THE QUADRATIC FORMULA IN BANACH SPACE

by

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A THESIS

submitted to

OREGON STATE COLLEGE

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1956

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Date thesis is presented April 10, 1956

Typed by Louis Rall

ACKNOWLEDGMENT

The author is forever indebted to Professor A. T. Lonseth for his unwavering confidence, valuable suggestions, and penetrating criticisms.

Financial support for this thesis was provided by the Office of Ordnance Research, U. S. Army, under terms of Contract DA-04-200-ORD-177, "Numerical Solution of Integral Equations".

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CHAPTER I

INTRODUCTION

The generalization of the simple linear equation

(1.01) $$ax + b = 0$$

to the more abstract setting of a Banach space in order to obtain results concerning finite and infinite systems of linear equations, as well as linear integral equations, has resulted in rich mathematical theories and results of practical importance. In an attempt to deal with the nonlinear systems of equations and integral equations which arise from mathematical physics, it seems natural to consider such a generalization of the algebraic equation of degree n,

(1.02) $$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0.$$

In particular, for n = 2, equation (1.02) reduces to

(1.03) $$ax^2 + bx + c = 0,$$

the familiar quadratic equation. For the solution of this equation in the case $a \neq 0$, there is available the convenient formula

(1.04) $$x = \frac{1}{2a}\{-b \pm (b^2 - 4ac)^{1/2}\}.$$

The purpose of the present study is to give a natural generalization of equation (1.03) to Banach space, to examine its properties, and to determine the extent of the validity of the quadratic formula (1.04). As such equations may arise from the retention of second order terms in the formulation of mathematical descriptions of physical systems, results concerning their solution could be of considerable practical importance.

The remainder of this chapter is devoted to the consideration of the basic definitions and results concerning Banach spaces and linear operators on them. For the purpose of illustration, a number of examples of such spaces and operators of importance in applied mathematics are readily available (1, pp. 78–95, 7, pp. 5–15, and 10, pp. 100–108). The following set of postulates (10, pp. 92–93) serve to characterize a complex Banach space. If the word "complex" is replaced everywhere by "real", they also serve to define a real Banach space.

Definition (1.A). A set X is called a complex linear space if for the elements of X there exist two uniquely defined operations, an addition and a multiplication by complex numbers such that $(x + y) \varepsilon X$ for all $x, y \varepsilon X$, and $(\lambda x) \varepsilon X$ for all $x \varepsilon X$ and all complex numbers λ. There is for these operations the following rules:

1°. $x + y = y + x$.

2°. $(x + y) + z = x + (y + z)$.

3°. If $x + y = x + z$, then $y = z$.

4°. $\lambda(x + y) = \lambda x + \lambda y$.

5°. $(\xi\eta)x = \xi(\eta x)$.

6°. $(1)x = x$.

On the basis of Definition (1.A), it is possible to derive the existence of an element $0 \varepsilon X$ such that $x + 0 = 0 + x = x$ for all $x \varepsilon X$, and $(0)x = 0$ for the multiplication of x by zero (10, p. 93).

Definition (1.B). A complex linear space X is called a complex normed linear space if to every element $x \varepsilon X$ there corresponds a non-negative real number $\|x\|$ (the norm of x) such that:

1°. $\|\lambda x\| = |\lambda| \, \|x\|$.

2°. $\|x + y\| \leq \|x\| + \|y\|$.

3°. $\|x\| > 0$ for $x \neq 0$.

Definition (1.C). A complex normed linear space X is said to be complete if for every sequence $\{x_n\}$ of elements of X which satisfies the condition

$$\lim \|x_n - x_m\| = 0 \text{ as } m,n \rightarrow \infty,$$

there exists an element $x \varepsilon X$ such that

$$\lim \|x - x_n\| = 0 \text{ as } n \rightarrow \infty.$$

The sequence $\{x_n\}$ in this case is said to converge to x.

Definition (1.D). A complex Banach space X is a complete complex normed linear space.

A Banach space thus has properties in common with the real and complex number systems, which are the most simple examples of Banach spaces. In what follows, the symbol X and the word "space" will always denote a Banach space.

Definition (1.E). An operation P from a space X to a space Y is a set of ordered pairs (x,y) of elements $x \varepsilon X$, $y \varepsilon Y$ such that there exists an $(x,y) \varepsilon P$ for all $x \varepsilon X$, and if $(x,y) \varepsilon P$ for a given $x \varepsilon X$, $(x,\bar{y}) \not\varepsilon P$ if $\bar{y} \neq y$. The statement $(x,y) \varepsilon P$ is symbolized by

(1.05) $$y = Px,$$

where P will be called an operator for mapping X into Y, or simply an operator from X to Y. If Y = X, P is said to be an operator in X.

Examples of operators in a space X are the identity operator I defined by $Ix = x$ for all $x \varepsilon X$, and the null operator 0 defined by $0x = 0$ for all $x \varepsilon X$.

Definition (1.F). If R and S are operators in X, their sum (R + S) is defined by

(1.06) $$(R + S)x = Rx + Sx$$

for all $x \varepsilon X$, and their product (RS) by

(1.07) $$(RS)x = R(Sx)$$

for all $x \varepsilon X$.

Definition (1.G). An operator P in X is continuous if $\lim \|x - x_n\| = 0$ as $n \to \infty$ implies $\lim \|Px - Px_n\| = 0$ as $n \to \infty$; it is additive if $P(x + y) = Px + Py$ for all

$x, y \in X$; and it is homogeneous (of first degree) if for all numbers λ, $P(\lambda x) = \lambda(Px)$.

An additive and continuous operator is homogeneous (1, p. 36).

Definition (1.H). An operator P in X is bounded if there exists a non-negative real number M such that

$$\|Px - Py\| \leq M \|x - y\| \tag{1.08}$$

for all $x, y \in X$. The greatest lower bound of the numbers M satisfying equation (1.08) is called the bound of P, and is denoted by $\|P\|$.

It follows that if R and S are bounded operators in X,

$$\|R + S\| \leq \|R\| + \|S\| \tag{1.09}$$

and

$$\|RS\| \leq \|R\| \|S\| \tag{1.10}$$

(8, p. 194). An additive operator is bounded if and only if it is continuous (1, p. 54).

Definition (1.I). An additive and continuous (or bounded) operator L in X is said to be linear. If for a given linear operator L, a linear operator L^{-1} exists such that

$$L^{-1}L = LL^{-1} = I, \tag{1.11}$$

L^{-1} is called the inverse of L.

A necessary and sufficient condition for the existence of the inverse of a linear operator is given in the following theorem (9, p. 979).

