AN ABSTRACT OF THE DISSERTATION OF

Lakana Nilklad for the degree of Doctor of Philosophy in Mathematics Education presented on April 5, 2004.
Title: College Algebra Students' Understanding and Algebraic Thinking and Reasoning with Functions.

Abstract approved: [signature]

Margaret L. Niess

The purpose of study was to investigate college algebra students' understanding of function concepts. In addition, their solution strategies and algebraic thinking and reasoning were explored. Twenty-four volunteer students from one college algebra recitation class participated in the study to access their understanding of functions. Five out of 24 volunteer students were selected to participate in problem-solving interview sessions to provide a rich description of their understandings of functions and their algebraic thinking and reasoning.

A Function Understanding Questionnaire was administered to gather these college students' understandings of functions after they completed the college algebra course. The questionnaire consisted of four questions asking students to identify their understanding of: (1) the definition of function, (2) multiple representations of functions, (3) the use of functions in doing mathematics, and (4) the use of functions in the real-world situations. Formal interviews prior to, during, and after instruction on functions with the five students were conducted, and their work on homework problems, quizzes and tests were explored to clarify these college students' understanding of functions and to explore their solution strategies and algebraic thinking and reasoning while solving problems.

Overall, instruction supported students' understanding of functions. The students' definitions of a function improved toward a more formal definition. In addition, students had a better understanding of multiple representations, function

transformations, and the application of functions to new mathematical situations and to real-world situations after completing the course.

Algebraic reasoning includes the ability to approach and solve mathematical problems in multiple ways. The students in this study were able to use different methods to solve mathematical problems when they were encouraged to do so. However, the instruction did not encourage this activity. Perhaps for this reason, their algebraic thinking and reasoning abilities did not seem to progress as much.

In concert with the recommendation of the several mathematics education organizations, more research needs to deal with the development of algebraic thinking and reasoning. In addition, research involving the communication of mathematical ideas and connection of mathematical understanding, thinking, and reasoning to other mathematics disciplines, to different subject areas, and to real-world situations are recommended.

College Algebra Students' Understanding and
Algebraic Thinking and Reasoning with Functions.

By
Lakana Nilklad

A DISSERTATION
Submitted to
Oregon State University

in partial fulfill of
the requirements for the
degree of

Doctor of Philosophy

Presented April 5, 2004
Commencement June 2004

Doctor of Philosophy dissertation of Lakana Nilklad
presented on April 5, 2004

APPROVED:

Major Professor, representing Mathematics Education

Chair of the Department of Science and Mathematics Education

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorized release of my dissertation to any reader upon request.

Lakana Nilklad, Author

ACKNOWLEDGEMENTS

I would like to express my appreciation to several people who have provided help, support, and endless encouragement throughout this study. I am perpetually grateful to my major advisor, Dr. Margaret Niess, for her wisdom, guidance, patience, and belief in me. Without her assistance I would not have achieved this success.

Special thanks are also extended to my other committee, Dr. Dianne Erickson, Dr. Barbara Edwards, Dr. Larry Flick, and Dr. Bruce Cobentz who have given me valuable advice and support. Also, I would like to thanks Dr. Norman Lederman, Dr. Larry Enoch, and Dr. Edith Gummer for their instruction and direction.

In addition, my deepest appreciation goes to the faculty, staff, and fellow graduate students in both the Department of Science and Mathematics Education and the Department of Mathematics. Appreciation is also extended to the instructor, the Graduate Teaching Assistant, and the College Algebra students who were involved in this study. Without them this study would have been impossible.

I express gratefulness to my family and friends for their support and encouragement throughout my life. My special appreciation goes to two of my best friends, Mary Skoda and Brian Sullivan, who gave me endless help, support, and encouragement.

Finally, acknowledgement is made to the Royal Thai Government for providing me the opportunity and scholarship for my studies at Oregon State University.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

LIST OF FIGURES

Figure | Page

LIST OF FIGURES (Continued)

LIST OF TABLES

COLLEGE STUDENTS' UNDERSTANDING AND ALGEBRAIC THINKING AND REASONING WITH FUNTIONS

CHAPTER I

THE PROBLEM

Traditionally, algebra has served as a gatekeeper (Kaput, 1995; Kieran, 1989; NCTM, 1989) restricting access to further study in mathematics at the college level (i.e. discrete mathematics, calculus, linear algebra). Also, algebra provides a wide range of real world situations and careers because many of its concepts directly support higher-level mathematics courses (McGrone, 1985). Entry to many professional fields requires algebraic knowledge. Employees must be able to apply algebraic concepts as tools for translating problems or situations to mathematical models (Herscovics, 1989). Therefore, algebra provides a foundation for study in many mathematical and scientific disciplines such as engineering, computer science, and other science major areas. Recognition of the use of algebraic ideas along with the thinking and reasoning processes needed for considering their applications has stimulated an interest in assuring that all students have access to the powerful ideas of algebra.

In the 1980's many mathematics educators and teachers claimed that the teaching and learning of algebra did not support students' understanding of algebraic concepts (Kieran, 1989). They indicated that algebraic teaching and learning was focused on symbolic manipulation and calculation including simplification of algebraic expressions, solving equations and inequalities, and using algebraic rules without regard to making any connections to other mathematical knowledge and students' sense-making of their real world. As a result in 1989, several organizations including the National Research Council [NRC] and the National Council of Teachers of Mathematics [NCTM] called for a reform in the algebra curriculum to enhance an emphasis on algebraic thinking and reasoning within a functional approach to algebra.

By 1994, the NCTM's Algebra Working Group recommended another change in the algebra curriculum, a change to promote learning for understanding through the development of algebraic thinking and reasoning across the grade levels. These reform efforts proposed to encourage more students to achieve success with algebra by decreasing the emphasis on symbolic manipulation and computation while promoting a better understanding of algebraic concepts, emphasizing algebraic thinking, and reasoning (Lacanpagne, Blair, & Kaput, 1995; NCTM, 1991, 2000). The proposed reconstruction of the mathematics curriculum, to provide greater support for the development of algebraic thinking and reasoning, required students to have earlier access to a variety of tools for representing algebraic ideas and an opportunity to apply those ideas in many contexts as their skills and understandings grew (Kaput, 1995).

With the NCTM recommendations (2000) in the *Principles and Standards for School Mathematics,* an algebra-for-all curriculum focused on students at all grade levels and development of their algebraic thinking and reasoning. In these standards, it was stated that algebraic instructional programs should enable all students to "understand patterns, relations, and functions and represent and analyze mathematical situations and structures using algebraic symbols" (p. 296). Additionally, the NCTM Standards (2000) proposed a reasoning standard in the school mathematics curriculum as:

> Reasoning and proof should be a consistent part of students' mathematical experiences in pre-kindergarten through grade 12. Instructional programs from pre-kindergarten through grade 12 should enable students to
>
> - recognize reasoning and proof as fundamental aspects of mathematics;
> - make and investigate mathematical conjectures;
> - develop and evaluate mathematical arguments and proofs;
> - select and use various types of reasoning and methods of proof. (p. 56)

Application of this reasoning to all mathematical concepts and understanding including algebra proposed to support students in understanding mathematical concepts, recognizing connections and relationships between concepts, and applying this understanding to new problems and daily situations.

At the college level, the American Mathematical Association of Two-Year Colleges [AMATYC] (1995) recommended three standard categories in the *Crossroads* *in Mathematics: Standards for Introductory College Mathematics before Calculus:* intellectual development, content, and pedagogy. Intellectual development addressed reasoning indicating that students are expected to expand their thinking and reasoning skills while they develop convincing arguments and explore the meaning and role of mathematical identities provided in multiple representations (i.e. graphical, numerical (tabular), symbolic). The content standard emphasized algebraic symbolism, stressing that students are expected to develop algebraic thinking and reasoning by translating problem situations into symbolic representations and use those representations to solve problems.

For years the algebra curriculum has been debated. Mathematics educators and researchers have focused on a central topic (NCTM, 1989) and all have agreed that function concepts are "gatekeepers" to the majority of college mathematics (Kaput, 1995; Kieran, 1989). The NCTM (1989) called for the inclusion of function-related activities as early as fourth grade, continuing to a higher-level mathematics curriculum. The NRC (1989) stated in *Everybody Counts* "if undergraduate mathematics does nothing else, it should help students develop function sense" (p. 51). The AMATYC (1995) claimed that students at the college level needed to demonstrate understanding of the concepts of functions using multiple representations including numerical (tabular), graphical, symbolic, and verbal.

Although functions are central concepts in algebra, many research studies of high school and college levels have shown that these concepts are some of the most difficult for students to understand. For example, many students have difficulty translating functions among different representations (Sierpinska, 1992) as well as applying basic algebraic concepts at different levels (Leinhardt, Zaslavsky, & Stein, 1990). In addition to the students' difficulties identified by several researchers, many educators also indicated that teaching and learning algebraic concepts in a traditional method that emphasized computational skills does not support students in understanding those concepts. Therefore, the shift from an emphasis on computational skills to an emphasis on thinking and reasoning skills was considered a key aspect of the current curriculum reform of all branches of mathematics education including algebra (NCTM, 2000).

The importance of fostering algebraic thinking and reasoning has been widely documented (Kaput, 1995; NCTM 2000). Yet, a shortage of research exists that focuses on the role of these thinking and reasoning processes. This specific area of thinking and reasoning, algebraic thinking and reasoning, is defined similarly. Langrall and Swafford (1997) described algebraic thinking and reasoning as "the ability to operate on an unknown quantity as if the quantity is known" (p. 2). Driscoll (1999) provided a definition for this view of algebraic thinking and reasoning as "the capacity to represent quantitative situations so that relations among variables become apparent" (p. 1). Whereas, Herbert and Brown (1997) described algebraic thinking and reasoning as follows:

> the use of mathematical symbols and tools to analyze different situations by (a) extracting information from a situation, (b) representing that information mathematically in words, diagrams, tables, graphs, and equations, and (c) interpreting and applying mathematical findings such as solving for an unknown, testing conjectures, and identifying functional relationships to the same situation and to new, related situations. (p. 340)

The study of functions in algebra is an area where algebraic thinking and reasoning is critical because the importance of translation among representations is an essential basis for several mathematical topics. For example, with respect to a linear function, students are expected to connect a graphical representation, a numerical (tabular) representation and a symbolic representation of the function. Numerous student misconceptions of functions have been documented. For instance, students hold misconceptions that functions are only linear, or that they are only continuous, (Becker, 1991; Slavit, 1994), students thought that a point or points on the Cartesian coordinate system were not functions. Several students accepted only one-to-one functions as functions and refused to recognize many-to-one representations as functions. They also did not consider constant functions and piecewise-defined functions as functions. (Vinner, 1983, 1992; Selden & Selden, 1992).

In order to develop the algebra curriculum and instructional strategies, students' development of algebraic concepts and misconceptions must be considered. For example, if all the examples that teachers use in discussing functions are only linear, students may incorrectly equate the concept of functions with linear functions. As Noddings (1990) noted, "In order to teach well, we need to know what our students are thinking, how they produce the chain of little marks we see on their papers, and what they can do (or want to do) with the material we present to them" (p.15). To support students' progress in studying algebra, an investigation must consider how they understand algebraic concepts by including their thinking and reasoning with particular algebraic concepts such as functions.

Statement of the Problem

Function concepts are both focal points and unifying ideas in the study of advanced mathematics. It is important for students to thoroughly understand functions before they begin calculus and higher-level mathematics courses. Nevertheless, studies of the teaching and learning of functions in both national and international contexts have indicated that the general level of students' understanding of functions is not high. The report on the students' performance on algebra and functions in the 1996 NAEP assessment indicated that students

> Although there was some progress in 1996 in exceeding the 1990 and 1992 performance levels, many students still do not perform well on the [algebraic] topics the usually receive the most emphasis in algebra courses. (Blume & Heckman, 2000, p. 298)

A comparison of this assessment result with the earlier NAEP assessments indicated that students' understanding of the function concepts and their ability to apply the concepts had not significantly progressed over time (Carpenter, Coburn, Reys, & Wilson, 1975; Carpenter, Corbit, Kepner, Lindquist, & Reys, 1980; Carpenter, Lindquist, Mathews & Silver, 1983). In addition, the results of the Second International Mathematics Study [SIMS] also suggested that students had limited understanding of function concepts (McKnight et al., 1985). Besides these reports, many research studies have indicated that students have difficulties in understanding functions (Bergeron & Herscovics, 1982; Even, 1990, 1992; Sierpinska, 1992; Vinner, 1983) including difficulties in making connections among different representations (i.e. connection among formulas or symbols, graphs, diagrams, and word descriptions of relationships), difficulties in making a translation between representations, difficulties in interpreting graphs of functions, and difficulties in manipulating symbols related to functions (Carlson, 1997; Sierpinska, 1992; Zaslavsky, 1997).

To understand the functions, students need to connect the concepts of functions that they have. Tall (1992) indicated key aspects of functional understanding not only include a basic understanding of functions (each value of x corresponds to precisely one value of y) but also an understanding of:

- variable relationships (functions vs. relations, dependent and independent variables)
- functional representations (i.e. graphical, tabular, and symbolic)
- functional manipulation or procedures (i.e. function operation, function composition)
- functional process (input and output process).

To assess students' understanding about functions, all aspects related to understanding characteristics described above must be considered.

The literature related to the teaching and learning of algebra has implied that students leave algebra courses with an inadequate understanding of functions (Beckers, 1991; Selden & Selden, 1992; Slavit, 1994, Vinner, 1983). Furthermore, Hauger (1995) claimed that those students encountering difficulty with calculus do so because of their lack of knowledge of functions. Students' lack of functional knowledge and their misconceptions caused them difficulty in forming and manipulating relationships between quantities. In addition, a lack of an ability to translate a function among its various representations also created students' difficulties in understanding functions. Confronted with this evidence, mathematics teachers should frequently ask themselves "What is causing the problems?" In most cases, discussions around this question eventually lead to discussions around others: What should students know and be able to do in algebra and what is the best way to instill this algebraic knowledge?

A number of researchers in the area of mathematics education have been interested in the teaching and learning of several algebraic topics such as equations, inequalities, and algebraic word problems. They have also indicated an interest in topics related to the understanding of functions at the secondary and college levels. Studies conducted in countries such as Australia, England, Israel, and the United States used questionnaires to gather information on function conceptions and misconceptions, as well as in-depth interviews to identify students' misconceptions about functions. Ongoing research of students' understanding of function concepts and investigations of what students are thinking while attempting particular algebraic problems (i.e. problems involving function concepts) and how they develop their thinking in ways that enhance students' understanding of functions (Haimes, 1996; Noddings, 1990) are needed. And yet it is students' understanding of algebraic concepts such as functions that keeps them from continued learning in mathematics. This research is important in addressing knowledge of functions as a gatekeeper to advanced mathematics.

Several studies have indicated students' difficulties and misconceptions about functions (Barnes, 1988; Carlson, 1997; Demarois & McGowen, 1996). Their lack of adequate knowledge and understanding of functions causes them difficulties in higher-level mathematics courses. However, knowing only students' difficulties and misconceptions about functions does not provide the reasons for their misconceptions. To understand how students have constructed or developed those misconceptions, understanding what is constructed in their heads or what they are thinking while attempting the problems is essential, particularly with functions requiring translation among representations. Identifying how students reason about functions requires more than an analysis of the algorithms they use for solving the problems. This information does not show students' thinking and reasoning processes. In order to facilitate students' learning in more empowering ways, it is essential to understand their thoughts (Steffe, 1991). One of the methods to help understand the processes students use in thinking and reasoning is a thinking aloud strategy; this strategy was called thinkback by Lochhead (2001). Asking students to think out loud as they solve mathematical function problems offers a means of getting at their thinking and reasoning processes.

Even though several researchers in the last two decades have emphasized the importance of thinking and reasoning processes in mathematics classrooms (House, 1999; Russell, 1999), there is little information involving students' mathematical thinking and reasoning at the college level, particularly related to students' algebraic thinking and reasoning. Thus, this study was conducted to explore college students' thinking and reasoning as they learn about functions. More specifically this study investigates how the students used their understanding to extract information from a situation, represent that information in multiple forms, and interpret and apply the findings to new situations. The college level is the focus for the study because function concepts are essential for more advanced college mathematics courses at this level. The specific research questions addressed in this study are:

1. What is college algebra students' understanding of functions?
2. What solution strategies and thinking and reasoning processes do college algebra students use as they attempt mathematical problems involving functions?

Significance of the Study

Mathematical reasoning is an ongoing focal point for teaching and learning mathematics in all grade levels. NCTM (2000) has provided content and process recommendations for teaching and learning school mathematics in its publication *Principles and Standards for School Mathematics*. In this recommendation, algebra, including functions, is one of the five main content standards, and reasoning is one of the five process standards. Reasoning in mathematics, including algebraic reasoning, is an important skill needed in mathematics, particularly from pre-kindergarten through college levels. Additionally, it has been claimed that the ability to reason is essential to understanding mathematics in all grade levels (NCTM, 2000).

Without adequate information about how students construct and understand meaning for particular algebraic concepts, function concepts in this individual study, mathematics educators and teachers have difficulty designing effective instructional tasks to support and develop students' knowledge and understanding of those concepts. Algebraic thinking and reasoning suggests an individual's engagement in making sense of some facets of algebraic ideas. Specifically, knowing what concepts of functions students possess, what they are thinking while working on problems associated with functions, what successful and unsuccessful strategies they attempt function problems, and what their reasons are for using such strategies provides teachers with important guidelines for developing instructional practices. This knowledge enables them to develop alternative ways of teaching to help students challenge their misconceptions of functions.

CHAPTER II

REVIEW OF THE LITERATURE

Introduction

The goal of reforming algebra in school mathematics at all grades, from pre-kindergarten through college levels, has been a major challenge in the algebra curriculum for years (Kieran & Wagner, 1989). Within the algebra curriculum, over the past century the content of algebra has changed little, while teaching and learning processes have gradually changed from teacher-centered to student-centered and from symbolic and artificial problems to real world situation problems. During the 50 years before the NCTM's recommendations (1989), the teaching of algebra in various schools was almost uniform. That is, algebra was traditionally taught through expression simplification and word problems, where the actual problems seldom changed over the years. In addition, the problems lacked real-life applications, were arbitrary, and artificial. However, the main purpose of algebra instruction has changed from computational skills to the development of algebraic conceptual understanding and thinking and reasoning skills regarding NCTM's recommendations.

Kieran (1992) distinguished two approaches for teaching and learning algebra: *procedural* and *structural*. Kieran stated that "Procedural refers to arithmetic operations carried out on numbers to yield numbers.... structural, on the other hand, refers to a different set of operations that are carried out, not on numbers, but on algebraic expressions" (p. 392). Traditionally, the procedural approach to algebra is presented by providing a few examples and exercises that can be solved by substitution of an algebraic expression or by using arithmetic techniques. However, this deception is dropped when expressions are to be simplified and equations are to be solved by structural approaches (Kieran, 1992).

Focusing on mathematical understanding, Skemp (1987) defined and described two types of understandings: *instrumental understanding* and *relational understanding.*

He described instrumental understanding as an understanding of the processes required to solve a particular problem. The processes might be simple algorithmic applications, without a clear concept of why the applications work. Relational understanding, according to Skemp, incorporated a deeper understanding of solution processes and relationships of the relevant concepts. In 1987, Skemp revised his definitions of understandings to include a new type of understanding he named *formal understanding.* Formal understanding was defined as "the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning" (p. 166). In addition Skemp indicated that "To understand something means to assimilate it into an appropriate schema" (p.29). Schema, for Skemp, was a group of connected concepts, each of which were formed by abstracting invariant properties from other concepts. To develop ability in mathematical thinking and reasoning, the three types of understanding, instructional, relational, and formal, are necessary.

Pirie and Kieran (1989) described mathematical understanding as follows:

> Mathematical understanding can be characterized as leveled but nonlinear as recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication.... Indeed each level of understanding contained within succeeding levels. Any particular level is dependent on the forms and processes within and further, is constrained by those without. (p. 8)

Pirie and Kieran (1994) also stated "mathematical understanding is a process, grounded with a person, within a topic, within a particular environment" (p. 39).

Knowing how to calculate numbers does not lead anyone to understand mathematics. In this similar view, Kaput (1995) stated that calculation skills in mathematics study are insufficient; in addition, students must be able to reason effectively. Algebra is considered one strand of the mathematics that facilitates reasoning about relationships within problematic situations. Kaput also recommended that algebra should be re-conceptualized as a strand woven through many grade levels, serving as a sense-making tool in elementary through higher levels. In the *Curriculum and Evaluation Standards for School Mathematics,* NCTM (1989) recommended that experiences with patterns and relationships provide a basis for algebraic concepts in elementary school level and can be extended to focus on analyses, representations, and generalizations of functional relationships in a secondary school level. Agreeing with this idea, Philips (1995) stated that, as students learn relationships among the quantities in patterns, they gradually construct knowledge about important mathematical concepts, such as functions, learn to reason, and communicate about the important content and processes of algebra.

The purpose of this chapter is to provide a review of research concerning students' understanding of function concepts, their difficulties and misconceptions in learning function concepts in algebra classrooms, as well as to provide views of students' algebraic thinking and reasoning strategies. Since formal algebra has been taught from secondary school through college levels, most of the studies related to functions reviewed in this section included both secondary and college levels. The main concepts in the chapter are organized as follows:

- Development of function understanding;
- Difficulties and common misconceptions in learning function concepts;
- Students' mathematical and algebraic reasoning.

Development of Function Understanding

Function concepts are not new in learning mathematics as Markovit, Eylon, Bruckheimer (1986) mention in Godfrey's words written in 1912:

> The fact is that we have been teaching functionality for years, whether we have realized it or not. Every schoolboy now learns to plot graphs; this is nothing but the study of functionality in its visible form. (p. 18)

Understanding of function concepts contains many aspects including being able to apply the concept in fields other than mathematics and use the concepts in different contexts within mathematics itself. Markovits, Eylon, Bruckheimer (1986) stated that there were two states of understanding: passive, such as classifying and identifying; active, such as doing something or giving examples. To understand how students understand function concepts, Markovits, Eylon, and Bruckheimer constructed algebraic problems related to graphical and symbolic representations of functions and gave these problems to 400 students in ninth- grade algebra classes after they had studied the relevant concepts. Their study investigated students':

(1) ability to classify relations into functions and nonfunctions. The findings showed that students correctly distinguished relations and functions. However, some difficulty was experienced when the relations were given in a symbolic representation, particularly when the relation was a constant function or a piecewise-defined function. About half of the students reasoned the correspondence was not a function because "every preimage has more than one image" (p. 20). From these reasons students seemed not to realize that different constraints of a correspondence applied to different parts of a function domain.

(2) ability to give an example of a relation that is a function and one that is not. The researchers said that the students performed the problems reasonably well. About half of the students gave examples graphically. Some students had difficulty with many-to-one and one-to many relationships.

(3) ability to identify preimages, images and (preimage, image) pairs for a given function. The findings indicated that students understood that points on a curve of a graphical representation represented (preimage, image) pairs of the function and points not on the curve did not. In the symbolic representation, most of the students used the correct procedure to identify a preimage by checking whether the number belonged to the domain. To identify if a given number was an image of a function, few students used the three steps correctly: checking if the number belonged to the range, calculating the preimage, and checking if this preimage belonged to the domain. About half of the students only checked whether the number belonged to the range.

(4) ability to find the image for a given preimage and vice-versa. The findings indicated that more than half of the students did not realize that when the function was given in graphical form, preimages were located on the x-axis and images on the y-axis. However, when the function was given in a symbolic form, the students were able to find the image for a given preimage, but they had some difficulties with the reverse.

(5) appreciation that the same function can be represented in several forms and the ability to identify identical functions. The students understood that a function could be represented in several forms, and most of them used symbolic and graphical forms to represent a given function. Some difficulties occurred when they were asked to identify functions identical to a given function.

(6) ability to transfer from one representation to another. The finding showed that students lacked this ability. Less than a third of the students answered the problems correctly. The findings also showed that when the function was familiar, students had more difficulty in transforming a graphical form to a symbolic form than from symbolic to graphical.

(7) ability to identify functions satisfying given constraints. The finding showed that students had difficulty in understanding that the set of images of a function could be a subset of the preimages.

(8) ability to give examples of function satisfying some given constraints. The finding showed that with a graphical form, most of the students gave a correct function, while from a symbolic form, correct answers were provided only if the constraints allowed familiar functions.

Breidenbach, Dubinsky, Hawks, and Nichols (1992) revealed that college students, even those who took an adequate number of mathematics courses including the calculus sequence, did not have a strong understanding of function concepts. They observed 62 college students, mainly sophomores and juniors preparing to be high school, middle school and elementary school teachers. They investigated students' understanding of functions in the discrete mathematics course from three situations. The first situation was called an ordinary school environment. The students were asked to respond to the question, "What is a function?" then they were asked to give examples of a function. The students were asked to respond to the same question, "What is a function?" then to give three different types of examples within a computer environment. The third situation, an instructional treatment, included computer experiences, small group problem-solving, and discussions addressing functions. The researchers interpreted students' understanding of functions in three categories: prefunction, action, and process. They defined prefunction when a student did not demonstrate a concept of function. They said students had a process concept of functions if they were able to "think about the transformation as a complete activity beginning with objects of some kinds, doing something to these objects, and obtaining new objects as a result of what they have done" (p. 251).

The results of this study indicated that most of the students had a prefunction concept at the beginning of the course. After some time in the general computer environment, they progressed from prefunction through action to process function concepts. The researchers claimed that the instructional treatment had helped students construct an understanding of function concepts.

Difficulties and Misconceptions in Learning Function Concepts

Of all algebraic content taught in the mathematics classroom, functions are considered to be one of the most important topics, a topic that permeates every branch of mathematics and occupies a core position in the development of mathematics understanding. However, research indicates that students have a low understanding of the concepts of functions.

Numerous recommendations concerning mathematics understanding have been based on the difficulties students encounter while learning and solving problems involving mathematical function concepts. Several mathematics education committees such as the National Committee on Mathematical Requirements of the Mathematics Association of America have recommended a study of functions to begin in secondary school mathematics (Cooney & Wilson, 1993). The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) also called for the inclusion of function-related activities starting as early as elementary school. In addition, many mathematics educators claimed that function concepts are important in modern mathematics and essential for relating mathematics, science and technology (Leinhardt, Zaslavsky, & Stein, 1990; Selden & Selden, 1992). Breidenbach et al. (1992) claimed that an understanding of function concepts provided a necessary background enabling students to better understand calculus and advanced mathematics.

Several research studies have responded to the calls for the algebraic curriculum reforming. Many of those studies revealed common misconceptions and difficulties that secondary and college students experienced in comprehending the concepts of functions. Leinhardt et al. (1990) defined student misconceptions as "incorrect features of student knowledge that are repeatable and explicit" (p. 30). They noted that misconceptions about functions and graphs were often correlated to previous mathematical conceptual learning. Consequently, they stated that "the function concepts may be limited because of a lack of variety of instructional examples, or a translation may be performed inaccurately because of confusion over symbolic notation" (p. 30). This study considered the research related to studies of students' misconception about function because they wanted to help mathematics teachers and educators understand students' development process of function concepts. The students' function misconceptions and limited understanding of functions were related to their cognitive process of learning function (Rho, 2000).

Several research studies have focused on determining common misconceptions that students have about functions. In order to explain students' understanding and misconception of functions, Markovits et al. (1988) investigated ninth- and tenth-grade students' difficulties and misconceptions with linear functions. The findings indicated that these students had difficulty with the terminology associated with functions (i.e. preimage, image, domain, range, and image set). The students also had difficulties distinguishing between the range and the image sets. Several students disregarded the specified domain and range entirely and assumed the domain and range to be the whole set of real numbers. These students did not immediately understand the connection between the components in the written definition (i.e. domain, range, and rule of correspondence) and the corresponding components in the graphical representation. Constant functions, functions represented by disconnected graphs, and piecewise-defined functions were also not clearly understood. Many students did not think that those types of functions were functions. In some cases, students' difficulties related to the misconception that all functions had to be linear, possibly because of the extended time studying linear functions; and numerous students had a misunderstanding that linearity was a property assumed by all functions.

Another study related to tenth- and eleventh-grade students' conceptions of functions was conducted by Vinner (1983). In this study all participants had previously studied functions. The study indicated the function concept images held by the students were:

- A function had to be given by one rule. If two rules were given using two disjoint domains, then there were two functions.
- A function could be given by several rules relating to disjoint domains, provided these domains were intervals. However, a correspondence with one exception point was not considered a function.
- For every y in the range, the function had only one x in the domain corresponding to that y.
- A function was only a one-to-one correspondence.

In addition to the studies investigating types of misconceptions of functions, studies involving students' understanding of functions were conducted. For example, Carlson (1997) designed a study to guide mathematics teachers and curriculum developers by providing insights on how high ability college algebra students developed their understanding of the main aspects of the concepts of functions. The subjects of this study consisted of 30 students who had just completed a Function-Integrated College Algebra course with an "A" grade. A 25-problem written exam, covering many concepts of functions, taught in the college algebra course, was developed for measuring students' understandings of functions. The follow-up interview questions were designed to determine how students gained specific function knowledge.

Of all students, five of them who performed at different levels on the written exam were scheduled for follow-up interviews. They were asked to verbally describe their written responses, and provide clarification and justification for their solution strategies. Both quantitative and qualitative results were provided. The researcher concluded that high-performing college algebra students had limited understanding of the function concepts. They had little understanding of function language and were unable to use function notations to represent real world relationships. During the interviews, the students could not express a quantity as a function of another and were unable to verbalize the meaning of $f(x + a)$ when a function $f(x)$ was given. Although these students were able to algebraically evaluate functions for specific inputs, construct graphs of simple algebra functions, and interpret points on a graph, they demonstrated an inability to interpret graphical function information for intervals of the domain.

Furthermore, the analysis of the interview results showed that these students viewed the evaluation of a function as nothing more than a set of memorized steps. They did not view a function more generally as processes; rather they viewed them as a sequence of procedural operations to be completed. They did not understand graphical solution(s) to the equation $f(x) = g(x)$, nor algebraically constructing an equation by equating the expressions for the two functions. From the analysis, the researcher summarized that these students did not:

- understand the language of functions
- know how to represent real world relationships using algebraic and graphic function representations
- know how to interpret graphical information for intervals of the domain
- know how to interpret graphical information representing "rate of change"
- understand the general nature of a function. (They mistakenly thought all functions must be definable by a single algebraic formula)
- understand the role of the independent and dependent variable in an algebraic function representation
- distinguish between "solutions of an equation" and "roots of the function." (Carlson, 1997, p. 56)

The researcher claimed that even the most able students, at the completion of college algebra, still had many misconceptions and were unable to access much of the information explicitly taught during the course.

Tall and Bakar (1992) explored students' visual images about functions. The researchers used the term "mental prototype" to describe the visual images that students had about functions. The researchers investigated high school students' understanding of functions with a group of "A-level" mathematics students in England. The students were asked to:

- explain what they thought a function was in a sentence and give a definition of a function if possible.
- determine whether each given sketch represented a function, and if not, use a diagram to demonstrate their reason.

These same tasks were also given to 109 first year university students (the level of outstanding was not provided), in addition to the following task:

- Determining if the given symbolic expressions or procedures represent y as a function of x.

The results showed that none of the high school students gave satisfactory definitions, but all gave explanations, including the following:

- A function is like an equation which has variable inputs, processes the inputted number and give an output;
- A "machine" that will put out a number from another number that is put in;
- An expression that gives a range of answers with different value of x;
- A form of equation describing a curve/path on a graph;
- A way of describing a curve on a Cartesian graph in terms of x and y coordinates;
- An order which plots a curve or straight line on a graph;
- A mathematical command which can change a variable into a different value;
- A set of instructions that you can put numbers through;
- A process that numbers go through, treating them all the same to get an answer;
- A process which can be performed on any number and is represented in algebraic form using x as a variable;
- A series of calculations to determine a final answer, to which you have

submitted a digit;
- A term which will produce a sequence of numbers, when a random set of numbers is fed into the term. (Tall & Bakar, 1992, p. 40-41)

Figure 1 shows the graphs used in the questionnaire that were presented.

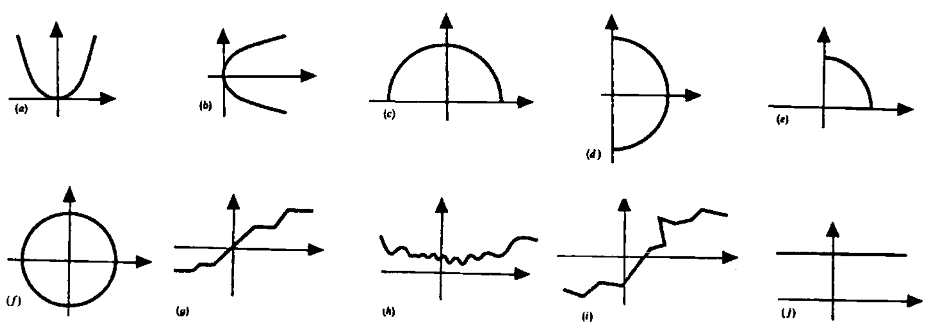


Figure 1. Graphs in questionnaire used in Tall and Bakar's study.

The results indicated that all of the high school students and 97% of the university students believed graph (a) was a function. About half of high school and university students answered the rest correctly except graph (b), in which about 91% of the university students gave a correct answer. When functions were presented symbolically, the university students were asked to state which of a number of symbolic expressions or procedures could represent y as a function of x. The results showed that 38 of the 109 students explicitly mentioned, at least once in their responses, that for each x there must be one y and almost half of the students stated that a constant such as $y = 4$ was not a function.

In Vinner's (1983) study, students who thought of functions as involving a single formula considered a piecewise-defined function as several different functions rather than as one function. The difficulties the students faced also seemed to be the "strangeness" of the expressions and the fact that the presentations did not fit the students' mental prototypes. Comparing student performance on the equation $y = 4$ and the graph of $y = constant$, the researchers found 28% of the students responded correctly in the affirmative to both questions; 41% of the students responded negatively to both questions; 29% of the students stated that the graph corresponded to a function but the symbolic representation did not.

Another study related community college students' understandings of function concepts was conducted by DeMarois and McGowen (1996). The purpose of this study was to explore students' understandings of the definition of a function and the notational aspect of the function concepts. Written questions focused on students' interpretation of function notations, were used as a pretest and a posttest. In the pretest, none of the questions on functions was included, because it was assumed the students had not previously experienced the function concepts. The posttest focused on students' interpretations of function notations. The results indicated that students had little understanding of symbolic function notations. A large percentage of students were able to identify $f(x)$ as a function notation, while only a small percentage of the students were able to distinguish between $af(b)$ and $bf(a)$ function notations.

Investigations of function concepts have also focused on specific aspects of functions. The concept of the x-intercept could be considered as one of the common misconceptions or misunderstandings in learning about functions. Moschkovic (1999) conducted a specific study to investigate first year algebra students' (ninth and tenth graders) use of the x-intercept in equations of the form of $y = mx + b$.

The study focused on mathematical processes, rather than on results or answers and supported group-work, as well as encouraged students to discuss their ideas with others. In addition to videotape recording of discussions, written assessments (a pretest and a posttest) were also used to explore the students' use of the x-intercept in the linear function. A 30-item written assessment asked students to predict what would happen to the line $y = x$ graph on a coordinate grid, if the equation was changed or what they would do to a line ($y = x$) to graph a second line on a coordinate grid.

The responses on the pretest showed that 13 out of 18 students used the x-intercept when working with equations in the form $y = mx + b$. Twelve students used the x-intercept in place of b and five students used it in the place of m in the equation they generated. Six of the students described lines as moving left to right (or right to left) along the x-axis as a result of changing the value of b in an equation. Responses involving the use of the x-intercept were coded as graphical and algebraic uses. Algebraic uses of the x-intercept were coded as use of the x-intercept either for parameter m or for parameter b. The categories of the x-intercept used are described in Table 1:

Table 1
Categories of the x-intercept (Moschkovic, 1999, p. 177)

Categories	Description
x graphical intercept	A student described a line as moving 'left or right' or 'on the x-axis' as a result of a change in b in an equation.
x-intercept for b	A student used the x-intercept of a line for b in an equation or responded that the number b in an equation corresponded to the x-intercept of a line.
x-intercept for m	A student used the x-intercept for m in an equation or responded that the number m in an equation corresponded to the x-intercept of a line.

The results revealed that 14 students (72%) showed at least one instance of any use of the x-intercept (graphical and algebraic) and 10 students (55%) showed two or more instances of any use of the x-intercept. Students' answers and explanations for the written assessments showed that students used this conceptual understanding in different settings. The students' explanations showed that their responses involving the use of the x-intercept were not due to carelessness, but reflected an underlying conception. The response on the posttest showed that the use of the x-intercept was robust, because students continued using the x-intercept even after participating in the discussion sessions. An analysis of the videotapes showed that the use of the x-intercept was refined in the following ways:

- Students moved from using the x-intercept for b when $m = 1$ to using the x-coordinate of the x-intercept as b in the equation, to using the opposite of the x-coordinate of the x-intercept as b.

- Students came to use the x-intercept only in the contexts in which it was applicable. For example using the opposite of the x-coordinate of the x-intercept for b only when $m = 1$.
- Most of the students refined their description of lines so that they focused on vertical, rather than horizontal, translation as a result of changing b. (Moschkovic, 1999, p. 178-179)

The two case studies discussed in this study showed that students' use of the x-intercept was context dependence. Using the opposite of the x-intercept was productive for generating the algebraic y-intercept when $m = 1$, while it was not applicable when $m = 2$. This study also illustrated that students' use of the x-intercept was part of the process of making sense of the connections between the two representations: symbolic and graphical.

The above studies demonstrate that both secondary school and college students in general have not succeeded in developing a good understanding of functions. The studies indicated a number of students at these levels had similar misconceptions about functions such as: a function represented by a disconnected graph was not a function; a function had to be one-to-one (students thought that neither constant nor piecewise-defined functions were functions), and that all functions were linear. Some studies indicated that secondary school students experienced difficulties with: terminology associated with functions, definitions of functions (the students were given more than one definition of functions), function notations, and function representations. For the specific content of functions, a study by Moschkovic (1999) showed that secondary students' use of the x-intercept was a robust conception because the form of the function students often used, $y = mx + b$, does not address the x-intercept.

Students' Algebraic Reasoning and Solution Strategies

"Mathematical reasoning must stand at the center of mathematics learning" (Russell, 1999). And mathematics teachers should know that students learn mathematics from reasoning to developing sense-making and justification at elementary school level, using thinking and reasoning to make conjectures and apply inductive and deductive reasoning at middle school level, and using thinking and reasoning form, validate, and prove mathematical assertion at high school level (NCTM, 1989). In addition, NCTM (2000) stated that teaching and learning at all grades should enable students to select and use various types of reasoning to develop their mathematical concepts.

As stated earlier teaching and learning of algebra in the schools did not significantly change until the NCTM's recommendation in *The Standards and Evaluation for School Mathematics* (NCTM, 1989). The standards called for students to justify their answers and solutions, to make and evaluate mathematical conjectures, to use counter examples efficiently, to draw conclusion from deductive and inductive reasoning. For a specific branch of mathematics, algebra, algebraic reasoning includes "engaging in these activities with planned or unplanned use of, or conclusion about, properties of, uses of, and operations on algebraic entities (variable, equation, function)" (Zbiek, 1998, p. 35). Following the NCTM's recommendation, the algebraic curriculum shifted directions. In the past, algebra evoked an emphasis on computational skills such as solving equations and inequalities. However, this perspective began to change after the publication of *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). Moreover, *Principles and Standards for School Mathematics* (NCTM, 2000) challenged that algebraic thinking and reasoning is important at all grade levels. In order to develop the role of algebra in the mathematics curriculum, the Mathematical Science Education Board [MSEB] and the NCTM co-sponsored a national symposium "*The Nature and Role of Algebra in the K-14 Curriculum*" (NCR, 1998). The primary goals were to (a) promote an informed dialogue on issues concerning the K-14 algebra curriculum, (b) provide examples of students' algebraic thinking and reasoning to synthesize research and to consider how these factors impact algebra in school mathematics, and (c) provide a forum for those involved in algebra-related curriculum projects at elementary school through postsecondary school levels to share their visions of curricula, teaching, and assessment.

Algebraic reasoning has frequently been referred to as an important form of mathematical reasoning. A few research studies were conducted with the concepts of functions in general and some others investigated more specific students' concepts. For example: McDermott, Rosenquits, and van Zee (1987) conducted a study related to college physics and mathematics students that involved making connections between velocity and position represented as graphs. The researchers reported on one particular task that asked students to use the graph of the position of two different objects (A and B) to decide which one was moving faster. Students responded that B was moving faster than A, because on the graph B was higher than A. However, in actuality object A was moving faster than B throughout the entire time since graph A had a greater slope than the graph of B. Another study presented by Clement (1989) was conducted with college mathematics major students. They were presented a graph of the velocity of two moving cars. The students were given graph of speed vs. time for two cars (Figure 2).

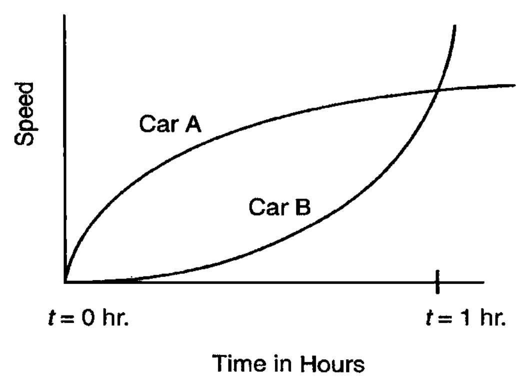


Figure 2. Graph of speed vs. time of two cars.

The findings of this study indicated that the students were able to read the speed of the cars at the specific time, but when they were asked to describe the relationship between the cars' positions at t = 1 hour, the students stated that Car B was passing Car A or that the cars were next to one another at that certain time. In other words, students used the fact that these two cars were at the same points on the velocity graph to conclude that the cars were at the same position.

It is also suggested that multiple representations provide an environment for students to understand mathematical and algebraic concepts. Therefore, it is necessary to understand how students connect and use these representations. Ozgun-Koca (1998) conducted a study designed to investigate students' thinking and their preferences while choosing a type of representation for solving mathematical problems and to help teachers see some effects of students' thinking and reasoning processes when they were dealing with mathematical representations. Fourteen freshman college students in a remedial mathematics class were observed. One of the observations was set for students to work in groups with four activities. For the first part, students were presented a problem with no suggestion related to the representations. The purpose of this part was to see which type of representations students would choose to solve the problem. For the second part, the same problem was presented graphically to students and they were asked questions related to the graphical representation. The third part of the activity was the same for a tabular representation. In the final part, the problem was represented in symbolic form.

When asked about their preferences of function representations, students had reasons for using each representation. For example, to reach the only exact answer, the symbolic representation made them comfortable and confident. However, there was a student who was comfortable with having many possible answers, he stated that "[tables] show you *many possible solution*s [Italic added] to one problem that helps you graph the problem" (Ozgun-Koca, 1998, p. 12). Since tables showed information in a more organized way, some of the students selected tables as their choice of representations. The common reason for using graphical representations was the visual benefit, since graphs made it possible for them to see how functions behaved.

Multiple reasons affect students when selecting and using particular mathematical representations. To enable students to experience different representations and use the representation that is the most meaningful for them, teachers need to provide an environment with multiple representations instead of favoring a particular representation (Eisenberg, 1992).

Heid, Hollebrands, and Iseri (2002) conducted a study focused on a seventh-grade student, Kevin, working on a problem involving functions in technological environment. The functional problem that Kevin dealt with was more difficult than any other problems that he experienced in his mathematics class. Kevin was asked to construct functions that met five constraints (see Appendix I). First, he needed to construct a function that met the first constraint, then to modify the function so that it met the first two constraints, and so on. Working on this problem, Kevin was allowed to use a computer algebra system (CAS). The results of this study indicated that Kevin was able to work on the problem that he had not previously encountered. He was able to examine the conclusions that he had reached with the CAS. Generally, he used his understanding of the concepts of functions such as zeros of a function, a non-negative function and a domain of a function. In addition, Kevin was able to broaden his thinking and reasoning about functions through his use of the technology. He connected the graphical and symbolic representations through the use of technology tool. Ultimately, his thinking about the mathematics was supported by the technology in that he was able to use the technology to clarify and understand the ideas through a variety of mathematical representations to interpret and solve problems.

Zbiek (1998) explored the solution strategies used by 13 preservice secondary school mathematics teachers to develop and validate functions to real-world situations. The study indicated that at the beginning of the class (first three weeks of the 15-week course), the preservice teachers used previously proposed models of real-world situations and computing tools, such as curve fitting, function graphers, and symbolic manipulators, to expand their understanding of functions. They also created and manipulated multiple representations of functions to answer real-words phenomenal problems. During the last four weeks of the course, the preservice teachers collected data, created function models by using computer tools, and discussed the real-world possibility. The study indicated that students' solution strategies used to solve problems differed with respect to the relative influence of mathematical ideas and real-world knowledge.

Summary

The NCTM's visualization of learning mathematics has shifted toward investigating, conjecturing, formulating, representing, reasoning, and applying a variety of strategies to solutions of problems (NCTM, 1995, 2000). With respect to learning mathematics for understanding including algebra, mathematics educators and teachers now endorse an emphasis on students' constructing mathematical knowledge and mathematical thinking and reasoning. It is recommended that learning mathematics requires constructions; therefore, mathematics teachers need to accept the responsibility for establishing an environment in their classrooms to support students' learning.

As indicated in several studies at the secondary school level (Markovists et al., 1988; Vinner, 1983) and college level (Breidenbach et al., 1992; Carlson, 1997; DeMarois & McGowen, 1996; Tall & Bakar, 1992), students at both secondary school and college levels had difficulties in understanding function concepts and held function misconceptions. Several studies specified similar misconceptions among secondary school and college students, including preservice teachers, but rarely mentioned their thinking and reasoning processes related to these misconceptions. Even though there were some studies (Heid et al., 2002, Zbiek, 1998) discussing secondary school students' ability in selecting solution strategies and thinking and reasoning algebraically, more information is needed, particularly at the college level. Therefore, a vision of how college students think and reason algebraically needs to be investigated using several data sources to confirm or disconfirm the findings.

By providing opportunities and environments in which students can share their thoughts, such as asking questions in assignments about their rationale for their answers, asking them to speak aloud to demonstrate their thoughts as they are working on the problems, teachers can learn more about their thinking and reasoning, and their ways of understandings. These activities will empower teachers and educators to improve mathematics education in the future. In order to understand how students think and reason algebraically, a qualitative research case study is needed to obtain an in-depth understanding of students' cognitive thinking and reasoning processes. Classroom observations and interviews of students asking them to express their thinking verbally will provide in-depth understanding of the influence of instruction on students' algebraic thinking and reasoning.

