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In this paper, we study the continuity problem for multiparameter maximal functions of type

$$T^* f(x,y) = \sup_{s,t>0} \left| \int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} f(x - sx', y - ty') K(x', y') \, dx' \, dy' \right|$$

with respect to the norm of $L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$. The boundedness criterion for linear operators mapping from Hardy space to L^1 related with the atomic decomposition of Hardy space, the duality between Hardy space and BMO, and the Littlewood-Paley theory in the product space setting play central roles in establishing our results.

Multiparameter Maximal Operators and

Square Functions on Product Spaces

by

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1. INTRODUCTION

The present paper is devoted to establishing results of L^p boundedness for the maximal operator arising in connection with singular integrals on product spaces. From the motivational point of view, it will perhaps be best to start by describing certain classical situations as to the theory of multipliers which lead naturally to our topic in question. Consider a bounded function m in $\mathbf{R}^n$ and let T be the operator defined by $(Tf)^\wedge = m\hat{f}$. The function m is said to be a multiplier for the linear space $\mathcal{L}$ of functions f, if $f \in \mathcal{L}$ implies $Tf \in \mathcal{L}$. It follows via the Fourier transform that a large class of singular integrals including the Calderón-Zygmund operators and those which commute with translations can be equivalently viewed as multiplier operators. The primary benefit of this approach lies in the fact that we can make use of explicit analysis on multipliers (decay, smoothness, etc.) to study the continuity of singular integral operators with the aid of suitable interpolations. Although we have precedents, let us mention the classical theorem of L. Hörmander [39] which in a simplified version states that if $|D^\alpha m(x)| \leq C\,|x|^{-|\alpha|}, \quad |\alpha| \leq \left[\dfrac{n}{2}\right] + 1$, then m is a multiplier for L^p, $1 < p < \infty$.

On the other hand, the existence question for the pointwise limit of certain sequence of singular integrals turns out to be directly related to the continuity problem for the maximal multiplier operator

$$T^*f(x) = \sup_{t>0} |(T_t f)(x)|, \quad (T_t f)^\wedge(x) = m(tx)\hat{f}(x).$$

In this connection, a typical illustrating example is that of the maximal spherical average, studied by E.M. Stein [49] and by J. Bourgain [2],

$$\mathcal{M}^* f(x) = \sup_{t>0} \left| \int_{|y|=1} f(x - ty) \, d\sigma(y) \right|.$$

They substantiated its L^p boundedness for $p > \dfrac{n}{n-1}$ and in particular, the method of E.M. Stein's proof exploited the Littlewood-Paley theory, the known decay estimates for $m = \hat{\sigma}$, and the complex analytic interpolation theorem in the passage of $p \neq 2$, which has prevailed in many problems, often accompanied by techniques of decay mesurements such as Van der Corput's lemma and oscillatory integrals (see [14], [15], [22], [28], [48], [52], etc.). A significant generalization of the preceding results was implemented in the work of Rubio de Francia which gives us a direct motivation and which contains the seeds of some of the methods we shall employ later. His results can be summarized as follows: Let $k = \left[\dfrac{n}{2}\right] + 1$. If $m \in C^{k+1}(\mathbf{R}^n)$ and $|D^\alpha m(x)| \leq C |x|^{-a}$ for some $a > \dfrac{1}{2}$, $|\alpha| \leq k + 1$, then T^* is bounded in L^p for $\dfrac{2n}{n+2a-1} < p < \dfrac{2n-2}{n-2a}$. Moreover, if m is the Fourier transform of a compactly supported Borel measure and $|m(x)| \leq C |x|^{-a}$, then T^* is bounded in L^p for $p > \dfrac{2a}{2a+1}$. (See [46] for the details)

Let us turn now to the discussion of singular integrals on product spaces. It results mainly from the difficult natures in attempting to extend results of classical theory by iteration that to some extent the Harmonic Analysis on product spaces has been developed along very different lines. In analogy with the Euclidean case, for example, let T be the operator given by $(Tf)\widehat{\ }(\xi,\eta) = m(\xi,\eta)\hat{f}(\xi,\eta)$. In the light of the Hörmander multiplier theorem, only under the strong entailing decay assumption

$$\left| \partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta) \right| \leq C_{\alpha,\beta} |\xi|^{-|\alpha|} |\eta|^{-|\beta|} \quad \text{for sufficiently large } |\alpha|, |\beta|,$$

R. Gundy and E.M. Stein [38] were able to show that m is a multiplier for $L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$, $1 < p < \infty$. Since their proof was based on the idea of pointwise majorization of T by the Littlewood-Paley product g and g_λ functions, one can not obtain the (H^1, L^1)-inequality without imposing a considerable amount of smoothness hypothesis on m.

The complicated structures of Hardy spaces and BMO in the product space can be seen by the counterexample of L. Carleson [9] against the rectangle atomic decomposition conjectures about those spaces. Nevertheless, due to the boundedness criterion set up by R. Fefferman [29] and by J.L. Journé [41], it suffices to work on the rectangle atoms so as to establish the (H^1, L^1) estimates for L^2-bounded linear operators (we will explain the details in the section 3). By the use of this criterion, in a recent paper [17], L.K. Chen proved several results on multiplier operators one of which asserts that if

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right| \leq C|\xi|^{-|\alpha|}|\eta|^{-|\beta|} \quad \text{for} \quad |\alpha| \leq \left[\frac{n_1}{2}\right] + 1, |\beta| \leq \left[\frac{n_2}{2}\right] + 1,$$

then m is a multiplier for $L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$, $1 < p < \infty$, while R. Fefferman and K.C. Lin [33] acquired the same result with weaker conditions on m.

It is clear by now that there arises naturally the question of continuity properties for multiparameter maximal functions in the product space setting. To be more specific, for a given bounded fuction m on $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$, we define a family of operators $\{T_{s,t}\}_{s,t>0}$ by $(T_{s,t}f)\widehat{\ }(\xi, \eta) = m(s\xi, t\eta)\hat{f}(\xi, \eta)$ $(\xi \in \mathbf{R}^{n_1}, \eta \in \mathbf{R}^{n_2})$ and we shall deal in detail with the maximal operator T^* given by

$$T^* f(x,y) = \sup_{s,t>0} |(T_{s,t}f)(x,y)| \quad (x \in \mathbf{R}^{n_1}, y \in \mathbf{R}^{n_2}).$$

We are to use notations $\quad k_i = \left[\frac{n_i}{2}\right] + 1$, $i = 1, 2$. In consideration of the continuity of T^*, we establish the following theorem:

Theorem A. *Suppose that* $m \in C^{k_1+1,k_2+1}(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ *and*

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)\right| \leq C(1+|\xi|)^{-a}(1+|\eta|)^{-b},$$

with some $a,b > \dfrac{1}{2}$ *and all multi-indices* $\alpha, \beta,$ $|\alpha| \leq k_1+1, |\beta| \leq k_2+1.$ *Then we have*

$$\|T^*f\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})} \leq A_p\|f\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})}, \quad q_{a,b} < p < r_{a,b},$$

where

$$q_{a,b} = \max\left(\frac{2(n_1+2)}{n_1+2a+1}, \frac{2(n_2+2)}{n_2+2b+1}\right), \quad \text{and}$$
$$r_{a,b} = \min\left(\frac{2(n_1+1)}{n_1-2a+2}, \frac{2(n_2+1)}{n_2-2b+2}\right).$$

It is to be interpreted that if $a \geq \dfrac{n_1+2}{2}, b \geq \dfrac{n_2+2}{2},$ then we take $r_{a,b} = \infty$ and in case when $a \geq \dfrac{n_1+3}{2}, b \geq \dfrac{n_2+3}{2},$ we take $q_{a,b} = 1.$ Applying this theorem, we immediately obtain the following subsequent result:

Theorem B. *Assume that m is the Fourier transform of a compactly supported Borel measure and*

$$|m(\xi,\eta)| \leq C(1+|\xi|)^{-a}(1+|\eta|)^{-b}, \quad \text{for some} \quad a,b > \frac{1}{2}.$$

Then

$$\|T^*f\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})} \leq C_p\|f\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})}, \quad q_{a,b} < p < \infty.$$

As an illustration, taking into account the maximal elliptical average

$$\mathcal{S}^*f(x,y) = \sup_{s,t>0}\left|\int_{|\theta|=1} f(x-s\theta_1, y-t\theta_2)\,d\sigma(\theta)\right|, \quad \theta = (\theta_1,\theta_2),$$

where $d\sigma$ denotes the unit surface area measure, we get

$$\|\mathcal{S}^*f\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})} \leq C\|f\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})}$$

for

$$\max\left(\frac{4(n_1+2)}{3n_1+n_2+1}, \frac{4(n_2+2)}{3n_2+n_1+1}\right) < p < \infty, \quad \text{if} \quad n_1+n_2 > 3.$$

The main flow of the proof for the Theorem A will be essentially along the same charts as that of Rubio de Francia's. The proof will consist of a chain of lemmas and throughout this paper, C will denote a constant which might be different in each occurences. For an appropriate function $f(x,y)$ in $\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$, we shall denote by $\hat{f}^1(\xi,y)$, $\hat{f}^2(x,\eta)$ the Fourier transform of $f(x,y)$ acting only on x-variables, y-variables, respectively.

2. Square functions and L^2 estimates

We shall exploit the decay hypothesis as well as the smoothness of $m(\xi,\eta)$, by the use of certain square functions, in order to study the L^2-behavior of our maximal operator T^*. We set about making an appropriate decomposition. In the standard manner, we shall consider the partition of unity on $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ subordinate to the dyadic rectangles $\{R_{ij}\}_{i,j\in\mathbf{Z}}$,

$$R_{ij} = \left\{ (\xi,\eta) \mid 2^{i-1} \le |\xi| < 2^{i+1}, 2^{j-1} \le |\eta| < 2^{j+1} \right\},$$

which will enable us to write $m(\xi,\eta)$ as the sum of smooth dyadic building blocks. Specifically, we fix a radial Schwartz function ϕ on $\mathbf{R}$ so that

$$\mathrm{supp}(\phi) \subset \{\frac{1}{2} < |t| < 2\}, \qquad 0 \le \phi(t) \le 1 \qquad \text{for all} \quad t,$$

and

$$\sum_{k\in\mathbf{Z}} \phi(2^{-k}t) = 1 \qquad \text{for} \quad t \ne 0.$$

Clearly,

$$\sum_{i,j\in\mathbf{Z}} \phi(2^{-i}\xi)\phi(2^{-j}\eta) = 1,$$

where

$$\phi(\xi) = \phi(|\xi|), \quad \phi(\eta) = \phi(|\eta|) \qquad \text{for} \quad \xi \in \mathbf{R}^{n_1}, \eta \in \mathbf{R}^{n_2}.$$

With this decomposition, however, because of the restricted decay assumptions we imposed, we encounter some obstacles in analyzing the multiplier operators supported on the strip regions

$$\left\{ |\xi| \le 1, \eta \in \mathbf{R}^{n_2} \right\}, \quad \left\{ \xi \in \mathbf{R}^{n_1}, |\eta| \le 1 \right\}.$$

Ideally, we would like to utilize the classical results together with the smoothness of m near the origin to investigate the actions of our operators on the aforementioned regions. To implement this aim, we proceed as follows.

