-aged stand can be stated in a discrete-time optimal-control formulation with free terminal point and bounded control variables:

$$\max_{\{Y_{ij}\}} PNW = \sum_{i=0}^{n-1} \sum_{j=1}^{m} P_{ij} X_{ij} Y_{ij} + \sum_{j=1}^{m} P_{nj} X_{nj} \tag{1}$$

subject to, for $i = 0,\ldots,n-1$ and $j = 1,\ldots,m$,

$$X_{i+1j} = X_{ij} - X_{ij} Y_{ij} + f_{ij}[X_{i1}(1 - Y_{i1}),\ldots,X_{im}(1 - Y_{im})] \tag{2}$$

$$0.0 \leq Y_{ij} \leq 1.0, \tag{3}$$

and X_{0j}, $j = 1,\ldots,m$, is given;

where

X_{ij} = a state variable representing number of trees per acre in diameter class j at the beginnning of period i before cut

Y_{ij} = a control variable representing the percentage of the number of trees per acre in diameter class j which are cut at the beginning of period i

P_{ij} = the discounted net price per tree in diameter class j at the beginning of period i

f_{ij} = a continuous nonlinear function with continuous partial derivatives representing the change in number of trees per acre in diameter class j during period i

n = the number of periods in the planning horizon

m = the number of diameter classes.

The objective function, equation (1), is formulated to seek control-variable values that maximize the present net worth (PNW) of financial returns from harvesting, assuming that all remaining trees are harvested at the end of the planning horizon. No restrictions are placed on the form of the terminal diameter distribution. For a planning horizon greater than 150 years, the discounted value of the terminal diameter distribution is so small that it has no effect on the determination of the control variable values in earlier periods. In these cases, the value of the optimal management regime obtained from this formulation can be viewed as the present value of future income that could be obtained by managing the existing land and timber with an uneven-aged harvesting system indefinitely.

Constraint (2) represents the stand growth dynamics. The number of trees per acre in diameter class j at the beginning of period i + 1 equals the number of trees before cut in diameter class j period i less the amount cut plus the change in number of trees (f_{ij}) which is a function of the residual diameter distribution. The stand growth dynamics are used to compute the state variables for a given control variable sequence and a given initial stand diameter distribution. Constraint (3) ensures that control variables are fractions between 0.0 and 1.0. The initial diameter distribution is given.

Optimal control theory has been applied to stand-level management problems, but results have either not been presented or have been unstable and difficult to interpret. Adams and Ek (1974) developed a control theoretic formulation of a conversion strategy problem in which the control variables were the number of trees cut per diameter class in each period. Because both state and control variables were constrained to be non-negative, the development and execution of a solution procedure was impractical. When the control variables were changed to the percentage of trees cut per diameter class in each period and bounded between 0.0 and 1.0 (Adams and Ek 1975), no constraints were needed on state variables. No solution procedure was presented, however.

Bullard (1983) adapted the control theoretic formulation of Adams and Ek (1974) to mixed-species even-aged hardwood stands. He demonstrated that a problem with eight decision variables and two periods could not be solved with a nonlinear programming technique.

Rapera (1980) formulated the optimal control problem defined above and tested three solution procedures. He used the stand growth simulator given by Adams and Ek (1974) to model diameter class growth dynamics, using a stumpage value function that assigned premiums to large diameter trees. He found that the two procedures that used the gradient-based control-vector-iteration method (McDonough and Park 1975) converged to solutions within specified tolerances. However, values of optimal solutions varied by more than 25 percent depending on different initial values given to control variables. Harvest patterns ranged from periodic removal of all trees greater than 6 inches to the maintenance of a downward-sloping residual diameter distribution with a maximum tree size of 20 inches. Rapera concluded that many locally optimal solutions existed.

The solution procedure that we present is a multi-dimensional version of the method of steepest descent used by Dreyfus and Law (1977, p 102) to numerically solve single-variable discrete-time optimal-control problems. The method of steepest descent is a gradient-based method which finds solutions by setting first derivatives equal to zero. Since these conditions are necessary for any relative maximum or for any other types of stationary solutions, there is no way of distinguishing the absolute maximum. Thus, we never know if the globally optimal solution has been found. Nevertheless, the values and harvest patterns for solutions obtained for different initial values of control variables do not vary widely. As a result, we can make stronger conclusions about uneven-aged management regimes.

GRADIENT METHOD OF NUMERICAL SOLUTION

The solution method seeks to improve PNW by successive approximations of the control variables. Starting with an initial guess of the control variables $\{Y_{ij}^{0}\}$, $i = 0,\ldots,n-1$ and $j = 1,\ldots,m$, we seek formulas for finding a better sequence $\{Y_{ij}^{1}\}$, $i = 0,\ldots,n-1$ and $j = 1,\ldots,m$.

We define $T_i(X_{i1},\ldots,X_{im};\ Y_{i1}^{0},\ldots,Y_{im}^{0})$ as the PNW from period i through period n, starting in period i with state $X_{i1},\ldots,X_{im}$ and using the guessed control sequence. The function T_i satisfies the recurrence relation for periods $i = 0,\ldots,n-1$:

$$T_i = \sum_{j=1}^{m} P_{ij}X_{ij}Y_{ij}^{0} + T_{i+1}(X_{i1} - X_{i1}Y_{i1}^{0} + f_{i1},\ldots,X_{im}$$

$$- X_{im}Y_{im}{}^{0} + f_{im};\ Y_{i+11}{}^{0},\ldots,Y_{i+1m}{}^{0}) \qquad (4)$$

and the boundary condition:

$$T_n = \sum_{j=1}^{m} P_{nj}X_{nj} \qquad (5)$$

We use the stand growth dynamics to compute the state sequence $\{X_{ij}{}^{0}\}$, $i = 1,\ldots,n$ and $j = 1,\ldots,m$, determined by $\{Y_{ij}{}^{0}\}$ and $\{X_{0j}\}$. To improve PNW we need to know how it would behave if we changed $Y_{ij}{}^{0}$ but kept all other control variables fixed. Partial differentiation of (4) with respect to $Y_{ij}{}^{0}$ gives the answer:

$$\left.\frac{dT_i}{dY_{ij}{}^{0}}\right|_{0} = P_{ij}X_{ij}\Big|_{0} - X_{ij}\left.\frac{dT_{i+1}}{dX_{i+1j}}\right|_{0} + \sum_{k=1}^{m}\left.\frac{dT_{i+1}}{dX_{i+1k}}\right|_{0}\left.\frac{df_{ik}}{dY_{ij}{}^{0}}\right|_{0} \qquad (6)$$

where $\Big|_{0}$ means expressions are evaluated in terms of the particular sequence $\{Y_{ij}{}^{0},X_{ij}{}^{0}\}$. We can compute (6) if we know dT_{i+1}/dX_{i+1k}, $k = 1,\ldots,m$. Hence we take the partial derivative of (4) with respect to X_{ij}:

$$\left.\frac{dT_i}{dX_{ij}}\right|_{0} = P_{ij}Y_{ij}{}^{0} + (1 - Y_{ij}{}^{0})\left.\frac{dT_{i+1}}{dX_{i+1j}}\right|_{0} + \sum_{k=1}^{m}\left.\frac{dT_{i+1}}{dX_{i+1k}}\right|_{0}\left.\frac{df_{ik}}{dX_{ij}}\right|_{0} \qquad (7)$$

Partial differentiation of the boundary condition (5) yields:

$$\left.\frac{dT_n}{dX_{nj}}\right|_0 = P_{nj} \tag{8}$$

This allows us to compute partial derivatives (7) for all periods i working backward from period n. Equations (6), (7), and (8) allow the computation of dT_i/dY_{ij}^0, $j = 1,\dots,m$, at each period of the guessed control sequence. We use this information to compute the change in each control variable δY_{ij}^0 so that

$$Y_{ij}^1 = Y_{ij}^0 + \delta Y_{ij}^0 \tag{9}$$

for $i = 0,\dots,n-1$ and $j = 1,\dots,m$.

Suppose we let the change in each control variable Y_{ij}^0 be proportional to the size of the partial derivative dT_i/dY_{ij}^0:

$$\delta Y_{ij}^0 = p \left.\frac{dT_i}{dY_{ij}^0}\right|_0 \tag{10}$$

for $i = 0,\dots,n-1$ and $j = 1,\dots,m$. The resulting change in PNW can be approximated by

$$\delta PNW \cong \sum_{i=0}^{n-1} \sum_{j=1}^{m} \left.\frac{dT_i}{dY_{ij}^0}\right|_0 \delta Y_{ij}^0 = p \sum_{i=0}^{n-1} \sum_{j=1}^{m} \left(\left.\frac{dT_i}{dY_{ij}^0}\right|_0\right)^2 \tag{11}$$

For a specified improvement $\overline{\delta PNW}$ we can use (11) to compute p:

$$p = \frac{\overline{\delta PNW}}{\sum_{i=0}^{n-1} \sum_{j=1}^{m} \left(\frac{dT_i}{dY_{ij}^{0}} \Big|_{0} \right)^2} \tag{12}$$

and use equations (10) and (9) to compute the new decision sequence $\{Y_{ij}^{1}\}$. Whenever the new values of the control variables are outside the bounds given by equation (3), we set their values equal to the nearest bound. The new decision sequence is used in the dynamical equation (2) to compute the state sequence $\{X_{ij}^{1}\}$, and if $\{Y_{ij}^{1}, X_{ij}^{1}\}$ improve PNW by more than a predetermined limit, they are used as the starting point for the next iteration. If they do not improve PNW, we replace $\overline{\delta PNW}$ by $\overline{\delta PNW}/2$ and compute a new solution, repeating the process until PNW improves by the desired amount. The process terminates when we seek an increase $\overline{\delta PNW}$ that is smaller than a predetermined limit and even this does not give the desired improvement in PNW. No improvement in PNW is possible when all of the following conditions hold:

$$dT_i/dY_{ij} = 0.0 \quad \text{for } 0.0 < Y_{ij} < 1.0 \tag{13}$$

$$dT_i/dY_{ij} \leq 0.0 \quad \text{for } Y_{ij} = 0.0 \tag{14}$$

$$dT_i/dY_{ij} \geq 1.0 \quad \text{for } Y_{ij} = 1.0 \tag{15}$$

A control sequence $\{Y_{ij}\}$ which satisfies these conditions also satisfies Kuhn-Tucker necessary conditions for constrained optimization, which are satisfied by any relative minima, maxima, or other stationary solution. Thus we must compare the values of solutions that are obtained from several different starting guesses for the control sequence to estimate the globally optimal solution.

DEMONSTRATING THE GRADIENT METHOD

Stand Growth Projection. - We employ a whole-stand/diameter-class simulator for northern hardwoods reported by Ek (1974) and modified by Adams and Ek (1974). The simulator has been incorporated into static-optimization techniques by Adams and Ek (1974) and Adams (1976). The simulator computes f_{ij}, the change in the number of trees per acre in a 2-inch diameter class j during a 5-year growth period i as a function of the residual stand structure at the start of the growth period. This change is equivalent to upgrowth from diameter class j - 1 less upgrowth into diameter class j + 1 less mortality from diameter class j. Models for diameter class upgrowth and mortality are a function of site index, diameter class midpoint, number of trees in the class, and total stand basal area and number of trees. A separate ingrowth model is used to predict upgrowth into the smallest diameter class (6 inches).

The largest diameter class is 20 inches. Due to limits on the growth data used to construct the simulator, Adams and Ek (1974) assumed that only trees 18 inches in diameter and smaller were retained in the residual stand. Trees were allowed to grow up to 20 inches during a 5-year growth period, but were cut at the beginning of the next period. To make our results comparable to the solutions given by Adams (1976), we introduce the same constraint by setting the value of the control variable for the 20-inch diameter class equal to 1.0 in each period. The solution procedure is otherwise unaffected.

Optimal Management Regimes. - To demonstrate the gradient method we develop three management regimes for a site index 60 mixed hardwood stand. These are developed to maximize PNW from harvests taken in 5-year intervals over a 150-year planning horizon with a real interest rate of 4 percent and with three different functional relationships between tree diameter and stumpage value: (1) a sawlog objective where price per cubic foot is constant to 12 inches and increases rapidly to 20 inches, (2) a cordwood and sawlog objective where price per cubic foot increases at a constant rate with increasing tree size, and (3) no market premium for larger trees (Table 1). The initial diameter distribution is the profit-maximizing pre-harvest equilibrium diameter distribution developed for a sawlog objective and corresponding stumpage value function with a 4 percent real interest rate presented by Adams (1976). Net cubic foot volume per tree is from the cordwood volume

TABLE 1. Initial diameter distribution, volume, and stumpage value functions for a northern hardwood stand (site index 60).

Function	Diameter class midpoint (inches)							
	6	8	10	12	14	16	18	20
Initial diameter distribution (trees/acre)	69.9	49.3	37.5	29.7	24.1	10.7	2.1	0.3
Volume (ft^3/tree)	3.9	10.1	17.9	27.4	38.4	51.0	65.1	80.8
Stumpage value, regime 1 (\$/ft^3)	0.01	0.01	0.01	0.01	0.025	0.05	0.08	0.10
Stumpage value, regime 2 (\$/ft^3)	.03	.04	.05	.06	.07	.08	.09	.10
Stumpage value, regime 3 (\$/ft^3)	.05	.05	.05	.05	.05	.05	.05	.05

table given by Adams and Ek (1974).

Management regime 1, developed with a sawlog objective, can be divided into two 75-year blocks on the basis of harvest patterns and resulting sequences of residual diameter distributions (Table 2). All 6-inch and 77 percent of the 8-inch trees are cut at the start. These initial heavy cuts in the smallest diameter classes accelerate the growth of trees in the 10-inch and larger diameter classes. These trees are cut in future periods when they reach the three most valuable diameter classes. A downward-sloping diameter distribution is present at the end of 75 years due to natural regeneration (modeled by the ingrowth function) and low levels of cut in the smallest diameter classes. The sequence of residual diameter distributions shows the management of a pulse over a 75-year period (Figure 1).

During the second 75 years a downward-sloping diameter distribution is maintained by natural ingrowth, mortality, and cuts taken from the 20-inch diameter class. Due to the high value of 20-inch trees, the values of cuts stay between $24.00 and $32.00/acre even though the total number of trees harvested in each period is low.

Management regime 2, where price per cubic foot increases at a constant rate with increasing tree size, removes at the first harvest all trees in diameter classes greater than 6 inches (Table 3). Subsequent harvests take all 10-inch trees and 65 to 95 percent of the 8-inch

TABLE 2. A portion of management regime 1 (PNW = $171.42/acre).