Theorem (1.A). If L is a linear operator in X, L^{-1} exists if and only if there exists a linear operator P such that P^{-1} exists and $\|I - PL\| < 1$. If these conditions are satisfied,

$$L^{-1} = \sum_{n=0}^{\infty} (I - PL)^n P. \tag{1.12}$$

Here the notation R^n for an operator R in X is defined by $R^n = RR^{n-1}$ for all positive integers n, with $R^0 = I$ by definition. On the basis of Theorem (1.A), it is possible to derive another necessary and sufficient condition for the existence of the inverse of a linear operator L.

Theorem (1.B). If L is a linear operator in X, L^{-1} exists if and only if there exists a linear operator R such that R^{-1} exists and $\|R - L\| < 1/\|R^{-1}\|$.

Proof: If R exists, take $P = R^{-1}$. Then

$$I - PL = R^{-1}(R - L), \tag{1.13}$$

and from equation (1.10),

$$\|I - PL\| \leq \|R^{-1}\| \, \|R - L\| < 1, \tag{1.14}$$

so L^{-1} exists by Theorem (1.A). If L^{-1} exists, take $R = L$ so that R^{-1} exists, and

$$\|R - L\| = \|L - L\| = 0 < 1/\|L^{-1}\|. \tag{1.15}$$

There is no significant alteration in the above considerations if the operations and operators are defined from a space X to a space Y.

Definition (1.J). If L is a linear operator from a space X to a space Y, the set

(1.16) $$O\{L\} = \{x: Lx = 0\}$$

is called the null space of L.

Due to the linearity of L, O{L} is a linear subspace of X. From Theorem (1.A), L^{-1} cannot exist if O{L} contains any element of X different from 0.

Definition (1.K). If L and R are linear operators in X such that $R^2 = L$, R is called a square root of L, and is denoted by $L^{1/2}$.

Square roots of linear operators will arise in the generalization of the quadratic formula (1.04) to a Banach space X.

Theorem (1.C). If L is a linear operator in X and $L^{1/2}$ exists,

(1.17) $$L^{-1/2} = (L^{1/2})^{-1}$$

exists if and only if L^{-1} exists.

Proof: If $L^{-1/2}$ exists, $L^{-1} = (L^{-1/2})(L^{-1/2})$ exists as $L = (L^{1/2}L^{1/2})$. If now L^{-1} exists,

(1.18) $$(L^{-1}L^{1/2})L^{1/2} = L^{1/2}(L^{1/2}L^{-1}) = I,$$

and it follows that

(1.19) $$L^{-1/2} = (L^{-1}L^{1/2}) = (L^{1/2}L^{-1})$$

exists.

CHAPTER II

BILINEAR OPERATORS

The concept of a bilinear operator is fundamental to the generalization of the quadratic equation (1.03) to Banach space. The following theorem (6, pp. 32–33) plays a key rôle in the notion of a bilinear operator.

Theorem (2.A). Let (X) denote the set of all linear operators in a Banach space X. With addition and scalar multiplication of elements of (X) as given in Definitions (1.F) and (1.G), and $\|L\|$ defined for all $L \varepsilon (X)$ by Definition (1.H), (X) is a Banach space.

Definition (2.A). A bilinear operator B in a space X is a linear operator from X to (X).

If B is a bilinear operator in X, for all $x \varepsilon X$, Bx is a uniquely defined linear operator in X. Thus for all $y \varepsilon X$,

$$Bxy = (Bx)y = z \tag{2.01}$$

is a uniquely defined element of X. Thus to every ordered pair of elements of X a bilinear operator B corresponds a unique element of X. The rules for this correspondence are as follows: As Bx is linear,

$$Bx(y + z) = Bxy + Bxz, \tag{2.02}$$

and since B is also linear,

(2.03) $$B(x + y)z = Bxz + Byz.$$

Both B and Bx are homogeneous of first degree, so for all numbers ξ and η,

(2.04) $$B(\xi x)(\eta y) = (\xi\eta)Bxy.$$

The operators B and Bx are also bounded, and hence from equation (1.10),

(2.05) $$\|Bxy\| \leq \|B\| \|x\| \|y\| ,$$

where equations (2.02–05) hold for all $x,y,z \in X$.

Definition (2.B). Corresponding to every bilinear operator B in X are the bilinear operators B*, called the permutation of B, and $\overline{B}$, called the mean of B, which are defined respectively by

(2.06) $$B^*xy = Byx$$

and

(2.07) $$\overline{B}xy = \frac{1}{2}\{Bxy + Byx\} = \frac{1}{2}\{B + B^*\}xy$$

for all $x,y \in X$. A bilinear operator B such that $B = B^* = \overline{B}$ is said to be symmetric.

An important class of linear and bilinear operators in a space X are the first and second Fréchet derivatives of operators P in X (4, pp. 293–323). In what follows, the word "derivative" will always mean "Fréchet derivative".

Definition (2.C). By $o(\varepsilon)$ is meant any real function of the real variable ε such that

(2.08) $$\lim \frac{1}{\varepsilon}o(\varepsilon) = 0 \text{ as } \varepsilon \rightarrow 0.$$

Definition (2.D). If P is an operator in X, and for some point $x \epsilon X$ there exists a linear operator $(dP|x)$ such that

$$(2.09) \qquad \|P(x + \Delta x) - Px - (dP|x)\Delta x\| = o(\|\Delta x\|),$$

P is said to be once differentiable at x, and $(dP|x)$ is called the first derivative of P at x.

Definition (2.E). If for some $\lambda > 0$, P is once differentiable at all $\tilde{x}$ such that $\|x - \tilde{x}\| < \lambda$, and a bilinear operator $(d^2P|x)$ in X exists such that

$$(2.10) \qquad \|(dP|x+\Delta x) - (dP|x) - (d^2P|x)\Delta x\| = o(\|\Delta x\|),$$

P is said to be twice differentiable at x, and $(d^2P|x)$ is called the second derivative of P at x.