CHAPTER III

MATERIALS AND METHODS

Based on the trends and direction of the research in the past two decades on the teaching and learning of algebra in addressing the function domain, an important shift has occurred from verifying students' difficulties and misconceptions to considering their solution strategies and their thinking and reasoning as they work on the problems. In recognition of this shift, this study proposes to describe the nature of college students' understanding of function and their thinking and reasoning strategies and processes as they work with function problems. In particular, this study considers two research questions:

1. What is college algebra students' understanding of functions?
2. What solution strategies and thinking and reasoning processes do college algebra students use as they attempt mathematical problems involving functions?

This chapter presents the methodology used to conduct this study with college students enrolled in the College Algebra course at a northwest public university. The College Algebra course was selected for this study because the fundamental mathematical function concepts were taught in this course. Case study analysis of the verbal data collected through a Function Understanding Questionnaire, thinking and reasoning interview problems, and classroom observations conducted as the students participated in the instruction of function concepts were used. To answer the first question, the participants were asked to complete a Function Understanding Questionnaire illustrating their thinking and understanding about functions at the end of the instruction. College students' understanding of functions and abilities to work with functions, to characterize real world functional relationships, to operate with particular types of functional representations, and to translate among different representations of the same function were considered. To answer the second question, a subset of the students participated in thinking and reasoning problem-solving and interviews conducted every two weeks during the instruction of functions Winter term, 2003. Students' responses on interview problems allowed a comparison of their reasoning and thinking throughout the instruction. The interview data allowed the researcher to examine students' insights more carefully. Examining students' words and their responses supported the development of the students' reasoning and thinking as they developed their responses to mathematical function situations. Classroom observations provided additional information in explaining how students selected their solution strategies and their thinking and reasoning about specific problems.

Participants

Participants for this study were college students enrolled in a college algebra course in winter term of year 2003. Enrollment in this course implied that the student had achieved some minimum level of proficiency in manipulating prerequisite algebraic concepts through college level or an equivalent high school preparation. The topics covered in this course were intended to provide the students with the mathematical background needed to pursue higher levels of college mathematics courses.

Instructional Staffs

The instructional staff for this course included one instructor and one graduate teaching assistant [GTA.]. Both volunteers were experienced in teaching and assisting students learn mathematics at the college level, particularly in teaching college algebra.

The Instructor

The instructor for the course had a strong background in mathematics. She earned a Bachelors of Arts Degree in December 1999 with two majors: Mathematics and History. She earned a Master of Science degree in mathematics in June 2002 at the university where the study took place.

When she started college she planned to study mathematics but she had no idea what she wanted to do after graduation. After a couple years of college, she added a history major to her degree and started thinking about teaching. Originally she considered pursuing a Master of Arts in Teaching (MAT), but changed her mind because with that degree she was likely restricted to teaching at the high school level; she wanted to teach at the college level also. Instead she began her master's degree. While studying for the master's degree, she taught part-time at a nearby community college. This experience indicated that college level teaching was what she wanted to do.

Her experience with algebra began when she was in middle school. She took two years of algebra in eighth and ninth grades (home schooled). She never took college algebra. However, as a graduate student, she was able to be a graduate teaching assistant [GTA] for College Algebra course for five terms (2000-2002). After her graduation, she was hired as an instructor in the Mathematics Department. Winter term was her second term of teaching college algebra as an instructor. She also taught two other college mathematics courses including finite mathematics and calculus for a business major. She had also worked with college algebra students in a mathematics lab class at the community college since January, 2001.

The Graduate Teaching Assistant

A graduate student assisting in this class was currently studying mathematics and had experience in teaching college algebra before participating in this study. She earned a Bachelors of Science degree in Mathematics in June 1999 at a university in Turkey and a Master of Science degree in Mathematics in June 2001 from another university in the northwestern United States.

During her Master's degree program, she was a graduate teaching assistant in Discrete Mathematics for four terms. At that point she became interested in teaching.

She was originally planning to get a doctorate, but she wanted to learn more about Mathematics Education. So, she participated in a program that resulted in a Teaching Certificate in June 2002. While in this program, she was an instructor at the university and taught Intermediate Algebra. As a part of Teaching Certificate program, she taught at middle school and high school levels. While teaching at high school, she realized that teaching at the college level was more interesting and she decided to get a doctorate in Mathematics Education. As a doctoral student at the university where the study was conducted, she served as a graduate teaching assistant for mathematics classes. She also had experience in teaching college algebra before participating in this study. Prior to the study, she was a GTA for College Algebra course twice.

The responsibilities for the GTA were determined by the instructor. During the recitation period, the GTA reviewed the content taught in the lecture class the previous week, answered the students' questions within the lecture content (if any), directed students in lab project activities that related to some topics in the lecture class, gave students quizzes (depending on the assignment of the instructor), and graded students' homework and lab activity assignments. All tests (midterm and final tests) were constructed by the instructors from all sections of College Algebra and were computer graded with an answer key prepared by the instructors.

Student Participants

After the instructor and the GTA had been identified for this study, the specific class from which to select student participants was identified. The researcher introduced herself and presented a description of the study to all of the GTA's recitation classes during Winter term, 2003, describing the data to be collected and expectation of students who volunteered.

Subjects participating were drawn from volunteer students enrolled in the College Algebra course (MTH 111) of the volunteer instructor. Students who registered for the College Algebra Mathematics Excel class were excluded from the study because this course provided additional assistance in learning and solving mathematical problems involving function concepts. A Biographical Background Information Questionnaire provided students' information involving their history of mathematics courses taken at high school and college before taking the College Algebra course, their achievement, their proposed future mathematics courses, and their intended major.

From the pool of volunteers, 10 students planning to take mathematics courses beyond College Algebra were contacted and scheduled for participating in a one-hour interview. The interview was used to identify five students with the ability to demonstrate, verbalize, and explain their problem-solving strategies and their thinking and reasoning while working on mathematical problems. The students were asked to talk about their background, and their future plans including their plans for taking higher mathematics courses. They were also asked to participate in a think-aloud protocol while solving an algebraic problem. Each student was asked to describe his / her thoughts about how to complete the problem. The researcher selected only five students because the limitation of time needed for the in depth interviews that occurred every two weeks during the 10 weeks instruction of functions.

Data Collection Instruments

Many students are able to repeat verbatim equivalent definitions of a function without the slightest understanding of what the definitions represent (Walton, 1988). This conflict between students' words and actions required an investigation that used multiple data sources. To obtain a more complete understanding of students' understanding of functions, their thinking and reasoning processes, and their solution strategies as they solved the function problems, data were gathered from the Background Information Questionnaire, a Function Understanding Questionnaire, thinking and reasoning interview problems, classroom observation notes, the researcher's journal, and fieldnotes from classroom observations and interviews.

Background Information Questionnaire

Prior to the instruction on function concepts, the Background Information Questionnaire was administered to all volunteer students (24 out of 30 students in the recitation that the researcher observed) in the recitation class selected for this study (see Appendix A). The Background Information Questionnaire gathered self-reported biographical and background information of the volunteer students. The data included mathematics courses taken in high school and college before this course with achievement in these experiences, proposed future mathematics courses, intended major, and college grade level.

Function Understanding Questionnaire

To answer the first research question, all volunteer (n = 24) students in the recitation section of the GTA who agreed to participate in the study completed the Function Understanding Questionnaire (see Appendix E) after the studying a function unit. The questionnaire was used to develop a broad description of college students' understanding of functions. Before completing this questionnaire, all participant students were told that the questions were intentionally vague with many different ways to respond with no right or wrong answers; their answers were not used in determining their course grade.

Validity of this questionnaire was established by a panel of five experts prior to its use. The panel members determined whether the questionnaire focused on the understanding of students' understanding of functions and its applications, ensured that students were encouraged to describe their full understanding of functions, and assured that it assessed students' understanding with respect to the concepts of functions. The questionnaire was given to the panel members to review the appropriateness of the questions. The questions, with 80% agreement of the panel members, were administered at the end of the instruction.

Reliability for this questionnaire was established by administering the questionnaire to 20 volunteer college algebra students who were not subjects in the study. This administration determined whether the questions were understandable and students responses were consistent.

Thinking and Reasoning Interview Problems

A major goal of the study was to examine college algebra students' thoughts, thinking and reasoning processes, and solution strategies used as they worked on problems related to mathematical functions. The instruction on functions quite possibly influenced their thinking, solution strategies, and reasoning; therefore the researcher observed the instruction in both lecture and recitation classes and she interviewed the selected participants during and after instruction of functions. The five selected students participated in a one-hour problem-solving interview every two weeks. They were asked to solve mathematical problems related to functions with respect to the concepts of functions taught within the previous two weeks. A think-aloud protocol was used with each problem. Each participant was asked to verbalize his / her thoughts while working on the problems. The interview problems assessed students' thoughts, their solution strategies, and their reasoning as they solved each problem.

A panel of five experts was asked to validate the problems. This panel consisted of five professors: two mathematics educators, one mathematician, and two instructors. From this panel of five, the mathematician and two instructors had previously taught the college algebra course. The panel was given the set of problems and a set of objectives and expectations for the problems. Using a scale of 1 to 3, each panel member completed a Table of Specifications to describe the level at which the problem met the stated objectives for this study. A rating of 1 indicated the problem was not appropriate for the study while a rating of 2 indicated the problem was appropriate but needed to be revised; a rating of 3 indicated the problem was appropriate as written. The comments and suggestions for improvement and the results from the Table of Specifications helped the researcher modify the problems. Each problem used in this study achieved at least expert agreement of 80%. In other words, 4 of 5 experts agreed that each problem used in the study was appropriate.

The interview procedures and a set of prospective problems were piloted and audiotaped with approximately 40 students enrolled in the college algebra course the term prior to the study. Some of these students participated in the pilot study of Function Understanding Questionnaire. The pilot study was conducted to ensure that the students participating in the study were familiar with the mathematical tools needed to solve the problems and to determine the length of the interview. Additionally the pilot study was used to develop the researcher's probing protocol strategies as students attempted the problems. As the volunteers worked on the problems, they were asked to clarify the wording of the problems and their understanding of the problems. This method helped in presenting problems that encouraged students to think aloud.

Thinking and reasoning interview problems were used prior to, during, and at the end of the instruction on function concepts. The five selected students were scheduled for one-hour to work on problems to demonstrate their thinking and reasoning by responding to directed questions and mathematical function problems every other week during the instruction of College Algebra. Three types of interviews were conducted, each at a specific time during the term: (1) a pre-instruction problem-solving and interview, (2) an interview during instruction, and (3) a post-instruction, problem-solving interview. All interviews and problem-solving activities were videotape and audiotape recorded.

The main goal of the pre-instruction interview was to allow the participants to become comfortable with the recording equipment, to encourage them to speak aloud, and to familiarize them with the data collection process and the researcher. In the first interview, the participants were asked some questions related to their mathematical background information as well as provided an opportunity to describe their understanding of mathematical concepts and ideas obtaining from the class prior to the unit on functions. The interviews began by allowing students to describe their understanding of algebraic concepts in solving four algebraic problems. During the interview, the participants were asked to read each problem aloud and solve the problem by verbalizing their thinking and reasoning while attempting the problem. The participants were allowed to use any tools they wanted in working on mathematical problems. They were also encouraged to talk about their learning activities, homework assignments, as well as to ask the researcher questions. The videotapes focused on the participants' written work and visual movements.

During the function unit, the participants were interviewed and asked to work on function problems using the think-aloud strategy. These interview problems made it possible to focus attention on the participants' thinking and reasoning processes rather than just on correct and incorrect answers they produced. During these interviews, the participants were asked to read the problem aloud before solving the problem and to verbalize their solution method and their thinking and reasoning while attempting to solve the problems. Again participants were allowed to use typical tools for working with these problems (graphing calculators and graph paper). The video recorder focused on the participant's written work to show the physical manifestation of their thoughts and process. While each participant was working, the researcher asked relevant questions to probe and clarify the student's thinking and reasoning and to test emerging hypotheses.

In order to improve the quality of the participants' reports of their thinking, three methods to limit major distortions in their thought processes were addressed. First, the researcher urged the students to think aloud (stating everything that happened in their head) rather than to reflect on their thought processes. Second, since some students used more nonverbal representations than others, the researcher encouraged the drawing of diagrams or translations of images to language. Finally, the researcher used probing questions to encourage a sufficient amount of verbalization and clarification of thoughts (Ericsson & Simon, 1993).

The post-instructional interview was conducted after the function unit in the College Algebra was taught. During this time, the researcher, as an interviewer, continued to probe the student to explain his / her statements at all steps if they were not clear during the problem-solving process. Moreover, this interview provided an opportunity for the researcher to clarify her understanding of the participants' solution strategies. Thus, the goal of the post-instruction interview was to provide the researcher an opportunity to ask the participants to clarify or explain their knowledge and understanding, solution strategies, and thinking and reasoning of functions at any steps not sufficiently verbalized or written during the problem-solving process.

Classroom Observations

Classroom observations were used to gather data on the sequence of topics and instruction in the course. The lectures were for 250-300 students and the recitation class consisted of 30 –35 students. Alternatively, the data were used to describe and identify activities and the interaction students had in the instruction about functions. The classroom observations verified the focus of the instruction and the resources the students received. These data were considered in conjunction with student interviews to assist in interpretation of the students' reasons or explanations to the problems. Additionally, the classroom observations provided the researcher information to guide the ongoing interviews with students. The researcher also examined all course material provided to students in either lecture or recitation to understand how it influenced the students' development of their understanding of functions.

To minimize the influence of the observer in the classroom, the researcher attended lecture and recitation classes prior to the instruction on the function units. Observation data were collected with handwritten notes and through audiotapes recorded during the instruction. The handwritten notes of each class utilized an anecdotal record technique with a focus on key concepts and instructional methods used to guide students in learning about functions. All classroom observational data were transcribed and used for the ongoing development of the interview protocol.

Materials such as textbooks, graphing calculators and activities provided resources and references for students in this course. These course materials were examined for a potential role in the development of students' understanding of function concepts and solution strategies, algebraic thinking and reasoning processes.

The researcher used specific questions to guide her observations. For example:

- What did the instructor and students do during the class?
- What topics were taught? How was the material presented?
- What examples were used?

In this study, the researcher was interested in strategies that the instructor used for teaching the function concepts and how students reacted to the instructor during the class. The researcher asked structured focus questions. For example:

- What teaching strategies were used in the class?
- How did the instructor interact with students during the class?

More specifically, the observation in the recitation class focused on the five students participating in the interview session to see their behaviors in the class. In the recitation the researcher asked guided questions, for example:

- How did these students respond to the GTA?
- How did these students attend to the class instruction?

In general, observations of the classroom behavior of the students and the instructors provided the researcher with additional information pertaining to the participants' opportunity to learn and think about algebraic concepts, concepts of functions in this particular case, and the interpretation of "algebraic reasoning," which included their ability to extract, represent, interpret information, and even to develop confidence in symbolic generalizations

The Researcher

The primary instrument for gathering and analyzing data was the researcher (Merriam, 1998). Data were collected and analyzed from the perspective of the researcher; thus, the researcher's perspective, background, and experience provided a framework from which the study was conducted. The researcher first encountered a formal algebra course in middle school; functions were also addressed at high school and college levels. After earning her Bachelor's and Master's degrees in mathematics education, she taught secondary school mathematics in Thailand from seventh to twelfth grades for seven years before pursuing a Ph.D. in Mathematics Education. Moreover, the researcher had experience as a GTA in the college algebra course at the university where the study was conducted for three years and was an instructor for the course in the study for two terms.

Researcher's Fieldnotes and Journals

For this study, fieldnotes consisted of videotaped records of the thinking and reasoning problem-solving and interviews, and were used to maintain a description of the classroom observations as well as the interviews. In addition, written anecdotal records of the class meetings, transcriptions of all videotapes and audiotapes, and a written description of individual activities were recorded. The use of the videotapes and audiotapes were beneficial in providing various reactions (i.e. talking, gesture, and eye movement) and in providing guidance for the transcribed interviews, allowing the chronological order of the students' written solutions to complement their verbal response. The video and audio recordings were supplemented by the researcher's journal. This journal contained a written account of the researcher's comments, impressions and thoughts, reactions, and initial interpretation and working hypotheses.

Data Analysis

The purpose of this study was to gain an in-depth understanding of college level students' understanding of functions, their thinking and reasoning processes, and the solution strategies they used in solving mathematical function problems. In order to enhance the validity of the study findings, multiple data collection methods, including classroom observations, participant interviews, fieldnotes, and researcher's journal were used to confirm or disconfirm observations and categorical development, and to look for contradicted reasons.

To answer the first question, the data from the responses to the Function Understanding Questionnaire of 24 volunteers were analyzed to determine how college algebra students described their understanding of function concepts. The students' responses to each item in the questionnaire were considered. The responses were categorized into four categories:

- Prefunction (having a limited concept of functions)
- Process (using a process of input/output)
- Correspondence (indicating a relationship between variables)
- No concept (indicating no understanding of functions)

The first three categories were classified based on the study of Schwingendorf et al. (1992). In addition, the interviews of the five selected students were used to clarify and expand the information from the questionnaire that resulted in the identification in each category.

To answer the second question, the results from all interviews of the five students' interview session as well as the information of the classroom observations were described and analyzed. The students responses to the interview problems were considered with respect to their problem solving approaches and reasoning while solving the problems related to (1) classifying relations into functions and non-functions, (2) representing a function in several forms, (3) transforming a function from one representation to another, (4) applying a function understanding to unfamiliar symbolic problems and to real-world problems, (5) giving examples of functions satisfying some given constraints, and (6) identifying a function satisfying a given constraint (Markovit et al., 1986).

More specifically on algebraic thinking and reasoning, the students' responses were analyzed with respect to their ability to use mathematical symbols and tools to analyze mathematical problem situations by (1) extracting information, (2) representing information using multiple forms, and (3) interpreting and applying mathematical ideas to a new situation. The case study analysis assisted in creating categories and refining detailed descriptions of the categories of the solution strategies used by students and their thinking and reasoning processes in solving mathematical function problems. Each student's solution strategies and his / her thinking and reasoning processes were determined and noted whether they used patterns for each problem, or changed their strategies from one problem to another. Each individual's solution strategies and thinking and reasoning processes were compared, looking for similarities and dissimilarities. Besides identifying the solution strategies students used, their thinking and reasoning processes in solving mathematical function problems, this study also provided information involving these students' thinking before, during, and after instruction on functions. This description helped to frame the effect of instruction on their thinking and reasoning strategies.

Summary

This descriptive analysis was conducted to provide portraits of the students' understanding of functions, solution strategies, and processes of thinking and reasoning as they attempted algebraic problems involving function concepts. The data collection consisted of assessing the participants' (1) understanding of functions by analyzing their responses to the Function Understanding Questionnaire and (2) solution strategies in solving function problems by collecting verbal data through the administration of ongoing interviews before, during and after the instruction of the function unit. In addition, the verbal data were organized to support a comparison of the patterns of thinking and reasoning processes across various students, as well as student's thinking and reasoning before, during and after the instruction.

CHAPTER IV

RESULTS

This chapter presents a picture of college algebra students' knowledge and understanding of functions, solution strategies, and algebraic thinking and reasoning as they attempt mathematical function problems. The overview of the course and the format of the instruction on functions in the College Algebra class, both in lecture and recitation, where this study was conducted are presented. In addition, the profiles of the students are presented. The data present the knowledge of functions the students gained in this class. The instructional profiles provide a context supporting the results of this study. Also, a comparison of the students' function knowledge and understanding, solution strategies, and algebraic thinking and reasoning is presented. Finally, the results include college students' understanding of function concepts, solution strategies, and thinking and reasoning processes as they attempt mathematical problems involving functions. The main data sources of this study were from written responses to the Function Understanding Questionnaires administered to all volunteers at the end of the instruction and verbal responses to thinking and reasoning problem-solving interviews of five selected volunteer students. The five students who participated in the problem-solving interviews were assigned pseudonyms to protect their anonymity.

Course Overview

The College Algebra curriculum taught in this class considered equations and inequalities, their graphs, techniques for solving equations and inequalities, functions including specific functions, such as linear, quadratic, exponential, and logarithmic functions, their graphs, domains and ranges, their inverse, and applications of functions. The instruction of functions began in the fifth week of the ten-week course. Before the instruction on functions, students studied equations, inequalities, and their graphs. The content taught in this class addressed linear equations and inequalities, equations of circles in forms of $x^2 + y^2 = r^2$ and $(x-h)^2 + (y-k)^2 = r^2$. The content also included absolute value equations and inequalities, as well as polynomial, quadratic, rational, and radical equations. Prior to enrolling in the College Algebra course, the students were expected to have a minimum level of proficiency in manipulating prerequisite algebraic concepts through college level or an equivalent high school preparation.

The material used in this course included, the textbook, *College Algebra and Trigonometry* (Dwyer & Gruenwald, 2000), the *Study Guide* (Fein & Lee, 2000), and a graphing calculator. The instruction and homework assigned to students were organized based on the textbook and the *Study Guide*. The instruction consisted of three 50-minute lectures and one 50-minute recitation class each week in the ten-week course. The lectures were on Mondays, Wednesdays, and Fridays (9:00 - 9:50 a.m.) and the recitations were on Tuesdays (10:00 – 10:50 a.m.). The lecture class was a large group of about 250 students. The recitation class was a small group of 30 students.

The students were not required to bring their textbook to the lecture and recitation classes, but they were required to bring their *Study Guide* to the recitation class. The examples demonstrated in the class were from both the textbook used for this class and some other different college algebra books. The lecture instructor used her graphing calculator to demonstrate problem-solving methods in almost all her lectures depending on the content taught each day. The graphing calculator was used more often for numeric calculations, plotting graphs of functions, and finding zeros of a function. The figures from the graphing calculator were displayed through an overhead projector.

The large group of 250 students used a lecture instructional style. Even though the lecture instructor always gave students opportunities to ask her questions, the size of the class restricted the number of opportunities for students to ask questions. Only students who sat in the first three rows of the lecture class were able to ask the instructor questions. Before starting a new lesson, the instructor typically asked whether the students had questions from the previous lecture or from the units that had been taught, and then told the students what she planned to teach for that day. At the end of each lecture the instructor recommended that the students work on the homework assignments listed in the *Study Guide*. Although these homework assignments were not collected and graded, the questions provided practice for the quizzes, midterm and final examinations. One feature consistent throughout the instruction was the instructor's inspection of students' understanding of the concepts taught each day by asking them questions. Because of the size of the class, the instructor did not identify specific students to answer her questions. Typically there were no responses from the students resulting in the instructor answering the questions. Therefore, whether the students understood the lesson was not determined.

The recitation class was taught by the Graduate Teaching Assistance [GTA]. The general format for the recitation included answering students' questions related to the content taught by the instructor, demonstrating how to solve homework or quiz problems (based on students requested), and working on a lab project. The homework problems of each section were indicated in the *Study Guide*. The recitation included five lab projects and five quizzes. Fifty percent of the problems on each quiz were from the problems assigned for homework; the other half of the quiz was constructed by the lecture instructor and was similar to the homework assignments. Two of the five lab projects related to concepts of functions: quadratic functions and function transformations.

In the recitation class, the GTA never used a graphing calculator while she demonstrated examples. She always approached the problem using symbolic manipulation. For example, she showed how to find zeroes of a function ($y = f(x)$) or the x-intercept by setting $f(x) = 0$ and solving for x. Some students argued that they could graph the function $f(x)$ and find the zeroes of a function by using their graphing calculator. She agreed and said, "You can do it that way too." However, she did not show any examples using a graphing calculator to find the zeros of a function.

In general, the GTA did not teach new concepts in her recitation class. She started her class by asking students whether they had questions involving the content they had learned from their lecture class and whether they had questions from doing the homework assignment. The students seldom had questions from the lecture; they had more questions involving the problems in their homework assignments. The GTA used the blackboard to demonstrate the homework problem situations. After finishing the problems, the GTA assigned a lab project. The students worked individually for 10 minutes, and then were allowed to work in pairs or groups of three or four. The students selected their own partners. While the students worked on the assignment, the GTA walked around the classroom giving help if needed. The students might turn in their lab assignment at the end of recitation class or they might turn it in at the beginning of the next recitation class. Late homework was not accepted unless an acceptable reason was given.

The GTA's instruction was directed by the students' questions; sometimes the students did not ask questions related to the concepts they had learned from the lecture prior to the recitation class. Most of their questions came from homework and lab project problems. Lab projects were assigned every other week. In one of the recitation classes, the students learned more about function transformations because the lab project related to this concept was assigned to them one day after they learned this concept from the lecture class. In addition, the quadratic function lab, the students also learned about transforming functions from $y = ax^2 + bx + c$ to $y = a(x-h)^2 + k$) from the quadratic equation unit prior to the function unit

Besides attending the lecture and recitation classes, all students had the opportunity to get help from the instructor and the GTA during their office hours or by appointment. In addition, the students could get help on their mathematics problems from mathematics tutors in Math Learning Center (MLC) provided by the Mathematics Department from Monday to Thursday (9 a.m. - 8 p.m.) and Friday (9 a.m. - 5 p.m.). The five students selected to participate in the problem-solving interviews claimed that they never asked for help from the tutors in the Math Learning Center.

The assessment of the course included the five lab projects (assigned every other week), five quizzes (given every other week), one midterm, and one final exam. The lab projects and the quizzes were constructed by the instructor and were graded by the GTA. The instructor constructed the midterm and final exams and the solutions. These exams used a computer program provided by the computing center at the university.

Research Question 1: Understanding of Functions

The first research question of this study focused on identifying college algebra students' understandings of functions. To answer this question, the instruction on functions, student interviews, and student responses to the Function Understanding Questionnaire were considered.

The instruction on functions was over the fifth week through the tenth week of a ten-week term and was followed by a review session for the entire course so that students were prepared for their final exam. The content of functions included linear functions (slope of a line, identifying a linear function given a slope and a y-intercept, or given two points), multiple representations of functions, domain and range of functions, quadratic functions, one-to-one functions, composition of functions, inverse functions, exponential functions, and logarithmic functions. The function instruction consisted of 14 lectures (each 50-minute) and 10 recitation classes (each 50-minute).

Instructional Episodes

Episode One: The Definition of Functions

Prior to providing a formal definition of functions, the instructor introduced the function unit by writing "A function shows a relationship (correspondence) between two sets of objects." She provided an example of a table that represented a function using temperature data (see Figure 3).

Date	High Temp
Jan 1st	39°F
Feb 1st	42°F
Mar 1st	48°F
Apr 1st	56°F

Figure 3. The instructor's example of a table representation of a function.

During this first lecture on functions, the instructor provided a formal definition of functions: "A function from a set D to a set R is a correspondence that assigns to each element of D exactly one element of R." In explaining this definition, the instructor said:

> If I put one value [input] for x, I must get only one value for y. If I have more than one y as the output, it's not a function. For being a function, each input has exactly one output.

The instructor gave this verbal explanation; she did not draw any pictures or diagrams. Instead, she returned to the previous example, the temperature, to check if this example followed the definition. At this time she described and wrote the domain and range of the function on the overhead: "Domain is all the possible input values (x). Range is all the possible output values (y)."

The instructor described a second example of a function in a table form and asked students to determine whether it was a function. She also asked them to identify their reasoning in their responses.

Instructor: Consider this table; is it a function? [Writing the table showed in Figure 4]

Students: No. Because for input 4, there are two outputs -2 and 2.

D	4	1	0	1	4
R	-2	-2	0	1	2

Figure 4. A tabular representation of a non-function created by the instructor.

One of the students stated "an input of 1 also gave two outputs: -2 and 1." Everyone in the class agreed that it was not a function. The instructor further clarified the idea: "If we found one input that gives two different outputs, we say it's not a function."

In this same lecture, the instructor also provided an example in the form of graphical and symbolic representations.

Instructor: $x^2 + x - y = 0$. Is y a function of x?
Students: [No response]
Instructor: Is there exactly one y value for each x value?

The instructor rewrote the equation y in terms of x. However, she did not say why she made this change and the students did not ask for the reason.

Instructor: We will solve for y. [Writing $y = x^2 + x$, she did not write down every step of the reasoning.] Look! [Pointing to the equation] If you plug x into the equation and get y more than once, then it's not a function.

She did not insert numbers for x into the equation nor look at the outputs to show it was a function. However, she indicated through this example that a function could be determined using a vertical line test. As the first lecture, this discussion was the first time that she talked about the vertical line test. She then defined an equation as "a function if no vertical line crosses the graph more than once." She used the same equation displaying that it was a function using the vertical line test. She described:

> I will draw the graph of this equation [demonstrating a graph of $y = x^2 + x$ on the overhead]. There is no vertical line that crosses the graph more than once [drawing the vertical lines all over the graph that she had drawn previously]. Therefore, it is a function.

Following this explanation, one of the students asked "Is it not a function if a [vertical] line crossed the graph more than once?" The instructor said, "Right," then the instructor drew a graph of a circle on the overhead without showing a symbolic representation. She showed that this graph was not a function because many vertical lines crossed the graph more than once.

The instructor demonstrated one more example of an equation using its symbolic form ($y^2 - x^2 = 0$) to consider whether it represented a function. She displayed the process on the overhead, explaining:

> From $y^2 - x^2 = 0$, I will solve for y. So $y^2 = x^2$ and $y = \pm\sqrt{x^2}$. Any time that we see $\pm$ sign; we have no need to graph it. It's so obvious that there are two y values for one x value. Plug in one x we will get two y values. For example $x = 1$: $y = \pm\sqrt{1^2} = \pm\sqrt{1} = \pm 1$. When $x = 1$, $y = 1$ or -1. Therefore, it's not a function.

In summary, the instructor stated, "If the case is not obvious, then graph it and use the vertical line test." None of the students asked for further explanation. The instructor did not describe the domain and range of the function in the definition of a function. However, the definition of a function in the textbook included a reference to both domain and range.

> A function from a set D to a set R is a correspondence or rule that assigns to each element x of D exactly on element y of R. The set D is called the domain of the function and the elements x of D are the input values. The elements y of R that correspond to the input values are the output values. The set of all possible output values is called the range of the function. (p. 227)

The instructor did not suggest or direct the students to read this definition or any other content in the textbook.

In the following recitation class, the GTA did not refer to the definition of functions that was taught in the previous lecture class of that week. The students did not have any questions related to the function definition and the vertical line test. In addition, the GTA did not challenge students' understanding.

The homework assignment assessed students' understanding of a function definition through 16 problems. Five of the 16 problems required the students to determine whether the tabular, symbolic, and graphical representations defined a function.

One week after the students had been introduced to the function definition, they took the midterm examination with 20 problems. One of these problems asked students to determine whether a tabular representation defined a function. When the midterm was returned, one of the students asked the recitation GTA for an explanation of why the data in the table (see Figure 5) represented a function.

D	0	1	2	3	4	5
R	1	1	1	2	2	2

Figure 5. Table representation of function in midterm exam.

The GTA explained using an arrow diagram that each input had only one output. She also stated "For different inputs if we have the same output, it is okay." She gave examples in the form of an arrow diagram showing function and non-function (see Figure 6a and 6b).

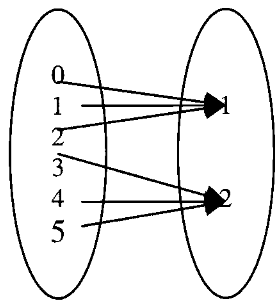


Figure 6a. An arrow diagram defining a function.

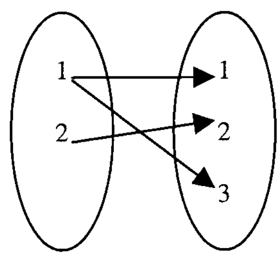


Figure 6b. An arrow diagram indicating a non-function.

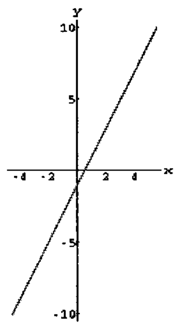
When the students took the final examination, none of the 25 problems assessed students' understanding of the definition of a function and no problems asked the students to identify whether a function was represented.

Episode Two: Multiple Representations of Functions

The instructor introduced multiple representations of functions to her students after presenting the definition of functions. She stated, "A function can be shown in many different forms such as an equation, a table, or a graph." Again, she showed the students the information in Figure 3 and said, "These tables represent functions."

The instructor showed how a function could be transformed from one representation to another using her graphing calculator with the display on the overhead.

> A function $f(x) = x^2$ can be transformed to a graph. It is easy if you do it by a graphing calculator. I input $y = x^2$ and a graph of a parabola will show up. We can also change from an equation to a table. For example, $y = 2x - 1$, I input this function to the calculator and make a table [pressing a table bottom on the calculator] and a table will show up [displaying the table on the overhead as shown in Figure 7].



x	y
-3	-7
-2	-5
-1	-3
0	-1
1	1
2	3
3	5

Figure 7. A graph and a table of a function $y = 2x - 1$ displayed by the instructor.

The instructor described the advantage of the transformation from one representation to another.

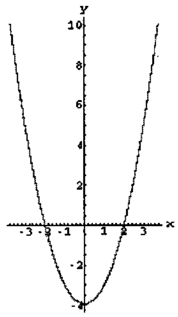
> If we want to know if this equation [writing $y = x^2$] is a function or not, we may graph it and use the vertical line test. We know that if it passes the vertical line test, then it is a function.

The instructor did not give many examples of the multiple representations of functions in the class. In addition, the GTA did not discuss multiple representations in her recitation class. However, the instructor and the GTA used multiple representations, especially graphical and symbolic, demonstrating problem-solving methods for many example problems in other sections. For example, the lecture instructor used a graphical and symbolic representation to demonstrate how to find zeros of a function (see Figure 8).

Example: Find zeros of the function $f(x) = x^2 - 4$
Solution: To find the zero we set $f(x) = 0$ and solve for x.

$$f(x) = 0$$
$$x^2 - 4 = 0$$
$$(x+2)(x-2) = 0$$
$$x = -2, x = 2$$

We can solve for the zeros of the function using its graph [the instructor graphed this function using her calculator and displayed the graph on the overhead]



From the graph, we see that the zeros of the function are –2 and 2, which correspond to the x-intercept of the graph

Figure 8. The example of finding zeros of a function using symbolic manipulation and graphical representation.

The graphical representation was used more often in presenting a function transformation section. The problems for the homework assignment asked students to determine whether the data in tabular and graphical forms represented a function. None of them asked students to transform one form to another. Nevertheless, the students were allowed to use all representations to help them solve all problems in other sections.

Episode Three: Transformations of Functions

Transformations of functions were introduced through two types of transformations: horizontal and vertical. The instructor provided an example of horizontal transformation using the quadratic function, $y = x^2$. She showed the graphs from her calculator on the overhead. She did not show the symbolic form of each equation but she demonstrated the example graphically (see Figure 9).

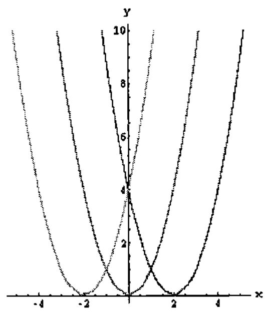


Figure 9. The horizontal transformation of $y = x^2$, two units to the left and two units to the right.

> If the graph moves to the right two units, all points on the graph $y = x^2$ will move. For example, (0,0) moves to (2,0). $x = 2, y = 0$; therefore, $y = (x-2)^2$. If the graph moves to the left two units, (0,0) moves to $(-2,0)$. When $x = -2$, $y = 0$; therefore, $y = (x-(-2))^2 = (x+2)^2$.

After discussing this example, the instructor wrote the formal notation of a horizontal transformation of the function $f(x)$ on the overhead "$f(x)$ moves *h* units ➔ $f(x-h)$."

Since the students had no questions, the instructor used the same function to discuss a vertical transformation. She used the calculator display to describe this transformation (see Figure 10).

> Let's see how to get the new function $f(x)$ if it moves up 3 units. (0,0) moves to (0,3); $x = 0$ and $y = 3$. I will try, $y = (0+3)^2 = 9$. This doesn't work. I try $y = (x-3)^2 = (0-3)^2 = 9$. This doesn't work either. Therefore, the vertical move, adding the number inside the parenthesis doesn't work. Let's try $y = x^2 + 3 = 0^2 + 3 = 3$. That works. So if the graph moves down 3 units, (0,0) moves to $(0,-3)$. The function is $y = x^2 - 3$ [showing that this function is correct by checking: $0^2 - 3 = -3$].

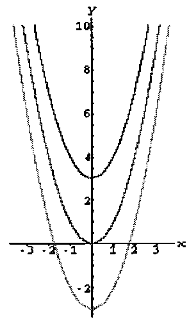


Figure 10. The vertical transformation of $y = x^2$, up and down 3 units.

Since students did not have questions, the instructor continued by illustrating a combination of vertical and horizontal transformations.

> From $f(x) = 3x^2$, there is a new function $g(x) = 3(x+2)^2 - 10$. We can see how it moves [writing her summary as shown in Figure 11].

$$g(x) = 3(x+2)^2 - 10$$

Move horizontally

Move vertically

Figure 11. Summary of horizontal and vertical transformation presented by the instructor.

The instructor concluded, “From the formula [the symbolic representation], we can make a rough sketch of its graph and from the graph, we can think about the formula.”

A transformation lab project supported the lecture on function transformations. The GTA provided the students an opportunity to ask her questions related to the transformations. Since there were no questions, students worked on the lab individually for about 10 minutes, and then they were allowed to discuss and compare their work in groups of three or four. While the students were working on the lab, the GTA walked around the classroom watching and listening to their discussion. If the students had questions, they raised their hands, and the GTA provided some help. The GTA did not directly answer students' questions. Instead, she asked students questions making them think and discuss their answer. Many students demonstrated that they understood the major concept of transformations. They correctly worked on the lab by themselves, although a few students needed some assistance either from the GTA or other students in their group.

The homework problems supported the students' understanding of graphing transformations. They were asked to determine how a function was transformed to another. One of the 20 problems on the midterm assessed students' understandings of the vertical and horizontal transformation of the function graphs. There were no problems in the final examination that dealt with transformations of functions.

Episode Four: One-to-One and Inverse Functions

The instructor approached the topic of one-to-one and inverse functions explaining "For functions $f(x)$ and $g(x)$, if $g \circ f(x) = x$ and $f \circ g(x) = x$, then *f* and *g* are inverse." She provided the notation for the inverse, "$f^{-1} = f$ inverse" and also stated and wrote. "If $f = g^{-1}$ and $g = f^{-1}$, then they are inverses of each other." The instructor emphasized that, "Only functions that are one-to-one have an inverse." After mentioning one-to-one functions, the instructor introduced the horizontal line test. She reviewed how to determine a function stating "Each input has exactly one output; and when you graph it, make sure that each vertical line crosses the graph once." She showed the example of $y = x^2$, drew the graph by hand on the overhead, and then drew vertical lines that were similar to Figure 12 indicating that $y = x^2$ was a function.

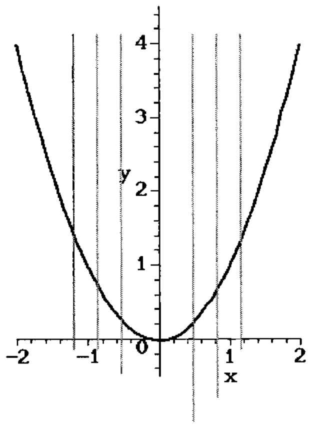


Figure 12. An example of the use of the vertical line test for a function.

The instructor related the basic idea of a function to a one-to-one function. She stated, "a one-to-one function is a function where each output corresponds to only one input." She clarified her description by using the graph $y = x^2$ similar to the graph in Figure 13.

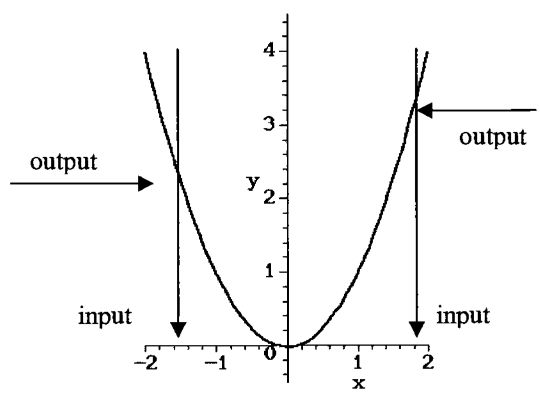


Figure 13. Instructor's example of a function that was not one-to-one.

One of the students argued "If there is a power on *x*, then it is not a one-to-one function." As a counterexample to the student's conjecture, with the function $y = x^3$, the instructor provided a symbolic representation showing that there was a power on *x*, and then showed its graph on the overhead showing that the student's conjecture was incorrect (see Figure 14).

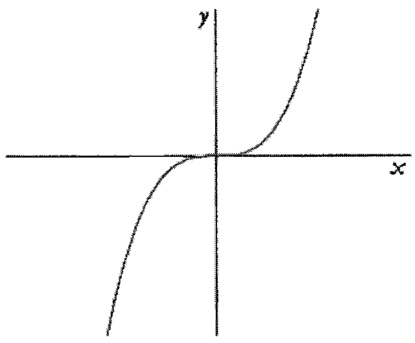


Figure 14. Graph of $y = x^3$ that is a one-to-one function.

After showing this example, the instructor summarized, "A function is one-to-one if and only if no horizontal line crosses the graph more than once." She recommended that if students wanted to determine a one-to-one function, they should draw a graph and use the horizontal line test. She stated, "If it is one-to-one, then it has an inverse and its inverse is a function."

In the recitation class, the GTA did not discuss the concepts of a one-to-one function and the function inverses and the students did not ask questions about either topic. There were 14 problems in a homework assignment related to determining a one-to-one function and finding function inverses. Six of the problems asked for only determining whether a function was one-to-one. Eight of the problems asked students to find the inverse if it was a one-to-one function. From the homework assignment, the students were expected to have experience working with several types of problems including application problems. One of the four problems on the quiz asked students to determine whether a given function (in a symbolical representation) was one-to-one. None of the problems on the midterm assessed students' knowledge of one-to-one and inverse functions; however, there were two problems in the final exam that asked them to find the inverse of functions.

Student Profiles Concerning Understanding of Functions

This section presents each student's understanding of functions as they progressed through the instruction. Five students participated in problem-solving interview sessions. They had various levels of mathematical knowledge, understanding, and problem-solving abilities. However, all of them had studied algebra and functions in high school.

Amy

Amy was majoring in zoology. She was a freshman, taking College Algebra as a requirement. This College Algebra course was her first college mathematics class. After taking College Algebra, she was required to take two additional mathematics courses (Elementary Functions and Differential Calculus) for her major. In high school, she took Prealgebra, Algebra I, Geometry, Algebra II, and Functions, Statistics and Trigonometry (a one-year course). She said that she did very well in high school mathematics. She always earned A's and B's except her last term of Function, Statistics and Trigonometry where she said that she had a conflict with her mathematics teacher.

Amy said she loved algebra because she could understand it well. She claimed that she understood how to deal with everything with algebraic problems. However, algebra was not her most favorite subject: "My most favorite is trigonometry because it is fun. I love trigonometry, cosine, secant, and cosecant, that kind of stuff. I love all that. Algebra is second." She was confident in her mathematics ability indicating that this College Algebra class was easy for her. She stated that she wished she had started with the Elementary Functions instead of College Algebra because of the easiness of the course.

When asked how she studied the material, Amy said she came to every class, both lecture and recitation, and did all the suggested homework problems. She never went to her instructor or GTA office to ask for help and she never asked for help from any tutors at the Math Learning Center. Most of the time she studied by herself. If she had any questions or did not understand some ideas, she asked questions in the recitation class.

Amy's Understanding of Functions Prior to Instruction

Even though Amy had studied algebra and functions in high school, she was unable to distinguish functions from equations at the beginning of the course. When asked if she could describe a function, she indicated, "A function is an equation." With this conceptual understanding, Amy indicated that a circle represented as $x^2 + y^2 = 1$ was a function. She said, " I know that this is a function. It's familiar to me." However, she was not able to explain why she thought $x^2 + y^2 = 1$ was a function.

Amy indicated a limited knowledge of multiple representations of functions prior to instruction. Her idea of functional representations related to symbolic and graphical representations. Initially, Amy said that a function could be presented as an equation of a line, a parabola (a quadratic equation) and a circle suggesting that she was more familiar with symbolic representations than any other representations. With probing, she was able to relate various representations by transforming one representation to another. When asked if given an equation, such as an equation of a line, whether she could graph this equation. She said that she could graph it by finding and connecting two points for the coordinates (x,y). When asked if the equation of the line and the graph of that line represented the same information, she agreed that they did. Despite questioning about other representations besides graphical, verbal, and symbolic representations, Amy said she believed that it had other representations but she could not recall them at this time.

Amy showed she was able to change a symbolic representation to a graphical representation [a graph of a circle]. The interviewer did not have an opportunity to investigate if she was able to change a graphical representation to a symbolic representation.

Amy was also asked to match the graphs in Figure 15 with a specific situation.

Situation: The kettle heats before the corn begins to pop. The corn starts to pop and continues popping until almost all the corn has popped. The amount of *unpopped* corn in the kettle is the dependent variable.

Graphs:

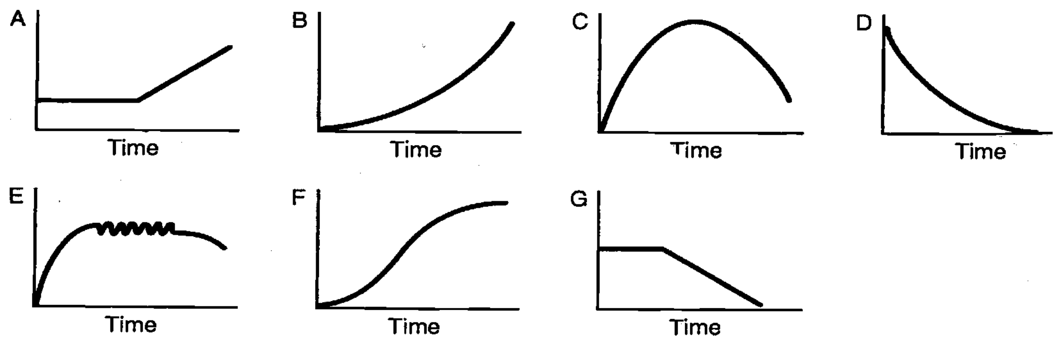


Figure 15. Pre-instruction interview graphs for the popcorn problem.

In attempting the problem, she deleted irrelevant graphs by describing the situation in her own words; then she connected her understanding to the correct graphical representation.

> The kettle heats before the corn begins to pop. The corn starts to pop and continues popping until almost all the corn has popped. The amount of *unpopped* corn in the kettle is the dependent variable. So, it's gonna heat up until popcorn started popping and the amount of unpopped popcorn is gonna get less as the time passes. So there is a lot of unpopped popcorn at the beginning. It means the amount of unpoped popcorn will be high and has the same amount for awhile during the kettle is heated at the beginning and less at the end. It can't be "A" because graph A shows that the amount of unpopped popcorn becomes larger instead of getting less at the end. It can't be B, C, D, E, and F either because the amount of unpopped popcorn should be at the same level while the kettle heats up. So it's gonna be G.