	Diameter class midpoint (inches)								Totals		
Year	6	8	10	12	14	16	18	20	trees/acre	ft²/acre	$/acre
A. Cut trees/acre											
0	69.9	37.8	0.0	0.0	0.0	0.4	0.6	0.3	109.0	29.3	13.6
5	6.4	1.1	.0	.0	.0	2.9	3.5	.8	14.7	13.6	33.4
10	3.5	.0	.0	.0	.0	2.5	3.9	.9	10.8	13.2	34.4
15	.4	.0	.0	.0	.0	2.0	4.3	1.0	7.7	12.7	35.3
20	.0	.0	.0	.0	.0	.5	4.8	1.1	6.4	11.6	34.4
25	.0	.0	.0	.0	.0	.4	5.4	1.1	6.9	12.2	36.7
30	.0	.0	.0	.0	.0	.0	5.7	1.2	6.9	12.9	39.0
35	.0	.0	.0	.0	.0	.0	5.4	1.4	6.8	12.8	39.1
50	2.1	.0	.0	.0	.0	.0	2.9	2.1	7.1	10.3	31.8
75	8.0	.6	.0	.0	.0	.0	.7	2.6	11.9	8.6	24.9
100	4.8	1.0	.0	.0	.0	.0	.7	3.0	9.5	9.3	28.2
145	.3	.8	.0	.0	.0	.0	.1	3.5	4.7	8.3	29.2
B. Residual trees/acre											
0	0.0	11.5	37.5	29.7	24.1	10.3	1.5	0.0	114.6	90.5	75.2
5	3.4	7.3	28.1	29.7	24.1	11.4	1.5	.0	105.5	86.4	76.8
10	8.0	5.6	20.5	27.2	24.2	12.4	1.6	.0	99.5	82.4	78.6
15	14.1	5.8	14.7	23.3	23.1	13.7	1.8	.0	96.5	78.7	80.4
20	19.2	7.4	11.0	19.1	21.8	15.6	1.9	.0	96.0	76.0	83.4
25	23.1	9.7	9.1	15.2	19.4	16.8	2.1	.0	95.4	72.8	84.2
30	26.4	12.1	8.5	12.1	16.6	16.5	2.4	.0	94.6	68.8	81.9
35	29.4	14.5	8.9	10.0	13.8	15.4	2.8	.0	94.8	65.0	77.8
50	37.4	21.6	12.6	8.7	8.5	10.1	4.1	.0	103.0	59.3	65.7
75	25.7	22.8	18.1	13.1	9.4	7.3	4.7	.0	101.1	62.0	63.7
100	23.5	15.8	14.8	13.5	11.5	9.3	5.9	.0	94.3	64.7	75.9
145	39.5	21.4	13.2	10.0	8.6	7.7	6.6	.0	107.0	62.1	72.4

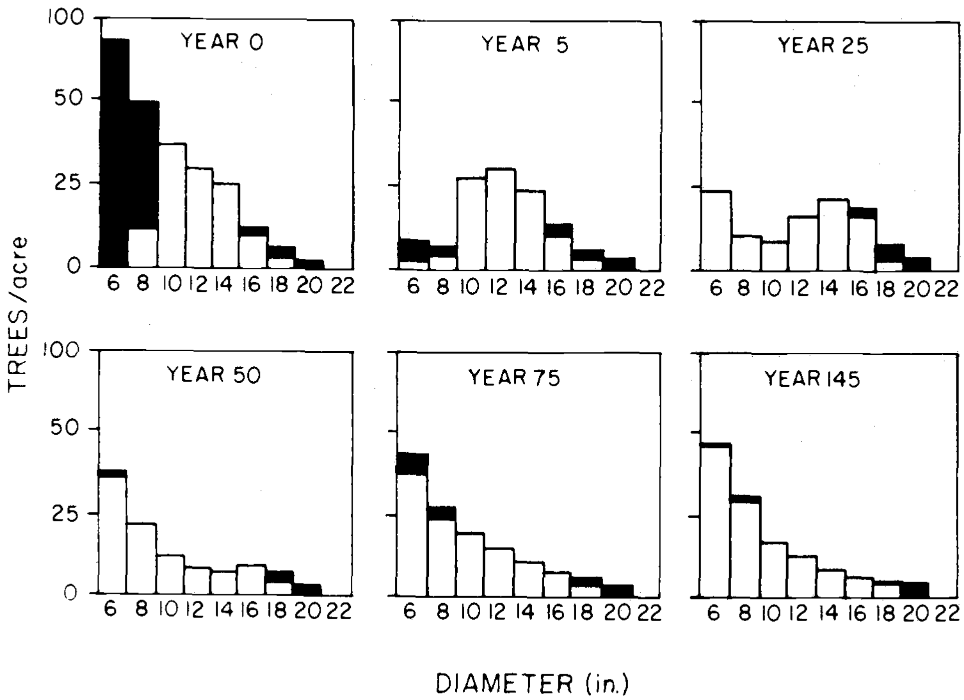

FIGURE 1. Residual diameter distributions at selected points of management regime 1 (shaded regions show the number of trees cut).

TABLE 3. A portion of management regime 2 (PNW = $299.13/acre).

	Diameter class midpoint (inches)								Total		
Year	6	8	10	12	14	16	18	20	trees/acre	ft^2/acre	$/acre
A. Cut trees/acre											
0	0.0	49.3	37.5	29.7	24.1	10.7	2.1	0.3	153.7	106.0	225.6
5	.0	14.9	.0	.0	.0	.0	.0	.0	14.9	5.3	6.3
10	.0	21.7	1.7	.0	.0	.0	.0	.0	23.4	8.5	10.3
15	.0	27.6	3.6	.0	.0	.0	.0	.0	31.2	11.6	14.3
20	.0	32.9	4.7	.0	.0	.0	.0	.0	37.6	14.1	17.5
25	.0	37.2	5.1	.0	.0	.0	.0	.0	42.3	15.8	19.6
30	.0	40.7	4.9	.0	.0	.0	.0	.0	45.6	16.9	20.8
35	.0	43.4	4.5	.0	.0	.0	.0	.0	47.9	17.6	21.5
50	.0	48.9	3.0	.0	.0	.0	.0	.0	51.9	18.7	22.3
75	.0	53.7	1.4	.0	.0	.0	.0	.0	55.1	19.5	22.8
100	.0	55.8	.7	.0	.0	.0	.0	.0	56.5	19.9	23.0
145	.0	57.0	.3	.0	.0	.0	.0	.0	57.3	20.1	23.1
B. Residual trees/acre											
0	69.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	69.9	13.8	8.6
5	116.5	4.2	.0	.0	.0	.0	.0	.0	120.7	24.3	15.7
10	149.1	9.8	.0	.0	.0	.0	.0	.0	158.9	32.7	21.8
15	172.3	13.5	.0	.0	.0	.0	.0	.0	185.8	38.6	26.1
20	189.6	14.9	.0	.0	.0	.0	.0	.0	204.5	42.4	28.8
25	203.2	14.4	.0	.0	.0	.0	.0	.0	217.6	45.0	30.2
30	214.2	13.1	.0	.0	.0	.0	.0	.0	227.3	46.7	31.0
35	223.5	11.5	.0	.0	.0	.0	.0	.0	235.0	47.9	31.5
50	244.0	7.1	.0	.0	.0	.0	.0	.0	251.1	50.4	32.1
75	262.8	3.1	.0	.0	.0	.0	.0	.0	265.9	52.7	32.8
100	271.3	1.5	.0	.0	.0	.0	.0	.0	272.8	53.8	33.1
145	276.3	.7	.0	.0	.0	.0	.0	.0	277.0	54.5	33.4

trees, leaving an increasing number of trees in the 6-inch diameter class and a small number of trees in the 8-inch diameter class.

A similar harvest pattern exists for management regime 3--no market premium for large diameter trees--which specifies that all trees in the 8-inch or greater diameter classes are cut at the start (Table 4). Upgrowth into the 8-inch diameter class is cut in each subsequent period. The residual diameter distributions include increasing numbers of trees in the 6-inch diameter class and no trees in the 8-inch or larger diameter classes.

Imposition of the maximum-tree-size constraint does not affect the solution obtained for management regimes 2 and 3 because the maximum residual tree sizes are 8 and 6 inches, respectively. The impact of the constraint on regime 1 depends on the value per cubic foot assigned to trees greater than 20 inches. If this is the same as the value of a 20-inch tree, the constraint would have no effect, but if it increases at the same rate for trees with increasing diameter, a pulse-transition harvest pattern with a maximum tree size greater than 20 inches would be obtained.

<u>Sensitivity Analysis.</u> - There are two problems connected with the use of the method of steepest descent: (1) the gradient method converges at a very slow rate as a stationary point is approached, and (2) there is no

TABLE 4. A portion of management regime 3 (PNW = $274.39/acre).

	Diameter class midpoint (inches)								Totals		
Year	6	8	10	12	14	16	18	20	trees/acre	ft²/acre	$/acre
A. Cut trees/acre											
0	0.0	49.3	37.5	29.7	24.1	10.7	2.1	0.3	153.7	106.0	181.2
5	.0	19.1	.0	.0	.0	.0	.0	.0	19.1	6.8	10.0
10	.0	29.3	.0	.0	.0	.0	.0	.0	29.3	10.2	14.9
15	.0	36.2	.0	.0	.0	.0	.0	.0	36.2	12.6	18.5
20	.0	41.2	.0	.0	.0	.0	.0	.0	41.2	14.4	21.0
25	2.4	44.8	.0	.0	.0	.0	.0	.0	96.5	16.1	23.4
30	5.3	47.2	.0	.0	.0	.0	.0	.0	52.5	17.5	25.2
35	6.8	48.5	.0	.0	.0	.0	.0	.0	55.3	18.3	26.1
50	6.8	50.2	.0	.0	.0	.0	.0	.0	57.0	18.9	27.0
75	3.5	52.4	.0	.0	.0	.0	.0	.0	55.9	19.0	27.5
100	1.5	54.4	.0	.0	.0	.0	.0	.0	55.9	19.4	28.2
145	.1	56.5	.0	.0	.0	.0	.0	.0	56.6	19.9	29.1
B. Residual trees/acre											
0	69.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	69.9	13.8	14.1
5	116.5	.0	.0	.0	.0	.0	.0	.0	116.5	22.8	23.3
10	151.6	.0	.0	.0	.0	.0	.0	.0	151.6	29.7	30.3
15	178.2	.0	.0	.0	.0	.0	.0	.0	178.2	35.0	35.6
20	198.8	.0	.0	.0	.0	.0	.0	.0	198.8	39.0	39.8
25	212.5	.0	.0	.0	.0	.0	.0	.0	212.5	41.7	42.5
30	220.3	.0	.0	.0	.0	.0	.0	.0	220.3	43.3	44.1
35	225.0	.0	.0	.0	.0	.0	.0	.0	225.0	44.2	45.0
50	233.4	.0	.0	.0	.0	.0	.0	.0	233.4	45.8	46.7
75	247.7	.0	.0	.0	.0	.0	.0	.0	247.7	48.7	49.6
100	260.7	.0	.0	.0	.0	.0	.0	.0	260.7	51.3	52.3
145	273.4	.0	.0	.0	.0	.0	.0	.0	273.4	53.8	54.9

assurance that a stationary point is a global optimum, because the production surface defined by the stand simulator is not convex (see Bullard (1983) for proof of non-convexity). We therefore performed sensitivity analysis to evaluate these drawbacks.

Solutions obtained with the gradient method depend on the tolerance used to terminate the algorithm. For a smaller tolerance, the method terminates closer to a stationary point and execution times are longer.

We terminated the algorithm when an additional iteration improved PNW by less than \$0.001. The range in values of $\overline{\delta PNW}$ examined was bounded by \$128.00 and \$0.1. Reducing the termination criterion to \$0.0001 caused differences of less than one tree per acre per diameter class in residual diameter distributions for the first eight periods of all three management regimes, but greater differences did occur in later periods. Objective function values improved by less than 1 percent.

Execution times for regimes 1, 2, and 3 were 72, 6, and 9 seconds, respectively, on a Control Data Corp. CYBER 73/16. Reducing the tolerance increased execution times by more than 100 percent.

We also analyzed a subset of solutions to each problem using different sets of starting values for the control variables. Two sets of starting values were the boundary conditions (0.0 and 0.99) and each remaining set utilized starting values randomly chosen between 0.0 and

0.99.

Management regime 1 was obtained with starting values set at 0.0. Solutions with different sets of starting values had differences of less than one tree per acre per diameter class in residual diameter distributions for the first eight periods. Markedly different diameter distributions were established and maintained in subsequent periods. The maximum difference between objective function values obtained with different starting values was less than 2 percent. Execution times varied from 200 to 400 seconds. Similar sensitivity results were obtained for management regimes 2 and 3.

Because of the stability of the solutions, we conclude that each of the three management regimes is very close to the globally optimal solution for the first eight periods. The long-term management regimes are dependent on starting values given to control variables because large discount factors make the marginal values of distant-period control variables smaller than the termination criterion, and the algorithm terminates before the values of these control variables reach a stationary point. The slow convergence rate for distant-period control variables shows that many near-optimal long-term management regimes exist but, due to the limits on execution time, prevented us from analyzing long-term stationary solutions.

CONCLUSIONS

Optimal Equilibrium Stand Structures. - Our solutions suggest two types of optimal equilibrium stand structures: (1) a downward-sloping diameter distribution if large value premiums are assigned to trees in larger diameter classes (management regime 1), and (2) a truncated diameter distribution if premiums for larger trees are gradual or absent (management regimes 2 and 3).

A comparison of management regime 1 with an investment-efficient management regime developed by Adams (1976) for the same stumpage value function and interest rate demonstrates that dynamically determined management regimes differ from equilibrium regimes determined with static optimization techniques. Furthermore, the PNW of management regime 1 ($171.42/acre) is 5 percent greater than that of equilibrium harvests taken over the same planning horizon plus the liquidation value of the terminal stand. Thus, a stand-specific management regime developed with a dynamic optimization technique will not converge to the equilibrium determined using a static optimization technique.

The difference in the values of dynamically and statically determined management regimes calls into question the use of static analysis for developing longrun equilibrium management regimes (Adams and Ek 1974, Adams 1976, Martin 1982) and for determining the value of an uneven-aged stand (Chang 1981, Hall 1983). A detailed comparison of the problems solved by dynamic and static optimization techniques is beyond the scope of this paper. A comparison of these problems is necessary, however, so that managers and analysts can select the proper decision-making tools.

The truncated equilibrium diameter distributions suggested by management regimes 2 and 3 result from the shapes of the ingrowth and price functions. The number of trees growing into the 6-inch diameter class is greatest for truncated residual diameter distributions. When the stumpage value function assigns equal or gradually increasing values per cubic foot for trees of increasing diameter, an incentive is given to liquidating the growing stock greater than 6 inches and capturing high levels of periodic ingrowth.

Management regimes 1, 2, and 3 are sensitive to the forms of the growth equations and parameters used in the equations. In particular, the maintenance of a truncated diameter distribution is dependent on the predicted high level of periodic ingrowth. Thus, these solutions should be viewed with caution until the accuracy of the growth model projections is validated.

<u>Expanding the Solution Algorithm and Problem Statement.</u> - The gradient method converges to the globally optimal harvest in the first period and beyond, though it does not

produce stable solutions for the long-term management regime. A procedure has been developed (Haight 1985) which takes advantage of the stability of the first-period solution to develop long-term managment regimes that are stable and independent of starting values given to control variables.

The problem statement, which we used to determine the optimal sequence of diameter distributions for a 5-year cutting cycle, can be expanded to consider both cutting cycle length and species composition problems. Sequences of diameter distributions with cutting cycles that are multiples of the simulator projection interval can be analyzed by setting control variables equal to zero in the periods between harvests. Sequences of diameter distributions for species classes can be developed by defining state and control variables and growth equations for species and diameter classes.

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A COMPARISON OF DYNAMIC AND STATIC ECONOMIC MODELS OF UNEVEN-AGED STAND MANAGEMENT

by

Robert G. Haight

ABSTRACT

Numerical techniques have been used to compute the discrete-time sequence of residual diameter distributions that maximize the present net worth (PNW) of harvestable volume from an uneven-aged stand. Results contradicted optimal steady-state diameter distributions determined with static analysis. In this paper, optimality conditions for solutions to dynamic and static harvesting problems are established. Comparison of these conditions shows that for a stand with any diameter distribution: (1) the optimal transition regime does not converge to the steady state that maximizes land expectation value (LEV) using the Faustmann equation; (2) the PNW of the optimal transition and steady-state regime is greater than the PNW of the statically determined steady-state regime; and (3) the optimal steady-state regime is invariant. A refined version of a recently published dynamic optimization algorithm is provided and demonstrated with a whole-stand/diameter-class simulator for hardwood stands in Wisconsin. Optimal management regimes are computed for comparison with a static equilibrium management regime and for analysis of the effect of cutting cycle length on

harvest pattern and PNW.

A COMPARISON OF DYNAMIC AND STATIC ECONOMIC MODELS OF UNEVEN-AGED STAND MANAGEMENT

INTRODUCTION

Harvesting decisions for an uneven-aged stand are dynamic because stand structure changes over time as a result of tree removals. Dynamic optimization has not been used to determine harvest schedules because of the large number of decision variables involved. Instead, studies in uneven-aged management have concentrated on the smaller problem of determining optimal equilibrium management regimes.

Equilibrium management regimes have been developed with static optimization, which maximizes the present value of equilibrium harvests minus the value of the residual growing stock. Adams (1976) developed investment-efficient diameter distributions for a fixed cutting cycle and Martin (1982) extended the analysis to consider alternative cutting cycles. Chang (1981) and Hall (1983) derived marginal production rules for optimal equilibrium growing stock levels and cutting cycle lengths. Chang (1981) demonstrated a method for solving these equations.

Solutions obtained with static analysis have been used to develop transition strategies and to evaluate alternative management systems. Adams and Ek (1974) demonstrated a method for determining optimal transition regimes that converge to steady-state diameter distributions in three periods or less. Chang (1981) and Hall (1983) showed that the objective function for determining the optimal equilibrium growing stock level and cutting cycle is equivalent to maximizing land expectation value (LEV) using the Faustmann (1849) equation. Chang (1981) then used LEV as a measure for comparing the profitability of uneven-aged and even-aged management. Martin (1982) measured the profitability of uneven-aged management on different sites using LEV.

In a departure from static analysis, Haight and others (1985) used a dynamic optimization procedure to determine the optimal sequence of diameter distributions and selection harvests for an existing stand. They reported a 150-year management regime for a stand with a pre-harvest investment-efficient diameter distribution. This regime exhibited a pulse-like transition away from the steady-state diameter distribution and had a higher present net worth (PNW) than the equilibrium management regime. These results called into question the use of static analysis for developing optimal steady-state stand structures and the use of LEV for evaluating the profitability of uneven-aged management.