Pick another auxiliary radial Schwartz function p on $\mathbf{R}$ such that

$$\operatorname{supp}(p) \subset \{|t| < \tfrac{1}{2}\}, \qquad p(t) = 1 \quad \text{if} \quad |t| \leq \tfrac{1}{3},$$

and consider the smooth Taylor polynomials associated with m

$$\Phi_1(\xi, \eta) = (1 - p(\xi)) p(\eta) \sum_{|\rho| \leq k_2} (\partial_\eta^\rho m)(\xi, 0) \frac{\eta^\rho}{\rho!},$$

$$\Phi_2(\xi, \eta) = (1 - p(\eta)) p(\xi) \sum_{|\sigma| \leq k_1} (\partial_\xi^\sigma m)(0, \eta) \frac{\xi^\sigma}{\sigma!},$$

$$\Phi_3(\xi, \eta) = p(\xi) p(\eta) \left\{ \sum_{|\rho| \leq k_2} (\partial_\eta^\rho m)(\xi, 0) \frac{\eta^\rho}{\rho!} \;+\; \sum_{|\sigma| \leq k_1} (\partial_\xi^\sigma m)(0, \eta) \frac{\xi^\sigma}{\sigma!} \right.$$

$$\left. - \sum_{|\sigma| \leq k_1, |\rho| \leq k_2} (\partial_\eta^\rho \partial_\xi^\sigma m)(0, 0) \frac{\xi^\sigma \eta^\rho}{\sigma! \rho!} \right\},$$

$$\Phi(\xi, \eta) = \Phi_1(\xi, \eta) + \Phi_2(\xi, \eta) + \Phi_3(\xi, \eta),$$

where $\quad p(\xi) = p(|\xi|), \quad p(\eta) = p(|\eta|)$. Setting $\quad \mu(\xi, \eta) = m(\xi, \eta) - \Phi(\xi, \eta)$, we have

$$\mu(\xi, \eta) = \sum_{i, j \in \mathbf{Z}} \phi(2^{-i}\xi) \phi(2^{-j}\eta) \mu(\xi, \eta)$$

$$= \sum_{i, j \in \mathbf{Z}} \mu_{ij}(\xi, \eta), \qquad \text{with}$$

$$\mu_{ij}(\xi, \eta) = \phi(2^{-i}\xi) \phi(2^{-j}\eta) \mu(\xi, \eta) \qquad \text{for integers} \quad i, j.$$

Note that

$$\operatorname{supp}(\mu_{ij}) \subset \left\{ (\xi, \eta) \mid 2^{i-1} < |\xi| < 2^{i+1}, 2^{j-1} < |\eta| < 2^{j+1} \right\}.$$

Let us define the Fourier multipliers on $L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ by

$$\left(T_{ij}^{s,t} \right)^{\widehat{\,}}(\xi, \eta) = \mu_{ij}(s\xi, t\eta) \hat{f}(\xi, \eta) \qquad \text{for} \qquad s, t > 0$$

and let T_{ij}^* denote its maximal operator

$$T_{ij}^* f(x, y) = \sup_{s, t > 0} \left| T_{ij}^{s,t} f(x, y) \right|.$$

With the kernel

$$K_{s,t}(x,y) = s^{-n_1} t^{-n_2} K\left(x/s, y/t\right), \qquad \hat{K} = \Phi,$$

we have

$$T^* f(x,y) \leq \sup_{s,t>0} \left| T_{s,t} f(x,y) - (K_{s,t} * f)(x,y) \right|$$

$$+ \sup_{s,t>0} \left| (K_{s,t} * f)(x,y) \right|$$

$$\leq \sum_{i,j \in \mathbf{Z}} T_{ij}^* f(x,y) + \sup_{s,t>0} \left| (K_{s,t} * f)(x,y) \right|.$$

Based on the results of one-parameter case, we observe the following simple facts regarding the latter term:

Lemma 2.1.

$$\left\| \sup_{s,t>0} \left| (K_{s,t} * f)(x,y) \right| \right\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \leq A_p \| f \|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})},$$

provided

$$\max\left(\frac{2n_1}{n_1 + 2a - 1}, \frac{2n_2}{n_2 + 2b - 1} \right) < p < \min\left(\frac{2(n_1 - 1)}{n_1 - 2a}, \frac{2(n_2 - 1)}{n_2 - 2b} \right).$$

Proof. First we write $*^1$, $*^2$ for the convolution operational symbols in each variables x, y and M_1, M_2 for the Hardy-Littlewood maximal functions acting only on x, y variables, respectively. Writing $\hat{K}^i = \Phi_i$, $i = 1, 2, 3$, we notice that

$$\sup_{s,t>0} \left| (K_{s,t} * f)(x,y) \right| \leq \sum_{i=1}^{3} \sup_{s,t>0} \left| (K_{s,t}^i * f)(x,y) \right|.$$

For each multi-index ρ, $|\rho| \leq k_2$, if we put

$$(L_s^\rho)\hat{}(\xi) = (1 - p(s\xi)) (\partial_\eta^\rho m)(s\xi, 0),$$

then for any Schwartz function f,

$$\left| (K_{s,t}^1 * f)(x,y) \right| \leq \sum_{|\rho| \leq k_2} C_\rho \left| \left[(\partial_y^\rho \check{p})_t *^2 (L_s^\rho *^1 f)(x, \cdot) \right](y) \right|.$$

By taking supremum, we see by the well-known property related to the Hardy-Littlewood maximal functions that

$$\sup_{s,t>0} \left| \left(K^1_{s,t} * f \right)(x,y) \right| \le \sum_{|\rho| \le k_2} C \sup_{s>0} M_2 \left(\left(L^\rho_s *^1 f \right)(x,\cdot) \right).$$

Since

$$\|f\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} = \left\| \|f(x,\cdot)\|_{L^p_{dy}} \right\|_{L^p_{dx}},$$

it follows readily that

$$\left\| \sup_{s,t>0} \left| \left(K^1_{s,t} * f \right)(x,y) \right| \right\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})}$$

$$\le \sum_{|\rho| \le k_2} C \left\| \sup_{s>0} M_2 \left(\left(L^\rho_s *^1 f \right)(x,\cdot) \right) \right\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})}$$

$$\le \sum_{|\rho| \le k_2} C \left\| \left\| M_2 \left(\sup_{s>0} \left| \left(L^\rho_s *^1 f \right)(x,\cdot) \right| \right) \right\|_{L^p_{dy}} \right\|_{L^p_{dx}}$$

$$\le \sum_{|\rho| \le k_2} C \left\| \left\| \sup_{s>0} \left| \left(L^\rho_s *^1 f \right)(x,\cdot) \right| \right\|_{L^p_{dy}} \right\|_{L^p_{dx}}$$

$$= \sum_{|\rho| \le k_2} C \left\| \left\| \sup_{s>0} \left| \left(L^\rho_s *^1 f \right)(\cdot,y) \right| \right\|_{L^p_{dx}} \right\|_{L^p_{dy}}.$$

As the function

$$(1 - p(\xi)) \left(\partial^\rho_\eta m \right)(\xi, 0) = \nu^\rho(\xi)$$

satisfies all of the hypotheses in the theorem of Rubio de Francia (see Theorem B, [46]), we have

$$\left\| \sup_{s>0} \left| \left(L^\rho_s *^1 f \right)(\cdot,y) \right| \right\|_{L^p_{dx}} \le B_p \|f(\cdot,y)\|_{L^p_{dx}}$$

for

$$\frac{2n_1}{n_1 + 2a - 1} < p < \frac{2n_1 - 2}{n_1 - 2a},$$

and in this range of p's,

$$\left\| \sup_{s,t>0} \left| \left(K^1_{s,t} * f \right)(x,y) \right| \right\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \le C \left\| \|f(\cdot,y)\|_{L^p_{dx}} \right\|_{L^p_{dy}}$$

$$= C \|f\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})}.$$

Similar treatments for other cases should complete the proof. $\square$

We shall now define

$$m_{ij}(\xi,\eta) = \sum_{|\alpha|=1} \left(\partial_\xi^\alpha \mu_{ij}\right)(\xi,\eta)\xi^\alpha, \qquad \left(Q_{ij}^{s,t}f\right)\widehat{\ }(\xi,\eta) = m_{ij}(s\xi,t\eta)\hat{f}(\xi,\eta),$$

$$\tilde{m}_{ij}(\xi,\eta) = \sum_{|\beta|=1} \left(\partial_\eta^\beta \mu_{ij}\right)(\xi,\eta)\eta^\beta, \qquad \left(\tilde{Q}_{ij}^{s,t}f\right)\widehat{\ }(\xi,\eta) = \tilde{m}_{ij}(s\xi,t\eta)\hat{f}(\xi,\eta),$$

$$\tilde{\mu}_{ij}(\xi,\eta) = \sum_{\substack{|\alpha|=1 \\ |\beta|=1}} \left(\partial_\eta^\beta \partial_\xi^\alpha \mu_{ij}\right)(\xi,\eta)\xi^\alpha\eta^\beta, \qquad \left(\tilde{T}_{ij}^{s,t}f\right)\widehat{\ }(\xi,\eta) = \tilde{\mu}_{ij}(s\xi,t\eta)\hat{f}(\xi,\eta),$$

for all integers i,j and $s,t > 0$.

Let us introduce the square functions in question

$$G_{ij}f(x,y) = \left(\int_0^\infty \int_0^\infty |T_{ij}^{s,t}f(x,y)|^2 \, \frac{dsdt}{st}\right)^{1/2},$$

$$S_{ij}f(x,y) = \left(\int_0^\infty \int_0^\infty |Q_{ij}^{s,t}f(x,y)|^2 \, \frac{dsdt}{st}\right)^{1/2},$$

and analogous $\tilde{G}_{ij}$, $\tilde{S}_{ij}$ corresponding to the operators $\tilde{T}_{ij}^{s,t}$, $\tilde{Q}_{ij}^{s,t}$, respectively. Now that

$$T_{ij}^{s,0}f \;=\; T_{ij}^{0,t}f \;=\; T_{ij}^{0,0}f \;=\; 0$$

for any Schwartz function f and $s,t > 0$, we have the following pointwise majorization for T_{ij}^*

$$\left(T_{ij}^*f\right)^2 \leq 2 \int_0^\infty \int_0^\infty |T_{ij}^{s,t}f| \, |\partial_s\partial_t T_{ij}^{s,t}f| \, dsdt$$

$$+ 2 \int_0^\infty \int_0^\infty |\partial_s T_{ij}^{s,t}f| \, |\partial_t T_{ij}^{s,t}f| \, dsdt$$

$$\leq 2(G_{ij}f)(\tilde{G}_{ij}f) \;\; + \;\; 2(S_{ij}f)(\tilde{S}_{ij}f),$$

a simple consequence of the Cauchy-Schwartz inequality. Here comes our key L^2 estimates for T_{ij}^*.

Lemma 2.2. *For integers* i, j,

$$\left\|T_{ij}^* f\right\|_{L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \leq C \Delta_{ij} \|f\|_{L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})}$$

where

$$\Delta_{ij} = \begin{cases} 2^{i(\frac{1}{2}-a)+j(\frac{1}{2}-b)}, & \text{for } i, j \geq 0 \\ 2^{i(\frac{1}{2}-a)+j}, & \text{for } i \geq 0, j < 0 \\ 2^{i+j(\frac{1}{2}-b)}, & \text{for } i < 0, j \geq 0 \\ 2^{i+j} & \text{for } i, j < 0. \end{cases}$$

Proof. Upon invoking Plancherel's theorem,

$$
\begin{aligned}
\|G_{ij}f\|_2^2 &= \int_0^\infty \int_0^\infty \int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} |T_{ij}^{s,t} f(x,y)|^2 \, dx \, dy \, \frac{ds \, dt}{st} \\
&= \int_0^\infty \int_0^\infty \int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} |\mu_{ij}(s\xi, t\eta)\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta \, \frac{ds \, dt}{st} \\
&= \int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} \left|\hat{f}(\xi,\eta)\right|^2 \underset{\substack{2^{i-1} < |s\xi| < 2^{i+1} \\ 2^{j-1} < |t\eta| < 2^{j+1}}}{\iint} |\mu_{ij}(s\xi, t\eta)|^2 \, \frac{ds \, dt}{st} \, d\xi \, d\eta \\
&\leq C \|\mu_{ij}\|_\infty^2 \|f\|_2^2, \quad \text{i.e.,} \quad \|G_{ij}f\|_2 \leq C \|\mu_{ij}\|_\infty \|f\|_2.
\end{aligned}
$$

Similar observations for other square functions lead us to

$$
\begin{aligned}
\|T_{ij}^* f\|_2^2 &\leq 2\|G_{ij}f\|_2 \|\tilde{G}_{ij}f\|_2 + 2\|S_{ij}f\|_2 \|\tilde{S}_{ij}f\|_2 \\
&\leq C \left(\|\mu_{ij}\|_\infty \|\tilde{\mu}_{ij}\|_\infty + \|m_{ij}\|_\infty \|\tilde{m}_{ij}\|_\infty\right) \|f\|_2^2 \qquad (2\text{-}1)
\end{aligned}
$$

and we need to estimate the appropriate L^∞-norms in order to prove the Lemma.

We shall deal only with $\|\mu_{ij}\|_\infty$ because other estimations are essentially similar.

(i) Case $i, j \geq 0$: In this case, by the support condition of Φ, we have

$$\mu_{ij}(\xi, \eta) = \phi(2^{-i}\xi)\phi(2^{-j}\eta)m(\xi, \eta).$$

The hypothesis on $m(\xi, \eta)$ implies instantly that $\|\mu_{ij}\|_\infty \leq C 2^{-ia-jb}$.

(ii) Case $i \geq 0$, $j < 0$: An inspection on the support of Φ shows that

$$\mu(\xi, \eta) = m(\xi, \eta) - \Phi_1(\xi, \eta)$$

$$= m(\xi, \eta) - p(\eta) \sum_{|\rho| \leq k_2} \left(\partial_\eta^\rho m \right) (\xi, 0) \frac{\eta^\rho}{\rho!}.$$

Note $\mu(\xi, 0) = 0$ and thus it follows from the Mean Value Theorem (MVT) that for some $|\bar\eta| \leq 1$,

$$\mu(\xi, \eta) = \sum_{|\tau|=1} \left[\left(\partial_\eta^\tau m \right) (\xi, \bar\eta) - \partial_\eta^\tau p(\bar\eta) \sum_{|\rho| \leq k_2} \left(\partial_\eta^\rho m \right) (\xi, 0) \frac{\bar\eta^\rho}{\rho!} \right.$$

$$\left. - p(\bar\eta) \sum_{|\rho| \leq k_2, \rho \geq \tau} \left(\partial_\eta^\rho m \right) (\xi, 0) \frac{\bar\eta^{\rho-\tau}}{(\rho-\tau)!} \right] \eta^\tau,$$

which immediately provides $\|\mu_{ij}\|_\infty \leq C 2^{-ia+j}$. Similarly, $\|\mu_{ij}\|_\infty \leq C 2^{i-jb}$ if $i < 0, j \geq 0$.