Amy's Understanding of Functions During Instruction

During the period of instruction, two problem-solving interviews related to function concepts were conducted. The first interview was completed in the sixth week of the term (two weeks after the instruction of functions) and the second interview was completed in the eighth week of the term (four weeks after the instruction of functions). Amy demonstrated an improvement of her understanding of functions as she worked on the problems over the two interview periods.

Instructional Interview One

In the first interview, Amy was asked to solve four problems (Instructional Interview Problem No. 1, 2, 3, and 4) related to multiple representations and real world situations. As she worked on the problems, Amy demonstrated a tentative understanding of functions related to working with radical equations and functions in the first interview. This incomplete understanding was evident when she attempted to determine whether the equation $x^2 + y^2 = 1$ was a function (Instructional Interview Problem No. 1c).

A: To determine if $x^2 + y^2 = 1$ is a function, I may need to make a leaf-plot. [What she called the leaf-plot was the same as a numerical representation but the information was vertically presented as a table.]

I: Would you please show me how you will do that?

A: I will plug numbers for x in the equation to see the value of y. OK. $1^2 + 0 = 1$. So when x = 1,y = 0. And 2^2 + something here = 1. Hmm... 2^2 + Not 1, not 2 but let's see $2^2 + y^2 = 1$ [writing the equation on a paper]. So $y^2 = -3$ and $\sqrt{y^2} = \sqrt{-3}$. It's a square root of –3. No it can't be this. It must be a negative of square root of 3 [$-\sqrt{3}$].

x	y
1	0
2	..
3	..
4	..

[Amy developed this table beginning with x values and then identifying the corresponding y values.]

I: How did you get that?

A: I moved the negative out.

I: Why?

A: Because you can't have a square root of negative numbers.

Amy correctly interpreted the information from the graphical representation as shown in Figure 16.

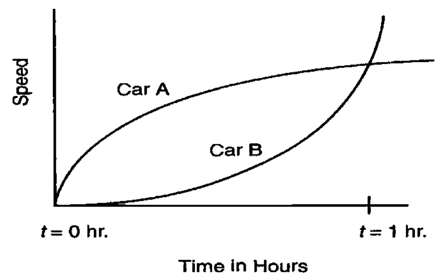


Figure 16. Graphical representation for Instructional Interview Problem No. 2.

I: Can you describe the graph?

A: OK. They [Car A and Car B] start traveling at the same point. Car A goes faster than Car B and keeps going faster until t = 1 hour. They have the same speed. So Car A is gonna be ahead of Car B because it goes faster from the beginning.

I: What does it mean at the point t = 1 hour?

A: That point means they are going at the same speed but they are not at the same position.

Amy was also able to transform a graphical representation to a numerical (tabular) representation and a graphical representation to a symbolic representation referring to the graph in Figure 17.

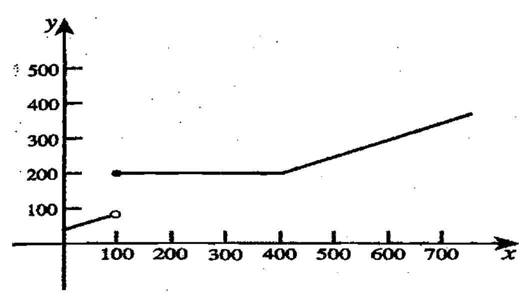


Figure 17. Graphical representation for Instructional Interview Problem No. 3a.

A: To represent this information in another way, I will make a table like [writing out this numerical representation of the data she extracted from the graph shown in Figure 17].

x	0	50	100	150	200	400	500	600
y	50	75	200	200	200	200	250	300

I: Any other representations that you can think about?

A: Maybe I can make an equation.

I: Would you please show how?

A: I'm not sure that there are two equations or three equations. OK. This is one line [pointing to the line in Figure 17 that started with $x = 0$], and these are two different lines because they have different slopes. So there are three equations. To find the equation of the line, I need a slope. The first line here has a slope $= \frac{75-50}{50-0} = \frac{25}{50} = \frac{1}{2}$. Then my equation is $y = mx + b$, which is $y = \frac{1}{2}x + 50$.

I: Would you please give me more explanation of how you got this equation?

A: OK. I get slope $m = \frac{1}{2}$ and b is the y-intercept, which equals 50. I plug these numbers into the slope-intercept form, $y = mx + b$. Then I get the equation.

I: How about the other two lines?

A: This line here [pointing at the horizontal line in Figure 17] has the same y so the equation is $y = 200$. And the last one here, I need a slope. Again the slope $= \frac{300-250}{600-500} = \frac{50}{100} = \frac{1}{2}$. Umm… [pausing for awhile].

I: So what is the equation?

A: This confuses me. I don't have a y-intercept. So I guess the y-intercept equals zero. Then the last equation is $y = \frac{1}{2}x$.

Noting that the graph did not cross the y-axis, Amy did not attempt to extend the graph and check if the graph passed the origin of the Cartesian coordinate system. She summarized the problem and assumed that the y-intercept was zero. As a result, she made a correct identification, but the identification was correct for the wrong reason. Her response did not rely on symbolically determining the function because at this point her understanding of finding a linear function was limited to the point-slope form.

Amy had a partial understanding of transforming functional representations. She was able to make a symbolic representation corresponding to each piece of a graph.

However, when asked to write the collective symbolic representation of this information in function notation [a piecewise-defined function notation], she was unable to perform the task. The instruction to this point, at most, provided two examples of piecewise-defined functions; both of these functions were given in a symbolic representation [$f(x) = \begin{cases} \frac{-2}{3}x + 2, x \leq -3 \\ 3 - x^2, x > -3 \end{cases}$] showing how to graph and evaluate specific values, such as $f(2), f(0), f(-4)$. Amy may not have had enough instructional support to develop symbolic representations from graphs of piecewise-defined functions.

Amy was able to describe how she constructed a symbolic representation for a linear function when some properties were given.

I: One of your homework problems asked you to find an equation of the line with slope $= \frac{2}{3}$ and a y-intercept $= 2$. Would you please show or tell me how you find it?

A: I used a slope-intercept formula: $y = mx + b$ where m is a slope and b is a y-intercept. So the equation is $y = \frac{2}{3}x + 2$.

Amy's homework problems showed that she was able to find linear equations using different forms, using two points on the line and using the point-slope form. However, she did not demonstrate an understanding of finding a linear equation using a point-slope form to find a linear equation in the Interview Problem No. 3a.

Amy indicated that she was able to transform a symbolic representation to a graphical representation, and she also demonstrating algebraic reasoning from a symbolic representation to a graph when she worked on the Ball Dropped Problem (Instructional Interview Problem No. 4).

> A ball dropped from the top of a tall building has height from the ground represented by $s = -16t^2 + 145$ feet after t seconds…OK. I know that is a graph of a parabola, which is upside down [opened downward]. So let's see. I will put this equation into my calculator to see if it is correct. OK. $y_1 = -16x^2 + 145$.

When solving function problems used in the first interview during the instruction of functions, Amy was able to make connections among the representations and function concepts. For example, she connected:

- a graphical representation with the vertical line test for determining a function. Amy explained that to identify a function, she could use the vertical line test with a graph of the relation. She indicated if the vertical line hit the graph more than once, then the relation was not a function.
- the concept of a function, its graph, a horizontal line test, a one-to-one function, and an inverse of function. In describing her understanding of a one-to-one function, she used the horizontal test, indicating that if the graph of a function passed the horizontal line test, it was a one-to-one function. Amy also understood that if it was a one-to-one function, then it had an inverse.

Amy identified applications for graphical representations in real situations. With the Piecewise-Defined Function Problem, she was able to construct a real situation that corresponded to the graph provided in the problem (see Figure 18).

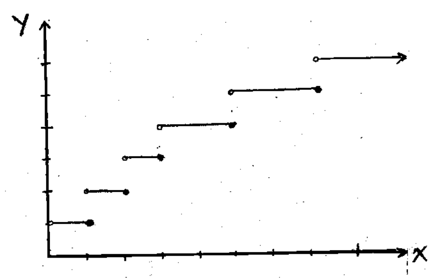


Figure 18. Graphical representation for Instructional Interview Problem No. 3b.

> From this graph, it shows something starts here; they don't know the exact point [because there is a white circle at the beginning point] but around zero and then starts to go and stop at this point. Ah...And then starts to go and then stop, starts to go and stop again and keeps going and going. They don't know exactly what the starting point is. And the last one keeps going straight. So something starts and then stops at the certain time like umm. I don't know. Umm... It's like a plane ticket. Like paying one price you can go from one spot to another and pay another price to go from another spot to another, and so on.

When asked about the line with the arrow, Amy said, "After that point [pointing at the most right-hand side of the arrow], y will be the same whatever x is." She did not consider an explanation for a real situation that would explain this particular point.

Amy was also able to apply her understanding of functions (i.e. the one-to-one functions and the inverse functions) when she worked with real situation problems. For example, when working with the Ball Dropped Problem (Instructional Interview Problem No. 4), she identified that the function of a dropped ball was one-to-one even though the symbolic representation of the function was a parabola, which in the general case was not one-to-one. Amy's algebraic reasoning with this problem was correct. Her explanation was that the ball was being dropped; therefore, it was only moving in one direction downwards. This decision helped her conclude the correct answer that the function had an inverse.

Instructional Interview Two

When solving function problems used in the second interview four weeks after the instruction of functions began, Amy's understanding had improved. When working on the quadratic functions in the Interview Problems No. 5 and No. 6, Amy correctly described the variables in the quadratic symbolic representations. She described that the variable a told her if a parabola [a quadratic graph] was opened upward where a is positive or downward where a is negative). Variables h and k told her if the graph moved to the right (h is positive) or the left (h is negative) and moved up (k is positive) or moved down (k is negative) respectively. She described those variables correctly; however, it is hard to conclude that she had instrumental or relational understanding among these variables. When asked how she knew these ideas, she said she learned it from one of the labs in her recitation class. The translation between the two forms of the quadratics function ($y = ax^2 + bx + c$ and $y = a(x-h)^2 + k$) was not considered much in the lecture and recitation classes.

Amy used her number sense to identify the domain of a function. For example, when she worked on the Equivalent Problem (Instructional Interview Problem No. 8), given $f(x) = \frac{x^2 - 4}{x + 2}$, Amy stated that the functions $f(x) = \frac{x^2 - 4}{x + 2}$ and $g(x) = x - 2$ were not the same because the domain of $f(x)$ could be all real numbers except –2. She said that when $x = -2$, the function *f(x)* had zero as a denominator; division by zero was undefined, but the domain of $g(x)$ could be all real numbers.

Amy's Post-Instruction Understanding of Functions

Following the instruction of functions, Amy possibly shifted her understanding of functions. Her response to the question "In your opinion, what is a function?" was expanded from "a function is an equation" to "a function is an equation where its graph passes the vertical line test." Her explanation also included a concept of a one-to-one function. Besides stating "A one-to-one function is a function that passes the horizontal line test," Amy expanded on her explanation of what she meant by passing the horizontal line test: "When you draw horizontal lines on the graph of a function, there is no line that hits the graph more than once." Additionally, she provided examples of what she meant as shown in Figure 19.

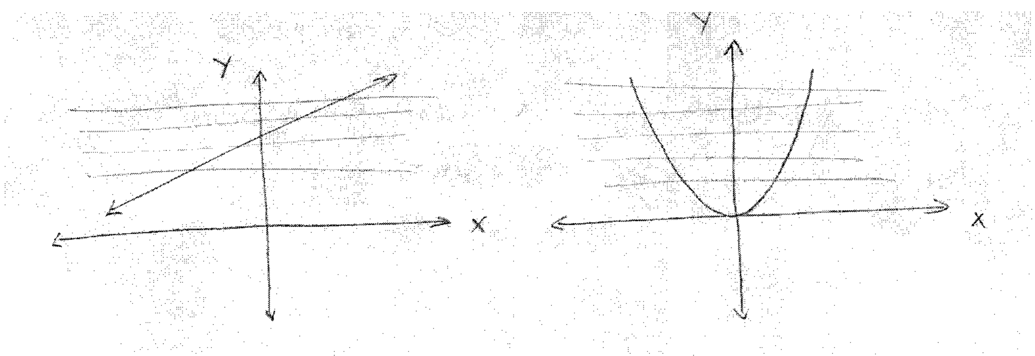


Figure 19. Amy's examples of one-to-one (left) and not one-to-one functions (right).

> This line is a one-to-one function because when I draw horizontal lines, there is no line that hits the graph more than once. But this parabola is not a one-to-one function because it does not pass the horizontal line test.

Amy was able to tell if a function had an inverse and was able to show how to find an inverse function. She stated that a function had an inverse if it was one-to one because after reversing the domain and range, it was still a function.

Amy indicated that the function concepts taught in this course were easy for her to understand. She claimed that she knew more about functions than she had known prior to and during the instruction. Prior to instruction, most of her understanding relied on her recall of mathematics from high school. At the end of the ninth week when the instruction on functions was completed, she was able to relate her understanding obtained from the class for solving new problems involving function concepts. When she was asked to solve the Construction Function Problem (Post-instructional Interview Problem), she solved this problem using some concepts of functions, including an undefined function, zeros of a function, a nonnegative function, and a function that contained a certain point.

> OK. The function is undefined at –3, which means the function has to be a fraction and at the bottom of the fraction [denominator] has to be 0 when $x = -3$, and on the graph it has to stop at x = –3 so that would be $x + 3$. Next the function has a zero at $\frac{1}{2}$, so it means when y = 0, $x = \frac{1}{2}$. Next the function contains the point (4,7), so when x= 4, y has to be 7. A function is always nonnegative, so y will never be less than zero. [She wrote $y \geq 0$.] OK. The top part of the function has to be something that when $x = \frac{1}{2}$, y has to be zero. So on top, x has to be $\frac{1}{2}$ in order to get $y = 0$. [She set an equation $x = \frac{1}{2}$ then worked backward to get the equation, she multiplied both sides by 2, then subtracted 1. Her final expression was $2x - 1$.] So the top part of the function is $2x - 1$. Let me check $x = \frac{1}{2}$, then $y = 0$. It works. And the function has to be nonnegative. So you can put the absolute value sign for the whole thing because no matter what [x] you have, except at –3, you always have positive numbers if you take the absolute value for the whole thing.

However, Amy demonstrated a misunderstanding about domain and range of this function at this time. She incorrectly claimed that the lowest domain corresponding to the lowest range.

A: The function is always nonnegative and the function's domain is $[-5, \infty)$, so –5 is the lowest for x that we get. So that means when $y = 0$, x has to be 5. Wait when $y = 0$, x has to be $\frac{1}{2}$ too.

I: Why do you think when y = 0, x has to be –5?

A: Because the lowest y is 0 and the lowest x is –5.

At the end of the instruction, Amy was able to explain her opinion of a *mathematical function* more accurately and in more than one way. Amy provided a function description more precisely than what she gave at the beginning of the course: "When you put in one input in the equation, you will get only one output. If there is more than one output, it is not a function." When asked if she could give examples of functions, she not only provided a mathematical function, ("$y = x^2$ is a function because when you graph it, it is a parabola and it passes the vertical line test.") but also a real situation example: "A person and date of birth is a function because each person has only one date of birth, which means one input and one output."

During the post-instructional interview, Amy concentrated on two types of representations: verbal and symbolic. The representations that she used for solving each problem in the interview consistently led her to a correct solution. The verbal representation that Amy used was the description or explanation of the information given in the Function Construction Problem (Post-instructional Interview Problem). Saying "the function is undefined at –3," Amy clarified her thought by using a symbolic representation in the process of constructing the function based on the constraints given.

> For the constraint of "the function is undefined at –3," this function has to be a rational function and its denominator has to be $x+3$. So it must be something on the top [numerator] and x+3 at the bottom [denominator] like this [writing $\frac{\Delta}{x+3}$].

Even though this problem did not encourage a particular representation, Amy worked on the problem addressing two types of representations: verbal and symbolic. She focused on verbal and symbolic representations after instruction, and the interviewer did not probe her to use other representations. Furthermore, the instructor focused on symbolic manipulation in solving problems during the last two weeks of the term, perhaps influencing Amy's choice of representations.

On another level, Amy viewed functions as sets of procedures. Amy correctly identified a function with the vertical line test, a graph of a function with the horizontal line test, and a function and its inverse. In addition, Amy expanded function ideas while attempting to solve the Function Construction Problem (Post-instructional Interview Problem). She connected the concept of undefined [from a problem constraint: A function is undefined at –3], the zeroes of a function [from a problem constraint: A function has a zero at $\frac{1}{2}$], and a nonnegative function [from a problem constraint: A function is always nonnegative]. From these connections, Amy immediately stated that the function was a rational function that had $x-\frac{1}{2}$ as the numerator and $x+3$ as the denominator. In addition, she quickly claimed that this function was an absolute value function because it was always nonnegative. Amy's understanding of the domain and range of a function did not support her in finishing the construction of the function needed in the Post-instructional Interview Problem. She was unable to construct the part of the function that had $[-5,\infty)$ as the domain. Since the lecture instructor discussed a domain of radical functions in class, the interview probed Amy by asking her whether she could find the domain of these functions: $f(x)=\sqrt{x+2}$ and $f(x)=\sqrt{x-3}$. Amy responded correctly that the domain was "negative 2 to positive infinity" and "3 to positive infinity" respectively. However, when she was asked to construct a function that had a domain from negative 5 to positive infinity, she did not think about a radical function that she had experienced in class. Given a radical function, Amy was able to find its domain but not vice versa, suggesting that she was unable to think in the opposite direction for this situation. Looking back to what had been taught in class, the interviewer found that the examples related to constructing functions when domains were provided were not discussed. Even though there were exercise problems in their textbook, those problems were not assigned as homework.

After instruction, when the researcher asked Amy to apply her knowledge and understanding of functions, she described real world problems such as the population growth and decay problems as well as the money investment problems. She also stated,

> I can use function knowledge to find how much money I will earn if I deposit a certain amount of money in my account in a certain period of time. Or I can find how long I should deposit money in my account if I need a certain amount of money. Also I can find the number of population in a certain year by using the formula.

Amy did not show her work on these types of problems, but she did refer to situations that the instructor provided in the class. Consequently, it is questionable whether she was able to extend her understating beyond what she studied in class or whether she remembered doing application problems that were taught or demonstrated in the class.

Ross

Ross majored in Exercise and Sport Science; he was a freshman taking College Algebra as a requirement for his major. The College Algebra course was his second college mathematics class. He took Intermediate Algebra as his first college mathematics course, but he thought that wasted his money because it was too easy for him. He said, "It is a good refresher but I think I can do College Algebra without taking that class." After taking College Algebra, he is required to take one more mathematics course (Elementary Functions) for his major. Ross was home-schooled from second to ninth grade. In high school, he took Algebra I, Algebra II, Geometry and Trigonometry. Ross said that he really liked mathematics, and because of this College Algebra class, he wondered if he should change his major to mathematics. With respect to the amount of time he spent on this class, he said, "I spend the most of my time on the difficult classes. I don't spend lots of my time on mathematics [College Algebra] at all because it is very easy." Ross said that he was very confident about his mathematics ability. On the other hand, he was asked the same question near the end of the term, he said "No. I'm not confident in this class just because I haven't done the homework assignments but I know I can review it very easy. I think I am good at math." When asked how he studied the material, he said he read the content before coming to the class. He never went to his instructor or the GTA office to ask for help, and he never asked for help from any tutors at the Math Learning Center. In fact, most of the time he studied by himself. If there were any questions or any parts of the content that he could not understand, then he asked his GTA in the recitation class.

As with Amy, Ross often participated in the class by sitting in the front row in both lecture and recitation classes, by answering questions, and by presenting his ideas if asked. When he was asked to rank the difficulty of this class from 1 to 10 (1 is very hard and 10 is very easy), he said that for him this class was right in the middle: "I think it's not hard. But it's not like you can pass without looking over your notes and doing exercises. I know if I spend enough time studying it, I will get a good grade."

Ross's Understanding of Functions Prior to Instruction

Prior to the instruction on functions in this College Algebra course, Ross claimed some knowledge about functions. He stated that "To me a function is like a relationship between two things like x and y." When asked if he could give examples of the functions, he indicated his familiarity with linear functions by providing examples of linear functions.

I: Would you please give an example of functions?
R: I think $y = x + 3$ or $y = 2x - 1$ are functions.
I: How do you know they are functions?
R: I remember that each of these is an equation of a line and a line is a function.
I: Do you have any different reason that indicates these are functions?
R: No.

Even though Ross had studied functions in high school, he indicated a limited knowledge and understanding of multiple representations with respect to functions. He asked for an explanation. The researcher described multiple representations as:

> Same data or information can be represented in different forms. Those forms are called multiple representations. For example, I can represent the number of students enrolled in Collage Algebra distinguishing their major by using a table, pie graph or bar graph.

Ross represented a function as "a relationship of x and y in an equation," indicating that he had an idea of symbolic representations. Further conversation guided him to describe more about multiple representations.

I: You said that a function could be represented as an equation of x and y. If you have an equation indicating their relationship, what can you do with the equation in order to clarify their relationship in another form?
R: I can graph it.
I: Anything else that you can do?
R: No.
I: Can you find a value of y if I give a value of x?
R: Yes, I can.
I: How?
R: I plug the number for x in the equation and calculate a value for y.
I: Would you please tell me again, what can you do if you have an equation indicating the relationship between x and y?
R: I can graph the equation and I can find the value of y related to the value of x.

This conversation suggested that Ross had some ideas about changing a function representation from one form to another.

Ross's Understanding of Functions During Instruction

During the instruction of function, Ross's knowledge and understanding of functions was investigated through two interviews conducted after the second week and fourth week of the instruction of functions.

Instructional Interview One

In the first interview, Ross demonstrated a misunderstanding about functions when he was asked if a graph of a single point represented a function (Instructional Problem 1 Part A, question c). His misunderstanding was that a function had to have "an ongoing line."

R: Umm. This graph showed a dot. It is not a function. I think a function is gonna have an ongoing line but this is just a dot.
I: What do you mean by "an ongoing line?"
R: A line that passes through a dot in this case.
I: What criteria do you use for determining if these graphs represent graphs of functions?
R: Well, I just check how many outputs for one input.
I: How about this graph [pointing to a graph of a dot]? Can you apply your criteria to this problem?
R: There is only one input and one output. I'm not positive but I don't think it is a function because there is no ongoing line.

However, in a later problem in the interview, he was asked to determine if a graph of three points on the Cartesian coordinate system represented a graph of a function (Pre-Instruction Problem 1 Part A, question i). Ross recalled that his instructor showed an example of three numbers from one set corresponding to three different numbers in the other set. This recollection helped him to connect his thoughts to the graph of three points. Eventually, he stated that the graph of three points represented a graph of a function.

Besides developing his knowledge and understanding of functions, Ross also demonstrated his understanding of one-to-one and inverse functions during the first interview. Ross indicated that a one-to-one function was a relation between inputs and outputs and that each input had only one output and each output had only one input. His statement was similar to the explanation in the textbook and the one that the instructor gave in class: "If a function is one-to-one, then for each y-value, there will be only one x-value." Ross's understanding encouraged him to check a one-to-one function using the horizontal line test. He indicated a function was one-to-one if a horizontal line crossed the graph of a function no more than once.

Ross was able to tell if a function had an inverse. When he was asked to talk about what he knew about inverses of functions, Ross said,

> Not all functions have inverses; only a one-to-one function has an inverse because when the inputs and outputs are switched, each input still has only one output.

He also gave two examples showing that a one-to-one function had an inverse. On the other hand, if a function was not a one-to-one function, it had no inverse.

> I can give examples like $\{(1,2),(3,4),(5,6)\}$, this function has an inverse because after I switch domain and range, $\{(2,1),(4,3),(6,5)\}$, and it still is a function. But if I have $\{(2,1),(3,2),(4,2)\}$, after I switch domain and range, $\{(1,2,),(2,3),(2,4)\}$, this is not a function.

Even though the example that Ross gave to indicate his understanding of one-to-one functions was simple, it differed from the ones given in class. The examples given in class were in a graphical representation and were related to the horizontal line test explanation.

Ross used four different representations when working on function problems during the first interview: verbal, numerical (tabular), graphical, and symbolic (equations). He showed that he was most comfortable with numerical representations. Each time he was asked to determine if a graph represented a function, he checked the inputs and outputs of the graph. When he saw the graph of a circle, he realized that he could use the vertical line test to determine if it was a graph of a function.

> R: This is a graph of a semi-circle on the *x*-axis. I would say it is a function. The reason is the same as the previous problems. There is one output for every input. The next one is also a graph of a semi-circle, but it is on the *y*-axis so it is not a function because there are two outputs for one input. And this is a graph of a circle. It is not a function. I checked by using the vertical line test. There is more than one output for every input.
>
> I: Do you use a different strategy to determine a function?

> R: Yes. I was never aware of that [the vertical line test] until I saw the graph of a circle.

Working on the Car Problem (Instructional Interview Problem No. 2), Ross interpreted a verbal representation for Car A and Car B traveling as described in the graph showed in Figure 16.

> From this graph, Car A is faster than Car B because it shows that the line of Car A is going higher than Car B's. Car B is barely going up at the beginning but when the graph shows up further, it shows that Car A's speed doesn't go up much because the line is curving back. It doesn't have much incline and Car B…umm… It's inclining and getting steeper. It shows that Car B goes faster as the graph goes up and Car A is getting slower because the incline is not as steep as the incline of Car B.

From his graphical transcription in Interview Problem No. 2, Ross seemed to possess conceptual understanding of graphical representations. He stated, "Car B goes faster as the graph goes up." He did not indicate that he compared the speed of Car B to that of Car A or to the speed of Car B itself at the previous time interval. Similarly, he stated the speed of Car A was slower.

Ross showed that he was able to connect with his different understandings about functions such as the concepts of functions, their graphs, the vertical and the horizontal line test, one-to-one functions, and inverses of functions. Ross showed he understood the use of the horizontal line test to examine if a function had an inverse by connecting his knowledge of the horizontal line test with a one-to-one function. He said,

> I can check if a function has an inverse or not by using the horizontal line. If no horizontal line crosses the graph of a function more than once, then it is a one-to-one function. And because it is a one-to-one function, then the function has an inverse.

Ross was not able to apply his understanding of a graphical representation to a real situation at the first interview during the instruction. For the Instructional Interview Problem No.3b, he was not able to describe a real situation that corresponded to the graph in the problem. Additionally, when working Instructional Interview Problem No.

4, he was unable to identify that the function of a ball dropped was a one-to-one function because he considered only the graph of the function from the initial functional representation (a parabola opened downwards). He said it was not a one-to-one function because it failed the horizontal line test; therefore, this function did not have an inverse.

R: I will put this function in my calculator to see what the graph looks like [putting the function $y_1 = -16x^2 + 145$ into his calculator].

R: It's a parabola opened downwards. So it does represent a function because there is no vertical line that crosses the graph more than once. And I will say it doesn't have an inverse because if you use the horizontal line test, it crosses the graph twice at some points. It's supposed to cross the graph once then it will have an inverse.

I: You said that the graph that you got from this function [$y_1 = -16x^2 + 145$] has no inverse. If you think about the situation that the ball was dropped from the tall building, do you think the graph of the ball dropped still looks like what you have now?

R: Yes.

This lack of connection to the real situation of this problem led him to the incorrect notion that the Ball Dropped function did not have an inverse since he thought that a ball dropped function was not a one-to-one function.

Instructional Interview Two

At the second interview session, Ross was asked to work on problems using symbolic and graphical representations of the quadratic function. He demonstrated his understanding of the relationship between these two representations and of variables in the quadratic standard form (Instructional Interview Problem No. 5).

> *a* indicates what direction the parabola faces. If it's positive, it faces up and if it's negative it faces down... *h* is going to indicate how many places that the vertex shifts to the right or to the left...[*k*] tells me how many places that the vertex shifts up or down.

As with Problem No. 5, Ross described his understanding of quadratic function when solving Problem No. 6.

> Umm... I'll start with a. It's gonna be negative because it's facing downwards. Umm... h is going to be negative too in this standard form because it is on the left of y-axis. So I just guess like $-1.-2,-3,-4$, and -5. I guess like -5 and the vertex is positive on y-axis because it is in the positive area of y-axis [Quadrant 2]. So k is gonna be... let's say 1. So, I can't remember a bigger number makes it wider or fraction makes it smaller. I'm pretty sure that fractions make it smaller but I can check on this by using my calculator [using his calculator to check the impact of a on the width of a parabola].

After trying several values for a, Ross realized that the smaller the absolute value of a, the wider the parabola was.

When working on the Salary Problem (Instructional Interview Problem No. 7), Ross recognized that if the number of years [N] in the salary function of A [salary = 30000 + 2500N] and that of B [salary = 30000 + 1800N] were the same, it was impossible for B to earn more money than A unless B worked longer. With this understanding, he recognized two different ideas: the number of years N was different or N was the same. These two different ideas suggested that Ross connected the context to a real situation where the person who worked longer possibly earned more than the one who worked fewer years.

When working on the Equivalent Function Problem (Instructional Interview Problem No. 8), Ross simplified a function and incorporated the domain constraints. He knew that the expression $\frac{x^2-4}{x+2}$ could be simplified to $x-2$. However, to determine if $f(x)=\frac{x^2-4}{x+2}$ and $g(x)=x-2$ were the same, he correctly made a decision to check their domains. He recognized the importance of checking the domains of the functions when considering whether they were the same.

Ross's Post-Instruction Understanding of Functions

Following the instruction on functions, Ross showed that he had developed his knowledge and understanding of functions. His concept of functions changed from "a function is a relationship between two sets of numbers or objects" to "a function is a relationship or correspondence between two sets of numbers or objects, each element in the first set corresponds to only one element of the second set." His statement was similar to one of the definitions provided in class by the instructor, which stated that "A function is a correspondence between sets of objects named a set *D* and a set *R*, to each element of a set *D* that corresponds to exactly one element of a set *R*." With regard to determining a function, he seemed to be more comfortable checking inputs and outputs than in using any other approaches. When asked how he could determine a function, Ross stated that he checked inputs and outputs to see if for each input, there was only one output. When asked if he could use other methods, he said he could use the vertical line test with a graphical representation.

After instruction with functions, Ross used verbal and symbolic representations more often than graphical and numerical (tabular) representations. Ross did not use the numerical (tabular) and graphical representations during the process of constructing the function to check if his function worked with the given constraints. The reasons for not using these two representations might be that (1) at the end of the instruction, the instructor often used symbolical manipulation strategies to demonstrate example problems given in the class or (2) the instructor never demonstrated that they could check their work using the numerical (tabular) or graphical representations. The interviewer did not challenge him to use these two representations as he worked on the problems.

Ross used several function concepts as he solved a problem after the instruction (Post-Instruction Interview Problem). He constructed a single function using concepts of undefined functions, zeros of a function, nonnegative functions, function domains and a function containing a particular point. While attempting the Function Construction Problem (Post-Instructional Problem), Ross described his understanding of the concepts and connected those concepts to produce the function.

I: From the problem constraints, would you please give me a function that you will construct?

R: From all these fives constraints [see Post-Instruction Interview Problem, Appendix I], my function will be a fraction [a rational function]. The

> function will have $x - \frac{1}{2}$ for the numerator because that will give a zero when $x = \frac{1}{2}$. And $x + 3$ will be the denominator and that will make a function undefined at $x = -3$. A function is always nonnegative so I think I can either square the function or use the absolute value sign. And I have to think about how to make the domain start from –5 to infinity and when $x = 4$, I need $y = 7$. That is what I can think of right now but I have to check again to see if all these constraints work for my function.

Ross's responses to the Function Understanding Questionnaire indicated that he was able to apply his knowledge and understanding of functions to some types of real world situation problems such as population growth, growth rate, compound interest, and other types of business profits.

> I can use what I know about functions to solve problems like population growth rate, compound interest, compound continuous interest, and the maximum profit from business. These kinds of problems she [the instructor] showed us in the class. I'm positive that I can do these problems if I see them again.

The application problems that Ross referred to were similar to the problems that the instructor demonstrated in the class and that existed in the textbook.

Emma

Emma majored in Exercise and Sport Science. She was a freshman in her first term in this university. She transferred from one of the universities in the southern part of the U.S. She took the course because it was a major requirement and her first college mathematics class. After taking College Algebra, she was required to take one more mathematics course (Elementary Functions) for her major. In high school, she took Geometry, Advanced Algebra, Precalculus I, and Precalculus II. But she dropped Precalculus II before completing it. No reason was provided.

Most of the time Emma participated in the recitation class by sitting in the back of the class and at about the middle in the lecture class. She was quiet and seldom asked or answered any questions posed in the recitation class. Emma said she always studied by herself and before going to the lecture so she knew what she needed to pay attention to. If she got to the point that she could not understand, she waited until the instructor discussed it in the lecture class and then she reviewed it after the class. She always did her homework assignments. She thought the way she studied was helpful for her. She never went to see her instructor or her GTA during their office hours. She never went to the Math Learning Center that provided help for students who needed it. She said, "I never ask for help from other people in doing math. I usually study by myself."

At the end of the instruction, Emma was asked to rank the difficulty of the College Algebra course from 1 to 10 (1 means very hard and 10 means very easy) she responded with an 8 indicating the ease of the class. She said that she was familiar with most of the content of this course and also felt that most of the topics in this class she had learned previously.

Emma's Understanding of Functions Prior to Instruction

Prior to the instruction on functions, Emma had learned some concepts of functions. When asked what she knew about functions, Emma said "It is a relationship between x and y; that's what I can think about it right now." Further conversation led to a more thorough understanding of her knowledge. Emma thought a function only related to two sets of numbers.

I: Would you please tell me what x and y represent in a function?
E: Umm… two groups of numbers, I guess.
I: So do you think a function is only involving the relationship of numbers?
E: I think it is.

When she was asked to give some examples of functions, Emma explained

> $x + y = 10$, x and y are related, like when $x = 1$ then $y = 9$; when $x = 2$, then $y = 8$; and when $x = 5$, then $y = 5$ and so on.

Even though Emma had studied functions before taking this College Algebra course, she had a limited understanding of functions. Initially, at the beginning of this interview she demonstrated a limited knowledge of multiple representations of functions. When asked how a function could be represented, she stated

E: A function can be represented as a diagram [an arrow diagram] from the first group of numbers to the second group of numbers."
I: Are there any other forms that a function can be represented?
E: Umm... I think there are but I can't remember.

Her answers were based on what she remembered from her previous knowledge obtaining from high school. The interviewer continued the conversation probing her as follow:

I: Recently, you gave an example of a function. Do you remember what you said?
E: I said $x + y = 10$.
I: What do you call that form?
E: Um... Equation
I: So do you think besides a diagram what form that a function can be represented?
E: An equation.

While Emma did not explicitly describe a function, it was difficult to say whether she was able to distinguish the difference between a function and an equation. She was correct in stating that a function $x + y = 10$ was an equation. Still, she did not say that all equations represented functions. Emma did not recognize the phrase *symbolic representation* as a type of multiple representations of functions. When asked if she ever heard this terminology, she said "No or maybe; I cannot remember if I have heard it." The interview probed further about multiple representations.

I: Suppose you have an equation. What can you do with this equation?
E: What can I do with the equation?
I: I mean can you change the equation to another form that represents the same data?
E: I see. I think I can graph it. Umm... I think I can make a table too.

This conversation showed that she had some familiarity with at least three types of representations: graphical, tabular, and symbolic (she referred to as an equation).

Emma's Understanding of Functions During Instruction

During the instruction of functions, Emma's knowledge and understanding of functions was investigated through two interviews conducted two and four weeks after the instruction on functions began.

Instructional Interview One

Working on the problems during this first interview, Emma showed that she had some knowledge and understanding of functions. For example, when she was asked to determine whether the relations written in different representations including graphical, numerical (tabular), and symbolic representations were functions, she did most of the problems correctly except when she considered whether a graph of a single point was a function (Pre-instructional Problem No. 1A, question c and i). She said that,

> I'm not sure about this one because it is a point. What should I do with this? Let's see. I would say it is not a function because it's just a point. It does not look like a graph to me. I'm not sure because we never had an example of a function of just one point.

Similarly, when working with the graph of three points (Pre-Instructional Interview Problem 1A, question i), Emma also stated that it was not a function. Emma did not think the graph of points was a function because "It has only three points; there is not even a line." However, Emma was not able to correct herself with the case of the graph of a point or three points even though the interviewer encouraged her to use different approaches to consider the graphs such as the vertical line test (her typical method) and checking inputs and outputs.

Emma said that $y = 4$ was a function but it was not a function of y in terms of x. When asked about her understanding of a function of y in terms of x, she explained,

> A function of y in terms of x is a function that shows terms of x and y in the equation. Therefore, $y = 4$ was not a function of y in terms of x because all y

> values are always 4. It doesn't depend on x. No matter what x is, the y value is gonna be 4.

During this interview, Emma described further understandings of functions including a concept of a one-to-one function and a concept of an inverse of a function. When asked about her understanding of a one-to-one function, Emma initially thought of a function as passing the horizontal line test. When probed, Emma described that a function that passed the horizontal line test indicated one output corresponding to only one input. Emma understood that a function had an inverse if it was a one-to-one function. When asked how "a function has an inverse if and only if it is a one-to-one" worked, she explained that if it is a one-to-one function then when its domain and range are switched, the result was still a function.

Working on the problems in the first interview during the instruction of functions, Emma dealt with four representations: verbal, numerical (tabular), graphical, and symbolic. When determining a function, she only used the vertical line test. To investigate her understanding of this concept, the interviewer asked her to clarify how the vertical line test worked. Emma stated that the vertical line test would tell the number of outputs for a particular input. She provided her reason and examples:

> If a vertical line hits a graph of an equation twice, there are two outputs for that one input and this graph is not a graph of a function. If all vertical lines hit the graph at only one point, there is only one output for one input and this graph is a graph of a function.

Working on the Car Problem (Instructional Interview Problem No. 2), Emma showed that she understood the graphical representation of the cars' speeds. She described the information using a graphical representation (see Figure 16). She transformed the information from a graphical to a verbal representation correctly.

> From the graph, I know that they [Car A and Car B] start traveling at the same point. And Car A travels faster than Car B because its graph is above the graph of Car B all the time until they got to time at 1 hour. At t = 1 hour they have the same speed because their graphs intersect each other at that point. After t = 1 hour, Car B travels faster than Car A because its graph is above the graph of Car A.

Emma related her understanding of function concepts when solving function problems during the first instructional interview. She demonstrated her understanding of the graphical representation and the vertical line test. Emma understood how to relate the vertical line test concept to the graphical representation and the definition of function. By using the vertical line test and determining if the graph passed the test, she knew that it was a graph of a function because there was only one output for each input. Emma indicated her understanding of the use of the horizontal line test to examine if a function has an inverse.

> A function has an inverse if it is a one-to-one function. And a function is one-to-one if it passes the horizontal line test.

When working on the Instructional Interview Problem No. 3b, Emma was the only one of all the interviewed students who called this function a step function. When asked how she knew about this function (since the instructor never talked about this type of function in class), she said she remembered it from her mathematics classes in high school.

With the Ball Dropped Problem (Instructional Interview Problem No. 4), Emma, described her understanding of a graphical representation to a real situation, indicating that a ball was dropped from the tall building so it traveled one way from top to bottom. This understanding led her to answer the question correctly.

E: I remember this kind of question. The ball drops from the building, and it is a function. OK I can check it. I need to graph it. I put in $-16x^2 +$ 145. Oh I have a strange window right now, so I need to change it. Let's see [setting Xmin = 0]

I: Why do you make Xmin = 0?

E: Umm.. Because you can't have negative of time and the graph is in terms of time [putting in Xmax = 10; Ymin = 0 and Ymax = 150].

I: Is it a function?

E: Yes because it passes the vertical line test. And yes, it has an inverse because it is a one-to-one function because it passes the horizontal line test.

Emma demonstrated how she symbolically worked with the square root of an algebraic equation when she attempted to find an inverse of the function. With the Ball Dropped Problem, Emma's understanding of inverse functions with real situations guided her to the two correct solutions. Emma believed this function had an inverse because the function was one-to-one and demonstrated her familiarity with symbolic manipulation.

E: I need to find the inverse of $s = -16t^2 + 145$. Therefore, I need to switch s and t. That will be $t = -16s^2 + 145$, then $t - 145 = -16s^2$, and then I divide by -16: $\frac{t}{-16} - \frac{145}{-16} = \frac{-16s^2}{-16}$. So I have $\frac{t}{-16} + \frac{145}{16} = s^2$ because negative is canceled out so I have positive 145 over 16. And I need to put a square root $\sqrt{\frac{t}{-16} + \frac{145}{16}} = \sqrt{s^2} = s$. So s inverse is [writing in symbolic notation] $s^{-1}(t) = \sqrt{\frac{t}{-16} + \frac{145}{16}}$.

I: Generally, when we take a square root like $x^2 = 4$, what is x?

E: $x = \pm 2$.

I: In this situation when you take square root of s^2, how come you do not use $\pm$?

E: No we don't because we can't have negative for time and height. So we don't use the negative value.

Emma showed that she understood how to solve the symbolic representation problem related to taking a square root. She indicated that she had more than an algorithmic understanding of finding an inverse of function. She understood when a quadratic function had two real solutions and when it had only one real solution.

Instructional Interview Two

Emma showed progress with her knowledge and understanding of function concepts during the second interview. She was asked to work on the problems using symbolic and graphical representations of the quadratic function in Instructional

Interview Problem No. 5. In particular she talked about variables represented in the quadratic standard form as follows:

> So this is a graph of a parabola and I know from the equation that a has to be negative because the parabola is opened down. The vertex of a parabola is (h, k) so h has to be positive because a x value for the vertex is in positive area [Quadrant 4]. Umm… k is a y value on the vertex, it should be negative number.

She also considered the values of variables a, h and k in Instructional Interview Problem No. 6.

E: [Reading the problem]: "Give a reasonable symbolic representation for $g(x)$ in the form $g(x) = a(x-h)^2 + k$. Explain why your representation is reasonable." So, the original is just $y = x^2$, and it is a parabola and umm... I know from the last problem that a flipped over parabola [open downward] a has to be a negative number and the smaller $|a|$ umm… the wider a parabola is. So for this one, let's see. I will say $a = -5$ again, just for starting out with. And then my vertex is …Let's see. Umm… x corresponds to the x value and the x value goes much further along the x-axis than the y-value goes along the y-axis. In this case, I'll try $h = -4$ and $k = 1$ [putting $y_1 = -.5(x-4)^2 + 1$ in her calculator and graphing it].

Working on the Salary Problem (Instructional Interview Problem No. 7), Emma interpreted her thinking in two different possible ways; A and B might either work in the same numbers of years or different numbers of years.

E: There are four different groups and N represents the number of years since the date of contract. Each salary represents the salary that will be earned during the given years. OK. B will never earn more per year than A because they both have the same starting money [$30,000] and A always earns more per year than B. So it is not possible for B to earn more.

I: Is there any case that B can earn more than A?

E: The only way B would earn more was if he were employed for more years than A. Umm... that is the only one possible case otherwise he will never earn more than A.

For the last question in this problem (Will D ever catch up C?), Emma used her graphing calculator comparing the graphical representation of the salary of C and D to see their intersection point. She was able to explain when D earned more money than C from the graph.

I: Can you tell when D is gonna catch C?
E: Yes I can
I: How can you find? Would you please show me?
E: I'm gonna calculate [using a function menu in the graphing calculator] and find the intersection; first curve, second curve, and they intersect at (10, 42000), so that would mean at the 10th year they would earn the same amount of money and after that D would have a higher salary than C.

Working on the Salary Problem, Emma used the real situation to guide her in setting the calculator window for the number of years and amount of money, which in the real situation would never be negative numbers. Also she found the intersection of two linear equations, in this case the intersection of the salary equation of A and B, using her graphing calculator rather than completing the symbolic manipulation.

Emma's Post-Instruction Understanding of Functions

At the end of the College Algebra course, Emma's concept of function had developed from the notion that a relation of two sets of numbers written in terms of variables x and y to a relationship of two sets of numbers or objects, each element in the first set related to only one element of the second set. When asked how she could determine a function, Emma stated that she could either (1) use the vertical line test with the graphical representation to see if any vertical line crossed the graph more than once or (2) check the inputs and outputs to see if each input had only one output.

From the researcher's examination of her homework, quizzes, and midterm exams, the researcher concluded that Emma was able to connect her knowledge and understanding of functions obtained from the class to solve similar problems, new problems, or real world problems related to function concepts. Her responses to the Function Understanding Questionnaire revealed that she had gained confidence in her ability to work on mathematical function problems. Also, when asked to distinguish a relation and a function, Emma said she could use either numerical inputs and outputs or the graphical representation of that relation with the vertical line test depending on which one she felt was easier for her.

After the instruction of functions, Emma used two types of representations: verbal and symbolic more often than graphical and numerical (tabular). She used a verbal representation to explain the information of the constraints in the Function Construction Problem (Post-Instructional Interview Problem) verbally clarifying the constraint statements and writing symbolic representations in the process of constructing a function. For example:

> The function is undefined at –3 means umm... if $x = -3$ we cannot match that x with a y value. The function is undefined if it is a fraction and the denominator equals zero. That would be $x + 3$ at the bottom of the fraction.

Emma knew how to find the zeros of a function. She set the equation equal to zero; then she solved for the x value(s). For this problem she identified the function had a zero at $\frac{1}{2}$.

> That would mean $y = 0$ when $x = \frac{1}{2}$, so that would be $x - \frac{1}{2}$. And that would be the numerator.

Emma did not use either numerical (tabular) or graphical representations during the process of constructing the function for checking if her function worked for each given constraint. She checked if the function passed each constraint of the problem by entering the numbers into the function.

Attempting the problem after the instruction of functions, Emma connected several function concepts to solve this problem given at the end of the instruction interview (Post-Instructional Interview Problem). She connected the conceptions of an undefined function, zeros of a function, a nonnegative function, a function domain, and a function containing a particular point to form a single function required in the problem. She described her understanding of the problem constraints before she started solving the problem.

> The function is undefined at -3 and has a zero at $\frac{1}{2}$. So this function is in the form of a fraction. The top part [numerator] gives the zero and the bottom part [denominator] gives an undefined value. The function is nonnegative so umm… it is an absolute value function, I guess. The function has a domain from -5 to infinity so x values can't be any numbers but -5 to infinity. The last one, the function contains point (4,7) so when $x = 4$, the y value is 7.

The problem used for the interview at the end of the instruction on functions was not a real world situation problem. However, the interviewer investigated whether Emma was able to apply her knowledge of functions to a real world situation by examining her homework assignment and quizzes. The evidence indicated that she correctly solved the application problems. The application problems assigned for homework related to the population growth and compound interest. In addition, Emma indicated that she was able to apply her function knowledge and understanding to the real situations in her response to the Function Understanding Questionnaire. She claimed that she was able to use functions for some real world situations such as population growth and decay, and the investment rates (compound and continuously compounded interests).

