

AN ABSTRACT OF THE THESIS OF

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The degenerate nature of the metric on null hypersurfaces creates many difficulties when attempting to define a covariant derivative on null submanifolds. This dissertation investigates these challenges and provides a technique for defining a connection on null hypersurfaces in some cases. Recent approaches using decomposition to define a covariant derivative on null hypersurfaces are investigated, with examples demonstrating the limitations of the methods. Motivated by Geroch's work on asymptotically flat spacetimes, conformal transformations are used to construct a covariant derivative on null hypersurfaces. In addition, a condition on the Ricci tensor is given to determine when this construction can be used. All of the results are motivated through a sequence of examples of null surfaces on which the covariant derivative is defined. Finally, a covariant derivative operator is given for the class of spherically symmetric hypersurfaces.

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Covariant Derivatives on Null Submanifolds

by

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Don Hicketier, Author

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COVARIANT DERIVATIVES ON NULL SUBMANIFOLDS

1 INTRODUCTION

The covariant derivative, ∇ , is one of the primary tools used to describe the geometric structure on a manifold. Separate points on a manifold have different tangent spaces and the covariant derivative defines a relationship between the tangent spaces. For this reason, the covariant derivative is often called the *connection* on the manifold. The covariant derivative determines the *Riemann* and *Ricci curvature* tensors as well as the *scalar curvature*, all of which describe geometric properties of the manifold.

A non-degenerate metric uniquely determines a metric-compatible, torsion-free connection on the manifold, called the Levi-Civita connection; the corresponding connection coefficients, called *Christoffel symbols*, depend only on the metric. For a non-degenerate submanifold, it is straightforward to obtain a Levi-Civita connection from the metric on the full manifold. However, if the metric on the submanifold is degenerate, the Christoffel symbols can not be computed, so an alternate strategy is needed. The remainder of this dissertation discusses and extends previous attempts to produce connections in the degenerate case.

1.1 Connections on Null Submanifolds

Degenerate metrics often occur when considering submanifolds defined with a Lorentzian metric. A Riemannian metric is positive definite, i.e., has signature $++ \dots +$, while Lorentzian metrics are not positive definite. For example, the metric of a spacetime in

relativity has Lorentz signature, that is, $- + + +$. A manifold defined with a Lorentzian metric is called a Lorentzian manifold. Vectors on a Lorentzian manifold are classified as *spacelike*, $v \cdot v > 0$, *timelike*, $v \cdot v < 0$, and *lightlike*, $v \cdot v = 0$. If there is a lightlike vector n that is also orthogonal to the submanifold, then the submanifold is classified as *lightlike* or *null*. Null submanifolds have degenerate metrics, thus introducing great difficulties in defining a covariant derivative.

The need for a well-defined covariant derivative on a null submanifold arises in the study of asymptotically flat spacetimes. The boundary of an asymptotically flat spacetime can be studied according to several different structures corresponding to approaching infinity along spacelike, timelike or null directions. In Geroch [1], both the spatial and lightlike cases are addressed. Notation, structure and methods are given to define these infinite boundaries as submanifolds in Lorentzian space. Moving in a lightlike direction results in a null submanifold. According to Geroch:

In the null case, one has no unique derivative operator, and so one works more with Lie and exterior derivatives, and with other differential concomitants. As a general rule, it is considerably more difficult in the null case to write down formulae which say what one wants to say.

Geroch's discussions of null boundaries of asymptotically flat spacetimes motivated the techniques used here to find a preferred derivative operator on null submanifolds and will be summarized in Chapter 4.

1.2 Statement of the Problem

Given a null submanifold, is it possible to define a preferred torsion-free, metric-compatible covariant derivative? Is it possible to determine, a priori, when such a connection may be found? Attempts to define a covariant derivative on the null boundary of

an asymptotically flat spacetime are the primary motivations for this dissertation. Two results will be given to address these questions. Chapter 5 provides a construction, based on the Lie derivative, to define the preferred covariant derivative on some null submanifolds. The second result, given in Chapter 6, uses the Ricci tensor to determine when the construction in Chapter 5 is possible.

1.3 Standard Methods to Define a Covariant Derivative

The results of this dissertation are motivated by numerous examples using standard techniques to define a connection. Chapter 2 demonstrates the use of Gauss decomposition to induce the connection on a non-degenerate submanifold. Using an analogous decomposition, the connection on a null submanifold is addressed, summarizing work by Duggal and Bejancu [2]. The chapter concludes with a simple example where the Duggal method fails, thus motivating the need for an alternate method to construct a covariant derivative.

Chapter 3 provides another series of examples that introduce the method of the pullback to define the covariant derivative on null surfaces. Through the examples, limitations of the pullback method will be demonstrated. Motivated by work of Geroch [1] on the asymptotic structure of spacetime, conformal transformations will be investigated in order to define a connection on the null surfaces. Chapter 4 will give a more detailed summary of the Geroch approach.

Again based on the work of Geroch, Chapter 5 establishes an existence condition, using Lie derivatives of the degenerate metric, to determine when the pullback method can be used or whether an additional conformal transformation may be necessary to determine the connection. In the case where the conformal transformation is needed, an equation for the conformal factor is given. The conformal transformation is applied to

the original manifold, the connection coefficients are then recalculated and pulled back to the submanifold. This process requires one to work with the metric on the degenerate submanifold for the Lie derivative, compute the conformal factor, carry out the conformal transformation on the original manifold itself, and finally compute and pullback the connection coefficients. This is a tedious process that requires one to obtain information from both the null submanifold and the original manifold in which the null surface is defined.

Chapter 6 provides a simpler method applied on the original manifold to determine when and if the conformal transformation method is possible. This check, involving a limit of the difference of Ricci tensors on the original manifold, simplifies the previous results by working only on the original manifold and a conformal transformation of the manifold. This result eliminates the need to bounce back and forth between the two metric structures. Other than needing to know the metric on the submanifold, all calculations are done in the 4-dimensional space of the overlying manifold.

The results of the previous two chapters are tested in Chapter 7 on two specific spacetimes used regularly in relativity. In particular, the horizon of the Schwarzschild geometry and the class of all spherically symmetric spacetimes are investigated. In both cases, the existence of a well-defined covariant derivative is shown. Lastly, Chapter 8 provides a summary of the results, with areas of possible future research included.

2 CONNECTION ON SUBMANIFOLDS, TRADITIONAL APPROACHES

For a Riemannian manifold M , let $\Gamma(TM) = C^\infty(M, TM)$ denote the set of smooth vector fields on M . The covariant derivative ∇ on M is an affine connection from $\Gamma \times \Gamma \rightarrow \Gamma$ such that

1. $\nabla_{\alpha X + \beta Y} Z = \alpha \nabla_X Z + \beta \nabla_Y Z$,
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,
3. $\nabla_X (\alpha Y) = (X\alpha)Y + \alpha \nabla_X Y$

where X, Y, Z are vector fields in Γ with α and β functions on M . These three conditions do not determine the connection uniquely. The Levi-Civita connection is the unique affine connection satisfying the additional conditions

4. ∇ is torsion free: $[X, Y] = \nabla_X Y - \nabla_Y X$,
5. ∇ is metric compatible: $Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.

Given coordinates (x^1, x^2, \dots, x^n) , the connection coefficients can be defined by

$$\nabla_{\partial/\partial x^j} \partial/\partial x^i = \Gamma^k_{ij} \partial/\partial x^k \quad (2.1)$$

The coefficients for the Levi-Civita connection are given by the Christoffel symbols,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kh} \left(\frac{\partial g_{ih}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h} \right). \quad (2.2)$$

If (Σ, q) is a submanifold of (M, g) given by $\varphi : \Sigma \rightarrow M$, and if $q = \varphi^*g$ is a nondegenerate metric on Σ , then a connection ∇ on M induces a natural connection D on Σ . A traditional approach to defining this connection is to split TM into the direct sum

$$TM = T\Sigma \oplus (T\Sigma)^\perp, \quad (2.3)$$

where $(T\Sigma)^\perp$ is the orthogonal complement of $T\Sigma$ in TM .

For $X, Y \in \Gamma(TM)$, $\nabla_X Y$ can be separated on Σ into tangential and orthogonal components of TM which define the induced connection, $D_X Y$, and the second fundamental form, $II(X, Y)$. Explicitly, we have

$$D_X Y = (\nabla_X Y)^\parallel \quad (2.4)$$

$$II(X, Y) = (\nabla_X Y)^\perp = \nabla_X Y - D_X Y. \quad (2.5)$$

If ∇ is the Levi-Civita connection on M , then D turns out to be the Levi-Civita connection on Σ . The decomposition

$$\nabla_X Y = D_X Y + II(X, Y) \quad (2.6)$$

is called Gauss' formula [3].

2.1 An example: $ds^2 = -dt^2 + q_{ij} dx^i dx^j$

To demonstrate Gauss' formula, consider the line element

$$ds^2 = -dt^2 + q_{ij} dx^i dx^j \quad (2.7)$$

with corresponding metric

$$g = \begin{bmatrix} -1 & 0 \\ 0 & q_{ij} \end{bmatrix} \quad (2.8)$$

where all components are functions of $t = x^0, x^1, x^2, x^3$.

Let $\Sigma = \{t = \text{constant}\}$ with induced line element

$$ds^2 = q_{ij} dx^i dx^j \quad (2.9)$$

where all components are now functions of x^1, x^2, x^3 . The Levi-Civita connection ∇ on

M is given by

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Gamma^0_{ij} \frac{\partial}{\partial x^0} + \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad (2.10)$$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^0} = \Gamma^0_{0j} \frac{\partial}{\partial x^0} + \Gamma^k_{0j} \frac{\partial}{\partial x^k} \quad (2.11)$$

$$\nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^i} = \Gamma^0_{i0} \frac{\partial}{\partial x^0} + \Gamma^k_{i0} \frac{\partial}{\partial x^k} \quad (2.12)$$

$$\nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = \Gamma^0_{00} \frac{\partial}{\partial x^0} + \Gamma^k_{00} \frac{\partial}{\partial x^k} \quad (2.13)$$

where the Christoffel symbols are defined by equation (2.2).

Since $0 = \nabla_V \left(\frac{\partial}{\partial x^0} \cdot \frac{\partial}{\partial x^0} \right) = 2 \left(\nabla_V \frac{\partial}{\partial x^0} \cdot \frac{\partial}{\partial x^0} \right)$, we have

$$\nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} \cdot \frac{\partial}{\partial x^0} = -\Gamma^0_{00} = 0, \quad (2.14)$$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^0} \cdot \frac{\partial}{\partial x^0} = -\Gamma^0_{0j} = -\Gamma^0_{j0} = 0. \quad (2.15)$$

Using $\nabla_V \left(\frac{\partial}{\partial x^0} \cdot \frac{\partial}{\partial x^i} \right) = \nabla_V \frac{\partial}{\partial x^0} \cdot \frac{\partial}{\partial x^i} + \nabla_V \frac{\partial}{\partial x^i} \cdot \frac{\partial}{\partial x^0} = 0$ with $V = \frac{\partial}{\partial x^0}$ yields

$$\nabla_{\frac{\partial}{\partial x^0}} \left(\frac{\partial}{\partial x^0} \cdot \frac{\partial}{\partial x^i} \right) = -\Gamma^0_{i0} + q_{ik} \Gamma^k_{00} = 0, \quad (2.16)$$

which implies

$$\Gamma^k_{00} = 0, \quad (2.17)$$

where the last equation follows from (2.15) when q is nondegenerate. Thus, equations (2.10)–(2.13) simplify to

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} + \Gamma^0_{ij} \frac{\partial}{\partial x^0} \quad (2.18)$$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^0} = \Gamma^k_{0j} \frac{\partial}{\partial x^k} \quad (2.19)$$

$$\nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^i} = \Gamma^k_{i0} \frac{\partial}{\partial x^k} \quad (2.20)$$

$$\nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = 0 \quad (2.21)$$

Comparing the first term of (2.18) with (2.6) allows us to read off the coefficients for the induced connection D on Σ . Furthermore, due to the assumed form (2.8) of the

metric g on M , these connection coefficients are

$$\Gamma^k{}_{ij} = \frac{1}{2}q^{kh} \left(\frac{\partial q_{ih}}{\partial x^j} + \frac{\partial q_{hj}}{\partial x^i} - \frac{\partial q_{ij}}{\partial x^h} \right). \quad (2.22)$$

Comparing with (2.1), we see that D is the Levi-Civita connection on Σ ,

$$D_X Y = X^i \Gamma^k{}_{ij} Y^j \frac{\partial}{\partial x^k}. \quad (2.23)$$

The second term of (2.18) now gives an expression for the second fundamental form,

$$II(X, Y) = X^i \Gamma^0{}_{ij} Y^j \frac{\partial}{\partial x^0}. \quad (2.24)$$

A closer look at $\Gamma^0{}_{ij}$ will produce a more explicit expression of II for this example.

Returning to M ,

$$\Gamma^0{}_{ij} = \frac{1}{2}g^{0h} \left(\frac{\partial}{\partial x^j} g_{ih} + \frac{\partial}{\partial x^i} g_{hj} - \frac{\partial}{\partial x^h} g_{ij} \right). \quad (2.25)$$

Again using the assumed form (2.8) of g , (2.25) simplifies to

$$\Gamma^0{}_{ij} = \frac{1}{2} \frac{\partial q_{ij}}{\partial x^0}. \quad (2.26)$$

Finally notice (2.24) is defined only in terms of $\partial/\partial x^0 \in \Gamma((T\Sigma)^\perp)$. With $x^0 = t$ and letting $n = \partial/\partial t$, equation (2.18) produces the decomposition

$$\nabla_X Y = D_X Y + \frac{1}{2} \left(\frac{dq}{dt}(X, Y) \right) n. \quad (2.27)$$

Thus, the second fundamental form can be expressed as a multiple of the unit normal, n ,

$$II(X, Y) = B(X, Y) n \quad (2.28)$$

and B_{ij} is given by (2.26).

2.2 Duggal Lightlike Hypersurfaces, Definitions and Notation

Difficulties arise when the metric q , on Σ , is degenerate. In this case, the Christoffel symbols are not even defined on Σ . Furthermore, if Σ is lightlike, TM cannot be decomposed into the direct sum of $T\Sigma$ and $(T\Sigma)^\perp$, since there are vectors in $T\Sigma$ that are also in

$(T\Sigma)^\perp$, as well as vectors that are in neither space. Despite these difficulties, Duggal [2] introduces a decomposition that produces equations similar to the Gauss formula (2.6) as we now describe.

Given a lightlike submanifold Σ of M with tangent space $T\Sigma$, the goal is to create a decomposition of TM by producing a vector bundle similar to $(T\Sigma)^\perp$. Choose a *screen manifold* $Scr(T\Sigma) \subset T\Sigma$ such that

$$T\Sigma = Scr(T\Sigma) \oplus T\Sigma^\perp, \quad (2.29)$$

where $T\Sigma^\perp = \{V \in \Gamma(T\Sigma) : V \cdot W = 0 \forall W \in \Gamma(T\Sigma)\}$ is the orthogonal complement of $T\Sigma$ in TM .

Given a screen manifold, $Scr(T\Sigma)$, Duggal proves the existence of a unique complementary vector bundle, $tr(T\Sigma)$, to $T\Sigma$, called the *lightlike transversal vector bundle* of Σ with respect to $Scr(T\Sigma)$.

Theorem 2.1 (Duggal). *Let $(\Sigma, q, Scr(T\Sigma))$ be a lightlike hypersurface of a semi-Riemannian manifold (M, g) . Then there exists a unique vector bundle $tr(T\Sigma) \subset TM$, of rank 1 over Σ , such that for any nonzero $\xi \in \Gamma(T\Sigma^\perp)$ there exists a unique $N \in \Gamma(tr(T\Sigma))$ such that*

$$N \cdot \xi = -1, \quad N \cdot N = 0, \quad N \cdot W = 0 \forall W \in Scr(T\Sigma). \quad (2.30)$$

Thus, $tr(T\Sigma) \perp Scr(T\Sigma)$ and since $tr(T\Sigma)$ is 1-dimensional, $\Gamma(tr(T\Sigma)) = Span(N)$. By construction we have $tr(T\Sigma) \cap T\Sigma = \{0\}$ and have decomposed TM to

$$TM = Scr(T\Sigma) \oplus (T\Sigma^\perp \oplus tr(T\Sigma)) = T\Sigma \oplus tr(T\Sigma) \quad (2.31)$$

where TM is restricted to Σ .