(iii) Case $i, j < 0$: Here we have

$$\mu(\xi, \eta) = m(\xi, \eta) - \Phi(\xi, \eta)$$

$$= m(\xi, \eta) - p(\eta) \sum_{|\rho| \leq k_2} \left(\partial_\eta^\rho m \right) (\xi, 0) \frac{\eta^\rho}{\rho!} - p(\xi) \sum_{|\sigma| \leq k_1} \left(\partial_\xi^\sigma m \right) (0, \eta) \frac{\xi^\sigma}{\sigma!}$$

$$+ p(\xi) p(\eta) \sum_{|\sigma| \leq k_1, |\rho| \leq k_2} \left(\partial_\eta^\rho \partial_\xi^\sigma m \right) (0, 0) \frac{\eta^\rho}{\rho!} \frac{\xi^\sigma}{\sigma!}.$$

As $\mu(\xi, 0) = 0$, MVT implicates

$$\mu(\xi, \eta) = \sum_{|\tau|=1} \left[\left(\partial_\eta^\tau m \right) (\xi, \bar\eta) - \partial_\eta^\tau p(\bar\eta) \sum_{|\rho| \leq k_2} \left(\partial_\eta^\rho m \right) (\xi, 0) \frac{\bar\eta^\rho}{\rho!} \right.$$

$$- p(\bar\eta) \sum_{|\rho| \leq k_2, \rho \geq \tau} \left(\partial_\eta^\rho m \right) (\xi, 0) \frac{\bar\eta^{\rho-\tau}}{(\rho-\tau)!} - p(\xi) \sum_{|\sigma| \leq k_1} \left(\partial_\eta^\tau \partial_\xi^\sigma m \right) (0, \bar\eta) \frac{\xi^\sigma}{\sigma!}$$

$$+ p(\xi) \partial_\eta^\tau p(\bar\eta) \sum_{|\sigma| \leq k_1, |\rho| \leq k_2} \left(\partial_\eta^\rho \partial_\xi^\sigma m \right) (0, 0) \frac{\xi^\sigma}{\sigma!} \frac{\bar\eta^\rho}{\rho!}$$

$$\left. + p(\xi) p(\bar\eta) \sum_{\substack{|\sigma| \leq k_1 \\ |\rho| \leq k_2, \rho \geq \tau}} \left(\partial_\eta^\rho \partial_\xi^\sigma m \right) (0, 0) \frac{\xi^\sigma}{\sigma!} \frac{\bar\eta^{\rho-\tau}}{(\rho-\tau)!} \right] \eta^\tau$$

$$= \sum_{|\tau|=1} \left(\sum_{|\kappa|=1} H(\bar\xi, \bar\eta) \xi^\kappa \right) \eta^\tau,$$

on account of MVT in ξ-variables for the inside expression, where H is certain continuous function of $\tilde{\xi}$, $\tilde{\eta}$ in the compact domain $\{|\tilde{\xi}| \leq 1, |\tilde{\eta}| \leq 1\}$. Consequently, $\|\mu_{ij}\|_\infty \leq C2^{i+j}$.

Putting all of the above estimates and other corresponding estimates into (2.1), we finish the proof. $\square$

We remark that Lemma 2.1 and Lemma 2.2 provide instantaneously the L^2-boundedness of T^* upon summing the suitable geometric series's;

$$
\begin{aligned}
\sum_{i,j \in \mathbf{Z}} \|T_{ij}^* f\|_2 &= \left(\sum_{i,j \geq 0} + \sum_{i \geq 0, j < 0} + \sum_{i < 0, j \geq 0} + \sum_{i,j < 0} \right) \|T_{ij}^* f\|_2 \\
&\leq C \sum_{i,j \geq 0} 2^{i(\frac{1}{2}-a)+j(\frac{1}{2}-b)} \|f\|_2 + C \sum_{i \geq 0, j < 0} 2^{i(\frac{1}{2}-a)+j} \|f\|_2 \\
&\quad + C \sum_{i < 0, j \geq 0} 2^{i+j(\frac{1}{2}-b)} \|f\|_2 + C \sum_{i,j < 0} 2^{i+j} \|f\|_2 \\
&\leq C \|f\|_2.
\end{aligned}
$$

3. THE $\left(H^1, L^1\right)$-THEORY

Let us focus now on the continuity question for T^* with respect to the norm of $L^p\left(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}\right)$, $p \neq 2$. In the classical theory of singular integrals on $\mathbf{R}^n$, by the Marcinkiewicz interpolation theorem and the standard duality argument, it suffices to establish the weak-type $(1,1)$ estimate

$$|\{x \in \mathbf{R}^n : |Tf(x)| > \alpha\}| \leq \frac{C}{\alpha}\|f\|_1, \quad \text{for} \quad \alpha > 0,$$

where T is an L^2-bounded sublinear singular integral operator. As far as the method of the above estimate is concerned, it is the Calderón-Zygmund decomposition lemma that provides the most fundamental ingredient (see [8] and [51]). A great deal of extension and refinement of the Calderón-Zygmund lemma has been afforded through the consolidation of the theory of Hardy spaces, certain characterizations of which are shown to be intimately linked with maximal functions, singular integrals, and Littlewood-Paley theory (refer E.M. Stein and G. Weiss [53], C. Fefferman and E.M. Stein [25] for the definition and description of $H^p(\mathbf{R}^n)$, $p > 0$). We shall discuss briefly only those subjects involving H^p spaces on product domains that are most relevant to our purpose.

To begin with, let us define $H^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ in the sprit of the real-variable characterizations of C. Fefferman and E.M. Stein [25] for the space $H^p(\mathbf{R}^n)$. Fix two arbitrary Schwartz functions ϕ_i on $\mathbf{R}^{n_i}$ with $\int \phi_i = 1$, $i = 1, 2$, and write

$$\phi_{s,t}(x,y) = s^{-n_1} t^{-n_2} \phi_1(x/s)\phi_2(y/t), \quad s, t > 0.$$

We also choose any nonzero radial Schwartz function ψ on $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ satisfying $\int \psi(x,y)\,dx = 0$ and $\int \psi(x,y)\,dy = 0$. Further, $\Gamma(x,y)$ will denote the product cone in the bi-half space, $\Gamma(x,y) = \Gamma(x) \times \Gamma(y)$, where $\Gamma(x)$, $\Gamma(y)$ are the cones with vertices at $(x,0)$, $(y,0)$ in $\mathbf{R}_+^{n_1+1}$, $\mathbf{R}_+^{n_2+1}$, separately. Given a function f on

$\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$, we define its Littlewood-Paley-Stein S function to be

$$S_\psi^2(f)(x,y) = \iint_{\Gamma(x,y)} |\psi_{s,t} * f(x_1, x_2)|^2 \, \frac{dx_1 \, dx_2 \, ds \, dt}{s^{n_1+1} t^{n_2+1}}$$

(One should note that for $1 < p < \infty$, $\|S_\psi(f)\|_p \sim \|f\|_p$ as a direct consequence of Littlewood-Paley theory). We use the bi-Poisson kernel $P_{s,t}(x,y)$, the product of one-parameter Poisson kernels, to define the biharmonic extension of f by

$$u(x,y,s,t) = (P_{s,t} * f)(x,y)$$

and form the Lusin area integral as

$$A^2(u)(x,y) = \iint_{\Gamma(x,y)} |\nabla_1 \nabla_2 u(x_1, x_2, s, t)|^2 \, \frac{dx_1 \, dx_2 \, ds \, dt}{s^{n_1-1} t^{n_2-1}}.$$

For any tempered distribution f on $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$, $p > 0$, we say that

$$f \in H^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$$

$$\iff f^*(x,y) = \sup_{s,t>0} |\phi_{s,t} * f(x,y)| \in L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$$

the nontangential maximal function

$$\iff N(f)(x,y) = \sup_{\Gamma(x,y)} |\phi_{s,t} * f(x_1, x_2)| \in L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$$

$$\iff S_\psi(f) \in L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}) \iff A(u) \in L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}),$$

where one may define the 'norm' $\|f\|_{H^p}$ to be any one of

$$\|f^*\|_{L^p} \sim \|N(f)\|_{L^p} \sim \|S_\psi(f)\|_{L^p} \sim \|A(u)\|_{L^p}$$

(cf. R. Gundy and E.M. Stein [38], A. Chang and R. Fefferman [11], [12], [13]).

In other words, the space $H^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ is an extremely nice subspace of $L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ which is stable under the action of any reasonable singular integral or maximal operator, in which the Littlewood-Paley characterizations continue to hold, and in which each element can be realized as the boundary distributional

values of its harmonic extension to $\mathbf{R}_+^{n_1+1} \times \mathbf{R}_+^{n_2+1}$ (For the precise description, consult the exciting article of R. Fefferman [31]). In particular, for $1 < p < \infty$, $H^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ is naturally isomorphic to $L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ and when $p = 1$, we have the characterization

$$
\begin{aligned}
H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}) = \{ & f \in L^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}) : R_j^{(1)} \in L^1, R_k^{(2)} \in L^1, \\
& R_j^{(1)} R_k^{(2)} \in L^1, j = 1, 2, \dots, n_1, k = 1, 2, \dots, n_2 \},
\end{aligned}
$$

where $R_j^{(1)}, R_k^{(2)}$ are the Riesz transforms associated to $\mathbf{R}^{n_1}, \mathbf{R}^{n_2}$, respectively.

Let us turn now to a concise description of the atomic decomposition for the space $H^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$, which turned out not to be a routine iterative extension of the classical results of R. Coifman [19] and R. Latter [42], due to a counterexample of L. Carleson [9].

According to A. Chang and R. Fefferman [11], [12], for $0 < p \leq 1$, an H^p function f on $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ can be decomposed into atoms a_Ω supported on open sets Ω of finite measure such that $a_\Omega = \sum_{S \in \mathcal{M}(\Omega)} \alpha_S$, where $\mathcal{M}(\Omega)$ denotes the maximal class of dyadic rectangles (product of cubes) $S \subset \Omega$. The rectangle atoms α_S are supported in a 2-fold enlargement of S and have a certain number of vanishing moments in each variables separately. Moreover,

$$
\|a_\Omega\|_{L^2}^2 \leq |\Omega|^{1 - \frac{2}{p}} \quad \text{and} \quad \sum_{S \in \mathcal{M}(\Omega)} \|\alpha_S\|_{L^2}^2 \leq |\Omega|^{1 - \frac{2}{p}}.
$$

The relevance of the preceding decomposition stems from the fact that it yields a number of important results through its simple applications. As a useful example, an elaboration of the technique for the proof of the atomic decomposition leads to the analogous result of the Calderón-Zygmund lemma in the product space setting.

Theorem 3.1 (A. Chang and R. Fefferman[12]).

For $\alpha > 0$, $f \in L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$, $1 < p < 2$, there exist $g \in L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ and $b \in H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ satisfying

$$f = g + b, \quad \|g\|_2^2 \le \alpha^{2-p} \|f\|_p^p, \quad \text{and} \quad \|b\|_{H^1} \le C\alpha^{1-p} \|f\|_p^p.$$

On account of the Marcinkiewicz interpolation theorem, we immediately attain

Theorem 3.2.

Let T be a sublinear operator with the property that

$$\|Tf\|_{L^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \le B_1 \|f\|_{H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})},$$

$$\|Tf\|_{L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \le B_2 \|f\|_{L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})}.$$

Then we have

$$\|Tf\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \le C\, B_1^{1-\theta} B_2^\theta \|f\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \quad \text{for} \quad \frac{1}{p} = 1 - \frac{\theta}{2}, \, 0 < \theta < 1.$$

For the classical interpolation results on $H^p(\mathbf{R}^n)$, refer R. Coifman and G. Weiss [20], p.596.

These results are significant from the point of view of the Calderón-Zygmund machinery in attempting to establish the L^p-continuity of a singular integral operator. We are able to shift the focus of our attention from the 'weak' L^1 theory to the 'strong' (H^1, L^1)-theory where we are endowed with extremely nice functions and above all with atoms.