If $(dP|x)$ and $(d^2P|x)$ exist, they are unique, and $(d^2P|x)$ is symmetric (6, pp. 81–82). Differentiation in the sense of Fréchet obeys the following rules (7, pp. 159–166):

$$(2.11) \qquad (d\{R + S\}|x) = (dR|x) + (dS|x),$$

$$(2.12) \qquad (d\{RS\}|x) = (dR|Sx)(dS|x),$$

and

$$(2.13) \qquad \|P(x+\Delta x) - Px - (dP|x)\Delta x\| \leq \frac{1}{2} \max_{(\tilde{x})} \|(d^2P|\tilde{x})\| \, \|\Delta x\|^2,$$

$$\tilde{x} = x + \lambda\Delta x, \; 0 \leq \lambda \leq 1.$$

Given a bilinear operator B, a linear operator L, and an element y in a space X, consider the operator Q defined by

$$(2.14) \qquad Qx = Bxx + Lx + y$$

for all $x \varepsilon X$. From Definitions (2.D) and (2.E) it follows at once that

(2.15) $$(dQ|x) = Bx + B^*x + L = 2\bar{B}x + L,$$

and

(2.16) $$(d^2Q|x) = B + B^* = 2\bar{B}$$

for all $x \varepsilon X$. If, corresponding to equation (2.14), the operators Q^* and $\bar{Q}$ in X are defined by

(2.17) $$Q^*x = B^*xx + Lx + y$$

and

(2.18) $$\bar{Q}x = \bar{B}xx + Lx + y$$

for all $x \varepsilon X$, it is evident that

(2.19) $$Qx = Q^*x = \bar{Q}x,$$

(2.20) $$(dQ|x) = (dQ^*|x) = (d\bar{Q}|x),$$

and

(2.21) $$(d^2Q|x) = (d^2Q^*|x) = (d^2\bar{Q}|x)$$

for all $x \varepsilon X$. There is thus no loss of generality in the assumption, which will be made throughout, that the bilinear operator B in equation (2.14) is symmetric, for a non-symmetric B may be replaced by its mean without altering the value of the operator Q or its derivatives.

Consideration will now be given to the solution of the linear equation

(2.22) $$Bx + L = 0,$$

where B is a bilinear, and L a linear operator in X. If equation (2.22) has a solution x, it can have another solution $\bar{x}$ if and only if $(x - \bar{x}) \varepsilon O\{B\}$. The null space $O\{B\}$ of B is determined from the null spaces of the linear operators Bx in X by means of the following theorem.

Theorem (2.B). If B is a symmetric bilinear operator in X,

$$O\{B\} = \prod_{x \varepsilon X} O\{Bx\}, \tag{2.23}$$

the product denoting the intersection of the sets $O\{Bx\}$.

Proof: If $y \varepsilon O\{B\}$, then $Byx = (Bx)y = 0$ for all $x \varepsilon X$, and $y \varepsilon \prod_{x \varepsilon X} O\{Bx\}$. If now $y \varepsilon \prod_{x \varepsilon X} O\{Bx\}$, $Bxy = (By)x = 0$ for all $x \varepsilon X$, so $By = 0$ and $y \varepsilon O\{B\}$.

From the way in which the symmetry of B was used in the proof of Theorem (2.B), it follows that for a general bilinear operator B in X,

$$O\{B\} = \prod_{x \varepsilon X} O\{B^* x\}. \tag{2.24}$$

Theorem (2.C). Equation (2.22) can have a solution $x \varepsilon X$ only if

$$O\{B\} \subseteq O\{L\}, \tag{2.25}$$

and it can have at most one solution if $O\{B\bar{x}\} = \{0\}$ for some $\bar{x} \varepsilon X$.

Proof: From Theorem (2.B), $O\{B\} \subseteq O\{Bx\}$ for all $x \varepsilon X$. The linear operator $-L$ thus cannot be represented in the form Bx unless equation (2.25) is satisfied. If $O\{B\bar{x}\} = \{0\}$ for some $\bar{x} \varepsilon X$, $O\{B\} = \{0\}$ as $0 \varepsilon O\{B\}$ and also

$O\{B\} \subset \{0\}$, in which case equation (2.22) has at most one solution.

A number of examples of bilinear operators are given by Kantorovich (7, pp. 155–166); these examples also include first and second derivatives of nonlinear operators. Certain classes of bilinear operators in vector and function spaces will be considered in Chapter V.

CHAPTER III

THE QUADRATIC EQUATION

The expression

(3.01) $$Qx = Bxx + Lx + y = 0,$$

where the symmetric bilinear operation B in X, the linear operator L in X, and the element $y \varepsilon X$ are given, is called a quadratic equation in X. Any point $x \varepsilon X$ such that $Qx = 0$ will be called a solution, or root, of equation (3.01).

Theorem (3.A). Equation (3.01) has a solution $x \varepsilon X$ if and only if there exists a linear operator M in X such that

(3.02) $$Bx + L = M$$

and

(3.03) $$Mx + y = 0.$$

Proof: This is a restatement of the definition of a solution $x \varepsilon X$ of equation (3.01), as $Qx = (Bx + L)x + y$ for all $x \varepsilon X$.

Theorem (3.B). If $Q\tilde{x} = Q\hat{x} = 0$, then

(3.04) $$(dQ|\frac{\tilde{x}+\hat{x}}{2})(\tilde{x} - \hat{x}) = 0.$$

Proof: From equation (2.15),

(3.05) $$(dQ|\frac{\tilde{x}+\hat{x}}{2}) = B\tilde{x} + B\hat{x} + L.$$

Thus

$$(dQ|\tfrac{\bar{x}+\hat{x}}{2})(\bar{x}-\hat{x}) = B\bar{x}\bar{x} + B\hat{x}\bar{x} + L\bar{x} - B\hat{x}\hat{x} - B\bar{x}\hat{x} - L\hat{x}$$

$$= Q\bar{x} - Q\hat{x}$$

$$= 0,$$

which establishes equation (3.04).

Theorem (3.C). If $Q\bar{x} = 0$ and $(dQ|\bar{x})^{-1}$ exists, the solution $\bar{x}$ of equation (3.01) is unique in the sphere

(3.06) $$\|x - \bar{x}\| < 1/\{\|B\|\ \|(dQ|\bar{x})^{-1}\|\}.$$

Proof: By Theorem (1.B), $(dQ|x)^{-1}$ exists for

(3.07) $$\|(dQ|x) - (dQ|\bar{x})\| < 1/\|(dQ|\bar{x})^{-1}\|,$$

or, from equation (2.15), for

(3.08) $$2\ \|B(x - \bar{x})\| < 1/\|(dQ|\bar{x})^{-1}\|.$$

As $\|Bz\| \leq \|B\|\,\|z\|$, equation (3.08) will be satisfied if

(3.09) $$\|x - \bar{x}\| < \tfrac{1}{2}\{\|B\|\ \|(dQ|\bar{x})^{-1}\|\}^{-1},$$

or, equivalently, $(dQ|y)^{-1}$ fails to exist only if

(3.10) $$\|y - \bar{x}\| \geq \tfrac{1}{2}\{\|B\|\ \|(dQ|\bar{x})^{-1}\|\}^{-1}.$$

If now $\hat{x} \neq \bar{x}$ is a solution of equation (3.01), by Theorems (3.B) and (1.A), $(dQ|\frac{\bar{x}+\hat{x}}{2})^{-1}$ fails to exist, so from equation (3.10),

(3.11) $$\tfrac{1}{2}\ \|\hat{x} - \bar{x}\| \geq \tfrac{1}{2}\{\|B\|\ \|(dQ|\bar{x})^{-1}\|\}^{-1}.$$

As the inequality (3.11) is valid for every solution $\hat{x} \neq \bar{x}$ of equation (3.01), there are no solutions $x \neq \bar{x}$ of $Qx = 0$ in the sphere defined by equation (3.06).