> Now I think I can solve problems like population growth and decay, the bank accounting like compound interest and continuously compounded interest, and the maximum profit and the minimum cost of a business.

The application problems that Emma claimed she was able to solve were the same type of problems that were demonstrated in class and in the textbook as examples. Her work on the homework assignments and quizzes also indicated that she was able to apply her understanding of functions to those real world situations. When asked to give examples of applying functions to the real situations, Emma gave only the examples that were discussed in class.

Lindsey

Lindsey majored in Exercise and Sport Science. She was a freshman taking College Algebra as a requirement. She also was required to take one additional mathematics course. She said that she did not like mathematics. The College Algebra course that she was enrolled in this winter term was her first college mathematics class. In high school, she took Integrated Mathematics for three years; therefore she had some experiences with algebra, geometry, and trigonometry. In her senior year she took precalculus for one semester. Lindsey neither liked nor enjoyed mathematics classes; however, she did earn good grades in her high school mathematics classes. When asked about her favorite mathematics class, Lindsey said that she did not like trigonometry at all, was not a fan of geometry, and disliked algebra the least. She felt more confident working with algebra problems than others types of mathematics problems.

When asked how she studied the material, Lindsey said she reviewed the lecture notes when she worked on the homework problems and checked her solutions with the solutions at the back of her textbook. She did not ask for individual help from her instructor or the GTA or tutors at the Math Learning Center. However, she did ask her GTA questions sometimes in the recitation classes. Most of the time she studied by herself. Lindsey stated that if she did not understand the content she probably would ask for a tutor.

Typically, Lindsey sat in the middle of the class during the recitation, asking some questions but rarely presenting any ideas if asked in class. When asked to rank the difficulty of this class from 1 to 10, where 1 is very hard and 10 is very easy, she selected 7 to indicate this class is not too difficult for her.

Lindsey's Understanding of Functions Prior to Instruction

Lindsey had studied algebra and functions in high school. She described her understanding: "A function is a relationship between two numbers." She focused on two sets of numbers represented by x and y rather than two sets of objects in a general case because the examples she had seen most often in class were related to sets of numbers. When asked to give some examples of functions, she provided a relation of positive integers ($\{(1,2),(2,4),(3,6),(4,8)\}$). When asked why she thought this was a function, she reasoned, "each x had only one y." Initially, she did not give any examples in the form of equations or symbolic representations. When asked whether she thought some equations were functions, she agreed that some equations were functions; however, she said not sure which statement between "Not all equations are function" and "Not all functions are equations" was true. In fact, both of them were correct. When asked for examples of an equation as a function, she wrote $y = 2x + 1$. She explained that the equation she gave was a line and she remembered that a graph of lines other than vertical lines, were functions. She was unable to provide more information about lines versus vertical lines. What she said was what she remembered from her high school mathematics class. Her comment about her understanding of function prior to the instruction appeared to be instrumental with little connection of one concept to another.

Lindsey had some knowledge about multiple representations of functions prior to instruction. She correctly worked on problems (Pre-Instructional Interview Problem No. 1 and No. 3) using different representations, including verbal and graphical representations. Lindsey said that she remembered a function could be represented as an equation, a graph, and a table.

Lindsey's Understanding of Functions During Instruction

Lindsey solved mathematical function problems in interviews twice during the instruction of functions. The first interview took place two weeks after the instruction began and the second interview took place two weeks after the first interview.

Instructional Interview One

In the first interview, Lindsey was asked to solve four problems (Instructional Interview Problem No. 1, 2, 3 and 4) involving multiple representations and real world situations. As she worked on the problems, Lindsey demonstrated that her understanding was more procedural than conceptual; she was able to tell how and what the solution was, however, she was unable to explain the reasoning behind her answer. She also showed her procedural knowledge of how to solve a radical and constant function.

L: For $y = 4$, I will say this is a constant function. All y values are 4 no matter what x value is. And its graph is a horizontal line like this (drawing a Cartesian coordinate system and a horizontal line at $y = 4$). For $x^2 + y^2 = 1$ (writing an equation), I have to subtract x^2 from both sides.

I: Why do you have to do that?

L: To rewrite the equation y in terms of x so I can know if it is a function. [Her strategy was: $y^2 = -x^2 + 1$, $\sqrt{y^2} = \sqrt{-x^2 + 1}$, $y = \sqrt{-x^2 + 1}$] I will graph this to see its graph [drawing the graph as it appeared in her calculator].It is a semicircle on the x-axis so it is a function because it passes the vertical line test.

Lindsey followed the symbolic manipulation process rewriting the equation y in terms of x but she did not think of negative value when she took the square root of real number. With probing, she recognized that she did not perfectly follow the algorithm for taking the square root of real numbers:

I: When you take a square root of a real number, for example $x^2 = 4$, what will you get for x?

L: Plus or minus 2. Oh, yeah same as this problem. I have two y values $y = \pm\sqrt{-x^2 + 1}$. So this would not be a function because there are two y values for one x. For example, if x equals 2, y would be plus or minus the square root of two squared plus one that gives me plus or minus square root of 5 [writing $\pm\sqrt{2^2 + 1} = \pm\sqrt{5}$].

At this time Lindsey did not determine a function by using its graph. Instead she gave an example showing that there were two *y* values for one *x*; therefore it was not a function.

Lindsey was able to use different representations when working on function problems during this interview. These problems included verbal, tabular (numerical), graphical, and symbolic representation (equations). She was able to verbally interpret the information from the graphical representation when she worked on Instructional Interview Problem No. 2 (see Figure 16).

> L: I'm looking at the graph at t = 1 hour. It looks like Car A and Car B are going at the same speed. Because umm... they [graphs of Car A and Car B] intersect at t = 1 hour. But it doesn't say anything about the position.
> I: Is there any information from the problem that tells you about their position?
> L: Umm (pausing)

The interviewer encouraged her to think out loud interpreting her idea.

> I: Can you tell me about the speed of these two cars before one hour?
> L: Umm... Before one hour, Car A was going faster than Car B but Car B increased its speed more than Car A did. Umm...Car A was faster at the whole time before one hour. That would tell me that Car A was in front of Car B. So even though they had the same speed at t = 1 hour, Car A was in front of Car B because Car A got more distance. I mean Car A went faster than Car B at that amount of time.

Lindsey correctly responded to the problem by talking about the acceleration of the cars, even though this information was not directly provided. The question was "What is the relationship between the acceleration of Car A and that of Car B?" Lindsey's response was:

> It looks like Car B has a higher acceleration than Car A. The graph shows that Car A moves almost steadily. Look at this line (pointing to graph of Car B), it's getting steeper than that line [the graph that represented the speed of Car A). So Car B has a bigger acceleration than Car A has.

When determining a function from a graphical representation, Lindsey constantly used the vertical line test.

> Umm… If I use the vertical line test to determine a function, it works except a point right here (pointing at the dark and white points in Figure 17). But I remember that one [kind of point] includes the value at that point and the other doesn't. So either way the vertical line test still works. So it is function because no vertical line hits two points on the graph.

Lindsey demonstrated an ability to transform a graphical representation to a numerical (tabular) representation. She also seemed to understand transforming a graphical representation to a symbolic representation referring to the graph in Figure 17 (Instructional Interview Problem No. 3a). However, she was not able to complete writing the symbolic representation for this problem. One of the reasons might be that there was not much discussion in class about how to write a piecewise-defined function if the graph was given.

L: To represent this information in another way, I can write it as a table like this:

x	y
0	40
50	50
100	not sure
200	200
300	200
400	200
500	250
600	300

I: Any other representations that you can think about?

L: Maybe I can write it as an equation like $y = \begin{cases} \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{cases}$ because the graph breaks apart so each part has a different equation.

I: Would you please show me how to find the equation for each piece?

L: I'll try. From the graph I think there are three equations. To find an equation of the line, I need a slope. The first line here (pointing at the line on the most left hand of Figure 17) has a slope $= \dfrac{50-40}{50-0} = \dfrac{10}{50} = \dfrac{1}{5}$. Then my equation is $y = mx + b$, which is $y = \dfrac{1}{5}x + 40$.

I: What are *m* and *b*?

L: m is a slope of the first line, which is $\frac{1}{5}$ and my b is the y-intercept, which is 50. $y = mx + b$ is the slope-intercept formula of a linear equation.

I: How about another two lines?

L: This line here [pointing at the horizontal line in Figure 14] has the same y all the way through until $x = 400$ so the equation is $y = 200$. And the last one here, I need a slope again. So slope $= \frac{300-250}{600-500}$ $= \frac{50}{100} = \frac{1}{2}$ Umm...(pause)

I: So what is the equation?

L: I can't use the slope-intercept formula because I don't have the y-intercept. Umm... I will use a slope and a point. I can't remember what it's called. But I know how to do it. I use $y - y_1 = m(x - x_1)$; x_1 and y_1 is a point that the line passes through. So I pick (600, 300) as the point.

I: Why do you pick this point?

L: I can pick any point that this line passes through, but this point is obviously on this line. I put in $y - 300 = \frac{1}{2}(x - 600)$, and then I rewrite the equation $y = \frac{1}{2}x - 300 + 300 = \frac{1}{2}x$

I: Can you write the equation represented by this graph?

L: That would be $y = \begin{cases} \frac{1}{5}x + 40 \\ 200 \\ \frac{1}{2}x \end{cases}$

Lindsey showed her procedural understanding finding a linear equation using the point-slope form. However, she did not specify the domain for each equation. After she was asked, she wrote the equation as: $y = \begin{cases} \frac{1}{5}x + 40; 0 \le x < 100 \\ 200; 100 \le x \le 400 \\ \frac{1}{2}x; 400 < x \le 750 \end{cases}$

Lindsey was the only one of the five students who was able to write the symbolic representation for the graph of Problem No. 3a correctly.

Lindsey provided a real situation represented by the graph of Interview Problem No. 3b (see Figure 18).

> Umm... this is like the entrance price to a park with a certain number of customers. I say the age is on the x-axis and price is on the y-axis. You pay a different price depending on your age. For example, three dollars for people who are 10 years old or under, \$5 for people who are older than 10 years old to 15 years old, and so on.

Lindsey did not talk about the line with the arrow in the graph for this problem; therefore the interviewer asked how she thought about that line. She explained, "This line means people over this age pay the same price."

In this interview Lindsey did not have a chance to show that she correctly found a linear equation if two points were given. However, her homework showed that her procedures and solution were correct.

When solving function problems in the first interview, Lindsey changed from one representation to another and connected her understanding of function concepts. She was also able to apply her knowledge and understanding of functions (i.e. one-to-one functions and inverse of functions) when she worked with real situation problems. For example, when working with the Ball Dropped Problem (Instructional Interview Problem No. 4), she justified that the function of the dropped ball was a one-to-one function, even though the symbolic representation of the function was a parabola. She explained that in this situation where the ball was dropped from the building, its graph was only half of a parabola upside down. Her understanding suggested that the function had an inverse.

Instructional Interview Two

The second interview took place four weeks after the instruction of functions. Lindsey demonstrated her developing understanding of representations and function concepts that guided her solution to problems. When working on the Quadratic Standard Form Problem and Quadratics Standard Form with Reference Graph Problem

(Instructional Interview Problem No. 5 and No. 6), Lindsey correctly described her understanding of variables *a*, *h*, and *k* in quadratic, symbolic representations. Her understanding was the same as other students.

> I know that a parabola faces up if *a* is a positive number and faces down if *a* is a negative number. Umm… the vertex of the parabola moves to the right and up if both *h* and *k* are positive numbers. The vertex moves left and down if both *h* and *k* are negative numbers. The vertex moves left and up if *h* is a negative number and *k* is a positive number. The vertex moves right and down if *h* is a positive number and *k* is a negative number.

Lindsey showed that sometimes she did not connect the problem to a real world situation. However, with some assistance, she expanded her understanding. In the Salary Problem (Instructional Interview Problem No. 7), she responded to the first part of the problem as follows

> L: A's salary is 30000 plus 2500 times number of years that he have worked and B's is 30000 plus 1800 times the number of years he have worked too. So it doesn't appear that B will ever earn more than A because they both have the same amount of money to start with and A earns more per year. So B will never earn more than A.
> I: Is there any chance that B will earn more than A?
> L: Umm… I don't think so. *N* is number of years that they work and they are the same, right? [Lindsey considered only the case that A and B have worked for the same number of years.]
> I: What if the numbers of years is different?
> L: If they start working at different times… Oh yeah, if B works longer than A, then B possibly earns more.

Lindsey worked on application problems including the real world situation problems in the textbook that had been suggested by the instructor. After working on each problem by herself, she checked the solutions with the answers provided at the back of the textbook. Most of the time she got the right answers. If her solutions were not the same as those provided, she asked her GTA in the recitation class.

Lindsey was able to solve a system of two equations. However, her initial acceptance was incorrect sometimes. As she worked on the Instruction Interview

Problem No. 8, which asked her to determine whether $f(x) = \frac{x^2 - 4}{x + 2}$ and $g(x) = x - 2$ were the same, she stated they were the same after finishing reading the problem. Her reason was that $\frac{x^2 - 4}{x + 2}$ could be simplified to $\frac{(x+2)(x-2)}{x+2} = x - 2$. When she was asked to show how they were the same, she found that they were different because $f(x)$ had a domain restriction [x cannot be –2] while the domain of $g(x)$ could be all real numbers.

Lindsey's Post-Instruction Understanding of Functions

Lindsey showed that she had some understanding about functions from her high school mathematics class. Following the instruction of functions, she showed that her understanding of functional concepts had expanded. She claimed that most of the function concepts taught in this class were similar to her high school mathematics courses except there were more application problems.

Lindsey stated that the function concepts taught in this College Algebra course were not too difficult because most of the content she had learned in high school. However, she claimed that she had learned the concepts better than in high school, particularly applying function concepts to real situations. At the end of the instruction on functions, she used her knowledge and understanding of functions to solve the Post-instruction Interview Problem. In response to the Construction Function Problem (Post-Instructional Interview Problem), she used her conceptual understanding of functions including an undefined function, zeroes of a function, a nonnegative function, and a function that contained a certain point. She verbally described each constraint required for constructing a function in this problem.

L: The first constraint, function is undefined at –3, implies that the function has to be a fraction and it is undefined.

I: Why do you think it is a fraction?

L: I remember from the class that if the function is in a form of a fraction [a rational function], the function will be undefined at x, which makes the denominator equal to zero.

Lindsey continued describing each constraint of a function given in this problem.

> A function is undefined at –3, which means at $x = -3$ the denominator equals zero. So there is no y value when $x = -3$.

She continued with the other constraints

> Next a function has a zero at $\frac{1}{2}$. This means when $x = \frac{1}{2}$, $y = 0$ and zeros of a function can be found by setting the numerator equal to zero and then solving for x. "The function is always nonnegative" means the y values never be any negative numbers and the graph of this function will be above the x- axis. "The domain of a function is –5 to positive infinity" means x values start from –5 and gets bigger with no limit. The last one is "the function that contains the point (4,7)." This means when $x = 4$, $y = 7$.

Not surprisingly, at the end of the instruction, Lindsey showed that she had more confidence when she worked with mathematical function problems than prior to instruction. She said that she did more types of mathematical function problems than she had ever done in high school. Lindsey responded to the Function Understanding Questionnaire administered at the end of the instruction revealing her conceptual understanding of mathematical functions was more accurate than originally in the first interview during the instruction of function. At the end of the instruction of function Lindsey stated:

> A function is a relationship between two sets of numbers. This relationship can be shown in either an equation or table. Umm...it can be shown as a graph too. We can check a function by using the vertical line test or by looking at the inputs and outputs.

When she was asked to give examples of a function, she gave a simple linear function.

> $y = x$ is a function. I can check inputs and outputs. There is one y for every x. I can also check by drawing a graph and then using the vertical line test.

When asked how she applied her knowledge of functions to the real situations, Lindsey said that she could apply mathematical function knowledge to solve real world problems such as the investment problems and population growth and decay problems. She did not show her work on a real world situation problem during this interview, but she showed her work on the application problems in the textbook that were assigned. She claimed that she worked on the problems by herself and she did the problems correctly.

Kyle

Kyle, a senior who was majoring in Liberal Studies and Social Science took College Algebra because it was a core course for his major. Before taking this College Algebra class, he took two mathematics classes: Introduction to Contemporary Mathematics and Mathematics for Management, Life, and Social Sciences. This College Algebra course was his last mathematics class for his degree. He took Algebra I, Algebra II, and Geometry in high school. He had studied functions in high school but believed that it did not include logarithmic functions. He said he did not hate mathematics. However, this term he enrolled in many classes required for his major, so this College Algebra class was not his priority. He said that he spent less time studying for this class. He believed that if he knew mathematics, he would know how to solve and get an answer for a problem. He said he liked mathematics but he just did not have much time to study and do homework. He said this was his last year so he spent more time on the classes that related to his degree than this class. He believed if he understood mathematics, it was easy for him to get an "A" grade. He felt this would be the first mathematics class that he would not do well in. When asked how he studied mathematics, he stated that if he did not understand it or had some problems, he usually figured it out on his own. He had never asked for help from his instructor or GTA. He went to the Math Learning Center (MLC) one time when we worked on the first lab homework because he did not know how to solve equations using his calculator. That was the only time that he got help from MLC. Most of the time he studied on his own.

He claimed that he did not feel comfortable with this class, as he did not spend much time on it. He spent about one or two hours per week doing homework. He was sure that if he studied more he probably would rate this class easy, but he only studied about one or two hours a week. Consequently, he ranked the difficulty of this class at about 6 (1 represented the most difficulty and 10 represented the easiest). He stated that this class seemed more time consuming; the concepts were not very hard but needed a lot of practice.

Kyle's Understanding of Functions Prior to Instruction

Before taking this College Algebra class, Kyle had studied algebra and functions in high school. He had a partial knowledge of functions. Prior to the instruction of functions in this class, he addressed a function as two sets of numbers. He stated, "A function is a relationship between two sets of numbers," and he provided an example of a function using an arrow diagram. At that time he did not discuss one-to-one functions.

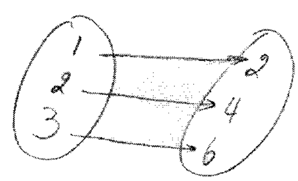


Figure 20. Kyle's example of a function.

Kyle did have an understanding of multiple representations of functions. He said that a function could be presented as a diagram, ordered pairs, and an equation. He also confirmed that a function could be changed from one representation to another. He changed the function represented in his arrow diagram (Figure 20) to a numerical representation (a set of order pairs, $\{(1,2),(2,4),(3,6)\}$).

Kyle demonstrated his understanding of a transformation among multiple representations by correctly matching the graphs in Figure 15 corresponding to a specific situation. For example a situation provided in the problem was:

> A balloon was blown up in class and then let go. It flew around the room. The amount of air in the balloon is the dependent variable.

He described his understanding of the situation as the following:

> Umm… the amount of air is on the *y*-axis. It's blown up so it means air is increasing and then starts decreasing because it flew around the room. The *y*-axis represents the amount of air; this will start with none at the beginning because there is no air in the balloon and as blowing it up and letting it go, so the air is decreasing again.

Kyle's Understanding of Functions During Instruction

Kyle participated in two interviews conducted in the sixth and eighth weeks during the instruction of functions.

Instructional Interview One

In the first interview, Kyle was asked to solve four multiple representations and real world situation problems (Instructional Interview Problem No. 1, 2, 3 and 4). While working on the problems, Kyle demonstrated a partial understanding of functions. He had a limited understanding of determining a function by either using the vertical line test or checking inputs and outputs. He could not explain how the vertical line test worked. He knew that if he had a graphical representation, then the vertical line test worked for verifying whether it was a function. He correctly worked on algebraic symbolic manipulation. He completely demonstrated the symbolic manipulation of y in terms of x for the equation $x^2 + y^2 = 1$, which few students could do.

Kyle precisely used four different representations when working on function problems during this interview: verbal, tabular (numerical), graphical, and symbolic (equations). He was able to verbally interpret the information from the graphical representation. He correctly described the graphical representation provided in the Instructional Interview Problem No. 2 (see Figure 16).

> This graph shows the relationship between the speed of two cars and time. Car A and Car B start traveling from time = 0. Car A goes faster than Car B. But Car B speeds up more because this line [pointing to the graph of Car B] has more incline. At t = 1 hour, they have the same speed because their graphs intersected here [pointing at the intersection point]. After one hour Car B goes faster than Car A.

Even though he correctly transformed the graph to a verbal representation, he could not relate a conceptual idea of speed, time, and position of the cars. He stated that these two cars were at the same position at t = 1 hour. His reason was: "They intersected at the same point; that should mean they are at the same point at that time."

Kyle sometimes gave quick responses to a problem. At the beginning he said that the graphical representation provided in Instructional Interview Problem No. 3a (see Figure 17) was not a function because it failed the vertical line test. He said that when $x = 100$, there were two outputs. When asked to state the outputs at $x = 100$, he realized that he was wrong.

> I was wrong. There is only one output when $x = 100$, that is 200 because the white dot does not include the value at the edge. Therefore, this graph represents a function.

Kyle stated that this information could be represented in a linear form $y = mx + b$. He demonstrated a procedural understanding in his use of an algorithm to find the linear form of the first line.

> To write a linear formula, I need to find a slope. I need two points for a slope. I say points (0, 50) and (50, 75). Slope, m, is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{25}{50} = \frac{1}{2}$. The y-intercept is 50. Therefore, the linear formula is $y = \frac{1}{2}x + 50$.

Kyle had a misconception that lines with the same slope had the same equation. While trying to construct an equation for the third line of the graph, he found its slope was $\frac{1}{2}$. (He used (400,200) and (600, 300) as the two points.) He said that "The slope is the same as the first line, so they are the same function." His knowledge of finding a linear function was limited to the point-slope form. When asked to prove that the first line and the third line were the same, he extended the graph of the third line until it intersected the *y*-axis. It intersected the *y*-axis at the (0,0) point. Therefore, he concluded that the *y*-intercept $b = 0$. The equation of this line was $y = \frac{1}{2}x$, which differed from the first line. When he was asked about what he knew from his explanation, he said that two lines with the same slope did not mean that they were the same line and recognized that the lines with the same slope would be parallel.

Kyle was able to make a symbolic representation corresponding to each piece of a graph. However when asked to write the symbolic representation of this information in function notation [a piecewise-defined function notation], he was unable to complete the task. He was not the only student who was unable to do this task. A lack of this ability might be that he did not have much experience with this kind of problem.

Kyle was unable to identify applications for graphical representations in a real situation. He could not construct or think about any real situation that could be represented by the graph provided in the Instructional Interview Problem No.3b. (see Figure 18). He was sure that this graph represented a function and this information could be represented in a symbolic form similar to the Problem 3a. But he could not write a piecewise defined function notation.

Kyle transformed a symbolic representation to a graph representation using a graphic calculator when he worked on the Ball Dropped Problem (Instructional Interview Problem No. 4).

> A ball dropped from the top of a tall building has height from the ground represented by $s = -16t^2 + 145$ feet after t second. Umm… I will check if it is a function by graphing it [entering the equation $y = -16x^2 + 145$ into his calculator]. It is a graph of a parabola facing down. So it is a function.

Kyle thought that the horizontal line test was used for testing a one-to-one function. After receiving a probe from the interviewer, he understood that if a function passed the horizontal line test, it could be implied that a function had an inverse.

K: How can I know if it has an inverse?
I: How do you check whether it is a function?
K: I use the vertical line test.
I: Is there any test that you can use with a function?
K: I can use the horizontal line test. But that checks a one-to-one function, doesn't it?
I: So is this a one-to-one function?
K: No.
I: Why not?
K: Because it doesn't pass the horizontal line test.
I: Tell me one more time what do you know about this equation.
K: I know that it is a function and it is not a one-to-one function.
I: Does it tell you anything about an inverse?
K: Umm… I think it does not have an inverse.
I: Why?
K: Because it is not a one-to-one function. When I switch x and y for finding its inverse, it is not a function.

Kyle demonstrated that he was able to apply the horizontal line test to determine the inverse of a function. However, he was unable to relate the problem context to a real situation. While working on the Ball Dropped Problem (Instructional Interview Problem No. 4), he did not recognize that the graph of a ball dropped was a one-to-one function because it was half of a parabola graph starting from the top and dropping to the ground. Without a connection between the problem context and a real situation, he made mistakenly claimed that this function had no inverse.

Basically, Kyle applied the following function concepts that were covered in the lecture class when he solved the problems.

- The vertical line test: He used the vertical line test for determining a function.
- The definition of a function (provided by the instructor): He used a function definition, a correspondence that assigns to each element of set D exactly one element of set R, to determine a function when the relation was represented by a table.
- The horizontal line test: He used the horizontal line test to determine a one-to-one function. He also applied the horizontal line test to determine whether a function had an inverse.

Instructional Interview Two

When solving function problems in the second interview (two weeks later), Kyle's understanding of functions had improved. While working on the problems involving quadratic functions (Instructional Interview Problem No. 5 and No. 6), he reasonably selected numbers for each variable, *a*, *h*, and *k*, to construct a symbolic representation that matched with the graphical representation provided in the problems. Even though he described the effect of those variables correctly, he seemed to have memorized their effects from the class, rather than understanding the relationships among these variables.

During the second interview, Kyle seldom used a graphing calculator or any other tools to help solve the problems. He responded to the application problem, the Salary Problem (Instructional Interview Problem No. 7), using logical reasoning and symbolic manipulation. To solve part *a*, *b*, and *c* of the Salary Problem, he responded using the symbolic representation. His responses were correct.

When asked whether he did any application problems in the textbook, he said that he did not do all of the homework assignments because they were not graded. He did some assignments when he studied for quizzes. He did not work on the application problems often, especially the problems related to functions.

Kyle responded to Instructional Interview Problem No. 8, considering whether function $f(x)$ and $g(x)$ were the same, without thinking about their domains. When asked when these functions were the same, he said, "When I put the same value of x in each function, I get the same value of y."

Kyle's Post-Instruction Understanding of Functions

By the end of instruction, Kyle had gain confidence about his knowledge and understanding of function concepts. He explained his conceptual understanding of a mathematical function.

> A function is a relation between two things, I say x and y, that one x relates to one y. When you draw its graph and use the vertical line test, it passes the test. I mean each vertical line crosses the graph only one point.

His understanding of functions included a concept of a one-to-one function. He stated:

> A one-to-one function is a function that has only one output for one input and one input for one output. A one-to-one function can be checked by using the horizontal line test.

When asked for further explanation of how the horizontal line test worked, Kyle responded:

> To check a one-to-one function by using the horizontal line test, you need to draw a graph of an equation. Then draw horizontal lines and check how many points that each line crosses the graph. For being a one-to-one function, there is no horizontal line that crosses the graph more than one point.

To examine if he understood a one-to-one function, Kyle was asked to give examples of both a one-to-one function and a function that was not one-to-one. As he explained, he used symbolic and graphical representations.

> An example of a one-to-one function is like umm… $y = x$. Its graph is a line like this [using his hand makes a diagonal from top right to bottom left]. The horizontal lines will cross this graph at only one point. An example of a function that is not a one-to-one is $y = x^2$. Its graph is a parabola opened upward. It does not pass the horizontal line test because the horizontal lines cross the graph at more than one point.

Kyle related the concept of a one-to-one function to an inverse of a function. He stated that if a function was one-to-one then it had an inverse. But when asked how he checked whether a function given in symbolic form had an inverse, he was unable to respond without assistance.

Kyle said that he had learned some function concepts taught in this class in high school. He said that he could recall some but not until he heard the instructor talk about it in class. He said the functions taught in this class were not difficult but they were not easy either. Sometimes he felt he was behind the class because he missed some lecture classes.

At the end of the instruction on functions, Kyle was asked to solve the Construction Function Problem (Post-instructional Interview Problem). His understanding of the concept of functions, including an undefined function, zeroes of a function, a nonnegative function, and a function that contained a certain point, were connected and used for solving this problem.

During the post-instructional interview, Kyle used two types of representations: verbal and symbolic more than any other representations. Prior to instruction on functions, he used a verbal and symbolic representation more often and during instruction, he used all four common representations (verbal, graphical, numerical (tabular) and symbolic). Kyle verbalized his understanding of an undefined function constraint.

> The function is undefined at –3. This means that when $x = -3$, we cannot find the value of y. And I remember that it happens when it is divided by zero.

He described the second constraint; the function with a zero at $\frac{1}{2}$ means "the value of y equals zero when $x = \frac{1}{2}$." He initially was unclear about the meaning of "nonnegative" in the third constraint. He clarified the word "nonnegative" as "positive. But when he was probed by asking him whether zero (0) was negative or positive, he said "neither." Kyle restated that nonnegative included positive and zero. Kyle's explanation of the last two constraints where the function's domain was $[-5, \infty)$ and the function containing the point $(4,7)$, was that the x values in this function included –5 and larger numbers and when x was 4 and y was 7.

Kyle connected the concepts of undefined, zeroes of a function, a nonnegative function, and a function containing a specific point. With these connections, he correctly stated that this function was a rational function that had $x - \frac{1}{2}$ as the numerator and $x + 3$ as the denominator. Kyle had a limited idea of a nonnegative function. He thought that this function was only an absolute value function because it was always a nonnegative function. He did not consider a quadratic function such as $y=(x-h)^2$ that also provided a nonnegative function.

After instruction, Kyle did not have a chance to work on any real world problems. The post-interview problem was not a real world problem. However, he responded to the question "How are mathematical functions useful in thinking about real world situations?" so that he could use mathematical functions knowledge in funding the best option for money investment or buying mobile phone service. His examples of using mathematical function knowledge differed from the ones that the instructor discussed in class. When asked about the application problems presented in the classroom, he remembered that the instructor gave examples related to interest rates of banking including compounding and continuously compounding interest rates. His homework assignments related to the application problems indicated that he did not work on the application problems provided in the textbook.

Questionnaire: College Algebra Students' Understanding of Functions

Twenty-four volunteers including the five students described in the profiles participated in a Function Understanding Questionnaire at the end of the instruction on functions. The questionnaire was designed to gather data to answer the first research question: What is college algebra students' understanding of functions? The questionnaire was administered to the students in the last recitation class of the winter term, 2003. The students were asked to respond in writing to four open-ended questions related to their understanding of mathematical functions.

The first question asked the students to describe their understanding of the concept of mathematical functions. They were allowed to use diagrams, pictures, or examples to clarify their thoughts. Similar to the study conducted by Schwingendorf et al. (1992), the students' responses were classified into four categories: prefunction, process, correspondence, and no concept. If students indicated a limited concept of functions such as identifying a function is an equation, they were classified in the "prefunction" category. They were placed in the "process" category if they indicated a use of a process of getting an output from entering an input. If students indicated a relationship between two variables, they were grouped in the "correspondence" category. Finally, if students indicated no understanding of functions, they were placed in a "no concept" group. The students' responses are shown in Table 2.

None of the students used any diagrams, or pictures to clarify their thoughts. They described their thoughts with words. The results indicated that of all students, the majority of the students, including the five interviewed students, described their understanding of mathematical function as a correspondence or a relationship. They described a function as a relationship between two variables or two sets of objects that each element of one set was assigned to only one element of the second set. This definition was similar to the one that the instructor defined it in class, which was "A function from a set D to a set R is a correspondence that assigns to each element of D exactly one element of R." 11 o 24students some, including Amy and Lindsey, identified a function as an equation that passed the vertical line test. Eight students described a mathematical function as an equation that helped them find an answer to mathematical application problems.

Four students had a procedural understanding and were placed in the "process" category. They described a function as a process a process for obtaining. They said that a mathematical function was like a machine that performs an operation on a number to give a result or a machine that whenever one put in an input and an output will result.

Eight students did not have a fully developed idea of functions. They were categorized in the "prefunction" group. The students in this group stated that a function was an equation. This statement is correct sometimes; however, whether these students knew that not all equations were functions is unknown.

Finally, 1 of the 24 volunteer students did not have any concept of functions. He expressed his idea in words that mathematical functions did not mean much to him.

Table 2

Student responses to the first question

Question 1: In your opinion, what does a mathematical function mean to you? Describe your understanding of mathematical functions. You may use diagrams, picture, or examples to clarify your thoughts.

Categories of Response	Percent of Students	Examples of Responses
Prefunction	33% (8 students)	- A function is an equation. - A function is an equation with two variables *x* and *y*.
Process	17% (4 students)	- A mathematical function is like machine that we put a number in then we get another number out. - A function is a tool to calculate an output. If we know *x* then we can find *y*. - A mathematical function is a way to manipulate number.
Correspondence	46% (11 students)	- A mathematical function is a relationship between two variables that one element of one set was assigned to only one element of the second set.(*Emma* and *Ross* were included) - A function is a relationship between *x* and *y* and every *x* has only one *y*, or it passes the vertical line test. (*Kyle* was included) - A function is and equation that passes the vertical line test (*Amy* was included.) - A relationship of two numbers or objects (*x* and *y*) is a function if every *x* has only one *y*, and to check a function, using the vertical line test.(*Lindsey* was included.) - Not all equations are functions. Equations that pass the vertical line test are functions.
No Concept	4% (1 student)	- A mathematical function means nothing to me.

The second question asked the students to describe their understanding of multiple representations for functions. They were also asked to give examples of the different representations. Table 3 presents a summary of the students' responses.

Table 3

Student responses to the second question

Question 2: Can a mathematical function be represented in multiple ways? If so, give examples of each type of the mathematical function representations

Categories of Responses	Percent of Students	Examples of Responses
Multiple representations (without examples)	79% (19 students)	- Table, graph, equation (*Kyle* was included.) - A function can be represented in different ways such as graphs, tables, and equations (*Amy, Emma, Lindsey*, and *Ross* were included).
Multiple representations (with only examples of a symbolic representation)	4% (1 student)	- A function can be represented in multiple ways depending on the variables. For example: $D = rt$, $\frac{D}{t} = r$, and $\frac{D}{r} = t$.
Multiple representations (without reasonable reasoning and examples)	8% (2 students)	- I learned that a function can be represented in many different ways. - A function can be represented in many different ways.
One representation (an equation)	8% (2 students)	- A function can be represented in one way that is an equation. For example: $y = x$. - A function may be represented by $y = x^2$.

The results indicated that the majority of the students (19 of 24) agreed that a function could be represent in different ways, including tables, graphs, and equations. (The majority of the students used a term "equation" instead of "symbolic representation.) However, none of these students provided examples of the different types of representations. The five interviewed students' responses were in this category and provided written responses to the questionnaire that a function could be represented in different forms including a table, a graph, and an equation.

Two students admitted there were many different ways to represent a function. One of them stated that he knew there were different ways to represent a function because he learned it in class, both of these students did not state types of representations and did not provide any examples.

One student said that there were multiple ways to represent a function but he misunderstood the forms of representations. He described that the representation of the function depended on variables; the example he gave was an equation that represented the same relationship among three variables [$D = rt$, $\frac{D}{t} = r$, $\frac{D}{r} = t$].

Two students incorrectly identified that there was only one way to represent a function. They misunderstood an equation was the only way that a function could be represented.

From the classroom observation, the researcher noticed that the instructor did not provide examples that used more than two mathematical representations to display the same data. She also did not spend time translating one representation to another.

The third question asked the students to describe the usefulness of functions in thinking about or doing mathematics and in particular algebra. They also were asked to give examples and were allowed to use diagrams or pictures to clarify their examples. The majority of the students in this study including Amy, Emma, Lindsey, Kyle, and Ross identified that functions were useful for doing mathematics (see Table 4).

Most of the students responded that mathematical functions are useful in doing mathematics. Twenty of 24 students indicated that mathematical functions were useful in doing mathematics, helping them solve mathematical problems in a simple way. One of the students in this group wrote, “If you look at a problem you may not see how it is applied, but with a function you can graph it, which gives some visual pictures.” Two of 24 students agreed that functions are useful in seeing the relationship between groups of objects. For example, one of the two students clarified, “With the functions you will be able to see the relationship between things or numbers.” Two students agreed that functions helped in interpreting real-world situations in mathematics and provided an actual visual of an equation and how to use the function further in a real life situation. None of the students responded to this question by providing examples corresponding to their thinking. In addition, none of the students used diagrams or pictures to clarify their examples.

Table 4

Student responses to the third question

Question 3: In your opinion, how are mathematical functions useful in thinking about or doing mathematics and in particular algebra? Give some examples. You may use diagrams and picture to clarify your examples.		
Categories of Responses	Percent of Students	Examples of Responses
Functions are useful (with relevant reasoning)	83% (20 students)	- With some mathematical problems you may not see how it is applied, but with a function you may graph it, which gives some visual pictures (*Amy, Emma, Lindsey, Kyle*, and *Ross* were included).
Functions are useful (without relevant reasoning)	13% (3 students)	- With functions you will be able to see relationship between things or numbers like if you know x then you can find y. - Functions are useful for solving math problems. That what we study for this class.
Functions are of no use	4% (1 student)	- Functions confuse me.

As other students, the five students did not provide examples how functions are useful in doing mathematics in their written questionnaire. However, while they were interviewed, they claimed that they used multiple representations to help them solve problems. For example, Amy, Emma, Lindsey, and Ross said that they changed problem situations from words to an equation (symbolic representation), and then solved the equation. Kyle said that he could change a problem given in words to a table, which helped him see the relationship among pieces of information in the problem.

For the fourth question the respondents were asked to state their opinions on how mathematical functions are useful in thinking about real-world situations. The majority of the students, including the five students, stated that they were able to apply functions to real-world situations. The responses to this question are presented in Table 5.

Table 5

Student responses to the fourth question

Question 4: In your opinion, how are mathematical functions useful in thinking about real world situations? Give some specific examples. You may use diagrams and picture to clarify your examples.		
Categories of Responses	Percent of Students	Examples of Responses
Functions are useful for real situations (provided real situations)	79% (19 students)	- I can use mathematical functions solve real-world problems like population growth, interest rate. - I use functions to compare price when I buy stuff in stores. - It [a function] helps me solve compounded interest problems (*Amy, Emma, Lindsey, Kyle,* and *Ross* were included).
Functions are useful (did not provide real situations)	8% (2 students)	- Mathematical functions are useful in a real life.
Functions are of no use	13% (3 students)	- I perceive them as uselessness and I don't think that I've ever used them to help me to solve any of my life's problems. - I don't think I can use them in any real situations.

The results indicated that the majority of the students (19 of 24), including Amy, Emma, Lindsey, Kyle, and Ross, confirmed that mathematical functions were useful in real-world situations such as in business and accounting (compounded interest of an investment, stock) and jobs in science areas (population growth and decay, relationship among time distance, and velocity/speed). Some of this group also mentioned the usefulness of functions in simple daily life, for example, comparing prices to determine the better deal. Two of the 24 students said that mathematical functions were useful in the real life situations; however, they did not give an example of the usefulness of these functions. And three students said they had never seen mathematical functions related to real-world situations and never thought that they could apply functions to solve their real life problems.

Mainly, the students' responses to the Function Understanding Questionnaire described the development of an understanding about functions that were more thoroughly described by the five students' profiles. Most of the students illustrated their understanding of a function definition towards correspondence level. The students who understood a function as a correspondence also understood multiple representations of functions indicating different types of representations. The students in this group appreciated the usefulness of functions other mathematics disciplines and real-world situations. Similarly, the one student who indicated that a mathematical function had no meaning to him also indicated that a function could be represented in only a form of an equation. He was the only student who thought that functions had no use. He, together with two other students in the prefunction group, believed that functions could not be applied to a real-world situation.

Analysis of Students' Understanding of Functions

This section provides a summary and analysis of these college students' knowledge and understanding of functions. The comparative analysis was based on the 24 students' responses to the Function Understanding questionnaire and the multiple interviews of the five students: Amy, Emma, Kyle, Lindsey, and Ross.

The students' responses to the Function Understanding Questionnaire indicated that after they had completed the College Algebra course, the majority of all volunteer students had a more complete understanding of functions compared to the National

Assessment of Education Progress [NEAP] report (Brown et al., 1988) and several research studies conducted in the 1990s (Becker, 1991; Demarois & McGowen, 1996; Selden & Selden, 1992). Most of the students in this study were able to classify functions and equations. Their definition of functions represented a more formal definition after instruction. Some students stated that functions were equations and this statement was correct sometimes. Perhaps if the researcher had an opportunity to question them further, these students may have recognized that all equations are not functions. Only one of 24 students claimed that a mathematical function had no meaning to him.

Besides the responses of the 24 students' responses to the questionnaire, the interviews with the five students more fully supported the development of the students' understanding of functions. At the beginning of the course, Amy, Emma, Kyle, Lindsey, and Ross had naïve yet different levels of knowledge and understanding of functions. Their knowledge of functions prior to the instruction was from their high school mathematics courses especially from Algebra II and precalculus classes. But not all of them took these classes. With this college level algebra course, their knowledge and understanding of functions including the definition, the multiple representations of functions, the use of the vertical and the horizontal line test, one-to-one functions, and inverse of functions developed. All five students defined functions as the relationships of two sets of objects in which the graph passed the vertical line test. They were able to classify the given relations in multiple forms into functions and non-functions. However, they had demonstrated some difficulty when working with symbolic representations, particularly when the relationship was a circle. Kyle, Lindsey, and Ross correctly identified that a symbolic representation of a circle was not a function, but they spent more time on this question than on other questions. Amy and Emma incorrectly responded to this question in the first during the instructional interviews because of deficient knowledge of working with radical equations. These five students had no difficulty with linear and constant functions, because they easily graphed and used the vertical line test to determine whether they were functions.

With respect to concepts of a one-to-one function and an inverse of a function, all five students had the same idea that a function is one-to-one if each output (range) corresponds to only one input (domain). They asserted that a one-to-one function could be determined by using the horizontal line test. They also claimed that if a function was one-to-one, then it had an inverse.

All five students understood that a function could be represented in different forms. They were familiar with the common forms of representations: symbolical, numerical (tabular), and graphical. None of them were familiar with the words "multiple representations." They demonstrated that they were able to transform one representation to another. All five students had no difficulty changing graphical representations (graphs of lines) to symbolic representations for linear equations. The only difficulty they displayed was in transforming the graphical representation of a piecewise-define function to the symbolic representation.

Although they all improved on their knowledge and understanding of functions, they did have different levels of understanding. All of these students gave a correct function (symbolic representation) of familiar functions that could be obtained from each of those given constraints. For example for the constraint of "a function is always nonnegative," all of them provided an absolute value function. Amy, Emma and Kyle had difficulty finding a function that had a domain from -5 to positive infinity. Amy and Kyle were unable to define the function. Emma found one after some time. When several constraints were given and they were asked to construct only one function based on those constraints, not all of these five students were able to finish the problem. Amy, Emma, and Kyle had difficulty working with the constraint that gave them a domain –5 to positive infinity. Amy and Kyle were unable to give a correct function to satisfy this constraint, they were able to find a function that satisfied only four of the five given constraints. Emma got a function with the domain constraint, but she could not connect it to her previous function. Therefore, she created her function without the domain constraint. Only Ross and Lindsey were able to complete the work on this problem. Ross and Lindsey seemed to have enough knowledge and understanding of functions to solve the problems beyond those in their homework assignment, those that were demonstrated as examples in classes, and those that were on their quizzes and examinations.

Research Question 2: Solution Strategies and Algebraic Thinking and Reasoning

The second research question concerned solution strategies and algebraic reasoning that College Algebra students used as they solved problems. The primary source of data for responding to this question was the multiple interviews of the five students as they demonstrated their solution methods and reasoning. Instructional episodes provide examples, explanation, methods, and techniques that the instructor and the GTA demonstrated in their respective classrooms in order to provide the context for lecture and recitation.

Instructional Episodes: Solution Strategies and Thinking and Reasoning

Episode One: Identifying Functions

After introducing the definition of a function to the students in the class, the lecture instructor provided them with examples of strategies to determine functions given in tabular, graphical, and symbolic forms. For a numerical (tabular) representation, the instructor suggested that students check "If each element of x is assigned to exactly one element of y, then this [relationship] is a function."

Instructor: Consider this table [showing a table on the overhead].

x	1	2	3	4
y	-2	2	-2	2

Is it a function?

Students: Yes.
Instructor: Why do you think it is a function?
Student 1: Because each x has only one y.
Instructor: -2 and 2 happen twice.
Student 1: It doesn't matter. Each x must be matched with only one y.

Instructor: Right. Now let's see this table [showing a table on the overhead].

x	1	2	3	1
y	−2	2	−2	2

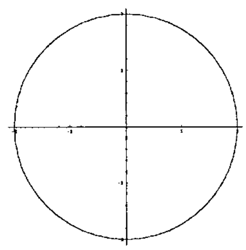
Is this a function?

Student 2: No. Because when x equals 1, it was assigned to two different y.

Instructor: Correct. Any questions? [There was no response from the students.]

To determine whether a graphical representation identified a function, the instructor described the use of the vertical line test.

Instructor: Is this a function [showing a graph similar to the graph below on the overhead]?



Students: No

Instructor: Why is it not a function?

Student 3: Because it does not pass the vertical line test.

Instructor: What does that mean?

Student 3: When you draw vertical lines, they cross the graph more than one point, which means that x [input] has more than one y [output]

Instructor That's correct. Any questions? [No responses.]

The lecture instructor also presented a method to determine if a symbolic representation identified a function by drawing a graph of an equation and then applying the vertical line test. The instructor demonstrated one more example of an equation in its symbolic form ($y^2 - x^2 = 0$) to consider whether it represented a function. Basically, the method she used was to re-write an equation of y in terms of x and enter the expression of x variable to her graphing calculator. After obtaining a graph, she applied the vertical line test. However, in this one example, the instructor displayed the process on the overhead, explaining:

> From $y^2 - x^2 = 0$, I will solve for y. So $y^2 = x^2$ and $y = \pm\sqrt{x^2}$. Any time that we see $\pm$ sign; we have no need to graph it. It's so obvious that there are two y values for one x value. Plug in one x to this equation, we will get two y values. For example, $x = 1 : y = \pm\sqrt{1^2} = \pm\sqrt{1} = \pm 1$. When $x = 1$, $y = 1$ or -1. Therefore, it's not a function.

The students did not seem to have any difficulty determining functions. Some homework problems were related to determining functions, and none of the students asked the GTA to discuss these problems. In addition, the GTA did not question the students' reasoning processes. However, when she reviewed for the first midterm, she summarized that,

> A relation is a function if each x. I mean each element of the first set is matched with one y or one element of the second set.

This was the first time the GTA had used the word "relation" in the class, and no one asked any questions related to what a relation was. The GTA continued summarizing "If you want to know if a graph represented a function you may use the vertical line test." The GTA did not show any examples after the summary.

Episode Two: Multiple Representations

After introducing the idea that a function could be represented in different forms, the lecture instructor provided examples of transforming a function from one representation to another. All the examples displayed in the lecture class were transformed from a symbolic to a graphical or to a tabular representation using a graphing calculator. The instructor demonstrated some examples involving multiple representations and those were presented in the instruction for the previous section on understanding of functions.