However, a major drawback of this atomic decomposition is the fact that atoms are supported on arbitrary open sets, or putting in another way, rectangle atoms do not span H^p spaces. With a view to circumventing this deficiency, R. Fefferman exploited his own version of Journé's geometric lemma [40] to set up a particularly

valuable result which can be regarded as a far-reaching extension of the Calderón-Zygmund theory to the two parameter product case. To describe in detail, we first clarify the meaning of a rectangle atom. If a function $a(x,y)$ supported on the rectangle $R = I \times J$ satisfies that

$$\|a\|_{L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \leq |R|^{1/2 - 1/p},$$

and for certain integer-valued $N(p)$,

$$\int_I a(x,y) x^\alpha \, dx = 0, \quad \int_J a(x,y) x^\beta \, dy = 0, \quad |\alpha|, \, |\beta| \leq N(p),$$

then a is called a rectangle atom on $H^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ (for $p = 1$, $N(p) = 0$). For $\gamma > 0$, we shall denote the concentric γ-fold dilation of R by $\tilde{R}_\gamma$ and its complement by $^c\tilde{R}_\gamma$.

Theorem 3.3 (R. Fefferman [29]). *Let T be a bounded sublinear operator on $L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$. Suppose that for any rectangle atom $a(x,y)$ supported on the rectangle R and $\gamma \geq 2$, we have*

$$\int_{^c\tilde{R}_\gamma} |Ta(x,y)|^p \, dx\, dy \leq C\, \gamma^{-\delta} \quad \text{with some} \quad \delta > 0.$$

Then T is bounded from $H^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ to $L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$.

For the purpose of studying the behaviors of our square functions, we shall examine the above criterion in detail. From now on, we assume that $a(x,y)$ is an $H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$-atom supported in $R = I \times J$ and write $l(I), l(J)$ for the side lengths of I, J, respectively. Since our square functions have similar structures, we shall mainly concentrate on the function G_{ij} and keep track of any necessary adjustment for other square functions later. Following R. Fefferman and K.C. Lin [33], we split $^c\tilde{R}_\gamma$ into three subsets

$$^c\tilde{R}_\gamma = (I \times {}^c\tilde{J}_\gamma) \cup ({}^c\tilde{I}_\gamma \times J) \cup ({}^c\tilde{R}_\gamma - I \times {}^c\tilde{J}_\gamma - {}^c\tilde{I}_\gamma \times J)$$

$$= {}^c\tilde{R}_\gamma^1 \cup {}^c\tilde{R}_\gamma^2 \cup {}^c\tilde{R}_\gamma^3, \quad \text{whenever } \gamma \geq 2,$$

and note

$$\int_{c\tilde{R}_\gamma} |G_{ij}a(x,y)|\, dx\, dy \;=\; \left(\int_{c\tilde{R}_\gamma^1} + \int_{c\tilde{R}_\gamma^2} + \int_{c\tilde{R}_\gamma^3}\right) |G_{ij}a(x,y)|\, dx\, dy.$$

To begin with, we work on the first integral.

Lemma 3.1. *For an arbitrarily small $\varepsilon > 0$, we have*

$$\int_{c\tilde{R}_\gamma^1} |G_{ij}a(x,y)|\, dx\, dy \;\leq\; C\,\Gamma_{ij}\,\gamma^{-\delta}$$

for some $\delta > 0$, where

$$\Gamma_{ij} \;=\; \begin{cases} 2^{i(-a)+j(1-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i,j \geq 0 \\ 2^{i(-a)+j(1+\varepsilon-k_2+\frac{n_2}{2})}, & \text{for } i \geq 0, j < 0 \\ 2^{i+j(1-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i+j(1+\varepsilon-k_2+\frac{n_2}{2})}, & \text{for } i,j < 0. \end{cases}$$

Proof. We first write

$$\int_{c\tilde{R}_\gamma^1} |G_{ij}a(x,y)|\, dx\, dy \;=\; \int_{c\tilde{R}_\gamma^1} |y|^{-k_2+\varepsilon}|y|^{k_2-\varepsilon}|G_{ij}a(x,y)|\, dx\, dy.$$

Apply the Cauchy-Schwartz inequality to see that it is bounded by

$$\left(\int_{c\tilde{R}_\gamma^1} |y|^{-2k_2+2\varepsilon}\, dx\, dy\right)^{1/2} \left(\int_{c\tilde{R}_\gamma^1} |y|^{2k_2-2\varepsilon}|G_{ij}a(x,y)|^2\, dx\, dy\right)^{1/2}$$

$$\leq C\,\gamma^{-1/2+\varepsilon}|I|^{1/2}|J|^{1/2-k_2/n_2+\varepsilon/n_2}$$

$$\cdot\left\{\left(\int_0^{l(J)}\int_0^\infty\int_{c\tilde{R}_\gamma^1} + \int_{l(J)}^\infty\int_0^\infty\int_{c\tilde{R}_\gamma^1}\right)|y|^{2k_2-2\varepsilon}|T_{ij}^{s,t}a(x,y)|^2\, dx\, dy\, \frac{ds\, dt}{st}\right\}^{1/2}$$

$$= C\,\gamma^{-\frac{1}{2}+\varepsilon}|I|^{1/2}|J|^{1/2-k_2/n_2+\varepsilon/n_2}\{(\text{I})^{1/2} + (\text{II})^{1/2}\}. \tag{3-1}$$

Let us look at the first expression. If we set

$$\hat{K}_{ij} = \mu_{ij} \quad \text{and} \quad K_{ij}^{s,t}(x,y) = s^{-n_1}t^{-n_2}K_{ij}(x/s, y/t),$$

then as

$$T_{ij}^{s,t}a(x,y) \;=\; \int_J \left(\int_I K_{ij}^{s,t}(x-x',y-y')a(x',y')dx' \right) dy',$$

Minkowski's integral inequality coupled with Plancherel's theorem on the x variables shows that the integrand expression with respect to $\dfrac{dsdt}{st}$ in (I) is dominated by

$$\int_{c\tilde{J}_\gamma} |y|^{2k_2-2\varepsilon} \left(\int_J \left\{ \int_I \left| \int_I K_{ij}^{s,t}(x-x',y-y')a(x',y')\,dx' \right|^2 dx \right\}^{1/2} dy' \right)^2 dy$$

$$\leq \int_{c\tilde{J}_\gamma} |y|^{2k_2-2\varepsilon} \left(\int_J \left\{ \int_{\mathbf{R}^{n_1}} \left| t^{-n_2}\hat{K}_{ij}^1\left(s\xi, \frac{y-y'}{t}\right)\hat{a}^1(\xi,y') \right|^2 d\xi \right\}^{1/2} dy' \right)^2 dy$$

$$\leq C|J| \int_{c\tilde{J}_\gamma} \int_J \int_{\mathbf{R}^{n_1}} \left| |y|^{k_2-\varepsilon}t^{-n_2}\hat{K}_{ij}^1\left(s\xi, \frac{y-y'}{t}\right)\hat{a}^1(\xi,y') \right|^2 d\xi dy' dy, \qquad (3\text{-}2)$$

where the last inequality followed after applying Hölder's inequality. Using the fact that $|y| \approx |y-y'|$ for $y \in {}^c\tilde{J}_\gamma, y' \in J$, integrating and changing variables accordingly, we notice that (I) is controlled by

$$\int_0^\infty \int_J \int_{\mathbf{R}^{n_1}} |\hat{a}^1(\xi,y')|^2 \int_{\mathbf{R}^{n_2}} \left| |y|^{k_2-\varepsilon}\hat{K}_{ij}^1(s\xi,y) \right|^2 dy d\xi dy' \frac{ds}{s}, \qquad (3\text{-}3)$$

except the multiplicative constant $C|J|^{2k_2/n_2-2\varepsilon/n_2}$. Now use a Fourier transform formula related to the Riesz potentials to get

$$\int_{\mathbf{R}^{n_2}} \left| |y|^{k_2-\varepsilon}\hat{K}_{ij}^1(s\xi,y) \right|^2 dy = \int_{\mathbf{R}^{n_2}} \left| |y|^{-\varepsilon}\left(|y|^{k_2}\hat{K}_{ij}^1(s\xi,y) \right) \right|^2 dy$$

$$= C \int_{\mathbf{R}^{n_2}} \left| |\eta|^{-n_2+\varepsilon} * \sum_{|\beta|=k_2} \left(\partial_\eta^\beta \mu_{ij} \right)(s\xi,\eta) \right|^2 d\eta$$

$$\leq C \sum_{|\beta|=k_2} \int_{\mathbf{R}^{n_2}} \left| |\eta|^{-n_2+\varepsilon} * \left(\partial_\eta^\beta \mu_{ij} \right)(s\xi,\eta) \right|^2 d\eta$$

$$\leq C \sum_{|\beta|=k_2} \left(\int_{\mathbf{R}^{n_2}} \left| \left(\partial_\eta^\beta \mu_{ij} \right)(s\xi,\eta) \right|^p d\eta \right)^{2/p}, \qquad (3\text{-}4)$$

where $\dfrac{1}{2} = \dfrac{1}{p} - \dfrac{\varepsilon}{n_2}$, by an application of the Hardy-Littlewood-Sobolev fractional integration theorem (see [51]).

We now estimate the right side term of (3-4). In the case $i, j \geq 0$, compute

$$
\left(\partial_\eta^\beta \mu_{ij}\right)(s\xi, \eta) = \phi(2^{-i}s\xi) \sum_{\alpha+\gamma=\beta} \frac{\beta!}{\alpha!\gamma!} \left(\partial_\eta^\alpha \phi(2^{-j}\eta)\right) \left(\partial_\eta^\gamma m(s\xi, \eta)\right)
$$

$$
= \phi(2^{-i}s\xi) \sum_{\alpha+\gamma=\beta} \frac{\beta!}{\alpha!\gamma!} 2^{-j|\alpha|} \left(\partial_\eta^\alpha \phi\right)(2^{-j}\eta) \left(\partial_\eta^\gamma m\right)(s\xi, \eta),
$$

to observe that

$$
\left|\left(\partial_\eta^\beta \mu_{ij}\right)(s\xi, \eta)\right| \leq C |s\xi|^{-a} |\eta|^{-b} \chi_{\{2^{i-1}<|s\xi|<2^{i+1}, 2^{j-1}<|\eta|<2^{j+1}\}}
$$

and subsequently the summand in (3-4) is at most

$$
C|s\xi|^{-2a} \chi_{\{2^{i-1}<|s\xi|<2^{i+1}\}} \left(\int_{2^{j-1}<|\eta|<2^{j+1}} |\eta|^{-bp} d\eta\right)^{2/p}
$$

$$
\leq C|s\xi|^{-2a} \chi_{\{2^{i-1}<|s\xi|<2^{i+1}\}} 2^{2j(-b+\frac{n_2}{p})}.
$$

It follows that (3-3) is bounded above by

$$
C|J|^{2k_2/n_2-2\varepsilon/n_2} 2^{-2ai+2j(-b+\frac{n_2}{p})} \int_J \int_{\mathbf{R}^{n_1}} \left|\hat{a}^1(\xi, y')\right|^2 d\xi dy'
$$

$$
\leq C|I|^{-1}|J|^{-1+2k_2/n_2-2\varepsilon/n_2} 2^{-2ai+2j(-b+\frac{n_2}{p})},
$$

by using the atomic properties of $a(x, y)$. As to the case $i \geq 0, j < 0$, we notice that

$$
\left(\partial_\eta^\beta \mu_{ij}\right)(s\xi, \eta) = \phi(2^{-i}s\xi) \sum_{\alpha+\gamma=\beta} \frac{\beta!}{\alpha!\gamma!} 2^{-j|\alpha|} \left(\partial_\eta^\alpha \phi\right)(2^{-j}\eta) \left(\partial_\eta^\gamma \mu\right)(s\xi, \eta),
$$

and Leibniz's formula shows

$$
\left(\partial_\eta^\gamma \mu\right)(s\xi, \eta) = \left(\partial_\eta^\gamma m\right)(s\xi, \eta)
$$

$$
- \sum_{\gamma_1+\gamma_2=\gamma} C_\gamma \left(\partial_\eta^{\gamma_1} p\right)(\eta) \left[\sum_{|\rho|\leq k_2, \rho\geq\gamma_2} \left(\partial_\eta^\rho m\right)(s\xi, 0) \frac{\eta^{\rho-\gamma_2}}{(\rho-\gamma_2)!}\right].
$$

Since $\left(\partial_\eta^{\gamma_1} p\right)(0) = 0$ except when $\gamma_1 = 0$, we have $\left(\partial_\eta^\gamma \mu\right)(s\xi, 0) = 0$, and MVT implies the existence of some $\bar\eta$ such that

$$\left(\partial_\eta^\gamma \mu\right)(s\xi, \eta) = \sum_{|\tau|=1} \left[\left(\partial_\eta^{\gamma+\tau} m\right)(s\xi, \bar\eta) - F(s\xi, \bar\eta)\right] \eta^\tau,$$

where F is the appropriate partial derivative of the latter term satisfying

$$|F(s\xi, \bar\eta)| \leq C |s\xi|^{-a} \qquad \text{for} \quad |\bar\eta| \leq 1.$$