In the first three parts of this paper, I resolve the dynamic and static optimization problems for the diameter distribution model of uneven-aged management. In parts 1 and 2, I formulate the problems and derive marginal production rules that are satisfied by optimal solutions. In part 3, I compare the marginal production rules and steady-state solutions to each problem and discuss how solutions to each problem should be used.

In part 4, I use the solution procedure given by Haight and others (1985) to develop an algorithm that provides stable solutions for transition and equilibrium management regimes. In the next two parts, I demonstrate the algorithm by computing optimal management regimes for comparison with a statically determined equilibrium regime and for analysis of the effect of increasing the cutting cycle length. Part 7 provides a summary and discussion.

PROBLEM 1: DYNAMIC OPTIMIZATION

Consider the problem of determining the optimal sequence of selection harvests for an uneven-aged stand:

$$\max_{\{S_{ij}\}} PNW = \sum_{i=0}^{\infty} \sum_{j=1}^{m} \frac{(X_{ij} - S_{ij})P_j}{(1 + r)^{it}} \tag{1.1}$$

subject to, for all $i = 0,\ldots,\infty$ and $j = 1,\ldots,m$,

$$X_{i+1j} = S_{ij} + f_{ij}(S_{i1},\ldots,S_{im}) \tag{1.2}$$

$$S_{ij} \geq 0 \tag{1.3}$$

$$X_{ij} - S_{ij} \geq 0 \tag{1.4}$$

$$X_{0j} \tag{1.5}$$

where

X_{ij} = a state variable representing the number of trees per acre in diameter class j at the beginning of period i before cutting

S_{ij} = a control variable representing the number of trees per acre in diameter class j at the beginning of period i after cutting

P_j = the net price per tree in diameter class j

f_{ij} = a continuous nonlinear function with continuous partial derivatives representing the change in number of trees per acre in diameter class j during period i

m = the number of diameter classes

r = the discount rate

t = the number of years in a time period.

The objective function, equation (1.1), is formulated to seek the set $\{S_{ij}\}$ representing the residual number of trees in each diameter class and time period that maximizes the PNW of harvests over an infinite planning horizon. This formulation is equivalent to the PNW maximization problem presented and numerically solved by Haight and others (1985), in which the control variable set represented the percentage of the number of trees in each diameter class that were cut at the beginning of each time period. I use residual numbers of trees as control variables in the dynamic problem presented here to facilitate comparison with the static optimization problem.

The motion equation set (1.2) represents the diameter-class growth dynamics. The change in number of trees per acre f_{ij} is the sum of functions for ingrowth from smaller diameter classes, upgrowth into larger diameter classes, and mortality. Each of these components is a continuous and differentiable nonlinear function of the residual diameter distribution $S_{i1},...,S_{im}$. Uneven-aged stand simulators which have this structure have been developed for Wisconsin hardwood stands (Adams and Ek 1974) and for Arizona ponderosa pine stands (Hann 1980).

Equation sets (1.3) and (1.4) restrict the residual numbers of trees and harvest levels to be non-negative. The initial stand structure is given by equation set (1.5).

Knapp (1983) derived Kuhn-Tucker conditions which are satisfied by optimal transition and steady-state solutions to a general dynamic optimization problem. I apply his analysis to the dynamic uneven-aged management problem given by equation sets (1.1) to (1.5) to obtain marginal production rules that are satisfied by optimal transition and steady-state management regimes.

Kuhn-Tucker Conditions. - I define an optimal value function $T_i^*(X_{i1},...,X_{im})$ as the discounted value of the optimal sequence of selection harvests in periods i to ∞

when starting period i in state $X_{i1},\ldots,X_{im}$. Using the principle of optimality (Dreyfus and Law 1977, p.101), T_i^* satisfies the functional equation:

$$T_i^*(X_{i1},\ldots,X_{im}) = \max_{\{S_{i1},\ldots,S_{im}\}} \left[\sum_{j=1}^{m} \frac{(X_{ij} - S_{ij})P_j}{(1 + r)^{it}} + T_{i+1}^*(S_{i1} + f_{i1},\ldots,S_{im} + f_{im}) \right] \qquad (1.6)$$

subject to

$$S_{ij} \geq 0 \quad \text{and} \quad X_{ij} - S_{ij} \geq 0, \quad j = 1,\ldots,m.$$

For $S_{i1}^*,\ldots,S_{im}^*$ to solve the constrained maximization problem (1.6), I can define Lagrange multipliers $u_{i1}^*,\ldots,u_{im}^*$ and $v_{i1}^*,\ldots,v_{im}^*$ that satisfy the following Kuhn-Tucker conditions for $j = 1,\ldots,m$:

$$\frac{-P_j}{(1 + r)^{it}} + \frac{dT_{i+1}^*}{dX_{i+1j}} + \sum_{k=1}^{m} \frac{dT_{i+1}^*}{dX_{i+1k}} \frac{df_{ik}}{dS_{ij}} + u_{ij}^* - v_{ij}^* = 0 \qquad (1.7a)$$

$$u_{ij}^* S_{ij}^* = 0 \quad \text{and} \quad v_{ij}^*(X_{ij} - S_{ij}^*) = 0 \qquad (1.7b)$$

$$S_{ij}^* \geq 0 \quad \text{and} \quad X_{ij} - S_{ij}^* \geq 0 \qquad (1.7c)$$

$$u_{ij}^* \geq 0 \quad \text{and} \quad v_{ij}^* \geq 0 \qquad (1.7d)$$

The dependence of the optimal control variable values $S_{i1}^*,\ldots,S_{im}^*$ on the values of the state variables $X_{i1},\ldots,X_{im}$ is recognized with the following equation set:

$$\frac{dT_i^*}{dX_{ij}} = \frac{P_j}{(1 + r)^{it}} + v_{ij}^* \qquad j=1,\ldots,m \qquad (1.7e)$$

The optimal value function T_i^* and conditions (1.7a) to (1.7e) are defined for each time period $i = 0,\ldots,\infty$. An optimal management regime must satisfy this complete set of interperiodic conditions.

Marginal Production Rules. - To more easily derive and interpret the marginal production rules I first define functional relationships for the Lagrange multipliers u_{ij}^* and v_{ij}^*. Examination of equation sets (1.7a), (1.7b) and (1.7c) yields the following conditions for the Lagrange multipliers, $j = 1,\ldots,m$:

$$u_{ij}^* = \begin{cases} 0 & \text{for} \quad S_{ij}^* > 0 \\ -dT_i^*/dS_{ij} & \text{for} \quad S_{ij}^* = 0 \end{cases} \qquad (1.8a)$$

$$v_{ij}^* = \begin{cases} 0 & \text{for} \quad X_{ij} - S_{ij}^* > 0 \\ dT_i^*/dS_{ij} & \text{for} \quad X_{ij} - S_{ij}^* = 0 \end{cases} \qquad (1.8b)$$

Each Lagrange multiplier is a shadow price, which measures the change in the value of the objective function corresponding to a small change in the constraint. When $S_{ij}^* = 0$, the multiplier u_{ij}^* is the increase in benefits that would result from having one less unit of growing stock in diameter class j in period i. Similarly, when $X_{ij} - S_{ij}^* = 0$, the multiplier v_{ij}^* is the increase in benefits due to an increase in the residual number of trees in diameter class j period i. When a constraint is not binding in the optimal solution, the corresponding shadow price is zero.

Equation (1.7e) gives the value of an additional tree in diameter class j at the beginning of period i, dT_i^*/dX_{ij}. When $X_{ij} - S_{ij}^* > 0$, the marginal tree is cut with value $P_j/(1 + r)^{it}$. When $X_{ij} - S_{ij}^* = 0$, the marginal value includes the shadow price for an additional unit of growing stock in diameter class j at the beginning of period i.

With these definitions, condition (1.7a) can be rearranged to obtain marginal production rules that are defined over the set of diameter classes, j = 1,...,m, in period i:

$$\frac{P_j}{(1 + r)^{it}} - u_{ij}^* + v_{ij}^* = \sum_{k=1}^{M} \left(\frac{P_k}{(1 + r)^{(i+1)t}} + v_{i+1k}^* \right) \frac{df_{ik}}{dS_{ij}}$$

$$+ \frac{P_j}{(1 + r)^{(i+1)t}} + v_{i+1j}^* \qquad (1.9)$$

The left-hand side of equation (1.9) represents the opportunity cost of leaving an additional tree in diameter class j. When $0 < S_{ij}^* < X_{ij}$, the marginal input cost equals the discounted tree price. When $S_{ij}^* - X_{ij} = 0$, marginal cost equals the sum of the discounted tree price and the shadow price (discounted) of an additional tree. When $S_{ij}^* = 0$, marginal cost equals the discounted sum of the tree price and a price reduction due to an additional tree. The price reduction means that we would prefer to harvest more trees to improve the objective function value.

The right-hand side of equation (1.9) gives the marginal revenue product of leaving an additional tree in diameter class j. This equals the discounted value growth resulting from the additional tree summed over all diameter classes plus the discounted value of the tree. The value of the growth in period i + 1 is the tree price if $X_{i+1k} - S_{i+1k}^{*} > 0$. If $X_{i+1k} - S_{i+1k}^{*} = 0$, the value of the marginal growth includes v_{i+1k}^{*}, the shadow price for an additional tree in diameter class k.

The rule for resource production states that the residual number of trees in diameter class j period i is increased until the marginal input cost of one more tree equals the marginal revenue product associated with leaving the tree. An optimal management regime for an existing stand has satisfied this rule simultaneously over all diameter classes and all periods in the planning horizon. The marginal production rules and definitions of the shadow prices are summarized in the Appendix.

PROBLEM 2: STATIC OPTIMIZATION

To derive the static optimization problem, I modify the original dynamic formulation given by equations (1.1) to (1.5) by including an equilibrium constraint:

$$\max_{\{S_{ij}\}} PNW = \sum_{i=0}^{\infty} \sum_{j=1}^{m} \frac{(X_{ij} - S_{ij})P_j}{(1 + r)^{it}} \tag{2.1}$$

subject to, for all $i = 0,\ldots,\infty$ and $j = 1,\ldots,m$,

$$X_{i+1j} = S_{ij} + f_{ij}(S_{i1},\ldots,S_{im}) \tag{2.2}$$

$$S_{ij} \geq 0 \tag{2.3}$$

$$X_{ij} - S_{ij} \geq 0 \tag{2.4}$$

$$X_{0j} \tag{2.5}$$

$$X_{i+1j} = X_{ij} \quad i = 1,\ldots,\infty \quad \text{and } j = 1,\ldots,m \tag{2.6}$$

Constraint (2.6) requires that an equilibrium harvest regime be achieved after one harvest from an existing stand. With this constraint I can rearrange the objective function (2.1) to obtain

$$\max_{\{S_{ij}\}} PNW = \sum_{j=1}^{m} (X_{0j} - S_{0j})P_j + \sum_{j=1}^{m} \frac{(X_{1j} - S_{0j})P_j}{(1 + r)^t - 1} \tag{2.7}$$

Substituting the growth dynamics (2.2) into (2.7) results in the static optimization problem:

$$\max_{\{S_{0j}\}} PNW = \sum_{j=1}^{m} (X_{0j} - S_{0j})P_j + \sum_{j=1}^{m} \frac{f_{0j}(S_{01},\ldots,S_{0M})P_j}{(1 + r)^t - 1} \tag{2.8}$$

subject to, for $j = 1,\ldots,m$,

$$S_{0j}(1 + r)^t/[(1 + r)^t - 1] \geq 0 \quad (2.9)$$

$$f_{0j}(S_{01},\ldots,S_{0m})/[(1 + r)^t - 1] \geq 0 \quad (2.10)$$

$$X_{0j} - S_{0j} \geq 0 \quad (2.11)$$

$$X_{0j} \quad (2.12)$$

The objective function (2.8) seeks a steady-state diameter distribution $S_{01},\ldots,S_{0m}$ that maximizes the value of the initial harvest plus the present value of equilibrium harvests in perpetuity. This is equivalent to the forest value objective function given by Chang (1981), except that I consider diameter class growth dynamics.

In constrast to Chang's (1981) formulation, three sets of constraints are needed to determine the optimal steady-state management regime. Constraint set (2.9) requires a non-negative residual stocking level and constraint sets (2.10) and (2.11) require non-negative harvest levels. Constraint sets (2.9) and (2.10) are multiplied by the constants $(1 + r)^t/[(1 + r)^t - 1]$ and $1/[(1 + r)^t - 1]$ to simplify the derivation and interpretation of marginal production rules. The initial diameter distribution is given in equation set (2.12).

When the initial harvest constraints (equation set (2.11)) are not binding and the value of the initial stand structure is ignored, the problem is equivalent to maximizing LEV. Solutions are investment-efficient diameter distributions that have been developed by Adams (1976) and Martin (1982).

<u>Kuhn-Tucker</u> <u>Conditions.</u> - For $S_{01}^*, \ldots, S_{0m}^*$ to solve the constrained maximization problem (2.8) to (2.12), I can define Lagrange multipliers $a_1^*, \ldots, a_m^*$, $b_1^*, \ldots, b_m^*$ and $c_1^*, \ldots, c_m^*$ that satisfy the following Kuhn-Tucker conditions, for j = 1,...,m:

$$-P_j + \sum_{k=1}^{m} \frac{(P_k + b_k^*)}{(1+r)^t - 1} \frac{df_{0k}}{dS_{0j}} + \frac{a_j^*(1+r)^t}{(1+r)^t - 1} - c_j^* = 0 \qquad (2.13a)$$

$$a_j^* S_{0j}^* = 0, \quad b_j^* f_{0j} = 0, \quad \text{and} \quad c_j^*(X_{0j} - S_{0j}^*) = 0 \qquad (2.13b)$$

$$S_{0j}^* \geq 0, \quad f_{0j} \geq 0, \quad \text{and} \quad X_{0j} - S_{0j}^* \geq 0 \qquad (2.13c)$$

$$a_j^* \geq 0, \quad b_j^* \geq 0, \quad \text{and} \quad c_j^* \geq 0 \qquad (2.13d)$$

<u>Marginal</u> <u>Production</u> <u>Rules.</u> - The Lagrange multipliers a_j^*, b_j^* and c_j^* are shadow prices, which measure the change in value of the objective function corresponding to a change in the associated constraint. When $S_{0j}^* = 0$, $a_j^*(1 + r)^t/[(1 + r)^t - 1]$ is the increase in PNW resulting from a decrease in residual stocking in diameter class j. When $f_{0j} = 0$, $b_j^*/[(1 + r)^t - 1]$ is the increase in PNW due to an increase in the number of trees in

diameter class j in the equilibrium stand structure. Finally, when $X_{0j} - S_{0j} = 0$, c_j^* is the increase in PNW due to an increase in residual stocking in diameter class j. The shadow prices a_j^* and b_j^* must be multiplied by the scaling factors used in constraints (2.9) and (2.10) to obtain these definitions. Whenever a constraint is nonbinding in the optimal solution, the corresponding shadow price is zero.

Rearranging (2.13a) gives the marginal production rules for an optimal equilibrium diameter distribution, j = 1,...,m:

$$P_j - a_j^* + c_j^* = \sum_{k=1}^{m} \frac{(P_k + b_k^*)}{(1 + r)^t} \frac{df_{0k}}{dS_{0j}} + \frac{P_j}{(1 + r)^t} + \frac{c_j^*}{(1 + r)^t} \qquad (2.14)$$

The marginal input cost of leaving an additional tree in diameter class j is the tree price. Marginal input cost is reduced by a_j^* if $S_{0j}^* = 0$ and is increased by c_j^* if $X_{0j} - S_{0j}^* = 0$. Marginal revenue product equals the discounted value growth that results from the additional tree in diameter class j plus the discounted value of the tree. Prices of the growing stock are increased or reduced according to whether the growing stock and harvest constraints are binding.

The marginal production rule stated by equation set (2.14) is to increase the number of trees in diameter class j until the marginal input cost of one more tree equals the marginal revenue product associated with leaving the tree. An optimal steady-state diameter distribution satisfies this rule simultaneously over all diameter classes. Production rules and shadow price definitions are summarized in the Appendix.

PROBLEM COMPARISON

The dynamic and static optimization problems both seek to maximize PNW, but the static problem includes an equilibrium harvest constraint. For any initial stand, problem 1 seeks the optimal sequence of diameter distributions and selection harvests over an extended planning horizon, while problem 2 seeks the optimal equilibrium diameter distribution that can be established with one harvest from the stand. Although problems 1 and 2 have different constraint sets, the set of feasible solutions to problem 2 is a subset of the feasible solutions to problem 1.