The lecture instructor demonstrated several problems related to functions represented in different forms in the previous lectures. For example, to determine whether an equation identified a function, the instructor solved the problem by graphing an equation and applied the vertical line test. In addition, while solving most of the story problems in this class, the instructor transformed the problem situations given in words to symbolic forms (equations) and solved for solutions from those equations.

> Deposit $300 in 2002. How much will you have in 2005 with an annual interest rate of 2% if the interest is compounded 6 times a year?
>
> The instructor presented her solution method as follows:
>
> Instructor: Deposit $300 means the principle P equals $ 300 [writing $P = 300$]
> Annual interest rate r equals 2%, so r equals .02 [writing $r = .02$].
> Interest is compounded 6 times a year, so n equals 6 [writing $n = 6$].
> From 2002 to 2005 is three years, so t equals 3 [writing $t = 3$].
> [She showed $P = 300$, $r = .02$, $n = 6$, $t = 3$ on the overhead.]
> The formula for compounded interest is [writing $B = P(1+\frac{r}{n})^{nt}$,
>
> then entering the numbers to the equation]
> $B = 300(1+\frac{.02}{6})^{(6)(3)}$ [using a calculator to calculate the solution].
> $B = 318.52$
> In 2005 the balance will be $ 318.52.

The instructor did not engage the students as she presented this solution method.

The GTA did not directly discuss multiple representations of functions in her class. However, she solved problems by transforming situations given in the problems from a verbal to a symbolic form before she solved for solutions.

> Suppose that the number of board feet of lumber in a ponderosa pine varies directly as the cubes of its circumference at waist height. If a ponderosas pine with a circumference of 100 inches yields 1500 board feet lumber, how much can be obtained from one with a circumference of 120 inches?
>
> The GTA presented her solution method as follows:
>
> GTA: Let B represent the number of board feet of lumber
> and C represent the circumference
> What is the relationship between B and C?

Ross: $B = kC^3$

GTA: Yes. Therefore, $B = kC^3$

We need to find a constant value *k*. [entering the number into the equation]. What do we know about *B* and *C*?

Ross: $B = 1500$ and $C = 100$.

GTA: So, the equation is $1500 = k(100)^3$

$0015 = k$ [asking student to calculate the k value].

What will we do next? [No responses.]

Therefore, this function is represented by $B = .0015C^3$

We look for *B* when $C = 120$ inches [entering 120 for *C* in the equation].

$B = .0015(120)^3 = 2592$ [asking students to calculate the solution using their calculators].

When the GTA provided this example on the board, she challenged the students' thinking by asking some questions. Ross was the only student who responded to her questions. Sometimes there were no responses; the GTA reacted to her own questions.

Even though the assessment, including suggested homework problems, quizzes, midterms, and final examinations, did not request that they needed to transform one form of representation to another, the students were allowed to use all representations as needed. The students typically solved many problems by converting a verbal to a symbolic representation or converting from a symbolic to a graphical representation.

Episode Three: Transformations of Functions

The instructor provided a few examples of function transformation after introducing concepts of horizontal and vertical transformations.

Instructor: What is the relationship between these two graphs? [showing the graph of *f(x)* and *g(x)* graphing by her calculator on the overhead, which were similar to the graph shown in Figure 21].

Student 1: The graph of *g(x)* moves from the graph of *f(x)* to the right 2 units.

Instructor: If an equation for *f(x)* is $y = x^2 + 1$, what is the equation for *g(x)*?

Student 1: Umm... it should be $y = (x-2)^2 + 1$

Instructor: Yes, because it moves to the right 2 units so *f(x)* is changed to $f(x-2)$. Any questions? [No responses.]

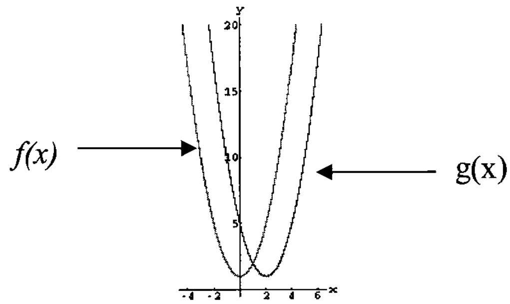


Figure 21. A quadratic function used for discussing vertical and horizontal transformations.

Since there were no questions, the instructor showed another example (see Figure 22). Basically, there was only one student who responded to the instructor's questions while she demonstrated the example.

Instructor: What is the relationship between *f(x)* and *h(x)*?

Students 1: *h(x)* shifts downward for about 3 units and shifts to the right for about 1 unit form *f(x).*

Instructor: Okay, let's say $f(x) = |x|$ and *h(x)* moves downward 3 units and to the right 1 unit from *f(x)*, what is the equation for *h(x)*?

Student 1: $h(x) = |x - 1| - 3$

Instructor: Correct. Any questions?

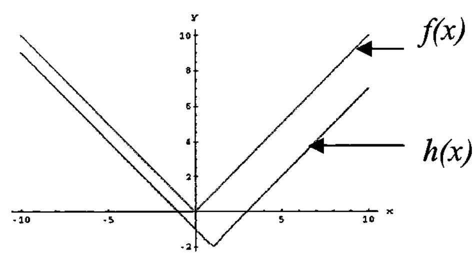


Figure 22. An absolute value function used for discussing vertical and horizontal transformation.

Although there were no questions, the lack of questions did not necessarily mean that all of the students understood the concept of transformation of functions. The instructor never discussed a real-world situation that related to the transformation concept. However, two problems in the suggested homework of this section asked the students to apply the concept to real-world situations. There was one problem concerning the vertical and horizontal transformation in final examination, but it was a symbolic representation transformation.

In the recitation, the students were assigned to work on a lab relating to the transformation of functions. The problems in the lab merely related to a symbolic transformation. Although there were two application problems (real-world situation problems) involving the transformation concept in the suggested homework, none of the students asked their GTA for the discussion of these two problems during their recitation class.

The GTA approached the lab by reviewing two types of transformations: horizontal and vertical. She described the transformations as:

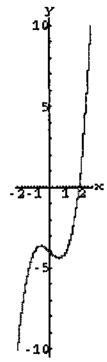
> No matter what the graph of *f(x)* looks like, if *f(x)* changes to $f(x-5)$, the graph of $f(x-5)$ will move or shift from the graph of *f(x)* to the right 5 units. If *f(x)* changes to *f(x*+5), the graph of *f(x*+5) will moves from the graph of *f(x)* to the left for 5 units. However, if *f(x)* changes to *f(x)* + 5, the graph of *f(x)* will move upward for 5 units, and if it changes to $f(x)-5$, the graph of *f(x)* moves downward 5 units.

After summarizing the transformations, the GTA let the students work on the lab and provided some help if needed. The GTA helped students by clarifying the problem, probing or questioning to make students think and obtain the answer by themselves. Many students had difficulty with the horizontal transformation. They misunderstood, thinking that $f(x+h)$ always meant *h* was positive.

Episode Four: One-to-One and Inverse Functions

After presenting the students with the definition and explanation of an inverse function, the lecture instructor demonstrated how to determine whether a function had a function inverse by checking a one-to-one function using the horizontal line test.

> Let's see if this function is a one-to-one function [showing an equation $y = x^3 - x - 4$]. I will draw the graph of this equation [drawing the graph by using her calculator and showing the graph on the overhead]



> From the graph, I will check for a one-to-one function using the horizontal line test.

The instructor drew horizontal lines over the graph and stated that this function was not a one-to-one function because it did not pass the horizontal line test.

> This function is not a one-to-one function because there are some horizontal lines that cross the graph more than one point.

After introducing the horizontal line test as well as a one-to-one function and making sure that the students had no questions involving these concepts, the lecture instructor provided an example of finding an inverse of a function $f(x) = \sqrt{2x+5}$. The instructor displayed her process for solving this problem.

> First I will ask myself "Is this a one-to-one?" If yes, I will find an inverse; if no, an inverse doesn't exist. Let's go ahead and graph it. I will use the horizontal line test [drawing vertical lines over the graph of the function]. It passes, so it is one-to-one. Then, I will find the inverse. First, I replace $f(x)$ with y; so $y = \sqrt{2x+5}$. Second, I switch x and *y*, so $x = \sqrt{2y+5}$. This step also means we switch the domain and range of this function. Third, I solve for *y*. I will square both sides of the equation, first. What I have is $x^2 = 2y+5$, and then I subtract 5 from both sides and divide both sides by 2. So I have $y = \dfrac{x^2-5}{2}$. The last step: I replace y with $f^{-1}(x)$. Therefore, $f^{-1}(x) = \dfrac{x^2-5}{2}$.

After demonstrating the process of finding the inverse of this function, the lecture instructor warned the students that they had to consider the domain of the inverse function because it was to become the range of the original function. She showed how to find the domain of the inverse by using the original graph.

> From the graph of the function $f(x)$, we know that the range is from 0 to positive infinity. We also know that the domain of the inverse is the range of $f(x)$; therefore, the domain of the inverse is 0 to positive infinity. My final answer is $f^{-1}(x)=\dfrac{x^2-5}{2}$, $x \geq 0$. The answer [for the inverse of the function] has to include the domain of the inverse.

After providing the students opportunities to ask her questions, the lecture instructor moved on to the next example where the domain of the inverse was all real numbers. While demonstrating the example, the instructor did not ask the students any questions on how the problem was solved, so the students did not have a chance to indicate their understanding. They listened to the instructor and were given a chance to ask questions at the end of this example.

> Find the inverse of *f* for $f(x)=\dfrac{-x+3}{4}$. I will check for a one-to-one function. What kind of function is it? [She did not wait for students to answer; she drew a graph]. It is a linear function. Therefore, it's one-to-one. I will find *f* inverse. I will replace $f(x)$ with *y*: $y=\dfrac{-x+3}{4}$. Switch *x* and *y*, this gives me an inverse. Then I write y in terms of x.
>
> [She wrote on the overhead $x=\dfrac{-y+3}{4}$
>
> $$4x=-y+3$$
> $$4x-3=-y$$
> $$y=-4x+3$$
> $$f^{-1}(x)=-4x+3]$$

Then she continued to explain the check of the domain.

> Last thing, I will check the domain. From the graph, the range of $f(x)$ is negative infinity to positive infinity [writing $(-\infty,\infty)$], so this is the domain of $f^{-1}(x)$. There is no restriction on the domain of the inverse; therefore, I can write $f^{-1}(x)=-4x+3$.

The instructor and the GTA did not give examples of real-world problems that included one-to-one or inverse functions. They also did not provide examples of how to use the one-to-one and inverse function concepts to solve real-world problems. There were several real-world situation problems related to one-to-one and inverse function concepts in the textbook used for this course; nevertheless, those problems were not suggested in this section. There were no real-world situation problems that related to inverse functions.

Episode V: Applications to Real-World Situations

Not many applications or real-world problems were demonstrated in the lecture class. In addition the GTA rarely discussed those types of problems unless the students asked for them. When approaching the application problems, the instructor read the problem situation, interpreted information, constructed the relevant information symbolically, and then solved for the solution. Sometimes, the instructor encouraged students' thinking by asking questions. When there were no responses, the instructor continued showing her solution method. The following example shows how the instructor approached and solved a real-world problem.

> A computer purchased for $2,500 in 1992 is worth $1,500 in 1995. Assuming linear depreciation estimate the value of the computer in 1997.

After the instructor posed the problem on the overhead and read the problem, she approached and solved the problem as follows:

Instructor: To solve this problem we need a linear equation. Instead of using big numbers, we reduce it to small numbers. The year of 1992 is represented by $t = 0$ and 1995 is represented by $t = 3$. Now, we have two points: (0, 2500) and (3, 1500). With these two points we can find the slope of this linear equation [writing $m = \dfrac{1500-2500}{3-0} = \dfrac{-1000}{3} = -333.3$, and using her calculator to get the last number]. We can use one of these two points to find the linear equation. The better choice is (0, 2500) because the number is smaller. Enter those numbers into linear

> equation $y - 2500 = -333.3(t - 0)$ [writing the final equation, $y = -333.3t + 2500$]. Any question? [No response.]
> Use this equation to predict the price in 1997, which means $t = 5$ [writing $y = -333.3(5) + 2500 = \$833.5$]. Any question? [No response.]

In this example, the instructor did not clarify what t represented and why she reduced the big number of years to smaller numbers. In addition, the instructor did not refer to a point-slope formula of a linear function. Since the students had no questions, the researcher wondered how many students followed that approach and understood why their instructor had used such a strategy. All the way through obtaining the solution, the instructor described what needed to be done; the students were asked whether they had questions twice.

Besides approaching real-world problems using symbolic manipulations skills, the instructor solved some problems using a graphing calculator.

> A ball is thrown into the air from the top of the building with an initial velocity of 96 feet/second. Its height (in feet) after t seconds is $s(t) = -16t^2 + 96t + 256$. When is the ball at its maximum height?

The instructor solved this problem using her graphing calculator. She demonstrated her calculator screen on the overhead. She entered the ball's height function to the calculator in order to get its graph. The graph was similar to the one in Figure 23.

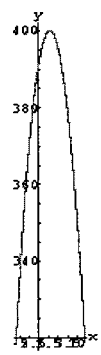


Figure 23. The graph used for a ball thrown example problem.

From the graph, the instructor looked for the highest point of the graph.

Instructor: My graph looks like this [showing the graph on the overhead]. My maximum point is (3, 400) [using her calculator to find maximum point]. Which one is the answer? [Some students said 3 was the answer. On the other hand, some other students said 400 was the answer.]

Instructor: Check back. What is the question?

Students: When is the ball at its maximum height? So the answer must be 3.

Instructor: Yes. If the question is what is the maximum height? Then the answer is 400 feet.

Working on the next example, the instructor engaged students with more questions in thinking through the problem.

> How long will it take for money to triple in an account paying 5% interest compounded annually?

The instructor provided the formula of compound interest without asking students any questions. She approached the problem as follows:

Instructor: The formula for compound interest is $B = P(1+\frac{r}{n})^{nt}$ [showing the formula on the overhead]. What do we know from the problem?

Students: No responses.

Instructor: Any ideas for B [Balance] and P [Principal]?

Student 1: Set a variable for P and multiply that variable by 3 for B.

Instructor: Yes. Any questions up to this point? [No responses.] We need to triple the initial deposit so we multiple by 3.
Let P = initial deposit, $B = 3P$ [writing on the overhead and speak out at the same time]. Any ideas for n and r?

Student 1: $n = 1$ because it is compounded annually, and $r = 5\%$. [Same student responded to her question.]

Instructor: Good. So we have $3P = P(1+\frac{.05}{1})^{1.t}$ [writing on the overhead].
Make it simple, so $3P = P(1.05)^t$ and divide both sides by P [the instructor did not ask students for a method to solve this problem and wrote $3 = (1.05)^t$ on the overhead]. Any questions up to this point? [No responses.] We want the value of t, which is in the power place, so we need to use either log or natural log [ln]. I will use natural log.

From this step, the instructor did not ask students to suggest a process for solving this problem. She solved the problem completely.

Instructor: $\ln(3) = \ln(1.05)^t$. Use log properties, so $\ln(3) = t(\ln(1.05))$
$t = \dfrac{\ln(3)}{\ln(1.05)} \approx 22.5$ [using a calculator]. So it will take about 22.5 years to triple the initial deposit. Any questions about this problem? [No responses.]

The GTA approached problems similarly to the instructor. She sometimes asked students for more suggested strategies. In a recitation some students asked her to show how to solve an investment problem. The GTA approached the problem as follows:

> An investor deposits $500 into a saving account on January 1, 2000. Her only transaction is a deposit of $200 on January 1, 2002. The balance in the account as of January 1, 2005 is $1000. Assuming the bank pays interest compounded annually and that the interest rate has been constant, compute the interest rate.

GTA: What is the formula for compounded interest?

Student 1: $B = P(1+\frac{r}{n})^{nt}$ [the GTA wrote the formula on the backboard]

GTA: What is *B*, *P*, *r*, and *t* [No responses.] *B* is balance, *P* is initial amount, *r* is rate, and *t* is time in year. How many times the money were deposited?

Student 2: Two times.

GTA: How many years had the first deposit been done before the second deposit?

Student 2: Two years.

GTA: Let's say $B_1 = 500(1+r)^2$. The second deposit was in 2002 and at that time the money from the first deposit plus interest was there, and it was deposited until 2005. That would be three more years. So the balance in 2005 is $B_2 = (200 + 500(1+r)^2(1+r)^3 = 1000$
Do you think it is too complicated?

Students: Yes

GTA: Do you have any suggestion for a different method?

Students 3: I did the first deposit for 5 years and the second deposit for 3 years, the sum is 1000.

GTA: So you have $B = 500(1+r)^5 + 200(1+2)^3 = 1000$

Students 3: Yes.

GTA: Are you comfortable with this equation?

Students: Yes.

GTA: Okay, we will use this equation. Next, we graph this equation and trace for $y = 1000$, the answer will be x, which represents interest rate. [She did not show a graph and did not give student a solution.]

The GTA did not give an explanation of the similarity or difference of her equation with the one suggested by the student.

Summary of the Instructor and GTA's Approaches to the Problems

Basically, the lecture instructor and the GTA's demonstrated only one approach for each problem solution. They did not show alternative ways to solve the problems in order to get the correct answer. In addition, they did not direct students to check whether the problem solutions were correct. The instructor and the GTA never showed students a method that could be used for solving some problems but that was not good for solving others. Most of the time the instructors explained how to solve the problems. They described what they did while they were working on those problems on the overhead or blackboard and did not include much student involvement in the solution.

The lecture instructor always used a graphing calculator for graphing and evaluating numeric expressions in her class, whereas, the GTA never used the device in her recitation class. The GTA sketched graphs of function on the blackboard by hand, and she asked students who had calculators to do numeric calculations if needed.

The instructor and the GTA sometimes encouraged students' thinking and reasoning by asking them questions. Unfortunately, few students responded to those questions. With such little interaction, it was difficult to determine the students' development of thinking and reasoning through the instruction.

Student Profiles: Solution Strategies and Thinking and Reasoning

Amy

Basically, Amy extracted and represented information from each problem situation, interpreted a solution, and applied the solution strategies to a problem in a new situation. For some problems, she used various strategies to approach the problems. Amy worked on each problem with confidence. Sometimes she used a method that gave her a correct answer quickly. At other times, the method she used at first did not work. She was able to change to a new strategy and completely explain her solution strategy as well as her thinking and reasoning processes. Her favorite strategy for determining a function in a symbolic representation was to enter numbers in the equation and check its inputs and outputs. Amy said that she chose either this strategy or graphing and using the vertical line test strategy to determine if that equation was a function. When asked why she selected only those two approaches, Amy said that those methods just came to her mind when she saw the equations.

Amy's Solution Strategy and Algebraic Thinking and Reasoning Prior to Instruction

Prior to instruction, Amy was able to describe her solution strategies and thinking and reasoning processes when working with mathematical problems. She was confident in her mathematical ability. When she solved each problem, she read the problem, described her method for approaching the problem, and answered the questions quickly. She did not draw any pictures or diagrams for the problems unless she was asked to do so. She used a graphing calculator to evaluate numerical expression. Amy extracted information from the graphical representation such as the dependent and independent variables represented on the xy-axis and the changes in the graph, and transformed the graphical representation to a verbal representation and vice versa. In addition to working with those two representations, Amy was able to analyze and interpret the information provided as a graphical representation (Pre-Instructional Interview Problem No.1, see Figure 24).

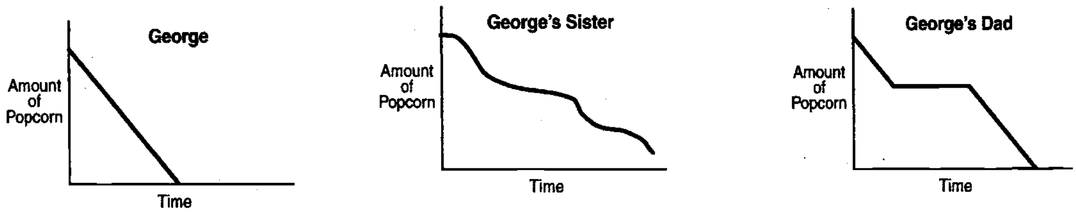


Figure 24. Graphical representation of eating popcorn over time.

> From the graph, I believe George ate consistently over the period of time because his graph showed as a graph of a linear equation. And George's sister ate at a different speed. She might eat fast sometimes. There was sometime she slowed down, sometimes she stopped eating and then speeded up again. She didn't eat consistently because there is a straight line like this [circling the point that shows a straight line]. She didn't finish her popcorn because the *y*-axis showed that her popcorn was not at zero. For George's dad, he started eating steadily and then stopped for a certain amount of time and then started eating again until he finished his popcorn. Because of this straight line [pointing at the horizontal straight line on the graph] it means he didn't eat any because the amount of popcorn was still the same.

Besides working with those two representations, Amy relied on her mathematical knowledge of percent with the information provided in the problem situation.

> I don't agree because you do 20% off and then do another 10% off so that like, let's say you have a $100 item and get 20% off of it which means you pay $80 and after that you get 10% off which means $70; never mind. I do agree with them. [She incorrectly calculated the value without the use of a calculator and paused for a few seconds.] Oh! No, I'm wrong. 10% of $80 is $8 so 80 minus 8 equals 72 instead of $70 and the total of 30% off is $70.

Amy responded to the problem quickly. At first glance she concluded that 10% of $80 was $10. However, as she talked, she soon realized that 10% of $80 was not $10, but in fact it was $\frac{10}{100} \bullet \$80$ or $8.

In some cases, Amy focused her thinking on specific points that kept her from completing a strategy leading to a solution. She believed that, the Population Growth

Problem (Pre-Instruction Interview Problem No. 4) did not give her enough information to find the solution.

> I don't know who is right; I just don't know. I don't have enough information to say who is right because you don't know who lives where when the population had been growing and they both can be right. Town A might have grown 3000 in one year then stopped when Town B could have grown steadily to 3000 in five years then stopped. They both could be right. They both could have grown in population at the same time or they both could have gotten the same amount of population growth at different times. Umm... So they both could be right. I don't know. I don't have enough information.

She stated that these two towns might increase the amount of population differently in each year; on the other hand, they might have the same increase in population in the same year. With this understanding, Amy had incorrectly concluded that there was insufficient information for finding the solution; this conclusion halted further consideration of the problem. Her response suggested that she was unable to relate the difference between the actual amount and the percentage of change.

Amy's Solution Strategies and Algebraic Thinking and Reasoning During Instruction

Amy demonstrated her ability to solve mathematical function problems while working on the problems during the instruction interviews. Classroom and recitation observations, homework, and quizzes were used to add to the description of the strategies she considered and used most.

Instructional Interview One

As she did prior to instruction on functions, Amy immediately responded to the problems when she finished reading them. She used a verbal explanation rather than drawing a picture or diagram to describe her solution strategies and reasoning. Amy used different strategies to identify functions. In some cases, she used the vertical line test while for others she looked at the numerical inputs and outputs. For all graphical representation problems, she used the vertical line test to determine if the graph was a function.

> A: I will use the vertical line test to determine each graph. As it [each vertical line] goes along, each x has only one y. So yes, this [(a)] is a function.
>
> I: Would you please tell me more what you mean, "Each x has one y?"
>
> A: There is no vertical line that hits the graph more than once. So it means each input x had only one output y. If there is a [vertical] line that hits the graph more than once, it means there is more than one output for that one input and that is not a function.

On the other hand, when the problems used a numerical (tabular) representation, Amy looked at the input and output values. She checked if there were any pairs that had the same x but different y values; if she identified a pair that had the same x but different y, she indicated that it was not a function. With the symbolic representations (equations), her initial strategy was typically a *guess and check* approach (entering numbers for the variable in the equation to see if any input had more than one output). She identified $y = x^2$ as a function by calculating numbers in her head for x in the equation.

> A: $y = x^2$ is a function.
>
> I: How do you know that?
>
> A: I just plug numbers in my head; like I plug $x = 2$ or $x = 3$. You can see it never has the same y with a different x. So it is a function because there is one x for one y.

When the symbolic representation was easy to graph, Amy graphed the equation and used the vertical line test.

> For $y = 4$, yes this is a function. Because when you draw a graph, it is a horizontal line. Each x has only one y and $y = 4$.

With all three questions that used symbolic representations, Amy first tried entering numbers for the variables to determine if these were functions. However, if this effort did not work, she tried drawing the graph and using the vertical line test.

Amy's response to the Car Problem (Instructional Interview Problem No. 2) revealed that she was able to extract information from the given graphical representation and represent that information in her own words; however, her limited background knowledge on motion restricted her work on this problem. She stated that Car A and Car B had the same speed at t = 1 hour because their graphs intersected at that time. At the beginning she was unable to connect the relationship between the speed and the distance of the two cars. She incorrectly concluded that they were also at the same position. However, after thinking about the graph in Figure 16, Amy was able to describe it correctly.

A: At t = 1 hour, they are at the same spot [drawing a dotted line at t = 1 hour to the intersection point].

I: What does it mean by the same spot?

A: They are going at the same speed at 1 hour because the lines cross and at that point t = 1 hour. So they have the same speed.

I: What does the question ask you?

A: The position. So this is speed [pointing at the vertical axis]; this is the time [pointing at the horizontal axis]. And they go the same direction. I don't understand.

I: Can you describe the graph to me like when they start and how they go? Which car goes faster?

A: OK. They start at the same point and Car A goes faster than Car B. Car A still goes faster until t = 1 hour they have the same speed. So Car A is gonna be ahead of Car B because it goes faster from the beginning.

I: What does it mean at the point t = 1 hour?

A: That point means they are going at the same speed but not the same position. So Car A is going to be further than Car B.

Amy was able to make connections between mathematical concepts. For example, in the Car Problem (Instructional Interview Problem No. 2), she related the speed and the acceleration of the cars correctly responding that Car B had a greater acceleration than Car A because its graph had a steeper slope than that of Car A. She correctly stated that Car B would be able to catch Car A at some time after t = 1 hour because the speed of Car A seemed to level off and Car B speeded up after one hour.

Amy's ability to connect and apply her understanding of functions to real situations also developed during instruction. This development was evident as she worked with the Ball Dropped Problem (Instructional Interview Problem No. 4). Although the symbolic representation of this problem was a parabola, Amy understood the context that time and distance would never be negative numbers. Therefore, she set the calculator widow for Xmin and Ymin at zero and showed how to find the function's inverse correctly.

> A ball is dropped from the building, so x and y have never been negative numbers because x represents time and y represents height. So the graph is one way down [half of a parabola]. Then it does have an inverse because it passes the horizontal line test.

Instructional Interview Two

Amy's responses to the two problems related to the graphical representations (Instructional Interview Problem No 5 and No. 6) revealed that she was in some way able to connect a quadratic function to its graphical and symbolic representations. Amy was able to construct and provide a reasonable symbolic representation of a quadratic function [$y = a(x-h)^2 + k$] from information given in the form of the graphical representation.

A: *a* tells me how the graph looks [open upwards or open downwards], (*h*,*k*) is the vertex of it; *h* tell how many spaces it moves to the left or right, and *k* tell how many spaces it moves up or down.
I: Would you please tell me more about the effect of *a*, *h* and *k*?
A: *a* tells how it opens, like if *a* is positive then it opens upwards and if it is negative, then it opens downwards. *h* is positive; it moves to the right and when it is negative, it moves to the left. For *k*, if it is positive it moves up and if it is negative, it moves down.

Amy related information provided in the Instructional Interview Problem No. 6 to the graph that she needed to find the symbolic representation. From the reference graph [$f(x) = x^2$], she said she needed a negative number to make a parabola open downward (identified from the class). By entering several numbers for checking the impact of the variable *a* in her calculator, she knew that if she used a negative number between –1 and 0 for the variable, a would made a parabola graph opening downward and wider than $a = \pm 1$ for the graph of $g(x)$. She also memorized (demonstrating an instructional understanding) that she needed a negative number for h and a positive number for k to be the vertex of $g(x)$. She stated that:

> $g(x)$ is wider than $f(x)$ and it's upside down, so a for $g(x)$ has to be a negative number and the number for a has to be smaller than 1 [she meant $|a| < 1$], and the vertex is right here [pointing the 2nd quadrant], so h is negative and k is positive.

When she was asked about her understanding of quadratic functions, Amy explained that the standard form of this function [$y = a(x-h)^2 + k$] gave the information how the graph opened upward or downward depending on whether a was a negative or positive number and where the vertex *(h,k)* was. She explained that she knew about the effects of these variables from a lab in the *Study Guide.* This lab asked students to describe the effect of the coefficient a on the general shape of a parabola $y = a(x-4)^2 + 2$ with $a =$ $-2, -1, -\frac{1}{3}, \frac{1}{3}$, 1, 2. Amy also remembered the effects of h and k from the example given in class. She seemed to remember what she did with the variable in a quadratic function rather than showing understanding of how they worked. Furthermore, the instruction on quadratic functions did not emphasize the translation between $y = ax^2 + bx + c$ to $y = a(x-h)^2 + k$. Thus, it was reasonable that Amy could not transform one form to the other.

When working on the Salary Problem (Instructional Interview Problem No. 7), Amy viewed the problem from a symbolic representation rather than thinking about the context or a real situation. In this problem, she entered a number for N in each function to see how much money each person could earn with the same number of years after the date of the contract.

A: Let $N = 100$. That is a good number for N.
I: What is N?

A: *N* is the number of years. Oh! Then 100 is not a good number because you can't work 100 years. OK. Let's go with 5; then 5 is a good number for *N*.

At this point she recognized she was wrong in calling *N* a constant. She said, "*N* is the number of years and can be changed. Then *N* is not a constant. *N* is a variable." After entering some numbers for *N* in the functions of A and B, she wanted to use a different strategy.

A: Let's think another way.
I: Which way?
A: I'm gonna graph it to see if B earns more than A.
I: How do you know if B earns more than A?
A: Graph of B will be higher than graph of A.

This evidence showed that Amy was able to extract and interpret information from the graphical representation. From the graphs, she was able to clarify her thinking and reasoning that B would not earn more than A because the graph of B never went higher. As the time passed, they went apart from each other meaning that A earned much more than B as the number of years became longer.

Similarly, her initial response to the Equivalent Function Problem (Instructional Interview Problem No. 8), was to enter numbers for the variable x to see if the solution for $f(x) = \frac{x^2 - 4}{x + 2}$ and $g(x) = x - 2$ was the same rather than considering the domains of the functions $f(x)$ and $g(x)$. After obtaining the same solution from entering two numbers for x, Amy assumed that the functions were the same. The researcher asked her to try $x = -2$. When entering -2 for x in the function $f(x)$ and $g(x)$, she found that these two functions had different values, and she also demonstrated that she initially thought $\frac{0}{0} = 0$. However, she was able to reconstruct a correct understanding of the meaning of "*undefined*."

A: So $\frac{(-2)^2-4}{(-2)+2}=\frac{0}{0}=0$ and for $g(x)=-2-2=-4$

I: Are you sure that $\frac{0}{0}=0$?

A: Oh wait, it [$\frac{0}{0}$] is undefined because it has zero at the bottom [denominator]; we can't divide any number by zero, and $g(x)=-4$. [$f(x)$ was undefined and $g(x)=-4$].

In this process, Amy reconstructed her understanding of *undefined* with assistance from the interviewer. When asked what she meant by "*undefined*," her explanation was "It [$f(x)$] has no solution."

Amy's Post-Instruction Solution Strategies and Algebraic Thinking and Reasoning

By the end of the instruction, Amy had begun working on problems more carefully. She slowly read the given information and clarified each part of the information rather than looking at the whole problem as she did earlier. She was able to identify the portion of the information that related to what she needed to identify in a problem solution. Working on the Post-Instructional Interview Problem, Amy was able to extract information from each constraint, represent it symbolically, and make a connection with other function concepts she understood. From the first constraint, the function undefined at –3, Amy knew immediately that she needed to construct a function in the form of a fraction [a rational function] that had $x+3$ as its denominator; her reason was that the function was undefined if it was divided by zero and $x+3$ was zero when $x=-3$. She also understood the meaning of a "zero of a function," describing her understanding as the "function has a zero at $\frac{1}{2}$, so it means when $y=0$, $x=\frac{1}{2}$." For the second constraint, when the zero of function was $-\frac{1}{2}$, Amy justified her thinking.

> So on top, x has to be $\frac{1}{2}$ in order to get $y = 0$ [setting an equation $x = \frac{1}{2}$ then working backward to get the equation, and multiplying both sides by 2, then subtracting both sides by 1 to get a final equation of $y = 2x - 1$].
> So the top part of the function is $2x-1$. Let me check [entering $x = \frac{1}{2}$ into the equation $y = 2x - 1$]. I get $y = 0$. It works.

For the third constraint, the function was always nonnegative; Amy verbalized her thinking and reasoning:

> The function has to be nonnegative. So you can put the absolute value for the whole thing because no matter what [x] you have except –3, you always have positive numbers if you find the value of the numbers in this absolute value.

Amy continued working on the fourth constraint as follows:

> A: Next, its domain is from –5 to infinity. OK, this is from –5, but you can't have –3. I don't understand how it can't be undefined at –3.
>
> I: OK. If I change this domain to $[-5,-3) \cup (-3,\infty)$, does that make sense?
>
> A: Yeah, that makes more sense. Then yes, that would work. Right now what I have is x can be any numbers except –3. What should I put in to get x to start from –5. Umm… I can't put –5 or 5 into this because zero wouldn't be at $\frac{1}{2}$. I should do something with 5.
>
> I: Why?
>
> A: Because I will get $x = -5$ if I do something with $x + 5$, which is the same as when I have $x - \frac{1}{2}$, I get $x = \frac{1}{2}$ [pausing for awhile]. I am confused. I'm gonna work on the last condition then I will get back to this.

Amy worked on the last constraint of this problem [a function contains point (4,7)] demonstrating that she understood and knew how to check if a specific point was contained in a function.

> A: OK. The last constraint when $x = 4$ then y has to be 7. So if $x = 4$, 2 times 4 minus 1 is 7, and 4 plus 3 is 7. 7 divided by 7 is 1. But this somehow has to be 7. So 7 divided by 7 is 1. I need 7. 14 divided by 7 is 2. So I need the top to be 49 because 49 divided by 7 is 7. 7 times 7 is 49

so I square it. That would work. So I have $\left|\frac{(2x-1)^2}{x+3}\right|$, so if $x = -3$ it's undefined, and if $x = \frac{1}{2}$, it's zero. And the function's always nonnegative and when $x = 4$, $y = 7$. Wow! That works.

I: How about it's domain?

A: The domain is from –5 to infinity. What should I do for this? Right now x can be any number except –3. I can't change anything from this [the function that she had constructed so far] otherwise it won't work. I can't do it. OK. I give up.

The interviewer probed her thinking about the domain of some functions such as $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, and $f(x) = \sqrt{x-1}$. Amy was able to indicate those functions' domain; however, she was unable to work on the domain constraint of the function in the Post-Instructional Interview Problem. After trying to deal with this constraint, she said, "I give up." This evidence suggested that she did not have an understanding of radical functions to help her find the function when its domain was given. However, she was able to find a domain of a radical function if a symbolic representation was provided.

Ross

From the beginning through the end of the College Algebra course, Ross was able to clarify his solution strategies and algebraic reasoning processes. As the instruction continued, he had more confidence in solving functional problems. He used more than one approach to solve problems and interpret his thinking. For example, in determining a function, Ross used the numerical inputs and outputs, checking with a numerical representation, the vertical line test with a graphical representation, and the graph and the vertical line test with the a symbolic representation. Basically, he used his graphing calculator to draw graph of functions. He was able to identify, gather, and represent information from each problem situation, and interpret a solution and apply his solution to new problems. Before being taught the concepts of functions, Ross was able to gather information from the graphs and correctly interpret the information using verbal, symbolical, and graphical or numerical (tabular) representations. He was able to extract information describing the situation and matched the description to its graph. In an attempt to solve the problems, Ross also was able to relate his reasoning and solution strategies showing how he obtained the correct solution.

Ross's Solution Strategies and Algebraic Thinking and Reasoning Prior to Instruction

Before concepts of functions were taught in the College Algebra class, Ross was asked to solve two graphical and verbal representation problems and two problems related to general algebraic problem solving. These problems helped to clarify his ability in explaining and clarifying his solution strategies and thinking and reasoning processes. Ross worked carefully when dealing with the problems. He read each problem slowly at least two times to make sure that he gathered the relevant information. During this interview, Ross demonstrated his ability in extracting information from graphical representations and transforming those types of representations to a verbal representation and vice versa. He correctly transformed each graphical representation in Figure 25 to a verbal representation. For example, he transformed the graphical representation of George's sister eating popcorn as:

> For the graph of George's sister, it looks like she ate her popcorn very fast at the start and then she slowed down and then fast again and slows down because the line goes down at a steeper angle and then it seems to spread out. For me it indicates that she must be slowing down her eating of popcorn and the line is steep again. It appears to me that George ate his popcorn a lot faster than his sister does and he finished his popcorn before the end of the movie. And his sister did not finish her popcorn at the end of the movie.

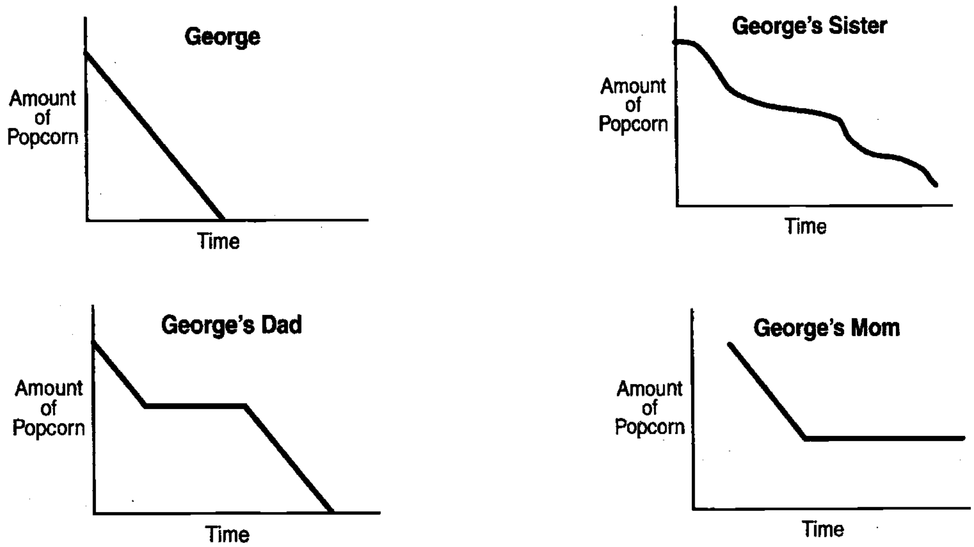


Figure 25. Graphs of popcorn remaining over a period of time.

Ross also described his reasoning before he matched the scenarios represented in words with their graphs. The balloon situation that Ross was asked to match to the seven potential graphs is shown in Figure 15. He attempted the problem by reading the problem slowly and clarifying his thinking and reasoning for each part.

> [Ross read the situation: A balloon was blown up in class and then let go. It flew around the room. The amount of air in the balloon is the dependent variable.] Umm… so the *y*-axis is gonna be the amount of air. And the balloon is empty to begin with because at the beginning there is no air in the balloon. It needs to be started from the bottom and then it fills up with air. So the graph needs to go up and the air is gonna be released and then the graph needs to go down. Umm… so I think graph C indicates this one because it starts from empty and then it goes up and then goes down.

Besides transforming and matching graphical to verbal representations, Ross described how he solved algebraic problems using his knowledge of computation with percents. As well as with some of the information provided in the problem situation. He mentally solved the problem by selecting a number, which was easy to covert to percents without using any tools (i.e. pencil and paper, calculator).

R: No. I don't agree that they are the same. Because umm... for example, I am thinking of $100 to start with.
I: Why do you pick $100?
R: Because $100 is easy to work with percent. So 20% off of $100 means whatever the item is, it will cost $80. I will show two scenarios. First, from $100, 20% off that money will give you $80 and 10% off of $80 will give you $72. Second, 30% off of $100 will give you $70. So the 20% off and the 10% off of $100 after a 20% discount would cost more than taking 30% off at one time.
I: If you are a customer which scenario will you pick?
R: I will pick the second one; taking 30% off at one time.

Ross's Solution Strategies and Algebraic Thinking and Reasoning During Instruction

Ross's strategies for solving mathematical function problems were observed during two instructional interviews. In addition to the interviews, his solution strategies and algebraic thinking and reasoning processes were investigated through various data sources, such as lecture and recitation classroom observations, his homework, and his quizzes.

Instructional Interview One

Ross used two different approaches, determining inputs and outputs and the vertical line test to identify functions embedded in three representations (graphical, numerical (tabular), and symbolic). With graphical representation problems, Ross checked inputs and outputs to identify whether each graph represented a function. When he saw the graph of a circle, he switched his method to the vertical line test to determine if the graph represented a function.

R: This is a graph of a circle. It is not a function. I will check by using the vertical line test. There is more than one output for every input.
I: This time you used the vertical line test to determine if the graph represents a function instead of checking its input and output.
R: Right. I am never aware of that [using the vertical line test to identify a function] until I see the graph of a circle.

Ross seemed comfortable determining a function using the vertical line test because after using the vertical line test the first time (with the graph of a circle), he continued using it with the rest of the graphical representation problems. However, he did not use the vertical line test when he determined if the graph of a point (Pre-Instruction Interview Problem 1, question c) was a function. He said he did not apply the vertical line test to this problem because there was no line or curve on the graph. He was not sure if it was a function. When he was asked to identify if a graph of three points (Pre-Instruction Interview Problem 1(i)) was a function, he extended his thinking and reasoning about this problem to answer that the graph of a point represented a function.

> This is a graph of three points. It doesn't have a line between points. Perhaps it is a function. Perhaps it is the same as the graph of a point. Umm… I can remember an example from the class. She [the instructor] gave an example of three numbers and that would be the domain and then there were three outputs, probably like three points right here. I remember that a function can be anything when you put in one input x and there is only one output y. I will say this is a function and I want to change my answer of the graph of a point too. I will say it is a graph of a function because it has only one output for one input.

Working with problems that provided the information in a numerical (tabular) representation, Ross looked at the input and output values (Function Multiple Representation Problem Part B) to determine if it was a function.

> R: I'm checking if the table below represents a function. And I am trying to explain my reasoning. Umm… so if I put in 2 only 333.8 comes out. And every one of these only has one output for every one input. Umm… so that would be function. And if one more data point (8, 430.6) is added to the table, will this…umm… I don't think that gives me enough information. There is no x that corresponds… oh never mind [realizing that he read the information incorrectly] that is 8 then comma. OK. If I plug 8 in I would get 430.6 out. Yes, it is a function.
>
> I: Would you please tell me why it is a function?
>
> R: Even though it already has 430.6 for the y output, it has a different x. I don't know if I am saying it correctly. The thing that you're gonna be concerned with… umm… If there are two outputs for… umm…each x input that would not be a function. This table still has one output for each x input. So it is a function still.

When the symbolic representation (an equation) was presented, Ross's strategy for identifying a function was to draw a graph and then use the vertical line test.

> For $y = 4$, so I graph $y = 4$ that would imply that when $x = 0, y = 4$ [graphing it by hand]. Because $y = 4$ would be a straight line across the y-axis at 4 horizontally and when you draw the vertical line for every x there would be only one output for each x input.

If the given equation was not in the form expressed in terms of x, Ross attempted the problem by trying to simplify that equation.

R: For $x^2 + y^2 = 1$. Umm... I will rewrite the equation so I can tell if it is a function. I try to get y at one side; first I will use the whole equation $x^2 + y^2 = 1$, and I'm gonna subtract x^2 from both sides, that will be $y^2 = -x^2 + 1$

I: Why do you want y to be alone?

R: That is how I can check it by graphing in a calculator. So I take a square root of both sides: $\sqrt{y^2} = \pm\sqrt{-x^2 + 1}$. So I'm gonna have $y = \pm\sqrt{-x^2 + 1}$ [remembering that when taking a square root, he will get both positive and negative value].

R: Um... Is it all right if I use my calculator?

I: Go ahead.

R: Um... To me, it doesn't look like a function but I'm not sure so I'm gonna plug it in to see if its graph passes the vertical line test. So I put in $y_1 = \sqrt{-x^2 + 1}$. And I'm gonna go with $y_2 = -\sqrt{-x^2 + 1}$. The graphs are like a circle and we know that a circle is not a function because we can check by using the vertical line test and we can see that there are two points for every x. So it is not a function. Oh! Yes, it's not necessary to graph it. This equation was not a function by looking at the symbolic representation, which was obvious that there were several inputs for x that gave two output y values, positive and negative values.

Ross extracted information from a graphical representation and transformed a graphical representation to a verbal representation when working on the Car Problem (Instructional Interview Problem No. 2); however, he was unable to describe the relationships among the speed, time and position of the cars.

R: It shows that early on Car A travels faster than Car B and then it slows down gradually where Car B shows that it's start out very slow and then it accelerates faster and faster as the graph grows up.
I: At each single time, how are their speeds?
R: Um… from this graph Car A is faster than Car B because it shows in the line of Car A is going higher than Car B's. Car B is barely going up at all in the beginning but when the graph shows up further it shows Car A's incline and Car B umm… its incline gets steeper. It shows that Car B goes faster as the graph goes up and Car A is getting slower because the incline is not as steep as the incline of Car B.

Ross indicated that the graph of Car B was steeper than that of Car A; therefore, Car B was faster than Car A. In fact, the rate of change of Car B was greater than that of Car A; however, Car A still was faster than Car B until one hour when they had the same speed. And after one hour Car B was faster than Car A. Even though he described the graphical representation correctly, he had two misunderstandings about an average speed of these two cars. First, he misunderstood that the diagonal line connecting the starting point and the point at a time equal to 1 hour represented the average speed (see Figure 26). Second, he misunderstood that the same average speed made the cars be at the same position. This idea guided him to an incorrect answer that those two cars were at the same position at t equal one hour.

I: So what would you answer for the first question?
R: It's um… they are at the same place, same position… umm… because the cars … umm… I don't know. There is no information to determine that. Wait! You can clearly see that Car A goes a lot faster; let's say right here is 10 minutes [marking a dot for showing 10 minutes after the starting point on the x-axis]. Umm... Car B would barely be moving at all. How can I explain this? [He read the problem again then he drew the line connecting the origin and the intersection point (see Figure 21).] OK this line shows the average speed of Car A and Car B, I guess. And I see that the averages of both cars have to be the same.
I: Why?
R: Umm.. Because both of these lines eventually intersect at the same point right here at $t = 1$. So they both have the same average speed at that point. So, both cars have the same average speed and they are at the same position.