We can use (2.31) to decompose the connection ∇ on M as follows. Let $X, Y \in \Gamma(T\Sigma)$ and $V \in \Gamma(tr(\Sigma))$. Then, since $tr(T\Sigma)$ has *rank 1*, we can write

$$\nabla_X Y = D_X Y + B(X, Y) N \quad (2.32)$$

$$\nabla_X V = -A_N X + \tau(X) N, \quad (2.33)$$

where $D_X Y, A_N X \in \Gamma(T\Sigma)$. Equation (2.32) can be thought of as the Gauss formula for the lightlike hypersurface and (2.33) as the lightlike Weingarten formula. Under this decomposition, $D_X Y$ is a connection on Σ , but, as discussed in Duggal [2], this connection is not, in general, metric-compatible.

2.2.1 An example with null coordinates

To investigate the Duggal decomposition, consider the line element

$$ds^2 = -2du dv + q_{ij} dx^i dx^j \quad (2.34)$$

and metric

$$g = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & q_{ij} \end{bmatrix} \quad (2.35)$$

with $i, j = 1, 2$. Let ∇ be the Levi-Civita connection on (M, g) . The lightlike submanifold $\Sigma = \{u = 0\}$ has degenerate metric

$$q = \begin{bmatrix} 0 & 0 \\ 0 & q_{ij} \end{bmatrix}. \quad (2.36)$$

Choose a screen $Scr(T\Sigma) = Span(\{X_1, X_2\})$ by setting

$$X_k = \frac{\partial}{\partial x^k} + \alpha_k \xi \quad (2.37)$$

where $\xi = \eta \frac{\partial}{\partial v} \in \Gamma(T\Sigma^\perp)$ and α_k is a function of v, x^1 and x^2 . All possible screens can be obtained by choosing different α_k . For $N \in \Gamma(tr(T\Sigma))$ satisfying Theorem 2.1, the vectors $\{\xi, X_1, X_2, N\}$ form a basis for $\Gamma(TM)$. Using this basis, the covariant derivatives

take the form

$$\nabla_{X_j} X_i = \gamma^0_{ij} \xi + \gamma^k_{ij} X_k + B_{ij} N \quad (2.38)$$

$$\nabla_{X_j} \xi = \gamma^0_{0j} \xi + \gamma^k_{0j} X_k + B_{0j} N \quad (2.39)$$

$$\nabla_{\xi} X_i = \gamma^0_{i0} \xi + \gamma^k_{i0} X_k + B_{i0} N \quad (2.40)$$

$$\nabla_{\xi} \xi = \gamma^0_{00} \xi + \gamma^k_{00} X_k + B_{00} N \quad (2.41)$$

and

$$\nabla_{X_j} N = -A^0_j \xi - A^k_j X_k + \tau_j N \quad (2.42)$$

$$\nabla_{\xi} N = -A^0_0 \xi - A^k_0 X_k + \tau_0 N, \quad (2.43)$$

with 0 used for the v index. Equations (2.38)–(2.43) give the Gauss-Weingarten decomposition of ∇ on M ; it remains to find the coefficients explicitly.

The properties of the basis vectors $\{\xi, X_1, X_2, N\}$ are now used to find equations for the connection coefficients and thus determine the decomposition of ∇ as given by equations (2.32) and (2.33). Since the metric q is degenerate on Σ , the γ 's in (2.38)–(2.41) cannot, a priori, be defined by equation (2.22). As will be shown, only γ^k_{ij} follow the Christoffel symbol definition given in (2.22).

2.2.2 Derivation of coefficients for ∇

Recall that for $\xi \in \Gamma(T\Sigma^\perp)$, $\xi \cdot \xi = 0$, $\xi \cdot X_i = 0$ and by (2.30), $\xi \cdot N = -1$ with $N \cdot X_i = 0$ and $N \cdot N = 0$. For any vector $V \in \Gamma(TM)$ we have $0 = \nabla_V(\xi \cdot \xi) = 2(\nabla_V \xi \cdot \xi)$. Letting $V = X_j$ and taking the inner product with equation (2.39) gives

$$\nabla_{X_j} \xi \cdot \xi = -B_{0j} = 0. \quad (2.44)$$

Repeating with $V = \xi$ in (2.41) yields

$$B_{00} = 0. \quad (2.45)$$

Carrying out a similar argument with $\nabla_V(N \cdot N) = 2 (\nabla_V N \cdot N) = 0$ and letting $V = X_j$ in (2.42) yields

$$A^0_j = 0 \quad (2.46)$$

and $V = \xi$ in (2.43) gives

$$A^0_0 = 0. \quad (2.47)$$

The metric compatibility condition 5 for the Levi-Civita connection gives $0 = \nabla_V(\xi \cdot X_i) = \nabla_V \xi \cdot X_i + \xi \cdot \nabla_V X_i$. Setting $V = \xi$ and using (2.41) and (2.40) gives

$$B_{i0} = g_{ik} \gamma^k_{00}. \quad (2.48)$$

Letting $V = X_j$ and using (2.39) and (2.38) gives $g_{ik} \gamma^k_{0j} + -B_{ij} = 0$, so that

$$B_{ij} = g_{ik} \gamma^k_{0j}. \quad (2.49)$$

It is clear from (2.37) that the commutator of X_k with any coordinate vector field will be parallel to ξ . Thus, both $[\xi, X_i]$ and $[X_i, X_j]$ are parallel to ξ . Subtracting (2.39) from (2.40) and using the torsion-free condition 4, $[\xi, X_i] = \nabla_\xi X_i - \nabla_{X_i} \xi$, simplifies to

$$[\xi, X_i] = (\gamma^0_{i0} - \gamma^0_{0i}) \xi + (\gamma^k_{i0} - \gamma^k_{0i}) X_k + (B_{i0} - B_{0i}) N. \quad (2.50)$$

Since $[\xi, X_i]$ is parallel to ξ , the last two terms of (2.50) must be zero, giving

$$\gamma^k_{i0} = \gamma^k_{0i} \quad (2.51)$$

$$B_{i0} = B_{0i}. \quad (2.52)$$

The nondegeneracy of g along with equations (2.44) and (2.48) implies

$$B_{i0} = B_{0i} = 0 \quad (2.53)$$

$$\gamma^k_{00} = 0. \quad (2.54)$$

Substituting the results from (2.45)–(2.47), (2.53) and (2.54), the Gauss and Weingarten formulas can now be simplified to

$$\nabla_{X_j} X_i = \gamma^0_{ij} \xi + \gamma^k_{ij} X_k + B_{ij} N \quad (2.55a)$$

$$\nabla_{X_j} \xi = \gamma^0_{0j} \xi + \gamma^k_{0j} X_k \quad (2.55b)$$

$$\nabla_{\xi} X_i = \gamma^0_{i0} \xi + \gamma^k_{i0} X_k \quad (2.55c)$$

$$\nabla_{\xi} \xi = \gamma^0_{00} \xi \quad (2.55d)$$

and

$$\nabla_{X_j} N = -A^k_j X_k + \tau_j N \quad (2.56a)$$

$$\nabla_{\xi} N = -A^k_0 X_k + \tau_0 N \quad (2.56b)$$

Duggal defines the last term in equation (2.55a) to be the second fundamental form, $II(X, Y) = B(X, Y) N$. Thus, equations (2.55a)–(2.55d) decompose ∇ to a form similar to Gauss' formula, (2.32). Once the coefficients in (2.55a)–(2.55d) are known, a connection D on Σ has been constructed. The properties of the Levi-Civita connection ∇ on Σ are used to find the remaining coefficients.

Using $0 = \nabla_V(\xi \cdot N) = \nabla_V \xi \cdot N + \xi \cdot \nabla_V N$ with $V = X_j$ in equations (2.55b) and (2.56a) yields

$$\gamma^0_{0j} = -\tau_j \quad (2.57)$$

and with $V = \xi$ in (2.55c) and (2.56b) gives

$$\gamma^0_{00} = -\tau_0. \quad (2.58)$$

Now using $0 = \nabla_V(X_i \cdot N) = \nabla_V X_i \cdot N + X_i \cdot \nabla_V N$ with $V = X_j$ in (2.55a) and (2.56a) gives

$$\gamma^0_{ij} = -g_{ik} A^k_j \quad (2.59)$$

and with $V = \xi$ in (2.55c) and (2.56b) yields

$$\gamma^0_{i0} = -g_{ik} A^k_0. \quad (2.60)$$

Switching indices in equation (2.55a) and subtracting, the commutator $[X_j, X_i] = \nabla_{X_j} X_i - \nabla_{X_i} X_j$ simplifies to

$$[X_j, X_i] = (\gamma^0_{ij} - \gamma^0_{ji}) \xi + (\gamma^k_{ij} - \gamma^k_{ji}) X_k + (B_{ij} - B_{ji}) N. \quad (2.61)$$

Using the fact that $[X_j, X_i]$ is parallel to ξ , equation (2.61) gives

$$\gamma^k_{ij} = \gamma^k_{ji} \quad (2.62)$$

$$B_{ij} = B_{ji}, \quad (2.63)$$

establishing symmetry in the lower indices for both γ^k_{ij} and B_{ij} .

The remaining coefficients are found using *Koszul's formula* [5],

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \quad (2.64)$$

This formula is derived through repeated sums and differences using the Levi-Civita torsion-free condition 4 and metric compatibility 5 on the connection. Letting $X = \xi$, $Y = X_j$, and $Z = X_i$ in equation (2.64) gives

$$2g(\nabla_\xi X_j, X_i) = \xi g(X_j, X_i) + X_j g(\xi, X_i) - X_i g(\xi, X_j) + g([\xi, X_j], X_i) - g([\xi, X_i], X_j) - g([X_j, X_i], \xi) \quad (2.65)$$

Using the properties that for $V \in \Gamma(T\Sigma)$ the commutator $[V, X_i]$ is parallel to ξ and that $\xi \cdot \xi = 0$ and $X_i \cdot \xi = 0$ implies the last three terms of (2.65) vanish,

$$g([\xi, X_j], X_i) - g([\xi, X_i], X_j) - g([X_j, X_i], \xi) = 0 \quad (2.66)$$

Now evaluating the remaining terms of (2.65) gives

$$2g_{ik} \gamma^k_{j0} = \xi g_{ji} = \xi g_{ij} \quad (2.67)$$

A simpler form of B_{ij} is found by substituting (2.67) into (2.49) and using the lower index symmetry from (2.51),

$$B_{ij} = g_{ik} \gamma^k{}_{0j} = g_{ik} \gamma^k{}_{j0} \quad (2.68)$$

$$B_{ij} = \frac{1}{2} \xi g_{ij}. \quad (2.69)$$

This equation is very similar to the coefficient for the second fundamental form found in the previous example with nondegenerate metric on Σ .

Finally, Koszul's formula with $X = X_j$, $Y = X_i$, $Z = X_h$ and the symmetry condition (2.62) produces the familiar definition of the Christoffel symbols

$$\gamma^k{}_{ij} = \frac{1}{2} q^{kh} (X_j q_{ih} + X_i q_{hj} - X_h q_{ij}) \quad (2.70)$$

In summary, the nonzero coefficients in (2.55a)–(2.55d) are given by

$$\gamma^k{}_{ij} = \frac{1}{2} q^{kh} (X_j q_{ih} + X_i q_{hj} - X_h q_{ij}) \quad (2.71)$$

$$\gamma^0{}_{ij} = -g_{ik} A^k{}_j \quad (2.72)$$

$$\gamma^0{}_{0j} = -\tau_j \quad (2.73)$$

$$\gamma^k{}_{0j} = \frac{1}{2} g^{ik} \xi g_{ij} \quad (2.74)$$

$$\gamma^0{}_{00} = -\tau_0 \quad (2.75)$$

$$B_{ij} = \frac{1}{2} \xi g_{ij}. \quad (2.76)$$

Using these coefficients, the induced covariant derivative on $T\Sigma$ becomes

$$D_{X_j} X_i = \gamma^0{}_{ij} \xi + \gamma^k{}_{ij} X_k \quad (2.77)$$

$$D_{X_j} \xi = \gamma^0{}_{0j} \xi + \gamma^k{}_{0j} X_k \quad (2.78)$$

$$D_\xi X_i = \gamma^0{}_{i0} \xi + \gamma^k{}_{i0} X_k \quad (2.79)$$

$$D_\xi \xi = \gamma^0{}_{00} \xi \quad (2.80)$$

Duggal goes on to prove that under certain conditions there is a unique induced connection on Σ ,

Theorem 2.2 (Duggal). *Let $(\Sigma, q, \text{Scr}(T\Sigma))$ be a lightlike hypersurface of (Σ, g) . Then the induced connection D is unique, that is, D is independent of $\text{Scr}(T\Sigma)$, if and only if the second fundamental form II vanishes identically on Σ . Furthermore, in this case, D is torsion free and metric compatible.*

For the example in null coordinates, Theorem 2.2 implies that if $B_{ij} \neq 0$, or equivalently, $\xi g_{ij} \neq 0$ for all i, j , then there is a need for a new method to define D . The following examples give one case where the conditions of Theorem 2.2 are met and one where the conditions are not satisfied.

2.2.3 Duggal and the Null Plane

The line element for Minkowski space \mathbb{M}^4 in null rectangular coordinates takes the form

$$ds^2 = -2 du dv + dx^2 + dy^2. \quad (2.81)$$

The metric on \mathbb{M}^4 is

$$g = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.82)$$

The null plane is defined by $\Sigma = \{u = 0\}$ with metric

$$q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.83)$$

Clearly q is degenerate.

For the null plane, the tangent space $T\Sigma$ is spanned by $\left\{ \frac{\partial}{\partial v}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$ with

$T\Sigma^\perp = \text{Span} \left\{ \frac{\partial}{\partial v} \right\}$. The screen manifold is constructed such that $\text{Scr}(T\Sigma) \subset T\Sigma$ and $\text{Scr}(T\Sigma) \oplus T\Sigma^\perp = T\Sigma$. This screen is not unique. In general, a basis for $\text{Scr}(T\Sigma)$ is

$$\left\{ \alpha \frac{\partial}{\partial v} + \frac{\partial}{\partial x}, \beta \frac{\partial}{\partial v} + \frac{\partial}{\partial y} \right\}. \quad (2.84)$$

Let

$$X_1 = \alpha \frac{\partial}{\partial v} + \frac{\partial}{\partial x}, \quad X_2 = \beta \frac{\partial}{\partial v} + \frac{\partial}{\partial y}, \quad \xi = \eta \frac{\partial}{\partial v} \quad (2.85)$$

where $X_1, X_2 \in \Gamma(\text{Scr}(T\Sigma))$ and $\xi \in \Gamma(T\Sigma^\perp)$.

Recall Theorem 2.1: For $\xi \in \Gamma(T\Sigma^\perp)$ there is a unique $N \in \Gamma(\text{tr}(T\Sigma))$ such that

$$N \cdot \xi = -1, \quad N \cdot N = 0, \quad N \cdot W = 0 \forall W \in \Gamma(\text{Scr}(T\Sigma)). \quad (2.86)$$

For the null plane, $\text{tr}(T\Sigma) \subset TM$ has basis

$$\left\{ \frac{\partial}{\partial v}, \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right\}. \quad (2.87)$$

so N must be of the form

$$N = A \frac{\partial}{\partial v} + B \left(\frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right). \quad (2.88)$$

One can now find A and B so that N satisfies the conditions of (2.86).

Using $N \cdot \xi = -1$ gives

$$-\eta B = -1 \quad (2.89)$$

$$B = 1/\eta \quad (2.90)$$

and $N \cdot N = 0$ yields

$$-2AB + B^2\alpha^2 + B^2\beta^2 = 0 \quad (2.91)$$

$$A = \frac{B(\alpha^2 + \beta^2)}{2} = \frac{\alpha^2 + \beta^2}{2\eta}. \quad (2.92)$$

Thus N is of the form

$$N = \frac{1}{\eta} \left(\frac{\partial}{\partial u} + \left(\frac{\alpha^2 + \beta^2}{2} \right) \frac{\partial}{\partial v} + \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) \quad (2.93)$$

It is not difficult to check $X_1 \cdot N = 0$ and $X_2 \cdot N = 0$.

