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In this thesis, we will study certain generalizations of the classical Shannon Sampling Theorem, which allows for the reconstruction of a π-band-limited, square-integrable function from its samples on the integers. J. R. Higgins provided a generalization where the integers can be perturbed by less than 1/4, which includes nonuniform and nonperiodic sampling sets. We generalize Higgins' theorem by allowing for sampling sets that are perturbations of the set of zeros of a π-sine-type function.

A second type of generalization allows for functions f that, while still band-limited, need not be square-integrable but may have polynomial growth when restricted to the real line. We investigate two ways to achieve this goal, again using nonuniform sampling sets. The first is an approximate method that uses the multiplication of f by a smooth and rapidly decaying auxiliary function. The second method is exact and uses oversampling by finitely many additional points. It is also shown that oversampling by finitely many points is not only economical and may lead to faster convergence of the series, but also enables the perturbed sampling points to go beyond a quarter from the integers.

Furthermore, oversampling by finitely many points is applied to control the error stemming from a quantization of the sampled function values.

The final topic considered is the so-called peak value problem, where one seeks to find an upper bound for the infinity norm of a function from knowledge of the supremum of its sampled values. We generalize an existing approach by first proving and then applying a nonuniform version of the Valiron-Tschakaloff sampling theorem.

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Nonuniform Sampling Of Band-limited Functions

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Hussain Y. Al-Hammali, Author

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Academic

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Personal

I would like to thank my parents for their patience, support, and love.

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NONUNIFORM SAMPLING OF BAND-LIMITED FUNCTIONS

1 INTRODUCTION

In this thesis we consider extensions of the well-known Whittaker-Kotelnikov-Shannon (WKS) sampling theorem. In its original form, this theorem allows the reconstruction of a band-limited function from its sampled values and reads as follows: *If a function f is band-limited to $[-\pi, \pi]$, i.e., it is represented as*

$$f(t) = \int_{-\pi}^{\pi} g(x)e^{-ixt}dx \ \ , \ t \in \mathbb{R}, \tag{1.1}$$

for some function $g \in L^2(-\pi, \pi)$, then f can be reconstructed from its samples, $f(k)$, $k \in \mathbb{Z}$. The reconstruction formula is

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\frac{\sin \pi (t-k)}{\pi (t-k)}, \ t \in \mathbb{R}. \tag{1.2}$$

The series converges absolutely, in the L^2-sense, and uniformly on $\mathbb{R}$.

A function that has the representation (1.1) is called a band-limited function with bandwidth π. One can say that the WKS sampling theorem marks the beginning of Information Theory. Information Theory is an interdisciplinary field with applications in electrical engineering, computer science, communications, and applied mathematics. It concerns compressing and restoring data, and studying the reliability of data reconstruction. Some early works in this field are by Whittaker [46], Ogura [34], Kotelnikov [24], and Shannon [41].

The cardinal series (1.2) is an example of the more general formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(\lambda_n)\,\varphi_n(t) \tag{1.3}$$

where

$$\varphi_n(t) = \frac{\varphi(t)}{(t-\lambda_n)\,\varphi'(\lambda_n)} \tag{1.4}$$

and

$$\varphi(t) = \lim_{N\to\infty} \prod_{|k|<N} \left(1 - \frac{t}{\lambda_k}\right). \tag{1.5}$$

In (1.2) one has $\lambda_k = k$ and $\varphi(t) = \frac{1}{\pi}\sin(\pi t)$.

A major theme of this thesis is to establish the formula (1.3) for more general choices of the λ_k's and for f in more general function spaces. In particular, we consider functions that may have polynomial growth on the real line and are band-limited in the distributional sense.

Higgins [17] proved a generalization of the WKS sampling theorem where the sampling points, the integers, are replaced by points that are within one quarter from the integers. In **Chapter 2** we will generalize this result and consider sampling points that lie within a certain distance from the zeros of a sine-type function that are not necessarily real. The integers are an example of the set of zeros of a π-sine-type function.

In **Chapter 3** we will consider an approximate reconstruction of functions that have polynomial growth when restricted to the real line. The idea of the reconstruction is to control the growth of such a function by multiplying the function by a smooth cut-off function that decays faster than the reciprocal of any polynomial. The reconstructed function can be made arbitrarily close to the original function on compact subsets of $\mathbb{C}$. The reconstruction is accomplished by means of the Paley-Wiener-Schwarz theorem in conjunction with slight "ratio-type" oversampling. By ratio-type oversampling we mean a sampling set of higher density, e.g., using $\lambda\mathbb{Z}$, with $0 < \lambda < 1$ instead of $\mathbb{Z}$.

Chapter 4 considers the reconstruction of functions from the same class studied in Chapter 3. However, this time, we will use oversampling that is not of ratio-type as in Chapter 3. Instead, only finitely many points will be added to the sampling set. The main idea is to use quotient division to create an auxiliary function with reduced growth that lies in $\mathcal{PW}_\sigma^2$, the space of functions that have the representation (1.2). Then we use a sampling theorem for the space $\mathcal{PW}_\sigma^2$ for the reconstruction. After that, we unravel the quotient to obtain the desired reconstruction. A similar theme can be applied by using the Taylor polynomial that requires the derivatives of the function at zero which constitutes the additional information in this case.

Furthermore, a method using contour integration will be used to obtain a reconstruction with sampling points that may be perturbed by more than a quarter from the integers. An estimate of the canonical product (1.5) by Hinsen [19], and a theorem by Phragmén-Lindelöf [27, p. 39], will be used in the proof.

In **Chapter 5** we will control the error in the reconstruction that occurs in the quantization process for the Paley-Wiener Space $\mathcal{PW}_\sigma^1$, the functions (1.1) with $g \in L^1(-\sigma, \sigma)$. The quantization error occurs in the sense that $f(\lambda_k)$ is replaced by $f(\lambda_k) + \delta_k$ where $|\delta_k| < \delta$ for sufficiently small $\delta > 0$ in the sampling series. This requires us to find a sampling series for the space $\mathcal{PW}_\sigma^2$ that has a sufficiently fast decay. Then we use that sampling series to reconstruct functions in the space $\mathcal{PW}_\sigma^1$ by means of the density property of $\mathcal{PW}_\sigma^2$ in $\mathcal{PW}_\sigma^1$. Also, we will treat the problem over compact subsets of $\mathbb{R}$ by using a sampling series deduced in Chapter 4.

Chapter 6 will be concerned with finding an estimate of the infinity norm of band-limited functions that are bounded on the real line by knowing their values on a set $\Gamma \subset \mathbb{R}$. This is called *Peak Value Problem.* We will show that such an estimate exists if Γ is the zero set of a π-sine-type function. This answers a question posed by Boche and Mönich [32, page 2218]. We will prove a nonuniform version of the Valiron-Tschakaloff sampling theorem and then apply it to the peak value problem.

2 A SAMPLING THEOREM BY PERTURBING THE ZEROS OF A SINE-TYPE FUNCTION

One generalization of the Whittaker-Kotelnikov-Shannon (WKS) sampling theorem is a result by Higgins. It claims that if we perturb the sampling set, the integers, by less than a quarter, we can have a sampling series of Lagrange-type for the class of band-limited functions with band-width π. In this chapter, we are interested in generalizing the result by Higgins and this will be by perturbing the zeros of a π-sine-type function instead of the integers.

Theorem 2.1 (Higgins [17], see also Seip [39]). *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that*

$$|\lambda_k - k| \leq D < \frac{1}{4},$$

and let $\varphi(t)$ be defined as in (1.5). Then for all $f \in \mathcal{PW}_\pi^2$, we obtain

$$f(t) = \sum_{n=-\infty}^{\infty} f(\lambda_k) \frac{\varphi(t)}{\varphi'(\lambda_k)\,(t-\lambda_k)}. \tag{2.1}$$

The convergence is uniform over $\mathbb{R}$.