Because the problems have different constraint sets, they have differences in marginal production rules. Problem 1 defines interperiodic production rules for both transition and equilibrium management regimes, while problem 2 defines production rules for equilibrium management regimes only. Thus, only the production rules for steady-state stand structures can be compared. The conditions for the equilibrium stand structures are given in the Appendix. I make two comparisons. The first comparison is made with the assumption that the initial diameter distribution is arbitrarily large. In the second comparison this assumption is dropped.

Comparison 1. - Will an investment-efficient diameter distribution satisfy the marginal production rules for a dynamic equilibrium stand structure? An investment-efficient diameter distribution is found by solving problem 2 with an arbitrarily large initial stand structure. Thus, $c_j^* = 0$, $j = 1,...,m$. In this case, the marginal input cost and marginal revenue product of the conditions for the static equilibrium do not contain Lagrange multipliers for binding non-negative harvest constraints. As a result, whenever an equilibrium harvest regime found by solving the static optimization problem contains no harvest in diameter class k, so that $b_k^* = 0$, the management regime will not satisfy the conditions for a dynamic equilibrium. The converse of this is also true.

Suppose an investment-efficient diameter distribution contains positive harvest levels in all diameter classes so that $b_k^* = 0$, $k = 1,...,m$, and all trees growing into the largest diameter classes are cut so that $a_m^* > 0$. In this case the rules for static and dynamic equilibrium production are equivalent. The converse of this is also true.

How often will investment-efficient diameter distributions contain positive harvests in all diameter classes? Getz (1980) investigated the structure of maximum sustained-yield harvest policies for age-structured populations with linear growth dynamics and arbitrary stock-recruitment functions. He showed that, at most, two age classes are harvested, the younger partially and the older completely. The remaining age classes that contain individuals are not harvested. The structure of the Adams and Ek (1974) simulator is similar to the age-class population model analyzed by Getz (1980), except that movement of trees to the next larger diameter class is predicted with a nonlinear function. Nevertheless, dynamic equilibria obtained by numerically solving problem 1 using the Adams and Ek (1974) simulator exhibit bimodal harvest policies. If optimal equilibrium harvest policies for diameter class simulators with nonlinear growth dynamics are always bimodal, dynamic and static equilibrium regimes will be equivalent only when the residual diameter distributions contain trees in the smallest diameter class.

These results restrict the use of statically determined steady-state diameter distributions. Adams (1976) and Martin (1982) solve problem 2 with an assumption that the first period harvest constraints are not binding. They call their solutions investment-efficient diameter distributions; these are used as goals in transition management. Because investment-efficient diameter distributions do not satisfy marginal production rules for dynamic equilibrium stand structures, they should not be used as goals for long-term management. They should be used only when the objective is to convert the stand to an equilibrium structure in one harvest.

Chang (1981) used static analysis to determine the optimal steady-state growing stock and cutting cycle without considering diameter class growth dynamics. The management objective was to maximize forest value, which he defined as the PNW of harvests when converting an existing stand to an equilibrium growing stock-level in one cut. This objective is equivalent to maximizing LEV with the Faustmann equation where the opportunity cost on the residual growing stock represents the regeneration costs of the stand. As a result, Chang (1981) concluded that the forest value model is the only correct method of determining equilibrium growing stock and cutting cycle.

The static optimization problem that I present is equivalent to Chang's (1981) forest value model, except that it seeks the optimal equilibrium stand structure and includes constraints on the diameter-class growth dynamics and bounds on the residual diameter distribution. Investment-efficient diameter distributions that solve problem 2 can also be viewed as maximizing LEV using the Faustmann equation. However, investment-efficient management regimes do not maximize the present value of returns to the initial growing stock. Thus, maximizing

LEV using the Faustmann equation is not the correct criterion for determining optimal steady-state stand structures.

Why does the dynamically determined harvest regime converge to a steady state that does not maximize LEV? An intuitive explanation can be made by viewing an optimal management regime as having transition and equilibrium phases. Since the equilibrium phase takes place later in the planning horizon its value is discounted more than the value of the transition phase. As a result, it is possible to convert a stand to an equilibrium phase that has less than maximum LEV with a transition regime. The value of the transiton regime would more than offset the discounted losses during the equilibrium phase.

Comparison 2. - Here we make no assumptions about the form of the initial stand structure. Does the initial stand structure affect the form of the equilibrium found by solving problems 1 or 2? The rules for dynamic equilibrium production do not contain any conditions regarding the initial stand structure. Thus, under the assumption that the stumpage price function and interest rate are constant over time, optimal transition strategies for different stands will converge to the same equilibrium.

In constrast, the equilibrium which is found by solving the static-optimization problem does depend on the initial stand structure since the objective is to maximize PNW of harvests when converting to an equilibrium structure in one cut.

For a given stumpage value function and interest rate, is there an initial stand structure that results in the same equilibrium solution to both problems 1 and 2? Suppose the initial stand has the pre-harvest dynamic equilibrium structure. Equating the Lagrange multipliers for the rules of static and dynamic production shows that the optimal solution to problem 2 is the dynamic-equilibrium harvest regime. For any other initial stand structure the management regime found by solving problem 2 will not be the same as the optimal transition and equilibrium regime found by solving problem 1.

This result changes the method for measuring the profitability of uneven-aged management. Both dynamic and static optimization problems can be used to develop management regimes for an existing stand. The steady-state regime found by solving problem 2 will not, except in special cases, be the optimal solution to problem 1. Therefore, the optimal transition and steady-state management regime will have a PNW greater than the PNW of the optimal steady-state management regime which solves problem 2. This makes economic sense, since the dynamic formulation is not constrained to achieve a steady state in any future time period. Thus, the profitability of uneven-aged management should be measured by the PNW of the optimal transition and equilibrium management regime found by solving problem 1.

SOLUTION PROCEDURE FOR DYNAMIC OPTIMIZATION

In the analysis presented above, the dynamic optimization formulation and marginal production rules for optimal transition and equilibrium harvesting were used to point out the limitations of investment-efficient stand structures which solve the traditional static optimization problem. In this section we develop an algorithm for solving the dynamic harvesting problem that improves the unconstrained nonlinear programming approach used by Haight and others (1985).

Haight and others (1985) used a gradient-based procedure called the method of steepest descent. The algorithm starts with an initial guess of the control variable values and seeks to improve the objective function value by successive approximations of the control variables. Each approximation is found by moving in the direction of the gradient, and the process terminates when improvements are less than a set tolerance. The solution obtained at termination is an approximation to a stationary point that satisfies Kuhn-Tucker conditions for optimality.

Haight and others (1985) developed optimal management regimes with 5-year cutting cycles over a 150-year time horizon, with the assumption that all remaining trees were harvested at the end of the time horizon. For time horizons 150 years or longer and a 4 percent real discount rate, the discounted value of the terminal diameter distribution was so small that it had no effect on the determination of the control variable values in earlier periods.

The discount factors for distant-period harvests did affect the ability of the gradient method to reach stationary solutions. Because the marginal values of distant-period control variables were much smaller than the termination criterion, the algorithm terminated before reaching a stationary point. As a result, final solutions for distant-period control variables were sensitive to values initially given to the control variables. Nevertheless, the gradient method produced stable solutions for the first several periods.

In order to obtain stationary solutions for transition and equilibrium management regimes, I have developed an algorithm that takes advantage of the stability of the first-period solution and the sequential nature of the problem. The method uses repeated application of the gradient method described by Haight and others (1985).

In equation (1.6), I defined the optimal value function $T_i^*(X_{i1},...,X_{im})$ as the discounted value of the optimal sequence of selection harvests in periods i to ∞ when starting period i in state $X_{i1},...,X_{im}$. Using equation (1.6) we can write the optimal value function for period 0 as follows:

$$T_0^* = \max_{\{S_{01},\ldots,S_{0m}\}} \left[\sum_{j=1}^{m} (X_{0j} - S_{0j})P_j + T_1^*(S_{01} + f_{01},\ldots,S_{0m} + f_{0m}) \right] \quad (4.1)$$

subject to, for j = 1,...,m,

$$S_{0j} \geq 0$$

$$X_{0j} - S_{0j} \geq 0$$

$$X_{0j}$$

The gradient method can be used to find a stationary solution $S_{01}^*,\ldots,S_{0m}^*$ to the constrained maximization problem defined by (4.1) for a finite-period time horizon (n) 150 years or longer. Using $S_{01}^*,\ldots,S_{0m}^*$ we can write the optimal value function for period 1:

$$T_1^* = \max_{\{S_{11},\ldots,S_{1m}\}} \left[\sum_{j=1}^{m} \frac{(X_{1j} - S_{1j})P_j}{(1 + r)^t} + T_2^*(S_{11} + f_{11},\ldots,S_{1m} + f_{1m}) \right] \quad (4.2)$$

subject to, for j = 1,...,m,

$$S_{1j} \geq 0$$

$$X_{1j} - S_{1j} \geq 0$$

$$X_{1j} = S_{0j}^* + f_{0j}(S_{01}^*, \ldots, S_{0m}^*)$$

A stationary solution $S_{11}^*, \ldots, S_{1m}^*$ to the constrained optimization problem defined by (4.2) can be obtained by multiplying each side of the objective function by the constant $(1 + r)^t$ and applying the gradient method for an n-period time horizon. Multiplication of the objective function by $(1 + r)^t$ shifts the discount factors back one period and allows the use of the same termination criterion. Repeating this process for a predetermined number of periods (L) gives a stationary solution for the L-period management regime.

The algorithm is summarized as follows:

1. Set the number of periods in the planning horizon equal to L and the current period (K) equal to 0. To operate the gradient method specify the number of periods in the time horizon (n) and the termination criterion. To start the gradient method in period 0 specify the initial diameter distribution and the starting values for the control variables.
2. Solve the n-period control problem using the gradient method. Save the control variable values for period 0 and the vector of state variables for period 1 from this solution.
3. If $K < L$, replace K by $K + 1$ and set the initial diameter distribution equal to the vector of state

variables that were saved in step 2. Reset the starting values for the control variables and go to step 2.

If K = L, stop. The optimal control sequence for an L-period management regime has been determined.

I demonstrate this algorithm with a whole-stand/diameter-class simulator for mixed hardwood stands in Wisconsin. The simulator has been incorporated into static-optimization algorithms (Adams and Ek 1974, Adams 1976, Martin 1982) and a dynamic-optimization program (Haight and others 1985). The simulator includes nonlinear functions f_{ij} for the change in number of trees per acre in a 2-inch diameter class j during a 5-year growth period i. Growth function arguments are the residual stand structure at the start of the growth period. The smallest diameter class is 6 inches.

The algorithm applies the gradient method in each period of an L-period time horizon. This method requires (1) predetermined values for the number of periods (n) over which the gradient method will be applied and (2) the termination criterion, which is the minimum acceptable improvement in the objective function value used to obtain the next approximation of the control variable values.

After sensitivity analysis, I set the time horizon equal to 31 5-year periods and the termination criterion equal to $0.01/acre PNW. With these parameters the algorithm produced stable solutions for transition and equilibrium management regimes. Management regimes developed with random values given to control variables varied by less than 1 tree per acre per diameter class in residual diameter distributions over a 31 period time horizon. Increasing the number of periods (n) and reducing the termination criterion did not change the solutions but increased execution times. Execution times for the examples in the next sections varied between 1 and 3 hours on an IBM PC-XT microcomputer with an INTEL 8087 coprocessor.

COMPARISON OF DYNAMIC AND STATIC HARVEST REGIMES

Adams (1976) used static optimization to develop investment-efficient diameter distributions for hardwood stands (site index 60) managed with 5-year cutting cycles. He used a stumpage value function that assigned a constant price per cubic foot for trees between 6 and 12 inches in diameter at breast height and a rapidly increasing price for larger trees. Each distribution was constrained to a predetermined value of residual growing stock. Because of limits on the data base from which the simulator was constructed, the maximum diameter of residual trees was 18 inches.

For comparison, I use the dynamic optimization algorithm presented here to develop two management regimes for a hardwood stand, site index 60. The regimes are developed to maximize the PNW of selection harvests taken on a 5-year cutting cycle over a 150-year time horizon. The value of the terminable diameter distribution is not included in the objective function. I use the same stumpage value function (Table 5) and a 4 percent real interest rate.

The initial diameter distribution for management regime 1 is the pre-harvest investment-efficient diameter distribution, which has a marginal value growth of 4 percent (Adams 1976). Management regime 2 is developed for an initial diameter distribution with fewer trees in each diameter class.

Management regime 1 exhibits a pulse-transition harvest pattern during the first 55 years (Table 6). All 6-inch and 76 percent of the 8-inch trees are cut at the start. In the remaining periods, trees are harvested primarily from the two largest and most valuable diameter classes. Except for the first period, harvest values vary between $30.1 and $40.5 per acre. The pulse-transition is nearly the same as the harvest pattern presented by Haight and others (1985) for the same initial diameter distribution.

Downward-sloping diameter distributions are maintained between years 60 and 150. Harvests take place in the 6-, 18-, and 20-inch diameter classes. By year 135, harvests stabilize so that during the last four periods residual diameter distributions differ by fewer

TABLE 5. Initial diameter distributions and stumpage values for a northern hardwood stand (site index 60).

Diameter class midpoint (inches)	Initial diameter distribution: Regime 1	Regime 2	Stumpage value	
	---- Trees/acre ---		$/ft^3	$/tree
6	69.9	15.0	0.01	0.04
8	49.3	11.5	.01	.10
10	37.5	8.5	.01	.18
12	29.7	6.5	.01	.26
14	24.1	5.5	.025	.97
16	10.7	4.5	.05	2.80
18	2.1	1.5	.08	5.04
20	.3	.0	.10	7.90

TABLE 6. Management regime 1 (PNW = $171.29/acre).

Year	Diameter class midpoint (inches)								Totals		
	6	8	10	12	14	16	18	20	Trees	Ft²	$
	Trees per acre cut										
0	69.8	37.6	0.0	0.0	0.0	0.5	0.6	0.3	108.9	29.3	13.7
5	4.1	.0	.0	.0	.0	2.8	3.5	.9	11.4	13.0	33.2
10	3.4	.0	.0	.0	.0	2.9	3.9	.9	11.2	13.7	35.5
15	1.2	.0	.0	.0	.0	1.7	4.1	.9	8.0	12.0	33.4
20	.0	.0	.0	.0	.0	.6	4.8	1.1	6.6	11.9	35.1
25	.0	.0	.0	.0	.0	.0	5.3	1.1	6.4	11.9	35.9
30	.0	.0	.0	.0	.0	.0	5.9	1.3	7.2	13.4	40.5
35	.0	.0	.0	.0	.0	.0	5.5	1.3	6.8	12.7	38.5
40	.0	.0	.0	.0	.0	.0	5.1	1.5	6.7	12.5	38.3
45	.0	.0	.0	.0	.0	.0	4.0	1.7	5.7	10.9	33.9
50	.0	.0	.0	.0	.0	.0	3.5	2.0	5.6	10.7	33.9
55	.0	.0	.0	.0	.0	.0	2.8	2.0	4.8	9.4	30.1
60	8.7	.0	.0	.0	.0	.0	2.2	2.0	13.0	10.1	27.7
65	10.2	.0	.0	.0	.0	.0	1.8	2.0	14.1	9.7	25.9
70	10.0	.0	.0	.0	.0	.0	1.7	2.0	13.8	9.5	25.3
75	8.7	.0	.0	.0	.0	.0	1.7	1.9	12.4	9.0	24.6
80	8.6	.0	.0	.0	.0	.0	1.8	1.9	12.3	9.1	24.7
85	7.9	.0	.0	.0	.0	.0	2.0	1.8	11.8	9.1	25.1
90	6.5	.0	.0	.0	.0	.0	2.2	1.8	10.6	9.2	25.9
95	5.2	.0	.0	.0	.0	.0	2.5	1.7	9.5	9.3	26.9
100	4.6	.0	.0	.0	.0	.0	2.7	1.7	9.0	9.5	27.7
105	4.1	.0	.0	.0	.0	.0	2.9	1.7	8.7	9.7	28.5
110	4.1	.0	.0	.0	.0	.0	3.0	1.7	8.9	9.9	29.0
115	4.3	.0	.0	.0	.0	.0	3.0	1.7	9.1	10.0	29.4
120	3.5	.0	.0	.0	.0	.0	3.0	1.7	8.3	9.9	29.4
125	3.6	.0	.0	.0	.0	.0	2.9	1.7	8.3	9.7	29.0
130	3.8	.0	.0	.0	.0	.0	3.0	1.8	8.7	10.0	29.7
135	5.4	.0	.0	.0	.0	.0	2.8	1.7	10.0	10.0	28.6
140	5.5	.0	.0	.0	.0	.0	2.7	1.8	10.0	9.9	28.4
145	6.7	.0	.0	.0	.0	.0	2.6	1.8	11.2	10.0	28.1
150	4.8	.0	.0	.0	.0	.0	2.6	1.8	9.3	9.6	27.9
	Residual trees per acre										
0	0.0	11.6	37.5	29.7	24.1	10.1	1.5	0.0	114.7	90.5	75.1
5	5.7	7.5	28.2	29.7	24.1	11.4	1.5	.0	108.4	87.0	76.9
10	10.0	6.4	20.6	27.2	24.2	12.1	1.6	.0	102.4	82.6	77.6
15	15.1	6.8	15.1	23.3	23.5	13.8	1.8	.0	99.7	79.7	81.1
20	20.0	8.3	11.5	19.2	21.8	15.6	1.9	.0	98.6	76.9	83.5
25	23.9	10.4	9.7	15.4	19.4	16.7	2.2	.0	98.1	74.0	85.0
30	27.1	12.8	9.1	12.4	16.7	16.6	2.2	.0	97.3	69.7	81.4
35	30.2	15.2	9.5	10.4	14.0	15.5	2.6	.0	97.7	66.1	77.8
40	33.3	17.5	10.4	9.3	11.7	13.9	2.9	.0	99.2	62.8	72.4

(continued)

TABLE 6. Continued.