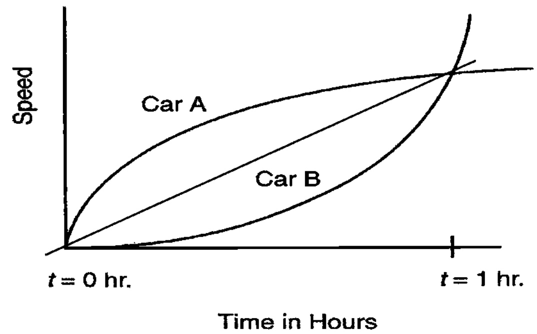


Figure 26. Ross's diagonal line showing the cars' average speeds.

When asked what he meant by "average speed," Ross described it as an average of the speeds of the two cars by adding the speed of the two cars and dividing by two.

Ross demonstrated his understanding and ability to transform among the functional representations. For example, as he worked with the Piecewise Graphical Problem (Instructional Interview Problem No. 3a) he transformed a graphical representation to a numerical (tabular) representation.

R: From the graph, I can make a table to represent the information. Do you want me to make a table for this?

I: Yes, please.

R: So when $x = 0$, y looks more like 50. Actually, it's in the middle between 0 and 100. Well, when $x = 50$, y is about 75; when $x = 100$, $y = 200$ and I use the top line because the bottom line doesn't include $x = 100$ because there is an open dot at the end of the line. When $x = 200$, $y = 200$; $x = 400$, $y = 200$; $x = 500$ y grows up to 250; $x = 600$, $y = 300$ [making a table].

x	0	50	100	200	400	500	600
y	50	75	200	200	200	250	300

Ross also was able to transform a numerical (tabular) representation to a symbolic representation.

I: As you just have finished representing the information in a table, can you represent the same information in another form?

R: I should be able to make an equation.

I: Can you show me how to do that?

R: Yeah, they've got two lines; no I guess it would be three lines. I need to make an equation for each line. I don't know if I can do it. I'll try my best. I need two points in order to make a linear equation. Umm... What am I doing?

I: You want to find the equation of a line, don't you?

R: Right. A slope! Yes, I need a slope. I need to use my table. When $x = 0$, $y = 50$ and $x = 50, y = 75$. Then I would go with rise over run [remembering this formula from high school; the instructor never used them in the class]. So $\frac{y_2 - y_1}{x_2 - x_1} = \frac{75}{50} = \frac{2}{3}$. So that would be a slope. The equation is $y = \frac{3}{2}x + b$, and b will be 50.

I: What is b?

R: b is 50 because it starts at 50 points on the y-axis.

I: What does b represent?

R: Umm... the origin. No, that is the y-intercept. So this is an equation for the first part [writing $y = \frac{3}{2}x + 50$]. For the second part, umm... $y = 200$ when x equals 200 and then I think it would show like $100 \le x \le 400$.

I: How about the first part? That equation will be used when x equals what?

R: When x greater than or equal to zero because I don't see a round dot on that line and x is less than 100 [writing $0 \le x < 100$].

Ross attempted the third part of the function using the same strategy as he did for the first part of the equation. He computed a slope, but he could not finish finding the equation because of his limited knowledge of finding a linear function using the point-slope form [$y - y_1 = m(x - x_1)$] even though he had studied it in class.

R: The third part, umm...first I need to find a slope, so I will do that in terms of $y_2 - y_1$ again. For the point (600, 300) and (500, 250). So $\frac{300 - 250}{600 - 500} = \frac{50}{100} = \frac{1}{2}$. So slope equals $\frac{1}{2}$. So $y = \frac{1}{2}x$ and it starts at 200.

I: Why do you think it starts at 200?

R: Umm ... no. It doesn't cross the y-axis at all. So I don't know how to find b.

Even with assistance, Ross could not remember finding a linear equation using the point-slope approach. Therefore, the interviewer assumed that he had three linear equations for each line in the graph and asked him to write a function notation including the three linear equations. Ross did not know how to write a piecewise-defined function notation.

I: Let's assume you got the third piece. This equation will be used when x equals what?
R: The third part is used when x is greater than 400.
I: Can you represent all three parts in a function notation?
R: I can't remember it.

Even though the interviewer probed Ross, he still could not write the notation of the piecewise-defined function including these three pieces of linear equations.

Basically, Ross showed that he was more comfortable working on functional problems using symbolic representations. He was uncomfortable with connecting the problem to real situations. However, he developed his ability connecting his understanding of functions with real situations during instruction while he worked with the Ball Dropped Problem (Instructional Interview Problem No. 4). With this problem, he was unable to relate the problem situation to the real situation. He said that the function for the Ball Dropped Problem [$f(t) = -16t^2 + 145$] did not have an inverse because its graph was a parabola upside down and it was not a one-to-one function. However, if he related this problem to a real-world situation, he would have known that the graph of the ball movement was a semi-parabola upside down and the function would be one-to-one. Two weeks later, Ross was able to apply the problem to a real situation when he worked with the Salary Problem by relating the problem context to the fact that groups of people might work the same or different length of years.

Instructional Interview Two

By the second interview, Ross' responses to graphical representation problems (Instructional Interview Problem No. 5 and Instructional Interview Problem No. 6), revealed that he was able to connect a quadratic function and its graphical and symbolic representations. Ross constructed and clarified his reason for selecting a symbolic representation of a quadratic function [$y = a(x-h)^2 + k$] from the information given in the form of a graphical representation.

> a indicates what direction the parabola faces, if it's positive, it faces up and if it's negative it faces down. And this is facing up so you can assume that a is positive. And a also tells how big the parabola is. Umm… I don't know exactly how big it is. I'm gonna guess something like 2.The h is going to indicate how many places that the vertex shifts to the right or to the left. And it is on the right, so h is gonna be a positive number in this standard from. We can guess somewhere around 2 or 3. And k is gonna show me how many places that the vertex will be along the y-axis. It [k] tells me how many places that the vertex shifts up or down from (0,0), and it's negative. So it's gonna be under the zero. I guess it's –1.

When working with the Instructional Interview Problem No. 6, Ross did not relate a reference graph $f(x)$ to the graph $g(x)$ requiring a symbolic representation. He worked to find a reasonable symbolic representation in the same way he did in the Instructional Interview Problem No. 5, which considered the variables a, h and k, respectively.

> First a is gonna be negative because it's facing downwards. Umm… h is going to be negative in this standard form because it is on the left side. So I just guess like –5. And the vertex is positive on the y-axis because it is in the positive area, above the x-axis. So k is gonna be 1 or 2. Let's say 1. So, I can't remember whether a bigger number makes it wider or a fraction makes it smaller. I'm pretty sure that a smaller number makes it smaller but I can check this with my calculator.

He was unable to remember the effect of the size of a on the width of a parabola. After he checked with his calculator, he knew that the smaller the absolute value of a, the wider the parabola.

Ross showed that he could solve the Salary Problem (Instructional Interview Problem No. 7) using different approaches. The approach that he used more often was to use a graphing calculator to graph each salary function to see if they had any intersection points, or to enter the number of years (N) to check which person earned more money for that particular number of years. He considered two different ideas in the process of finding a solution. He considered whether the number of years (*N*) was equal or different for each person. To answer the first question of this problem, he considered the salary functions and compared their slope and *y*-intercept. He compared the fixed amount of money earned by A and B [*y*-intercept] and unfixed money as depending on numbers of years [slope].

R: B will never earn more than A because I can see that term with the variable *N* of A [2500*N*] is larger than that of B [1800*N*]. So the money of A will increase more than the money of B. And the constant amount of money of A is larger than that of B. There is no potential at all if *N* is the same for both groups.
I: You said if *N* was the same, what if the number *N* is different?
R: If N is different, let's say B works longer than A like 5 years and A works only 1 year. Then B will earn more than A.

To answer the question whether D earned more money than C, Ross considered two parts of the salary equations. Ross used a mental reasoning process to explain his thinking.

> There are two different parts. C has a higher constant [27000] but D has higher money increasing for each year [2100*N*]. When the number of years *N* increases, D is gonna have more amount of money than C. Eventually D will catch up to C.

When asked if he could find when D earned more than C, Ross transformed a symbolic representation to a graphical representation using his graphing calculator. He also showed his knowledge of the slope-intersection form of a linear function and set an appropriate window for the graphing calculator to match a real situation.

I: Can you find when D will earn more than C?
R: I think I can find it by using their graphs.
I: How?
R: I will set $y_1 = 27000 + 1500N$ for C and $y_2 = 21000 + 2100N$ for D. Then I graph these functions. Umm...you know what? I can change these functions around. I can change the function to $y_1 = 1500x + 27000$.
I: Why do you do that?
R: I can change it to a linear function form. I can make it in the form of

$y = mx + b$ [putting the function of y_1 and y_2 in his calculator]. The graph does not show up. So I have to make a new window. My Xmin = 0 and Xmax = 12, Ymin = 30000 and Ymax = 50000

I: Why do you select this window?

R: Because each point on the x-axis will never be a negative number.

I: Why?

R: Because the x-axis represents the number of years after the date of the contract, which will not be a negative number and I think D will earn more money than C sometimes in the future. If it takes longer than 12 years, I will change my Xmax. The y-axis represents the amount of money, which I think should be more than 30000 when D earned more than C [setting a new window, graphing the two functions and explaining the graph].

R: You may not see the graphs very well. But it looks like at the 10^{th} year they cross each other. And after the 10^{th} year, D will earn more money than C.

I: What does it mean at the point that the graphs cross each other?

R: It means at that point they earn the same amount of money.

Ross incorrectly attempted the Equivalent Function Problem (Instructional Interview Problem No. 8) by looking at the initial feature. He was able to correct himself at the end.

R: Yes. They [$f(x) = \dfrac{x^2 - 4}{x + 2}$ and $g(x) = x - 2$] are the same because umm...basically *g(x)* is a simplification of *f(x)*. If you change *f(x)* and *g(x)* to *y* and put any number for *x* then *y* is gonna be equal for both functions. For example if the input for *x* equals 1 in both equations, the answer will be the same for *y*. Oh! Except...

I: Except what?

R: I'm wrong. They are not the same. Because umm... The function $f(x) = \dfrac{x^2 - 4}{x + 2}$. There is a term in the denominator $x + 2$, when $x = -2$ would be undefined but there isn't any number to make g(x) undefined. So they aren't the same and that is my reason they are not the same.

Ross's Post-Instruction Solution Strategies and Algebraic Thinking and Reasoning

At the end of the instruction on functions, Ross was asked to solve one problem involving function concepts. He worked on the problem carefully. He read the problems slowly and most of the time he read the problems twice. Working on the Function Construction Problem (Post-Instructional Interview), he read the given constraints and verbalized them one by one. Ross was able to gather information given in the Post-Instruction Interview Problem. He symbolically represented each constraint of the problem and used the variety of concepts related to functions for finding the problem solution. He clarified each piece of the constraints that related to the solution that he was asked to find.

> The function is undefined at –3, which means we don't want –3. So when $x = -3$ the denominator would be undefined, that would be $x + 3$ at the bottom of the fraction. The function has a zero at $\frac{1}{2}$, so $x - \frac{1}{2}$ on will be at the top of a fraction because it makes this function be zero when $x = \frac{1}{2}$. The function is always nonnegative so all of these terms have to be squared [squaring for the numerator: $y = \frac{\left(x - \frac{1}{2}\right)^2}{(x+3)^2}$].

Ross connected the undefined constraint to the domain constraint identifying that it was impossible to have a function's domain from –5 to positive infinity.

R: The function domain is -5 to infinity. Umm…wait this doesn't make sense because it's undefined at –3 but –5 to positive infinity includes –3 but x can't be –3.
I: What do you think its domain should be?
R: It should be –5 to positive infinity except –3.

The interviewer changed the function domain to –5 to positive infinity except –3 $[[-5,-3)\cup(-3,\infty)]$ and asked Ross to continue working on the problem.

R: Umm… OK. $x - \frac{1}{2}$ over $x + 3$ and I want to have another term to get x from –5. OK x is gonna be a square root because x is from –5. Umm… x equals negative 5 [writing $x = -5$]. So it has to be $x + 5$. OK. This term has to be a square root because the number under the square root sign cannot be a negative number.

I: With the term of $\sqrt{x+5}$, can x be –5 or not?
R: Yes, it can be –5.
I: What will you do with this term?
R: I will combine all parts together.

Even though Ross knew how to find the expression indicating the x value from -5 to infinity, he had difficulty putting this expression into the function he had created.

Initially, he combined all parts that he had as $y = \dfrac{\left(x-\frac{1}{2}\right)^2}{(x+3)^2\sqrt{x+5}}$ and attempted the problem using a guess and check method to see where he should have added the expression $\sqrt{x+5}$ into the function that he created previously.

I: What happens if x = –5?
R: Umm. x equals –5, Oh, right it's gonna make this zero and that would make this [a function] undefined at -5.
I: Do you say the domain includes –5?
R: Right, so I've got to change that. It gonna be plus 6.
I: So if you use $\sqrt{x+6}$, so what can x be?
R: x can be any number greater than or equal to –6.
I: So it still does not start from –5, is that right?
R: Wait, if I change this [$\sqrt{x+5}$] to be in the numerator part; that will work
I: Why do you think you should enter this term to the numerator?
R: Because if I put it into the numerator so x can be –5 and greater than -5. So now what I have is $\dfrac{(x-1/2)^2\sqrt{x+5}}{(x+3)^2}$. So, I think this is the function that we're looking for. And the last one, the function contains the point (4, 7).

Ross attempted the last constraint by using the symbolic representation of the function and numerical calculation by entering 4 for the value of x in the function that he created previously.

I: What does it mean that the function contains (4, 7).
R: Umm… well this means when $x = 4$, y is gonna be 7. So I'm gonna plug in $x = 4$ to see what happens. Square root of 4 plus 5 is 3 and 4 minus $\frac{1}{2}$

is 3.5. Then 4 plus 3 is 7. So I have $\frac{(3.5)^2(3)}{7^2}=\frac{36.75}{49}$. This is not gonna work [making a calculation using his calculator].

I: What do you mean it doesn't work?

R: It's not equal to 7. I need to do something. I will multiply this number by something.

I: So what are you gonna multiply by?

R: 36.75 over 49, I'm gonna set the equation $\frac{36.75}{49}x=\frac{7}{1}$ Then solve for x [writing $x=\frac{7}{1}\bullet\frac{49}{36.75}=9\frac{1}{3}=\frac{28}{3}$ and checking his calculation using his calculator].

R: So I need to put this whole thing in parenthesis and multiply by $\frac{28}{3}$ and that is the answer right there, $y=\frac{28}{3}\frac{\sqrt{x+5}(x-1/2)^2}{(x+3)^2}$. That's the answer.

While solving the Function Construction Problem, Ross always checked his understanding and his work by re-reading the given constraints that helped him determine if he had correct information and if he had correctly understood the problem.

Emma

Emma worked slowly on each problem. When she finished reading each problem direction or situation, she paused, and thought about the information for a few seconds. She then described her thinking and gave her reasoning for what she had done. She read some parts of the problems twice if she was not sure that she had all the information she needed for finding the solutions. Emma used different strategies to determine a function. She used the vertical line test, checked numerical inputs and outputs, drew the graph and then used the vertical line test if the relations were given in the form of a graph, table, and symbol respectively. She used her graphing calculator to draw graphs of functions and calculate numerical expression.

Emma's Solution Strategies and Algebraic Thinking and Reasoning Prior to Instruction

Emma solved four mathematical problems related to multiple representations and algebraic problem-solving before the concepts of functions had been taught. Working on these problems, Emma communicated her thinking and reasoning and solution strategies while solving the problems. Emma approached the problems by slowly reading each problem and clarifying her solution strategies including her thinking and reasoning. She gathered information from the problems and verbalized the information using her own words before describing her solution strategies and reasoning used for solving the problems.

Emma transformed a graph to a verbal representation (Pre-Instructional Interview Problem No. 1), solved algebraic problems (Pre-Instruction Interview Problem No. 2 and No. 4), and matched the verbal and graphical scenarios (Pre-Instructional Interview Problem No. 3). Emma translated a graphical representation into a verbal representation (see Figure 25).

> Let's see George's Mom. She didn't start eating the same time as others [pause] because the graph of the amount of popcorn doesn't start until [pause] it didn't intersect with the y-axis and that made me think that. Why do I think that? Umm... Because [pause] It's hard to explain. I don't know why just because the line showed that it didn't start from zero. There was a gap that made me think she didn't do anything during that time. And when she started eating, she ate at a consistent rate because it showed a straight line going down and then she just stopped eating because the straight line showed popcorn stayed at the same amount. She didn't finish hers.

With Pre-Instruction Interview Problem No. 3, Emma clarified her reasoning as to why each scenario's graph should match the verbal scenarios given in the problem. For example, she depicted the roller coaster situation (see Figure 15).

> [Emma reading the situation: We rode the roller coaster steadily to the top, then went faster and faster as we went down the other side. The speed of the roller coaster is the dependent variable.] The speed of the roller coaster is a dependent variable of the graph, that is, the variable on the vertical axis. So he went at the steady rate to the top, the speed should be on the *y*-axis. OK and then they went faster and faster down to the other side [pause a few seconds]. I' m thinking

> about the graph that should be beginning with the speed at the consistent rate so the line should be horizontal. Umm.. And then after that, the speed of the roller coaster is supposed to be faster and faster down to the other side. This should be a line where the slope is in a positive direction. Because it said the speed was faster and faster. Let's see, umm… The slope is rise over run and should be [pause] Umm… I think it is A.

When a scenario and a graph did not seem to match or seem to make sense, Emma drew the graph for the scenario herself. For example, she described her thinking and reasoning of this scenario:

> At the beginning of spring, the grass grew slowly and I seldom had to mow the lawn. By midsummer it was really growing, so I mowed twice a week. In fall, I only mowed once in a while. The number of lawn mowing to dates is the dependent variable.

From this situation, she described her thoughts and reasons as:

E: Umm... so the number of lawn mowing is on y-axis. The y value should be fairly low at the beginning of the graph because he did not mow very much and then at the middle of the graph it should be fairly high because they have to mow more often. Then it should be lower than before they have to mow once in awhile. So the graph of once in-a-while has to be low. Let's see [pause] Umm… no the graph was looking like exactly the same as I was thinking. Umm… the one that looks closest is an "E" but it doesn't look like it has to be E.

I: So if you think these graphs do not make sense, you may draw a graph by your own.

E: On my own? Do you think I should?

I: Yes if you think these graphs do not match with the scenario.

E: OK. If I have to pick one, I decide to say C but I don't like it at all.

I: So you can draw your own graph.

E: OK. I have time on the horizontal axis and number of mowing on y-axis. The graph should be low at the beginning and then it seems like it should have a little consistent rate and then by midsummer it was growing a lot and I would think that transition would have [pause] then would have been a transition period that can be noticed that it was growing slow and then growing fast so I should have a line that had slope up in a positive direction and then it should level out during midsummer and then [pause] in Fall only mow once-in-a-while and I think it should be a downward transition into the Fall and then in the Fall, it should level out again. And this is my graph (see Figure 27).

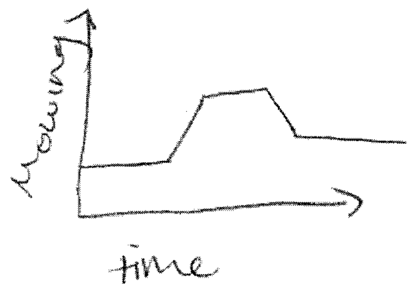


Figure27. Emma's graph for the lawn-mowing situation.

Based on the scenario, Emma verbally described how the grass grew and the number of mowing times correctly. However, the graph that she drew for the scenario related to how the grass was growing rather than to the number of mowings. For example, she said "[the grass is] growing fast so I should have a line that has a slope going up in a positive direction." She drew the line with a positive slope showing how the grass was growing, not the number of mowings to dates. Emma used the graph to describe her thinking of the number of mowings and how the grass was growing within the same graph.

Prior to the instruction on functions, Emma also solved an algebraic problem using her knowledge of percents related to information provided in the problem (Pre-Instructional Interview Problem No. 2). Using pencil and paper she explained and showed her work for obtaining the solution.

E: I do not agree with the manager because 10% discount should come After what price the item is. It has 20% discount to start with, then you take 10% off from 80% this left. So that would be [pause]

I: You may write it down if you want to show if it is the same.

E: OK. Let's see. It starts at 80% of the original cost and then 10% off of $80, which is 8, and so the total price you need to pay is 80 subtracts 8 which gives you 72%. So the total discount here is… Let's see 28% that is my answer. They are different.

I: This means that you do not agree with the interviewer.

E: No.

I: Do you think one is better, between getting 30% discount the first time and getting 20% off and then an additional 10% off?

E: Getting 30% discount at one time would be better for the person who buys things.

Emma mentally responded to the Population Growth Problem (Pre-Instructional Interview Problem No. 4) without writing any mathematical symbols, expressions, or equations. She described her reasoning verbally. Her logic in approaching this problem was reasonable and correct.

E: Town A starts at 5000 and grows up to 8000 so it's increasing by 3000 and Town B starts at 6000 and goes up to 9000, and that's increasing by 3000 as well. Umm... Brian claimed that they increased by the same amount. He was right because they both just grew by 3000 people. Linda claimed that Town A had grown more and she also was right because Town A grew by a greater percentage than Town B did.

I: How do you know that Town A grew by a higher percentage than Town B did?

E: Umm. . .because they both grew by the same number and Town A started out with less people and so if they both grow by the same amount it's gonna be a higher percentage for Town A just because they're originally small. So it just makes sense to me like this.

Emma's Solution Strategies and Algebraic Thinking and Reasoning During Instruction

To investigate her solution strategies and algebraic thinking and reasoning, multiple data sources were included classroom (lecture and recitation) observations, homework, quizzes, and interview problems to identify her solution strategies. Her reasoning for using those strategies was investigated through two interviews conducted during the instruction of functions.

Instructional Interview One

In the first interview Emma was asked to solve four problems related to functions. The first problem asked Emma to determine if the relations were functions. The relations were given in the form of a graph, a table, and a mathematical symbol. Emma used the vertical line test to determine if the graphs represented functions. She stated:

> I would say it [a graph of a relation] is a function if it passes the vertical line test. I mean if a line doesn't intersect the graph or I can draw any lines, those lines wouldn't intersect the graph more than one point. I would say it is not a function if it fails the vertical line test.

When asked to determine if some information provided in a tabular representation was a function, Emma converted it into a graph using the table data. She stated, "For every x there would be one y so it is a function." Yet, even though she thought about the graphical representation, she did not use the vertical line test to check if it was a function, instead she checked the inputs and outputs. When determining whether an equation was a function, Emma graphed the equation and considered its graph. However, she did not use the vertical line test but she stated it was a function because it had one output for every input.

> I just graph this [$y = x^2$]. It would be a parabola and I know that it's a function because there is one y value for every x and I know this is a function of y in terms of x because the value of y depends on x. And this is [$y = 4$], its graph would be a horizontal line and that is a function because there is one y value for every x value.

At the beginning of the first interview, Emma could not symbolically solve for y in the equation of a circle $x^2 + y^2 = 1$. She realized that she needed to rewrite this equation in order to make a graph in her calculator. However, her procedure was incorrect. She was able to check that her procedure for taking a square root did not work but she could not correct herself. Her approach of checking her thinking by taking the square root and by testing some values caused her to use a different strategy.

E: For $x^2 + y^2 = 1$, I want to put this equation into my calculator so I can see what its graph looks like.
I: How do you do that?
E: Let me think about this. I can't put in $x^2 + y^2 = 1$, so I would take the square root of both sides [writing $\sqrt{x^2 + y^2} = \sqrt{1}$]. So that would be $x + y = 1$. So then I can say y = 1-x. I don't know if it's right.
I: Do you think it is right?
E: I don't know, but if I can do that it seems too easy.

I: Let see if you have $4^2 + 3^2$ what is that equal to?

E: It would be 16 + 9 = 25 and the square root of 25 is 5. I will check with $\sqrt{4^2 + 3^2} = \sqrt{25}$, then 4 + 3 = 5. No, that doesn't work. Big mistake, so I can't do that.

After she realized that her method of taking the square roots did not work, she tried making a table.

I: How can you solve for *y*?

E: I'm not sure. Let's see. I'm not sure about this one. I'm going to pick some point to see what *x* and *y* look like. When $x = -2$, so $(-2)^2 + y^2 =$ 1, $4 + y^2 = 1$, $y^2 = -3$. That is not possible because it's negative. I will try $x = 1$. Let's see $1^2 + y^2 = 1$, $1 + y^2 = 1$, $y^2 = 0$, so $y = 0$. And then if I try $x = 0$ then $0^2 + y^2 = 1$, $y^2 = 1$, and y = 1.

I: y^2 equals 1, so what is *y*?

E: *y* would be 1 because 1^2 equals 1.

I: Are you sure?

E: Yes, $\sqrt{1} = 1$ [checking $\sqrt{1}$ by using her calculator]. I'm trying to graph but I don't know how it works. Hopefully this works [drawing the graph by hand]. Umm… that should be a function because it has one *y* for every *x*.

I: What does the graph look like?

E: It looks like a half circle [a semicircle along the *x*-axis]. So it is a function.

In fact, she had studied the equations including circle equations and their graphs in the first week of this course, but she could not remember that $x^2 + y^2 = 1$ was an equation of a circle. Even though she could not take square root of a two-variable equation, she had a definite strategy (using specific values to test if it was true with those values). Her lack of knowledge of calculating a square root led her to an incorrect answer, and she said that $x^2 + y^2 = 1$ was a function.

In addition to the Multiple Representation Problem (Instructional Interview Problem No. 1), Emma worked on three other problems in the first interview during the instruction of functions. She extracted information and transformed a graphical representation to a verbal representation when working on the Car Problem

(Instructional Interview Problem No. 2). She was able to relate the information to a graph that showed the relationship between the time, speed, and distance.

E: The graph shows that at the beginning Car A goes faster than Car B. But after one hour Car B goes faster than Car A. They are tricky. Ah...I try to think... I think Car A would be ahead of Car B at t = 1 hour.
I: Why?
E: Because it's been going faster the whole time. That would mean it would cover more distance in the same amount of time. At t =1 hour they have the same speed or velocity because they [the graph of Car A and that of Car B] intersect at t = 1 hour. The next question asks about the relationship of the acceleration of Car A and Car B at t = 1 hour. Ah.. So the acceleration of Car A is slower at one hour than the acceleration of Car B because Car B has a steeper slope and its speed is increasing quicker than Car A.
I: So its slope tells you how it speeds up?
E: Yes, slope does. The steeper slope, the faster it is.

In the third problem, the Piecewise-Defined Function problem (see Figure 17), Emma was asked to determine if the graphical representation was a function. She used the vertical line test to check if it was a function. She stated:

> I think it is a function because it passes the vertical line test. At these points [pointing at the dark and white dots at $x = 100$], one is counted and the other is not. I don't know for sure which one is counted but they are not counted at the same time, so it passes the vertical line test.

When asked how she could represent this data differently, Emma said she could represent it as an equation (symbolic representation).

E: There are three pieces of linear equations here.
I: Can you describe what each piece looks like?
E: Well, let's see. It would deal with x values which are less than 100 and umm...I can estimate, that looks like about $\frac{1}{3}$ of 100. x would be 0. Then y, let's see... that would be 33. And then this would be $x = 50$ and y umm...about 66. Then I need to find its slope. That would be $66-33$ divided by $50-0$, which is 50 and that is .66 [using her calculator]. So $y = .66x$, plus the y-intercept, which is 33 [writing $y = .66x + 33$].
E: So that is the first piece.

I: When will we use this piece?
E: When x is less than 100.
I: Do you include negative numbers too because you said when x is less than 100?
E: From the graph, it looks like they start from zero. So this will be used when x is greater than or equal to zero and x is less than 100.

For the second piece, Emma knew that the y values for the horizontal line were for a particular interval of x values.

> And then the second piece is $y = 200$ when x is greater than or equal to 100 and less than or equal to 400.

To find the linear equation for the third piece, Emma attempted the problem in the same way that she did for the first piece. However, she did not finish finding the equation of the third line using the point-slope form. She visually estimated the figure of the lines and declared that the first and third lines were the same line and should have had the same equations.

E: And I need to find another two points in order to find the slope of the third piece. That looks like 300 for y and 600 for x, and $x = 700$. I would say $y = 350$ [writing (700, 350) and (600, 300)]. I am going to find the slope right now. $m = \dfrac{350-300}{700-600}$. They look like the same line as the first line. If they are the same then they will have the same equation as the first one that I found.
I: Are they exactly the same?
E: From the graph, they look the same but from the number I got they are different. But maybe the number I got is not right. So I would say they are the same equations.

Emma was not confident when asked to write the piecewise-defined function notation

I: Can you write it in the form of function notation?
E: That would be $f(x) = \begin{cases} y = .66x + 33, 0 \le x < 100 \cup x > 400 \\ y = 200, 100 \le x \le 400 \end{cases}$
I don't know for sure that I could write it that way but it works [seeming like she still did not clearly know how to write a piecewise defined function].

Emma also was able to transform the graphical representation to a numerical (tabular) representation.

I: Are there other different forms of representations that this information can be presented?

E: Another form? I could make a table of points. That is the only way that I can think right now.

I: Can you show me how the table looks?

E: Umm... that looks like:

x	0	50	100	200	300	400	500	600	700
y	33	66	200	200	200	200	250	300	350

In Instruction Interview Problem 3b, Emma said that she remembered this type of function, called a step function, but she could not provide any real situations that could be represented by this graph.

At the end of the first interview, Emma had developed her understanding of solving a problem symbolically using the square root and relating it to a real situation. This was evident as she worked with the Ball Dropped Problem (Instructional Interview Problem No. 4, see her work in the instructional interview of understanding of functions section).

Instructional Interview Two

Emma's responses to Instructional Interview Problem No. 5 and Instructional Interview Problem No. 6 revealed that she could connect a quadratic function with its graphical and symbolic representations. Emma constructed and clarified her reason for selecting real numbers for variables *a, h,* and *k* in symbolic representation of a quadratic function [$y = a(x-h)^2 + k$] from the information given in Instructional Interview Problem No. 5.

E: This is a graph of a parabola and I know from the equation that *a* has to be positive because the parabola opened up [writing a positive sign above the variable *a*]. Umm…the vertex is (*h, k*). So *h* has to be positive.

I: Why do you think *h* has to be positive?

E: Because the value for h is in the positive area [Quadrant 4] on this graph. Umm... so let's see I would say that $a = 4$ and then I'm gonna guess that $h = 2$ and k is the y value on the vertex, I would say $k = -1$. And then I will put it in my calculator to see what I will get [entering $y_1 = -4(x-2)^2 - 1$ into her calculator and graphing it].

I: How was it?

E: It looks not so close. Let's see what happens if I change a to 6.

Emma did not remember that the size of variable a would effect the shape of a parabola.

I: What do you expect to see in the graph when $a = 6$?

E: I want to see if it's wider. I'm putting these graphs on the same axes to see how it changes [entering $y_2 = 6(x-2)^2 - 1$,then graphing it]. So it looks like my second graph is skinnier instead of getting wider. But I'm looking for the number that makes the graph wider. So let's try 1 because a big number makes it skinnier, so a smaller number would make it wider [entering $y_3 = -1(x-2)^2 - 1$, then graphing it]. That's closer.

I: What happened when you changed a?

E: The width of the graph is changed. $a = 1$ makes it wider than $a = 4$ or 6. So I'm saying .5 [entering $y_4 = -.5(x-2)^2 - 1$, then graphing it]. Umm... that looks pretty good. So my last equation that I came up with is $y = -.5(x-2)^2 - 1$.

When working with a quadratic function (Instructional Interview Problem No. 6), Emma related the graph of *g(x)* to the reference graph *f(x)* and found the symbolic representation for *g(x)* quicker than when she worked on the Instructional Interview No. 5. She related her understanding of Problem No.5 to Problem No. 6. Obviously, one of her reasoning approaches was graphing a function, which was much easier with the graphing calculator. So she was able to do more trial and error to check if she was thinking correctly.

E: The original graph is $y = x^2$, and it is a parabola and umm... I know from the last problem that the flipped over parabola... a has to be a negative number and the smaller $|a|$ umm... the wider a parabola is. So for this one, let's see. I will say $a = -5$ just for starting out with. And then my vertex is ... let's see... I'll try $h = -6$ and $k = 1$.

I: How did you consider those the numbers for h and k?

E: Umm... h corresponds to the x value and the x value umm... is much further from the origin than the y-value in this case [putting in $y_1 = -.5(x-6)^2 + 1$ and graphing it]. That looks pretty close to me.

I: Do you want to adjust anything?

E: Umm....it looks good. I will say $g(x) = -.5(x-6)^2 + 1$. Let me check the graph of $f(x)$ [entering $y_2 = x^2$, then graphing it]. Their widths are almost the same. I need to make $g(x)$ wider. I will change a to $-.25$ [entering $y_3 = -.25(x-6)^2 + 1$, then graphing it].

I: How was it?

E: This is much better. So I think $g(x) = -.25(x-6)^2 + 1$ is good for $g(x)$.

Emma attempted the Salary Problem (Instructional Interview Problem No. 7), without changing a symbolic representation given in the problem to another representation. She stated the answer and described her method and reason for her answer.

> Will B ever earn more per year than A? No, they won't because they both have the same starting money and A always earns more per year than B. So it is not possible for B to earn more.

When asked if it was possible that B can earn more than A, she stated:

> The only way B would earn more was if they were employed for more years than A. Umm... that is the only one possible case otherwise he will never earn more than A.

As with the other questions (b and c), Emma provided her solutions using two different ideas: (1) the same number of years and (2) a different numbers of years. When attempting the last question of this problem, Emma used her graphing calculator. She showed that she was familiar with changing the graphic window to make the graph easier to interpret.

E: Will D ever catch up C? I think I may have to graph this to find out because C starts out with more money but he earns less per year.

I: So what will you graph?

E: First, I'm gonna graph C's salary to see what it looks like. I put in $y_1 = 27000 + 1500x$ and then graph. I need to change my window. I change Xmin to zero because it's never going to be negative number and

> I put Xmax as 50 years. And scale for 5 years interval and Ymin they're never gonna have negative salary so Ymin = 0 then Ymax = 100000, I put scale for 10000. Then see what it looks like. It is a line going up. Then I put in $y_2 = 21000 + 2100x$ and it looks like D will eventually catch C because the two lines umm... intersect and after that...Oh yeah both lines have a constant slope and D has the steeper slope. So eventually it's gonna hit the line of C. D will eventually have a higher salary.

She was asked to find when D would earn more than C, Emma used her graphing calculator to find the intersection point rather than solving using symbolic manipulation.

I: Can you tell when D is gonna catch C?
E: Yes I can.
I: How can you find it? Would you please show me?
E: I'm gonna go to calculate [using a function menu in the graphing calculator] and find the intersection; they intersect at (10, 42000), so that would mean at the 10th year they would earn the same amount of money and after that D would have a higher salary.

It seemed like she knew how to find the solution using symbolic manipulation. When asked if she could solve this problem without using her graphing calculator, she said she would set the salary function of C to that of D and solve for the *x* variable. However she did not illustrate this symbolic procedure.

Emma's Post-Instruction Solution Strategies and Algebraic Thinking and Reasoning

At the end of the instruction on functions, Emma continued working with the problems in the same manner as in the beginning. She read the problem and interpreted her thoughts slowly. Working on the Function Construction Problem (Post-Instructional Interview), she read the problem and the given constraints all at once and then verbalized each constraint one by one. Emma gathered information from each constraint, verbalized each constraint based on her understanding, and then represented them symbolically.

> The function is undefined at -3 that means ah… if it's divided by something and the denominator can't be zero. So this would be something divided by $x + 3$. When $x = -3$ that would be undefined because the denominator would be zero.

Emma showed that she understood and knew how to find the zeros of a function.

I: When you were asked to find a zero of the function, what would you do?
E: Umm… I set the equation equal to 0 then solve for x.
I: What part of the function will you set equal to 0?
E: The top part [numerator].
I: The function has a zero at $\frac{1}{2}$, that [$\frac{1}{2}$] is the x or the y value?
E: OK. That would be $y = 0$ and $x = \frac{1}{2}$. So this would be $x - \frac{1}{2}$. This would be the numerator [writing $\frac{x-\frac{1}{2}}{x+3}$].

Working with the third constraint, the function is always nonnegative, Emma thought about an absolute value function.

> The function is always nonnegative, so I'll just put the absolute value sign around that. Then I would have $y = \left|\frac{x-\frac{1}{2}}{x+3}\right|$.

Emma thought beyond the problems demonstrated in the class and textbook. She could create a function that had a domain from -5 to positive infinity. The example problems used in class required students to find the domain by giving the symbolic representation of functions. Emma showed that she was able to think beyond what she was taught. However, she did not know how to relate that function to the function that she had previously constructed even though the interviewer probed her.

E: The function's domain is $[-5,\infty)$ Umm… [long pause]
I: What kind of function will give you a domain from negative 5 to infinity?

E: Let's see. If I have a function from negative 5 to infinity umm… maybe $\sqrt{x+5}$. That would work because x+5 has to be greater than or equal to 0, the x greater than or equal to negative 5. But I have no idea how to combine these two functions together [$\frac{x-\frac{1}{2}}{x+3}$ and $\sqrt{x+5}$].

I: How many arithmetic operations have you learned so far?

E: I believe there are addition, subtraction, multiplication, and division.

I: Can you apply one of those to this function?

E: Let's see… umm... it's undefined at –3, that is still undefined at –3; it has zero at $\frac{1}{2}$; if I put x = $\frac{1}{2}$ that would give y = 0, and it's always nonnegative. So this absolute value would work. If I add this $\sqrt{x+5}$ it would not have zero at $\frac{1}{2}$ any more because I add this. So if I um… [long pause]

I: What are you thinking?

E: I'm thinking that I don't know what to do. May I go for the last constraint?

I: Yes

E: The function contains the point (4,7). So this means when I put in 4 for x, I would get 7 out for y. If I put 4 for x, that would be 3.5 divided by 7, which is .5 that was not what I need. If I multiply that by 14 then I get 7 [checking by using her calculator; .5 x 14 = 7]. Umm... so this function would be $\left|\frac{x-\frac{1}{2}}{x+3}\right| \bullet 14$. With this function, I should get 7 when I put 4 for x.

As some of the other students thought, Emma thought of a radical function when she was asked to identify a function with a domain from –5 to infinity

> Let's see [checking by putting in 4 for x in her last function]. Yeah. Except its domain from –5 to infinity which I don't know how to do, it is undefined at –3, has zero at $\frac{1}{2}$, the function is always nonnegative and contains point (4,7). Umm... I know that $\sqrt{x+5}$ has a domain from –5 to infinity, but I really don't know how to combine this term to the function [constructed earlier]. I have to make the function have a domain from negative 5 to infinity.

Emma always checked to be sure her constructed function met all constraints of the problem and used that technique for looking at her progress on the problem. Yet, she still could not solve for the function domain constraint.

Lindsey

Lindsey's approach to solving problems included extracting and representing information from each problem situation, interpreting a method she used for a solution, and applying the strategies to a new problem. To determine a function she used the vertical line test if a graphical representation was given, and checked inputs and outputs if the numerical representation was given. When a symbolic representation was given, she either checked inputs and outputs or drew a graph and then used the vertical line test depending on her familiarity with the equations. With some assistance, she was able to relate this context to a real world situation and responded to the question correctly. She used her graphing calculator to plot graphs of functions and calculate numerical expressions. At the end of the instruction on functions, Lindsey had developed her algebraic thinking and reasoning processes. She was better able to extract information from a situation, represent that information in different ways, interpret the findings or solutions and apply the findings to a new situation. With her improved knowledge and thinking and reasoning skills she was easily able to choose appropriate strategies to solve the mathematical function problems provided in the textbook as well as the problems on the quizzes and exams.

Lindsey's Solution Strategy and Algebraic Thinking and Reasoning Prior to Instruction

Prior to instruction on functions, Lindsey comfortably solved mathematical problems. She thought the problems were not difficult and she enjoyed solving them. When she solved each problem, she read the problem, described her method for solving the problem, and answered the questions. Most of the time she mentally solved the problem. She seldom wrote anything on the paper. She did not draw any pictures or diagrams to help her solve the problems. Working on the first and third interview problems, Lindsey gathered information from graphical representations and transformed the graphical representations to verbal representations and vice versa. In addition to working with those two representations, Lindsey was able to analyze and interpret the information provided as a graphical representation in Pre-instructional Interview Problem No.1 (see Figure 25).

> From the graph below, the *y*-axis is the amount of popcorn and the *x*-axis is the time. I think George eats popcorn at the steady rate and finishes the popcorn pretty quickly because the amount of popcorn goes to zero when not much time passes like he finishes his popcorn at about half of the time. I mean he finishes his popcorn at about halfway through the movie showing compared to the graphs of other people.

Lindsey correctly described the popcorn graph of other people. For example she correctly talked about the graph of George's sister.

> For George's sister's graph, the amount of popcorn still is on the y-axis and time is on x-axis, but her graph is a straight line. It's not steady decreasing because it's a curvy line. She doesn't eat a lot all the time. She eats it pretty fast sometimes because the graph looks steep and then she slows down a little bit and then eats it pretty fast again and then slow down and she's never finished her popcorn because the graph never hits zero.

Besides interpreting a graphical representation verbally, Lindsey was able to transform a verbal description to a graphical representation. She was asked to match verbal to graphical representations. On each problem, she considered the problem sentence by sentence, converting the information to a potential graph. She tried to exclude a graph that did not relate to the situation (see Figure 15):

> It said we rode the roller coaster steadily to the top. So I look at all these graphs, a couple of these shows they rode steadily [Graph A and Graph G] and it went faster and faster to the other side, so umm… A doesn't work [crossing out A] because it is a horizontal line so this means the roller coaster did not change its speed. First it steadily goes to the top [repeating the first statement again]. B goes steadily but it doesn't go down to the other side. OK. Let's see. The speed

is on the y-axis [writing "speed" on the y-axis of graph A which did not mean that she picked A] and then it's going to get higher and higher when the time has gone because it's going up. So it couldn't be A or C because C's speed is less and less instead of more and more. D's speed gets less and less. Graph E goes back and forth and then starts decreasing. E goes up gradually and then levels off. G stays the same and then goes lower, so it has to be B.

She worked on this problem looking for possible graphs by excluding them with the new information from the next sentence she read. Limited time kept the interviewer from investigating whether she could transform the data between other representations.

Lindsey also worked on another mathematical problem (prior to Instruction Interview Problem No. 2) using her knowledge of percent with the information provided in the problem situation.

L: An employee's gonna get 20% discount on everything and an additional 10% off on a clearance sale. He said that this discount would make a total of 30% on the clearance sale. I don't agree because at first they have to take 20% off and then take 10% off of that. It does not add up to 30%.

I: Will it be more or less than 30%?

L: Let me see [doing a calculation on a piece of paper]. So they just had like x dollars and then took 20% off so I subtract 20% from 100%, so I'm gonna multiply by .80 for getting the amount of money after 20% is dropped and that is $.80x$ then they would take a 10% off of that, so I multiply by .90 which is $.72x$ and then if they take 30% off it will be .70x. So 30% off would be more than 20% off and an additional 10% off of that.

In her solution, Lindsey used a variable x to represent a general sale price. When asked why she picked x for the sale price, Lindsey said that she could use this form to find the amount of money she needed to pay if she knew the sale price. Otherwise, she recognized a need to calculate the solution every time the sale price changed.

As she worked on the Population Growth Problem (Pre-instruction Interview Problem No. 4), she considered two different points of views. However, she could not finish calculating the rate of growth.

L: They are saying that in 1980 Town A has 5000 people and in 1990, Town A has 8000 people [writing 1980➔5000 and 1990➔8000 on a piece of paper]. The difference is 3000 people and then the same thing for town B, start with 6000 people and then in 1990 is 9000 people. So they gain 3000 [writing 1980➔6000 and 1990➔9000]. But umm... it looks like they have increased the same amount of people [pause].

I: So whom do you agree with?

L: Umm... I guess they gain the same amount, so I believe Brian was right.

I: Do you think Linda was wrong?

L: I'm thinking why she thinks Town A has grown more. Umm… maybe it tarts it out with few people [5000] and so these few people have to grow more to get up to 3000 more rather than 6000 grow to get up to 3000 more.

I: So you think Linda's claim was also right?

L: Yes, she could be right.

I: Can you prove that she was right?

L: I don't know how to do it with math. I'm just thinking [pause]. I think Brian was right because they grow at the same amount. But Linda probably looks at it differently but really they gain the same amount. But their rate may be different. But I cannot remember how to find the rate of growth of this population.

Lindsey's Solution Strategies and Algebraic Thinking and Reasoning During Instruction

Lindsey demonstrated her ability to solve mathematical function problems while participating in two instruction interviews. Several data sources, including classroom and recitation observations, homework, and quizzes, were considered to develop a further description of her solution strategies and algebraic thinking and reasoning.

Instructional Interview One

Lindsey spent a few seconds thinking about the information she had from a problem situation before responding to the problem. She typically used a verbal explanation rather than drawing a picture or diagram to describe her thinking and solution strategies. Lindsey used different methods including the vertical line test and checking the numerical inputs and outputs to determine a function. For graphical representation problems, she used the vertical line test to determine if the graph was a function. For a numerical (tabular) representation, she determined whether it was a function by checking numerical inputs and outputs. For the symbolic representation problems, she drew a graph using her graphing calculator to draw graphs of some functions and then used the vertical line test.

L: I will use the vertical line test to determine these graphs.
I: How do you explain whether it is a function?
L: It is a function if no vertical lines intersect the graph more than one point. Otherwise, it is not a function.
A: What does the intersected point mean?
L: It is the value of x that matches with the value of y. One intersected point from a vertical means there is one y value for that x value. And two intersected points mean there are two y values for that x values. In this case it is not a function.

With her understanding and reasoning, Lindsey correctly responded to the first part of the interview problems, which asked her to determine if the given graphs represented functions. Lindsey had difficulty determining whether a graph of a dot and a graph of three dots were functions. She said she had never seen any examples in class like these before. She responded to the problems by stating that they were not functions because there were no lines, and she did not think that the vertical line test could be applied to these problems. With the symbolic representations (equations), her strategy was either drawing a graph and using the vertical line test or recalling factual knowledge.

L: $y = x^2$ is a function. [She did not use her graphing calculator to draw a graph of this function.]
I: How do you know that?
L: I remember that a graph of this equation is a parabola facing upwards and if I think of using the vertical line test with this graph, each [vertical] line crosses the graph at only one point. So it is a function because there is one x for one y. Same as $y = 4$, when you draw a graph, it is a horizontal line. So each vertical line crosses the graph only once. This means each x has only one y. So it is a function.

Lindsey was able to connect her knowledge of mathematics and physics. For example in the Car Problem (Instructional Interview Problem No. 2), she related the speed and the acceleration of the cars, correctly responding that Car B had a higher acceleration than Car A because its graph had a steeper slope than that of Car A.