Thus $\left|\left(\partial_\eta^\gamma \mu\right)(s\xi, \eta)\right| \leq C |s\xi|^{-a} |\eta|$ and

$$\left|\left(\partial_\eta^\beta \mu_{ij}\right)(s\xi, \eta)\right| \leq C \, 2^{-jk_2} |s\xi|^{-a} |\eta| \, \chi_{\{2^{i-1} < |s\xi| < 2^{i+1}, \, 2^{j-1} < |\eta| < 2^{j+1}\}}.$$

Consequently,

$$\left(\int_{\mathbf{R}^{n_2}} \left|\left(\partial_\eta^\beta \mu_{ij}\right)(s\xi, \eta)\right|^p d\eta\right)^{2/p} \leq C 2^{2j\left(1 + \frac{n_2}{p} - k_2\right)} |s\xi|^{-2a} \chi_{\{2^{i-1} < |s\xi| < 2^{i+1}\}},$$

whence (3-3) is less than or equal to

$$C |I|^{-1} |J|^{-1 + 2k_2/n_2 - 2\varepsilon/n_2} \, 2^{-2ai + 2j\left(1 + \frac{n_2}{p} - k_2\right)}.$$

We continue this procedure for other cases to conclude that since $\dfrac{n_2}{p} = \dfrac{n_2}{2} + \varepsilon$,

$$(\mathrm{I})^{1/2} \leq C A_{ij}, \quad \text{where} \quad A_{ij} = \begin{cases} 2^{i(-a) + j\left(-b + \varepsilon + \frac{n_2}{2}\right)}, & \text{for } i, j \geq 0 \\ 2^{i(-a) + j\left(1 + \varepsilon - k_2 + \frac{n_2}{2}\right)}, & \text{for } i \geq 0, j < 0 \\ 2^{i + j\left(-b + \varepsilon + \frac{n_2}{2}\right)}, & \text{for } i < 0, j \geq 0 \\ 2^{i + j\left(1 + \varepsilon - k_2 + \frac{n_2}{2}\right)}, & \text{for } i, j < 0. \end{cases}$$

Toward the estimates involving the second part (II), we make use of the vanishing properties of the atom a to write down

$$T_{ij}^{s,t} a(x, y) = \sum_{|\tilde\beta|=1} \int_{I \times J} \int_0^1 (-y')^{\tilde\beta} \partial_y^{\tilde\beta} K_{ij}^{s,t}(x - x', y - \delta y') a(x', y') \, d\delta \, dx' \, dy'$$

and follow the same lines up to (3-2) in the above process to find that the inside expression of the parenthesis in (II) is dominated by

$$C|J|^{\frac{2}{n_2}+1} \int_{l(J)}^{\infty} \int_0^{\infty} \int_{c\tilde{J}_\gamma} \int_0^1 \int_J \int_{\mathbf{R}^{n_1}}$$
$$\left| |y|^{k_2-\varepsilon} t^{-n_2} \partial_y^{\tilde{\beta}} \hat{K}_{ij}^1 \left(s\xi, \frac{y-\delta y'}{t} \right) \hat{a}^1(\xi,y') \right|^2 d\xi\, dy'\, d\delta\, dy\, \frac{ds\,dt}{st}.$$

Once again $|y| \approx |y - \delta y'|$ so it follows from the change of variables that the above expression is majorized by

$$C|J|^{\frac{2}{n_2}+1} \int_{l(J)}^{\infty} \int_0^{\infty} t^{2k_2-2\varepsilon-n_2-2} \int_J \int_{\mathbf{R}^{n_1}} |\hat{a}^1(\xi,y')|^2$$
$$\int_{\mathbf{R}^{n_2}} \left| |y|^{-\varepsilon} |y|^{k_2} \partial_y^{\tilde{\beta}} \hat{K}_{ij}^1(s\xi,y) \right|^2 dy\, d\xi\, dy'\, \frac{ds\,dt}{st}. \tag{3-5}$$

Now we observe that

$$\int_{\mathbf{R}^{n_2}} \left| |y|^{-\varepsilon} \left(|y|^{k_2} \partial_y^{\tilde{\beta}} \hat{K}_{ij}^1(s\xi,y) \right) \right|^2 dy$$
$$\leq C \sum_{|\beta|=k_2} \int_{\mathbf{R}^{n_2}} \left| |\eta|^{-n_2+\varepsilon} * \partial_\eta^{\beta} \left(\eta^{\tilde{\beta}} \mu_{ij}(s\xi,\eta) \right) \right|^2 d\eta$$
$$\leq C \sum_{|\beta|=k_2} \left(\int_{\mathbf{R}^{n_2}} \left| \partial_\eta^{\beta} \left(\eta^{\tilde{\beta}} \mu_{ij}(s\xi,\eta) \right) \right|^p d\eta \right)^{2/p}, \tag{3-6}$$

where $|\tilde{\beta}| = 1, \frac{1}{2} = \frac{1}{p} - \frac{\varepsilon}{n_2}$, by the Hardy-Littlewood-Sobolev theorem of fractional integration. Easy adjustments of the preceding estimations will lead us to

$$(\mathrm{II})^{1/2} \leq C\tilde{A}_{ij}, \quad \text{where} \quad \tilde{A}_{ij} = \begin{cases} 2^{i(-a)+j(1-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i,j \geq 0 \\ 2^{i(-a)+j(2+\varepsilon-k_2+\frac{n_2}{2})}, & \text{for } i \geq 0, j < 0 \\ 2^{i+j(1-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i+j(2+\varepsilon-k_2+\frac{n_2}{2})}, & \text{for } i,j < 0. \end{cases}$$

Now we put our estimates into (3-1) to complete the proof of the Lemma. $\square$

Due to the symmetric nature of our hypothesis and the domains ${}^c\tilde{R}_\gamma^1, {}^c\tilde{R}_\gamma^2$, we easily attain the estimates for $\int_{c\tilde{R}_\gamma^2} |G_{ij}a(x,y)|\, dx\, dy$:

Lemma 3.2.

For an arbitrarily small $\varepsilon > 0$, we have

$$\int_{c\tilde{R}_\gamma^2} |G_{ij}a(x,y)|\, dx\, dy \leq C\,\tilde{\Gamma}_{ij}\,\gamma^{-\tilde{\delta}}$$

for some $\tilde{\delta} > 0$, where

$$\tilde{\Gamma}_{ij} = \begin{cases} 2^{i(1-a+\varepsilon+\frac{n_1}{2})-jb}, & \text{for } i,j \geq 0 \\ 2^{i(1-a+\varepsilon+\frac{n_1}{2})+j}, & \text{for } i \geq 0, j < 0 \\ 2^{i(1+\varepsilon+\frac{n_1}{2}-k_1)-jb}, & \text{for } i < 0, j \geq 0 \\ 2^{i(1+\varepsilon+\frac{n_1}{2}-k_1)+j}, & \text{for } i,j < 0. \end{cases}$$

Next we shall prove

Lemma 3.3.

For an arbitrarily small $\varepsilon > 0$, we have

$$\int_{c\tilde{R}_\gamma^3} |G_{ij}a(x,y)|\, dx\, dy \leq C\,\Lambda_{ij}\,\gamma^{-\sigma}$$

for some $\sigma > 0$, where

$$\Lambda_{ij} = \begin{cases} 2^{i(1-a+\varepsilon+\frac{n_1}{2})+j(1-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i,j \geq 0 \\ 2^{i(1-a+\varepsilon+\frac{n_1}{2})+j(1+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i \geq 0, j < 0 \\ 2^{i(1+\varepsilon+\frac{n_1}{2}-k_1)+j(1-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i(1+\varepsilon+\frac{n_1}{2}-k_1)+j(1+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i,j < 0. \end{cases}$$

Proof. For sufficiently small $\varepsilon > 0$, the Cauchy-Schwartz inequality furnishes us

$$\int_{c\tilde{R}_\gamma^3} |G_{ij}a(x,y)|\, dx\, dy$$

$$\leq C\gamma^{-1+\varepsilon}|I|^{1/2-k_1/n_1+\varepsilon/n_1}|J|^{1/2-k_2/n_2+\varepsilon/n_2}$$

$$\cdot \left\{ \left(\int_0^{l(J)}\int_0^{l(I)} + \int_{l(J)}^\infty\int_0^{l(I)} + \int_0^{l(J)}\int_{l(I)}^\infty + \int_{l(J)}^\infty\int_{l(I)}^\infty \right) \right.$$

$$\left. \int_{c\tilde{R}_\gamma^3} |x|^{2k_1-2\varepsilon}|y|^{2k_2-2\varepsilon}\left|T_{ij}^{s,t}a(x,y)\right|^2 dx\,dy\, \frac{ds\,dt}{st} \right\}^{1/2}$$

$$\leq C\gamma^{-1+\varepsilon}|I|^{1/2-k_1/n_1+\varepsilon/n_1}|J|^{1/2-k_2/n_2+\varepsilon/n_2}$$

$$\left[(\mathrm{I})^{1/2} + (\mathrm{II})^{1/2} + (\mathrm{III})^{1/2} + (\mathrm{IV})^{1/2} \right]. \tag{3-7}$$

In considering (I), we use the following majorization

$$\left|T_{ij}^{s,t}a(x,y)\right| = \left|\int_{I\times J} K_{ij}^{s,t}(x-x',y-y')a(x',y')\,dx'\,dy'\right|$$

$$\le |I|^{-1/2}|J|^{-1/2}\left(\int_{I\times J}\left|K_{ij}^{s,t}(x-x',y-y')\right|^2\,dx'\,dy'\right)^{1/2},$$

where we used the Cauchy-Schwartz inequality effectively. It now follows that (I) is bounded above by

$$|I|^{-1}|J|^{-1}\int_0^{l(J)}\int_0^{l(I)}\int_{c\tilde{R}_\gamma^3}|x|^{2k_1-2\varepsilon}|y|^{2k_2-2\varepsilon}$$

$$\int_{I\times J}\left|K_{ij}^{s,t}(x-x',y-y')\right|^2\,dx'\,dy'\,dx\,dy\,\frac{ds\,dt}{st}$$

$$\le C|I|^{-1+2k_1/n_1-2\varepsilon/n_1}|J|^{-1+2k_2/n_2-2\varepsilon/n_2}$$

$$\int_{\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}}\left||x|^{k_1-\varepsilon}|y|^{k_2-\varepsilon}K_{ij}(x,y)\right|^2\,dx\,dy. \qquad (3\text{-}8)$$

For the integral portion of (3-8)

$$A = \int_{\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}}\left||x|^{k_1-\varepsilon}|y|^{k_2-\varepsilon}K_{ij}(x,y)\right|^2\,dx\,dy,$$

observe that

$$A \le C\int_{\mathbf{R}^{n_2}}|y|^{2k_2-2\varepsilon}\int_{\mathbf{R}^{n_1}}\left||\xi|^{-n_1+\varepsilon}*^1\sum_{|\alpha|=k_1}\partial_\xi^\alpha \hat{K}_{ij}^1(\xi,y)\right|^2\,d\xi\,dy$$

by the well known Fourier transform formula, where $C = C(n_2,\varepsilon)$. Use the linearity of convolution to write

$$A \le C\sum_{|\alpha|=k_1}\int_{\mathbf{R}^{n_2}}|y|^{2k_2-2\varepsilon}\int_{\mathbf{R}^{n_1}}\left||\xi|^{-n_1+\varepsilon}*^1\partial_\xi^\alpha \hat{K}_{ij}^1(\xi,y)\right|^2\,d\xi\,dy$$

$$= C\sum_{|\alpha|=k_1}\int_{\mathbf{R}^{n_1}}\int_{\mathbf{R}^{n_2}}\left||\xi|^{-n_1+\varepsilon}*^1\left(|y|^{-\varepsilon}\cdot|y|^{k_2}\partial_\xi^\alpha \hat{K}_{ij}^1(\xi,y)\right)\right|^2\,dy\,d\xi.$$

Apply the Plancherel theorem in y-variables to see that

$$A \le C\sum_{|\alpha|=k_1,|\beta|=k_2}\int_{\mathbf{R}^{n_1}}\int_{\mathbf{R}^{n_2}}\left||\xi|^{-n_1+\varepsilon}*^1\left\{|\eta|^{-n_2+\varepsilon}*^2\partial_\eta^\beta\partial_\xi^\alpha\mu_{ij}(\xi,\eta)\right\}\right|^2\,d\eta\,d\xi$$

$$\le C\sum_{|\alpha|=k_1,|\beta|=k_2}\int_{\mathbf{R}^{n_2}}\left(\int_{\mathbf{R}^{n_1}}\left||\eta|^{-n_2+\varepsilon}*^2\partial_\eta^\beta\partial_\xi^\alpha\mu_{ij}(\xi,\eta)\right|^q\,d\xi\right)^{2/q}\,d\eta,$$

where $\dfrac{1}{q} = \dfrac{1}{2} + \dfrac{\varepsilon}{n_1}$. As $\dfrac{2}{q} > 1$, we invoke the Minkowski integral inequality to obtain

$$A \leq C \sum_{|\alpha|=k_1, |\beta|=k_2} \left\{ \int_{\mathbf{R}^{n_1}} \left(\int_{\mathbf{R}^{n_2}} \left| |\eta|^{-n_2+\varepsilon} *^2 \partial_\eta^\beta \partial_\xi^\alpha \mu_{ij}(\xi,\eta) \right|^2 d\eta \right)^{q/2} d\xi \right\}^{2/q}$$

$$\leq C \sum_{|\alpha|=k_1, |\beta|=k_2} \left\{ \int_{\mathbf{R}^{n_1}} \left(\int_{\mathbf{R}^{n_2}} \left| \partial_\eta^\beta \partial_\xi^\alpha \mu_{ij}(\xi,\eta) \right|^p d\eta \right)^{q/p} d\xi \right\}^{2/q},$$

with $\dfrac{1}{p} = \dfrac{1}{2} + \dfrac{\varepsilon}{n_2}$.