Year	Diameter class midpoint (inches) 6	8	10	12	14	16	18	20	Totals Trees	Ft²	$
	Residual trees per acre (continued)										
45	36.7	19.8	11.6	9.0	10.0	12.1	3.4	.0	102.8	61.3	69.0
50	40.3	22.1	13.1	9.2	8.9	10.4	3.4	.0	107.6	60.2	63.8
55	44.2	24.5	14.6	9.8	8.3	9.0	3.5	.0	114.2	60.8	60.6
60	39.6	27.1	16.3	10.6	8.2	8.1	3.5	.0	113.6	61.0	58.7
65	34.4	27.8	18.0	11.7	8.5	7.6	3.5	.0	111.7	61.5	57.9
70	30.3	27.1	19.3	12.8	9.1	7.4	3.4	.0	109.6	62.2	57.6
75	27.9	25.6	19.9	13.9	9.8	7.5	3.3	.0	108.3	63.2	58.3
80	25.8	24.1	20.0	14.8	10.6	7.9	3.2	.0	106.5	64.1	59.5
85	24.3	22.5	19.6	15.3	11.3	8.4	3.1	.0	104.7	64.8	61.1
90	24.0	21.1	18.9	15.5	11.9	8.9	3.0	.0	103.6	65.3	62.6
95	24.9	20.1	18.0	15.4	12.3	9.5	2.9	.0	103.3	65.6	63.9
100	25.9	19.6	17.2	15.1	12.5	9.9	2.9	.0	103.4	65.7	64.9
105	27.2	19.5	16.5	14.6	12.5	10.2	2.9	.0	103.7	65.6	65.6
110	28.1	19.7	16.1	14.1	12.4	10.4	2.9	.0	103.9	65.4	65.8
115	28.7	20.1	15.8	13.7	12.1	10.4	3.0	.0	104.1	64.9	65.7
120	30.1	20.5	15.8	13.3	11.8	10.3	3.0	.0	105.1	64.7	65.4
125	31.3	21.1	15.9	13.1	11.5	10.2	3.1	.0	106.4	64.6	65.2
130	32.2	21.8	16.1	13.0	11.2	10.0	3.0	.0	107.5	64.4	64.1
135	31.5	22.4	16.4	13.0	11.1	9.7	3.1	.0	107.6	64.3	63.7
140	31.1	22.7	16.8	13.1	11.0	9.6	3.1	.0	107.6	64.3	63.3
145	29.5	22.8	17.2	13.3	10.9	9.4	3.1	.0	106.5	64.1	63.0
150	30.1	22.5	17.4	13.6	11.0	9.4	3.1	.0	107.3	64.4	62.8

than 1.0 tree per acre per diameter class for all but the 6-inch diameter class.

In part 3, I found that the PNW of a management regime developed with a dynamic optimization technique will be greater than the PNW of an equilibrium management regime developed with static analysis for the same initial stand structure. In this example, the PNW of management regime 1 ($171.29/acre) is 5.1 percent greater than the PNW of equilibrium harvests taken on a 5-year cycle for 150 years from the initial investment-efficient diameter distribution ($163.05/acre).

I also applied the dynamic-optimization algorithm using investment-efficient diameter distributions given by Martin (1982) as initial stand conditions. The resulting management regimes, which were developed using the corresponding stumpage value functions, interest rates and cutting cycle lengths given by Martin (1982), improved PNW by 30 to 50 percent relative to the PNW of the corresponding static equilibrium management regimes.

I also determined in part 3 that dynamically determined management regimes do not converge to steady-state stand structures determined with static analysis. In management regime 1, the residual diameter distribution at year 150 is essentially a steady state. This distribution has a flatter shape (Figure 2) and a 4.7 percent higher growing-stock value than the investment-efficient distribution (Table 7). The value of harvests

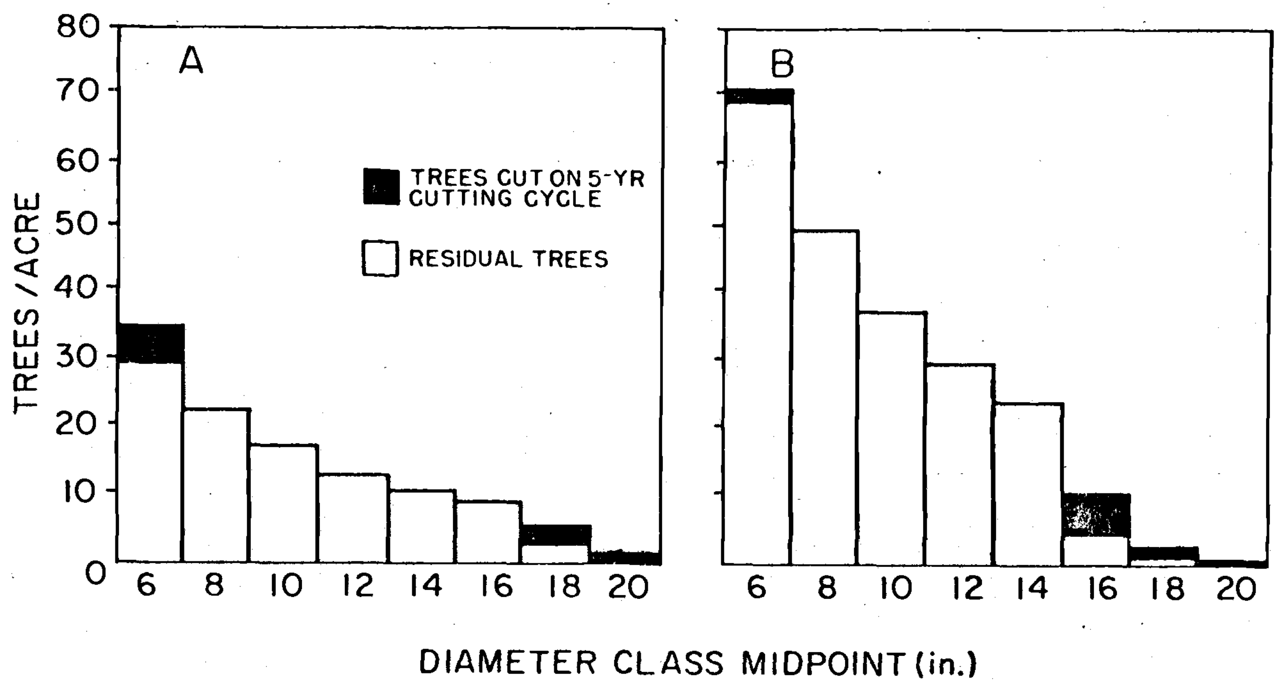

FIGURE 2. Residual diameter distributions for equilibrium harvests on a 5-year cutting cycle determined with (A) dynamic and (B) static optimization.

TABLE 7. Dynamically and statically determined steady-state residual diameter distributions and harvests (5-year cutting cycle).

Diameter class midpoint (inches)	Dynamic equilibrium diameter distribution		Investment-efficient diameter distribution	
	Residual	Cut	Residual	Cut
	Trees/acre			
6	29.7	5.2	68.5	1.4
8	22.5	.0	49.3	.0
10	17.4	.0	37.5	.0
12	13.6	.0	29.7	.0
14	11.0	.0	24.1	.0
16	9.4	.0	4.2	6.5
18	3.1	2.6	.5	1.6
20	.0	1.8	.0	.3
Value of residual ($/acre)	62.8		60.0	
Value of harvest ($/acre)	27.9		29.1	
Land expectation value, LEV ($/acre)	66.0		74.3	

taken from the dynamically determined steady state is 4.1 percent less than the value growth from the investment-efficient stand structure. As a result, the LEV of the dynamic equilibrium management regime is 11.1 percent less than the LEV of the statically determined regime.

Management regime 2, in which the initial stand structure includes fewer trees in each diameter class relative to the steady state found in regime 1, demonstrates that the dynamically determined steady state is independent of the initial stand structure (Table 8). During the first 25 years, harvests take trees from the 20-inch diameter class only, while the numbers of trees in the remaining classes increase. Between years 30 and 55, a weak pulse-like harvest pattern takes place with large numbers of trees cut from the 6-inch diameter class and small but increasing numbers of trees harvested from the 18-inch class. Downward-sloping diameter distributions are maintained for the remainder of the time horizon. The residual stand structure at year 150 is nearly stable and differs by fewer than 1 tree per acre per diameter class in all but the 6-inch diameter class from the steady-state structure in regime 1.

CUTTING CYCLE ANALYSIS

The dynamic optimization algorithm is easily expanded to analyze cutting cycles that are multiples of the 5-year projection interval. This is accomplished by constraining

TABLE 8. Management regime 2 (PNW = $82.03/acre).

Year	Diameter class midpoint (inches) 6	8	10	12	14	16	18	20	Totals Trees	Ft²	$
					Trees per acre cut						
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	.0	.0	.0	.0	.0	.0	.0	1.0	1.0	2.2	8.2
10	.0	.0	.0	.0	.0	.0	.0	1.9	1.9	4.1	15.0
15	.0	.0	.0	.0	.0	.0	.0	2.2	2.2	4.8	17.5
20	.0	.0	.0	.0	.0	.0	.0	2.3	2.3	5.0	18.4
25	.0	.0	.0	.0	.0	.0	.0	2.3	2.3	5.1	18.7
30	2.1	.0	.0	.0	.0	.0	.1	2.4	4.6	5.8	19.7
35	16.5	.0	.0	.0	.0	.0	.4	2.3	19.2	9.0	21.2
40	16.8	.0	.0	.0	.0	.0	.6	2.2	19.6	9.1	21.1
45	15.7	.0	.0	.0	.0	.0	.8	2.1	18.6	9.2	21.5
50	9.0	.0	.0	.0	.0	.0	1.2	2.0	12.2	8.3	22.3
55	8.6	.0	.0	.0	.0	.0	1.6	1.8	12.1	8.6	23.4
60	6.6	.0	.0	.0	.0	.0	2.0	1.7	10.5	8.8	24.6
65	5.4	.0	.0	.0	.0	.0	2.4	1.7	9.6	9.1	26.1
70	3.6	.0	.0	.0	.0	.0	2.8	1.6	8.1	9.3	27.5
75	4.3	.0	.0	.0	.0	.0	3.0	1.6	9.0	9.8	28.6
80	2.7	.0	.0	.0	.0	.0	3.2	1.6	7.7	10.0	29.8
85	3.0	.0	.0	.0	.0	.0	3.3	1.6	8.1	10.1	30.2
90	3.8	.0	.0	.0	.0	.0	3.3	1.7	8.8	10.3	30.4
95	2.4	.0	.0	.0	.0	.0	3.2	1.7	7.4	10.0	30.2
100	4.2	.0	.0	.0	.0	.0	3.1	1.7	9.1	10.1	29.8
105	5.4	.0	.0	.0	.0	.0	2.9	1.8	10.2	10.2	29.2
110	5.2	.0	.0	.0	.0	.0	2.7	1.8	9.9	9.9	28.6
115	5.7	.0	.0	.0	.0	.0	2.6	1.8	10.2	9.8	28.2
120	5.5	.0	.0	.0	.0	.0	2.6	1.8	9.9	9.6	27.8
125	6.4	.0	.0	.0	.0	.0	2.5	1.8	10.7	9.7	27.5
130	6.8	.0	.0	.0	.0	.0	2.5	1.8	11.1	9.8	27.4
135	5.4	.0	.0	.0	.0	.0	2.5	1.8	9.8	9.5	27.4
140	5.8	.0	.0	.0	.0	.0	2.6	1.7	10.2	9.6	27.6
145	5.5	.0	.0	.0	.0	.0	2.6	1.7	10.0	9.7	27.7
150	5.4	.0	.0	.0	.0	.0	2.7	1.7	9.9	9.8	28.0
					Residual trees per acre						
0	15.0	11.5	8.5	6.5	5.5	4.5	1.5	.0	53.0	31.5	30.4
5	24.3	11.4	8.7	6.7	5.4	4.6	3.0	.0	64.4	36.4	38.8
10	31.9	13.8	8.9	6.9	5.5	4.6	3.6	.0	75.5	40.3	42.7
15	38.7	17.3	9.7	7.1	5.7	4.7	3.8	.0	87.3	44.1	45.2
20	45.1	21.1	11.3	7.5	5.8	4.8	4.0	.0	99.8	48.4	47.3
25	50.9	25.1	13.3	8.2	6.0	4.9	4.1	.0	112.8	53.2	49.5
30	54.0	29.0	15.7	9.3	6.4	5.0	4.0	.0	123.8	57.8	51.2
35	42.4	32.4	18.2	10.7	7.1	5.3	3.8	.0	120.0	59.7	51.9
40	31.8	32.0	20.7	12.3	7.9	5.6	3.6	.0	114.2	61.3	53.3

(continued)

TABLE 8. Continued.

Year	Diameter class midpoint (inches)								Totals		
	6	8	10	12	14	16	18	20	Trees	Ft²	$
	------------- Residual trees per acre (continued) ---------------										
45	23.3	29.4	22.2	14.1	9.0	6.2	3.4	.0	107.8	62.4	55.0
50	22.2	25.6	22.4	15.6	10.2	6.9	3.2	.0	106.3	64.1	57.2
55	21.0	22.7	21.6	16.6	11.4	7.7	3.0	.0	104.2	65.3	59.6
60	21.3	20.4	20.2	16.9	12.4	8.6	2.9	.0	103.0	66.2	62.1
65	22.3	19.0	18.7	16.7	13.1	9.5	2.8	.0	102.3	66.7	64.3
70	24.6	18.2	17.3	16.1	13.4	10.2	2.8	.0	102.8	66.9	66.0
75	25.6	18.2	16.2	15.3	13.4	10.7	2.8	.0	102.5	66.7	67.2
80	27.9	18.5	15.5	14.4	13.1	11.0	2.8	.0	103.5	66.2	67.5
85	29.6	19.2	15.2	13.7	12.6	11.0	2.9	.0	104.4	65.7	67.3
90	30.3	20.1	15.1	13.1	12.1	10.8	2.9	.0	104.8	65.0	66.6
95	32.5	20.8	15.3	12.8	11.6	10.6	3.0	.0	106.8	64.7	65.7
100	32.8	21.9	15.7	12.6	11.2	10.2	3.0	.0	107.6	64.3	64.7
105	31.9	22.6	16.2	12.7	10.9	9.9	3.1	.0	107.5	64.0	63.8
110	31.6	23.0	16.7	12.8	10.7	9.6	3.1	.0	107.8	64.0	63.1
115	31.0	23.1	17.2	13.1	10.7	9.3	3.1	.0	107.8	64.0	62.6
120	30.7	23.1	17.5	13.4	10.8	9.2	3.1	.0	108.1	64.3	62.4
125	29.6	23.0	17.7	13.7	11.0	9.2	3.1	.0	107.5	64.4	62.4
130	28.3	22.7	17.8	14.0	11.1	9.2	3.0	.0	106.4	64.5	62.6
135	28.4	22.2	17.8	14.1	11.3	9.3	3.0	.0	106.4	64.8	62.9
140	28.1	21.8	17.6	14.2	11.5	9.4	3.0	.0	106.0	65.0	63.3
145	28.0	21.5	17.4	14.2	11.6	9.6	3.0	.0	105.7	65.1	63.8
150	28.0	21.5	17.2	14.2	11.7	9.7	3.0	.0	105.3	65.0	64.0

$X_{ij} - S_{ij} = 0$ for $j = 1,...,m$ in periods when no harvest is desired.