Finding N verifies Theorem 2.1, making way for the decomposition of ∇ on the null plane satisfying (2.32). To check that the induced metric on Σ produces a Levi-Civita connection independent of the screen, as stated in Theorem 2.2, it remains to verify that, for $\xi = \eta \frac{\partial}{\partial v}$, $B_{ij} = \frac{\eta}{2} \frac{\partial g_{ij}}{\partial v}$ vanishes on the null plane. For $u = 0$ on M ,

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.94)$$

Clearly $\frac{\partial g_{ij}}{\partial v} = 0$, so $B_{ij} = 0$. Thus a unique connection is defined using the screen distribution and transversal vector bundle.

2.2.4 Duggal and the Null Cone

The line element for Minkowski space \mathbb{M}^4 in null spherical coordinates takes the form

$$ds^2 = -2 du dv + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (2.95)$$

with $r = (v - u)/\sqrt{2}$. The metric is then given by

$$g = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \quad (2.96)$$

The null cone is defined by $\Sigma = \{u = 0\}$, so $r = v/\sqrt{2}$ and the degenerate metric on the cone is

$$q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{v^2}{2} & 0 \\ 0 & 0 & \frac{v^2}{2} \sin^2 \theta \end{bmatrix} \quad (2.97)$$

Similar to the null plane, the tangent space $T\Sigma$ for the null cone is spanned by $\left\{ \frac{\partial}{\partial v}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right\}$ with $T\Sigma^\perp = \text{Span} \left\{ \frac{\partial}{\partial v} \right\}$. The screen manifold is constructed with basis vectors

$$X_1 = \alpha \frac{\partial}{\partial v} + \frac{\sqrt{2}}{v} \frac{\partial}{\partial \theta}, \quad X_2 = \beta \frac{\partial}{\partial v} + \frac{\sqrt{2}}{v \sin \theta} \frac{\partial}{\partial \phi}. \quad (2.98)$$

For $\xi = \eta \frac{\partial}{\partial v} \in \Gamma(T\Sigma^\perp)$, we will find $N \in \Gamma(\text{tr}(T\Sigma))$. With basis for $\text{tr}(T\Sigma)$

$$\left\{ \frac{\partial}{\partial v}, \frac{\partial}{\partial u} + \frac{\alpha}{r} \frac{\partial}{\partial \theta} + \frac{\beta}{r \sin \theta} \frac{\partial}{\partial \phi} \right\}. \quad (2.99)$$

this amounts to finding A and B for

$$N = A \frac{\partial}{\partial v} + B \left(\frac{\partial}{\partial u} + \frac{\alpha}{r} \frac{\partial}{\partial \theta} + \frac{\beta}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \quad (2.100)$$

that satisfy (2.86). $N \cdot \xi = -1$ again gives

$$B = 1/\eta \quad (2.101)$$

and $N \cdot N = 0$ also yields

$$A = \frac{\alpha^2 + \beta^2}{2\eta}. \quad (2.102)$$

Thus N is

$$N = \frac{1}{\eta} \left(\frac{\partial}{\partial v} + \left(\frac{\alpha^2 + \beta^2}{2} \right) \frac{\partial}{\partial u} + \frac{\alpha}{r} \frac{\partial}{\partial \theta} + \frac{\beta}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \quad (2.103)$$

As with the null plane it is not difficult to check $X_1 \cdot N = 0$ and $X_2 \cdot N = 0$ on Σ .

For the null cone, finding N demonstrates the procedure in Theorem 2.1, but for a Levi-Civita connection, independent of screen, the hypotheses of Theorem 2.2 must be checked, namely $B_{ij} = \frac{1}{2} \xi g_{ij} = 0$. For $\xi = \eta \frac{\partial}{\partial v}$

$$\frac{\partial g}{\partial v} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}r & 0 \\ 0 & 0 & 0 & \sqrt{2}r \sin \theta \end{bmatrix} \quad (2.104)$$

where $r = (v - u)/\sqrt{2}$. On the null cone, $\Sigma = \{u = 0\}$,

$$\frac{\partial g}{\partial v} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & v \sin \theta \end{bmatrix} \quad (2.105)$$

so $B_{ij} \neq 0$. Thus, the method using the screen distribution and transversal vector bundle will not necessarily yield a Levi-Civita connection on the null cone. Alternate methods will be developed in this paper.

3 CONNECTIONS VIA THE PULLBACK

A series of surfaces are examined to motivate the methods used to define a covariant derivative on null surfaces. The plane and cone in Euclidean space will be used to demonstrate how the covariant derivative can be defined directly from the definition involving Christoffel symbols. While the Euclidean examples may not be very exciting, they do demonstrate the need for the other methods in the Minkowski space examples. Progression through the examples will also lay the groundwork for the final results.

3.1 Notation and Definitions

The preceding examples all used standard math tensor notation and component notation when a specific basis was given. There is yet a third notation commonly used by relativists, abstract index notation. This notation allows one to keep track of the tensor index without specifying components. The table below summarizes abstract index and component notation. For the remainder of this dissertation, abstract index notation will be used unless a coordinate basis is specified.

	Abstract Index	Component/Basis
Basis		(x^0, x^1, x^2, x^3)
vector	v^a	$v = v^i \partial/\partial x^i$
1-form	w_a	$w = w_i dx^i$
metric	g_{ab}	$g = [g_{ij}]$
Christoffel symbol	$\Gamma^a{}_{bc}$ (not a tensor)	$\Gamma^i{}_{jk} = \frac{1}{2}g^{im} \left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right)$
vector derivative	$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c$	$\nabla_X v = X^i \left[\frac{\partial v^j}{\partial x^i} + \Gamma^j{}_{ik} v^k \right] \partial/\partial x^j$ $\nabla_{\partial/\partial x^i} \partial/\partial x^j = \Gamma^k{}_{ji} \partial/\partial x^k$
1-form derivative	$\nabla_a w_b = \partial_a w_b - \Gamma^c{}_{ab} w_c$	$\nabla_X w = X^i \left[\frac{\partial w_j}{\partial x^i} - \Gamma^k{}_{ij} w_k \right] dx^j$ $\nabla_{\partial/\partial x^i} dx^j = -\Gamma^j{}_{ki} dx^k$

The calculations required to compute the Christoffel symbols given a metric g_{ab} are very involved and tedious. For all of the examples the computer algebra system Maple and the GRTensor package were used to check the computations of the Christoffel symbols as well as the Riemann and Ricci tensors.

3.2 Euclidean Space

In Euclidean space with rectangular coordinates the line element is given by

$$ds^2 = dz^2 + dx^2 + dy^2.$$

In \mathbb{E}^3 the metric is then

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All of the Christoffel symbols are zero and the covariant derivative is given by $\nabla_b v^a = \partial v^a/\partial x^b$ for vectors and $\nabla_b w_a = \partial w_a/\partial x^b$ for 1-forms.

3.2.1 Euclidean Plane

A plane in \mathbb{E}^3 with $x = z$ has line element

$$ds^2 = 2 dx^2 + dy^2.$$

The plane is a 2-surface with 2×2 metric

$$[q_{ij}] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

For this surface the covariant derivative can be computed directly. All of the Christoffel symbols are zero, and the covariant derivative is given by $D_b v^a = \partial v^a / \partial x^b$ and $D_b w_a = \partial w_a / \partial x^b$.

Since vectors on the plane can be thought of as vectors in \mathbb{E}^3 , the symbols v^a and w_a have been used for both spaces.

3.2.2 Cylindrical Coordinates in \mathbb{E}^3

The line element in cylindrical coordinates is given by

$$ds^2 = dz^2 + dr^2 + r^2 d\theta^2.$$

The metric is

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix}$$

On the cone the nonzero Christoffel symbols are

$$\Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^r_{\theta\theta} = -r. \quad (3.1)$$

Letting $\Theta = \partial / \partial \theta$, the covariant derivative is given by

$$\nabla_\Theta (d\theta) = -\Gamma^\theta_{r\theta} dr = -\frac{1}{r} dr \quad (3.2)$$

$$\nabla_\Theta (dr) = -\Gamma^r_{\theta\theta} d\theta = r d\theta \quad (3.3)$$

All other derivatives will be zero. In particular, the derivative $\nabla_{\Theta}(dz) = 0$ will be used in the next example

3.2.3 Euclidean Cone

In \mathbb{E}^3 the cone, C , is given by $r = z$. The line element is

$$ds^2 = 2 dr^2 + r^2 d\theta^2.$$

The metric is

$$[g_{ij}] = \begin{bmatrix} 2 & 0 \\ 0 & r^2 \end{bmatrix}$$

The nonzero Christoffel symbols are

$$\Gamma^{\theta}_{r\theta} = \frac{1}{r}, \quad \Gamma^r_{\theta\theta} = -\frac{r}{2} \quad (3.4)$$

The covariant derivative can be defined on 1-forms by

$$D_{\Theta}(d\theta) = -\Gamma^{\theta}_{r\theta} dr = -\frac{1}{r} dr \quad (3.5)$$

$$D_{\Theta}(dr) = -\Gamma^r_{\theta\theta} d\theta = \frac{r}{2} d\theta \quad (3.6)$$

One may wish to relate the derivative operator on the cone C to the space in which it is imbedded, \mathbb{E}^3 . Consider $v = r + z$ and $u = r - z$ in \mathbb{E}^3 with differentials $dv = dr + dz$ and $du = dr - dz$. Their respective Θ -derivatives are

$$\nabla_{\Theta}(dv) = \nabla_{\Theta}(dr) + \nabla_{\Theta}(dz) = r d\theta + 0 = r d\theta \quad (3.7)$$

$$\nabla_{\Theta}(du) = \nabla_{\Theta}(dr) - \nabla_{\Theta}(dz) = r d\theta - 0 = r d\theta \quad (3.8)$$

On the cone with $r = z$, let $\underline{v} = 2r$ and $\underline{u} = 0$ where the underline is used to show that \underline{u} and \underline{v} are now on the 2-dimensional cone and not in 3-dimensional Euclidean space. The differentials become $\underline{dv} = 2 dr$ and $\underline{du} = 0$ the Θ -derivatives are

$$D_{\Theta}(\underline{dv}) = D_{\Theta}(2 dr) = 2 \frac{r}{2} d\theta = r d\theta \quad (3.9)$$

$$D_{\Theta}(\underline{du}) = D_{\Theta}(0) = 0 \quad (3.10)$$

Returning to Euclidean 3-space, evaluate the Θ -derivatives at $r = z$ to check if the derivatives agree with those computed on the cone. In this case, let $\underline{\nabla}_{\Theta} dv$ represent the 3-space derivative evaluated at $r = z$ and then expressed in the 2-dimensional space of the cone,

$$\underline{\nabla}_{\Theta} (dv) = r d\theta \quad (3.11)$$

$$\underline{\nabla}_{\Theta} (du) = r d\theta \quad (3.12)$$

The derivatives of dv are the same, $\underline{\nabla}_{\Theta} dv = D_{\Theta} dv$, but the derivatives of du do not agree, $\underline{\nabla}_{\Theta} du \neq D_{\Theta} du$. This example demonstrates the difficulties in obtaining a well-defined derivative on submanifolds based on the space in which they are imbedded.

3.3 Minkowski Space

The next examples will concentrate on surfaces in Minkowski space. More challenges will present themselves when trying to obtain a well-defined covariant derivative on submanifolds. Surfaces in Minkowski space lead to degenerate metrics. Without an invertible metric it is not even possible to compute Christoffel symbols on the surface. Since our definition of covariant derivative is based on Christoffel symbols, it is not possible to directly define the derivative on the surface. Other methods must be investigated.

3.3.1 Rectangular Coordinates in \mathbb{M}^4

The line element in Minkowski space is given in rectangular coordinates by

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2$$

with metric

$$[g_{ij}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Like Euclidean space, all of the Christoffel symbols are zero and the covariant derivative is given by $\nabla_b v^a = \partial v^a / \partial x^b$ and $\nabla_b w_a = \partial w_a / \partial x^b$.

3.3.2 Minkowski Plane

Define a plane in \mathbb{M}^4 by $\Sigma = \{t = z\}$. The line element on this plane in Minkowski space is given in rectangular coordinates by

$$ds^2 = dx^2 + dy^2$$

with metric

$$[q_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly q_{ab} is not invertible. The covariant derivative, D , cannot be defined using Christoffel symbols since the inverse metric is needed to calculate Γ^a_{bc} . The properties of the derivative ∇ on \mathbb{M}^4 will be used to arrive at a definition of the derivative D on Σ . Before attempting this, more formal definitions and notation must be introduced to allow objects to be moved from one space to another.

3.3.3 Pullback

Let Σ be a submanifold of \mathbb{M}^4 with $\varphi : \Sigma \rightarrow \mathbb{M}^4$ an embedding of Σ into \mathbb{M}^4 . For vectors $v^a \in T\Sigma$, $\varphi_* v^a$ is a vector in $T\mathbb{M}^4$. That is, vectors are “pushed” from the tangent space of Σ onto the tangent space of \mathbb{M}^4 . On the other hand, 1-forms w_a defined on \mathbb{M}^4

are “pulled” back to the surface Σ . The *pullback* of w_a onto Σ is given by $\varphi^* w_a$ such that $(\varphi^* w_a)v^b = w_a(\varphi_* v^b)$.

Since φ^* acts on forms, the covariant derivative will be computed on 1-forms in \mathbb{M}^4 and then pulled back to Σ . As with the previous example, D will represent a covariant derivative on Σ and ∇ a covariant derivative on \mathbb{M}^4 . This approach attempts to define a covariant derivative on Σ by

$$D_a(\varphi^* w_b) = \varphi^*(\nabla_a w_b). \quad (3.13)$$

Less formal notation will be adopted to represent the pullback. Instead of φ^* for the pullback, an \leftarrow will be used. The pullback of 1-forms will be written $\varphi^* w_b = \underline{\leftarrow} w_b$. This notation will be less cumbersome and will lend itself to a more visual representation of pulling the object back to the surface Σ . With this new notation the pullback of the covariant derivative is written

$$D_a \underline{\leftarrow} w_b = \underline{\leftarrow} \nabla_a w_b. \quad (3.14)$$

This notation will also eliminate the confusion of where objects live. w_b is a 4-dimensional 1-form defined on \mathbb{M}^4 while $\underline{\leftarrow} w_b$ is the 3-dimensional 1-form pulled back to Σ . This notation will be used throughout the remainder of this document. Since the pullback is not the traditional method used to define D , care must be given to insure the derivative is well-defined.

3.3.4 Minkowski plane revisited

For the plane, the pullback method can be summarized by starting in \mathbb{M}^4 to calculate the Christoffel symbols, define ∇ on \mathbb{M}^4 , and then pull the derivative back to the plane by letting $z = t$. Since all of the Christoffel symbols vanish in \mathbb{M}^4 , the pullback of the

covariant derivative to the plane becomes

$$D_b \underline{w}_a = \underline{\nabla}_b \underline{w}_a = \underline{\partial} \underline{w}_a / \underline{\partial} x^b = \underline{\partial}_b \underline{w}_a. \quad (3.15)$$

In other words, using the pullback definition (3.14) the covariant derivative D on the plane is again given by partial differentiation.

Now to check that this derivative operator is well defined. Consider $v = t + z$ and $u = t - z$ in \mathbb{M}^4 with differentials $dv = dt + dz$ and $du = dt - dz$. Minkowski space with $t = z$ gives derivatives

$$\nabla_X dv|_{t=z} = \nabla_X dz + \nabla_X dz = 2 \partial_X dz \quad (3.16)$$

$$\nabla_X du|_{t=z} = \nabla_X dz - \nabla_X dz = 0 \quad (3.17)$$

Pulling these derivatives back to the plane gives

$$\underline{\nabla}_X \underline{dv} = 2 \underline{\partial}_X \underline{dz} \quad (3.18)$$

$$\underline{\nabla}_X \underline{du} = 0 \quad (3.19)$$

Using (3.15) on the plane with $\underline{dv} = 2 \underline{dz}$ and $\underline{du} = 0$ yields

$$D_X \underline{dv} = 2 \partial_X \underline{dz} \quad (3.20)$$

$$D_X \underline{du} = 0 \quad (3.21)$$

Indeed, both derivatives agree. The pullback method has produced a well-defined derivative operator on the Minkowski plane, a surface with degenerate metric.