We can point out that if the sampling series (2.1) is extended to complex values of t, then one may call it Paley-Wiener-Levinson theorem since it follows from the celebrated result by Paley-Wiener using the fact that Fourier transform defines an isometric isomorphism from $L^2[-\pi,\pi]$ onto the space $\mathcal{PW}_\pi^2$ defined in Definition 2.2 below.

Boche and Mönich [32] deduced a sampling series for a larger class of functions where the sampling set is the zeros of a π-sine-type function, see Appendix F. We need some definitions and results that will explain the theme of the generalization we are interested in. Functions of sine-type are defined as:

Definition 2.1. *An entire function f is of* ***exponential type*** *at most σ ($\sigma > 0$), if for any $\epsilon > 0$ there exists an A_ϵ such that*

$$| f(z) | \leq A_\epsilon e^{(\sigma+\epsilon)|z|}$$

for all $z \in \mathbb{C}$. In this case, we write, f is a function of exponential type $\leq \sigma$. This function is said to be of ***σ-sine-type*** *function if*

(i) The zeros of f are simple and separated (=uniformly discrete $\inf_{j \neq k} \mid \lambda_j - \lambda_k \mid \geq \underline{\delta}$ for all $k \in \mathbb{Z}$ for some $\underline{\delta} > 0$) and

(ii) There exist A, B and η such that

$$Ae^{\sigma|y|} \leq | f(x+iy) | \leq Be^{\sigma|y|} \tag{2.2}$$

for all $x, y \in \mathbb{R}$, such that $| y | \geq \eta$.

The function $\sin(\pi z)$ is an example of a π-sine-type function. An example of a π-sine-type function with non-equidistant zeros is provided by

$$\varphi_{\alpha,\beta}(z) = \cos(\pi z) - \beta \sin(\alpha \pi z) \ \text{ where } 0 < \alpha < 1 \text{ and } 0 \leq \beta \leq 1.$$

For more details see the Appendix A.

Let E denote the space of all entire functions and E_σ denote the class of all entire functions of exponential type $\leq \sigma$. The Paley-Wiener spaces $\mathcal{PW}_\pi^p$ are defined as follows:

Definition 2.2. *A function f is in the* ***Paley-Wiener space*** *$\mathcal{PW}_\pi^p$, $1 \leq p \leq \infty$ if $f(z) = \int_{-\pi}^{\pi} g(w) e^{izw} dw$, $z \in \mathbb{C}$ for some $g \in L^p[-\pi, \pi]$ where the norm is given by*

$$\|f\|_{\mathcal{PW}_\pi^p} := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(w)|^p \, dw \right)^{1/p}$$

for $1 \leq p < \infty$.

The ***band-limited function*** is a tempered distribution whose Fourier transform has a compact support. The functions in the spaces $\mathcal{PW}_\pi^2$ and $\mathcal{PW}_\pi^1$ are examples of band-limited functions.

A closely related function space is ***Bernstein space*** $\mathcal{B}_\sigma^p$ which consists of all functions in E_σ whose restrictions to the real line are in $L^p(\mathbb{R})$. The norm for $\mathcal{B}_\sigma^p$, $1 \le p \le \infty$ is given by $\| f \|_{\mathcal{B}_\sigma^p} = \| f \|_p$. Again the functions in this space are band-limited functions. That can be seen by Phragmén-Lindelöf and Paley-Wiener-Schwartz Theorem, see Theorem B.1 and Theorem E.3.

The two normed spaces $\mathcal{B}_\pi^2$ and $\mathcal{PW}_\pi^2$ are identical, and that can be seen by Plancherel, and Paley-Wiener Theorem, see Theorem E.1.

The Bernstein and Paley-Wiener spaces can be ordered as follows:

$$\mathcal{B}_\sigma^1 \subset \mathcal{B}_\sigma^{\mathbf{2}} \subset \ldots \subset \mathcal{B}_\sigma^\infty$$

and

$$\ldots \subset \mathcal{PW}_\sigma^2 \subset \mathcal{PW}_\sigma^1.$$

For more details about the inclusion ordering of the Bernstein and Paley-Wiener spaces see the Appendix E. The functions in the Bernstein space $\mathcal{B}_\sigma^2$ are band-limited in classical sense while the functions in $\mathcal{B}_\sigma^p$ with $p > 2$, $p = \infty$ are band-limited in distributional sense, with band-width σ, if they are not band-limited in the classical sense. This follows from the Paley-Wiener-Schwartz Theorem, see Theorem E.3.

Also, we obtain the following ordered inclusions:

$$\mathcal{B}_\sigma^2 = \mathcal{PW}_\sigma^2 \subset \mathcal{PW}_\sigma^1 \subset \mathcal{B}_{\sigma,0}^\infty \subset \mathcal{B}_\sigma^\infty.$$

where the elements f in $\mathcal{B}_{\sigma,0}^\infty$ are those in $\mathcal{B}_\sigma^\infty$ that satisfy $\lim_{|t|\to\infty} f(t) = 0$.

The WKS sampling series can be used for the class $\mathcal{B}_{\sigma,0}^\infty$ if we consider oversampling, see [32, Theorem 6]. For more general study of function reconstruction, we will consider the class of functions that is defined as

$$\widetilde{\mathcal{B}}_{\sigma,N} = \left\{ f \in E \mid |f(z)| \leq \gamma \left(1 + |z|\right)^N e^{\sigma|\mathrm{Im}\, z|}, \text{for some } N \in \mathbb{N} \text{ and } \gamma \in \mathbb{R} \right\}. \tag{2.3}$$

This class consists of all entire functions of exponential type $\leq \sigma$ that are bounded by some polynomial of degree $\leq N$ when restricted to the real line $\mathbb{R}$. The classes of functions that we are interested in for our study are listed below:

$$\mathcal{PW}_\sigma^2 \subset \mathcal{PW}_\sigma^1 \subset \mathcal{B}_{\sigma,0}^\infty \subset \mathcal{B}_\sigma^\infty \subset \widetilde{\mathcal{B}}_{\sigma,N}.$$

When we deal with perturbation, the question of stability of the reconstruction comes into the play. One kind of stability is defined from the sampling set point view. The set $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is called a ***set of stable sampling (or sampling)*** for the space $\mathcal{B}_\sigma^2$, if there exists a constant K such that

$$\|f\|_{L^2}^2 \leq K \sum_{k\in\mathbb{Z}} |f(\lambda_k)|^2$$

for all $f \in \mathcal{B}_\sigma^2$. And Λ is called a ***set of interpolating*** for $\mathcal{B}_\sigma^2$ if for each square-summable $\{c_k\}_{k\in\mathbb{Z}}$ there exists $f \in \mathcal{B}_\sigma^2$ with

$$f(\lambda_k) = c_k. \tag{2.4}$$

If f is the only solution in $\mathcal{B}_\sigma^2$ for (2.4), then Λ is called a ***complete interpolating set*** for $\mathcal{B}_\sigma^2$.

We are interested in the stability of the reconstruction for the space $\mathcal{B}_\sigma^\infty$. Due to Landau [25], if there exists a constant K such that

$$\|f\|_\infty \leq K \sup_{k\in\mathbb{Z}} |f(\lambda_k)|$$

for all $f \in \mathcal{B}_\sigma^\infty$, then $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is called a ***stable sampling set*** for the space $\mathcal{B}_\sigma^\infty$.