It follows from Theorem (3.A) that a solution x of $Qx = 0$ is determined if $(dQ|x)$ is known, for by (3.02),

(3.12) $$M = [L + (dQ|x)]/2.$$

The derivative of Q also enters into the consideration of quadratic equations by means of the addition formulas:

$$(3.13) \qquad Q(x+\bar{x}) = Bxx+(dQ|\bar{x})x+Q\bar{x} = B\bar{x}\bar{x}+(dQ|x)\bar{x}+Qx$$

and

$$(3.14) \qquad Q(x-\bar{x}) = B\bar{x}\bar{x} - (dQ|x)\bar{x} + Qx.$$

These formulas provide a scheme for the classification of quadratic equations in X.

Definition (3.A). The equation $Qu = Buu + Lu + v = 0$ is said to be of first kind if there is a $z \varepsilon X$ such that

$$(3.15) \qquad (dQ|z) = 0.$$

If the equation $Qu = 0$ is of first kind, set $u = x + z$, so that from equation (3.13),

$$(3.16) \qquad Q(x+z) = Bxx + Qz = 0.$$

For

$$(3.17) \qquad y = Qz = v - Bzz,$$

the problem of the solution of a quadratic equation of first kind is thus reduced to the consideration of the equation

$$(3.18) \qquad Q_1x = Bxx + y = 0,$$

which will be called the normal form of the quadratic equation of first kind. A quadratic equation of first kind thus has a solution $u = x + z$ if and only if $Q_1x = 0$, and z satisfies (3.15). It is to be noted that all equations (1.03) involving real or complex numbers are of first kind for $a \neq 0$.

Theorem (3.D). If $u \varepsilon X$ is a solution of a quadratic equation $Qu = 0$ of first kind, then $Q(u+w) = 0$ for all $w \varepsilon O\{B\}$.

Proof: For all $w \varepsilon O\{B\}$, and any $u \varepsilon X$,

$$B(u + w)(u + w) = Buu, \tag{3.19}$$

and by Theorem (2.C), as equation (3.15) is satisfied by some $z \varepsilon X$, $w \varepsilon O\{B\}$ implies $w \varepsilon O\{L\}$, and thus

$$L(u + w) = Lu \tag{3.20}$$

for all $u \varepsilon X$. As thus

$$Qu = Q(u + w) \tag{3.21}$$

for all $u \varepsilon X$ and all $w \varepsilon O\{B\}$, $Q(u+w) = 0$ if and only if $Qu = 0$.

Theorem (3.E). The roots of a quadratic equation of first kind occur in pairs: $Qu = Q(z + x) = 0$ if and only if $Q\bar{u} = Q(z - x) = 0$.

Proof: From equation (3.18), $Q_1 x = 0$ if and only if $Q_1(-x) = 0$, and the theorem thus follows from the satisfaction of equation (3.15) by some $z \varepsilon X$ in the case of the quadratic equation of first kind.

Definition (3.B). A quadratic equation

$$Q'u = B'uu + Lu + v = 0 \tag{3.22}$$

which is not of first kind is said to be of second kind if

$$(dQ'|z)^{-1} = S \tag{3.23}$$

exists for some $z \varepsilon X$.

If equation (3.22) is of second kind, set $u = x + z$, where z satisfies equation (3.23). From equation (3.13),

(3.24) $$Q'(x + z) = B'xx + S^{-1}x + Qz = 0$$

is satisfied by $(x+z)\varepsilon X$ if and only if

(3.25) $$Q_2 x = Bxx + Ix + y = 0,$$

where

(3.26) $$B = SB', \qquad y = SQz.$$

Equation (3.25) is called the normal form of a quadratic equation of second kind.

Theorem (3.F). The normal form of a quadratic equation of second kind is a quadratic equation of second kind.

Proof: From equation (3.25),

(3.27) $$(dQ_2|0) = (dQ_2|0)^{-1} = I,$$

so the normal form of a quadratic equation of second kind is of second kind if it is not of first kind. From equations (3.22) and (3.23),

(3.28) $$2B'z + L = S^{-1}.$$

If there exists a $\bar{z}\varepsilon X$ such that

(3.29) $$(dQ_2|\bar{z}) = 2SB'\bar{z} + I = 0,$$

it follows from equation (3.28) that

(3.30) $$(dQ'|z-\bar{z}) = 2B'(z-\bar{z}) + L = 0,$$

contrary to the assumption that the equation $Q'u = 0$ is of second kind.

Theorem (3.G). If $Q'u = 0$ is a quadratic equation of second kind, then for all $x, \bar{x} \varepsilon X$,

(3.31) $$(dQ'|x) \neq -(dQ'|\bar{x}).$$

Proof: If equation (3.31) is satisfied by any $x,\tilde{x}\varepsilon X$,

(3.32) $$(dQ'|\frac{x+\tilde{x}}{2}) = 0,$$

contrary to the assumption that $Q'u=0$ is of second kind.

This is in contrast with the quadratic equation of first kind, where if $Qu = Q(z+x) = 0$, by Theorem (3.E), $Q\bar{u} = Q(z-x) = 0$, and since $(dQ|z) = 0$,

(3.33) $$(dQ|z+x) = -(dQ|z-x).$$

Theorem (3.H). If u is a solution of the quadratic equation (3.22) of second kind, and $w \neq 0$ is an element of $O\{B'\}$, then $Q'(u+w) \neq 0$.

Proof: If $u = z+x$ and $u+w = x+z+w$ satisfy equation (3.22), where $w\varepsilon O\{B'\}$, then x and $x+w$ are solutions of equation (3.25). As $O\{B\} = O\{SB'\} = O\{B'\}$,

(3.34) $$Q_2(x+w) = Bxx + x + w + y = Q_2x + w,$$

and $Q_2(x+w) = Q_2x + w = 0$ if and only if $w = 0$.

Theorem (3.I). If u and $\tilde{u}$ are distinct solutions of a quadratic equation of second kind, then

(3.35) $$(dQ'|u) \neq (dQ'|\tilde{u}).$$

Proof: If $(dQ'|u) = (dQ'|\tilde{u})$, then

(3.36) $$2B'u + L = 2B'\tilde{u} + L,$$

and so $(u-\tilde{u})\varepsilon O\{B'\}$. However, by Theorem (3.H), if $Q'u=0$ and $Q'\tilde{u} = 0$, $u\neq\tilde{u}$, then $(u-\tilde{u})\notin O\{B'\}$, which proves (3.35).