3.3.5 Well-Defined Covariant Derivative

Since the pullback is not a traditional method used to define the covariant derivative, we must clarify what will be accepted as “well defined.” Consider a surface defined by $\Sigma = \{u = 0\}$ and any 1-form, w . For an arbitrary function f , the pullback of $w + f du$ onto Σ is $\underline{w + f du} = \underline{w}$ since $\underline{du} = 0$ on Σ . In this case

$$\underline{\nabla}_X (w + f du) = \underline{\nabla}_X w + (\partial_X f|_{u=0}) du|_{u=0} + (f|_{u=0}) \underline{\nabla}_X \underline{du} = \underline{\nabla}_X w \quad (3.22)$$

as long as $\underline{\nabla}_X du = 0$. Dealing with the du -term in pullback derivative certainly is not ideal, but it does provide the necessary condition to define a “well-defined” pullback derivative on Σ . The u -coordinate defines the surface, but is not part of the coordinate representation of Σ . For this reason, as long as the pullback derivative

$$D_X \underline{du} = \underline{\nabla}_X du = 0 \quad (3.23)$$

the derivative D will be considered well defined. On the other hand, if $\underline{\nabla}_X du \neq 0$ the covariant derivative D on Σ will not be well defined.

3.3.6 Minkowski Null Rectangular Coordinates

The choices $u = (t - z)/\sqrt{2}$ and $v = (t + z)/\sqrt{2}$ are called null coordinates in Minkowski space. This coordinate transformation turns out to be commonly used to represent spacetimes in relativity. Now the covariant derivative on \mathbb{M}^4 will be determined using null coordinates. The line element in null coordinates is

$$ds^2 = -2du dv + dx^2 + dy^2$$

with metric

$$[g_{ij}] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All of the Christoffel symbols are zero and the covariant derivative on 1-forms is given by $\nabla_a w_b = \partial_a w_b$. As expected this is exactly the same as the derivative in rectangular coordinates.

3.3.7 Minkowski Plane, Null Coordinates

Null coordinates give a very simple definition of the null plane, $\Sigma = \{u = 0\}$. In null coordinates the line element for the null plane is

$$ds^2 = dx^2 + dy^2$$

and the metric is

$$[q_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This appears to be the same degenerate metric as in rectangular coordinates, but they are different in their coordinates. In the rectangular case the coordinates on the plane are $[z, x, y]$ while in the null case the coordinates are $[v, x, y]$. Pulling ∇ back to the null plane yields the covariant derivative $D_{\underline{a}} \underline{w}_b = \underline{\nabla}_a \underline{w}_b = \underline{\partial}_a \underline{w}_b$. Once again this is the same derivative as computed in rectangular coordinates.

3.3.8 Minkowski Spherical Coordinates in \mathbb{M}^4

The line element in Minkowski space is given in spherical coordinates by

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

with metric

$$[g_{ij}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix}$$

The nonzero Christoffel symbols are

$$\Gamma^{\theta}_{r\theta} = \frac{1}{r}, \quad \Gamma^r_{\theta\theta} = -r \quad (3.24)$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin \theta \cos \theta, \quad \Gamma^r_{\phi\phi} = -r \sin^2 \theta \quad (3.25)$$

$$\Gamma^{\phi}_{r\phi} = \frac{1}{r}, \quad \Gamma^{\phi}_{\theta\phi} = \cot \theta \quad (3.26)$$

The nonzero components of the covariant derivative are

$$\nabla_{\Theta} d\theta = -\Gamma^{\theta}_{r\theta} dr = -\frac{1}{r} dr, \quad \nabla_{\Theta} dr = -\Gamma^r_{\theta\theta} d\theta = r d\theta \quad (3.27)$$

$$\nabla_{\Phi} d\theta = -\Gamma^{\theta}_{\phi\phi} d\phi = \sin\theta \cos\theta d\phi \quad \nabla_{\Phi} dr = -\Gamma^r_{\phi\phi} d\phi = r \sin^2\theta d\phi \quad (3.28)$$

$$\nabla_{\Phi} d\phi = -\Gamma^{\phi}_{r\phi} dr + -\Gamma^{\phi}_{\theta\phi} d\theta = -\frac{1}{r} dr - \cot\theta d\theta \quad (3.29)$$

All of the remaining components of the derivative vanish. In particular, $\nabla_{\Theta} dt = 0$.

3.3.9 Minkowski Cone

Using spherical coordinates in Minkowski space the cone, $\Sigma = \{t = r\}$, has line element

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

with degenerate metric

$$[q_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix}$$

Since q_{ab} is not invertible the Christoffel symbols needed to define D cannot be calculated. This forces a return to \mathbb{M}^4 to get the derivative ∇ and then pull the covariant derivative back to the cone by letting $t = r$. The nonzero components are of D on the cone are given by

$$D_{\Theta} \underline{d\theta} = \underline{\nabla_{\Theta} d\theta} = -\frac{1}{r} \underline{dr}, \quad D_{\Theta} \underline{dr} = \underline{\nabla_{\Theta} dr} = r \underline{d\theta} \quad (3.30)$$

$$D_{\Phi} \underline{d\theta} = \underline{\nabla_{\Phi} d\theta} = \sin\theta \cos\theta \underline{d\phi}, \quad D_{\Phi} \underline{dr} = \underline{\nabla_{\Phi} dr} = r \sin^2\theta \underline{d\phi} \quad (3.31)$$

$$D_{\Phi} \underline{d\phi} = \underline{\nabla_{\Phi} d\phi} = -\frac{1}{r} \underline{dr} - \cot\theta \underline{d\theta} \quad (3.32)$$

In this case there are no problems pulling ∇ back to the cone, but it turns out that this derivative D is not well defined. Recall that a well-defined derivative must satisfy

the condition $D_X \underline{du} = \underline{\nabla_X du} = 0$ on the surface given by $\Sigma = \{u = 0\}$. On the cone $u = t - r$. For $t = r$ the pullback of the 1-forms must be equal, $\underline{dr} = \underline{dt}$. Our condition for a well-defined derivative on Σ then becomes

$$D_X \underline{du} = \underline{\nabla_X du} = \underline{\nabla_X (dr - dt)} = \underline{\nabla_X dr} - \underline{\nabla_X dt} = D_X \underline{dt} - D_X \underline{dr} = 0. \quad (3.33)$$

So it remains to check if the pullback method yields $D_X \underline{dt} = D_X \underline{dr}$. Consider the derivative along θ ,

$$D_\Theta \underline{dr} = \underline{\nabla_\Theta dr} = r \underline{d\theta} \quad (3.34)$$

$$D_\Theta \underline{dt} = \underline{\nabla_\Theta dt} = 0. \quad (3.35)$$

Clearly $D_\Theta \underline{dr} \neq D_\Theta \underline{dt}$ even though $\underline{dr} = \underline{dt}$ on Σ . Thus D is not well defined on Σ . The lesson here is that one must be particularly careful when using the pullback to ensure a well defined derivative D .

3.3.10 Minkowski Null Spherical Coordinates

As with the plane, the null coordinates $u = (t - r)/\sqrt{2}$ and $v = (t + r)/\sqrt{2}$ can be used in spherical coordinates. Now $r = (v - u)/\sqrt{2}$ and $t = (v + u)/\sqrt{2}$. The line element in null spherical coordinates is

$$ds^2 = -2du dv + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 = -du dv + \frac{(v - u)^2}{2} d\theta^2 + \frac{(v - u)^2}{2} \sin^2 \theta d\phi^2$$

with metric

$$[g_{ij}] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(v - u)^2}{2} & 0 \\ 0 & 0 & 0 & \frac{(v - u)^2}{2} \sin^2 \theta \end{bmatrix}$$

For this coordinate system there are many more nonzero Christoffel symbols,

$$\Gamma^{\theta}_{u\theta} = \frac{-1}{v-u} \quad \Gamma^{\theta}_{v\theta} = \frac{1}{v-u} \quad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \quad (3.36)$$

$$\Gamma^{\phi}_{u\phi} = \frac{-1}{v-u} \quad \Gamma^{\phi}_{v\phi} = \frac{1}{v-u} \quad \Gamma^{\phi}_{\theta\phi} = \cot\theta \quad (3.37)$$

$$\Gamma^v_{\theta\theta} = \frac{-(v-u)}{2} \quad \Gamma^v_{\phi\phi} = \frac{-(v-u)}{2} \sin^2\theta \quad (3.38)$$

$$\Gamma^u_{\theta\theta} = \frac{v-u}{2} \quad \Gamma^u_{\phi\phi} = \frac{v-u}{2} \sin^2\theta \quad (3.39)$$

The nonzero components of ∇_X are given below.

$\nabla_X d\theta$ terms:

$$\nabla_{\Theta} d\theta = -\Gamma^{\theta}_{u\theta} du - \Gamma^{\theta}_{v\theta} dv = \frac{1}{v-u} (du - dv) \quad (3.40)$$

$$\nabla_{\Phi} d\theta = -\Gamma^{\theta}_{\phi\phi} d\phi = \sin\theta\cos\theta d\phi \quad (3.41)$$

$\nabla_X d\phi$ terms:

$$\nabla_{\Phi} d\phi = -\Gamma^{\phi}_{u\phi} du - \Gamma^{\phi}_{v\phi} dv = \frac{1}{v-u} (du - dv) \quad (3.42)$$

$$\nabla_{\Theta} d\phi = -\Gamma^{\phi}_{\theta\phi} d\theta = -\cot\theta d\theta \quad (3.43)$$

$\nabla_X dv$ terms:

$$\nabla_{\Theta} dv = -\Gamma^v_{\theta\theta} d\theta = \frac{(v-u)}{2} d\theta \quad (3.44)$$

$$\nabla_{\Phi} dv = -\Gamma^v_{\phi\phi} d\phi = \frac{(v-u)}{2} \sin^2\theta d\phi \quad (3.45)$$

$\nabla_X du$ terms:

$$\nabla_{\Theta} du = -\Gamma^u_{\theta\theta} d\theta = -\frac{v-u}{2} d\theta \quad (3.46)$$

$$\nabla_{\Phi} du = -\Gamma^u_{\phi\phi} d\phi = -\frac{(v-u)}{2} \sin^2\theta d\phi \quad (3.47)$$

It is important to note that $\nabla_X du$ contains nonzero terms other than du . As the null cone example will show, this creates problems when one tries to use the pullback (3.14) to define a derivative.

3.3.11 Minkowski Cone, Null Coordinates

The null cone is defined by $\Sigma = \{u = 0\}$. Now the line element for the null cone is

$$ds^2 = (v^2/2) d\theta^2 + (v^2/2) \sin^2 \theta d\phi^2$$

and the metric is

$$[q_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v^2/2 & 0 \\ 0 & 0 & (v^2/2) \sin^2 \theta \end{bmatrix}$$

Now with this degenerate metric the attempt to pull the derivative back from \mathbb{M}^4 with null coordinates will lead to problems when ∇ is pulled back to the null cone. As a reminder, a well-defined derivative D on the null cone must have $\overleftarrow{\nabla}_X du = 0$. Consider only the nonzero terms from $\nabla_X du = 0$ yields

$$\overleftarrow{\nabla}_\Theta du = -\frac{v-0}{2} \overleftarrow{d}\theta \neq 0 \quad (3.48)$$

$$\overleftarrow{\nabla}_\Phi du = \frac{(v-0)}{2} \sin^2 \theta \overleftarrow{d}\phi \neq 0. \quad (3.49)$$

As with the cone in spherical coordinates, the pullback method does not result in a well defined derivative operator D . This will be a problem for any null surface, $\Sigma = \{u = 0\}$, when there are nonzero components other than du in $\nabla_X du$.

3.3.12 Conformal Minkowski Cone

This example will use a conformal transformation to obtain a well-defined derivative, D , on the Minkowski cone. A conformal transformation on the 4-metric g_{ab} is of the form

$$\bar{g}_{ab} = \omega^2 g_{ab}. \quad (3.50)$$

The transformation is chosen so $\nabla_a du$ has no components other than du itself. When ∇ is pulled back to the cone, D will be a well-defined derivative. Let $\omega = 1/r$ where $r = (v - u)/\sqrt{2}$. The line element becomes

$$ds^2 = \frac{1}{r^2} (-du dv + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) = -\frac{2}{(v-u)^2} du dv + d\theta^2 + \sin^2 \theta d\phi^2$$

with metric

$$[\bar{g}_{ij}] = \begin{bmatrix} 0 & \frac{-1}{(v-u)^2} & 0 & 0 \\ \frac{-1}{(v-u)^2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2 \theta \end{bmatrix}$$

which gives nonzero Christoffel symbols:

$$\bar{\Gamma}^\theta_{\phi\phi} = -\sin \theta \cos \theta \qquad \bar{\Gamma}^\phi_{\theta\phi} = \cot \theta \qquad (3.51)$$

$$\bar{\Gamma}^v_{vv} = \frac{-2}{v-u} \qquad \bar{\Gamma}^u_{uu} = \frac{2}{v-u} \qquad (3.52)$$

The nonzero $\bar{\nabla}_a$ terms are

$$\bar{\nabla}_\phi d\theta = -\bar{\Gamma}^\theta_{\phi\phi} d\phi = \sin \theta \cos \theta d\phi \qquad (3.53)$$

$$\bar{\nabla}_\phi d\phi = -\bar{\Gamma}^\phi_{\theta\phi} d\theta = -\cot \theta d\theta \qquad (3.54)$$

$$\bar{\nabla}_v dv = -\bar{\Gamma}^v_{vv} dv = \frac{2}{v-u} dv \qquad (3.55)$$

$$\bar{\nabla}_u du = -\bar{\Gamma}^u_{uu} du = -\frac{2}{v-u} du \qquad (3.56)$$

Now there are no problems in the pullback to the cone. The first three derivatives pull back to the cone with $u = 0$. With $u = 0$ and $\underline{du} = 0$ the last term will vanish in the pullback,

$$D_u \underline{du} = \bar{\nabla}_u du = \frac{2}{v-0} \underline{du} = 0. \qquad (3.57)$$

The covariant derivative D now becomes

$$D_\phi \underline{d\theta} = \bar{\nabla}_\phi d\theta = \sin \theta \cos \theta \underline{d\phi} \qquad (3.58)$$

$$D_\phi \underline{d\phi} = \bar{\nabla}_\phi d\phi = -\cot \theta \underline{d\theta} \qquad (3.59)$$

$$D_v \underline{dv} = \bar{\nabla}_v dv = \frac{2}{v-0} \underline{dv} = \frac{2}{v} \underline{dv} \qquad (3.60)$$

3.4 Christoffel Symbols and the Pullback of ∇

In summary, we have seen three examples of null surfaces where using the pullback resulted in well-defined covariant derivatives; the plane, the plane in null coordinates and the cone with conformal null coordinates. There were two surfaces that did not yield well-defined derivatives; the cone in spherical and in spherical null coordinates. As defined earlier, D is not well defined when $D_a \underline{du} = \underline{\nabla}_a du \neq 0$. Based on the previous examples this occurs any time there are non- du terms in $\nabla_a du$. Looking more closely at this derivative reveals the Christoffel symbols on \mathbb{M}^4 will determine if the derivative is well defined on Σ :

$$\nabla_a du = \nabla_a \delta_b^u dx^b = \partial_a (\delta_b^u dx^b) - \Gamma^c_{ab} \delta_c^u dx^b = 0 - \Gamma^u_{ab} dx^b. \quad (3.61)$$

So $\nabla_a du$ will have non- du terms anytime $\Gamma^u_{ab} \neq 0$ on \mathbb{M}^4 . There is one exception to this condition. If $\Gamma^u_{au} \neq 0$, the pullback $\Gamma^u_{au}|_{u=0} \underline{du}$ will still vanish. Now, checking if the pullback of ∇ is well-defined amounts to looking for nonzero Christoffel symbols of the form Γ^u_{ab} .