In practice, we study function reconstructions over compact sets. Accordingly, there will be cases where we study stability over compact sets of $\mathbb{R}$ or $\mathbb{C}$. Oversampling can be used for stability. It provides a fast decay in the sampling functions that are used in the reconstruction formulae for larger function spaces.

The stability in sense of Riesz basis is another type of stability. It leads to stability in the sense of stable sampling set.

Definition 2.3. *A sequence of vectors $\{\varphi_k\}_{k\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a **Riesz basis** if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is complete in $\mathcal{H}$ and there exist constants A and B such that for all M, $N \in \mathbb{N}$ and arbitrary scalars c_k we have*

$$A \sum_{k=-M}^{N} |c_k|^2 \le \left\| \sum_{k=-M}^{N} c_k \varphi_k \right\|^2 \le B \sum_{k=-M}^{N} |c_k|^2 . \tag{2.5}$$

When we have to deal with perturbation, we use ***Kadec's 1/4-Theorem***, see [49, page 36].

Theorem 2.2 (Kadec)**.** *If $\{\lambda_k\}$ is a sequence of real numbers for which*

$$|\lambda_k - k| \le D < \frac{1}{4} \ , k = 0, \pm 1, \pm 2, \ldots$$

then $\left\{e^{i\lambda_k t}\right\}$ forms a Riesz basis for $L^2\left[-\pi, \pi\right]$.

2.1 Derivation Of The Main Result

Let us now consider the perturbation as $\lambda_k^* = \lambda_k + d_k$ and define φ^* as

$$\varphi^*(z) = \lim_{N\to\infty} \prod_{|k|\le N} \left(1 - \frac{z}{\lambda_k + d_k}\right) \tag{2.6}$$

where $(1 - z/(\lambda_k + d_k))$ is replaced by z if $\lambda_k + d_k = 0$. Here we assume $\{d_k\}_{k\in\mathbb{Z}} \in l^\infty$.

It is known that the sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ forms a complete interpolating sequence for $\mathcal{PW}_\pi^2$ if the system $\left\{e^{i\lambda_k t}\right\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2\left[-\pi, \pi\right]$. That can be seen by using the right-hand side of the inequality (2.5), inverse Fourier transform and the biorthogonal basis of the exponentials $\left\{e^{i\lambda_k t}\right\}_{k\in\mathbb{Z}}$ in $L^2[-\pi,\pi]$. The biorthogonal basis $\{h_k\}_{k\in\mathbb{Z}}$ satisfies

$$\left\langle e^{i\lambda_m(\cdot)}, h_n\right\rangle = \delta_{mn}. \tag{2.7}$$

A consequence of work by B. S. Pavlov [36], is that if $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$, then the canonical product $\varphi(z)$ in (1.5) defines an entire function of exponential type π, see also K. Seip and Y. Lyubarskii [40, 30]. If the system $\left\{e^{i\lambda_k t}\right\}_{k\in\mathbb{Z}}$ forms a Riesz basis for the space $L^2[-\pi,\pi]$, then the canonical product $\varphi(z)$ defines an entire function of exponential type π.

Here, we will study the perturbation of a sampling set. Perturbing the integers (which are zeros of a π-sine-type function) by less than a quarter guarantees that $\varphi^*(z)$ defines an entire function of exponential type π by Kadec and Pavlov mentioned above. However, we are interested in the more general case where we perturb the set of zeros of a π-sine-type function. Levin and Ostrovskii in [28, page 80] stated that if $\{d_k\}_{k\in\mathbb{Z}}$is a bounded sequence of complex numbers and $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a set of zeros of a π-sine-type function, then the canonical product $\varphi^*(z)$ in (2.6) is a function of exponential type π, but it need not be a π-sine-type function.

For later use, we summarize what have mentioned above as a Remark

Remark 2.1. *If $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$, then the canonical product $\varphi(z)$ in (1.5) defines an entire function of exponential type π. Consequently, if the system $\left\{e^{i\lambda_k t}\right\}_{k\in\mathbb{Z}}$ form a Riesz basis for $L^2[-\pi,\pi]$, then the function $\varphi(z)$ defines an entire function of exponential type π, see [49, Theorem 9, page 143].*

For later use, we state the following remark

Remark 2.2. *If $f\in\mathcal{PW}_\pi^2$ vanishes on a complete interpolating set Λ of $\mathcal{PW}_\pi^2$, then $f\equiv 0$.*

A. G. García [14] gave a new proof of the Paley-Wiener-Levinson Theorem where he used a theorem by Titchmarsh. Titchmarsh's theorem can be stated as follows, cf. [43, Theorem VI].

Theorem 2.3 (Titchmarsh)**.** *Let $g \in L^1[-\pi,\pi]$ and define the entire function f as*

$$f(z) = \int_{-\pi}^{\pi} g(w)e^{zw}dw.$$

Then, f has infinitely many zeros, $\{z_n\}_{n\in\mathbb{N}}$, with nondecreasing absolute values, such that

$$f(z) = Az^m e^{\left(\frac{a+b}{2}\right)z} \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)$$

for some $m \in \mathbb{N}\cup\{0\}$, where $[a,b] \subseteq [-\pi,\pi]$ is the smallest interval that contains the support of g. The infinite product is conditionally convergent.

For our application of Titchmarsh's theorem we also need the following lemma

Lemma 2.1. *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ be such that $\{f_k(z) = e^{i\lambda_k z},\ k \in \mathbb{Z}\}$ is a Riesz basis of $L^2[-\pi,\pi]$. Let $\{g_k\}_{k\in\mathbb{Z}}$ denote the corresponding biorthogonal basis. Let $[\alpha_k,\beta_k] \subseteq [-\pi,\pi]$ be the smallest interval that contains the support of g_k. Then $\alpha_k = -\pi$ and $\beta_k = \pi$.*

Proof. Assume that $[\alpha_k,\beta_k]$ is a proper subset of $[-\pi,\pi]$ for some $k \in \mathbb{Z}$, let $c = (\alpha_k+\beta_k)/2$, and consider the shifted function $h(z) = \overline{g_k}(z-c)$. Then h is supported in $[-(\beta_k - \alpha_k)/2,\ (\beta_k-\alpha_k)/2] = [-\beta\pi, \beta\pi]$ for some $\beta \in (0,1)$. The inverse Fourier transform of h now satisfies $\tilde{h}(\lambda_n) = \int h(x)e^{i\lambda_n x}dx = \int \overline{g_k}(x-c)e^{i\lambda_n x}\,dx = e^{i\lambda_n c}\int \overline{g_k}(x)e^{i\lambda_n x}\,dx = e^{i\lambda_n c}\langle f_n, g_k\rangle = e^{i\lambda_n c}\delta_{kn}$, i.e., $\tilde{h}$ vanishes on $\Lambda' = \Lambda - \{\lambda_k\}$. According to Beurling and Malliavin [4] the closure radii of Λ and Λ' are equal. Therefore, since $\beta < 1$ and $\tilde{h} \in \mathcal{PW}^2_{\beta\pi}$ vanishes on Λ', it must vanish identically, which is a contradiction. □

We restate the following theorem by Katsnel'son, see [23].

Theorem 2.4 (Katsnel'son)**.** *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be the set of zeros of a σ-sine-type function and let $\{d_k\}$ be a sequence of complex numbers satisfying the conditions*

$$|\mathrm{Re}\, d_k| \le dp, \sup_{k\in\mathbb{Z}} |\mathrm{Im}\, d_k| < \infty$$

where $p = \inf_k |\mathrm{Re}\lambda_k - \mathrm{Re}\lambda_{k+1}|$, $d < \frac{1}{4}$ *is a constant. Then, the sequence* $\left\{e^{i(\lambda_k + d_k)t}\right\}$ *is a Riesz basis in* $L^2(-\sigma, \sigma)$.