Management regime 3 (Table 9) is developed for the same initial stand structure, stumpage value function and interest rate as management regime 1; however, the cutting cycle is 10 years. Harvesting takes place in two pulse cycles during the first 90 years, with downward-sloping diameter distributions maintained thereafter. In contrast to management regime 1, the maximum diameter of residual trees is 16 inches and harvest values vary between \$46.0 and \$75.0 per acre.

A break-even analysis with fixed-entry costs can be used to compare the PNW of regimes with different cutting cycles. Similar to the results for thinning in even-aged stands (Brodie and others 1978), a fixed-entry cost for selection harvests on a given cycle lowers the PNW, leaving the management regime unchanged. With no fixed entry cost, the PNW of management regime 1 exceeds the PNW of regime 3 by 1.1 percent. An entry cost of \$0.75/acre would equate the values of the two management regimes. A similar comparison showed that an entry cost of \$2.37/acre would be necessary for a 15-year cutting cycle to be optimal. In general, higher entry costs favor extended cutting cycles.

TABLE 9. Management regime 3 (PNW = $169.41/acre).

Year	Diameter class midpoint (inches) 6	8	10	12	14	16	18	20	Totals Trees	Ft²	$
	Trees per acre cut										
0	69.8	34.0	0.0	0.0	0.0	3.7	2.1	0.3	110.1	35.3	30.0
10	8.9	.0	.0	.0	.0	6.5	6.8	1.7	24.0	26.7	66.7
20	.3	.0	.0	.0	.0	5.3	7.9	2.2	15.8	26.4	72.7
30	.0	.0	.0	.0	.0	1.0	8.5	2.7	12.3	22.5	67.6
40	.0	.0	.0	.0	.0	.0	9.3	3.4	12.7	23.9	74.1
50	.0	.0	.0	.0	.0	.0	8.4	3.3	11.7	22.2	69.0
60	8.4	.0	.0	.0	.0	.0	6.7	2.8	18.1	19.9	57.1
70	21.7	.0	.0	.0	.0	.0	5.7	2.4	29.8	19.6	48.6
80	21.2	.0	.0	.0	.0	.0	5.4	2.2	28.9	18.7	46.0
90	16.0	.0	.0	.0	.0	.0	5.8	2.2	24.2	18.5	48.3
100	7.4	.0	.0	.0	.0	.0	6.5	2.4	16.4	18.4	52.9
110	6.7	.0	.0	.0	.0	.0	7.1	2.7	16.5	19.7	57.3
120	7.6	.0	.0	.0	.0	.0	7.3	2.8	17.8	20.6	59.6
130	8.4	.0	.0	.0	.0	.0	7.2	2.8	18.5	20.6	59.2
140	10.2	.0	.0	.0	.0	.0	6.9	2.7	19.8	20.2	57.0
150	11.9	.0	.0	.0	.0	.0	6.6	2.6	21.2	19.8	54.8
	Residual trees per acre										
0	.0	15.2	37.5	29.7	24.1	6.9	.0	.0	113.5	84.5	58.8
10	8.2	9.3	22.0	27.5	24.2	9.1	.0	.0	100.5	77.2	61.6
20	20.0	9.3	13.1	20.1	22.0	11.0	.0	.0	95.9	69.2	61.7
30	27.9	13.5	10.2	13.5	17.3	14.3	.0	.0	97.0	65.0	64.8
40	34.0	18.2	11.1	10.2	12.5	13.5	.0	.0	99.9	59.6	58.2
50	40.7	22.9	13.6	9.8	9.6	10.8	.0	.0	107.7	56.6	48.7
60	40.4	27.8	16.8	11.2	8.8	8.6	.0	.0	113.8	57.2	43.2
70	28.8	30.3	19.8	13.3	9.5	7.9	.0	.0	109.9	58.8	42.7
80	20.6	27.4	21.3	15.4	10.9	8.3	.0	.0	104.2	60.7	45.4
90	18.3	22.9	20.3	16.5	12.4	9.3	.0	.0	100.0	62.1	49.2
100	23.8	19.8	18.1	16.2	13.2	10.4	.0	.0	101.6	62.9	52.4
110	27.4	20.1	16.4	14.9	13.1	11.0	.0	.0	103.1	62.6	53.6
120	29.0	21.5	16.0	13.8	12.4	10.9	.0	.0	103.9	61.5	52.6
130	29.9	22.8	16.5	13.4	11.6	10.5	.0	.0	104.9	60.5	50.7
140	29.4	23.9	17.2	13.5	11.3	9.9	.0	.0	105.5	60.2	49.1
150	27.9	24.2	17.9	13.9	11.3	9.6	.0	.0	105.1	60.3	48.5

SUMMARY AND CONCLUSIONS

I have presented the marginal production rules and a computational solution method for the optimal transition and equilibrium management regimes for an uneven-aged stand. Comparing the production rules for a dynamic equilibrium stand structure with marginal production rules for the steady state determined with static optimization techniques shows that solutions to the two problems will differ. Also, the PNW of the optimal transition and steady-state management regime will be greater than the PNW of the static equilibrium regime. Thus, for the diameter distribution model of uneven-aged management, maximization of LEV with the Faustmann equation is not the correct criterion for determining the optimal steady-state stand structure.

The solution method is the first to allow the simultaneous determination of optimal stand-specific transition and equilibrium management regimes. Numerical results show that while the steady-state solution is independent of the initial stand structure, the length and harvest pattern of the transition regime is not. Thus, generalizations about the relative profitability of uneven-aged and even-aged management cannot be made on the basis of steady-state management regimes alone, but must include the value of the stand-specific transition regime.

It would be very interesting to apply the solution method presented here to a simulator that projects growth and yield for both even-aged and uneven-aged stands. Then we could observe the effects of the initial stand structure and timing and cost of regeneration on the relative profitability of the two fundamental management systems.

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A NOTE ON THE INSEPARABILITY OF LAND AND TIMBER VALUES WHEN COMPARING EVEN-AGED AND UNEVEN-AGED HARVESTING SYSTEMS

by

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ABSTRACT

An economic model for harvesting forest stands is presented and used to contrast the two major timber harvesting systems: even-aged and uneven-aged management. The model treats land and existing trees as fixed inputs and seeks the sequence of harvests and plantings that maximizes forest value as defined for each management system. In contrast to even-aged management, the value of uneven-aged harvesting cannot be separated into independent components for stand value and land value. Thus, conclusions about the most profitable harvesting system depend on the joint productivity of the land and existing timber, in addition to the costs and prices associated with each harvesting system. Further, under certain cost and price assumptions, uneven-aged management will have a higher forest value for any initial stand structure. These results are demonstrated by developing optimal management regimes for Arizona ponderosa pine (*Pinus ponderosa* Laws.) stands.

A NOTE ON THE INSEPARABILITY OF LAND AND TIMBER VALUES WHEN COMPARING EVEN-AGED AND UNEVEN-AGED MANAGEMENT SYSTEMS

INTRODUCTION

Research on the valuation of forest land and timber has concentrated on the determination of the optimal rotation age for an even-aged plantation using the traditional Faustmann (1849) formulation. This basic economic model has been extended to determine the effects on the optimal rotation of changing future prices and costs (McConnell et al. 1983), taxation (Chang 1982), imperfect markets (Nautiyal and Fowler 1980) and the risk of fire (Reed 1984). Common to these analyses is the assumption that even-aged management, starting from bare ground, with no intermediate harvests is the preferred management system for the length of the planning horizon. Recently, however, stand growth simulators have been developed that predict with reasonable accuracy the effects of selective harvesting on stand structure and that can be used to develop optimal sequences of harvests for existing stands. As a result, the relative value of alternative harvesting systems applied to stands that initially contain timber can be determined.

When faced with an existing stand, a forest manager may choose between two fundamentally different harvesting systems: uneven-aged or even-aged management. Uneven-aged management is characterized by the cyclical harvest of trees from various size classes and the continual replacement of trees by natural regeneration or planting. Even-aged management is applied by either liquidating the stand and replacing it with a plantation that is managed with the optimal thinning regime and rotation age, or thinning the existing stand during a conversion period before replacement with a plantation. A problem not addressed by the recent literature is the valuation of a forest stand when either of the two management systems can be applied.

Faustmann (1849) considered the problem of valuing land that initially contains timber, assuming that the even-aged harvest system is applied. He defined forest value (FV) as the difference between the capital values of all incomes and expenditures that occur until infinity. Faustmann expressed FV as the sum of the stand value, which is the present net worth (PNW) of harvests taken during the conversion to clearcut, and land expectation value (LEV), which is the PNW of infinite-series plantation management. Since plantation management starting with bare land would eventually be practiced in perpetuity, LEV is independent and separable from the PNW of the conversion regime.

Studies in the valuation of forest stands that are managed with the uneven-aged harvesting system have concentrated on steady-state management. Land value is defined as the PNW of steady-state harvesting in perpetuity minus the value of the residual growing stock. When the value of the residual growing stock is viewed as a capital expenditure, this definition of land value is equivalent to Faustmann's definition for the LEV of bare land in even-aged management (Chang 1981, Hall 1983). It has been generally accepted that maximization of LEV is the correct criterion for defining the appropriate levels of growing stock and cutting cycles in uneven-aged stands (Adams 1976, Chang 1981, Martin 1982, Hall 1983, Hall and Bruna 1983). Further, when maximized, LEV is used to measure the profitability of uneven-aged management for comparison with the profitability of other uses of the land (Chang 1981, Martin 1982, Hall and Bruna 1983).

Departing from steady-state analysis, Haight (1985) formulated the uneven-aged harvesting problem as an unconstrained dynamic optimization problem that maximized the PNW of harvesting over an infinite planning horizon. In an analysis of the dynamic formulation he showed that the problem of maximizing LEV is equivalent to the dynamic problem with the constraint that steady-state management must be achieved after one harvest from the initial stand. As a result, for any initial stand structure, an unconstrained transition harvest regime will converge to a steady state that does not maximize LEV. Further, when the initial stand is in the steady state that maximizes LEV, a transition and steady-state regime can be found with a higher PNW. These results show that LEV is not the correct measure of the profitability of unconstrained uneven-aged management.

This paper is divided into three parts. In part 1, I develop a general economic model for managing existing timber stands and compare the even-aged and uneven-aged harvesting problems. The dynamic uneven-aged harvesting problem is equivalent to the FV problem defined by Faustmann for even-aged stands except that harvests are not constrained to clearcut in any period. Since the timber is not clearcut, FV cannot be separated into independent components for stand value and land value as Faustmann defined for even-aged management. As a result, comparisons of the profitability of the two harvesting systems cannot be made on the basis of LEV, but must be made on the basis of FV.

Since the FV criterion seeks to maximize the PNW of perpetual harvesting in an existing stand, conclusions about the most profitable harvesting system depend on the structure of the initial stand, in addition to the costs and prices associated with each harvesting system. Further, under certain assumptions about the costs and prices for each management system, uneven-aged management will have a higher FV for any initial stand structure. In parts 2 and 3, I demonstrate these results by developing and comparing optimal even-aged and uneven-aged management regimes for two kinds of Arizona ponderosa pine (Pinus ponderosa Laws.) stands.

GENERAL ECONOMIC MODEL FOR STAND MANAGEMENT

Consider the problem of determining the optimal sequence of harvests from an existing stand over a T-period planning horizon:

$$\max_{\{U(i),v(i)\}} PNW = \sum_{i=0}^{T-1} d^{i}[B(X(i),U(i))-C(v(i))] + d^{T}Q(X(T)) \qquad (1)$$

where

$X(i)$ = a state vector that contains elements $x_j(i)$, $j = 1,\ldots,m$, representing the number of trees in size class j before harvest at the beginning of time period i

$U(i)$ = a control vector that contains elements $u_j(i)$, $j = 2,\ldots,m$, representing the percentage of trees harvested from size class j at the beginning of time period i

$v(i)$ = a control variable for the number of trees planted at the beginning of time period i

B = a real-valued function giving the net revenue from harvest

C = a real-valued function giving the planting cost

Q = a real-valued function giving the value of the stand in period T

d = the discount factor.

The objective function is formulated to seek the set $\{U(i),v(i)\}$ representing the percentage of trees harvested from each size class and the number of trees planted in each time period that maximizes the PNW of harvests net planting costs over a T-period planning horizon. This formulation assumes constant prices and technology so that the revenue and cost functions are constant over the planning horizon. A finite time horizon is used so that numerical solution techniques can be applied. The discount factor d is equal to $1/(1 + r)$ where r is a positive annual discount rate. It is assumed that prices, costs and discount rate are in real terms.

The benefit function is defined for periods $i = 0,\dots,T-1$ as

$$B(X(i),U(i)) = \sum_{j=2}^{m} P_j x_j(i) u_j(i) \qquad (2)$$

where P_j is the price per tree in size class j. For merchantable size classes P_j is the stumpage price (i.e., net of harvest cost). For unmerchantable size classes P_j represents the cost of harvesting trees with no commercial value.

Recognizing that trees grow in size from one time period to the next, I impose the following transition equations as constraints on the objective function for periods $i = 0,\dots,T-1$:

$$x_1(i+1) = R + v(i) + (1 - p_1)s_1x_1(i) \tag{3}$$

$$x_j(i+1) = (1 - u_{j-1}(i))p_{j-1}s_{j-1}x_{j-1}(i)$$

$$+ (1 - u_j(i))(1 - p_j)s_jx_j(i) \quad j = 2,\ldots,m \tag{4}$$

where

R = a real-valued function for recruitment into the smallest size class based on natural regeneration

p_j = the proportion of trees transfering between size class j and j + 1 during the time period

s_j = the proportion of trees surviving in size class j during the time period.

Equation (3) represents the growth dynamics for the smallest size class and includes functions for recruitment based on natural regeneration (R) and planting (v). Equation (4) represents the general growth dynamics for size classes $j = 2,\ldots,m$. For each size class j, the number of trees in period i + 1 is the sum of the number of trees growing into the size class from below and the number of trees surviving and staying in the size class. In general, R, p_j and s_j are nonlinear functions of the variables X(i) and U(i). The units of time represent the growth period between harvests and are usually some integer multiple of five years.

In addition to the transition equations, the objective function includes feasibility constraints on the control variables for each period i = 0,...,T-1:

$$U(i) \geq 0, \quad (1 - U(i)) \geq 0, \quad \text{and} \quad v(i) \geq 0 \qquad (5)$$

Finally, the initial stand structure X(0) is given.

I define control variables for harvesting as percentages to facilitate numerical solution. When designing the solution method, transition equations (3) and (4) are incorporated into the objective function leaving constraint set (5). Since these constraints are defined only in terms of the control variables, an unconstrained nonlinear optimization technique based on gradients can be used to solve the harvesting problem.

This formulation differs from previous stand-level harvesting problems by including control variables for the harvest intensity by tree-size class and for the number of trees planted in each time period. As a result, I can define harvesting problems for uneven-aged management (problem 1), and by introducing appropriate constraints on the harvesting and planting variables, I can define even-aged harvesting (problem 2).

Problem 1 is defined without further constraints on the control variables. The solution represents the optimal uneven-aged harvest regime over a finite planning horizon with interplanting in each time period. Haight (1985) solved this problem without interplanting for Wisconsin hardwood stands and found that regimes with 5-year cutting cycles converged to steady-state harvesting with transition periods less than 150 years in length. When the optimal harvesting regime converges to steady-state harvesting in transition periods less than T, the PNW of perpetual uneven-aged management can be computed. The PNW of optimal transition and steady-state management is the FV of uneven-aged management.

Until now, the FV of uneven-aged management has been defined in terms of constrained harvesting problems. Chang (1981) first defined FV as the PNW of harvests when an initial stand is converted to steady-state harvesting in one cut. Michie (1985) extended this definition by including the PNW of transition harvests that reach a steady state in a specified number of periods. In contrast to these problem formulations, the uneven-aged management problem presented here is not constrained to reach a steady state in any period. As a result, this definition of FV gives the ultimate PNW of uneven-aged management.

Problem 2 is separated into problems of determining the optimal conversion harvest sequence that terminates in clearcut and the optimal plantation management regime. The plantation harvesting problem is independent of how the initial stand is converted to a plantation and is solved first.

The optimal single-rotation harvest regime with rotation age t is found by setting $T = t$ and $X(0) = 0$, and constraining $v(i) = 0$ for $i = 1,...,T-1$. Denote the value of this regime by PNW_t. In this formulation the planting density v(0) is determined simultaneously with the optimal thinning regime. The land expectation value for rotation age t, LEV_t, is found by the transformation $PNW_t/(1 - d^t)$. The value of the optimal infinite series plantation management regime, LEV^*, is the maximum of $[LEV_t]$ for all t.