As in the case of Lemma 3.1, simple computations yield

$$(\mathrm{I})^{1/2} \leq C|I|^{-1/2+k_1/n_1-\varepsilon/n_1}|J|^{-1/2+k_2/n_2-\varepsilon/n_2} E_{ij},$$

where

$$E_{ij} = \begin{cases} 2^{i(-a+\varepsilon+\frac{n_1}{2})+j(-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i,j \geq 0 \\ 2^{i(-a+\varepsilon+\frac{n_1}{2})+j(1+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i \geq 0, j < 0 \\ 2^{i(1+\varepsilon+\frac{n_1}{2}-k_1)+j(-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i(1+\varepsilon+\frac{n_1}{2}-k_1)+j(1+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i,j < 0. \end{cases}$$

For the part (II), using the Taylor polynomial as before, we notice

$$\left| T_{ij}^{s,t} a(x,y) \right| = \left| \sum_{|\tilde\beta|=1} \int_{I \times J} \int_0^1 (-y')^{\tilde\beta} \partial_y^{\tilde\beta} K_{ij}^{s,t}(x-x', y-\delta y') a(x',y') \, d\delta \, dx' \, dy' \right|$$

$$\leq C|I|^{-1/2}|J|^{1/n_2-1/2} \left(\int_{I \times J} \int_0^1 \left| \partial_y^{\tilde\beta} K_{ij}^{s,t}(x-x', y-\delta y') \right|^2 d\delta \, dx' \, dy' \right)^{1/2}$$

so that (II) is majorized by

$$C|I|^{-1+2k_1/n_1-2\varepsilon/n_1}|J|^{-1+2k_2/n_2-2\varepsilon/n_2}$$

$$\int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} \left| |x|^{k_1-\varepsilon}|y|^{k_2-\varepsilon} \partial_y^{\tilde\beta} K_{ij}(x,y) \right|^2 dx \, dy,$$

by using the fact that $|x| \approx |x - x'|, |y| \approx |y - y'|$ with pertinent change of variables. As in the case (I), the fractional integration provides that

$$\int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} \left| |x|^{k_1 - \varepsilon} |y|^{k_2 - \varepsilon} \partial_y^{\tilde{\beta}} K_{ij}(x, y) \right|^2 dx \, dy$$

$$\leq C \sum_{|\alpha| = k_1, |\beta| = k_2} \left\{ \int_{\mathbf{R}^{n_1}} \left(\int_{\mathbf{R}^{n_2}} \left| \partial_\eta^\beta \partial_\xi^\alpha \left(\eta^{\tilde{\beta}} \mu_{ij}(\xi, \eta) \right) \right|^p d\eta \right)^{q/p} d\xi \right\}^{2/q}$$

with $\dfrac{1}{q} = \dfrac{1}{2} + \dfrac{\varepsilon}{n_1}, \dfrac{1}{p} = \dfrac{1}{2} + \dfrac{\varepsilon}{n_2}$. Being our decay hypothesis independent of the orders of derivatives involved, it is not hard to see that we should be led to

$$(\mathrm{II})^{1/2} \leq C |I|^{-1/2 + k_1/n_1 - \varepsilon/n_1} |J|^{-1/2 + k_2/n_2 - \varepsilon/n_2} D_{ij},$$

where

$$D_{ij} = \begin{cases} 2^{i(-a + \varepsilon + \frac{n_1}{2}) + j(1 - b + \varepsilon + \frac{n_2}{2})}, & \text{for } i, j \geq 0 \\ 2^{i(-a + \varepsilon + \frac{n_1}{2}) + j(2 + \varepsilon + \frac{n_2}{2} - k_2)}, & \text{for } i \geq 0, j < 0 \\ 2^{i(1 + \varepsilon + \frac{n_1}{2} - k_1) + j(1 - b + \varepsilon + \frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i(1 + \varepsilon + \frac{n_1}{2} - k_1) + j(2 + \varepsilon + \frac{n_2}{2} - k_2)}, & \text{for } i, j < 0. \end{cases}$$

In view of akin formulations, we have analogous estimates for the third part

$$(\mathrm{III})^{1/2} \leq C |I|^{-1/2 + k_1/n_1 - \varepsilon/n_1} |J|^{-1/2 + k_2/n_2 - \varepsilon/n_2} \tilde{D}_{ij},$$

with

$$\tilde{D}_{ij} = \begin{cases} 2^{i(1 - a + \varepsilon + \frac{n_1}{2}) + j(-b + \varepsilon + \frac{n_2}{2})}, & \text{for } i, j \geq 0 \\ 2^{i(1 - a + \varepsilon + \frac{n_1}{2}) + j(1 + \varepsilon + \frac{n_2}{2} - k_2)}, & \text{for } i \geq 0, j < 0 \\ 2^{i(2 + \varepsilon + \frac{n_1}{2} - k_1) + j(-b + \varepsilon + \frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i(2 + \varepsilon + \frac{n_1}{2} - k_1) + j(1 + \varepsilon + \frac{n_2}{2} - k_2)}, & \text{for } i, j < 0. \end{cases}$$

Finally, we use the Taylor polynomial in both variables to obtain

$$(\mathrm{IV})^{1/2} \leq C |I|^{-1/2 + k_1/n_1 - \varepsilon/n_1} |J|^{-1/2 + k_2/n_2 - \varepsilon/n_2} \tilde{E}_{ij},$$

where

$$\tilde{E}_{ij} = \begin{cases} 2^{i(1 - a + \varepsilon + \frac{n_1}{2}) + j(1 - b + \varepsilon + \frac{n_2}{2})}, & \text{for } i, j \geq 0 \\ 2^{i(1 - a + \varepsilon + \frac{n_1}{2}) + j(2 + \varepsilon + \frac{n_2}{2} - k_2)}, & \text{for } i \geq 0, j < 0 \\ 2^{i(2 + \varepsilon + \frac{n_1}{2} - k_1) + j(1 - b + \varepsilon + \frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i(2 + \varepsilon + \frac{n_1}{2} - k_1) + j(2 + \varepsilon + \frac{n_2}{2} - k_2)}, & \text{for } i, j < 0. \end{cases}$$

We now combine all of our estimates and put into (3-7) to finish the proof of the Lemma. $\square$

Based on Lemma 3.1, Lemma 3.2, and Lemma 3.3, we now state

Corollary 3.1. *For all* $\gamma \geq 2$ *and sufficiently small* $\varepsilon > 0$,

$$\int_{^c\tilde{R}_\gamma} |G_{ij}a(x,y)|\,dx\,dy \leq C\,\Lambda_{ij}\gamma^{-\sigma} \qquad \text{for some}\quad \sigma > 0.$$

Furthermore in reference to other square functions, we are able to state the following results without any difficulty:

Corollary 3.2. *Let* $\gamma \geq 2$ *and* $\varepsilon > 0$ *be sufficiently small. Then there exists* $\tilde{\sigma}$, δ, $\tilde{\sigma} > 0$ *such that*

(1)
$$\int_{^c\tilde{R}_\gamma} |\tilde{G}_{ij}a(x,y)|\,dx\,dy \leq C\,\tilde{\Lambda}_{ij}\gamma^{-\tilde{\sigma}},$$

where
$$\tilde{\Lambda}_{ij} = \begin{cases} 2^{i(2-a+\varepsilon+\frac{n_1}{2})+j(2-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i,j \geq 0 \\ 2^{i(2-a+\varepsilon+\frac{n_1}{2})+j(2+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i \geq 0, j < 0 \\ 2^{i(2+\varepsilon+\frac{n_1}{2}-k_1)+j(2-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i(2+\varepsilon+\frac{n_1}{2}-k_1)+j(2+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i,j < 0; \end{cases}$$

(2)
$$\int_{^c\tilde{R}_\gamma} |S_{ij}a(x,y)|\,dx\,dy \leq C\,\Theta_{ij}\gamma^{-\delta},$$

where
$$\Theta_{ij} = \begin{cases} 2^{i(2-a+\varepsilon+\frac{n_1}{2})+j(1-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i,j \geq 0 \\ 2^{i(2-a+\varepsilon+\frac{n_1}{2})+j(1+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i \geq 0, j < 0 \\ 2^{i(2+\varepsilon+\frac{n_1}{2}-k_1)+j(1-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i(2+\varepsilon+\frac{n_1}{2}-k_1)+j(1+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i,j < 0; \end{cases}$$

(3)
$$\int_{^c\tilde{R}_\gamma} |\tilde{S}_{ij}a(x,y)|\,dx\,dy \leq C\,\tilde{\Theta}_{ij}\gamma^{-\tilde{\delta}},$$

where
$$\tilde{\Theta}_{ij} = \begin{cases} 2^{i(1-a+\varepsilon+\frac{n_1}{2})+j(2-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i,j \geq 0 \\ 2^{i(1-a+\varepsilon+\frac{n_1}{2})+j(2+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i \geq 0, j < 0 \\ 2^{i(1+\varepsilon+\frac{n_1}{2}-k_1)+j(2-b+\varepsilon+\frac{n_2}{2})}, & \text{for } i < 0, j \geq 0 \\ 2^{i(1+\varepsilon+\frac{n_1}{2}-k_1)+j(2+\varepsilon+\frac{n_2}{2}-k_2)}, & \text{for } i,j < 0. \end{cases}$$

Next observe that for each integer i, j,

$$
\begin{aligned}
|T_{ij}^* f| &\le \left\{ 2\, G_{ij} f \cdot \tilde{G}_{ij} f \;+\; 2\, S_{ij} f \cdot \tilde{S}_{ij} f \right\}^{1/2} \\
&\le \left(2\, G_{ij} f \cdot \tilde{G}_{ij} f \right)^{1/2} \;+\; \left(2\, S_{ij} f \cdot \tilde{S}_{ij} f \right)^{1/2} \\
&\le \left\{ 2^{(i+j)/2}\, G_{ij} f \;+\; 2^{-(i+j)/2}\, \tilde{G}_{ij} f \right\} \\
&\qquad + \left\{ 2^{(-i+j)/2}\, S_{ij} f \;+\; 2^{(i-j)/2}\, \tilde{S}_{ij} f \right\}.
\end{aligned}
$$

In view of Theorem 3.1, we are finally led to the (H^1, L^1)-inequality for the maximal operators T_{ij}^*: We are finally led to the (H^1, L^1)-inequality for the maximal operators T_{ij}^*:

Lemma 3.4. For any $l_j > \dfrac{n_j}{2} + 1, \quad j = 1, 2$, we have

$$
\| T_{ij}^* f \|_{L^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \le C\, \Omega_{ij}\, \| f \|_{H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})},
$$

where

$$
\Omega_{ij} = \begin{cases}
2^{i(\frac{1}{2} - a + l_1) + j(\frac{1}{2} - b + l_2)}, & \text{for } i, j \ge 0 \\
2^{i(\frac{1}{2} - a + l_1) + j(\frac{1}{2} + l_2 - k_2)}, & \text{for } i \ge 0, j < 0 \\
2^{i(\frac{1}{2} + l_1 - k_1) + j(\frac{1}{2} - b + l_2)}, & \text{for } i < 0, j \ge 0 \\
2^{i(\frac{1}{2} + l_1 - k_1) + j(\frac{1}{2} + l_2 - k_2)}, & \text{for } i, j < 0.
\end{cases}
$$

It follows immediately by interpolating Lemma 2.2 and Lemma 3.4 and by summing each corresponding geometric series that T^* is bounded in $L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ for $q_{a,b} < p \le 2$. In dealing with interpolation, we used the fact that the negative integer cases do not give any effect on the range of p's, which can be seen easily from the note

$$
\frac{1}{2} + l_j - k_j > 0 \qquad \text{for } \quad j = 1, 2.
$$

Moreover,

$$
\max\left(\frac{2 n_1}{n_1 + 2a - 1}, \frac{2 n_2}{n_2 + 2b - 1} \right) < q_{a,b},
$$

in comparison with the range of p's stated in Lemma 2.1.