For the cone in spherical coordinates, $u = 0$ implies $t = r$ and the Christoffel symbols were $\Gamma^r_{\theta\theta} = -r$, $\Gamma^r_{\phi\phi} = -r \sin^2 \theta$, $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^t_{\phi\phi} = 0$. Pulling the derivative back to Σ resulted in nonzero terms, $\underline{\nabla}_a du = \underline{\nabla}_a (dt - dr) \neq 0$

For the cone in null spherical coordinates it was much easier to look at the Christoffel symbols to realize the pullback will not give a well-defined derivative. Here one sees $\Gamma^u_{\theta\theta} = \frac{v-u}{2}$ and $\Gamma^u_{\phi\phi} = \frac{-(v-u)}{2} \sin^2 \theta$ so the pullback of ∇ will not be well-defined.

The fundamental question after all of the examples becomes: What are the necessary conditions on a null surface to give a well-defined covariant derivative? We began with a definition of well-defined that required $\underline{\nabla}_a du = 0$. A closer look at ∇ led us to check the Christoffel symbols of the form Γ^u_{ab} . Now the question becomes, if there are Christoffel symbols $\Gamma^u_{ab} \neq 0$, is there a conformal transformation that results in a well-defined deriv-

ative, and if so, what is the conformal factor? We address these questions in Chapter 5, but first some results on conformal transformations are presented from Geroch's work on asymptotically flat spacetimes.

4 ASYMPTOTICALLY FLAT SPACETIMES VIA GEROCH

Our prior constructions led to a well-defined covariant derivative on the conformal cone in section 3.3.12, but not for the null cone in section 3.3.11. It turns out the conformal transformation is key in defining a covariant derivative on null hypersurfaces using the pullback method. In a 1976 paper, Geroch [1] used conformal transformations to study the asymptotic structure of spacetimes in general relativity.

The boundary of a spacetime can be approached along spacelike, timelike, or lightlike paths. It is Geroch's work on the null case, with its degenerate metric, that is considered in this chapter. The null boundary of a spacetime is a hypersurface that is attached to the physical spacetime using a conformal transformation. This technique allows one to study the properties of the boundary as a submanifold.

While the covariant derivative was not the focus of his work, Geroch's techniques using conformal transformations turn out to provide the tools needed to define a connection on some null hypersurfaces. This chapter summarizes Geroch's presentation including the definition of an asymptotically flat spacetime, some equations that result from conformal transformations, and the use of an additional conformal transformation to guarantee that the null vector field is a Killing vector field. All of the calculations use abstract index notation.

4.1 Physical vs. Unphysical Spacetime

Let $(\widetilde{M}, \widetilde{g}_{ab})$ be a spacetime (i.e., a 4-manifold with smooth metric of Lorentz signature). By an *asymptote* of $(\widetilde{M}, \widetilde{g}_{ab})$ we mean a manifold M with boundary \mathcal{J} , together with a smooth Lorentz metric g_{ab} on M , a smooth function Ω on M and a diffeomorphism ψ from \widetilde{M} to $M - \mathcal{J}$ (by means of which we shall identify \widetilde{M} and $M - \mathcal{J}$),

satisfying the following conditions:

1. On \widetilde{M} , $g_{ab} = \Omega^2 \widetilde{g}_{ab}$.
2. On \mathcal{J} ,
 - (a) $\Omega = 0$,
 - (b) $\nabla_a \Omega \neq 0$, and
 - (c) $g^{ab}(\nabla_a \Omega)(\nabla_b \Omega) = 0$, where “ ∇_a ” denotes the gradient on M .

The metric g_{ab} is called the *unphysical metric* (to distinguish it from the physical metric \widetilde{g}_{ab}), while \mathcal{J} is called the *boundary* (at null infinity). Note that the definition requires the unphysical metric to be defined and have Lorentz signature also at points of the boundary. By contrast, the physical metric is not even defined on \mathcal{J} (and, indeed, according to conditions 1 and 2a could not be given sensible meaning there). It follows from the definition that Ω is nonzero on \widetilde{M} (by convention, we choose it positive there), and that \mathcal{J} is a null surface (since $\nabla_a \Omega$ on \mathcal{J} is normal to \mathcal{J} , nonzero, and null.) The *normal vector* will be defined by $n^a = g^{ab} \nabla_b \Omega$.

It is intended that the definition represent the intuitive idea of “the attachment to the spacetime manifold \widetilde{M} of additional ideal points at null infinity.” The additional points are of course those of \mathcal{J} , while the diffeomorphism ψ inserts \widetilde{M} in M ; thus M itself represents the physical spacetime manifold with points at infinity attached. Condition 1 states that the conformal factor rescales the physical metric to the unphysical, while condition 2a together with the requirement that the unphysical metric be well-behaved on \mathcal{J} states that “infinity is \widetilde{g}_{ab} -far away.” Condition 2b fixes the asymptotic behavior of Ω ; in effect, it states that Ω falls to zero “as $1/r$.” Finally, condition 2c states essentially that we are dealing with null infinity.

4.2 Conformal Transformations

A conformal transformation is most commonly thought of as a rescaling of length. The conformal transformation in our case is a rescaling of the metric in the spacetime, in effect bringing the boundary at infinity to a finite distance. Under this transformation it is now possible to study the structure of the null hypersurface \mathcal{J} . First we must consider the covariant derivative under the conformal transformation.

4.2.1 Conformal Coefficients for the Covariant Derivative

Let $(\widetilde{M}, \widetilde{g}_{ab})$ be an s -dimensional manifold with a non-degenerate metric of any signature. If Ω is a smooth, strictly positive function, then the metric $g_{ab} = \Omega^2 \widetilde{g}_{ab}$ is said to arise from \widetilde{g}_{ab} via a *conformal transformation*. Ω is called the *conformal factor*. Since now either \widetilde{g}_{ab} or g_{ab} could be used to raise and lower indices of tensor fields on \widetilde{M} , we adopt the convention that the indices of tensor fields with a “ \sim ” are to be raised and lowered with \widetilde{g}_{ab} those without the tilde use g_{ab} .

Each of \widetilde{g}_{ab} and g_{ab} gives rise to a Levi-Civita derivative operator, $\widetilde{\nabla}_a$ and ∇_a respectively. Setting $C^m{}_{ab} = \widetilde{\Gamma}^m{}_{ab} - \Gamma^m{}_{ab}$, where $\widetilde{\Gamma}^m{}_{ab}$ and $\Gamma^m{}_{ab}$ are the Christoffel symbols for $\widetilde{\nabla}_a$ and ∇_a respectively, we have

$$\begin{aligned} \widetilde{\nabla}_a \alpha^{b\dots c}{}_{d\dots e} &= \nabla_a \alpha^{b\dots c}{}_{d\dots e} + C^b{}_{am} \alpha^{m\dots c}{}_{d\dots e} + \dots + C^c{}_{am} \alpha^{b\dots m}{}_{d\dots e} \\ &\quad - C^m{}_{ad} \alpha^{b\dots c}{}_{m\dots e} - \dots - C^m{}_{ae} \alpha^{b\dots c}{}_{d\dots m} \end{aligned} \quad (4.1)$$

for any tensor field $\alpha^{b\dots c}{}_{d\dots e}$ on \widetilde{M} . While the Christoffel symbols are not tensors, $C^m{}_{ab}$ is none the less a tensor field, symmetric in its lower indices.

To find $C^m{}_{ab}$, we use metric compatibility and the Leibnitz rule for the derivative

operator:

$$\begin{aligned}
0 &= \tilde{\nabla}_a \tilde{g}_{bc} = \tilde{\nabla}_a (\Omega^{-2} g_{bc}) \\
&= -2\Omega^{-3} (\tilde{\nabla}_a \Omega) g_{bc} + \Omega^{-2} \tilde{\nabla}_a g_{bc} \\
&= -2\Omega^{-3} (\nabla_a \Omega) g_{bc} + \Omega^{-2} (\nabla_a g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{bd})
\end{aligned} \tag{4.2}$$

Assuming g_{ab} is invertible, we can symmetrize equation (4.2) in a and b , and solve for $C^c{}_{ab}$, giving

$$C^c{}_{ab} = -\Omega^{-1} (\nabla_a \Omega \delta_b^c + \nabla_b \Omega \delta_a^c - g_{ab} g^{cd} \nabla_l \Omega) \tag{4.3}$$

The *Riemann curvature tensor* is now expressed in terms of ∇_a using $\tilde{R}^d{}_{cab} \omega_d = -(\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a) \omega_c$, for any one-form ω_c . The relationship between the curvature tensors $\tilde{R}^d{}_{cab}$ and $R^d{}_{cab}$ is derived using equation (4.1)

$$\tilde{\nabla}_b \omega_c = \nabla_b \omega_c - C^k{}_{bc} \omega_k \tag{4.4}$$

$$\tilde{\nabla}_a \tilde{\nabla}_b \omega_c = \tilde{\nabla}_a (\nabla_b \omega_c - C^k{}_{bc} \omega_k) \tag{4.5}$$

$$\begin{aligned}
&= \nabla_a (\nabla_b \omega_c - C^k{}_{bc} \omega_k) \\
&\quad - C^l{}_{ab} (\nabla_l \omega_c - C^k{}_{lc} \omega_k) - C^l{}_{ac} (\nabla_b \omega_l - C^k{}_{bl} \omega_k)
\end{aligned} \tag{4.6}$$

Again using the symmetry of $C^k{}_{ab}$ and the Leibnitz property of ∇_a we get

$$\begin{aligned}
-\tilde{R}^d{}_{cab} \omega_d &= -R^d{}_{cab} \omega_d - (\nabla_a C^d{}_{bc} - \nabla_b C^d{}_{ac}) \omega_d \\
&\quad + (C^l{}_{ac} C^d{}_{bl} - C^l{}_{bc} C^d{}_{al}) \omega_d
\end{aligned} \tag{4.7}$$

Since ω_d is arbitrary, we have the relationship

$$\tilde{R}^d{}_{cab} = R^d{}_{cab} + (\nabla_a C^d{}_{bc} - \nabla_b C^d{}_{ac}) - (C^l{}_{ac} C^d{}_{bl} - C^l{}_{bc} C^d{}_{al}) \tag{4.8}$$

The *Ricci Tensor*, $\tilde{R}_{ab} = \tilde{R}^k{}_{akb}$, can now be expressed in terms of R_{ab} and $C^k{}_{ab}$,

$$\tilde{R}_{ab} = R_{ab} - (\nabla_a C^k{}_{kb} - \nabla_k C^k{}_{ab}) + (C^l{}_{ab} C^k{}_{kl} - C^l{}_{kb} C^k{}_{al}) \tag{4.9}$$

Equation (4.3) can be used to express the Ricci tensor in terms of Ω and g_{ab} . The results are given without derivation.

$$\begin{aligned}\tilde{R}_{ab} &= R_{ab} + (s-2)\Omega^{-1}\nabla_a\nabla_b\Omega + \Omega^{-1}g_{ab}\nabla^m\nabla_m\Omega \\ &\quad - (s-1)\Omega^{-2}g_{ab}(\nabla^m\Omega)(\nabla_m\Omega)\end{aligned}\quad (4.10)$$

The Ricci scalar curvature, $\tilde{R} = \tilde{g}^{ab}\tilde{R}_{ab} = \Omega^2 g^{ab}\tilde{R}_{ab}$, is given by

$$\tilde{R} = \Omega^2 R + 2(s-1)\Omega\nabla^m\nabla_m\Omega - s(s-1)(\nabla^m\Omega)(\nabla_m\Omega)\quad (4.11)$$

4.3 Lie Derivative under Conformal Transformations

The *Lie derivative* of an $(0, 2)$ tensor τ_{ab} is given by

$$\mathcal{L}_v\tau_{ab} = v^c\nabla_c\tau_{ab} + \tau_{cb}\nabla_av^c + \tau_{ac}\nabla_bv^c\quad (4.12)$$

Let the $(0, 2)$ tensor be a metric tensor, g_{ab} , and replace v^a with the normal vector field $n^a = g^{ak}\nabla_k\Omega = \nabla^a\Omega$. Assuming a metric-compatible ($\nabla_c g_{ab} = 0$) and torsion-free ($\nabla_a\nabla_b\Omega = \nabla_b\nabla_a\Omega$) derivative operator, we get

$$\begin{aligned}\mathcal{L}_n g_{ab} &= g_{cb}\nabla_a n^c + g_{ac}\nabla_b n^c \\ &= g_{cb}\nabla_a g^{ck}\nabla_k\Omega + g_{ac}\nabla_b g^{ck}\nabla_k\Omega \\ &= \delta_b^k\nabla_a\nabla_k\Omega + \delta_a^k\nabla_b\nabla_k\Omega \\ &= \nabla_a\nabla_b\Omega + \nabla_a\nabla_b\Omega \\ &= 2\nabla_a\nabla_b\Omega\end{aligned}\quad (4.13)$$

This result can now be substituted into equation (4.10) to give

$$\begin{aligned}\tilde{R}_{ab} &= R_{ab} + \frac{(s-2)}{2}\Omega^{-1}\mathcal{L}_n g_{ab} + \Omega^{-1}g_{ab}\nabla^m\nabla_m\Omega \\ &\quad - (s-1)\Omega^{-2}g_{ab}(\nabla^m\Omega)(\nabla_m\Omega)\end{aligned}\quad (4.14)$$

4.3.1 A Second Transformation, Killing Vector Field

A vector field, n^a , such that $\mathcal{L}_n g_{ab} = 0$ is called a *Killing vector field*. Using the “gauge freedom” in the original conformal transformation given in condition 1, we can make the Lie derivative above vanish. To do this, we seek to use a nonzero smooth function ω on \mathcal{J} to define a diffeomorphism $\phi : \mathcal{J} \rightarrow \mathcal{J}$ such that:

1. $\bar{g}_{ab} = \omega^2 g_{ab}$ and
2. $\mathcal{L}_{\bar{n}} \bar{g}_{ab} = 0$,

where $\bar{n}^a = \omega^{-1} n^a$. As shown by Geroch, this preserves the conditions outlined for \mathcal{J} .

Using equation (4.12) to calculate the Lie derivative gives:

$$\begin{aligned}
\mathcal{L}_{\bar{n}} \bar{g}_{ab} &= \omega^{-1} n^c \nabla_c (\omega^2 g_{ab}) + \omega^2 g_{cb} \nabla_a (\omega^{-1} n^c) + \omega^2 g_{ac} \nabla_b (\omega^{-1} n^c) \\
&= \omega^{-1} n^c (2\omega (\nabla_c \omega) g_{ab} + \omega^2 \nabla_c g_{ab}) + \omega^2 g_{cb} (-\omega^{-2} (\nabla_a \omega) n^c + \omega^{-1} \nabla_a n^c) \\
&\quad + \omega^2 g_{ac} (-\omega^{-2} (\nabla_b \omega) n^c + \omega^{-1} \nabla_b n^c) \\
&= n^c (2g_{ab} \nabla_c \omega - g_{cb} \nabla_a \omega - g_{ac} \nabla_b \omega) + \omega \mathcal{L}_n g_{ab}
\end{aligned} \tag{4.15}$$

Requiring that \bar{n}^a be a Killing vector field and solving for ω gives

$$n^c (2g_{ab} \nabla_c \omega - g_{cb} \nabla_a \omega - g_{ac} \nabla_b \omega) = -2\omega \square \Omega g_{ab} \tag{4.16}$$

where $\square \Omega = g^{ab} \nabla_a \nabla_b \Omega$ is the d'Alembertian. Using the inverse of the metric tensor gives

$$n^c \left(2\delta_a^c \nabla_c \omega - \delta_c^a \nabla_a \omega - \delta_c^b \nabla_b \omega \right) = -\omega \frac{2}{s} \square \Omega \delta_a^a \tag{4.17}$$

$$n^c (2n \nabla_c \omega - \nabla_c \omega - \nabla_c \omega) = -2\omega \square \Omega \tag{4.18}$$

$$n^c \nabla_c \ln \omega = \frac{-1}{s-1} \square \Omega \tag{4.19}$$

So ω is the solution to the ordinary differential equation (4.19) along each integral curve n^a .

In summary, given an asymptotically flat spacetime satisfying the original Geroch conditions, one can construct a second conformal transformation so as to ensure a Killing vector field, $\mathcal{L}_n g_{ab} = 0$.