These theorems for the quadratic equation of second kind show, by comparison with Theorem (3.D) for equations of first kind, that the properties of the equation of second kind are quite different from those of equation (1.03) for real and complex numbers.

Theorem (3.J). If the quadratic equation (3.01) has a solution $x \varepsilon X$, it has a solution $\bar{x} \neq x$ in X if and only if the quadratic equation

$$B\hat{x}\hat{x} + (dQ|x)\hat{x} = 0 \tag{3.37}$$

has a solution $\hat{x} \neq 0$.

Proof: Set $\bar{x} = x + \hat{x}$. By equation (3.13), $Q\bar{x} = 0$ if and only if $\hat{x}$ satisfies equation (3.37), provided that $Qx = 0$.

For the application of Newton's method to the solution of the quadratic equation (3.01), it is required that $(dQ|z)^{-1}$ exist for some $z \varepsilon X$ (2, pp. 827–831), so that this procedure applies only to some equations of first and second kinds.

Definition (3.C). A quadratic equation (3.01) for which $(dQ|x)^{-1}$ fails to exist and $(dQ|x) \neq 0$ for all $x \varepsilon X$ is said to be of third kind.

This definition is included for the sake of logical completeness; in what follows, only the quadratic equations of first and second kinds will be considered, and these in their normal forms.

CHAPTER IV

THE QUADRATIC FORMULA

The quadratic formula (1.04) will now be extended to quadratic equations (3.01) of first and second kinds, and it will be shown that the validity of this formula depends on a certain property of the bilinear operator B.

Definition (4.A). If B is a symmetric bilinear operator in X, the subset

$$F\{B\} = \{x\colon (Bx)^2 - B(Bxx) = 0\} \tag{4.01}$$

of X is called the factor set of B. If $F\{B\} = X$, B is said to be totally factorable.

It follows that $0 \varepsilon F\{B\}$ for all B, and that if $x \varepsilon F\{B\}$, $(\lambda x) \varepsilon F\{B\}$ for all numbers λ, as B and Bx are homogeneous.

Theorem (4.A). If $x \varepsilon F\{B\}$ and $y \varepsilon F\{B\}$, $(x+y) \varepsilon F\{B\}$ if and only if

$$BxBy + ByBx = 2B(Bxy). \tag{4.02}$$

Proof: For all $x,y \varepsilon F\{B\}$,

$$\begin{aligned}\{B(x+y)\}^2 - B\{B(x+y)(x+y)\} &= (Bx)^2 + BxBy + \\ &+ ByBx + (By)^2 - B(Bxx) - 2B(Bxy) - B(Byy) = \\ &= BxBy + ByBx - 2B(Bxy),\end{aligned} \tag{4.03}$$

so that the satisfaction of equation (4.02) is a necessary and sufficient condition that $(x+y) \varepsilon F\{B\}$ if $x,y \varepsilon F\{B\}$.

For a given symmetric bilinear operator B in X, and an element $z \varepsilon X$, consider the three associated bilinear operators, defined for all $x,y \varepsilon X$ by

(4.04) $$(Tz)xy = Bz(Bxy),$$

which is symmetric,

(4.05) $$(T'z)xy = Bx(Bzy),$$

and its permutation,

(4.06) $$(T'z)^*xy = By(Bzx).$$

Theorem (4.B). A symmetric bilinear operator B in X is totally factorable if and only if

(4.07) $$Tz = \overline{T'z}$$

for all $z \varepsilon X$.

Proof: If equation (4.07) holds, then for all x,y,z in X,

(4.08) $$2Bz(Bxy) = Bx(Bzy) + By(Bzx),$$

and, in particular, for x = y,

(4.09) $$Bz(Bxx) = (Bx)^2 z,$$

or

(4.10) $$\{(Bx)^2 - B(Bxx)\}z = 0,$$

and since equation (4.10) holds for each $x \varepsilon X$ and all $z \varepsilon X$, $x \varepsilon F\{B\}$ for all $x \varepsilon X$. If now B is totally factorable, from Theorem (4.A), equation (4.02) is satisfied for all $x,y \varepsilon X$, so equation (4.08) is satisfied for all $x,y,z \varepsilon X$, which implies that equation (4.07) holds for all $z \varepsilon X$.

It will now be shown that the quadratic formula for the solution of the quadratic equations (3.18) and (3.25) is valid only on the subset $F\{B\}$ of X.

Theorem (4.C). The quadratic equation of first kind

$$Q_1 x = Bxx + y = 0$$

has a solution $x \varepsilon F\{B\}$ if and only if $(-By)^{1/2}$ exists and

$$Bx = (-By)^{1/2}, \tag{4.11}$$

and

$$(-By)^{1/2} x + y = 0. \tag{4.12}$$

Proof: By Theorem (3.A), if $(-By)^{1/2}$ exists and x satisfies equations (4.11) and (4.12), then $Q_1 x = 0$. It thus follows that $Bxx = -y$, so that

$$B(Bxx) = -By = \{(-By)^{1/2}\}^2 = (Bx)^2, \tag{4.13}$$

and $x \varepsilon F\{B\}$. If now $x \varepsilon F\{B\}$ satisfies equation (3.18),

$$(Bx)^2 = B(Bxx) = -By, \tag{4.14}$$

so that $(-By)^{1/2} = Bx$ exists, and x satisfies equation (4.11), the satisfaction of equation (4.12) by x following at once from Theorem (3.A).

Theorem (4.D). The quadratic equation of second kind

$$Q_2 x = Bxx + Ix + y = 0$$

has a solution $x \varepsilon F\{B\}$ if and only if $(I - 4By)^{1/2}$ exists,

$$Bx = \frac{1}{2}\{-I + (I - 4By)^{1/2}\}, \tag{4.15}$$

and

$$\frac{1}{2}\{I + (I - 4By)^{1/2}\}x + y = 0. \tag{4.16}$$

Proof: If $(I - 4By)^{1/2}$ exists and x satisfies equations (4.15) and (4.16), then $Q_2 x = 0$ by Theorem (3.A). It thus follows that

$$B(Bxx) = -Bx - By, \tag{4.17}$$

while from equation (4.15),

$$\begin{aligned}(Bx)^2 &= \tfrac{1}{4}I - \tfrac{1}{2}(I-4By)^{1/2} + \tfrac{1}{4}I - By \\ &= \tfrac{1}{2}\{I - (I - 4By)^{1/2}\} - By \\ &= -Bx - By,\end{aligned} \tag{4.18}$$

so $x \varepsilon F\{B\}$. If now $x \varepsilon F\{B\}$ is a solution of equation (3.25),

$$\begin{aligned}I - 4By &= I + 4Bx + 4B(Bxx) \\ &= I + 4Bx + 4(Bx)^2 \\ &= (I + 2Bx)^2,\end{aligned} \tag{4.19}$$

so that $(I-4By)^{1/2} = I+2Bx$ exists and x satisfies equation (4.15). By Theorem (3.A), x also satisfies equation (4.16).