We now state the main result of this chapter. It is a generalization in the sense that the perturbed sampling points in Higgins's result are replaced by perturbed complex numbers with a certain distance from the zeros of a π-sine-type function.

Theorem 2.5. *Let* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *be a set of zeros of a* π*-sine-type function and let* $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ *be a sequence of complex numbers satisfying*

$$|\mathrm{Re}\lambda_k^* - \mathrm{Re}\lambda_k| \leq dp, \sup_{k\in\mathbb{Z}} |\mathrm{Im}\lambda_k^* - \mathrm{Im}\lambda_k| < \infty$$

where $d < \frac{1}{4}$ *and* $p = \inf_k |\mathrm{Re}\lambda_k - \mathrm{Re}\lambda_{k+1}|$. *Then, for all* $f \in \mathcal{PW}_\pi^2$, *we obtain*

$$f(z) = \sum_{k\in\mathbb{Z}} f(\lambda_k^*) \frac{\varphi^*(z)}{\varphi^{*\prime}(\lambda_k^*)(z - \lambda_k^*)},$$

where the convergence is uniform on any horizontal strip of $\mathbb{C}$ *of a finite width.*

Proof. By Theorem 2.4 we have the sequence $\left\{e^{i\lambda_k^*(\cdot)}\right\}_{k\in\mathbb{Z}}$ forms a Riesz basis over $L^2[-\pi, \pi]$. The sequence $\left\{e^{i\lambda_k^*(\cdot)}\right\}_{k\in\mathbb{Z}}$ possesses a complete biorthogonal sequence $\{g_k\}_{k\in\mathbb{Z}}$ in $L^2[-\pi, \pi]$, see Theorem D.1. The sequence $\{g_k\}_{k\in\mathbb{Z}}$ is a Riesz basis being biorthogonal to $\left\{e^{i\lambda_k^*(\cdot)}\right\}_{k\in\mathbb{Z}}$, see Theorem D.2. Let $h_k = \overline{g_k}$. It follows that

$$\int_{-\pi}^{\pi} h_n(x) e^{i\lambda_m^* x} dx = \left\langle e^{i\lambda_m^*(\cdot)}, g_n \right\rangle = \delta_{mn}.$$

We define the function G_n as

$$G_n(z) = \int_{-\pi}^{\pi} h_n(x) e^{izx} dx. \tag{2.8}$$

By using the biorthogonality condition (2.7) we obtain that

$$G_n(z) = \frac{\varphi^*(z)}{(z - \lambda_n^*)} K(z),$$

with $K(\lambda_n^*) \neq 0$. We claim that the function K has no zeros. The claim will be proved by the way of contradiction. Assume that $K(\mu) = 0$ for $\mu \neq \lambda_n^*$ and define the function H as

$$H(z) = G_n(z)\frac{(z-\lambda_n^*)}{(z-\mu)}.$$

Then, the function H belongs to $\mathcal{PW}_\pi^2 = \mathcal{B}_\pi^2$ since G_n is, and the factor $\frac{(z-\lambda_n^*)}{(z-\mu)}$ is asymptotically equal to 1. The function H vanishes on the complete interpolating set $\Lambda = \{\lambda_k^*\}_{k\in\mathbb{Z}}$ and thus $H \equiv 0$ by Remark 2.2. That implies that G_n is identically equal to zero, a contradiction. Therefore, the function K is different from zero everywhere. Now, by virtue of Theorem 2.3 and Lemma 2.1 we obtain

$$G_n(z) = \frac{\varphi^*(z)}{z-\lambda_n^*}K(z) = Az^m \prod_{n=1}^{\infty}(1-\frac{z}{z_k}).$$

Since φ^* has only simple zeros and $K(z)$ has no zeros, we have that $z_k \neq z_l$ for $k \neq l$. Furthermore, one has either $m = 0$ and $\lambda_0 \neq 0$, or $m = 1$ and $\lambda_0 = 0$. In either case $\{z_k, k = 1, 2, \ldots\} = \{\lambda_k^*, k \in \mathbb{Z}, k \neq n\}$. It follows that $K(z)$ is constant. By the biorthogonality condition we obtain that

$$1 = G_n\left(\lambda_n^*\right) = \lim_{z\to\lambda_n^*} G_n\left(z\right) = \varphi^{*\prime}\left(\lambda_n^*\right) K\left(\lambda_n^*\right)$$

and thus, we rewrite (2.8) as

$$\int_{-\pi}^{\pi} h_n(x)e^{izx}dx = \frac{\varphi^*(z)}{\varphi^{*\prime}\left(\lambda_n^*\right)\left(z-\lambda_n^*\right)} = \varphi_n^*(z).$$

Now, if $f \in \mathcal{PW}_\pi^2$, then

$$f(z) = \int_{-\pi}^{\pi} g(w)e^{izw}dw$$

for some g in $L^2[-\pi,\pi]$. We have $\{h_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$ and so

$$g(w) = \lim_{N\to\infty} \sum_{m=-N}^{N} c_m h_m(w)$$

in L^2 sense. It follows that $f(\lambda_m^*) = \langle e^{i\lambda_m^*(.)}, \overline{g} \rangle = \langle e^{i\lambda_m^*(.)}, \sum_{k\in\mathbb{Z}} \overline{c_k} g_k \rangle = c_m$ by biorthogonality. Now,

$$\begin{aligned}\left| f(z) - \sum_{m=-N}^{N} f(\lambda_m^*) \frac{\varphi^*(z)}{\varphi^{*\prime}(\lambda_m^*)(z-\lambda_m^*)} \right| &= \left| \int_{-\pi}^{\pi} g(w) e^{izw} dw - \sum_{m=-N}^{N} c_m \int_{-\pi}^{\pi} h_m(w) e^{izw} dw \right| \\ &= \left| \int_{-\pi}^{\pi} \left[g(w) - \sum_{m=-N}^{N} c_m h_m(w) \right] e^{izw} dw \right| \\ &\leq \sqrt{2\pi} \left\| g - \sum_{m=-N}^{N} c_m h_m \right\|_2 e^{\pi|\mathrm{Im} z|}.\end{aligned}$$

Thus, for $|\mathrm{Im} z| \leq M$ for some $M \in \mathbb{R}$, we have

$$\lim_{N\to\infty} \left| f(z) - \sum_{m=-N}^{N} f(\lambda_m^*) \frac{\varphi^*(z)}{\varphi^{*\prime}(\lambda_m^*)(z-\lambda_m^*)} \right| \leq \lim_{N\to\infty} \sqrt{2\pi} \left\| g - \sum_{m=-N}^{N} c_m h_m \right\|_2 e^{\pi|\mathrm{Im} z|} = 0,$$

which shows that the convergence is uniform over any horizontal strip of a finite width.

□

For completeness, we state the following theorem.

Theorem 2.6. [16, Theorem 3.12] *Under Fourier transformation, the pre-image of a Riesz basis for $L^2[-\pi,\pi]$ is a Riesz basis for $\mathcal{PW}_\pi^2$.*

With this result, we see that the sequence $\{\varphi_k^*\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $\mathcal{PW}_\pi^2$. Hence, the stability in the sense of (2.5) follows.

Example 2.1. *If $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ is a sequence of real numbers and $\{\lambda_k\}_{k\in\mathbb{Z}} = \mathbb{Z}$ which are zeros of a π-sine-type function, then $p = 1$ and $|\mathrm{Re}\lambda_k^* - \mathrm{Re}\lambda_k| \leq d < \frac{1}{4}$ and thus for all $f \in \mathcal{B}_\pi^2 = \mathcal{PW}_\pi^2$ we obtain that*

$$f(t) = \sum_{k\in\mathbb{Z}} f(\lambda_k^*) \frac{\varphi^*(t)}{\varphi^{*\prime}(\lambda_k^*)(t-\lambda_k^*)},$$

where the convergence is uniform over $\mathbb{R}$. This is the sampling series by Higgins, see Theorem 2.1. Furthermore, if we additionally set $d = 0$, then we obtain the WKS sampling theorem.