5 KILLING VECTORS AND CONFORMAL TRANSFORMATIONS

The examples in Chapter 3 used the pullback method to produce a well-defined covariant derivative operator on the null surfaces $\Sigma = \{u = 0\}$. It was shown that the pullback method works provided the Christoffel symbols of the form $\Gamma^u{}_{ab}$, except $\Gamma^u{}_{au}$, vanish. Geroch, on the other hand, requires the Lie derivative on the unphysical manifold satisfy a “divergence free” condition, $\mathcal{L}_n g_{ab} = 0$. If this condition is not satisfied, a second conformal transformation can be introduced to obtain a divergence free manifold. In this chapter, the necessary Lie derivative will be computed for several examples to verify that Geroch’s conditions hold. Finally, sufficient conditions on a null-surface, $\Sigma = \{u = 0\}$, will be derived for the existence of a well-defined derivative D on Σ . This result will be shown to be equivalent to Geroch’s conditions without needing to introduce a non-physical manifold by “attaching” a null surface to the physical manifold using the conformal transformation $\Omega = \nabla_a u$.

5.1 Lie Derivative

Recall from equation (4.12) that the Lie derivative of the metric tensor is given by

$$\mathcal{L}_n g_{ab} = n^c \nabla_c g_{ab} + g_{cb} \nabla_a n^c + g_{ac} \nabla_b n^c \quad (5.1)$$

where $\mathcal{L}_n g_{ab}$ is an $(0, 2)$ tensor. For a metric-compatible derivative operator, equation (5.1) reduces to

$$\mathcal{L}_n g_{ab} = g_{cb} \nabla_a n^c + g_{ac} \nabla_b n^c = \nabla_a (g_{cb} n^c) + \nabla_b (g_{ac} n^c) = \nabla_a n_b + \nabla_b n_a. \quad (5.2)$$

Notice that in (5.1), n is a normal vector of the form n^c while in (5.2) the right-hand side of the equation is taking a derivative of 1-forms, $n_b = g_{bc} n^c$. For a null surface with

$$n_b = \delta_b^u,$$

$$\nabla_a n_b = \nabla_a \delta_b^u = \partial_a \delta_b^u - \Gamma^c{}_{ab} \delta_b^u = -\Gamma^u{}_{ab}. \quad (5.3)$$

The symmetry of the Christoffel symbols implies $\nabla_a n_b = \nabla_b n_a$. The Lie derivative (5.2) can now be simplified to

$$\mathcal{L}_n g_{ab} = 2 \nabla_a n_b = -2 \Gamma^u{}_{ab}. \quad (5.4)$$

To find the Lie derivative on the surface Σ , use the pullback to get

$$\mathcal{L}_n q_{ab} = \underleftarrow{\mathcal{L}_n g_{ab}} = \underleftarrow{2 \nabla_a n_b}. \quad (5.5)$$

In coordinates, the 1-form n_b is du and the coefficient of the ij^{th} -component of $\mathcal{L}_n g$ is

$$(\mathcal{L}_n g)_{ij} = -2 \Gamma^u{}_{ij}. \quad (5.6)$$

The full tensor notation for the Lie derivative of the metric is then

$$\mathcal{L}_n g = -2 \Gamma^u{}_{ij} dx^i dx^j \quad (5.7)$$

5.1.1 Lie Derivative under Conformal Transformations

If the normal vector, n^a , is changed under the transformation to $\bar{n}^a = \omega^{-1} n^a$ and $\bar{g}_{ab} = \omega^2 g_{ab}$, the calculation of $\mathcal{L}_{\bar{n}} \bar{g}_{ab}$ will also be different.

Using equation (4.12) to calculate the Lie derivative of the metric gives:

$$\begin{aligned} \mathcal{L}_{\bar{n}} \bar{g}_{ab} &= \omega^{-1} n^c \nabla_c (\omega^2 g_{ab}) + \omega^2 g_{cb} \nabla_a (\omega^{-1} n^c) + \omega^2 g_{ac} \nabla_b (\omega^{-1} n^c) \\ &= \omega^{-1} n^c (2\omega (\nabla_c \omega) g_{ab} + \omega^2 \nabla_c g_{ab}) + \omega^2 g_{cb} (-\omega^{-2} (\nabla_a \omega) n^c + \omega^{-1} \nabla_a n^c) \\ &\quad + \omega^2 g_{ac} (-\omega^{-2} (\nabla_b \omega) n^c + \omega^{-1} \nabla_b n^c) \\ &= n^c (2(\nabla_c \omega) g_{ab} - g_{cb} \nabla_a \omega - g_{ac} \nabla_b \omega) + \omega (n^c \nabla_c g_{ab} + g_{cb} \nabla_a n^c + g_{ac} \nabla_b n^c) \\ &= n^c (2(\nabla_c \omega) g_{ab} - g_{cb} \nabla_a \omega - g_{ac} \nabla_b \omega) + \omega \mathcal{L}_n g_{ab} \end{aligned} \quad (5.8)$$

From the previous examples, it appears that if the pullback of the Lie derivative of the metric is zero, the pullback of ∇ gives a well-defined derivative, D , on Σ . The null

cone did not satisfy this property, but a conformal transformation was used to get the Lie derivative to vanish. The choice of $\omega = 1/r$ was given without justification. The next section will provide the conditions and equation for finding ω .

5.2 Killing Vectors and Well-Defined Covariant Derivative

Recall that a vector field v^a is *Killing* if $\mathcal{L}_v g_{ab} = 0$. Given a null surface, $\Sigma = \{u = 0\}$, the first theorem shows that if the vector n^a is Killing, then there is a well defined covariant derivative on Σ . The second theorem will give a condition on Σ such that a conformal transformation exists, after which the Killing condition is satisfied. A consequence of the proof will be an expression which can be used to compute ω . Together, the two theorems provide the conditions and techniques to define a covariant derivative operator on some null hypersurfaces. It should be pointed out that the following results are found in Geroch [1], but are not there applied to the covariant derivative.

5.2.1 Killing Null Vector

Theorem 5.1 (Covariant Derivative on Σ). *If $\mathcal{L}_n q_{ab} = 0$ on a null surface Σ , then the connection defined by the pullback is well-defined.*

Proof. Let n_a be a null 1-form and let w_b be any 1-form on the surface Σ . We would like to define a covariant derivative using the pullback, $D_a w_b = \nabla_a W_b$, where W_b is a 1-form such that $\underline{W}_b = w_b$, but we must show that this is well defined.

On $\Sigma = \{u = 0\}$, n_b is given in coordinates by du and $\underline{du} = 0$ on Σ . In abstract index notation $\underline{n}_b = 0$. Let V_b be a 1-form given by $V_b = W_b + k n_b$ where k is any function. V_b has the same pullback as W_b ,

$$\underline{V}_b = \underline{W}_b + \underline{k n}_b = w_b + 0. \quad (5.9)$$

Since V_b and W_b are identical on Σ , a well defined derivative operator must satisfy

$\overleftarrow{\nabla}_a (V_b - W_b) = 0$. Consider the pullback of the derivative of $V_b - W_b$,

$$\begin{aligned}
\overleftarrow{\nabla}_a (V_b - W_b) &= \overleftarrow{\nabla}_a (k n_b) \\
&= \left(\overleftarrow{\nabla}_a k \right) \overleftarrow{n}_b + \overleftarrow{k} \left(\overleftarrow{\nabla}_a n_b \right) \\
&= \left(\overleftarrow{\nabla}_a k \right) 0 + k|_{u=0} \left(\overleftarrow{\nabla}_a n_b \right) \\
&= k|_{u=0} \left(\overleftarrow{\nabla}_a n_b \right)
\end{aligned} \tag{5.10}$$

Since k is an arbitrary function, a well-defined covariant derivative must require $\overleftarrow{\nabla}_a n_b = 0$. Recall from Equation (4.12) using metric compatibility and defining $\mathcal{L}_n q_{ab} = \overleftarrow{\mathcal{L}}_n g_{ab}$,

$$\begin{aligned}
\mathcal{L}_n q_{ab} &= \left(n^c \overleftarrow{\nabla}_c g_{ab} + g_{cb} \overleftarrow{\nabla}_a n^c + g_{ac} \overleftarrow{\nabla}_b n^c \right) \\
&= \left(0 + \overleftarrow{\nabla}_a n_b + \overleftarrow{\nabla}_b n_a \right)
\end{aligned} \tag{5.11}$$

Now if $\overleftarrow{\nabla}_a n_b = 0$, as required for a well-defined connection, then $\mathcal{L}_n q_{ab} = 0$. Thus, if $\mathcal{L}_n q_{ab} = 0$, a well defined covariant derivative can be defined on the null surface Σ by $D = \overleftarrow{\nabla}$. \square

5.2.2 Conformal Killing Vector

Theorem 5.2 (Conformal Killing Vector). *If $\mathcal{L}_n q_{ab} = f q_{ab}$, then there exists a unique conformal factor ω , up to a constant function, such that $\mathcal{L}_{\bar{n}} \bar{q}_{ab} = 0$.*

Proof. For conformal transformation given by $\bar{n}^a = \omega^{-1} n^a$ and $\bar{g}_{ab} = \omega^2 g_{ab}$ the Lie derivative of the metric is given in (5.8) is

$$\mathcal{L}_{\bar{n}} \bar{g}_{ab} = n^c (2(\nabla_c \omega) g_{ab} - g_{cb} \nabla_a \omega - g_{ac} \nabla_b \omega) + \omega \mathcal{L}_n g_{ab}. \tag{5.12}$$

Pull the Lie derivative of the metric back to Σ with $\mathcal{L}_{\bar{n}} \bar{q}_{ab} = 0$ and $\mathcal{L}_n q_{ab} = f q_{ab}$ to get

$$n^c (2(\nabla_c \omega) q_{ab} - q_{cb} \nabla_a \omega - q_{ac} \nabla_b \omega) = -\omega f q_{ab} \tag{5.13}$$

$$2(n^c \nabla_c \omega) q_{ab} - (q_{cb} n^c) \nabla_a \omega - (q_{ac} n^c) \nabla_b \omega = -\omega f q_{ab} \tag{5.14}$$

$$2(n^c \nabla_c \omega) q_{ab} - n_b \nabla_a \omega - n_a \nabla_b \omega = -\omega f q_{ab} \tag{5.15}$$

On the null surface $n_a = q_{ac} n^c = 0$, leaving only

$$2(n^c \nabla_c) \omega q_{ab} = -\omega f q_{ab}. \quad (5.16)$$

Letting $\dot{\omega} = n^c \nabla_c \omega$ the equation simplifies to

$$\frac{\dot{\omega}}{\omega} = -\frac{f}{2} \quad (5.17)$$

This ordinary differential equation will have a unique solution, up to a constant function, yielding an ω such that the conformal transformation will result in $\mathcal{L}_{\bar{n}} \bar{q}_{ab} = 0$. \square

5.3 Well-Defined Covariant Derivative via Pullback

In combination, the two theorems above use the Lie derivative to determine if and when the pullback method will give a well defined covariant derivative on the null-surface using the pullback.

5.3.1 First Result

Theorem 5.3 (Covariant Derivative on Σ with conformal transformation). *If $\mathcal{L}_n q_{ab} = f q_{ab}$ on a null surface Σ , then the pullback method produces a well defined covariant derivative, D , on Σ .*

Proof. Since $\mathcal{L}_n q_{ab} = f q_{ab}$, Theorem 5.2 gives an ω such that under the conformal transformation $\mathcal{L}_{\bar{n}} \bar{q}_{ab} = 0$. Now by Theorem 5.1, define the covariant derivative by $D = \bar{\nabla}$. \square

5.4 The Examples and the Lie Derivative

Up to this point, the examples have been used to motivate the technique of using the pullback to define a covariant derivative on the null surfaces. Here we revisit the examples to show they satisfy the hypotheses of our theorem as well as the differential equation (4.19) for the conformal factor, ω .

5.4.1 Plane in \mathbb{M}^4

From 3.3.1 the covariant derivative in rectangular coordinates on \mathbb{M}^4 was shown to be $\nabla_a w_b = \partial_a w_b$. The Minkowski plane was defined by $\Sigma = \{t = z\}$. Letting $u = t - z = 0$ and $du = dt - dz$ gives $\nabla_{X^i} du = \nabla_{X^i} (dt - dz) = 0$. The Lie derivative of the metric is then $\mathcal{L}_n g_{ab} = 0$. Pulling this result back to the plane yields $\mathcal{L}_n q_{ab} = \underline{\mathcal{L}_n g_{ab}} = 0$. Notice the Lie derivative on the plane satisfies the Geroch condition needed to give a well-defined derivative on the Minkowski plane, $D_a = \partial_a$.

5.4.2 Plane in \mathbb{M}^4 with null coordinates

Using null coordinates on \mathbb{M}^4 , 3.3.7 showed the covariant derivative is $\nabla_a w_b = \partial_a w_b$. On the plane, $\Sigma = \{u = 0\}$, $\nabla_{X^i} du = 0$. The Lie derivative of the metric is again $\mathcal{L}_n g_{ab} = 0$. Pulling back to the plane gives $\mathcal{L}_n q_{ab} = \underline{\mathcal{L}_n g_{ab}} = 0$. The Geroch condition for a well-defined derivative is satisfied and as previously shown $D_a = \partial_a$ on the null plane.

5.4.3 Cone in \mathbb{M}^4

For the cone in \mathbb{M}^4 , the normal vector is given by $du = d(r - t) = dr - dt$. Now we need to compute the derivatives $\nabla_X (dr)$ and $\nabla_X (dt)$ on \mathbb{M}^4 and then pull the result back to the cone.

Recalling the Christoffel symbols from 3.3.8,

$$\begin{aligned} \Gamma^\theta_{r\theta} &= \frac{1}{r}, & \Gamma^r_{\theta\theta} &= -r \\ \Gamma^\theta_{\phi\phi} &= -\sin\theta \cos\theta, & \Gamma^r_{\phi\phi} &= -r \sin^2\theta \\ \Gamma^\phi_{r\phi} &= \frac{1}{r}, & \Gamma^\phi_{\theta\phi} &= \cot\theta. \end{aligned}$$

For the Lie Derivative, $\mathcal{L}_n g_{ab}$, only the $\nabla_{X^j} (dt)$ and $\nabla_{X^j} (dr)$ need be considered:

$$\nabla_{X^j} (dt) = \Gamma^t_{ij} dx^i = 0 \forall i, j \quad (5.18)$$

$$\nabla_{\Theta} (dr) = \Gamma^r_{\theta\theta} d\theta = -r d\theta \quad (5.19)$$

$$\nabla_{\Phi} (dr) = \Gamma^r_{\phi\phi} d\phi = -r \sin^2\theta d\phi. \quad (5.20)$$

The only nonzero components of $\mathcal{L}_n g$ are,

$$(\mathcal{L}_n g)_{\theta\theta} = -2\Gamma^r_{\theta\theta} = 2r \quad (5.21)$$

$$(\mathcal{L}_n g)_{\phi\phi} = -2\Gamma^r_{\phi\phi} = 2r \sin^2 \theta. \quad (5.22)$$

Pulling the Lie derivative back to the cone gives,

$$\mathcal{L}_n q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2r & 0 \\ 0 & 0 & 2r \sin^2(\theta) \end{bmatrix} = \frac{2}{r} \begin{bmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix} = \frac{2}{r} q \quad (5.23)$$

Clearly $\mathcal{L}_n q_{ab} \neq 0$, but the Lie derivative is proportional to q , $\mathcal{L}_n q_{ab} = (2/r) q_{ab}$.