4. THE (L^∞, BMO)-INEQUALITY

As was pointed out, the space $H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ serves as a good substitute for $L^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ for many purposes owing to the fact that a number of singular integrals are invariant in it. There is also a class of functions near $L^\infty(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ invariant under the Calderón-Zygmund operators, namely, the space of bounded mean oscillation, $BMO(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$. As a matter of fact, A. Chang and R. Fefferman [11] characterized this space as the dual of $H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ in the sense that every continuous linear functional on $H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ arises as

$$f \longrightarrow \int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} f(x,y)\varphi(x,y)\,dxdy$$

with a unique $\varphi \in BMO(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ (cf. see C. Fefferman and E.M. Stein [25]).

In accordance with the preceding observation, we immediately obtain the following analogue of Theorem 3.2:

Theorem 4.1.

Suppose that T is a sublinear operator such that

$$\|Tf\|_{L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \le M_1 \, \|f\|_{L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})},$$

$$\|Tf\|_{BMO(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \le M_2 \, \|f\|_{L^\infty(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})}.$$

Then we have

$$\|Tf\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \le C\, M_1^{1-\theta} M_2^\theta \, \|f\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \quad \text{for} \quad \frac{1}{p} = \frac{1-\theta}{2}, \, 0 < \theta < 1.$$

This duality kinship enables us to establish another seemingly powerful criterion on the (L^∞, BMO)-inequality, in the sprit of Theorem 3.1. For the precise statement, we need to define a size measurement which plays an important role in the product Fefferman-Stein sharp operator. Specifically, given a function $f(x,y)$ on

$\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ and a rectangle $R = I \times J$, we define the mean oscillation of f over R, $osc_R(f)$, by

$$osc_R(f) = \inf \left(\frac{1}{|R|} \int_R |f(x,y) - f_1(x) - f_2(y)|^2 \, dx \, dy \right)^{1/2}$$

where the infimum is taken over all pairs of functions f_1, f_2 depending only on the x, y variables, separately. It turns out that in order to check the boundedness of an L^2-bounded linear operator from L^∞ to BMO, it suffices to look into its mean oscillation over rectangles.

Theorem 4.1 (R. Fefferman [30]).

If T is a bounded linear operator on $L^2(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ such that for any rectangle R and $\gamma \geq 2$,

$$osc_R(Tf) \leq C \gamma^{-\delta} \quad \text{for some} \quad \delta > 0,$$

whenever f is an L^∞-function supported in $^c\tilde{R}_\gamma$ with $\|f\|_\infty \leq 1$, then

$$\|Tf\|_{BMO(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \leq C \|f\|_{L^\infty(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \quad \text{for all} \quad f \in L^\infty(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}).$$

Let us include a proof of this result for the sake of comprehending the duality kinship. Let $\tilde{T}$ be the adjoint operator of T. Taking the duality into consideration, it is sufficient to prove that $\|\tilde{T}(f)\|_{L^1} \leq C \|f\|_{H^1}$ for which it is in turn enough to establish

$$\int_{^c\tilde{R}_\gamma} \left| \tilde{T}a(x,y) \right| \, dx \, dy \leq C \gamma^{-\delta}$$

for a rectangle atom a supported on R in view of Theorem 3.3. Note that for any functions f_1, f_2, if we denote the unit ball in $L^\infty(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ by $\mathcal{B}_\infty$ and the set

of those elements supported outside ${}^{c}\tilde{R}_{\gamma}$ by $\tilde{\mathcal{B}}_{\infty}$, then

$$
\begin{aligned}
\int_{{}^{c}\tilde{R}_{\gamma}} \left|\tilde{T}a(x,y)\right| dxdy &= \sup_{f \in \mathcal{B}_{\infty}} \left|\int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} \tilde{T}a(x,y)\chi_{{}^{c}\tilde{R}_{\gamma}}f(x,y)dxdy\right| \\
&= \sup_{f \in \tilde{\mathcal{B}}_{\infty}} \left|\int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} a(x,y)\left(Tf(x,y) - f_1(x) - f_2(y)\right)dxdy\right| \\
&\leq \sup_{f \in \tilde{\mathcal{B}}_{\infty}} \|a\|_{L^2} \left(\int_{R} |Tf(x,y) - f_1(x) - f_2(y)|^2 \, dx\, dy\right)^{1/2} \\
&= \sup_{f \in \tilde{\mathcal{B}}_{\infty}} \left(\frac{1}{|R|}\int_{R} |Tf(x,y) - f_1(x) - f_2(y)|^2 \, dx\, dy\right)^{1/2}
\end{aligned}
$$

so that

$$
\int_{{}^{c}\tilde{R}_{\gamma}} \left|\tilde{T}a(x,y)\right| dxdy \leq \sup_{f \in \tilde{\mathcal{B}}_{\infty}} osc_R(Tf)
$$

and we are done.

Going through similar reasonings as in the section 3, we shall derive the following result:

Lemma 4.2. For every $\gamma \geq 2$ and arbitrarily small $\varepsilon > 0$,

$$
osc_R(G_{ij}f) \leq C\,\gamma^{-\sigma}\,\Lambda_{ij},
$$

for some $\sigma > 0$ whenever $\|f\|_{\infty} \leq 1$, $supp(f) \subset {}^{c}\tilde{R}_{\gamma}^1$, where Λ_{ij} is the same constant defined in Lemma 3.3.

Proof. Introducing

$$
L_{ij}^{s,t}(x,y) = K_{ij}^{s,t}(x,y)\,\chi_{\{s>0,l(J)<t<\infty\}},
$$
$$
h(x) = \left(\int_0^\infty \int_0^\infty \left|L_{ij}^{s,t} * f(x,0)\right|^2 \frac{dsdt}{st}\right)^{1/2},
$$

we observe that

$$osc_R(G_{ij}f) \leq \left(\frac{1}{|R|} \int_R |G_{ij}f(x,y) - h(x)|^2 \, dx \, dy \right)^{1/2}$$

$$\leq |I|^{-\frac{1}{2}}|J|^{-\frac{1}{2}} \left(\int_{I \times J} \int_0^\infty \int_0^\infty \left| K_{ij}^{s,t} * f(x,y) - L_{ij}^{s,t} * f(x,0) \right|^2 \frac{ds\,dt}{st} \, dx \, dy \right)^{1/2}$$

$$\leq |I|^{-\frac{1}{2}}|J|^{-\frac{1}{2}} \left\{ \left(\int_0^{l(J)} \int_0^{l(I)} + \int_0^{l(J)} \int_{l(I)}^\infty \right) \int_{I \times J} \left| K_{ij}^{s,t} * f(x,y) \right|^2 \, dx \, dy \, \frac{ds\,dt}{st} \right.$$

$$\left. + \left(\int_{l(J)}^\infty \int_0^\infty \int_{I \times J} \left| K_{ij}^{s,t} * f(x,y) - K_{ij}^{s,t} * f(x,0) \right|^2 \, dx \, dy \, \frac{ds\,dt}{st} \right) \right\}^{1/2}$$

$$\leq |I|^{-\frac{1}{2}}|J|^{-\frac{1}{2}} \{ (\mathrm{I})^{1/2} + (\mathrm{II})^{1/2} + (\mathrm{III})^{1/2} \}. \tag{4-1}$$

For the part (I), an application of Minkowski's integral inequality in interchanging the order of integrations shows that it is dominated by

$$\int_0^{l(J)} \int_0^{l(I)} \int_J \left\{ \int_{{}^c\bar{J}_\gamma} \left[\int_I \left(\int_{x-x' \in I} \left| K_{ij}^{s,t}(x',y') \right| dx' \right)^2 dx \right]^{1/2} dy' \right\}^2 dy \, \frac{ds\,dt}{st}.$$

Let us put

$$A = \left[\int_I \left(\int_{x-x' \in I} \left| K_{ij}^{s,t}(x',y') \right| dx' \right)^2 dx \right]^{1/2}.$$

By the translation-invariant nature of our estimates, we may assume that the center of the cube I lies at the origin. Thus if $x \in I$, then $\quad |x| \leq \frac{\sqrt{n_1}}{2} |I|^{1/n_1}$. We notice that

$$\int_{x-x' \in I} \left| K_{ij}^{s,t}(x',y') \right| dx'$$

$$\leq \left(\int_{\substack{|x'| \leq 2^{-11}\sqrt{n_1}|I|^{1/n_1} \\ x-x' \in I}} + \int_{|x'| > 2^{-11}\sqrt{n_1}|I|^{1/n_1}} \right) \left| K_{ij}^{s,t}(x',y') \right| dx'$$

$$= \left(\sum_{k \leq 10} \int_{A_k(x)} + \int_{|x'| > 2^{-11}\sqrt{n_1}|I|^{1/n_1}} \right) \left| K_{ij}^{s,t}(x',y') \right| dx',$$

where

$$A_k(x) = \left\{ x' \in \mathbf{R}^{n_1} : \quad x - x' \in I, \quad |x'| \leq 2^{-11}\sqrt{n_1}|I|^{1/n_1}, \right.$$

$$\left. 2^k|x'| < |x - x'| < 2^{k+1}|x'| \right\}.$$

It follows readily that

$$
\begin{aligned}
\mathrm{A} \quad &\leq \sum_{k \leq 10} \left[\int_I \left(\int_{A_k(x)} \left| K_{ij}^{s,t}(x', y') \right| dx' \right)^2 dx \right]^{1/2} \\
&+ \left[\int_I \left(\int_{|x'| > 2^{-11}\sqrt{n_1}|I|^{1/n_1}} \left| K_{ij}^{s,t}(x', y') \right| dx' \right)^2 dx \right]^{1/2} \\
&= \mathrm{A_S} + \mathrm{A_N}.
\end{aligned}
$$

In the first place, since

$$
A_k(x) \subset B_k = \left\{ x' \in \mathbf{R}^{n_1} : \quad |x'| \leq 2^{-k}\sqrt{n_1}|I|^{1/n_1} \right\},
$$

we have, for $\quad 0 < \lambda < n_1,$

$$
\begin{aligned}
&\int_{A_k(x)} \left| K_{ij}^{s,t}(x', y') \right| dx' \\
&\leq C'_{n_1,\lambda} 2^{k(n_1 - \lambda)} \int_{B_k} |x - x'|^{-n_1 + \lambda} |x'|^{n_1 - \lambda} \left| K_{ij}^{s,t}(x', y') \right| dx' \\
&= C_{n_1,\lambda} 2^{k(n_1 - \lambda)} I_\lambda(g_k)(x, y')
\end{aligned}
$$

where

$$
g_k(x', y') = \chi_{B_k} |x'|^{n_1 - \lambda} \left| K_{ij}^{s,t}(x', y') \right|,
$$

and $I_\lambda(g_k)(x, y')$ denotes the Riesz potential of g_k. We now appeal to the fractional integration theorem and then the Hölder inequality to observe that

$$
\begin{aligned}
\mathrm{A_S} &\leq C_{n_1,\lambda} \sum_{k \leq 10} 2^{k(n_1 - \lambda)} \| I_\lambda(g_k)(\cdot, y') \|_2 \\
&\leq C_{n_1,\lambda} |I|^{\lambda/n_1} \sum_{k \leq 10} 2^{k(n_1 - 2\lambda)} \left(\int_{\mathbf{R}^{n_1}} \left| |x'|^{n_1 - \lambda} K_{ij}^{s,t}(x', y') \right|^2 dx' \right)^{1/2} \\
&\leq C |I|^{1 - k_1/n_1 + \varepsilon/n_1} \left(\int_{\mathbf{R}^{n_1}} \left| |x'|^{k_1 - \varepsilon} K_{ij}^{s,t}(x', y') \right|^2 dx' \right)^{1/2},
\end{aligned}
$$

where we let $\lambda = n_1 - k_1 + \varepsilon$. On the other hand, it follows easily by the Cauchy-Schwartz inequality that

$$
\int_{|x'|>2^{-11}\sqrt{n_1}|I|^{1/n_1}} \left|K_{ij}^{s,t}(x',y')\right| dx'
$$
$$
\leq C|I|^{1/2-k_1/n_1+\varepsilon/n_1} \left(\int_{\mathbf{R}^{n_1}} \left||x'|^{k_1-\varepsilon}K_{ij}^{s,t}(x',y')\right|^2 dx' \right)^{1/2}.
$$

Therefore,

$$
\mathrm{A_N} \leq C|I|^{1-k_1/n_1+\varepsilon/n_1} \left(\int_{\mathbf{R}^{n_1}} \left||x'|^{k_1-\varepsilon}K_{ij}^{s,t}(x',y')\right|^2 dx' \right)^{1/2},
$$

and the same inequality holds for $\mathbf{A}$. We now make use of the Cauchy-Schwartz inequality in the integral over $^c\tilde{J}_\gamma$ and then change variables $x' \to sx''$, $y' \to ty''$ to arrive at

$$
\mathrm{(I)} \leq C\gamma^{-\sigma}|I||J| \iint_{\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}} \left||x'|^{k_1-\varepsilon}K_{ij}(x',y')\right|^2 dx'dy'
$$
$$
\leq C\gamma^{-\sigma}|I||J|E_{ij}^2,
$$

for some $\sigma > 0$, where the constant E_{ij} was defined in the proof of Lemma 3.3.