2.2 Numerical Example

In the following we have the plot of the function

$$f(x) = \frac{\sin(\pi\left(x - \frac{2}{10}\right))}{\left(x - \frac{2}{10}\right)} + \frac{\sin\left(\frac{\pi}{3}\left(x - \frac{23}{10}\right)\right)}{\left(x - \frac{23}{10}\right)}$$

that is in the space $\mathcal{PW}_\pi^2$. We consider sampling points that are the zeros of a π-sine-type function. The π-sine-type function is $g(x) = \cos \pi x - 0.5 \sin\left(\frac{\pi}{\sqrt{3}} x\right)$ that has zeros with $p = 0.7448$. The zeros form a nonuniform and nonperiodic sampling points. We perturb the first ten positive zeros. The perturbation achieved with a random function that maintains the quarter condition in Theorem 2.5. The number of the terms used for the reconstruction is $2N + 1 = 801$. The truncation error is $\|e\|_{L^\infty[-10,10]} = 0.0025$. The plot of the reconstruction and the exact function below coincides since the error is relatively small.

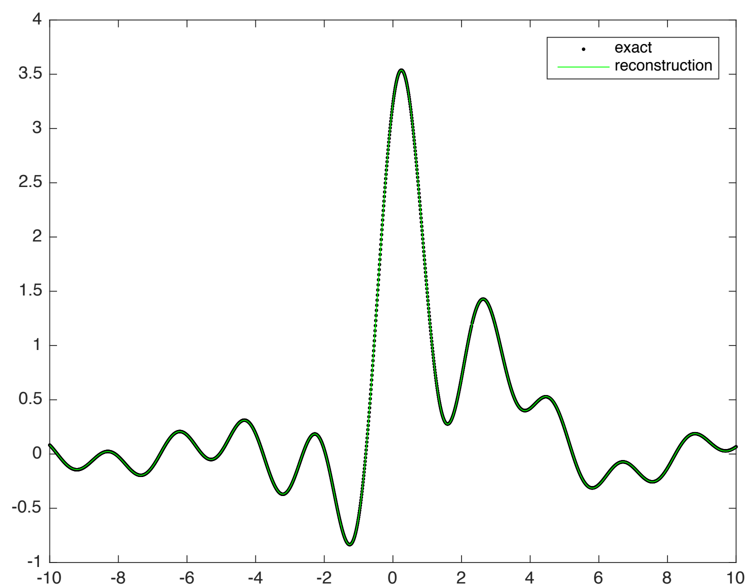

FIGURE 2.1: The graph of the function and its reconstruction where the sampling points are the zeros of a π-sine-type function.

3 APPROXIMATE RECONSTRUCTION OF BAND-LIMITED FUNCTIONS OF POLYNOMIAL GROWTH

In this chapter, we will study the problem of approximately reconstructing a band-limited function of polynomial growth from its nonuniform samples. The main result will be derived by using oversampling and the Paley-Wiener-Schwartz Theorem. We start with some background on the topic.

Campbell [8] is one of the early authors who studied a generalization of the sampling series. The generalization was in the sense that he considered functions that, essentially, have polynomial growth when restricted to the real line. Campbell's reconstruction required the support of the shifted distribution to be disjoint with a function λ that is in the space $\mathcal{D}$. The idea mainly is as follows:

Let $\langle g, h\rangle$ denote the result of applying a distribution $g \in \mathcal{D}'$ to a test function $h \in \mathcal{D}$. If g is a distribution with bounded support and f is defined as

$$f(t) = \left\langle g(w), \lambda(w)e^{-iwt}\right\rangle$$

where $\lambda \in \mathcal{D}$ and $\lambda\left(w\right) = 1$ on some open set containing the support of $g\left(w\right)$, then with use of the identity

$$\sum_{m} g\left(w - 2m\sigma\right) = \frac{1}{2\sigma}\sum_{n} f\left(\frac{n\pi}{\sigma}\right) e^{in\pi w/\sigma} \tag{3.1}$$

and the fact that $\lambda\left(w\right)$ is disjoint from the supports of the shifted distributions, that are $g\left(w - 2m\sigma\right)$, $m = \pm 1, \pm 2, \ldots$, we apply $\left\langle ., \lambda(w)e^{-iwt}\right\rangle$ to both sides of (3.1) to obtain

$$\left\langle g(w), \lambda(w)e^{-iwt}\right\rangle = \frac{1}{2\sigma}\sum_{n} f\left(\frac{n\pi}{\sigma}\right)\left\langle e^{in\pi w/\sigma}, \lambda(w)e^{-iwt}\right\rangle,$$

that is

$$f(t) = \frac{1}{2\sigma}\sum_{n} f\left(\frac{n\pi}{\sigma}\right)\int_{\mathbb{R}} \lambda\left(w\right) e^{-iw(t-n\pi/\sigma)}dw.$$

Several authors studied sampling expansions for distributions, both with uniform and nonuniform sampling points. Pfaffelhuber [37] obtained a reconstruction with uniform sampling points. The expansion is

$$f(t) = q_N(t)\cos(\sigma t) + \sum_{n\in\mathbb{Z}} \left[f(\lambda_n) - (-1)^n q_N(\lambda_n)\right] \left(\frac{t}{\lambda_n}\right) \frac{\sin\sigma(t-\lambda_n)}{\sigma(t-\lambda_n)}$$

where

$$q_N(t) = \sum_{k=0}^{N-1} \frac{t^k}{k!} \left(\frac{d}{dt}\right)^k \left[\frac{f(t)}{\cos\sigma t}\right]\Bigg|_{t=0}$$

Later, it is generalized by Hoskins and Pinto [20]. They replaced the function $\cos\sigma t$ by $\eta(t)$ which is the inverse Fourier transform of suitable distribution in the space $\mathcal{D}'$. An important tool in their paper was a result by Schwartz for decomposing a distribution, see [20]. Also, we should mention that Lee [26] did an early study where the class he considered was $B_k(W)$, the class of functions that are band-limited to $[-W, W]$ and satisfy $\int_{\mathbb{R}} |f(x)|(1+x^2)^{-k}dx < \infty$ when restricted to $\mathbb{R}$. He obtained the sampling series

$$f(z) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\sigma}\right) \frac{\sin\sigma(z - n\pi/\sigma)}{\sigma(z-n\pi/\sigma)} \frac{\sin^k\beta(z-n\pi/\sigma)}{\beta^k(z-n\pi/\sigma)^k},$$

where $\sigma > 2\pi W$ and $\beta < (\sigma - 2\pi W)/k$. The convergence is uniform over compact subsets of $\mathbb{C}$. Gilbert G Walter did an extensive study for the class of functions that are band-limited and have polynomial growth. In [44], he obtained a sampling expansion that includes a correction by a polynomial where the sampling set is uniform. While in [45], he derived a sampling expansion with nonuniform sampling points that are symmetric as $\lambda_k = -\lambda_{-k} \in \mathbb{C}$.