5.4.4 Cone in \mathbb{M}^4 with null coordinates

Using null coordinates on \mathbb{M}^4 , recall the Christoffel symbols from 6.3.2 are

$$\begin{aligned} \Gamma^{\theta}_{u\theta} &= \frac{-1}{v-u} & \Gamma^{\theta}_{v\theta} &= \frac{1}{v-u} & \Gamma^{\theta}_{\phi\phi} &= -\sin\theta \cos\theta \\ \Gamma^{\phi}_{u\phi} &= \frac{-1}{v-u} & \Gamma^{\phi}_{v\phi} &= \frac{1}{v-u} & \Gamma^{\phi}_{\theta\phi} &= \cot\theta \\ \Gamma^v_{\theta\theta} &= \frac{-(v-u)}{2} & \Gamma^v_{\phi\phi} &= \frac{-(v-u)}{2} \sin^2\theta \\ \Gamma^u_{\theta\theta} &= \frac{v-u}{2} & \Gamma^u_{\phi\phi} &= \frac{v-u}{2} \sin^2\theta \end{aligned}$$

The components of $\mathcal{L}_n g_{ab}$ come only from $\nabla_X du$ which has two nonzero components:

$$\nabla_{\Theta} du = -\Gamma^u_{\theta\theta} d\theta = \frac{v-u}{2} d\theta \quad (5.24)$$

$$\nabla_{\Phi} du = -\Gamma^u_{\phi\phi} d\phi = \frac{v-u}{2} \sin^2\theta d\phi \quad (5.25)$$

The only nonzero components of $\mathcal{L}_n g$ are,

$$(\mathcal{L}_n g)_{\theta\theta} = -2\Gamma^u_{\theta\theta} = (v-u) \quad (5.26)$$

$$(\mathcal{L}_n g)_{\phi\phi} = -2\Gamma^u_{\phi\phi} = (v-u) \sin^2\theta. \quad (5.27)$$

Pulling the Lie derivative back to the cone, $u = 0$, gives

$$\mathcal{L}_n q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v \sin^2(\theta) \end{bmatrix} = \frac{2}{v} \begin{bmatrix} 0 & 0 & 0 \\ 0 & v^2/2 & 0 \\ 0 & 0 & (v^2/2) \sin^2(\theta) \end{bmatrix} = \frac{2}{v} q \quad (5.28)$$

Again $\mathcal{L}_n q_{ab} \neq 0$, but the Lie derivative is proportional to q , $\mathcal{L}_n q_{ab} = (4/v) q_{ab}$. Recall for null coordinates $r = (v - u)/2$, so on the null cone, $u = 0$ and $r = v/2$ which is the same constant of proportionality for the Lie derivative in 5.4.3.

5.4.5 Conformal Factor for the Null Cone

The Lie derivative of the metric for the null cone was shown to be $\mathcal{L}_n q_{ab} = (2/v) q_{ab}$. Substituting $f = 2/v$ into (5.17) gives the ordinary differential equation

$$\frac{\dot{\omega}}{\omega} = -\frac{1}{v}. \quad (5.29)$$

In null coordinates $\dot{\omega} = n^c \nabla_c \omega = \partial \omega / \partial v$ giving

$$\frac{1}{\omega} \frac{\partial \omega}{\partial v} = -\frac{1}{v}. \quad (5.30)$$

Integrating with respect to v gives

$$\ln \omega = -\ln v + c = \ln \frac{1}{v} + c. \quad (5.31)$$

Solving for ω gives $\omega = C (1/v)$ where $C = e^c$. In the original example 3.3.12 the conformal transformation was $\omega = 1/r$ applied to first \mathbb{M}^4 and then the results were pulled back to Σ . As pointed out earlier on Σ , $1/r$ is evaluated at $u = 0$ giving

$$(1/r)|_{u=0} = (2/(v - u))|_{u=0} = 2/v. \quad (5.32)$$

Letting $C = 2$ gives the conformal factor, $\omega = 2/v$ which was used in the examples.

5.4.6 Conformal transformation on null cone in \mathbb{M}^4

Lastly consider the Lie Derivative for the null cone after the conformal transformation, $\omega = 1/r$ where $\bar{g}_{ab} = \omega^2 g_{ab}$. Recall from 3.3.12 the Christoffel symbols under that conformal transformation are

$$\bar{\Gamma}^{\theta}_{\phi\phi} = -\sin\theta \cos\theta, \quad \bar{\Gamma}^{\phi}_{\theta\phi} = \cot\theta, \quad (5.33)$$

$$\bar{\Gamma}^v_{vv} = \frac{-2}{v-u}, \quad \bar{\Gamma}^u_{uu} = \frac{2}{v-u}. \quad (5.34)$$

The components of $\mathcal{L}_n \bar{g}_{ab}$ come only from $\bar{\nabla}_U du$ which has just one nonzero component:

$$\bar{\nabla}_U du = -\bar{\Gamma}^u_{uu} du = -\frac{2}{v-u} du \quad (5.35)$$

leaving the only nonzero component of $\mathcal{L}_n \bar{g}_{ab}$,

$$(\mathcal{L}_n \bar{g})_{uu} = -2\bar{\Gamma}^u_{uu} = -\frac{4}{v-u} \quad (5.36)$$

The tensor notation for the Lie derivative is just $\mathcal{L}_n \bar{g} = -\frac{4}{v-u} du du$. Pulling this result back to Σ with $\underline{du} = 0$ yields $\mathcal{L}_n q = 0$. While the null cone did not satisfy the Geroch condition for a well-defined covariant derivative, the null cone with conformal transformation does satisfy the Geroch condition and the covariant derivative D can be defined by the pullback as in 3.3.12.

5.4.7 Lie Derivative conclusions

In the Minkowski space examples, the Lie derivative of the metric on Σ vanished for both the null plane and the conformal null cone where the covariant derivative D was well defined using the pullback. In both cases where the Lie Derivative of the metric did not vanish, the pullback method for the covariant derivative failed. In the case of the null cone, a conformal transformation produced the desired Lie derivative conditions and the choice of the conformal factor $\omega = 1/r$ was shown to satisfy (4.19).

6 RICCI TENSOR AND COVARIANT DERIVATIVE

6.1 Conformal Pullback and Geroch

The techniques from Geroch have addressed the fundamental question: What are the conditions on a null surface needed to construct a well-defined covariant derivative? Either a Killing normal vector, $\mathcal{L}_n q_{ab} = 0$, or $\mathcal{L}_n q_{ab} = f q_{ab}$, a conformal Killing vector, combined with a conformal transformation gives a well-defined covariant derivative on a null surface Σ using the pullback method.

One of the drawbacks of this method to define D is that the Lie derivative of the metric must first be computed on M and then pulled back to Σ to test the hypotheses of the theorems. If the hypotheses are met, we return to M , compute ∇ , then pull this derivative back to Σ , giving D . If the hypotheses are not met, there is the additional conformal transformation step that is required before ∇ can be pulled back to Σ . It would be nice if there was a test to tell if the pullback led to a well-defined covariant derivative on Σ and if a conformal transformation is required before pulling the ∇ back to Σ . Again the work of Geroch has pointed to exactly such a check, as explained in the following section.

6.2 Ricci Tensor and Covariant Derivative

In Chapter 4, an equation for the Ricci tensor under a conformal transformation was given in equation (4.10). For Ω a smooth, strictly positive conformal factor, and with metric $\tilde{g}_{ab} = \Omega^{-2} g_{ab}$, the Ricci tensor is expressed in terms of the Lie derivative of the metric as well as Ω and g in equation (4.14). This is significant since the Lie derivative of the metric is the primary indicator of when the pullback method will provide a well-

defined covariant derivative. This result will provide another test to determine when a well-defined derivative operator can be defined on a null hypersurface.

While the computations will be the same as Geroch, it should be pointed out that we are able to start with a metric g_{ab} , a null surface $\Sigma = \{u = 0\}$ and Ω defined by $\Omega = u$. In the sense of Geroch, we were creating an artificial “physical” space $(\widetilde{M}, \widetilde{g}_{ab})$ in order to determine if the pullback method will result in a well-defined covariant derivative. In the Geroch approach, the boundary at null infinity was separated from \widetilde{M} by the conformal transformation in order to define some structure of the null surface. The following approach begins with the null surface and uses the “physical” space to define the covariant derivative.

6.2.1 Second Result

Theorem 6.1 (Ricci Tensor and Covariant Derivative on Σ). *Given a manifold (M, g_{ab}) with invertible metric with Lorentzian signature, a null surface defined by $\Sigma = \{u = 0\}$, and a conformal transformation $\widetilde{g}_{ab} = \Omega^{-2} g_{ab}$ where $\Omega = u$. If $\Omega \left(\overleftarrow{R_{ab} - \widetilde{R}_{ab}} \right) = k q_{ab}$, where R_{ab} and \widetilde{R}_{ab} are the Ricci tensors from g_{ab} and \widetilde{g}_{ab} respectively, then the pullback gives a well-defined connection on Σ .*

Proof. From (4.14)

$$\begin{aligned} \Omega \widetilde{R}_{ab} &= \Omega R_{ab} + \frac{s-2}{2} \mathcal{L}_n g_{ab} + g_{ab} \nabla^m \nabla_m \Omega \\ &\quad - \frac{s-1}{\Omega} g_{ab} (\nabla^m \Omega) (\nabla_m \Omega). \end{aligned} \tag{6.1}$$

Setting $\Omega = u$ gives $g_{ab} (\nabla^m \Omega) (\nabla_m \Omega) = 0$. Solving for $\mathcal{L}_n g_{ab}$ gives

$$\mathcal{L}_n g_{ab} = \frac{-2}{s-2} \left(\Omega (R_{ab} - \widetilde{R}_{ab}) + g_{ab} \nabla^m \nabla_m \Omega \right) \tag{6.2}$$

Lastly, we must pull this result back to our surface Σ . If $\Omega \left(\overleftarrow{R_{ab} - \widetilde{R}_{ab}} \right) = k q_{ab}$ for some

function k , then $\mathcal{L}_n q_{ab}$ is

$$\begin{aligned}\mathcal{L}_n q_{ab} &= \overleftarrow{\mathcal{L}_n g_{ab}} \\ &= \frac{2}{n-2} (k + (\nabla^m \nabla_m \Omega)|_{u=0}) q_{ab}\end{aligned}\quad (6.3)$$

With $\Omega = u$ and $\overleftarrow{\Omega(R_{ab} - \tilde{R}_{ab})} = k q_{ab}$, Theorem 5.3 therefore implies there is a covariant derivative D on Σ . \square

6.3 Examples Revisited

Even though a connection has been shown to exist for the null plane and null cone, the examples will be revisited once again to verify the results of the previous theorem. In order to calculate the Ricci tensor, one must first compute the Riemann curvature tensor

$$R^a{}_{bcd} = \frac{\partial \Gamma^a{}_{bd}}{\partial x^c} - \frac{\partial \Gamma^a{}_{bc}}{\partial x^d} + \Gamma^a{}_{ck} \Gamma^k{}_{bd} - \Gamma^a{}_{dk} \Gamma^k{}_{bc}. \quad (6.4)$$

Now the Ricci Tensor is given by

$$R_{ab} = R^l{}_{alb} = \frac{\partial \Gamma^l{}_{ab}}{\partial x^l} - \frac{\partial \Gamma^l{}_{al}}{\partial x^b} + \Gamma^l{}_{ab} \Gamma^k{}_{lk} - \Gamma^l{}_{ak} \Gamma^k{}_{bl}. \quad (6.5)$$

These tensors require a great deal of computation so Maple was again used to check the tensors for all of the following examples.

6.3.1 Null Plane

Recall the line element for Minkowski space in rectangular null coordinates is

$$ds^2 = -2du dv + dx^2 + dy^2$$

with metric

$$[g_{ij}] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All of the Christoffel symbols are zero, so all the components of the Ricci tensor R_{ab} are also zero.

The null plane is defined by $\Sigma = \{u = 0\}$. Letting $\Omega = u$, $\tilde{g}_{ab} = (1/u^2)g_{ab}$ becomes

$$[\tilde{g}_{ij}] = \begin{bmatrix} 0 & -1/u^2 & 0 & 0 \\ -1/u^2 & 0 & 0 & 0 \\ 0 & 0 & 1/u^2 & 0 \\ 0 & 0 & 0 & 1/u^2 \end{bmatrix}$$

For this metric, the Christoffel symbols are not all zero, but all of the components of the Ricci tensor are zero. Thus all the components of $R_{ab} - \tilde{R}_{ab}$ are zero and the conditions of Theorem 6.1 are satisfied trivially, $\Omega \left(R_{ab} - \tilde{R}_{ab} \right) = 0 q_{ab}$.

6.3.2 Null Cone

The line element for Minkowski space in null spherical coordinates is

$$ds^2 = -2du dv + \frac{(v-u)^2}{2} d\theta^2 + \frac{(v-u)^2}{2} \sin^2 \theta d\phi^2$$

with metric

$$[g_{ij}] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(v-u)^2}{2} & 0 \\ 0 & 0 & 0 & \frac{(v-u)^2}{2} \sin^2 \theta \end{bmatrix}$$

All the components of the Ricci tensor for this metric are zero.

As with the null plane, the null cone is defined by $\Sigma = \{u = 0\}$. Letting $\Omega = u$, $\tilde{g}_{ab} = (1/u^2)g_{ab}$ becomes

$$[\tilde{g}_{ij}] = \begin{bmatrix} 0 & -1/u^2 & 0 & 0 \\ -1/u^2 & 0 & 0 & 0 \\ 0 & 0 & \frac{(v-u)^2}{2u^2} & 0 \\ 0 & 0 & 0 & \frac{(v-u)^2}{2u^2} \sin^2 \theta \end{bmatrix}$$

For the conformal metric, the Ricci tensor is

$$[\tilde{R}_{ij}] = \begin{bmatrix} 0 & \frac{2}{(v-u)u} & 0 & 0 \\ \frac{2}{(v-u)u} & 0 & 0 & 0 \\ 0 & 0 & \frac{-2(v-u)}{u} & 0 \\ 0 & 0 & 0 & \frac{-2(v-u)}{u} \sin^2 \theta \end{bmatrix} \quad (6.6)$$

Now

$$\Omega(R_{ab} - \tilde{R}_{ab}) = u(R_{ab} - \tilde{R}_{ab}) = \begin{bmatrix} 0 & \frac{-2}{(v-u)} & 0 & 0 \\ \frac{-2}{(v-u)} & 0 & 0 & 0 \\ 0 & 0 & 2(v-u) & 0 \\ 0 & 0 & 0 & 2(v-u) \sin^2 \theta \end{bmatrix} \quad (6.7)$$

In the pullback, $u = 0$, giving

$$\overleftarrow{\Omega}(R_{ab} - \tilde{R}_{ab}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2v \sin^2 \theta \end{bmatrix} = \frac{8}{v} \begin{bmatrix} 0 & 0 & 0 \\ 0 & v^2/4 & 0 \\ 0 & 0 & (v^2/4) \sin^2 \theta \end{bmatrix} = \frac{8}{v} [q_{ij}] \quad (6.8)$$

Again the hypotheses of the theorem are satisfied, guaranteeing the existence of a connection using the conformal pullback method.

7 FURTHER EXAMPLES

To this point, all of the examples were used to motivate the theorems. The following examples use the techniques developed to define a covariant derivative on the null surface. The first example is the horizon of the Schwarzschild geometry. The last example is a generalization to all spherically symmetric spacetimes.

7.1 Horizon of Schwarzschild geometry

The Schwarzschild metric is given by the line element

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

Letting

$$T = \sqrt{\left(\frac{r}{2M} - 1\right)} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \quad (7.1)$$

$$R = \sqrt{\left(\frac{r}{2M} - 1\right)} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \quad (7.2)$$

(when $r > 2M$) the metric becomes

$$ds^2 = \frac{32 M^3}{r} e^{-r/2M} (-dT^2 + dR^2) + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

To get null coordinates, the substitution

$$u = T - R \quad (7.3)$$

$$v = T + R \quad (7.4)$$

yields the Kruskal-Szekeres metric with line element

$$ds^2 = -\frac{32 M^3}{r} e^{-r/2M} du dv + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

where r is given implicitly by

$$u v = \left(1 - \frac{r}{2M}\right) e^{r/2M}. \quad (7.5)$$

With a little work it can be shown that all the components of the Ricci tensor are zero. Next, the Ricci tensor for the metric with conformal transformation $\tilde{g}_{ab} = \Omega^2 g_{ab} = (1/u^2) g_{ab}$ is computed. Since the difference $\Omega (R_{ab} - \tilde{R}_{ab})$ is being pulled back to Σ , only the \tilde{R}_{vv} , $\tilde{R}_{\theta\theta}$, and $\tilde{R}_{\phi\phi}$ components are needed. The resulting components are

$$\tilde{R}_{vv} = R_{vv} = 0 \quad (7.6)$$

$$\tilde{R}_{\theta\theta} = R_{\theta\theta} + \frac{r}{M} \quad (7.7)$$

$$\tilde{R}_{\phi\phi} = R_{\phi\phi} + \frac{r}{M} \sin^2 \theta \quad (7.8)$$

Pulling the expression, $\Omega (R_{ab} - \tilde{R}_{ab}) = u \underbrace{(R_{ab} - \tilde{R}_{ab})}_{\leftarrow}$, back to Σ , all of the components of the difference vanish to yield

$$\underbrace{\Omega (R_{ab} - \tilde{R}_{ab})}_{\leftarrow} = 0 q_{ab}, \quad (7.9)$$

trivially satisfying the necessary conditions of Theorem 6.1.