In this interview, Lindsey connected and applied her understanding to real situations but not without some assistance. As she worked with the Ball Dropped Problem (Instructional Interview Problem No. 4), she demonstrated her understanding that $s(t) = -16t^2 + 145$ was a function because its graph was a parabola opens up and it passes the vertical line test. However, she explained that it did not have an inverse because it was not a one-to-one function (failing the horizontal line test). In a general case, her conceptual idea of a one-to-one function and the horizontal line test were correct. On the other hand, in this situation, she did not consider relating the context of this problem to a real situation. If she did, she would have recognized that this function had an inverse.

L: I will graph this equation to see if it is a function or not [entering $y_1 = -16t^2 + 145$]. It's a parabola facing down so it's a function because it passes the vertical line test. But it's not a one-to-one function. So it has no inverse.
I: Why do you think it is not a one-to-one function?
L: Because it does not pass the horizontal line test. When I draw a horizontal line, it intersects the graph twice. This means there has two x values for one y value. It's not a one-to-one.

The interviewer encouraged her to think about this real situation by asking her to explain the graph in some detail.

I: In this situation would you please tell me what the xy-axis represents? L: The x-axis represents time after the ball dropped from the building and the y-axis represents the ball's height from the ground.
I: When you graph the function, what size of a window did you use?
L: I set Xmin = -10, Xmax = 25, Ymin = -10, and Ymax =150.
I: Is there anything not appropriate for using this window?
L: Umm… [pausing]
I: What is the difference between a ball being thrown up and then dropping to the ground and the ball dropped from the building to the ground?
L: Oh, I see. The ball dropped from the building should go from the top to the ground and the ball that is thrown up should go up and then down. So I should set my window for Xmin and Ymin from 0 [changing the

> calculator window to Xmin = 0 and Ymin = 0 and graphing it again]. It is a half of a parabola facing down. It's a one-to-one function. So it has an inverse.

Lindsey showed her understanding and a symbolic manipulative process of finding a function inverse based on this real situation:

> To find the inverse, I need to switch *x* and *y*. In this case it's *s* and *t*. So I have $t = -16s^2 + 145$, and then solve for *s*. I minus 145 from both sides [writing $t - 145 = -16s^2$] and divide both sides by -16 [writing $\frac{t-145}{-16} = \frac{-16s^2}{-16}$ so and I need to take a square root of both sides to get *s* [writing $\sqrt{\frac{t-145}{-16}} = \sqrt{s^2}$]. I don't think I need a negative value because it involves time and height. So I have $\sqrt{\frac{t-145}{-16}} = s$. This would be an inverse of this function.

Instructional Interview Two

Lindsey demonstrated that she was able to create a quadratic function corresponding to the given graph as she worked on Instructional Interview Problems No 5 and No. 6. She also showed that she was familiar with using a graphing calculator by adjusting the window of her calculator to match that graph.

> Since this parabola faces up so *a* is a positive number. I don't know what it is. Let's say 1 for now. The vertex has been moved to the right and down so *h* is a positive number and *k* is a negative number. Let's say the equation is $y = 1(x-5)^2 - 2$. Let's see how this graph looks [entering the equation into her calculator]. The graph is not quite the same as this picture [the given graph]. Let's see. My calculator window is standard window and the graph looks too small. I will change the window to Xmin $= -5$, Xmax = 15 and Ymin $= -5$, Ymax = 40 [graphing the graph after changing the window]. It looks pretty close to this graph [the given graph].

When asked why she adjusted the window of her calculator instead of adjusting the quadratic equation, Lindsey explained,

> I think all [*a*, *h* and *k*] are in the right position. But it looks a little bit different than I thought because my window differs from the given graph. And after I changed the window my graph looks the same as this one [the given graph].

Working on Instructional Interview Problem No. 6, Lindsey entered the function $f(x) = x^2$ into her calculator to see the width of the parabola when $a = 1$ so she could estimate the *a* variable for the function $g(x)$. Lindsey remembered the effects of *a*, *h* and *k* from her class and from the Interview Problem No. 5 that she needed a negative number between -1 and 0 to make the parabola wider and open downward.

> A negative number makes a parabola face down and the decimal number less than 1 makes it wider.

She also remembered that she needed a negative number for *h* and a positive number for *k* to be the vertex of $g(x)$. She stated that:

> $g(x)$ is wider than $f(x)$ and it faces down, so *a* for $g(x)$ needs to be a negative number and the number for *a* need to be a decimal between -1 and 0 because a = 1 or -1 gives the width of the graph about this [pointing to the given $f(x)$ graph]. The vertex is in the second quadrant, so I need a negative number for *h* and a positive number for *k*.

When working on the Instructional Interview Problem No. 7, Lindsey solved for the year that D could earn more than C. Lindsey was able to solve the problem from their graphs.

> L: To find when D will catch up C, I will use their graphs and look at the intersection point if these graphs [entering two functions: $y_1 = 27000 + 1500x$ and $y_2 = 21000 + 2100x$]. Umm... then for window, I make my Xmax = 100 and my Y is big, very big. Umm... It must be more than one of these [27000 or 21000], I think 40000 would be nice and big enough.
>
> I: Why do you think Y is very big?
>
> L: Because a fixed amount of money starts from 27000 for C and 21000 for D and they earn more every year so the amount of money is getting bigger and bigger [setting her window as

Xmin = -10; Xmax = 100, Ymin = -10 and Ymax = 40000; however, she could not see the graph well enough, so she tried to adjust her window to Xmin = -10; Xmax = 50; Ymin = 10; Ymax = 50000]. This looks much better. Then I just calculate the intercept. And it said $x = 10$ and $y = 42000$ [finding an intercept with the calculator].

I: Why did you change your Xmax to 50?

L: Because I just think that no one can work for 100 years.

I: If I want to set Xmin = 0, does this work?

L: Yes, because no one works less than zero years.

Besides solving the question graphically to find the intersection point, Lindsey also used symbolic manipulation.

I: Is there another way to find out when D is gonna catch up C?

L: Umm... I would set the salary [function] of C and D to equal each other and then solve for *N*.

I: Please show me how you would solve it.

L: I set 27000 + 1500*N* = 21000 + 2100*N* and get a term with *N* to one side and a term that has only number to the other side by subtracting 21000 and 1500*N* from both sides [writing 6000= 600*N*]. Then I divide both sides by 600 [writing $\frac{6000}{600} = \frac{600N}{600}$, therefore, 10 = *N*].

Lindsey understood algebraic simplification. She was asked to determine if $f(x) = \frac{x^2 - 4}{x + 2}$ and $g(x) = x - 2$ in the Equivalent Function Problem (Instructional Interview Problem No. 8) were the same. Initially, she thought they were the same because she simplified $\frac{x^2 - 4}{x + 2} = \frac{(x - 2)(x + 2)}{x + 2} = (x - 2)$. When she was asked to prove that this was true for every x value, she chose several numbers for x and entered those values to $f(x)$ and $g(x)$. She found that these functions were not the same when $x = -2$. Her explanation was:

> $f(x)$ and $g(x)$ are not the same because $f(x)$ has a domain restriction. I mean for the function $f(x)$, x cannot be –2. If $x = -2$, the function is undefined because the denominator equals zero. x values for $g(x)$ can be any real numbers. There is no restriction for the domain of $g(x)$.

Lindsey's Post-Instruction Solution Strategies and Algebraic Thinking and Reasoning

Lindsey worked on the post-instruction interview in the same manner as in previous interviews. She read the problems carefully, stated the information that she understood, and described her ideas about how to solve the problems. Working on the Function Instruction Problem (Post-instruction Interview Problem), she identified the portion of the information that related to a function that she needed to create. She extracted information from each constraint, represented it symbolically, and made a connection with other function concepts she had learned. From the first constraint, the function undefined at –3, Lindsey stated that she needed a function in the form of a fraction [a rational function], and a function was undefined if the denominator equaled zero. She said that "If $x = -3$, then $x + 3 = 0$. Therefore, the denominator of this function is $x+3$." For the second constraint, the function zero was $-\frac{1}{2}$, Lindsey justified her thinking and reasoning as:

> The numerator has to be zero when x = $\frac{1}{2}$ because the zero of a function is the value of x when we set the numerator = 0 [setting an equation $x = \frac{1}{2}$ then subtracting $\frac{1}{2}$ from both sides]. So the numerator is $x - \frac{1}{2}$.

After working on the first two constraints, Lindsey got $\frac{x - \frac{1}{2}}{x+3}$ as a part of the function. For the third constraint, the function was always nonnegative, Lindsey verbalized her thinking and reasoning:

> The function has to be nonnegative. So I can put the absolute value for the whole thing [writing $\left|\frac{x - \frac{1}{2}}{x+3}\right|$] because no matter what x is, I always have positive numbers for y.

Lindsey forgot to exclude –3 from the values of x that made the function nonnegative. However, the interviewer probed her to rethink her solution.

I: Can x be any number to make this function nonnegative?
L: Any numbers. Umm…but –3 because at $x = -3$ the function is undefined.

Lindsey continued working as follows:

L: A function has domain from –5 to positive infinity. What I have now works for these three constraints. I'd better not change what I have but I need something to make the domain of the function start from –5. For some reason, I should have x + 5. Then when I solve for x, I will get $x = -5$. I remember what it is. It is $\sqrt{x+5}$.
I: Why do you think it is $\sqrt{x+5}$?
L: I remember how to find the domain of this type of function. The number under the square root cannot be a negative number. So x + 5 has to be greater than or equal to zero [writing $x+5 \geq 0$] then $x \geq -5$.
I: What will you do with this expression?
L: I will combine it to the first part that I have [$\left|\dfrac{x-\frac{1}{2}}{x+3}\right|$].
I: How can you do that?

At this step, Lindsey began using a guess-and-check method.

[$\left|\dfrac{x-\frac{1}{2}}{x+3}\right|$]. I don't think the addition and subtraction would work because the zero of the function may be changed. I think either multiplication or division. Umm… I will try adding, subtracting, multiplying, or dividing to the first part will work. I will try multiplication [writing $\left|\dfrac{x-\frac{1}{2}}{x+3}\right| \bullet \sqrt{x+5}$]. I am going to check whether a function is undefined at –3, has a zero at $\frac{1}{2}$, and has a domain

> from –5 [entering the function $y = \left| \dfrac{x - \frac{1}{2}}{x+3} \right| \bullet \sqrt{x+5}$ into her calculator, then looking at the tabular representation menu]. Yes! This function works for all these constraints. Now I have only one constraint. If $x = 4$ then y has to be 7. From the table, when $x = 4$, $y = 1.5$ but I need y to be 7. I know that I cannot add or subtract any numbers because that will not work for the first four constraints. I will try to do multiplication again. So 1.5 times x equals 7 [writing $1.5x = 7$, then solving for x] then $x = 4.6666....$ So I will multiply 4.666…to the whole thing [writing $4.666\ldots \bullet \left| \dfrac{x - \frac{1}{2}}{x+3} \right| \bullet \sqrt{x+5}$]. So my final function is $y = 4.666\ldots \bullet \left| \dfrac{x - \frac{1}{2}}{x+3} \right| \bullet \sqrt{x+5}$.

Lindsey was another student who was able to provide a function that satisfied all the five constraints.

Kyle

Kyle worked on the mathematical function problems used in this study in the same manner through all the problems. He extracted and represented information from each problem situation, interpreted a solution, and applied the appropriate solution strategies to a new problem. With instruction, Kyle improved his thinking and reasoning processes. At the end of the course, Kyle had developed his algebraic thinking and reasoning ability. He was able to extract information from a situation, represent that information in multiple ways, interpret the findings or solutions, and apply the findings to new situations. He improved his thinking and reasoning skills and was able to solve mathematical function problems in the textbook for this class as well as the problems on the quizzes and exams.

Kyle's Solution Strategies and Algebraic Thinking and Reasoning Prior to Instruction

Prior to the instruction on functions, Kyle described his solution strategies and his thinking and reasoning when working with mathematics problems. He read each problem to obtain information and decided on a method to solve the problem. He typically did not use tools such as scratch paper to help him solve the problems; however, he sometimes used a calculator. He did not draw any pictures or diagrams. He gathered information from graphical representations (i.e. the dependent and independent variables represented on the Cartesian coordinate system) and transformed a graph to a verbal representation, and vice versa. In addition to working with those two representations, Kyle was able to analyze and interpret the information provided as a graph. For example, he transformed a verbal representation to a graph (see Figure 25):

> Look at George's graph. It looks like he steadily eats his popcorn. He eats faster than any other people because his time shows that he spends shorter time to finish his popcorn than his sister, his mom, and his dad. Actually, I don't think his mom eats all of her popcorn because the graph shows that her amount of popcorn is not zero. For his sister, she eats at different rate; sometimes she eats fast, sometimes she eats slowly. Her graph varies. She doesn't eat the whole popcorn in her bowl. She has some popcorn left over because her graph shows that she has still has some amount of popcorn left. George's dad, he starts eating at the constant rate because of the straight line. Umm… it looks like he stops eating right there or takes a break, or maybe the intermission of the movie [laughing] and then he starts eating at the constant rate again until he finishes his popcorn.

Besides working with those representations, Kyle worked on an algebraic problem related to his background knowledge of percent calculation (Pre-Instruction Interview Problem No. 2.). Looking at the information provided, he initially thought that it was true to say, "having a 20% clearance sale and getting an additional 10% off makes a total discount of 30%." After he checked himself, he found that his initial thinking was incorrect.

> It looks like it's true but we can check it. Let's say he buys a $60 sweater and there is 20% off. Then 60 times .20 [using a calculator], which means $12 and then 10% off because she is an employee; $48 times .10 it means she has

> another $4.80 off so she has $16.80 for her total discount. And 30% off of $60 should be 60 times .30. Oh! No. They are not the same. It's $18 for 30% off at one time.

Kyle made a table for the information he found for the Pre-instruction Interview No. 4. He said that the table helped him see the information easier than the words in the problem (see Table 7).

Table 6

Kyle's table representing information for Pre-Instruction Interview Problem No. 4

	Town A	Town B
1980	5000	6000
1990	8000	9000

Kyle described his thinking and reasoning:

> Looking at the table, I can definitely say that umm… they are both increasing by 3000 people. I guess Brian was right because both towns are increasing by 3000 people.

In some cases, Kyle focused his thinking on one specific point, ignoring all others. Getting one answer for the problem, he did not think whether there was a different answer for a different circumstance.

> I would say that Linda was absolutely wrong because both towns were increasing by the same amount of people but she said Town A had grown more. So she is definitely wrong and Brian was definitely right.

Thinking of the obvious feature of information (the differences between 5000 and 8000 and between 6000 and 8000), Kyle had concluded incorrectly that only Brian was correct. His conclusion about Linda stopped him from further consideration of the problem. His response suggested that he did not relate the difference between the actual amount and the percent change.

Kyle's Solution Strategies and Algebraic Thinking and Reasoning During Instruction

While working on the problems during the instruction interviews, Kyle demonstrated his ability to solve mathematical function problems. Information from classroom and recitation observations, homework assignments, and quizzes were used for clarifying the description of the strategies and algebraic thinking and reasoning he used as solving the problems.

Instructional Interview One

Kyle approached the first problem assigned in the first interview by reading the problem at a normal speed. He used different methods, relying on the function representations given in the problems for determining a function. He used the vertical line test when he determined whether a graphical representation represented a function. He checked numerical inputs and outputs when a numerical (tabular) representation was given. He drew a graph using his graphing calculator and then used the vertical line test when a symbolic representation was provided.

> I'm gonna run the vertical line test on each graph and if it doesn't intersect the graph more than one point, it means there is no more than one output for each input so it is a function. Therefore, graph (a) represents a function. For graph (b) if I draw vertical lines, the lines intersect the graph two points so this means there are two outputs for one input so it is not a function.

When he worked on problem (c), a graph of a point, he stated that he did not understand what the point represented. However, he continued using the vertical line test method. He correctly identified the graph of a point as a function. Working with the graph of three points, Kyle changed his method to checking inputs and outputs because the graph did not have a connection line between points. He said, "This is a weird graph." He had never seen this type of graph before, but with respect to the input / output approach, he said it was a function because there was one output for each input. Working on a numerical representation, he stated that the table represented a function.

K: I think it is a function because there is no repeating value for x.
I: What do you mean by repeating value?
K: I mean each x didn't have different values for *f(x)* or y. There is no repeated x value for different y values. That is what I mean. For example, if $x = 2; y = 333.8$ and $x = 2; y = 377.3$, this x is repeated with different y outputs. In this case it is not a function.

Kyle drew a graph using a graphing calculator and then used the vertical line test to determine if a symbolic form represented a function. He easily and quickly graphed the equation if it was given in the form of y in terms of x (i.e. $y = x^2$ or $y = 4$). He seemed to have difficulty when an equation was not written in the y equals form.

K: I do not know how to graph this ($x^2 + y^2 = 1$) in my calculator. Umm... let me think [pause]. Yeah. I remember. I need to solve for y.
I: How can you solve for y?
K: I subtract x^2 from both sides and then take a square root. And I remember that the number under a square root cannot be a negative number. [writing $y^2 = -x^2 + 1$, $y = \pm\sqrt{-x^2 + 1}$].
I: Why do you have two values for y?
K: I remember from my class that taking a square root of a positive number, like $x^2 = 4$, I will get two values, ± 2. One is positive and the other is negative. So I put $y_1 = \sqrt{-x^2 + 1}$ and $y_2 = -\sqrt{-x^2 + 1}$ into my calculator. [looking at the graph in the calculator]. It's a circle, so it's not a function because if I draw a vertical line, it intersects the graph two points.

After seeing the graph, he realized that the symbolic manipulation that he worked through was reasonable to conclude that this equation was not a function. He stated that,

> Actually, I don't think I need to graph it because it obviously shows that $y = \pm\sqrt{-x^2 + 1}$. There are two y values, one negative and one positive for one x input. So it is not a function.

Kyle responded to the second question of the Instruction Interview Problem (Car Problem). He stated that Car A and Car B had the same speed at t = 1 hour because they were at the same point. However, his response was not what the question requested. Therefore, the interviewer probed by asking him to describe the information he got from the graph. He described:

> From the graph, Car A and Car B start traveling from time = 0. Car A goes faster than Car B. But it seems like Car B is speeding up faster than Car A because the graph of Car B is steeper than the graph of Car A. At t = 1 hour they have the same speed. After 1 hour, Car B goes faster and faster while Car A goes at almost the same speed.

Even though Kyle described the graph correctly, he incorrectly responded to the question. He stated that Car A and Car B were at the same position. He explained that they were at the same point, which should mean they were at the same position. He correctly answered the second part of this problem that Car A and Car B had the same speed because their graphs intersected at the same point. He neither realized that he gave the same response to two different questions nor recognized that he changed the information represented on the y-axis. He sometimes considered the y-axis as speed, but at other times, he considered it as the cars' positions.

Kyle connected mathematics and physics concepts providing the correct answer to the third question. He related the speed and the acceleration of the cars and correctly identified that Car B had a greater acceleration than Car A because its graph had a steeper slope than that of Car A.

Kyle transformed the information represented by a graph to a symbolic form. He constructed three linear equations corresponding to each line in the graph of Instruction Interview Problem No.3a (see Figure 18). He showed his procedure for finding the equations. Working on the third part of this problem, Kyle incorrectly concluded that the first and the third lines had the same equation; in other words, they were the same line because they had the same slope. When he was asked to show that they were the same, he used the $y = mx + b$ formula.

> Umm... $y = mx + b$. The slope is $\frac{1}{2}$, so $y = \frac{1}{2}x + b$. I don't have b for the y-intercept. I will draw this line further to find the y-intercept [extending the most

right line until it intersected the y-axis]. The y-intercept is zero. So the equation is $y = \frac{1}{2}x + 0 = \frac{1}{2}x$. It is not the same as the first line.

Kyle did not use the point-slope form to find a symbolic representation of the third line. When asked whether he could find a symbolic form of a linear function if two points were given, he said he could not remember how to do it. He was not sure that he had learned about this in class. He was probably absent from the class when this concept was taught. Looking at his homework, the interviewer found that he had not done this exercise yet. There was no evidence showing that he was able to find a linear function using the point-slope form.

While working on the application problem (Instructional Interview Problem No. 4), Kyle was not able to apply the problem context to a real situation. As he worked with this problem, he drew the graph of the ball dropped function (using a graphing calculator). Seeing a parabola, he stated that it was a function because it passed the vertical line test. However, he did not connect the problem situation to a real world situation. Thus, he concluded that this function had no inverse because it was not a one-to-one function (did not pass the horizontal line test).

Instructional Interview Two

Kyle responded to the two problems related to the graphical representation (Instructional Interview Problem No 5 and No. 6) without first using a description of variables a, h, and k in a quadratic function [$y = a(x-h)^2 + k$]. When asked, he did describe the variables.

K: A parabola is opened upwards and it's not too wide or narrow so I 'm gonna try $a = 2$. Well, I think that the distance from the y-axis to the vertex is about 2 and from the x-axis to the vertex is about 1. I'm gonna graph this function $y = 2(x-2)^2 - 1$.

I: Would you please explain how do you get those numbers and what each of those numbers represented?

K: I got 2 for a because the parabola is opened up. If it's upside down, then a will be a negative number. And the distance from the y-axis to the

> vertex is represented by h, which has two directions. If it goes to the right, h is positive. If it goes to the left, h is negative. And the distance from the x-axis to the vertex is represented by k. It's positive if it goes up and negative if it goes down. So I got 2 for a, 2 for h, and –1 for k.

After explaining how he got the numbers for each variable, he graphed the function using his calculator.

> K: That's not the same as that graph (the given graph in the problem). Umm... I may need to change a because it makes the graph wider.
>
> I: What number will you select to make the graph wider?
>
> K: I'm not sure. I can't remember but I can try [putting $y_1 = 3(x-2)^2 - 1$ and $y_2 = (x-2)^2 - 1$]. Now I know that the smaller a makes the graph wider. $a = 1$ looks good to me. Let me try a smaller number than that. Let's say $a = .5$ [entering $y = .5(x-2)^2 - 1$]. The graph crosses the y-axis a little too high. Let's try one more time; a = .75 [entering $y = .75(x-2)^2 - 1$]. I think this is the best.

Kyle connected the information of the graph of $f(x)$ provided in the Instructional Interview Problem No. 6 to the graph that he needed to find the symbolic representation. From the reference graph [$f(x) = x^2$], he said:

> This is $y = x^2$ which means $a = 1$ and for the graph of *g(x)*, it's much wider so a must be a number smaller than 1, and it's upside down so a is a negative number. Let's try a = –.5. Umm... I think that $h = -5$ and $k = 1$ because the graph moves to the left more than moves up [entering $y = -.5(x-(-5))^2 + 1$] in his calculator]. Umm... It should be wider and the graph should cross the y-axis at about –5. So I will use a smaller number. Let's try half of it. So a = –.25 [entering $y = -.25(x-(-5))^2 + 1$]. That looks perfect.

Kyle did not remember the effect of size for a variable a in the symbolic representation of a quadratic function. He used the guess and check method to find his solution.

When working on the Salary Problem (Instructional Interview Problem No. 7), Kyle formed his response with symbolic representations. He initially thought about the effect of the length of the date of contract.

> Will B ever earn more per year than A? Umm... It depends on how many years from the date of contract. I mean if they both have the same amount of years from the date of contract, I can tell that A will earn more because 2500 multiplies by the number of years plus the fixed salary, which is \$30000, so A will earn more than B which has the same fixed amount of money plus 2800 multiplies by the same number of years. If the number of years from the date of contract of B is higher than that of A, then he probably earns more.

Kyle also focused on a symbolic representation for the last part of the problem.

K: Well, D's salary is 21000+ 2100*N* and C's is 27000+1500*N* which means that D makes 600 dollars more per year of the bonus. (He referred to the term with the variable *N* "bonus"). C makes \$6000 more for the fixed salary (the constant term) per year but they make \$600 less time the number of years. So it will take sometime for D to catch up to C.
I: Can you figure out how long it will take for D to catch up to C?
K: Oh yeah.
I: How long?
K: OK. They earn 6000 more for this (constant term) and 600 less for a bonus. So 6000 is divided by 600 [entering $\frac{6000}{600}$ in his calculator]. That would take them 10 years.

To find out whether two functions representing different forms of a symbolic representation were equivalent, Kyle did not consider a domain for each function. He used a factoring method to check whether $f(x) = \frac{x^2 - 4}{x + 2}$ and $g(x) = x - 2$ were the same. However, he initially thought that $\frac{0}{0}$ is 0 was true.

> Yes, they are the same. They factor $\frac{x^2 - 4}{x + 2} = \frac{(x+2)(x-2)}{(x+2)}$ and then they cancel like terms getting x–2. Then they are equal to each other.

When asked whether he could show that they were the same, he correctly entered some positive integers in both functions; however, he did not use any negative values.

I: Will you go with all real numbers?
K: It will take me years.

I: Let me pick one number for you
K: What?
I: –2
K: $\frac{(-2)^2-2}{-2+2}=\frac{0}{0}=0$, and –2–2 = –4. Wow! They are not the same.

Although he incorrectly said $\frac{0}{0}=0$, he knew that there was a number that made *f(x)* and *g(x)* different. When challenged to think about zero divided zero, he recognized this mistake.

I: Are you sure that $\frac{0}{0}=0$?
K: Oh, zero can't divide it. I can't find *f(x)* when x = –2 but I can find *g(x)* when x = –2. So they are not the same.

He challenged his own belief that these two functions were different for all negative numbers. After trying a few values, he was convinced that only a value of –2 was a problem.

K: Wait this doesn't work for all negative numbers or just –2. Let me try $x = 3$, $\frac{(-3)^2-4}{-3+2}=\frac{9-4}{-1}=\frac{5}{-1}=-5$ and –3 –2 = –5. It works. So it's just –2
I: Why do you think –2 don't work?
K: Because zero can't be a divisor so if x = –2, x + 2 will be zero, and $f(x)=\frac{x^2-4}{x+2}$ will be undefined.

Kyle's Post-Instruction Solution Strategies and Algebraic Thinking and Reasoning

By the end of the instruction, Kyle's conceptual knowledge of functions had improved. He worked on the problems in the same manner as he did in the previous interviews. He identified the portion of the information that related relevant information to achieve the solution. In the Function Construction Problem (Post-Instructional Interview Problem), he gathered information from each constraint, represented it symbolically, and made a connection with other function concepts. His first thought about an undefined function was in the form of a fraction (a rational function).

> Because it's undefined at –3, so the function is a fraction and $x + 3$ is at the bottom. When I use –3 for x, it makes the denominator be 0 and then that is undefined at $x = -3$ [writing $\frac{}{x+3}$].

He continued working on the other constraints.

> The function has a zero at $\frac{1}{2}$. Umm.... (pause). This means that when $x = \frac{1}{2}$, $y = 0$. It's the same as x-intercept. And when we look for x-intercept we set the top of a fraction equal to zero. I mean numerator. If I set the numerator equal to zero and then solve for x, I will get a zero of a function or the x-intercept.

His process was to work backwards by setting $x = \frac{1}{2}$ and making the equation equal to zero (subtracting $\frac{1}{2}$ from both sides). He set the expression for the numerator as $x - \frac{1}{2}$ [writing $\frac{x-\frac{1}{2}}{x+3}$].

K: Next a function is always nonnegative. I think it needs an absolute value sign for this.
I: What does nonnegative mean?
K: Positive, isn't it?
I: How about 0? Is it negative or positive?
K: Neither.
I: Do you think nonnegative mean positive?
K: I think positive or zero. But absolute value still works because $|0| = 0$ and absolute value of negative numbers is positive. So $\left|\frac{x-\frac{1}{2}}{x+3}\right|$ works for these three constraints.

At that moment, Kyle considered only an absolute value function that gave a nonnegative value. He was unable to find a part of a function that made the function domain start at –5 and continue to positive infinity. After thinking about the problem for a few minutes, he gave up and continued working on the last constraint.

> A function contains (4,7). Let's see, what happens if I put $x = 4$? 4 minus $\frac{1}{2}$ would be $3\frac{1}{2}$ or 3.5. And 4 + 3 is 7. So $\frac{3.5}{7} = \frac{1}{2}$. It's not 7. I need the result 7. If I multiply this $\frac{1}{2}$ by 2, I get 1. Then I multiply this by 7 again; I get 7. What am I doing? Umm… From $\left|\frac{x-\frac{1}{2}}{x+3}\right|$, I multiply it by 2 and 7 [writing $\left|\frac{x-\frac{1}{2}}{x+3}\right| \bullet 2 \bullet 7$]. I think this function passes all the constraints except that one [a function domain that is –5 to positive infinity].

Even though the interviewer encouraged and probed him to rethink the fourth constraint, Kyle was unable to identify a function that satisfied that constraint.

Analysis of Student Profiles: Solution Strategies
Algebraic Thinking and Reasoning

The purpose of this section is to analyze information from multiple data sources including the classroom observations, interviews, students' homework, quizzes, and filednotes. The analysis is based on solution methods and reasoning in order to answer the second research question. This section presents the strategies and techniques that the students used for obtaining the solution to the problems. The problems used for investigating students' solution strategies and algebraic thinking and reasoning while they worked with function problems consisted of 10 problems related to the main concepts of functions such as function definition, multiple representation, and application problems.

The results from working on multiple representations problems revealed that these five students had no difficulty determining graphical representations identifying functions. They also used the same strategies of applying the vertical line test to all graphs. The students reasoned, "[$y = x^2$] ... is a function because it passes the vertical line test" and "it is a function because there is no [vertical] line that crosses the graph more than once" The reason for not being a function was "... because the [vertical] lines cross the graph more than once." and "... because it does not pass the vertical line test." The students who incorrectly responded to the graphs of a point and three points did not apply either the vertical line test or the input/output checking approaches to those problems.

Similarly, all the students used the same strategy to determine whether the data represented in a table defined a function. They provided the reason related to their responses: "It is a function because there is only one output for each input" and "it is a function because each input has exactly one output." Some students identified the number of times that each domain was listed, "... because there is no x (input) that has been used more than once" or "... because there is no repeated x (domain) values."

Relations in symbolic form were the most difficult for the students. They were unable to immediately give their responses and reasoning. They needed more information to consider whether the relations were functions, either graphing and applying the vertical line test or entering numbers in the equation to check the corresponding inputs and outputs. After either graphing each relation or entering some numbers for x in the equation, they provided the same reason as they worked on the graphical and tabular representations. The results from the multiple representation problems indicated that the students had no difficulty determining functions given in different forms of relations if they were familiar with those relations. This result suggested that some students did not to think beyond what had been taught in class.

All the students used several approaches and reasoning strategies to obtain the problem solutions. Some used similar strategies. Since the participants were asked to read the problems aloud before solving each problem, they all started attempting each problem by reading the problem situation. The difference was how they read the problem. Amy read problems fast, whereas Emma read slowly, Lindsey and Kyle read at a normal speed, and Ross read faster than Emma but slower than normal speed. Ross read each problem at least two times. All others read it once except when they were not sure that they correctly interpreted the information from the problems. At the second time of reading, they read only the sentences that confused them. When rereading the problem, Amy read it slower than her first time. All participants made sure that they got all information right by restating the relevant data sentence by sentence using their own words. If there was something from their interpretation of what they read wrong or unclear, they were probed by the interviewer.

All the students' methods used to determine functions depended on the form of the function. When relations were present graphically, they preferred to use the vertical line test. If no vertical lines crossed the graphs more than once, they said those graphs defined functions. When the data were presented in tables, they looked for correspondence of the inputs and outputs. They stated that data in the table identified a function because "every input has only one output."

However, when the symbolic form was given, these five students used different strategies depending on their familiarity with the relationship. Amy initially entered numbers in the symbolic form of a quadratic function checking the corresponding inputs and outputs, whereas she approached a linear constant function by graphing and using the vertical line test. This process was easy for her. All other students (Emma, Kyle, Lindsey, and Ross) determined whether the given symbolic representations defined functions by graphing and using the vertical line test. During the interview, all the students indicated their preference for the graphical representation in determining functions.

The students had more success working on non-application problems (problems given in a symbolic form). Not all of the students were able to relate a problem context to a real situation the first time they worked with a real situation problem. Initially, Ross and Kyle worked on a real-world problem (the Ball Dropped Problem) in the same manner as they worked on the algebraic symbolic representation. Without connecting the problem situation to the real-world context, they provided an incorrect answer. Amy, Emma, and Lindsey were able to make a connection, and they were successful working on this problem. Through instruction, Ross and Kyle developed their sense of working with the real-world situation problems. Two weeks after working on the Ball Dropped Problem, they achieved correct solutions when they worked on other real-world problems including the Salary problems and problems assigned for their homework.

Lindsey and Emma indicated some flexibility when finding two given functions that had the same values. They demonstrated that they could find these values using a calculator or by solving an algebraic equation by hand. Other students did not show this flexibility. For example, Ross only solved problems using a calculator. All the students obtained the correct answer through their strategies.

Basically, the common approach used for solving each problem was making a connection between a problem situation and examples that they had experienced. The students used methods similar to those used by the lecture instructor and the GTA. For example, in finding zeros of a function, if a function was simple (such as a linear function or a quadratic function that were easy for them to factor), they used symbolic manipulation to obtain the solution. However, if a problem was more complicated, they used a graphing calculator to find the solution. All five students rarely checked their answers after they finished working on each problem unless they were challenged. The researcher noticed that both the lecture instructors and the GTA did not emphasize checking problem solutions in class. All the students attempted the problems as much as they could. If one strategy did not work, they tried another until they got a solution. All of the students were successful solving the problems assigned in the interview sessions. They got the correct solutions for those problems over 90% of the time after they solved them using different approaches.

The use of graphing calculators supported the students' confidence in doing mathematics. All of the students in the class, including the five interview students, used their graphing calculators as common tools to produce graphs and evaluate numerical expressions. The five interviewed students also used their graphing calculators for alternative solution methods or a guide to develop a strategy for solving the problems. For example, Amy used her graphing calculator to approach the Salary Problem after she was not successful using symbolic manipulation to solve the equations.

CHAPTER V

DISCUSSION AND CONCLUSION

Current research on learning and teaching algebra has shifted the direction from looking for students' misconceptions to focusing on students' strategies of problem solving, and noting their understanding and their thinking and reasoning processes. This study investigated college algebra students' understanding, solution strategies, and algebraic thinking and reasoning used as they solved mathematical function problems. To address the first research question about college students' understanding of functions, a questionnaire was administered to 24 volunteer college algebra students and interviews of five selected students were conducted to expand the description of students' knowledge and understanding. For the second research question, concerning solution strategies and algebraic reasoning, the five students were interviewed and observed as they approached and solved the problems. In addition, their thinking and reasoning for using particular approaches were investigated. Classroom observations were performed in order to describe the effect of instruction on their understanding of functions and thinking and reasoning.

This chapter provides the conclusions and discussion of the main findings in two areas: (1) students' understanding of functions and (2) their solution strategies and thinking and reasoning used while solving problems. Conclusions in response to the two research questions were drawn from data collected through students' interviews and responses to the questionnaire. In addition to the discussion and conclusions, the limitations of the study, implications, and recommendations for further research are addressed.

College Students' Understanding of Functions

The emphasis in the mathematics curriculum has shifted from memorization to understanding concepts and relationships within mathematics and connections between mathematics and other subjects American Mathematical Association of Two-Year

Colleges [AMATYC], 1995; National Council of Teachers of Mathematics [NCTM], 2000). This suggestion has also been applied to the college algebra curriculum, where the major topics include function concepts considered as the most important subject matter in both lower-and higher-level undergraduate collegiate mathematics courses (Selden & Selden, 1992, Romberg, Fennema, &, Carpenter 1993). Function concepts are primary ideas in the study of many other mathematical areas, in particular in the study of calculus concepts, such as limits, derivatives, and integration. The concepts of functions are also used outside the mathematics field and in real-life situations, such as bank accounting and the stock market. Because of the importance, the concepts of functions are the specific mathematical topics investigated in this study. Particularly, this study focuses on college level students' understanding of functions, their problem-solving strategies, and their algebraic thinking and reasoning processes.

How students connect their understanding of functions and how they use these ideas to solve problems is important in determining how to help them grow in their understanding. Hiebert and Carpenter (1992) classified mathematical knowledge and understanding in two categories: procedural and conceptual. Procedural understanding was defined as a sequence of actions or operations. Conceptual understanding was defined as the network connection of knowledge structures. In solving $x+5=9$, where students typically leave the unknown on the left-hand side and swap the number to the right-hand side, and change the sign, it is difficult to judge whether they have procedural or conceptual understanding. On the other hand, students with a conceptual understanding solve this problem in different ways depending on the relation they were using. For instance, they solved the problem by adding the additive inverse of 5 to both sides, subtracting 5 from both sides, or substituting numerical values for the unknown until creating a true equation. In addition, they are able to discuss a rationale for their actions.

Considering both procedural and conceptual understandings, this study suggested that the students improved their procedural understanding of functions to a more conceptual understanding. The students were able to describe their ideas of functions including the definition of functions, multiple representations of functions, the usefulness of function concepts, and the application of function understanding to real-world situations. The majority of the students demonstrated on their understanding of function to the correspondence level, viewing a function as a relationship between to sets of objects such that every element of the first set was matched with only one element of the second set (Schwingendorf et al., 1992). Twenty-three of the 24 volunteer students in this study realized that the same function could be represented in several forms. The most common forms that the students in this study considered were graphical, numerical (tabular), and symbolic.

The research related to function concepts in secondary and college levels has identified students' misconceptions, difficulties, and lack of success in understanding functions. For example, the students in Vinner and Dreyfus's (1989) and Selden and Selden's (1992) study indicated that students had a limited understanding of functions. The study conducted by Tall and Baker (1992), which assessed students' ability to identify functions given graphically and symbolically indicated that the students perceived graphs to represent functions if they were smooth (either a straight line or curve), not piecewise, and not constant. Most of the students in Tall and Baker's study did not consider the graph of a horizontal line to be a function. Slavit (1994) suggested that students had misconceptions that functions were not constant, they were linear, and they were continuous and smooth. Vinner (1983) and Selden and Selden (1992) illustrated college students' misconceptions about functions; they did not consider constant functions as functions because constant functions were one-to-one and the symbolic representation of constant functions did not contain an x-variable.

In this study, however, these college level students developed a broader advanced understanding of functions through instruction. The interviews with the five selected students indicated that the misconceptions found in the previous studies were not found in this study. The students in this study correctly described and worked with both piecewise-defined functions. Contrary to other studies (Sfard, 1992; Slvavit, 1994), the students in this study expanded their interpretation of functions to include one-to-one and many-to-one. In fact the students seemed to have a good understanding of one-to-one functions and inverse functions.

The interviews of the five students indicated that the symbolic and graphical forms were used more often than any other representations while these students solved problems. When interpreting the information in the problems, they were able to change from one representation to another, such as changing a verbal representation to a graphical or to a symbolic representation. When solving word or application problems, they changed the words represented in the problem situations to different types of representations, such as, tabular, graphical, or symbolical.

The results of this study may have reflected the impact of NCTM (1989, 2000) on algebra instruction. NCTM recommended a shift in the direction from memorizing to understanding concepts. The information from the interviews of the five students and the 24 students' responses to the questionnaire indicated that the students had a better understanding of functions than those students in prior studies. Students' misconceptions and difficulties found in early 1990s studies did not appear in this study. Thus, perhaps with the NCTM (1989), National Research Council [NCR] (1989) and AMATYC (1995) calls for new directions in teaching and learning mathematics, particular at high school levels, students have arrived in college with a better understanding of functions. Generally, most of the students in this study were able to identify the difference between equations and functions, were able to interpret mathematical situations in different forms, and were able to apply their understanding of those concepts to new situations including real-world situations.

Solution Strategies and Algebraic Thinking and Reasoning

For this study, algebraic thinking and reasoning refers to the ability to use mathematical symbols and analyze situations by extracting and representing information in words, diagrams, tables, graphs, and equations. Algebraic reasoning also includes interpreting and applying mathematical solution methods to new, related situations. Generally, the solution strategies in this study referred to the methods or techniques used for obtaining the problem solutions.

The results of this study indicated that students were adept at using a variety of strategies for identifying functions depending on the given forms. For example, they applied the vertical line test to determine whether a graphical representation identified a function. However, their automatic use of these strategies appeared more procedural than conceptual.

This study revealed that the students did not often solve functional problems in multiple ways unless they were challenged to do so. The information from classroom observations clarified that the students were neither instructed nor encouraged to use a variety of approaches solving the problems. Most of the time, the instructors demonstrated only one approach to solve each problem. The instructional strategies used in both lecture and recitation classes were traditional – the students listened to the instructors' explanations or description of the content and watched how their instructors solved the problems.

In order to support various methods, mathematics instructors need to consider using different ways to solve mathematical problems. Mathematics instructors need to help their students see that there is no one right method to solve mathematics problems (NCTM, 2000). Mathematics instructors should use and demonstrate various efficient methods for solving the problems. The classroom observations revealed that the instructors used traditional methods. They described all information given in each problem situation and how to get the solution. They did not focus on identifying more efficient methods or improving these methods. As a result the students did not experience a more rigorous process in solving the problems. Rather, they observed the instructors as they guided the thinking and the solution to the problems. The NCTM (2000) recommended that instructors' actions in their classroom need to encourage students to think, question, and solve problems, as well as discuss their ideas, strategies, and solutions. This questioning would include the search for more efficient solution strategies.

In this study the instructor determined the solution path and focused on completing particular methods for solving the problems. Furthermore, after obtaining the problem solutions, the students who were interviewed rarely checked their solutions unless they were specifically challenged to do so. The classroom observations provided a clearly noticeable reason; the instructors did not emphasize this process. In addition the students were not guided to use a variety of approaches to solve the problems. The NCTM (2000) recommended that mathematical instruction "should enable students to apply and adapt a variety of appropriate strategies to solve problems" (p. 334). Students should be encouraged to use their knowledge of strategies for identifying alternative approaches. "If the first approach to a problem fails, they can consider a second or third. If those approaches fail, they know how to reconsider the problem, break it down, and look at it from different perspectives" (NCTM, 2000, p. 334). Follow the NCTM's recommendation supports that students need to understand the problem better and make progress towards problem solutions.

The students enrolled in this college algebra course were required to use graphing calculators; however, no specific types of calculators were recommended. The students used a variety of graphing calculators. They were allowed to use any tools and techniques for finding solutions to the problems. Both instructors and students often used their calculators to identify graphical representations of functions and to find a solutions. The five interviewed students illustrated that they were able to use graphing calculators fluently. They easily graphed, looked at equations or tables, and solved the problems. For example, to find the solution of the problem involving two functions given in a symbolic form, the students used a graphical representation to identify of two functions had the same value; they searched for the intersection points of the two graphs. These results confirmed the study conducted by Even (1998), who investigated prospective secondary mathematics teachers' flexibility for working with multiple representations of functions. The results of Even's study also indicated that the students were able to use graphical representation to find problem solutions.

Earlier studies, such as Goldenberg's (1988), suggested that students lacked skills using graphing calculators, and needed instruction on the calculators in order to use them as learning tools. However, this study, particularly the information from the interviews, indicated that the students showed that they were able to use graphing calculator tools easily. They were skillful with these devices and were able to use them to identify solution as well as to talk about their solution. During the interview session, Amy, Emma, Lindsey, Kyle, and Ross showed that they were confident using their calculators for plotting graphs of functions and adjusting for an appropriate graphics window. The use of graphing calculators aided them in understanding the multiple representations of functions.

The results of this study supported AMATYC (1995) and NCTM (2000) recommendations for increasing students' understanding of functions through the use of technology. Using technology in mathematics classroom enables students to investigate many mathematics examples so that they are able to generate and test their conjectures and understanding. AMATYC recommended graphing calculators as tools that provide students with examples of functions in multiple representations. Graphing calculators can support students in the investigation of a large number of functions in multiple forms: graphical, tabular, and symbolic. With graphing calculators, students are able to investigate and make connections between properties of functions, and these connections help students develop their understanding of functions and multiple representations (Janiver, 1987, NCTM, 1989). "With calculators and computers students can examine more examples or representational forms than feasible by hand, so they can make and explore conjectures easily" (NCTM, 2000, p. 25). Computers and graphing calculators can generate precise values and graphs of functions. For example, using graphing calculators, students can easily obtain a graph of a function such as $f(x) = 3x^2 + 5x + 2$. Heid et al. (2002) described the use of technology, a computer algebra system [CAS] to support students' understanding of functions. The use of this technology allowed them to connect their understanding of functions through the use of multiple representations and to reason algebraically among various representations. In this study, graphing calculators were used as one of the problem solution tools for evaluating expressions, checking solutions, and transforming one functional representation to another. This study suggests that progress has been made in integrating graphing calculators in algebra instruction.

Limitations of the Study

As with any research study, the study had some limitations. The selection of participants, the sample size, the data collection instruments and procedures, and the limitation of the researcher all in some sense limited the results of this study.

One obvious limitation to the study was the selection of participants for the study. The individual college students in this study volunteered to participate in the study. They were not randomly selected. The students who volunteered might have had different mathematical backgrounds, experiences, and ability from those college students who did not volunteer.

This study is an in-depth description of a particular setting of a single college algebra class at a university located in Northwest area. The results of this study may not apply to another setting, different group of college students, or university.

Another limitation of the study involved the potential that the data collection instruments acted as a treatment. As mentioned previously, the Problem-Solving Interview Problems may have been influenced by the researcher's beliefs and biases, and may have been biased towards specific fields. For example, the Car Problem favored students with a physical science background. A variety of problem situations for the interview may have been biased toward specific student backgrounds. The interviews may have provided additional instruction for the five students. Thus, the results of this study may not have represented a typical understanding of functions of college algebra students who had experienced only lecture and recitation instruction.

The researcher also was a limitation in this study. The researcher's presence in the lecture classroom for the entire College Algebra course observations may have had a limited effect on student learning in the lecture class. While the students may not have noticed the researcher in the class, her presence might have had an effect on the instructor and the GTA's instructional strategies. Since they knew the purposes of this study, the way that they selected examples and questions posed to the students could have been affected. In the recitation classroom, which contained 30 to 35 students, the researcher's presence might have had a more noticeable effect on students' learning.

Finally, a limitation was directly related to the researcher's biases. The researcher collected the data and analyzed the data herself. Her background, beliefs, and experiences could have led to unintended biases in the development of the data collection instrument, the data collection, and the data analysis. The use of various techniques of data collection, examining and documenting all relevant data helped to minimize the effect of the researcher and strengthened the results of the data analysis. Another effort to minimize this limitation was the use of five mathematics educators to assist in the development of the questions for the questionnaire and for the interviews. In addition, multiple techniques were used in this study (observations, interviews, fieldnotes, and researcher's journal) to provide various ways of gathering the information.

Implications for the College Level Algebra Curriculum and Instruction

The results of this study have implications specifically for college level instruction on the concept of functions. This study highlighted college algebra students' understanding of functions, their solution strategies, and algebraic thinking and reasoning while they solved mathematical function problems. This information is useful for developing the curriculum and instruction on understanding of functions, using effective solution strategies, and developing algebraic reasoning.