3.1 Smooth Cut-off Function

We will study the functions in the class $\widetilde{\mathcal{B}}_{\pi,N}$, $N \in \mathbb{N}\cup\{0\}$. The class $\widetilde{\mathcal{B}}_{\pi,N}$ is defined as

$$\widetilde{\mathcal{B}}_{\pi,N} = \left\{ f \in E \mid |f(z)| \le \gamma(1+|z|)^N e^{\pi|\mathrm{Im} z|}, \text{for some } \gamma \in \mathbb{R} \right\}. \tag{3.2}$$

We will use the notion of Riesz basis of the exponentials. It is appropriate to use the result by Katsnel'son, see Theorem 2.4, since we will be oversampling and considering sampling points that are complex (not necessarily real).

We will use the method of multiplying the given function by a smooth cut-off function. As result, the new function will have a decay that is faster than a reciprocal of any polynomial. The main tool will be the celebrated theorem by Schwartz that was proved after the Paley-Wiener Theorem, see [38, page 198]. The theorem is stated as follows

Theorem 3.1 (Paley-Wiener-Schwartz)**.** *(a) If* $\phi \in \mathcal{D}(\mathbb{R}^n)$ *has its support in* rB*,* B *is the closed unit ball of* $\mathbb{R}^n$*, and if*

$$f(z) = \int_{\mathbb{R}^n} \phi(t)\, e^{-iz.t} dm_n(t)\,, \ (z \in \mathbb{C}^n) \tag{3.3}$$

then f *is entire, and there is a constant* $\gamma_N < \infty$ *such that*

$$|f(z)| \le \gamma_N (1 + |z|)^{-N} e^{r|Imz|}, (z \in \mathbb{C}^n, N = 0, 1, 2, \ldots) \tag{3.4}$$

(b) Conversely, if f *is an entire function in* $\mathbb{C}^n$ *which satisfies (3.4), then there exists* $\phi \in \mathcal{D}(\mathbb{R}^n)$*, with support in* rB*, such that (3.3) holds.*

For the following result we define f_δ as

$$f_\delta(z) = f(z)\hat{h}_\delta(z)$$

where $h_\delta(t) = \delta^{-1}h(t/\delta)$ for $h \in \mathcal{D}(B)$ with $\int h = 1$. The function f_δ converges to f pointwise as $\delta \to 0$.

We state and prove the following theorem

Theorem 3.2. *Let* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *be a sequence of complex numbers such that*

$$\left|\lambda_k - k\left(\frac{\pi}{\delta_0 + \pi}\right)\right| \le D < \frac{1}{4}\left(\frac{\pi}{\delta_0 + \pi}\right).$$

for some $\delta_0 > 0$. *Then for given* $f \in \widetilde{\mathcal{B}}_{\pi,N}$, $\mathcal{C} \subset \mathbb{C}$ *compact and* $\epsilon > 0$ *there is* $\delta = \delta(f, \mathcal{C}, \epsilon) < \delta_0$ *such that*

$$\left| f(z) - \sum_{k\in\mathbb{Z}} f_\delta(\lambda_k) \frac{\varphi(z)}{(z-\lambda_k)\varphi'(\lambda_k)} \right| < \epsilon.$$

for all $z \in \mathcal{C}$.

Proof. Let $\mathcal{C}$ be a compact set in $\mathbb{C}$ and let $f_\delta(z) = f(z)\hat{h}_\delta(z)$ where $h_\delta(t) = \delta^{-1}h(t/\delta)$ and $h \in \mathcal{D}(B)$ with $\int h = 1$. Let $\delta \leq \delta_0$. Now, by the Paley-Wiener-Schwartz Theorem we obtain

$$|f_\delta(z)| < \gamma_q (1+|z|)^{-q} e^{(\delta+\pi)|y|} \qquad q = 0, 1, 2, \ldots$$

and thus again by the same theorem we obtain

$$f_\delta(z) = \langle \varphi_\delta, e_{-z} \rangle = \int_{\mathbb{R}^n} \varphi_\delta(t) e^{-izt} dm_n(t)$$

for some $\varphi_\delta \in \mathcal{D}((\delta+\pi)B)$. Now, we use Theorem 2.5 to reconstruct $f_\delta \in \mathcal{B}^2_{\delta_0+\pi}$. The reconstruction of f_δ is

$$f_\delta(z) = \sum_{k\in\mathbb{Z}} f_\delta(\lambda_k) \frac{\varphi(z)}{(z-\lambda_k)\varphi'(\lambda_k)}$$

where the convergence is in L^2-sense. Using the fact that $|.| \leq \sqrt{2\pi}\,\|.\|_2\, e^{\pi|\mathrm{Im} z|}$ we can have uniform convergence over compact subsets of $\mathbb{C}$. From the definition above for f_δ we can have a sufficiently small $\delta = \delta(f, \mathcal{C}, \epsilon) < \delta_0$ such that

$$|f(z) - f_\delta(z)| < \epsilon$$

for all z in $\mathcal{C}$. Now,

$$\begin{aligned} \left| f(z) - \sum_{k\in\mathbb{Z}} f_\delta(\lambda_k) \frac{\varphi(z)}{(z-\lambda_k)\varphi'(\lambda_k)} \right| &= |f(z) - f_\delta(z)| \\ &\leq \epsilon \end{aligned}$$

and thus the conclusion follows where the convergence is uniform over compact subsets of $\mathbb{C}$.

□

3.2 Numerical Examples

In the following we have the plot of the function

$$f(x) = (x + 4.5)\left(\cos(\pi x) - \left(\frac{3}{4}\right)\sin\left(\frac{\pi}{\sqrt{2}}x\right)\right)$$

that is in the space $\widetilde{\mathcal{B}}_{\pi,1}$. The function has a polynomial growth of order 1. The sampling set is a perturbed sampling set from the scaled integers $\left(\frac{\pi}{\delta_0+\pi}\right)\mathbb{Z}$ with $\delta_0 = 0.2$. The perturbation is achieved by a random function that maintains the quarter condition in Theorem 3.2. The number of terms considered is $2N + 1 = 801$. The truncation error is $\|e\|_{L^\infty[-10,10]} = 0.3331$.

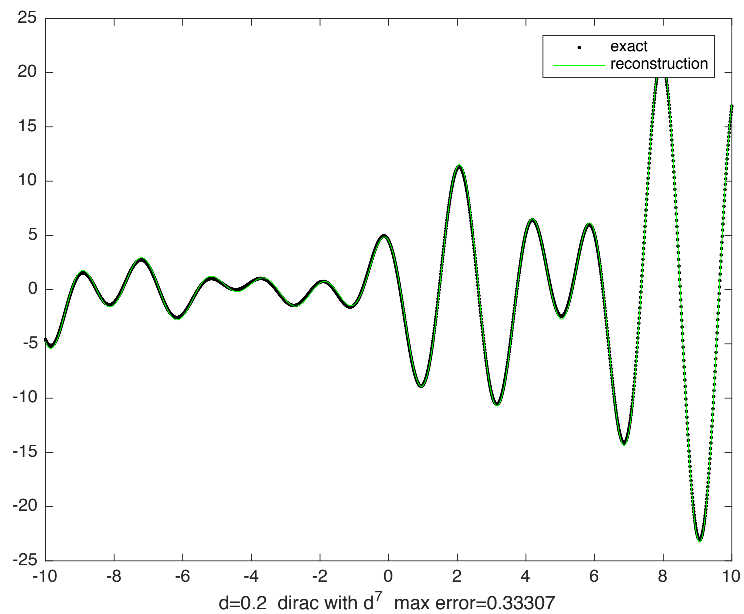

FIGURE 3.1: The graph of the function f of the polynomial and its reconstruction where the sampling points are within a quarter from the scaled integers $\left(\frac{\pi}{0.2+\pi}\right)\mathbb{Z}$.