It may be noted that Theorems (4.C) and (4.D) hold for equations involving additive and bi-additive homogeneous operators in a linear space, as the metric properties (boundedness and continuity) of linear and bilinear operators in a Banach space are not used in their proofs. However, the analytic problems of the solution of linear equations and the construction of square roots of linear operators are considered more conveniently in a Banach space. Unfortunately, the theory of the existence, multiplicity, and the construction of the square roots of linear operators is not in a satisfactory state at this time; in particular, the results obtained by Einar Hille

(6, pp. 124–126) hold provided that $LM = ML$ for all L, M in (X), which is not the case for a number of vector and function spaces X of interest in applied mathematics.

The multiplicity of solutions $x \varepsilon F\{B\}$ of quadratic equations of first and second kinds is related in a simple manner to the multiplicity of square roots of the linear operators $-By$ and $I - 4By$, respectively.

Theorem (4.E). If $O\{B\} = \{0\}$, the quadratic equation (3.18) of first kind can have no more solutions x in $F\{B\}$ than there are distinct linear operators M such that $M^2 = -By$.

Proof: If $Bx = B\bar{x} = (-By)^{1/2}$, $(x - \bar{x}) \varepsilon O\{B\}$, and $x = \bar{x}$ when $O\{B\} = \{0\}$. The Theorem now follows from equation (4.11).

Theorem (4.F). The quadratic equation (3.25) of second kind has at most one solution $x \varepsilon F\{B\}$ corresponding to each pair $M, -M$ of distinct linear operators M such that $M^2 = (-M)^2 = I - 4By$.

Proof: If for a given definition of $(I - 4By)^{1/2}$,

$$Bx = B\bar{x} = \tfrac{1}{2}\{-I + (I - 4By)^{1/2}\}, \tag{4.20}$$

then

$$(dQ_2 | x) = (dQ_2 | \bar{x}), \tag{4.21}$$

and by Theorem (3.I), at most one of the elements $x, \bar{x}$ can satisfy equation (3.25). If $x \varepsilon F\{B\}$ is a solution of equation (3.25), then for some definition of $(I - 4By)^{1/2}$, x

satisfies equation (4.15). There is thus no $\bar{x}\varepsilon X$ such that

$$B\bar{x} = \tfrac{1}{2}\{-I - (I - 4By)^{1/2}\} \tag{4.22}$$

for the same definition of the square root of $I-4By$, as **this would** imply that

$$(dQ_2|\bar{x}) = -(dQ_2|x), \tag{4.23}$$

in contradiction to Theorem (3.G).

Theorem (4.G). If $(-By)^{-1}$ exists, all solutions **$x\varepsilon F\{B\}$ of** $Q_1x = 0$ are unique in the spheres

$$||\bar{x} - x|| < 2/\{||B||\,||(Bx)^{-1}||\}. \tag{4.24}$$

If $(I-4By)^{-1}$ exists, all solutions $x\varepsilon F\{B\}$ of $Q_2x=0$ are **unique in** the spheres

$$||\bar{x} - x|| < 1/\{||B||\;||(I + 2Bx)^{-1}||\}. \tag{4.25}$$

Proof: By Theorem (1.C), $(-By)^{-1/2}$ and $(I-4By)^{-1/2}$ **exist provided** that $(-By)^{-1}$, $(-By)^{1/2}$, and $(I-4By)^{-1}$, $(I-4By)^{1/2}$ exist, respectively. If now $x\varepsilon F\{B\}$ is a sol- **ution of** $Q_1x = 0$, by Theorem (4.C),

$$2Bx = (dQ_1|x) = 2(-By)^{1/2}, \tag{4.26}$$

and equation (4.24) follows from Theorem (3.C). **Likewise,** **if $x\varepsilon F\{B\}$** is a solution of $Q_2x = 0$, from Theorem (4.D),

$$I + 2Bx = (dQ_2|x) = (I - 4By)^{1/2}, \tag{4.27}$$

equation (4.25) thus following from Theorem (3.C).

As a consequence of the definition of the set $F\{B\}$, **it follows** at once that $O\{B\} \subseteq F\{B\}$. For the roots of $Q_1x = 0$ **and** $Q_2x = 0$ in $O\{B\}$, the following two theorems **are immediate** consequences of previous definitions.

Theorem (4.H). The quadratic equation $Q_1x = 0$ has a solution $x \varepsilon O\{B\}$ if and only if $y = 0$, in which case $Q_1x = 0$ for all $x \varepsilon O\{B\}$.

Theorem (4.I). The quadratic equation $Q_2x = 0$ has a solution $x \varepsilon O\{B\}$ if and only if $y \varepsilon O\{B\}$, in which case $x = -y$ is the unique solution of $Q_2x = 0$ in $O\{B\}$.

CHAPTER V

QUADRATIC INTEGRAL EQUATIONS

In order to illustrate the results of the previous chapters, the Banach space X of real continuous functions $x = x(s)$, $0 \leq s \leq 1$, with the norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)| \tag{5.01}$$

will be considered. Linear operators in X are represented by kernels $L = L(s,t)$, $0 \leq s,t \leq 1$, where the function $y(s)$ such that $y = Lx$ is computed by

$$y(s) = \int_0^1 L(s,t)x(t)dt, \tag{5.02}$$

and it is required that $y \varepsilon X$ for all $x \varepsilon X$ in order that L be in (X). For $\|L\|$, the estimate

$$\|L\| \leq \max_{0 \leq s \leq 1} \int_0^1 |L(s,t)|dt \tag{5.03}$$

is valid (5, pp. 155–157 and 7, pp. 12–13).

The kernel of the identity operator I is defined to be $\delta(s,t)$; that is,

$$x(s) = \int_0^1 \delta(s,t)x(t)dt \tag{5.04}$$

for all $x \varepsilon X$. Bilinear operators in X are likewise represented by kernels $B = B(s,t,u)$, where the function $z(s)$ such that $z = Bxy$ is given by

$$(5.05) \qquad z(s) = \int_0^1\int_0^1 B(s,t,u)x(u)y(t)\,du\,dt.$$

It is evident that B is symmetric if and only if

$$(5.06) \qquad B(s,t,u) = B(s,u,t)$$

$0 \leq s,t,u \leq 1$, except perhaps for a set of measure zero (10, pp. 64–69). For $\|B\|$ the estimate

$$(5.07) \qquad \|B\| \leq \max_{0\leq s\leq 1} \int_0^1\int_0^1 |B(s,t,u)|\,du\,dt$$

is available (7, p. 158). The general quadratic equation in X thus has the form

$$(5.08) \quad \int_0^1\int_0^1 B(s,t,u)x(u)x(t)\,du\,dt + \int_0^1 L(s,t)x(t)\,dt + y(s) = 0,$$

a nonlinear integral equation.