The students relied on their calculators because they were required to have them for this course and they were allowed to use them at any time. Conversely, examining some students' quizzes and homework assignment, it was noticeable that, sometimes, they showed the correct problem solutions, but their procedures were incorrect. When asked how they got these solutions, they said they used the graphing calculator. Students can learn with appropriate uses of technologies (Dunham and Dick, 1994, Rojano, 1996). However, technologies should not be used as a replacement for basic understanding. All technologies, including graphing calculators, need to be used widely and responsibly to enrich students' learning of mathematics (NCTM, 2000). Specifically, graphing calculators have been recommended to assist students in their development of an understanding of functions (AMATYC, 1995, NCTM, 1989, 2000); still instruction and assessment must continue to emphasize symbolic manipulation skills, as well as use of the technologies.

The students in this study were successful in identifying functions in multiple representations. Instructors need to consider introducing function concepts using these multiple representations. As students progress in their understanding of these concepts, they need to be guided to connect the different representation both within mathematics and outside mathematics (Eisenberg, 1992). The results of this study indicated that students had little difficulty making connections between multiple representations within mathematical areas. Eisenberg (1992) stated, "Having the skills to visualize the graphs of functions is one important component in having a well-developed sense for functions, but many students are reluctant to (or cannot) connect the graph of a function with its analytic description" (p. 174). The incorporation of graphing tools in the curriculum does support students' visualization of functions because the graphing tools help them understand more abstract views of functions. Amit (1991) stated that high failure rates in beginning calculus could be avoided if students were supported in internalizing the visual connotation of concepts. Students should be instructed to select, apply, and translate among different mathematical representations to solve problems because "different representations support different ways of thinking about and manipulating mathematical objects. An object can be better understood when viewed through multiple lenses" (NCTM, 2000, p. 360). As a result, the use of multiple representations in different subject areas as well as in real-world situations needs to be emphasized. Mathematics teachers and educators need to consider demonstrating the importance of the use of multiple representations in mathematics, science, and other areas as well as in real-world situations.

The use of real-world problems was recommended to help students make more sense of abstract ideas (AMATYC, 1995, NCTM, 1989, 2000). In this study, the instructor provided only a few examples and homework assignments that used real-world situations. Providing students more experiences in using function concepts in real-world situations needs to be considered in order to make functional concepts less abstract and to support the development of connections of mathematical ideas to areas outside mathematics.

Recommendations for Future Studies

This study highlighted college algebra students' understanding of functions, the solution strategies they used, and their algebraic thinking and reasoning while they worked on mathematical function problems. The results from this study suggested related areas for further research.

First, the use of volunteers from different class sections and majors may produce prevailing tendencies that were not apparent in this group of participants. Additionally, a larger sample group or samples from a smaller lecture class may provide more widespread understanding of college students' understanding of functions. Testing a larger sample and using a different group of students could confirm the findings of this study. Studies with design similar to this study, but with larger sample size and/or different group of population, could be conducted to confirm the findings that college algebra students' understanding of functions has developed since 1990s. Since this study excluded students who enrolled College Algebra Excel Program, perhaps more research using college algebra students from the Excel Program may provide more information related to students' understanding of functions, solution strategies, and algebraic thinking and reasoning. The students in the Excel Program are encouraged to solve mathematical problems related to the concepts learned in their college algebra class by themselves within the groups of 3-5 students. The students in the class do not receive additional instruction, but they are challenged to demonstrate their algebraic thinking and reasoning. The results of such an investigation can provide a more understanding about helping college level students' algebraic thinking and reasoning.

Second, the *Principles and Standards for School Mathematics* (NCTM, 2000) recommended algebraic thinking and reasoning be included in the mathematics curriculum at all grade levels. In addition, *Crossroads in Mathematics: Standard for Introductory College Mathematics before Calculus* (AMATYC, 1995) suggested teaching in the college level should enable students to expand their reasoning skills. However, the results of this study indicated that the students at this college level did not have a significant development in their algebraic thinking and reasoning. One obvious effect on their development of algebraic thinking and reasoning skills was from the instructional strategies. Consequently, more research on the instructional strategies supporting students' development of algebraic thinking and reasoning needs to be conducted.

Third, this study was conducted from a single college algebra curriculum class. The class was instructed guiding by the *Study Guide* constructed by two mathematics professors and a textbook selected by the representatives of the mathematics department of the university where this study was conducted. More research needs to investigate students' solution strategies and algebraic thinking and reasoning from different college algebra curriculum in order to confirm or disconfirm the results of this study.

Fourth, the *Principles and Standards for School Mathematics* (NCTM, 2000) recommended mathematics instruction should enable students to organize and strengthen their mathematical thinking and reasoning through communication with their peers, teachers, and others. Communication provides a means for students to express their ideas and thinking (NCTM, 2000). The observation of this study showed that students did have many chances to exchange their mathematical ideas with others. Therefore, future research needs to effective strategies that provide and support students in communicating their mathematical thinking coherently and clearly to others. A small class of 25-30 students may be a good place to start by investigating specific instructional strategies that support communication skills. College Algebra Excel classes may be a good resource population for more research because they are small classes containing 15-25 students and because they focus on communicating their thinking.

Finally, the *Principles and Standards for School Mathematics* (NCTM, 2000) and *Crossroads in Mathematics: Standard for Introductory College Mathematics before Calculus* (AMATYC, 1995) recommended the connections of mathematical concepts within the mathematical discipline and other disciplines. Besides connecting the concepts of functions to some real-world situations, this study showed that students related the function concepts to other subject areas. Future studies should consider how students connect function concepts to other mathematical disciplines, to other subject areas, and to various real-world situations.

REFERENCES

Acheson, K. E., & Gall, M. D. (1997). Techniques in the critical supervision of teacher. New York, NY: Longman.

Algebra Working Group of the National Council of Teachers of Mathematics. (1994). A framework for constructing a vision of algebra: Working draft. Reston, VA: National Council of Teachers of Mathematics.

American Mathematical Association of Two-Year Colleges (1995). Crossroads in mathematics : standards for introductory college mathematics before calculus. Memphis, TN.

Amit, M. (1991). Applications of R-Rules as exhibited in calculus problem solving. Proceedings of the 15th International conference for the Psychology Mathematics Education, Assisi, Italy.

Barnes, M. (1988). Understanding the function concept: Some results of interviews with secondary and tertiary students, Research on Mathematics and Education in Australia, 24-33.

Becker, B.H. (1991). The concept of function: Misconceptions and remediation at the collegiate level. Unpublished doctoral dissertation, Illinois State University.

Bergeron, J. C., & Herscovics, N. (1982). Levels in the understanding of the function concept. In G. Van Barnveld & H. Karbbendam (Eds.), Conference on functions (pp. 1-46). Enschede, The Netherlands: Foundation for Curriculum Development.

Blubaugh, W. & Emmons, K. (1999), Graphing for all students, The Mathematics Teacher, 92(4), 323-326, 332-334).

Blume, G.W. & Heckman, D.S. (2000) Algebra and functions. In E. Silver & P. Kenney (Eds.) Results from the seventh mathematics assessment (pp. 269-306). Reston, VA: National Council of Teachers of Mathematics.

Breidenbach, D., Duninsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function, Educational Studies in Mathematics, 23 (3), 247-285.

Brown, C.A., Carpenter, T.P., Kouba, V.L., Lindquist, M.M., Silver, E.A., & Swafford, J.O. (1988). Secondary school results for the fourth NAEP mathematics assessment:Algebra, geometry, mathematical methods and attitudes. The Mathematics Teacher, 81(5), 337-347.

Carlson, M. P. (1997). Obstacles for college algebra students in understanding functions: What do high-performing students really know? The AMATYC Review, 19(1), 48-59.

Carlson, M. (1998). A cross-section investigation of the development of the function concept. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.) Research in Collegiate Mathematics Education 3 (pp. 114-126). Washington, DC. Conference Board of the Mathematical Science.

Carpenter, T. P., Coburn, T.G., Reys, R.D., & Wilson, J.W. (1975). Results and implications of the NAEP Mathematics Assessment: Secondary School. The Mathematics Teacher, 68(6), 453-470.

Carpenter, T.P., Corbit, M.K., Kepner, H.S., Lindquist, M.M., & Reys, R.E. (1980). Results of the NAEP Mathematics Assessment: Secondary School. The Mathematics Teacher, 73(5), 329-338.

Carpenter, T.P., Lindquist, M.M., Mathews, W., & Silver, E.A. (1983). Results of the NAEP Mathematics Assessment: Secondary School. The Mathematics Teacher, 76 (9), 652-659.

Cooney, T. J., & Wilson, M. R. (1993). Teachers' thinking about functions: Historical and research perspectives. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.) Integrating research of the graphical representation of functions. (pp. 131-158). Hillsdale, NJ: Lawrence Erlbaum.

DeMarois, P., & McGowen, M. (1996). Understanding of function notation by college students in a reform development algebra curriculum. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), Proceedings of the Eighteenth Annual Meeting North American Cahpter of the International Group for the Psychology of Mathematics Education (pp. 183-188). Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Driscoll, M. (1999). Fostering algebraic thinking: A guide for teacher, grade 6-10. Portsmouth, NH: Heinemann.

Dubinsky, E. & Harel, G. (1992). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), The Concept of Function: Aspects of Epistemology and Pedagogy (pp. 85-106). Washington, DC: Mathematics Association of America.

Dwyer, D. & Gruenwald, M. (2000). College Algebra and Trigonometry. Pacific Grove, CA: Brooks/Cole.

Dunham, P.H. & Dick, T.P. (1994). Research on graphing calculator. The Mathematics Teacher, 87, 440-445.

Eisenberg, T. (1992). On the development of a sense for function. In E. Dubinsky & G. Harel (Eds.), The Concept of Function: Aspects of Epistemology and Pedagogy (pp. 153-174). Washington, DC: Mathematics Association of America.

Ericsson, K. A., & Simon, H. A. (1993). Protocol analysis: verbal reports as data. Cambridge, MS: MIT Press.

Even, R. (1990). Subject-matter knowledge for teaching and the case of function. International Journal of Mathematics Education in Science and Technology, 14(3), 293-305.

Even, R. (1992). The inverse function: Prospective teachers' use of 'undoing'. International Journal of Mathematical Education in Science and Technology, 23(4), 557-562.

Even, R. (1998). Factors involved in linking representations of functions. Journal of Mathematical Behavior, 17(1), 105-121.

Fein, B., Lee, J. (2000. College Algebra Study Guide. Department of Mathematics, Oregon State University.

Gall, M. D., Brog, W.R., & Gall, J. P. (1996). Educational Research : An Introduction. White Plains, NY: Longman.

Goldenberg, P (1988). Mathematics, metaphors, and human factor: Mathematical, technical, and pedagogical challenges in the graphical representation of functions. Journal of Mathematical Behavior, 7(20), 135-174.

Haimes (1996). The implementary of a "Function" approach to introductory algebra: A case study of teacher cognitions, teacher action, and the intended curriculum. Journal for Research in Mathematics Education, 27(5), 582-602.

Hauger, G.S. (1995). Rate of change knowledge in high school and college students. Paper presented at the Annual Meeting of the American Educational Research Association. San Francisco, CA.

Heid, M.K., Hollebrands, K.F., & Iseri, L.W. (2002). Reasoning and justification, with examples from technology environments, Mathematics Teacher, 95(3), 210-216.

Herbert, K., & Brown, R. (1997). Patterns as tools for algebraic reasoning. Teaching Children Mathematics, 3(6), 340-344.

Herscovics, N. (1989). Cognitive obstacles in the learning of algebra. In S. Wanger & C. Kieran (Eds.), Research issues in the learning and teaching of algebra (pp. 60-86).Reston, VA: National Council of Teachers of Mathematics & Lawrence Erlbaum.

Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D.A. Grouws (Ed.), Handbook of Research on Mathematics Teaching and Learning (pp. 65-100). NY: Macmillan Publishing Company.

House, P. (1999). Mathematical reasoning: In the eye of the beholder. In L.V. Stiff & F.R. Curcio (Eds.), Developing mathematical reasoning in grade K-12: 1999 Yearbook. National Council of Teachers of Mathematics, Reston, VA.

Janvier, C. (1987). Representation and understanding: The notion of function as an example. In C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics (pp. 67-70). Hillsdale, NJ: Lawrence Erlbaum.

Kaput, J. J. (1995). Long-term algebra reform: Democratizing access to big ideas. In C.B. Lacampagne, W. Blair, & J. Kaput (Eds.). The algebra initiative colloquium, Vol. 1 (pp. 33-52). Washington, DC: U.S. Department of Education.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wanger & C. Kieran (Eds.), Research issues in the learning and teaching of algebra (pp. 33-56). Reston, VA: National Council of Teachers of Mathematics & Lawrence Erlbaum.

Kieran, C. (1992). The learning and teaching of school algebra. In D.A. Growns (Ed.), Handbook of research on mathematics teaching and learning (pp. 390-419). New York, NY: Simon & Schuster Macmillan.

Kieran, C., & Wagner, S. (1989). The research agenda conference on algebra: Background and issues. In S. Wagner & C. Kieran (Eds.), Research issues in the learning and teaching of algebra (pp. 1-10). Reston, VA: National Council of Teachers of Mathematics & Lawrence Erlbaum.

Krathwohl, D. R. (1997). Methods of educational and social science research (2nd Ed.). New York, NY: Longman.

Langrall, C.W., & Swafford, J.O. (1997). Grade six students' use of equations to describe and represent problem situations. Paper presented at the American Educational Research Association, Chicago, IL.

Lacanpagne, C. B., Blair, W., & Kaput, J. (1995). The algebra initiative colloquium (Vol. 1). Washington, DC: Department of Educational Research and Improvement. (ERIC Document Reproduction Service No. ED 385 436)

Larson, R., Hostetler, R., & Hodgkins, A. (2000). College Algebra: Concepts and Models. New York: Houghton Mifflin Company.

Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. Review of Educational Research, 60(1), 1-64.

Lochhead, J. (2001). Thinkback: A user's guide to minding the mind. New Jersey: Lawrence Erlbaum Associate.

Markovits, Z., Eylon, B. & Bruckheimer, M. (1986). Functions today and yesterday. For the Learning of Mathematics, 5(3), 2-7.

Markovits, Z., Eylon, B. & Bruckheimer, M. (1988). Difficulties students have with the function concept. In A.F. Coxford & A.P. Shulte (Eds.), The ideas of algebra, K-12 (pp. 43-60). National Council of Teachers of Mathematics, Reston, VA: Author.

McDermott, L., Rosenquist, M. & van Zee, E. (1987). Student difficulties in connecting graphs and physics: Examples from kinematics. American Journal of Physics, 55(6), 505-513.

McGrone, V. (1985). Improving Ninth Graders' Algebra Achievement--The Key to Future Success in Mathematics and Science--Through an Intensive Workshop Intervention Experience: Final report. (ERIC Document Reproduction Service No. ED259 899).

McKnight, C.C., Travers, K.J., Crosswhite, F.J. & Swafford, J. O. (1985). Eight-grade mathematics in U.S. schools: A report from the Second International Mathematics Study. Arithmetic Teacher, 32(8), 20-26.

Merriam, S. B. (1998). Qualitative research and case study applications in education: Revised and expanded from case study research in education. San Francisco, CA: Jossey-Bass Inc.

Moschkovic, J. N. (1999). Students' use of the x-intercept as an instance of a transitional conception. Educational Studies in Mathematics, 37(1), 167-197.

National Council of Teacher of Mathematics. (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, VA: Author

National Council of Teachers of Mathematics. (1991). Professional Standards for Teaching Mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). Assessment Standards for School Mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). Principles and Standards for School Mathematics. Reston, VA: Author.

Nation Research Council. (1989). Everybody Counts: A report to Nation on the Future Mathematics Education. Washington DC: National Academy Press.

Nation Research Council (1998). The Nature and Role of Algebra in the K-14 Curriculum. Proceedings of a National Symposium (May 27-28, 1997). Washington, DC: National Academic Press.

Nodding, N. (1990). Constructivism in mathematics education. In R. B. Davis, C. A. Mayer, & N. Noddings (Eds.), Constructivist views on the teaching and learning of mathematics (pp.7-18). National Council of Teachers of Mathematics, Reston, VA.

Ozgun-Koca, S.A. (1998). Students' use of representations in mathematics education. Paper presented at the Annual Meeting of the North America Chapters of the International Group for the Psychology of Mathematics Education. Raleigh, NC.

Phillips, E. (1995). Issues arounding algebra. In C.B. Lacampagne, W. Blair, & J. Kaput (Eds.), The algebra initiative colloquium, Vol. 2 (pp. 69-81). Washington, DC:U.S. Department of Education.

Pirie, S., Kieren, T. (1994). Beyond metaphor: Formalizing in mathematical understanding within constructivist environment. For the Learning of Mathematics 14(1), 39-43.

Rho, K. (2000). A case study on the change of Chinese students' function concept in virtual environment. Paper Present at the Annual Meeting of the American Educational Research Association. New Orleans, LA, April 2000. (ERIC Document Reproduction Service No. ED 440869)

Rojano, T. (1996). Developing algebraic aspects of problem solving within a spreadsheet environment. In N. Bednarz, C. Kieran, & L. Lee (Eds.), Approach to Algebra: Perspective for Research and Teaching. Boston, Kluwer Academic Publisher.

Romberg, T.A., Fennema, E., & Carpenter, T.P. (1993). Toward a common research perspective. In T.A. Romberg , E. Fennema, and T.P. Carpenter (Eds.), Integrating Research on the Graphical Representation of Functions (pp. 1-10). Hillsdale, NJ: Lawrence Erlbaum Associates.

Russell, S.J. (1999). Mathematical reasoning in the elementary grades. In L.V. Stiff & F.R. Curcio (Eds.), Developing mathematical reasoning in grade K-12: 1999 Yearbook. National Council of Teachers of Mathematics, Reston, VA.

Schwingendorf, K., Hawks, J. & Beineke, J. (1992) Horizontal and vertical growth of the students' conception of function. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 133-149). Washington, DC: Mathematical Association of America.

Selden, A., & Selden, J. (1992). Research perspective on conceptions of functions: Summary and overview. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 1-16). Washington, DC: Mathematical Association of America.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. Proceeding of the 13th International Conference for the Psychology of Mathematics Education, Paris, University of Paris.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 59-84). Washington, DC: Mathematical Association of America.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspect of epistemology and pedagogy (pp. 25-58). Washington, DC: Mathematical Association of America.

Skemp, R.R. (1987). The psychology of learning mathematics. Hillsdale, NJ: Lawrence Erlbaum Associates.

Slavit, D. (1994). The effect of graphing calculators on students' conceptions of function. Paper present at the annual meeting of the American Educational Research Association, New Orleans, LA.

Spradley, J. (1980). Participant observation. New York: Holt, Rinehast & Winston

Steffe, L. (1991). Operations that generate quantity. Learning and Individual Difference, 3, 61-86.

Tall, D. (1992). The transition to Advanced mathematical thinking: Functions, limits, infinity, and proof. In D.A. Growns (Ed.), Handbook of research on mathematics teaching and learning (pp. 495-51119). New York, NY: Simon & Schuster Macmillan.

Tall, D., & Bakar, M. (1992). Students mental prototypes for functions and graphs, International Journal of Mathematical Education in Science and Technology, 23(1),39-50.

Vinner, S. (1983). Concept definition, concept image and the notation of function. International Journal of Mathematics Education in Science and Technology 14(3), 293-305.

Vinner, S. (1992). The function concept as a prototype for problems in mathematics learning. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 195-213). Washington, DC: Mathematical Association of America.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. Journal for Research in Mathematics Education, 20, 356-366.

Walton, K.D. (1988). Examining functions in mathematics and science using computer interfacing. School Science and Mathematics, 88(5), 604-609.

Zaslavsky, O. (1997). Conceptual obstacles in the learning of quadratic functions. Focus on Learning Problems in Mathematics, 19(1), 20- 44.

Zbiek, R.M. (1998). How might technology enhance algebraic reasoning? In The Nature and Role of Algebra in the K-14 Curriculum (pp. 35-36). Washington, DC: National Academic Press.

APPENDICES

APPENDIX A

LETTER TO STUDENTS

Dear Students,

My name is Lakana Nilklad and I am a doctoral student in the Department of Science and Mathematics Education at Oregon State University. My doctoral research involves college students' algebraic thinking and reasoning. More specifically, the purpose of this study is to explore college students' knowledge and understanding of functions by focusing on their solution strategies and their thinking and reasoning processes as they solve function problems.

I am asking for your help in my investigation. If you volunteer to become involved in this study, you will be asked to complete a questionnaire providing information about your background of mathematics courses taken and knowledge of function. Nine participants will be asked to participate in two task-based interviews: pre-instructional interview and post-instructional interview. These two interviews will be video- and audio-taped recorded.

The questionnaire will last no longer than 10 minutes and each interview will last no longer than 60 minutes. In order to complete the pre- and post-instructional interview, the nine participants will be asked to solve five function word problems using a thinking aloud protocol. After solving the problems the participants will be asked questions related to the strategies they used to solve the problems.

All information gathered from the questionnaires, interviews, and classroom observations will be kept strictly confidential. In particular, responses given in the questionnaires and the interviews will not be shared with your instructor or graduate teaching assistant. The data will be coded to protect the participants; pseudonyms will be used so that participants will not be identifiable in any publication of the results of the study. All the information including audio- and video-tapes will be kept in a secure place. These tapes will be destroyed after the research project has been completed. None of this information will be made will affect your grade. The results from the study will provide information to support recommendations to mathematics educators and teachers in the redesign of instructional strategies and activities to enhance students' understanding of algebraic concepts, algebraic thinking and reasoning in general, and understanding of function concepts in a particular case.

Your participation in this project would be greatly appreciated. If you are interested in participating in this study, please fill out the form attached to this letter and return it to me at the end of this class. However, participating in this study, you must be 18 or older. Your prompt reply is greatly appreciated.

Participation is voluntary. There will be no compensation given to subjects participating in this study. You may refuse to participate or discontinue participation at any time without any penalty. Questions concerning this research should be directed to either:

Lakana Nilklad at (541) 753-6668 or nilkladl@ucs.orst.edu
Prof. Margaret Niess at: (541) 737-1818 or niessm@ucs.orst.edu

Questions concerning your rights as a human subject should be directed to the IRB Coordinator, OSU Research Office, (541) 737-3437 or irb@orst.edu

Sincerely,

Lakana Nilklad
Researcher

Dr. Margaret Niess
Major Professor

By signing this form below, I attest to the following:

1. I agree to participate in the research study in the following ways (check one):
   ___ providing the researcher with biographical information about myself and completing Function Knowledge Questionnaire.
   ___ providing the researcher with biographical information about myself, completing Function Knowledge Questionnaire, <u>and</u> participating in the interview part of the study.

2. I understand that participating in this study, I must be 18 or older and my participation in this study is voluntary, and that I may withdraw at any time with no penalty.

3. During the interviews, I have the right to refuse to answer any question(s).

4. The researcher has explained the purpose and procedures of this research study, and I have been given an opportunity to obtain answers to my questions.

5. I understand that the researcher will keep my responses confidential and will destroy all records upon the completion of the research.

6. I understand that participation will not affect my grade in the class.

7. I understand that the audio and videotapes will be transcribed by either the student researcher or by a paid transcriptionist.

8. I understand that the results of the function knowledge questionnaire and the video- and audio-taped interviews will not be shared with my instructor, nor affect my College Algebra course grade.

9. I understand that there will be no compensation, no foreseeable risks or direct benefits given to subjects participating in this study.

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

____
Name (printed)

______________________________ ____________________
Signature date

Biographical Information Questionnaire

Age ______________ Gender ___________ Major ____________

I am a: __ Freshman __ Sophomore __ Junior __ Senior __ Graduate
My intended major is ____

Please list all previous and current mathematics courses that you have taken at high school and / or college levels. If possible, please include the grade, which you earned for that course:

Level (High School / College)	Course	Grade	Year Taken

Please respond to the statements in the table below (check all apply)
I am taking the mathematics courses because __ I like mathematics.
__ the course is required for my major.
__ the course is an elective requirement.
__ the course is a Baccalaureate Core requirement.

Other mathematics course(s) that I plan to take is (are) 1. ________________________
2. ________________________
3. ________________________
4. ________________________
5. ________________________

Contact Information

Name: ___________________________ Telephone number: ________________

E-mail address: ____

The best way to contact me is by ___ phone ___ e-mail

The best times to contact me are

Day	Time

APPENDIX B

INSTRUCTOR AND GRADUATE TEACHING ASSISTANCE CONSENT FORM

By signing this form below, I attest to the following:

1. I agree to participate in the research study. The purpose of the research is to examine the students' thinking and reasoning processes, and solution strategies college students use working on function problems and effects of instructional strategies on their ability to think and reason while they work on the problems. My participation will allow the researcher to observe instruction in my class for entire course.

2. I understand that my participation in this study is voluntary, and that I may withdraw participation at any time with no penalty.

3. I understand that I may refuse to answer any question(s).

4. The researcher has explained the purpose and procedures of this research study, and I have been given an opportunity to receive answers to my questions.

5. I understand that the audiotape will be transcribed by either the student researcher or by a paid transcriptionist.

6. I understand that the researcher will keep my responses and the students' responses confidential and will destroy all records at the completion of the research. The only person who will have access to this information will be the researcher and the major professor. No names will be used in any data summaries.

7. I understand that I will not receive any compensation, foreseeable risks or direct benefits for my participation in this study.

Questions concerning this research should be directed to either:

Lakana Nillkad at (541) 753-6668 or nilkladl@ucs.orst.edu
Prof. Margaret Niess at: (541) 737-1818 or niessm@ucs.orst.edu

Questions concerning your rights as a human subject should be directed to the IRB Coordinator, OSU Research Office, (541) 737-3437

My signature below indicates that I have read and that I understand the procedures described above and give my informed and voluntary consent to participate in this study. I understand that I will receive a signed copy of this consent form.

____
Name (printed)

______________________________ ____________________
Signature date

APPENDIX C
STUDENT INTERVIEW PROTOCOL
(PRE-INSTRUCTIOAL INTERVIEW)

I. Introduction: Put a student [S] at ease with questions:

1. Tell me about the mathematics courses you had in high school.
2. What other college mathematics courses have you taken including any courses you are taking this term?
3. Have you studied algebra and / or functions previously?
4. Are you confident in your ability to succeed in this class?
   Would you consider this course as hard, not too difficult, or easy?
   Do you like or dislike taking this course?
5. Are there any questions you would like to ask me before we begin?

II. Explain the procedures to Students

In this experiment, I am interested in what you say to yourself while you solve these problems. In order to do this, I am asking you to *think aloud* while you work on the following problems. What I mean by "think aloud" is that I want you to say aloud *everything* that you say to yourself silently while solving the problem. Just act as if you are at home or your private place doing homework, speaking to yourself. I would like you to read each problem aloud and think aloud *constantly* from the time you are presented the problem until you have given your final response to the question. If you are silent for any length of time, I will remind you to keep thinking aloud. You do not need to plan what you will say or try to explain to me what you are saying. Just act as if you are in the room speaking to yourself. Do you have any questions?

III. Explain confidentiality

I want to assure you that the video- and audio-taped recording of this interview will be destroyed at the end of the study. Your identity will be kept confidential and your name will never be used. Also, comments made in this interview will not be shared with your instructor.

APPENDIX D

STUDENT INTERVIEW PROTOCOL

(DURING INSTRUCTIONAL INTERVIEW)

I. Introduction: Questions before Working on Problems:

The purpose of this component of the interview is to find out more about the aspects of this class you found beneficial in learning function concepts.

1. Describe learning activities you have done in order to meet the class requirements:
   a. For this class, you are expected to read each subsection of a unit prior to the lecture, review and work on suggested problems after the class. In addition, the instructor has a timeline as to when homework is to be completed and handed in. Do you to follow these suggestions or do you have a different process for learning the material?
   b. Do you use you textbook, lecture notes, a study guide, or handouts for studying for this class? If so, which and how?
   c. Have you used any outside resources including tutors, textbooks differ from the one required for this course, or some other materials related to functions for each section of function unit?

2. When you do not understand a concept in class, do you ask for help? If yes, whom do you ask? Describe particular times and the questions you asked.

3. Which of these activities or material will be helpful in understanding function concepts? Explain why each is or is not helpful.
   - Lecture
   - Lecture notes
   - Study guide
   - Recitation
   - Textbook
   - Supplemental material
   - Solutions manual
   - Working with others
   - Homework problems

II. Activities during instructional interview

At this time, we will do some activities similar to those we did before the study of functions. I will give you five function problems. You need to think aloud while you working on these problems. I will give you a practice problem to remind you of the think aloud process; first read the question as it is written, then proceed to solve the problem as if it was a homework problem, and think aloud. When you are finished, we can discuss you think-aloud process. After our discussion, you will continue working on the five problems in the same manner.

III. Explain confidentiality

Again, I want to assure you that the video- and audio-taped recording of this interview will be destroyed at the end of the study. Your identity will be kept confidential and your name will never be used. Also, comments made in this interview will not be shared with your teacher.

APPENDIX E
STUDENT INTERVIEW PROTOCOL
(POST-INSTRUCTIONAL INTERVIEW)

I. Introduction: Questions before Working on Problems:

The purpose of this component of the interview is to find out more about the aspects of this class you found beneficial in learning function concepts.

II. Activities during instructional interview

At this time, we will do some activities similar to those we did before the study of functions. I will give you five function problems. You need to think aloud while you working on these problems. I will give you a practice problem to remind you of the think aloud process; first read the question as it is written, then proceed to solve the problem as if it was a homework problem, and think aloud. When you are finished, we can discuss you think-aloud process. After our discussion, you will continue working on the five problems in the same manner.

Again, I want to assure you that the video- and audio-taped recording of this interview will be destroyed at the end of the study. Your identity will be kept confidential and your name will never be used. Also, comments made in this interview will not be shared with your teacher.

III. Questions after Working on Problems

1. How do you perceive your knowledge and understanding of functions?
2. Describe what you learned about function concepts in this class? Can you tell me any specifics?
3. What helped you learn about these function concepts the most?
4. The problems you just completed used everyday events. Did you find the context of the problem helpful in solving the problem? How?
5. Describe your understanding of functions. Does your thought about functions differ from the beginning of the course? What is the difference?

APPENDIX F

FUNCTION UNDERSTANDING QUESTIONNAIRE

1. In your opinion, what does a mathematical function mean to you? Describe your understanding of mathematical functions. You may use diagrams, picture, or examples to clarify your thoughts.

   [Additional space was provided]

2. Can mathematical functions be represented in multiple ways? If so, give examples of each type of mathematical function representation.

   [Additional space was provided]

3. In your opinion, how are mathematical functions useful in thinking about or doing mathematics and in particular algebra? Give some examples. You may use diagrams and picture to clarify your examples.

   [Additional space was provided]

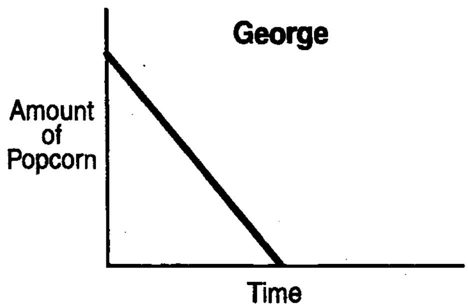
4. In your opinion, how are mathematical functions useful in thinking about real world situations? Give some specific examples. You may use diagrams and picture to clarify your examples.

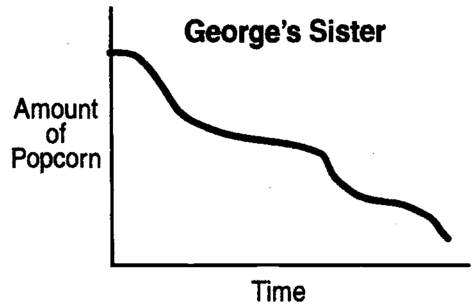
   [Additional space was provided]

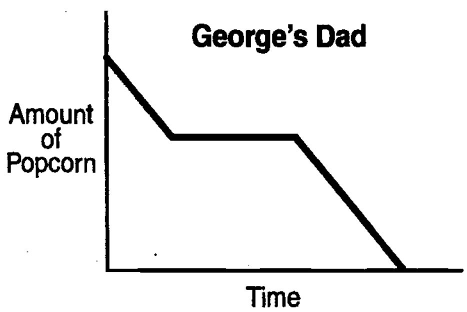
APPENDIX G

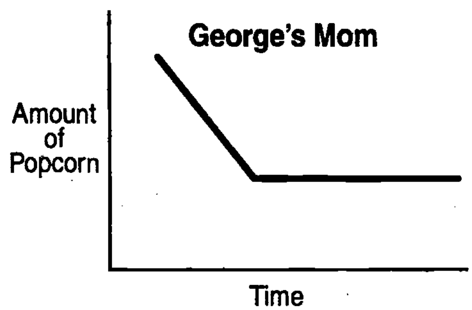
PRE-INSTRUCTIONAL INTERVIEW PROBLEMS

1. George and his family were watching a movie and eating popcorn. Each family member had a bowl with the same amount of popcorn. The graphs below all show the amount of popcorn remaining in the person's bowl over a period of time. From each graph describe what may have happened and provide your reasons (Blubaugh and Emmons, 1999).









2. An applicant for a position at the clothing store is told by interview that employees receive a 10 % discount on all purchases, including sale items. The manager gives and example the following scenario "Suppose that we're having a 20% clearance sale. Then we would get an additional 10 % off, for a total discount of 30%" Do you agree with the manager? Why or why not? Explain you reasoning (Retrived September, 12, 2002 from the World Wide Web: http://www.mathcount.org/Problems/problems.html).

3. Match each of the following seven scenarios with the most appropriate graph given. As you look at each graph from left to right, remember that time is advancing (Blubaugh & Emmons, 1999).

____ 1. We rode the roller coaster steadily to the top, then went faster and faster as we went down the other side. The speed of the roller coaster is dependent variable of the graph, that is, the variable on the vertical axis.

____ 2. The kettle heats before the corn begins to pop. The corn starts to pop and continues popping until almost all the corn has popped. The amount of *unpopped* corn in the kettle is the dependent variable.

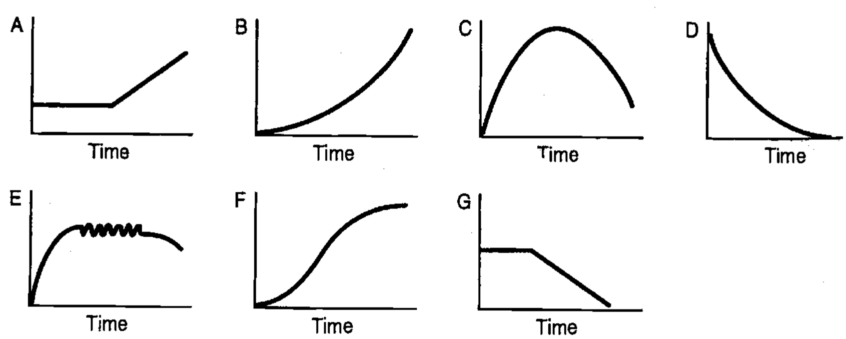
____ 3. A balloon was blown up in class and then let go. It flew around the room. The amount of air in the balloon is the dependent variable.

____ 4. At the beginning of spring, the grass grew slowly and I seldom had to mow the lawn. By midsummer it was really growing, so I mowed twice a week. In fall, I only mow once in a while. The number of lawn mowing to dates is the dependent variable.

____ 5. I turned the oven on. When it was hot, I put in the cake. The cake baked for about thirty minutes. I turned the oven off and removed the cake. The oven temperature is the dependent variable.

____ 6. We bought a pair of rabbits last year. They have had several liters, and we have so many rabbits that the pens are full. If more are born, we will have to give some away or find room for the new ones. The number of rabbits is the dependent variable.

____ 7. I Put water in the ice-cube tray and placed it in the freezer. The temperature of the water in the ice-cube tray is the dependent variable.



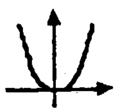
4. In 1980, the populations of Town A and Town B were 5,000 and 6,000, respectively. In 1990 populations of Town A and Town B were 8,000 and 9,000, respectively.
   Brian claims that from 1980 to 1990 the populations of the two towns grew by the same amount. Whereas, Linda claims that from 1980 to 1990 the population of Town A had grown more.

   Who was right? Use mathematics to explain how both might have justified their claim (Retrieved and developed September 12, 2002 from World Wide Web: http://www.mathcounts.org/Problems/problems.html).

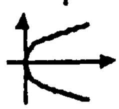
APPENDIX H

INSTRUCTIONAL INTERVIEW PROBLEMS

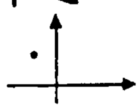
1. (a) From graphs below, identify those are function of y in term of x (Assume that y is the vertical axis and x is the horizontal axis). Explain your reasoning (Schwingendorf, Hawks, & Beineke, 1992).

(a) 

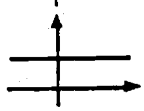
Yes Reason:
No

(b) 

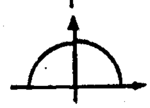
Yes Reason:
No

(c) 

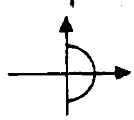
Yes Reason:
No

(d) 

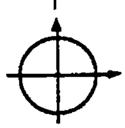
Yes Reason:
No

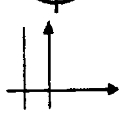
(e) 

Yes Reason:
No

(f) 

Yes Reason
No

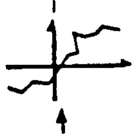
(g) 

(h) 

Yes Reason:
No

(i) 

Yes Reason:
No

(j) 

Yes Reason:
No

b. i) Does the table below represent a function? Explain your reasoning (Developed from Larson, Hostetler, & Hodkins, 2000)..

X	2	3	4	5	6	7	
f(x)	330.8	370.3	430.6	460.2	490.0	570.5	

Response and Reasoning:

[Additional space was provided]

ii) From the table above if one more data point (8, 430.6) is added, will this new set of data be a function? Explain your reasoning.

Response and Reasoning:

[Additional space was provided]

(c). From the symbolic representation given below, identify if they are function of y in term of x? Explain your reasoning.

(i) $y = x^2$

Response and Reasoning:

[Additional space was provided]

(ii) $y = 4$

Response and Reasoning :

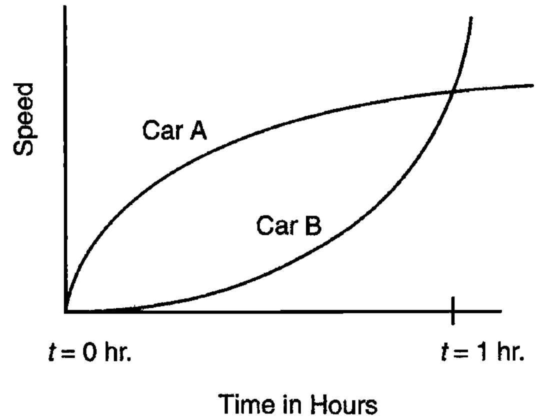
[Additional space was provided]

(iii) $x^2 + y^2 = 1$

Response and Reasoning:

[Additional space was provided]

2. The given graph represents velocity vs. time for two cars. Assume that the cars start from the same position and are traveling in the same direction (Carlson, 1998).



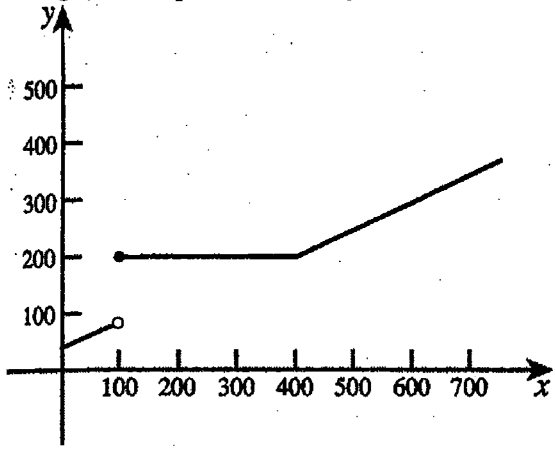
(a) What is the relationship between the position (location) of car A and that of car B at t = 1 hr? Explain.

(b) What is the relationship between the velocity of car A and that of car B at t = 1 hr? Explain.

(c) What is the relationship between the acceleration of car A and that of car B at t = 1 hr? Explain.

(d) How are the positions of the two cars related during the time interval between t = 0.75 and t = 1 hr.? (That is, is one car pulling away from the other?) Explain.

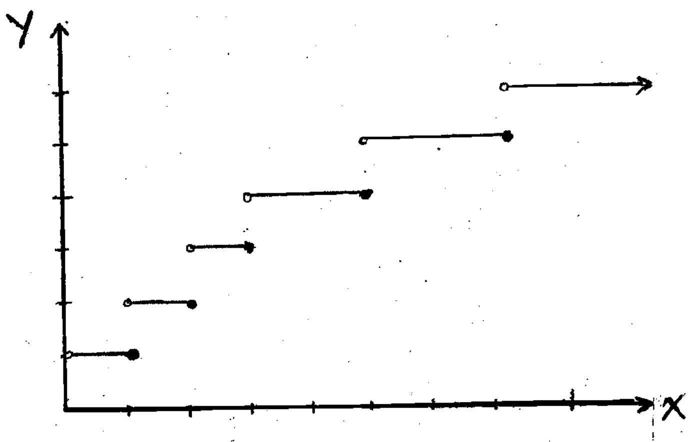
3. (a) Does this graph represent a function of y in term of x? Why or why not? Explain your reasoning (developed from Dwyer & Gruenwald, 2000).



Is it possible to represent this information in another way?

[Additional space was provided]

(b) Provide a situation that can be represented by this graph.



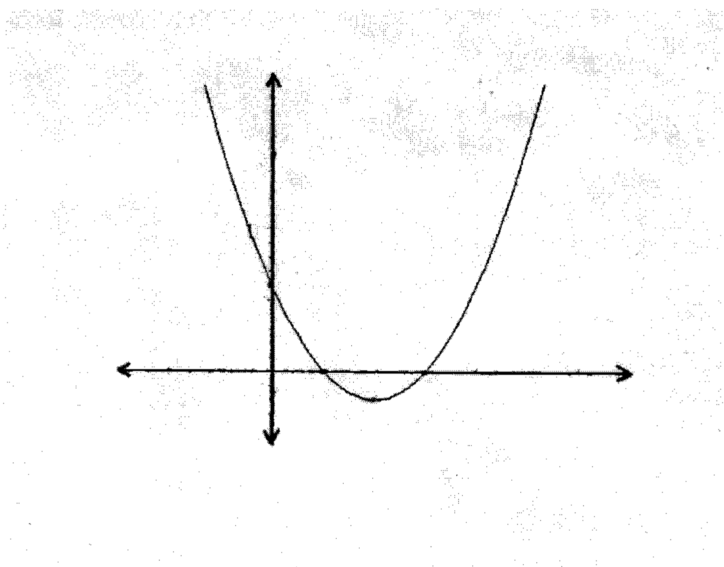
Does this graph represent a function of y in term of x? Why or why not? Explain your reasoning. Is it possible to represent this information in another way?

[Additional space was provided]

4. A ball dropped from the top of a tall building has height from the ground represented by $s = -16t^2 + 145$ feet after t seconds. Does this situation represent *s* as a function of *t*? Explain your reasoning. If it is a function, does it have an inverse? If yes, find its inverse. If no, explain why it does not have an inverse (developed from Dwyer & Gruenwald, 2000).

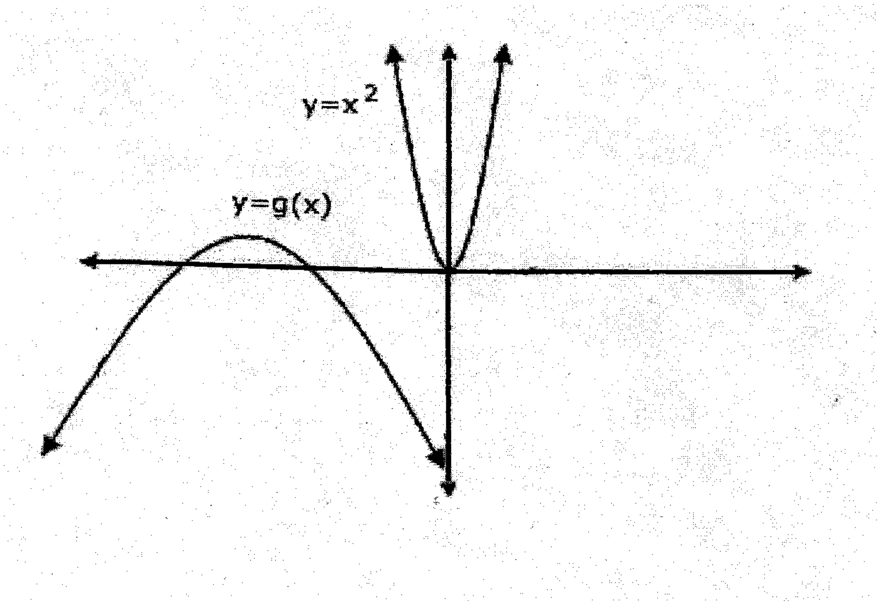
[Additional space was provided]

5. This is the graph of a function $y = a(x - h)^2 + k$. Give the reasonable numbers for a, h, and k. Explain your reasoning (Retrieved and developed October 24, 2002 from the World wide Web http://www.heinemann.com/math/nature.cfm).



[Additional space was provided]

6. Give a reasonable symbolic representation for *g(x)* in the form $g(x) = a(x - h)^2 + k$. Explain why your representation is reasonable (Retrieved and developed October 24, 2002 from the World wide Web http://www.heinemann.com/math/nature.cfm).



[Additional space was provided]

7. The following are formulas predicting future raises for four different employees. N represents the number of years from the date of contract. Each salary represents the salary that will be earned during the given years (Retrieved September 12, 2002 from the World Wide Web: http://mathforum.org/library/drmath/sets/college_modernalg.html)

A: salary = 30000 + 2500N
B: salary = 30000 + 1800N
C: salary = 27000 + 1500N
D: salary = 21000 + 2100N

(a) Will B ever earn more per year than A? Explain.

(b) Will C ever catch up A? Explain.

(c) Who will be making the highest yearly salary?

(d) Will D ever catch up C? Explain.

[Additional space was provided]

8. Rational expressions are simplified by dividing out equal factors of the numerator and denominator. For example,
$\frac{x^2-4}{x+2} = \frac{(x+2)(x-2)}{x+2} = x-2$. However, if we define
$f(x) = \frac{x^2-4}{x+2}$ and $g(x) = x-2$, are *f(x)* and *g(x)* the same? Explain your reasoning (Developed from Dwyer & Gruenwald, 2000).

APPENDIX I

POST-INSTRUCTIONAL INTERVIEW PROBLEMS

Use the following constraints, one at a time, in the order given, to create a function that meets all of the following constraints:

- The function is undefined at –3
- The function has a zero at 1/2
- The function is always nonnegative
- The function's domain is [-5, ∞)
- The function contains the point (4,7)

Explain your reasoning (Heid, Hollebrands, & Iseri, 2002):