4 FINITE POINTS OVERSAMPLING AND PERTURBING BEYOND A QUARTER

In this chapter, we will deduce a sampling series for a function that is band-limited in a distributional sense and has polynomial growth when restricted to the real line $\mathbb{R}$. The sampling set that we consider will be either a perturbed set from the integers or more generally from the set of zeros of a π-sine-type function. In the first section, we will consider a perturbed sampling set from zeros of a π-sine-type function. We will show that adding finite additional sampling points helps to obtain a reconstruction for those type of functions. This type of oversampling is not of ratio-type such as $\lambda\mathbb{Z}$ where $\lambda < 1$ in case of reconstructing a function with band-width π. It is an oversampling by adding a finite number of points that is determined with respect to the degree of the growth of the function to be reconstructed. In the second section, we will consider a perturbed set from the integers. We will show that adding one additional sampling point for the reconstruction allows the perturbation to go beyond a quarter.

4.1 Finite Points Oversampling

Our aim in this section to obtain a sampling series for the space $\widetilde{\mathcal{B}}_{\pi,N}$ with minimum oversampling. The sampling set is a perturbed sampling set either from the integers or more generally from zeros of a π-sine-type function. Let S_π denote the class of π-sine-type functions. For $f \in S_\pi$, Λ_f denotes the set of zeros of f. Let $\mathcal{S}$ denote the collection of all such Λ_f, i.e., $\mathcal{S} = \{\Lambda_f \mid f \in S_\pi\}$.

The desired perturbation that we will be studying is as follows: $\{\lambda_k\}_{k\in\mathbb{Z}}$ in $\mathcal{S}$ and $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ is any set satisfying

$$|\lambda_k^* - \lambda_k| < \delta \tag{4.1}$$

for all $k \in \mathbb{Z}$ and some $\delta > 0$.

K. Seip and J. R. Higgins, see [39, Theorem 2] and [18, Theorem 1], had results where they considered the space $\mathcal{B}^{\infty}_{\beta\pi}, 0 < \beta < 1$ (oversampling, ratio-type) where the perturbed sampling set is within a quarter from the integers (zeros of a π-sine-type function). Boche and Mönich, see [32, Theorem 4], had a sampling series for $\mathcal{B}^{\infty}_{\beta\pi}, 0 < \beta < 1$, where the sampling set is made of the zeros of a π-sine-type function. Moreover, Boche and Mönich [32, Theorem 2] had a result for the space $\mathcal{B}^{\infty}_{\pi,0} \subset \mathcal{B}^{\infty}_{\pi}$ where no oversampling required. But, the sampling set is zeros of a π-sine-type function. The result is stated as follows:

Theorem 4.1 ([32, Theorem 2])**.** *Let φ be a function of π-sine-type, whose zeros $\{\lambda_k\}_{k\in\mathbb{Z}}$ are all real and ordered increasingly. Furthermore, let φ_k be defined as in (1.4). Then, for all $T > 0$ and all $f \in \mathcal{B}^{\infty}_{\pi,0}$ we have*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(\lambda_k)\varphi_k(t) \right| = 0.$$

There is a progress where we have a perturbation from zeros of a π-sine-type function. It can be explained after the following result.

Theorem 4.2. (Levin-Ostrovskii, [28]) *If $\{\lambda_k\}_{k\in\mathbb{Z}}$ are zeros of a π-sine-type function and $\{d_k\}_{k\in\mathbb{Z}} \in l^p, 1 < p < \infty$, then*

$$\varphi(z) = \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{\lambda_k^*}\right)$$

where $\lambda_k^* = \lambda_k + d_k$, is a function of π-sine-type.

Theorem 4.2 allows a perturbation that leaves the function φ to be of a π-sine-type. If the set $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ is the zero set of a π-sine-type function, then the sampling series for $f \in \mathcal{B}^{\infty}_{\pi,0}$ by Boche and Mönich, Theorem 4.1, is applicable. However, this perturbation is different from the desired δ-perturbation, see (4.1).

With regard to $\lambda_k^* = \lambda_k + d_k$ where $|d_k| < \delta$, Boche and Mönich asked whether for sufficiently small δ the set $\{\lambda_k^*\}_{k\in\mathbb{Z}}$is again in $\mathcal{S}$. Obviously that will lead to the sampling series in Theorem 4.1 since $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ would then satisfy the hypothesis of their result. The question is not yet answered.

The sampling series by J. R. Higgins, see Theorem 2.1, of the functions in $\mathcal{B}_\pi^2$ does not directly apply to $\mathcal{B}_\pi^\infty$. To demonstrate this we give the following example

Example 4.1. *If* $f(t) = \sin(\pi t)$, *then* $f \in \mathcal{B}_\pi^\infty$. *If* $\{\lambda_k^*\}_{k\in\mathbb{Z}} = \mathbb{Z}$, *then* $f(k) = 0$ *and therefore*

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f(\lambda_k^*)\varphi_k(t) \right| \neq 0$$

when $t = 1/2$.

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $\Lambda^* = \{\lambda_k^*\}_{k\in\mathbb{Z}}$, and $\mu_1, \ldots, \mu_r$ be additional sampling points such that $\mu_i \neq \mu_j$ for $i \neq j$ and $\{\mu_1, \ldots, \mu_r\} \cap \Lambda^* = \emptyset$. With $\Lambda_\mu^* = \Lambda^* \cup \{\mu_1, \ldots, \mu_r\}$, we define $S_N f$ as

$$S_N f(t) = \sum_{\tilde{\lambda}_k \in \Lambda_\mu^*, |\tilde{\lambda}_k| \leq N} f\left(\tilde{\lambda}_k\right) G_k\left(t\right)$$

where the G_k's are suitable sampling functions. The main question is as follows:

Question 1. *Let* $\{\lambda_k\}_{k\in\mathbb{Z}} \in \mathcal{S}$. *Then, does there exist* $\delta > 0$ *such that given any sampling sequence* $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ *with* $|\lambda_k^* - \lambda_k| < \delta$ *for all* $k \in \mathbb{Z}$, *we have that*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} |f(t) - (S_N f)(t)| = 0$$

holds true for all $T > 0$ *and all* $f \in \mathcal{B}_\pi^\infty$*?*

Before answering the question we start by motivating a result that generalizes a result obtained by Gilbert G Walter [44]. Then we will try to obtain a result for a larger class of functions. That class of functions is the one defined in (3.2).

We will use a result we proved in Chapter 2, see Theorem 2.5. That result will enable us to consider sampling points that are complex (not necessarily real) instead of the result by Higgins, see Theorem 2.1, that uses only real sampling points. The proof is considerably easier than the one by Gilbert G Walter since we avoided using a result for decomposing a distribution by Schwartz. The result will be for a perturbation of the zeros of a π-sine-type function.