A characterization of totally factorable bilinear operators B in X will now be obtained from Theorem (4.B). From equation (4.04),

$$(5.09) \qquad (Tz) = \int_0^1\int_0^1 B(s,w,v)z(v)B(w,t,u)\,dv\,dw,$$

while from equation (4.05),

$$(5.10) \qquad (T'z) = \int_0^1\int_0^1 B(s,w,u)B(w,t,v)z(v)dvdw,$$

and from equation (4.06),

$$(5.11) \qquad (T'z)^* = \int_0^1\int_0^1 B(s,w,t)B(w,u,v)z(v)dvdw.$$

Theorem (5.A). A bilinear operator B in X is totally factorable if and only if

$$(5.12) \quad \int_0^1 B(s,w,v)B(w,t,u)dw = \frac{1}{2}\int_0^1 B(s,w,u)B(w,t,v)dw + \frac{1}{2}\int_0^1 B(s,w,t)B(w,u,v)dw,$$

$0 \le s,t,u,v \le 1$, except for at most a set of measure zero.

Proof: This is a direct consequence of equations (5.09–11) and Theorem (4.B), as the validity of equation (4.07) for all $z \varepsilon X$ is equivalent to the totaly factorability of B.

It follows from Theorem (5.A) that a bilinear operator B with a kernel B(s,t,u) which is a symmetric function of s,t,u is totally factorable. Simple examples of quadratic equations of first and second kinds will now be considered.

Example (5.A). The equation

(5.13) $$16\int_0^1\int_0^1 stu\, x(u)x(t)\,du\,dt - s = 0$$

is a quadratic equation of first kind in normal form with $B(s,t,u) = 16stu$ and $y(s) = -s$. As B is totally factorable, the considerations of Theorem (4.C) apply. Here

(5.14) $$(-By) = 16\int_0^1 stu^2 du = \frac{16}{3}st.$$

Two square roots of $(-By)$ are $4st$ and $-4st$, as, for example

(5.15) $$\int_0^1 (4sr)(4rt)\,dr = \frac{16}{3}st\,.$$

Corresponding to these definitions of $(-By)^{1/2}$ are the solutions $x(s) = s^2$ and $x(s) = -s^2$, as may be verified from equations (4.11) and (4.12). As $O\{B\}$ consists of all functions $w(s)$ such that

(5.16) $$\int_0^1 sw(s)\,ds = 0,$$

an example of which is

(5.17) $$w(s) = 3s - 5s^3,$$

all functions

(5.18) $$x_\lambda(s) = s^2 + \lambda(3s - 5s^3),$$

$-\infty < \lambda < +\infty$, are solutions of equation (5.13) by Theorem (3.D).

Example (5.B). The equation

$$(5.19)\quad 81\int_0^1\int_0^1 stu\, x(u)x(t)\,du\,dt + x(s) + s(\ln s - 1) = 0$$

is of second kind, as if for some $z \varepsilon X$,

$$(5.20)\quad 162\int_0^1 stu\, z(u)\,du + \delta(s,t) = 0,$$

operation on $x(s) = -1$ yields

$$(5.21)\quad 1 = -162\int_0^1 s\, z(u)\,du,$$

which cannot be satisfied by any $z \varepsilon X$ as the function on the right side of equation (5.21) is zero for $s = 0$. As Theorem (4.D) applies to equation (5.19), and

$$(5.22)\quad I - 4By = \delta(s,t) + 144st,$$

consider $(I-4By)^{1/2} = \delta(s,t) + 18st$ and $(I-4By)^{1/2} = -\delta(s,t) - 18st$. By Theorem (4.F), one of these definitions of $(I-4By)^{1/2}$ must be rejected. From equation (4.15) and the consideration of equations (5.20) and (5.21), only $(I-4By)^{1/2} = \delta(s,t) + 18st$ is possible. Corresponding to this value of $(I-4By)^{1/2}$, a solution of equation (5.19) is

$$(5.23)\quad x(s) = -s \ln s,$$

as may be verified from equations (4.15) and (4.16).

CHAPTER VI

AN EQUATION OF CHANDRASEKHAR

An important example of a quadratic integral equation is the one derived by Chandrasekhar in his study of radiative transfer (3, pp. 87–126). Here it is required to find x(s) such that

$$(6.01)\qquad x(s) = 1 + \tfrac{1}{2}\pi_0 x(s)\int_0^1 \frac{s}{s+t}x(t)\,dt,$$

where π_0 is a constant called the albedo, and $0 \leq \pi_0 \leq 1$. The attempt to solve this equation was one of the motivating factors in the present study of the theory of the quadratic equation in Banach space.

Equation (6.01) may be put into the form (5.08) with B(s,t,u) satisfying equation (5.06) by setting $\lambda = \pi_0/4$,

$$(6.02)\qquad B(s,t,u) = -\lambda\{\frac{s}{s+u}\delta(s,t) + \frac{s}{s+t}\delta(s,u)\},$$

$$(6.03)\qquad L(s,t) = \delta(s,t),$$

and

$$(6.04)\qquad y(s) = -1.$$

As it stands, equation (6.01) is of second kind if it is not of first kind. If for some $z \varepsilon X$, $I + 2Bx = 0$, then

(6.05) $$\{1 - 2\lambda\int_0^1 \frac{s}{s+u} z(u)du\}\delta(s,t) = 2\lambda z(s)\frac{s}{s+t}.$$

Operation on $\bar{x}(s) = 1$ yields

(6.06) $$1 = 2\lambda\{\int_0^1 \frac{s}{s+u} z(u)du + sz(s)\ln(\frac{s+1}{s})\},$$

which cannot be satisfied by any $z\varepsilon X$, as the function on the right side of equation (6.06) is zero for $s = 0$.