The optimal conversion harvest regime for an initial stand structure X(0) depends on LEV^*. The optimal regime for conversion period length n is found by setting T = n and constraining v(i) = 0 for i = 0,...,T-1. Denoting the value of each conversion regime by PNW_n, the FV of even-aged management, which is the sum of the values of conversion and plantation harvesting, is the maximum of $[PNW_n + d^n LEV^*]$ for all n.

In contrast to problem 2, the uneven-aged management problem does not include constraints for clearcut so that the FV of the optimal transition and steady-state regime cannot be separated into stand and land value components. As a result, FV is the correct measure for comparing the profitability of the two management systems. Further, since FV depends on the initial stand structure, conclusions about which of the two management systems is most profitable for a given set of planting costs and stumpage prices depend on the joint productivity of the land and existing timber.

Under certain cost and price assumptions it is possible to make generalizations about the relative FV's of the two harvesting systems, independent of the structure of the initial stand. Suppose cost per tree planted and stumpage price per tree are independent of the numbers of trees planted and cut, and costs and prices are the same for each management system. In this case, both management systems have the same production functions, but, in contrast to the even-aged harvesting problem, uneven-aged management does not have constraints on the harvesting variables. As a result, for any initial stand structure, uneven-aged management will have a higher FV than even-aged harvesting.

STAND GROWTH DYNAMICS AND OPTIMIZATION METHOD

To demonstrate the results presented in part 1, I have developed optimal harvesting prescriptions for two ponderosa pine stands in Arizona. The general form of the stand growth dynamics is given by equations (3) and (4). Equations for upgrowth and mortality were obtained from a stand simulator for ponderosa pine (Hann 1980), and the equation for recruitment based on natural regeneration was developed by Larson (1975). These equations were based on repeated observations in even-aged and uneven-aged plots on the Fort Valley Experimental Forest northwest of Flagstaff, Arizona. As a result, the growth dynamics can be used to predict with reasonable accuracy the effects of harvesting in both even-aged and uneven-aged stands.

Stand structure is described with a diameter distribution that gives the numbers of trees per acre in 1-inch diameter classes. The smallest and largest diameter classes are 1-inch and 40-inch, respectively. Upgrowth equations predict the percentages of trees in each diameter class that grow into the four adjoining larger diameter classes during a 5-year period. These equations are nonlinear functions of the total basal area of trees in diameter classes above and below the subject class. The mortality percentage for each diameter class is a nonlinear function of the predicted 5-year diameter growth for the surviving trees in the class. As a result, transformations of the residual stand structure are used as explanatory variables in both upgrowth and mortality equations.

I define two height classes for trees less than 0.5 inches in diameter: seedlings (0 to 3 ft.) and saplings (3 to 6 ft.). The number of trees added to the seedling class due to annual natural regeneration is the product of three components: total number of cones produced, number of seeds per cone, and percent seed germination rate. The total number of cones produced is computed with an equation for cone production for each diameter class that is a function of total stand basal area. Trees less than 12 inches in diameter do not produce cones. The number of seeds per cone is a constant that may take on values between 5 and 70, depending on the size of the seed crop.

We used a constant 5 seeds per cone to obtain the most conservative estimates of natural regeneration. The percent germination rate is a function of the total crown cover in the stand.

Larson (1975) developed these regeneration models for annual predictions. I multiplied annual seedling production by 5 to obtain an estimate of the average 5-year seedling production.

Neither Hann (1980) nor Larson (1975) developed equations for predicting seedling and sapling height growth and survival rates. Consequently, I represent the percentages of trees that grow out of the seedling and sapling classes and the percentage of trees that survive in each class during a 5-year period as constants. The movement probability for each height class is 0.48 and was obtained by extrapolating height versus age curves for ponderosa pine (Minor 1964). Seedling and sapling survival rates for natural regeneration are 0.01 and 0.50, respectively, and are based on field observations given by Harrington and Kelsey (1979) and Sackett (1984). Survival rates for seedlings and saplings resulting from artificial regeneration are 0.70 and 0.99 and reflect the plantation survival rates given by Heidmann et al. (1982).

The methods that we use to solve the discrete-time optimal-control harvesting problems given in part 1 were developed by Haight et al. (1985) and Haight (1985). The algorithm involves repeated application of a conjugate gradient procedure. The uneven-aged management problem is to find the optimal sequence of harvests on a 20-year cycle for 200 years for a given initial stand structure. The algorithm starts by using the gradient method to compute a solution to an 80-year harvesting problem. The solution is an approximation to a stationary point that satisfies Kuhn-Tucker necessary conditions for constrained optimization. The values for the harvesting variables in the initial period, which are invariant for different sets of random starting values given to control variables, are saved. The resulting diameter distribution at year 20 is used as the initial stand structure for the next application of the gradient method over a new 80-year time horizon. The procedure is repeated until a 200-year harvesting regime is obtained. This algorithm produces solutions that satisfy Kuhn-Tucker optimality conditions and that are invariant for different sets of random starting values given to control variables. Solution times for the uneven-aged regimes given in the next section vary between 10 and 13 hours on an IBM PC-XT microcomputer with an INTEL 8087 coprocessor.

The even-aged harvesting problems are solved in two stages: conversion to clearcut and plantation management. The conversion harvesting problem is to find the optimal sequence of harvests on a 20-year cycle for a given conversion period length (n) and initial stand structure. The algorithm starts by solving the n-year harvesting problem with the gradient method. The values of the initial-period control variables are saved and the stand structure at year 20 is used as the starting condition for the next application of the gradient method, which solves an (n - 20)-year conversion problem. This process is repeated until the n-year conversion harvest regime has been obtained. I computed 20- to 100-year conversion regimes with execution times that varied between 1 and 4 hours.

The plantation harvesting problem is to find the optimal planting density for bare land and the subsequent sequence of harvests on a 20-year cycle for a given rotation period. The probems are solved in the same manner as conversion harvesting. Execution times for rotation ages between 80 and 140 years varied between 3 and 6 hours.

OPTIMAL HARVESTING REGIMES

In this section I describe optimal even-aged and uneven-aged harvesting regimes for two ponderosa pine stands, site index 85. Stand 1 has an uneven-aged structure with a downward-sloping diameter distribution, and stand 2 has an even-aged structure with a bell-shaped distribution (Table 10). Tree volumes, which are measured in gross scribner thousand board foot (MBF) units, are computed from height versus diameter equations (Hann 1981) and volume equations (Hann and Bare 1978). The management

TABLE 10. Initial diameter distributions and stumpage values for two ponderosa pine stands (site index 85).

Diameter class midpoint (inches)	Initial diameter distribution stand 1	stand 2	stumpage value	Diameter class midpoint (inches)	Initial diameter distribution stand 1	stand 2	stumpage value
	----Trees/acre----		$/tree		----Trees/acre----		$/tree
seedlings[a]	100.00	0.00	0.00	16	1.26	1.09	7.77
saplings	50.00	.00	.00	17	1.04	1.28	9.60
1	24.00	.00	- .25	18	.84	1.19	11.75
2	20.00	.00	- .25	19	.70	1.35	14.18
3	16.40	.00	- .50	20	.58	1.39	16.89
4	13.20	.00	- .50	21	.47	1.40	19.91
5	11.00	.00	-1.00	22	.38	1.28	23.25
6	8.80	.00	-1.00	23	.32	1.34	26.91
7	7.30	.00	.06	24	.26	1.18	30.92
8	6.00	.00	.26	25	.00	1.02	35.27
9	4.96	.00	.59	26	.00	.91	39.99
10	4.04	.15	1.07	27	.00	.67	45.08
11	3.36	.30	1.70	28	.00	.79	50.55
12	2.76	.50	2.50	29	.00	.78	56.40
13	2.28	.88	3.49	30	.00	.55	62.65
14	1.85	.94	4.68	31	.00	.00	69.30
15	1.53	1.07	6.08				

a. Seedling and sapling classes include trees 0 to 3 feet and 3 to 6 feet in height, respectively.

regimes are developed to maximize FV with a three percent real discount rate and with a stumpage value function that assigns $50.00/MBF for trees 6.5 inches in diameter and greater. Precommercial thinning costs are assigned to trees between 0.0 and 6.5 inches in diameter. Tree prices are given in Table 10. For convenience, regimes are listed by collapsing the diameter distribution into 2-inch diameter classes. Each class is described by its midpoint.

The optimal plantation regime, which is independent of the initial stand structure, is listed in Table 11. The regime is developed with no fixed or variable planting costs to obtain a baseline value for plantation management. With no planting costs, 511 trees are planted. Beginning in year 40, trees 11.5 inches in diameter and greater are cut leaving residual stands with maximum diameters of 9.5 to 11.5 inches. Natural regeneration begins after year 40 and continues until clearcut. Precommercial thinning takes place in year 100 removing trees in the two smallest diameter classes. The stand is clearcut in year 120, and the LEV is $81.15/acre.

The optimal uneven-aged management regime for stand 1 is developed with an arbitrarily large fixed cost for planting in any period so that interplanting does not take place (Table 12). Trees 13.5 inches in diameter and greater are harvested during the first 100 years leaving residual stands with maximum diameter of 11.5 to 13.5

TABLE 11. Optimal plantation management regime for a ponderosa pine stand, site index 85 (LEV = $81.15/acre).

Diameter class midpoint (inches)	Year						
	0	20	40	60	80	100	120
	Trees per acre cut						
seedlings[a]	0.0	0.0	0.0	0.0	0.0	0.0	8205.0
saplings	.0	.0	.0	.0	.0	.0	40.0
1.5	.0	.0	.0	.0	.0	9.4	18.0
3.5	.0	.0	.0	.0	.0	1.5	4.8
5.5	.0	.0	.0	.0	.0	.0	2.7
7.5	.0	.0	.0	.0	.0	.0	7.2
9.5	.0	.0	.0	.0	.0	.0	12.4
11.5	.0	.0	2.1	34.6	17.5	25.4	17.3
13.5	.0	.0	.0	19.6	28.1	16.8	18.9
15.5	.0	.0	.0	6.0	11.3	3.5	17.1
17.5	.0	.0	.0	.5	.9	.1	8.0
19.5	.0	.0	.0	.0	.0	.0	.7
21.5	.0	.0	.0	.0	.0	.0	.0
total trees	0	0	2	61	58	57	107
total sq.ft.	0	0	1	53	58	53	78
total $	0	0	3	194	240	250	325
	Residual trees per acre						
seedlings	511.0	25.0	0.0	1448.0	2349.0	4653.0	0.0
saplings	.0	101.0	10.0	6.0	10.0	23.0	.0
1.5	.0	137.8	41.3	13.3	7.0	.0	.0
3.5	.0	14.3	65.1	29.6	11.8	5.0	.0
5.5	.0	.0	71.3	47.4	24.0	9.9	.0
7.5	.0	.0	52.0	53.7	35.8	18.1	.0
9.5	.0	.0	18.1	48.1	38.8	25.5	.0
11.5	.0	.0	.0	.0	18.5	27.8	.0
13.5	.0	.0	.0	.0	.0	.0	.0
total trees	0	152	248	192	136	86	0
total sq.ft.	0	3	42	51	48	41	0
total $	0	-42	-93	-17	38	71	0

a. Seedling and sapling classes include trees 0 to 3 feet and 3 to 6 feet in height, respectively.

TABLE 12. Optimal uneven-aged management regime for stand 1 with no interplanting allowed (FV = $322.03/acre).

Diameter class midpoint (inches)	Year						
	0	20	40	60	80	100	200
	Trees per acre cut						
seedlings[a]	0.0	0.0	0.0	0.0	0.0	0.0	0.0
saplings	.0	.0	.0	.0	.0	.0	.0
1.5	.0	.0	.0	.0	.0	.0	.0
3.5	.0	.0	.0	.0	.0	.0	.0
5.5	.0	.0	.0	.0	.0	.0	.0
7.5	.0	.0	.0	.0	.0	.0	.0
9.5	.0	.0	.0	.0	.0	.0	.0
11.5	.0	.0	.0	.0	.0	.0	.0
13.5	1.9	9.5	15.1	8.2	6.7	4.6	6.1
15.5	2.8	6.8	11.2	13.9	13.4	9.2	11.1
17.5	1.8	4.0	5.0	6.3	9.6	7.3	7.8
19.5	1.3	1.0	.5	.6	2.1	2.1	1.9
21.5	.9	.0	.0	.0	.0	.1	.0
23.5	.6	.0	.0	.0	.0	.0	.0
25.5	.0	.0	.0	.0	.0	.0	.0
total trees	9	21	32	29	32	23	27
total sq.ft.	15	27	39	39	45	34	38
total $	102	142	195	208	256	195	215
	Residual trees per acre						
seedlings	100.0	5462.0	5864.0	7245.0	9182.0	8242.0	8087.0
saplings	50.0	22.0	26.0	33.0	46.0	38.0	39.0
1.5	44.0	11.4	15.3	19.0	26.8	25.3	23.1
3.5	29.6	25.9	11.3	13.9	17.5	24.3	16.8
5.5	19.8	27.8	14.9	11.1	13.5	17.2	14.8
7.5	13.3	23.9	22.7	11.2	11.3	14.4	14.6
9.5	9.0	18.2	22.7	15.7	10.3	12.1	14.3
11.5	6.2	13.1	19.2	18.5	11.3	10.4	13.8
13.5	2.3	.0	.0	8.9	6.4	4.8	6.4
15.5	.0	.0	.0	.0	.0	.0	.0
total trees	124	120	106	98	97	109	104
total sq.ft.	21	33	36	36	27	27	32
total $	-16	2	38	62	28	51	65

a. Seedling and sapling classes include trees 0 to 3 feet and 3 to 6 feet in height, respectively.

inches. No cutting takes place in smaller diameter classes. A pulse of trees grows through the diameter distribution until a downward-sloping distribution is obtained at year 100. The harvest and residual structures at year 200 are essentially a steady state. The FV, which is the PNW of the 200 year transition harvest plus steady-state harvesting, is $322.03/acre.

To obtain the optimal even-aged harvesting regime for stand 1, I computed optimal conversion harvesting regimes that terminate in clearcut in years 0 to 100. The PNW's of these regimes, which are plotted in Figure 3, rise at a decreasing rate and become asymtotic to the FV of the optimal uneven-aged management regime. Adding the discounted LEV of plantation management to the PNW of each conversion regime does not change the shape of the curve (Figure 3). As a result, uneven-aged management is the preferred management system.

The optimal uneven-aged management regime for stand 2 does not allow interplanting in any period (Table 13). Trees from the initial stand are harvested when they are 19.5 inches in diameter or greater up to year 40. Heavy natural regeneration results in a two-aged stand in year 20. In year 40, an even-aged stand is in place. After year 40, trees are harvested when they reach 13.5 inches in diameter or greater leaving residual stands with maximum diameters of 13.5 inches. A pulse of trees grows through the diameter distribution between years 40 and 120

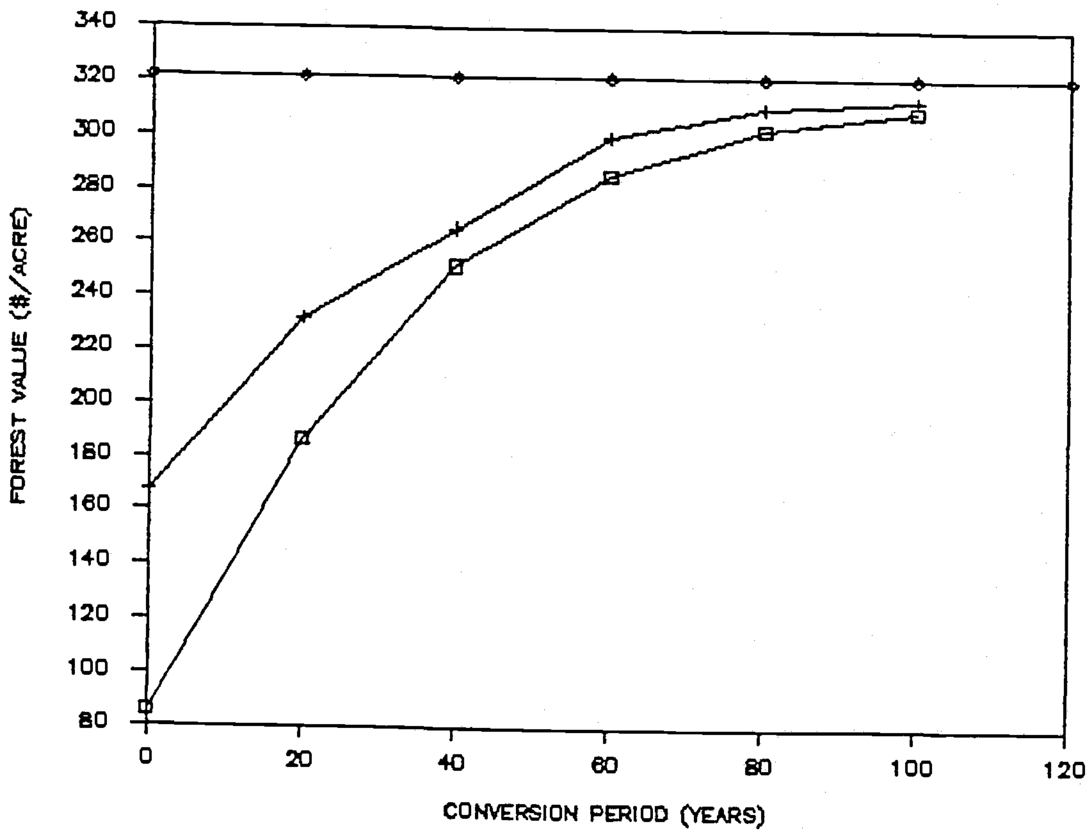

FIGURE 3. Values of conversion (—□—), conversion plus plantation (— + —), and perpetual uneven-aged management (—◇—) regimes for stand 1.