Now, letting $\omega = 1/r(u, v)$ the conformal metric $\bar{g}_{ab} = (1/r^2) g_{ab}$ becomes

$$ds^2 = -\frac{32 M^3}{r^3} e^{-r/2M} du dv + d\theta^2 + \sin^2 \theta d\phi^2.$$

The corresponding Christoffel symbols are

$$\bar{\Gamma}^u_{uu} = -\frac{(6M+r) \frac{\partial r}{\partial u}}{2Mr} = \left(\frac{2M}{r^2} (6M+r) e^{-r/2M}\right) v \quad (7.10)$$

$$\bar{\Gamma}^v_{vv} = -\frac{(6M+r) \frac{\partial r}{\partial v}}{2Mr} = \left(\frac{2M}{r^2} (6M+r) e^{-r/2M}\right) u \quad (7.11)$$

$$\bar{\Gamma}^\phi_{\theta\phi} = \cot \theta \quad (7.12)$$

$$\bar{\Gamma}^\theta_{\phi\phi} = -\sin \theta \cos \theta \quad (7.13)$$

The only nonzero $\bar{\nabla}_a$ terms will be

$$\bar{\nabla}_u du = -\bar{\Gamma}^u{}_{uu} du \quad (7.14)$$

$$\bar{\nabla}_v dv = -\bar{\Gamma}^v{}_{vv} dv \quad (7.15)$$

$$\bar{\nabla}_\phi d\phi = -\bar{\Gamma}^\phi{}_{\theta\phi} d\theta \quad (7.16)$$

$$\bar{\nabla}_\phi d\theta = -\bar{\Gamma}^\theta{}_{\phi\phi} d\phi \quad (7.17)$$

Again, there are no difficulties in the pullback to Σ . This time, the first two terms will vanish in the pullback with $u = 0$ and $\underline{d}u = 0$. The last two derivatives pull back with $u = 0$ to define a covariant derivative D on Σ given by

$$D_\phi \underline{d}\phi = \underline{\bar{\nabla}}_\phi d\phi = -\cot\theta \underline{d}\theta \quad (7.18)$$

$$D_\phi \underline{d}\theta = \underline{\bar{\nabla}}_\phi d\theta = \sin\theta \cos\theta \underline{d}\phi \quad (7.19)$$

7.2 Spherically Symmetric Spacetimes

Spherically symmetric space times can be written with the line element

$$ds^2 = h du dv + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

where h and r are both functions of the null coordinates u and v . The corresponding metric is given by

$$[g_{ij}] = \begin{bmatrix} 0 & h(u,v)/2 & 0 & 0 \\ h(u,v)/2 & 0 & 0 & 0 \\ 0 & 0 & r^2(u,v) & 0 \\ 0 & 0 & 0 & r^2(u,v) \sin^2\theta \end{bmatrix}$$

As in previous examples, the null surface is defined by $\Sigma = \{u = 0\}$. Unlike the previous examples, the Ricci tensor will be calculated first. The nonzero components are

$$R_{uu} = -\frac{2 \left(h \frac{\partial^2 r}{\partial u^2} - \frac{\partial h}{\partial u} \frac{\partial r}{\partial u} \right)}{h r} \quad (7.20)$$

$$R_{uv} = \frac{r \frac{\partial h}{\partial u} \frac{\partial h}{\partial v} - h r \frac{\partial^2 h}{\partial v \partial u} - 2 h^2 \frac{\partial^2 r}{\partial v \partial u}}{h^2 r} \quad (7.21)$$

$$R_{vv} = -\frac{2 \left(h \frac{\partial^2 r}{\partial v^2} - \frac{\partial h}{\partial v} \frac{\partial r}{\partial v} \right)}{h r} \quad (7.22)$$

$$R_{\theta\theta} = -\frac{4 r \frac{\partial^2 r}{\partial v \partial u} - h + 4 \frac{\partial r}{\partial v} \frac{\partial r}{\partial u}}{h} \quad (7.23)$$

$$R_{\phi\phi} = -\frac{\left(4 r \frac{\partial^2 r}{\partial v \partial u} - h + 4 \frac{\partial r}{\partial v} \frac{\partial r}{\partial u} \right) \sin^2 \theta}{h} \quad (7.24)$$

With $\Omega = u$ the metric \tilde{g}_{ab} is given by

$$[\tilde{g}_{ij}] = [(1/u^2) g_{ij}] = \begin{bmatrix} 0 & h(u, v)/(2u^2) & 0 & 0 \\ h(u, v)/(2u^2) & 0 & 0 & 0 \\ 0 & 0 & r^2(u, v)/u^2 & 0 \\ 0 & 0 & 0 & (r^2(u, v) \sin^2 \theta)/u^2 \end{bmatrix}$$

Here the Ricci tensor has nonzero components

$$\tilde{R}_{uu} = \frac{2 \left(-u h \frac{\partial^2 r}{\partial u^2} - r \frac{\partial h}{\partial u} + u \frac{\partial h}{\partial u} \frac{\partial r}{\partial u} \right)}{u h r} \quad (7.25)$$

$$\tilde{R}_{uv} = \frac{u r \frac{\partial h}{\partial u} \frac{\partial h}{\partial v} - u h r \frac{\partial^2 h}{\partial v \partial u} + 2 h^2 \frac{\partial r}{\partial v} - 2 u h^2 \frac{\partial^2 r}{\partial v \partial u}}{u h^2 r} \quad (7.26)$$

$$\tilde{R}_{vv} = -\frac{2 \left(h \frac{\partial^2 r}{\partial v^2} - \frac{\partial h}{\partial v} \frac{\partial r}{\partial v} \right)}{h r} \quad (7.27)$$

$$\tilde{R}_{\theta\theta} = -\frac{4 u r \frac{\partial^2 r}{\partial v \partial u} - u h + 4 u \frac{\partial r}{\partial v} \frac{\partial r}{\partial u} - 8 r \frac{\partial r}{\partial v}}{u h} \quad (7.28)$$

$$\tilde{R}_{\phi\phi} = -\frac{\left(4 u r \frac{\partial^2 r}{\partial v \partial u} - u h + 4 u \frac{\partial r}{\partial v} \frac{\partial r}{\partial u} - 8 r \frac{\partial r}{\partial v} \right) \sin^2 \theta}{u h} \quad (7.29)$$

To compute the pullback of $\Omega(R_{ab} - \tilde{R}_{ab}) = u(R_{ab} - \tilde{R}_{ab})$ only the R_{vv} , $R_{\theta\theta}$ and $R_{\phi\phi}$ components are needed since all other components vanish on $\Sigma = \{u = 0\}$.

$$u(R_{vv} - \tilde{R}_{vv}) = 0 \quad (7.30)$$

$$u(R_{\theta\theta} - \tilde{R}_{\theta\theta}) = \frac{-8r \frac{\partial r}{\partial v}}{h} = \left(\frac{-8 \frac{\partial r}{\partial v}}{r h} \right) r^2 \quad (7.31)$$

$$u(R_{\phi\phi} - \tilde{R}_{\phi\phi}) = \frac{-8r \frac{\partial r}{\partial v}}{h} \sin^2 \theta = \left(\frac{-8 \frac{\partial r}{\partial v}}{r h} \right) r^2 \sin^2 \theta \quad (7.32)$$

Evaluating the difference on $\Sigma = \{u = 0\}$ yields

$$\overleftarrow{\Omega}(R_{ab} - \tilde{R}_{ab}) = \left(\frac{-8 \frac{\partial r}{\partial v}}{r(0, v) h(0, v)} \right) q_{ab} \quad (7.33)$$

where

$$[q_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & r^2(0, v) & 0 \\ 0 & 0 & r^2(0, v) \sin^2 \theta \end{bmatrix}$$

Thus the conditions of Theorem 6.1 are satisfied, and the conformal transformation $\tilde{g}_{ab} = (1/\Omega^2) g_{ab}$ should lead to $\mathcal{L}_{\tilde{n}} \tilde{q}_{ab} = f \tilde{q}_{ab}$. More importantly, the conformal transformation $\bar{g}_{ab} = (\omega^2) g_{ab}$ with $\omega = r(u, v)$ will yield $\mathcal{L}_{\bar{n}} \bar{q}_{ab} = 0$, the condition needed to use the pullback to produce a well-defined covariant derivative on $\Sigma = \{u = 0\}$.

With the conformal transformation, the new metric for the spherically symmetric space time is

$$[\bar{g}_{ij}] = \begin{bmatrix} 0 & h(u, v)/(2r^2(u, v)) & 0 & 0 \\ h(u, v)/(2r^2(u, v)) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2 \theta \end{bmatrix}$$

The corresponding Christoffel symbols are

$$\bar{\Gamma}^u{}_{uu} = -\frac{2h\frac{\partial r}{\partial u} + r\frac{\partial h}{\partial u}}{rh} \quad (7.34)$$

$$\bar{\Gamma}^v{}_{vv} = -\frac{2h\frac{\partial r}{\partial v} + r\frac{\partial h}{\partial v}}{rh} \quad (7.35)$$

$$\bar{\Gamma}^\phi{}_{\theta\phi} = \cot\theta \quad (7.36)$$

$$\bar{\Gamma}^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \quad (7.37)$$

The only nonzero $\bar{\nabla}_a$ terms will be

$$\bar{\nabla}_u du = -\bar{\Gamma}^u{}_{uu} du \quad (7.38)$$

$$\bar{\nabla}_v dv = -\bar{\Gamma}^v{}_{vv} dv \quad (7.39)$$

$$\bar{\nabla}_\phi d\phi = -\bar{\Gamma}^\phi{}_{\theta\phi} d\theta \quad (7.40)$$

$$\bar{\nabla}_\phi d\theta = -\bar{\Gamma}^\theta{}_{\phi\phi} d\phi \quad (7.41)$$

As with the earlier examples, there are no difficulties in pulling the components of $\bar{\nabla}$ back to $\Sigma = \{u = 0\}$. The first term will vanish in the pullback with $u = 0$ and $\underline{du} = 0$. The last three derivatives pull back with $u = 0$ to define a covariant derivative D on Σ given by

$$D_v \underline{dv} = \underline{\bar{\nabla}}_v dv = -\left(\frac{2h\frac{\partial r}{\partial v} + r\frac{\partial h}{\partial v}}{rh}\right)\bigg|_{u=0} \underline{dv} \quad (7.42)$$

$$D_\phi \underline{d\phi} = \underline{\bar{\nabla}}_\phi d\phi = -\cot\theta \underline{d\theta} \quad (7.43)$$

$$D_\phi \underline{d\theta} = \underline{\bar{\nabla}}_\phi d\theta = \sin\theta\cos\theta \underline{d\phi} \quad (7.44)$$

8 DISCUSSION AND CONCLUSIONS

8.1 Overview

The first portion of this dissertation, Chapter 2, discussed the need for an alternate method to define a covariant derivative on a null hypersurface. Due to the degenerate metric on the null submanifold, traditional decomposition methods are not possible. The work of Duggal provided a framework for constructing a covariant derivative defined similarly to the traditional Gauss formula, but depending on the choice of decomposition. It was shown that the null cone did not satisfy the conditions necessary for the resulting derivative operator to be independent of the chosen decomposition.

In Chapter 3, a sequence of examples were presented to motivate a technique using the pullback to define a covariant derivative on a null hypersurfaces. Through these examples, it was shown that care must be taken to ensure the derivative is well defined. This technique, motivated by Geroch's work with asymptotically flat spacetimes, led to the use of conformal transformations to eliminate the possible ambiguities in the derivative. Chapter 4 provided a summary of Geroch's use of conformal transformations to study the boundary of asymptotically flat spacetimes as well as to produce normal vectors that are Killing.

Chapter 5, outlined a pullback technique similar to that of Geroch and provided conditions when a further conformal transformation is needed to arrive at a well-defined connection. An equation for the conformal factor was provided.

Chapter 6 gave a test to see if the conformal pullback method will result in a connection on the null hypersurface. This result allowed one to take a limit of the difference of two Ricci tensors to test whether the conformal pullback method will work. The conformal factor could now be checked in the equation from the first result.

Finally, the conformal pullback method was shown to work for the horizon of the Schwarzschild geometry and then generalized to all spherically symmetric spacetimes. It would be interesting to test this method on other null hypersurfaces. In particular it would be nice to find the set of all null hypersurfaces where the pullback method produces a well-defined covariant derivative.

8.2 Summary

Due to the degenerate metric, working with null surfaces offers some very challenging obstacles. Traditional tools such as Christoffel symbols are not defined due to the degenerate metric. The Gauss decomposition fails, since there is a non-zero null vector that is in the hypersurface itself. The work of Duggal and Benjacu attempts to overcome this difficulty by defining a screen manifold and a lightlike transversal vector bundle to decompose the manifold and the null hypersurface. Even with all of this structure, it was demonstrated that the null cone still does not satisfy the hypotheses necessary to produce a covariant derivative independent of the screen.

A technique using the pullback of 1-forms instead of vectors to define the covariant derivative was developed. Care must be taken when using this technique, since pulling forms back to the null surface can result in derivative operators that are not well defined. Upon closer inspection, it was shown that this technique will work as long as the null vector field is a Killing vector field.

Motivated by the work of Geroch on asymptotically flat spacetimes, conformal transformations were used not only to give a well-defined derivative on null hypersurfaces, but also to provide a test to determine whether the null surface admits such a definition. An equation for the necessary conformal transformation was presented. This method was extended by giving a test using the Ricci tensor to determine whether the conformal pullback

method will give a covariant derivative on a given null hypersurface.

8.3 Summary of Theorems

This dissertation gives a procedure for defining a connection on a null hypersurface. Given a Lorentzian manifold (M, g_{ab}) , a null hypersurface given by $(\Sigma = \{u = 0\}, q_{ab})$ and satisfying $\Omega \left(\overleftarrow{R_{ab} - \tilde{R}_{ab}} \right) = k q_{ab}$ has a well-defined covariant derivative given by the conformal pullback method, which can be constructed by the following procedure:

Theorem 6.1: If $\Omega \left(\overleftarrow{R_{ab} - \tilde{R}_{ab}} \right) = k q_{ab}$ where $\Omega = u$, then there exists an f such that $\mathcal{L}_n q_{ab} = f q_{ab}$.

Theorem 5.2: If $\mathcal{L}_n q_{ab} = f q_{ab}$, then there is a conformal transformation $\bar{g}_{ab} = \omega^2 g_{ab}$ such that $\mathcal{L}_{\bar{n}} \bar{q}_{ab} = 0$

Theorem 5.1: If $\mathcal{L}_{\bar{n}} \bar{q}_{ab} = 0$, the connection on Σ given by $D_a = \overleftarrow{\nabla}_a$ is well defined.

8.4 Future Work, Further Questions

What information do the covariant derivatives constructed here provide for null hypersurfaces? What are the implications of using $\overleftarrow{\nabla}$ defined via a conformal transformation? In the case of the null cone, the conformal transformation results in a null cylinder. What does it mean to use ∇_{cyl} on the cone?

All of the examples considered here are axially symmetric. Are there examples of non-symmetric null surfaces on which this construction works? One such class of examples to be investigated are Einstein metrics.

Is there a similar technique for Riemannian spaces? Since the traditional Gauss decomposition works, this technique is not really needed. Furthermore, it is not clear if there even exist analogous conditions, since n is not null. But, suppose there exists a

conformal factor ω leading to a Killing vector field, what would $\bar{\nabla}$ mean in that case?

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