For $\xi = 2\lambda = \pi_0/2$ and all $x\varepsilon X$,

(6.07) $$I + 2Bx = \{1 - \xi\int_0^1 \frac{s}{s+u} x(u)du\}\delta(s,t) - \xi x(s)\frac{s}{s+t}.$$

Also,

(6.08) $$I - 4By = \{1 - 2\xi s \ln(\frac{s+1}{s})\}\delta(s,t) - 2\xi\frac{s}{s+t},$$

and for

(6.09) $$K(s,t) = \{s\ln(\frac{s+1}{s})\}\delta(s,t) + \frac{s}{s+t},$$

(6.10) $$I - 4By = \delta(s,t) - 2\xi K(s,t).$$

From equation (5.03), $\|K\| \leq 2\ln 2$, so that for $\xi < \frac{1}{4\ln 2}$, $(I - 4By)^{1/2}$ may be expanded in terms of its Fréchet derivatives:

(6.11) $$(I - 4By)^{1/2} = I - \xi K - \frac{\xi^2}{2}K^2 - \frac{\xi^3}{2}K^3 - \ldots$$

(6, pp. 85–89). By Theorem (4.D), equation (6.01) has a solution $x \epsilon F\{B\}$ for this definition of $(I-4By)^{1/2}$ only if

$$(6.12)\qquad \{\int_0^1 \frac{s}{s+u}x(u)du\}\delta(s,t) + x(s)\frac{s}{s+t} = \{s \ln (\frac{s+1}{s})\}\delta(s,t) +$$

$$+ \frac{s}{s+t} + \frac{\xi}{2}\Bigg[\{s \ln (\frac{s+1}{s})\}^2\delta(s,t) + \frac{s^2}{s+t}\ln (\frac{s+1}{s}) +$$

$$+ \frac{st}{s+t}\ln (\frac{t+1}{t}) + \int_0^1 \frac{sr}{(s+r)(r+t)}dr\Bigg] + \ldots .$$

Hence, operating on $\bar{x}(s) = 1$, $x(s)$ satisfies (6.12) only if

$$(6.13)\qquad \{s \ln (\frac{s+1}{s})\}x(s) + \int_0^1 \frac{s}{s+u}x(u)du = s \ln (\frac{s+1}{s})^2 +$$

$$+ \frac{\xi}{2}\Bigg[2\{s \ln (\frac{s+1}{s})\}^2 + 2\int_0^1 \frac{st}{s+t}\ln (\frac{t+1}{t})dt\Bigg] + \ldots ,$$

or

$$(6.14)\qquad Kx = \{K + \frac{\xi}{2}K^2 + \frac{\xi^3}{2}K^3 + \ldots\}(1),$$

a linear integral equation. As the function on the right of (6.14) is analytic in ξ, assume that

$$(6.15)\qquad x(s) = x_0(s) + \xi x_1(s) + \xi^2 x_2(s) + \ldots .$$

This gives for $x_0(s)$, $x_1(s)$, ... a sequence of linear integral equations

(6.16)
$$\begin{aligned} Kx_0 &= K(1) \\ Kx_1 &= \tfrac{1}{2}K^2(1) \\ Kx_2 &= \tfrac{1}{2}K^3(1) \\ \cdots \quad & \cdots \quad \cdots \\ Kx_n &= \frac{1\cdot3\cdot5\cdots(2n-1)}{(n+1)!}K^{n+1}(1) \\ Kx_{n+1} &= \frac{1\cdot3\cdot5\cdots(2n+1)}{(n+2)!}K^{n+2}(1) \\ \cdots \quad & \cdots \quad \cdots \quad \cdots \quad \cdots \end{aligned}$$

These equations have the obvious solutions

(6.17)
$$\begin{aligned} x_0(s) &= 1 \\ x_1(s) &= \tfrac{1}{2}K(1) = s \ln\left(\frac{s+1}{s}\right) \\ \cdots \quad & \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_n(s) &= \frac{1\cdot3\cdots(2n-1)}{(n+1)!}K^{n+1}(1) \\ &= \frac{2n-1}{n+1}Kx_{n-1} \\ \cdots \quad & \cdots \quad \cdots \quad \cdots \quad \cdots \end{aligned}$$

As thus

(6.18)
$$\|x_n\|/\|x_{n-1}\| \leq 2\frac{2n-1}{n+1}\ln 2,$$

the uniform convergence of the series (6.15) is assured for $\xi < \frac{1}{4 \ln 2}$. It is not difficult to show that x(s) as given by (6.15) and (6.17) satisfies equation (6.01).

Set

(6.19)
$$J(s,t) = \frac{s}{s+t} = K(s,t) - \{s \ln\left(\frac{s+1}{s}\right)\}\delta(s,t).$$

Then

$$(6.20)\qquad \xi Jx = \xi K(1) + \tfrac{1}{2}\xi^2K^2(1) + \tfrac{1}{2}\xi^3K^3(1) + \ldots - \\ - \xi s \ln\left(\frac{s+1}{s}\right)\{1 + \tfrac{\xi}{2}K(1) + \tfrac{\xi^2}{2}K^2(1) + \ldots\},$$

and

$$(6.21)\qquad x(\xi Jx) = \xi\{K(1) - s\ln\left(\frac{s+1}{s}\right)\} + \xi^2\{\tfrac{1}{2}K^2(1) + \\ + \tfrac{1}{2}(K(1))^2 - K(1)s\ln\left(\frac{s+1}{s}\right)\} + \ldots \\ = \tfrac{1}{2}\xi K(1) + \tfrac{1}{2}\xi^2K^2(1) + \ldots ,$$

so

$$(6.22)\qquad x(\xi Jx) + 1 = 1 + \tfrac{\xi}{2}K(1) + \tfrac{\xi^2}{2}K^2(1) + \ldots = x;$$

that is, $x(s)$ satisfies equation (6.01) formally for all values of $\xi = \pi_0/2$. It follows that $x(s)$ as defined by (6.15) and (6.17) is a solution of equation (6.01) for $0 \leq \pi_0 < \frac{1}{2\ln 2}$. On the basis of (6.21), it is to be expected that a continuation of this solution to $\frac{1}{2\ln 2} \leq \pi_0 \leq 1$ is possible. (It is known that $\pi_0 = 1$ is a limiting case (3, p. 107).) Such an extension would go hand in hand with a continuation of the expansion (6.11) of $(I - 4By)^{1/2}$.

Numerical evaluation of the functions $x_1(s)$, $x_2(s)$, ... may be carried out readily with the aid of some type of high-speed digital computer. In this connection it should be observed that $x_2(s)$, $x_3(s)$, ... are all defined in terms of integral transforms of

(6.23) $$x_1(s) = s \ln\left(\frac{s+1}{s}\right),$$

which has an infinite derivative at s = 0; a suitable rule of numerical integration must take this into account. The numerical solution of equation (6.01) presented by Chandrasekhar (3, pp. 123–126) was obtained by the use of a polynomial rule of integration. Although the evaluation of x(s) was carried out by Chandrasekhar along different lines than the method presented here, the above considerations raise some doubt concerning the accuracy of the tabulated values (3, p. 125).

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