TABLE 13. Optimal uneven-aged management regime for stand 2 with no interplanting allowed (FV = $504.94/acre).

Diameter class midpoint (inches)	Year							
	0	20	40	60	80	100	120	200
	--Trees per acre cut--							
seedlings[a]	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
saplings	.0	.0	.0	.0	.0	.0	.0	.0
1.5	.0	.0	.0	.0	.0	.0	.0	.0
3.5	.0	.0	.0	.0	.0	.0	.0	.0
5.5	.0	.0	.0	.0	.0	.0	.0	.0
7.5	.0	.0	.0	.0	.0	.0	.0	.0
9.5	.0	.0	.0	.0	.0	.0	.0	.0
11.5	.0	.0	.0	.0	.0	.0	.0	.0
13.5	.0	.0	.0	.0	.0	.0	.0	6.5
15.5	.0	.0	.0	1.4	9.3	9.6	7.3	12.0
17.5	.0	.0	.0	.1	5.8	8.6	7.2	8.3
19.5	.0	.4	.6	.0	2.0	3.9	3.7	1.9
21.5	2.7	2.3	1.3	.0	.2	.3	.4	.0
23.5	2.5	2.2	1.5	.0	.0	.0	.0	.0
25.5	1.9	.8	.5	.0	.0	.0	.0	.0
27.5	1.5	.0	.0	.0	.0	.0	.0	.0
29.5	1.4	.0	.0	.0	.0	.0	.0	.0
31.5	.0	.0	.0	.0	.0	.0	.0	.0
total trees	10	6	4	2	17	22	19	29
total sq.ft.	34	16	11	3	26	36	30	41
total $	351	150	99	15	158	222	191	229
	--Residual trees per acre--							
seedlings	0.0	10844.0	5567.0	1243.0	8411.0	10842.0	9739.0	8318.0
saplings	.0	53.0	28.0	3.0	35.0	51.0	45.0	41.0
1.5	.0	15.6	18.1	6.4	8.7	26.5	27.6	24.3
3.5	.0	.3	20.3	11.6	3.4	14.5	25.2	17.0
5.5	.0	.0	12.1	11.3	6.7	4.6	16.6	14.5
7.5	.0	.0	5.2	13.9	9.7	4.1	11.8	13.6
9.5	.2	.0	.9	14.1	10.7	6.9	6.0	13.4
11.5	.8	.0	.0	9.5	11.8	8.8	4.4	13.6
13.5	1.8	.1	.0	4.7	11.7	9.5	6.0	6.7
15.5	2.2	.8	.0	.0	.0	.0	.0	.0
17.5	2.5	1.6	.0	.0	.0	.0	.0	.0
19.5	2.8	1.7	.0	.0	.0	.0	.0	.0
21.5	.0	.0	.0	.0	.0	.0	.0	.0
total trees	10	20	57	72	63	75	98	103
total sq.ft.	15	7	6	26	30	23	21	31
total $	93	44	-24	33	72	46	4	36

a. Seedling and sapling classes include trees 0 to 3 feet and 3 to 6 feet in height, respectively.

with a downward-sloping diameter distribution maintained thereafter. By year 200, a steady state harvesting regime similar to the steady state for stand 1 is achieved. The FV of the transition and steady-state harvesting regimes is $504.94/acre.

For stand 2, the PNW's of optimal conversion regimes that terminate in clearcut in years 0 to 100 are plotted in Figure 4. Values of the conversion regimes rise and become asymtotic to the FV for optimal uneven-aged management. In this case, adding the discounted LEV of plantation management to the PNW of each of the conversion regimes creates two even-aged management regimes that have higher FV's than the FV for uneven-aged management. Conversion to even-aged management in period 0, which produces a FV of $526.15/acre, is the optimal management regime. These results demonstrate that the management system that maximizes FV for a given set of planting costs and stumpage prices depends on the joint productive capability of the land and existing timber.

I also found in section 2 that for any initial stand structure uneven-aged harvesting will have a higher FV than conversion to even-aged management when the two systems are performed with the same planting costs and stumpage prices. To demonstrate this result, I developed a second uneven-aged management regime for stand 2 with the assumption that planting in each period is costless (Table 14). In the first period a two-aged stand is

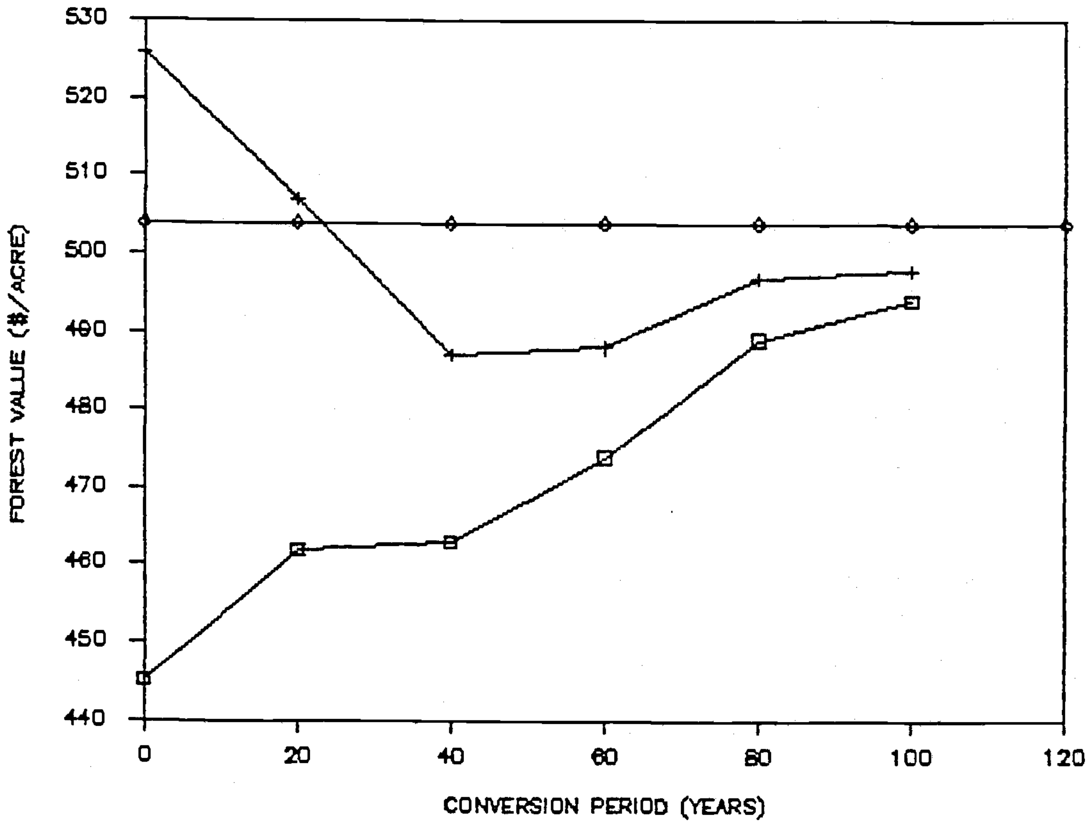

FIGURE 4. Values of conversion (—□—), conversion plus plantation (— + —), and perpetual uneven-aged management (—◇—) regimes stand 2.

TABLE 14. Optimal uneven-aged management regime for stand 2 with interplanting allowed (FV = $545.87/acre).

Diameter class midpoint (inches)	Year 0	20	40	60	80	100	160	200
	Trees per acre cut							
seedlings[a]	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
saplings	.0	.0	.0	.0	.0	.0	.0	.0
1.5	.0	.0	.0	.0	.0	.0	.0	.0
3.5	.0	.0	.0	.0	.0	.0	.0	.0
5.5	.0	.0	.0	.0	.0	.0	.0	.0
7.5	.0	.0	.0	.0	.0	.0	.0	.0
9.5	.0	.0	3.4	20.4	42.5	.0	.0	.0
11.5	.0	.0	.7	37.5	26.7	16.8	.0	.0
13.5	.0	.0	.0	18.1	6.8	28.1	.0	13.7
15.5	.0	.8	.0	3.4	.3	12.3	8.7	9.9
17.5	.0	1.6	.0	.1	.0	1.0	8.2	4.4
19.5	.0	2.1	.0	.0	.0	.0	3.7	.5
21.5	2.7	2.3	.0	.0	.0	.0	1.0	.0
23.5	2.5	2.2	.0	.0	.0	.0	.1	.0
25.5	1.9	.8	.0	.0	.0	.0	.0	.0
27.5	1.5	.0	.0	.0	.0	.0	.0	.0
29.5	1.4	.0	.0	.0	.0	.0	.0	.0
31.5	.0	.0	.0	.0	.0	.0	.0	.0
total trees	10	10	4	80	76	58	22	29
total sq.ft.	34	22	4	61	67	59	36	35
total $	351	192	20	193	242	247	229	176
	Residual trees per acre							
seedlings	538.0	11221.0	1152.0	709.0	1291.0	3228.0	10935.0	5900.0
saplings	.0	159.0	18.0	3.0	5.0	13.0	54.0	25.0
1.5	.0	171.6	60.6	20.8	6.7	4.8	32.2	17.6
3.5	.0	4.1	92.2	45.3	18.4	6.1	26.6	19.3
5.5	.0	.0	89.9	67.3	36.5	14.5	16.0	21.0
7.5	.0	.0	55.9	68.2	49.2	26.2	10.2	21.5
9.5	.2	.0	10.6	36.0	48.6	33.7	6.4	19.0
11.5	.8	.0	.0	.0	.0	17.4	5.2	16.8
13.5	1.8	.0	.0	.0	.0	.0	6.5	.0
15.5	2.2	.0	.0	.0	.0	.0	.0	.0
17.5	2.5	.0	.0	.0	.0	.0	.0	.0
19.5	2.8	.0	.0	.0	.0	.0	.0	.0
21.5	.0	.0	.0	.0	.0	.0	.0	.0
total trees	10	176	309	238	159	103	103	115
total sq.ft.	15	4	45	53	47	40	22	34
total $	93	-37	-136	-60	1	43	7	19

a. Seedling and sapling classes include trees 0 to 3 feet and 3 to 6 feet in height, respectively.

created by planting 538 trees and harvesting the existing trees that are 21.5 inches in diameter or greater. In year 20 the remaining merchantable trees are cut leaving a fully stocked even-aged stand. In the next seven periods a pulse of trees grows through the diameter distribution with harvesting taking place when trees reach 9.5 inches in diameter or greater. By year 160 a downward-sloping distribution is obtained, which is followed by a weaker pulse cycle to year 200. No planting takes place after the first period.

In contrast to the first uneven-aged management regime for stand 2, planting in period 1 allows the initial stand to be completely cut by year 20 and creates an even-aged stand with higher density in year 20. As a result, net revenue is higher in every period except one for the first 160 years. The PNW of the 200-year uneven-aged management regime with interplanting is 8.1% greater than the FV of the uneven-aged regime without interplanting, and 4.2% greater than the FV of the best even-aged regime.

CONCLUSIONS

Traditional harvesting theory in forest management has concentrated on the determination of the value of bare land. For even-aged harvesting, the theory has assumed that timberland is initially without timber and the decision variable is the optimal plantation rotation age.

For uneven-aged harvesting, the theory has assumed that the land contains a stand of timber that produces a steady-state harvest. The decision variables are the optimal level of growing stock and cutting cycle length. These analyses have ignored problems of timberland valuation under the more general assumptions that stands may initially contain timber and selective harvesting may be employed. Consequently, I have developed an economic model for harvesting and planting that treats land and timber as fixed inputs.

Explicit in the traditional models for stand management is the assumption that steady-state harvesting takes place from time zero to infinity. The model for harvesting that we present allows the determination of optimal conversion regimes that terminate in clearcut and plantation management and the determination of optimal transition harvesting that converges to steady-state uneven-aged management. Thus, regimes that maximize forest value, which is the present value of net revenues that acrue to the land and existing timber under even-aged or uneven-aged management, can be determined.

Analysis of the problem formulation for uneven-aged management shows that land value cannot be separated from the value of the optimal transition regime. As a result, forest value is the correct criterion for comparing the profitability of even-aged and uneven-aged harvesting systems. Since forest value for each management system depends on the joint productivity of the land and existing timber, conclusions about the most profitable land use will depend on the structure of the initial stand. Further, under certain cost and price assumptions, uneven-aged harvesting with interplanting will produce the highest forest value for any initial stand structure. Thus, the dynamic model for stand harvesting provides results that differ significantly from the results of steady-state harvesting models.

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OVERALL CONCLUSIONS

The studies in this dissertation have shown that it is feasible to obtain numerical solutions for the optimal sequence of diameter distributions and selection harvests for uneven-aged stand management. The algorithms presented here are the first to produce harvest regimes that are stable and interpretable on the basis of economic principles. These techniques can help answer stand-level management questions by providing optimal harvest policies that are baselines for comparison with other selection harvest regimes and for comparison with other uses of the land and timber.

These studies have also provided a general economic model for stand-level management that is amenable to mathematical analysis. The mathematical properties of the economic model can provide insights into the general characteristics of optimal steady-state management regimes.

The economic model and solution algorithm presented here open a new avenue for research in stand-level management. The analysis of the model can be extended in several ways. For example, it would be interesting to determine the degree to which the bimodality of the optimal equilibrium harvest policy (Getz 1980) is influenced by the removal of the assumptions of linearity in the objective function and diameter-class growth dynamics. The stability properties of equilibrium solutions can be used to develop an algorithm for determining the optimal sequence of harvests with the constraint that an equilibrium regime is attained after a specified time period.

The solution algorithms developed here can be used for a wide range of simulation studies. For example, the question of which harvesting system provides higher economic benefits can be addressed. The simulator for Arizona ponderosa pine is ideal for this study because it was developed to make accurate projections in even-aged and uneven-aged stands. Yet to be developed are functions for harvesting, planting and grazing that more accurately reflect management costs.

The restart algorithm is ideal for simulating feedback harvest policies that respond to random jumps in stumpage prices. These policies may be good approximations of optimal stochastic harvest policies. They also may be used to compare the flexibility of even-aged and uneven-aged harvesting systems in uncertain timber markets.

Finally, the algorithms presented here need to be adapted to single-tree/distance-independent simulators. This modeling approach is the dominant method for projecting the growth and yield in mixed conifer stands in the western United States.

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APPENDIX

APPENDIX

The marginal production rules and shadow price definitions for equilibrium harvest regimes in dynamic and static optimization problems are shown below.

__Problem 1: Dynamic Optimization__

$$P_j - u_{ij}^* + v_{ij}^* = \sum_{k=1}^{m} \frac{P_k + v_{ik}^*}{(1 + r)^t} \frac{df_{ik}}{dS_{ij}} + \frac{P_j}{(1 + r)^t} + \frac{v_{ij}^*}{(1 + r)^t}$$

where

u_{ij}^* = the increase in PNW that would result from having one less unit of growing stock in diameter class j in the residual stand

v_{ij}^* = the increase in PNW that would result from having one more unit of growing stock in diameter class j in the residual stand.

__Problem 2: Static Optimization__

$$P_j - a_j^* + c_j^* = \sum_{k=1}^{m} \frac{P_k + b_k^*}{(1 + r)^t} \frac{df_{0k}}{dS_{0j}} + \frac{P_j}{(1 + r)^t} + \frac{c_j^*}{(1 + r)^t}$$

where

$$\frac{a_j^*(1 + r)^t}{[(1 + r)^t - 1]}$$ = the increase in PNW that would result from having one less unit of growing stock in diameter class j in the residual stand

$$\frac{b_j^*}{[(1 + r)^t - 1]}$$ = the increase in PNW that would result from having one more unit of growing stock in diameter class j in the residual stand

c_j^* = the increase in PNW that would result from having one more unit of growing stock in diameter class j in the residual stand.