Next in dealing with part II, we observe the following simple dominations :

$$
\int_{I\times J} \left|K_{ij}^{s,t} * f(x,y)\right| dxdy
$$
$$
\leq C|I| \int_{I\times J} \left[\int_{^c\tilde{J}_\gamma} \left(\int_{\mathbf{R}^{n_1}} \left|K_{ij}^{s,t}(x',y')\right|^2 dx' \right)^{1/2} dy' \right]^2 dxdy
$$
$$
\leq C\gamma^{-\sigma}|I|^2|J|^{2-2k_2/n_2+2\varepsilon/n_2}s^{-n_1}t^{2k_2-2\varepsilon-n_2}
$$
$$
\times \iint_{\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}} \left||y'|^{k_2-\varepsilon}K_{ij}(x',y')\right|^2 dx'dy',
$$

and in view of the first estimation process in the proof of Lemma 3.1, we end up with $\mathrm{(II)} \leq C\gamma^{-\sigma}|I||J|E_{ij}^2$.

To the end of estimating (III), we first note that

$$
K_{ij}^{s,t} * f(x,y) - K_{ij}^{s,t} * f(x,0)
$$

$$
= \int_{I \times {}^c\tilde{J}_\gamma} \left\{ K_{ij}^{s,t}(x - x', y - y') - K_{ij}^{s,t}(x - x', -y') \right\} f(x',y') \, dx' dy'
$$

$$
= \sum_{|\tilde{\beta}|=1} \int_{I \times {}^c\tilde{J}_\gamma} \int_0^1 y^{\tilde{\beta}} (\partial_{y'}^{\tilde{\beta}} K_{ij}^{s,t})(x - x', \delta y - y') f(x',y') \, d\delta dx' dy'
$$

by Taylor's formula. First of all, we split the regions of integration with regards to the measure $\dfrac{ds}{s}$ into two intervals $(0, l(I))$ and $[l(I), \infty)$ as usual, and then we apply the same techniques as in the preceding case, while our estimates will be essentially slight variants from those of the second case in the proof of Lemma 3.3, to finish the proof. $\square$

In case when $\operatorname{supp}(f) \subset {}^c\tilde{R}_\gamma^2 = {}^c\tilde{I}_\gamma \times J$, we would get $osc_R(G_{ij}f) \le C\gamma^{-\tilde{\sigma}} \Lambda_{ij}$, for some $\tilde{\sigma} > 0$, upon considering the auxiliary function

$$
\tilde{h}(y) = \left(\int_0^\infty \int_0^\infty \left| \tilde{L}_{ij}^{s,t} * f(0,y) \right|^2 \frac{dsdt}{st} \right)^{1/2},
$$

where

$$
\tilde{L}_{ij}^{s,t}(x,y) = K_{ij}^{s,t}(x,y) \chi_{\{l(I) < s < \infty, t > 0\}}.
$$

If $\operatorname{supp}(f) \subset {}^c\tilde{R}_\gamma^3$, then we subtract the function $h(x) + \tilde{h}(y) + C$ where

$$
C = \left(\int_0^\infty \int_0^\infty \left| P_{ij}^{s,t} * f(0,0) \right|^2 \frac{dsdt}{st} \right)^{1/2},
$$
$$
P_{ij}^{s,t}(x,y) = K_{ij}^{s,t}(x,y) \chi_{\{l(I) < s < \infty, l(J) < t < \infty\}},
$$

from the formula of $osc_R(G_{ij}f)$ to end up with akin estimates.

On account of Theorem 4.1 and inspection on the method of our proof, we are led to the following proposition :

Lemma 4.2. *For all integers* i, j *and for any* $l_j > \dfrac{n_j}{2} + 1, \quad j = 1, 2,$ *we have*

$$\|T_{ij}^* f\|_{BMO(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \leq C \, \Omega_{ij} \, \|f\|_{L^\infty(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})},$$

where Ω_{ij} *is the constant defined in Lemma 3.4.*

We should note again that the negative integer cases are irrelevant to our purposes so far as interpolation and summation of geometric series's are involved.

5. The L^p-boundedness of T^* for $2 < p < r_{a,b}$

If we interpolate Lemma 2.2 with Lemma 4.2 and sum each suitable geometric series, we would obtain the L^p-boundedness of T^* for p satisfying

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{n_1 + 2}\left(a - \frac{1}{2}\right), \frac{1}{n_2 + 2}\left(b - \frac{1}{2}\right)\right\},$$

which does't provide the better range of p's asserted in the main Theorem A. As in the paper of Rubio de Francia [46], we shall acquire the improved range $2 < p < r_{a,b}$ in the following approach. Given $m \in L^\infty(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$, we let $\|m\|_{ML^p}$ be the norm of T_m as an operator in L^p, where $(T_m f)^\smallfrown = m\,\hat{f}$. Under the same setting as in the section 2, we have the following proposition (compare Lemma 7 in [46]).

Lemma 5.1. *For* $2 < p < \infty$, $f \in L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$,

$$\left\|T_{ij}^* f\right\|_p \le C_p \left\{\|\mu_{ij}\|_{ML^p}^{1/p'}\|\tilde{\mu}_{ij}\|_{ML^p}^{1/p} + \|\mu_{ij}\|_{ML^p}^{p/(p-2)}\|m_{ij}\|_{ML^p}^{1/p}\|\tilde{m}_{ij}\|_{ML^p}^{1/p}\right\}\|f\|_p$$

for all integers i, j.

Proof. For $p > 2$ we notice that

$$\left|T_{ij}^* f\right|^p \le \int_0^\infty \int_0^\infty \left|\partial_s \partial_t \left(\left|T_{ij}^{s,t} f\right|^p\right)\right| \, ds\,dt$$

$$\le p \int_0^\infty \int_0^\infty \left|T_{ij}^{s,t} f\right|^{p-1} \left|\tilde{T}_{ij}^{s,t} f\right| \, ds\,dt$$

$$+ p(p-1) \int_0^\infty \int_0^\infty \left|T_{ij}^{s,t} f\right|^{p-2} \left|Q_{ij}^{s,t} f\right| \left|\tilde{Q}_{ij}^{s,t} f\right| \, ds\,dt$$

$$\le p \left(\int_0^\infty \int_0^\infty \left|T_{ij}^{s,t} f\right|^p \frac{ds\,dt}{st}\right)^{1/p'} \left(\int_0^\infty \int_0^\infty \left|\tilde{T}_{ij}^{s,t} f\right|^p \frac{ds\,dt}{st}\right)^{1/p}$$

$$+ p(p-1) \left(\int_0^\infty \int_0^\infty \left|T_{ij}^{s,t} f\right|^p \frac{ds\,dt}{st}\right)^{(p-2)/p}$$

$$\left(\int_0^\infty \int_0^\infty \left|Q_{ij}^{s,t} f\right|^p \frac{ds\,dt}{st}\right)^{1/p} \left(\int_0^\infty \int_0^\infty \left|\tilde{Q}_{ij}^{s,t} f\right|^p \frac{ds\,dt}{st}\right)^{1/p},$$

a consequence of repeated Hölder's inequality, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Since $\dfrac{(p-2)}{p} + \dfrac{2}{p} = 1$, integrating over $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ and applying Hölder's inequality again, we see that

$$\left\| T_{ij}^* f \right\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})} \leq C_p \left\{ I_{ij}^{1/p'} \, \tilde{I}_{ij}^{1/p} + I_{ij}^{p/(p-2)} \, J_{ij}^{1/p} \, \tilde{J}_{ij}^{1/p} \right\}$$

where

$$I_{ij} = \left(\int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} \int_0^\infty \int_0^\infty \left| T_{ij}^{s,t} f \right|^p \frac{ds\,dt}{st} \, dx \, dy \right)^{1/p}$$

with analogous representations for $\tilde{I}_{ij}, J_{ij},$ and $\tilde{J}_{ij}$. For fixed integers i, j, we select Schwartz functions ψ_k in $\mathbf{R}^{n_k}$ such that $\int \psi_k \, dx_k = 0$, $k = 1, 2$, and

$$\hat{\psi}_1(\xi_1) = 1 \qquad \text{if} \quad 2^{i-1} \leq |\xi_1| \leq 2^{i+1},$$

$$\hat{\psi}_2(\xi_2) = 1 \qquad \text{if} \quad 2^{j-1} \leq |\xi_2| \leq 2^{j+1}.$$

Setting $\quad \psi_{s,t}(x_1, x_2) = s^{-n_1} t^{-n_2} \psi_1(x_1/s) \psi_2(x_2/t)$, we have

$$I_{ij}^p = \int_0^\infty \int_0^\infty \int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} \left| T_{ij}^{s,t} (\psi_{s,t} * f)(x,y) \right|^p \, dx \, dy \, \frac{ds\,dt}{st}$$

$$\leq \|\mu_{ij}\|_{ML^p}^p \int_0^\infty \int_0^\infty \int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} \left| \psi_{s,t} * f(x,y) \right|^p \, dx \, dy \, \frac{ds\,dt}{st}$$

$$\leq \|\mu_{ij}\|_{ML^p}^p \int_{\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}} \left(\int_0^\infty \int_0^\infty \left| \psi_{s,t} * f(x,y) \right|^2 \frac{ds\,dt}{st} \right)^{p/2} dx \, dy$$

$$\leq C_p \|\mu_{ij}\|_{ML^p}^p \|f\|_{L^p(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})}^p \qquad \text{whenever} \quad p > 2,$$

where the last inequality is a direct consequence of the iterative Littlewood-Paley theory in the product space formulation. We notice that in the same manner it is possible to get similar estimates for $\tilde{I}_{ij}, J_{ij},$ and $\tilde{J}_{ij}$ without any difficulty and therefore the proof is validated. $\quad \square$

We are now ready to finish the proof of Theorem A. For $p = 2$,

$$\|\mu_{ij}\|_{ML^2} = \|\mu_{ij}\|_\infty \leq C \, 2^{-ia-jb}$$

and the (BMO, L^∞)-norm of T_{ij} is at most

$$C_\varepsilon \, 2^{i(1-a+\varepsilon+\frac{n_1}{2})+j(1-b+\varepsilon+\frac{n_2}{2})} \quad \text{for sufficiently small} \quad \varepsilon > 0.$$

Interpolation yields

$$\|\mu_{ij}\|_{ML^p} \le C_\varepsilon \, 2^{i(-a+\varepsilon+(\frac{n_1}{2}+1)\theta)+j(-b+\varepsilon+(\frac{n_2}{2}+1)\theta)}$$

with $\quad \theta = 1 - \dfrac{2}{p}.$ Carrying out the same procedures to the other multipliers, we shall obtain

$$\|\tilde{\mu}_{ij}\|_{ML^p} \le C_\varepsilon \, 2^{i(1-a+\varepsilon+(\frac{n_1}{2}+1)\theta)+j(1-b+\varepsilon+(\frac{n_2}{2}+1)\theta)},$$

$$\|m_{ij}\|_{ML^p} \le C_\varepsilon \, 2^{i(1-a+\varepsilon+(\frac{n_1}{2}+1)\theta)+j(-b+\varepsilon+(\frac{n_2}{2}+1)\theta)},$$

$$\|\tilde{m}_{ij}\|_{ML^p} \le C_\varepsilon \, 2^{i(-a+\varepsilon+(\frac{n_1}{2}+1)\theta)+j(1-b+\varepsilon+(\frac{n_2}{2}+1)\theta)},$$

when $i, j \ge 0$ and consequently,

$$\left\|T_{ij}^* f\right\|_p \le C_\varepsilon \, 2^{i(-a+\varepsilon+(\frac{n_1}{2}+1)\theta+\frac{1}{p})+j(-b+\varepsilon+(\frac{n_2}{2}+1)\theta+\frac{1}{p})} \left\|f\right\|_p$$

the series of norms of which converges if and only if

$$a > \frac{1}{p} + \left(\frac{n_1}{2} + 1\right)\theta \quad \text{and} \quad b > \frac{1}{p} + \left(\frac{n_2}{2} + 1\right)\theta,$$

that is,

$$p < \min\left(\frac{2(n_1 + 1)}{n_1 - 2a + 2}, \frac{2(n_2 + 1)}{n_2 - 2b + 2}\right) = r_{a,b}.$$

The proof of our main theorem is now complete.

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