Let $\mu_1, \mu_2, \ldots, \mu_{N+1}$ be distinct and different from the sampling set $\{\lambda_k\}_{k\in\mathbb{Z}}$ and let f be in the class $\widetilde{\mathcal{B}}_{\pi,N}$. We define $\widetilde{\varphi}_N$ as

$$\widetilde{\varphi}_N(z) = \left(\prod_{i=1}^{N+1}(z-\mu_i)\right)\varphi(z).$$

Now, we define the function Q_{N+1} as follows: for $z \neq \mu_j$ let

$$\begin{aligned} Q_0(z) = f(z), \quad Q_1(z) &= \frac{Q_0(z) - Q_0(\mu_1)}{z-\mu_1}, \\ Q_2(z) = \frac{Q_1(z) - Q_1(\mu_2)}{z-\mu_2}, \ldots, \; Q_{N+1}(z) &= \frac{Q_N(z) - Q_N(\mu_{N+1})}{z-\mu_{N+1}} \end{aligned} \tag{4.2}$$

where $Q_j(z) = Q'_{j-1}(\mu_j)$ for $z = \mu_j$, $j = 1, \ldots, N+1$. By back substitution we can see that $Q_{N+1}(z)$ will require the values of f at $\mu_1, \mu_2, \ldots, \mu_{N+1}$. Furthermore, the function $Q_{N+1}(z) \in \mathcal{B}^2_\pi$ and thus we can apply Theorem 2.5 to obtain

$$Q_{N+1}(z) = \sum_{k=-\infty}^{\infty} Q_{N+1}(\lambda_k)\frac{\varphi(z)}{(z-\lambda_k)\varphi'(\lambda_k)}$$

which means that

$$\frac{Q_N(z) - Q_N(\mu_{N+1})}{(z-\mu_{N+1})} = \sum_{k=-\infty}^{\infty} Q_{N+1}(\lambda_k)\frac{\varphi(z)}{(z-\lambda_k)\varphi'(\lambda_k)}$$

$$Q_N(z) = Q_N(\mu_{N+1}) + \sum_{k=-\infty}^{\infty} Q_{N+1}(\lambda_k)\,(\lambda_k - \mu_{N+1})\,\frac{(z-\mu_{N+1})\,\varphi(z)}{(\lambda_k-\mu_{N+1})\,(z-\lambda_k)\,\varphi'(\lambda_k)}$$

If we continue with this back substitution we will end up with

$$f(z) = Q_0(\mu_1) + Q_1(\mu_2)\,(z-\mu_1) + \ldots + Q_N(\mu_{N+1})\,(z-\mu_1)\ldots(z-\mu_N)$$
$$+ \sum_{k=-\infty}^{\infty} Q_{N+1}(\lambda_k)\,(\lambda_k-\mu_1)\ldots(\lambda_k-\mu_{N+1})\,\frac{(z-\mu_1)\ldots(z-\mu_{N+1})\,\varphi(z)}{(\lambda_k-\mu_1)\ldots(\lambda_k-\mu_{N+1})\,(z-\lambda_k)\,\varphi'(\lambda_k)}.$$

By using the closed form of the product and the definition of $\widetilde{\varphi}_N$, the expression above can be written as

$$f(z) - Q_0(\mu_1) - \sum_{k=1}^{N} Q_k(\mu_{k+1}) \prod_{j=1}^{k} (z-\mu_j) = \tag{4.3}$$

$$\sum_{k=-\infty}^{\infty} \left(Q_{N+1}(\lambda_k) \prod_{j=1}^{N+1} (\lambda_k - \mu_j) \right) \frac{\widetilde{\varphi}_N(z)}{(z-\lambda_k)\,\widetilde{\varphi}'_N(\lambda_k)}. \tag{4.4}$$

We denote the polynomial in the left hand side

$$Q_0(\mu_1) + \sum_{k=1}^{N} Q_k(\mu_{k+1}) \prod_{j=1}^{k} (z-\mu_j)$$

by q_N. The polynomial q_N is a polynomial of degree N that is of Newton form. It interpolates the function f at the points $\mu_1, \ldots, \mu_{N+1}$. We compute the coefficients of the series in the right hand side of (4.4) by back substitution of (4.2) as follows

$$Q_{N+1}(z) = \frac{Q_N(z) - Q_N(\mu_{N+1})}{z-\mu_{N+1}} =$$
$$\frac{Q_{N-1}(z) - Q_{N-1}(\mu_N) - (z-\mu_N)\,Q_N(\mu_{N+1})}{(z-\mu_N)\,(z-\mu_{N+1})}$$
$$= \frac{Q_{N-2}(z) - Q_{N-2}(\mu_{N-1}) - Q_{N-1}(\mu_N)\,(z-\mu_{N-1}) - Q_N(\mu_{N+1})\,(z-\mu_{N-1})\,(z-\mu_N)}{(z-\mu_{N-1})\,(z-\mu_N)\,(z-\mu_{N+1})}$$
$$\vdots$$
$$= \frac{Q_0(z) - Q_0(\mu_1) - \sum_{k=1}^{N} Q_k(\mu_{k+1}) \prod_{j=1}^{k} (z-\mu_j)}{\prod_{j=1}^{N+1} (z-\mu_j)} = \frac{f(z) - q_N(z)}{\prod_{j=1}^{N+1} (z-\mu_j)}. \tag{4.5}$$

Thus,

$$Q_{N+1}(\lambda_k) \prod_{j=1}^{N+1} (\lambda_k - \mu_j) = Q_0(\lambda_k) - Q_0(\mu_1) - \sum_{k=1}^{N} Q_k(\mu_{k+1}) \prod_{j=1}^{k} (\lambda_k - \mu_j)$$

$$= f(\lambda_k) - Q_0(\mu_1) - \sum_{k=1}^{N} Q_k(\mu_{k+1}) \prod_{j=1}^{k} (\lambda_k - \mu_j)$$

which is the left hand side of (4.4) evaluated at λ_k. Therefore, the sampling series in (4.4) became

$$\begin{aligned} & f(z) - Q_0(\mu_1) - \sum_{k=1}^{N} Q_k(\mu_{k+1}) \prod_{j=1}^{k} (z - \mu_j) \\ &= \sum_{k=-\infty}^{\infty} \left(f(\lambda_k) - Q_0(\mu_1) - \sum_{k=1}^{N} Q_k(\mu_{k+1}) \prod_{j=1}^{k} (\lambda_k - \mu_j) \right) \frac{\widetilde{\varphi}_N(z)}{(z - \lambda_k)\, \widetilde{\varphi}_N'(\lambda_k)} \end{aligned} \tag{4.6}$$

The left hand side of (4.6) is the difference of function f and its corresponding polynomial (interpolating polynomial) of degree N. If $f(z)$ is a polynomial of degree N, then the interpolating polynomial in (4.6) is the function itself. So the left hand side of (4.6) is as $f(z) - p_N(z) = f(z) - f(z) = 0$ and the right hand side is also identically equal to zero. We will prove the following result.

Theorem 4.3. *Let $f \in \widetilde{\mathcal{B}}_{\pi,N}$ and let $\{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_0 \neq 0$ be a set of zeros of a π-sine-type function and let $\{\lambda_k^*\}_{k\in\mathbb{Z}}$, $\lambda_0^* \neq 0$ be a sequence of complex numbers satisfying*

$$|\mathrm{Re}\lambda_k^* - \mathrm{Re}\lambda_k| \leq dp, \ \sup_{k\in\mathbb{Z}} |\mathrm{Im}\lambda_k^* - \mathrm{Im}\lambda_k| < \infty$$

where $d < \frac{1}{4}$ and $p = \inf_k |\mathrm{Re}\lambda_k - \mathrm{Re}\lambda_{k+1}|$. Then,

$$f(z) - p_N(z) = \lim_{M\to\infty} \sum_{k=-M}^{M} \left(f\left(\lambda_k^*\right) - p_N(\lambda_k^*)\right) \left(\frac{z}{\lambda_k^*}\right)^{N+1} \frac{\varphi^*(z)}{(z - \lambda_k^*)\varphi'(\lambda_k^*)},$$

where p_N is the Taylor polynomial of f about zero of order N. Similarly,

$$f(z) - q_N(z) = \lim_{M\to\infty} \sum_{k=-M}^{M} \left(f\left(\lambda_k^*\right) - q_N(\lambda_k^*)\right) \prod_{j=1}^{N+1} \left(\frac{z - \mu_j}{\lambda_k^* - \mu_j}\right) \frac{\varphi^*(z)}{(z - \lambda_k^*)\varphi'(\lambda_k^*)},$$

where q_N is the interpolating polynomial of f at the points $\{\mu_j\}_{j=1}^{N+1}$. The convergence in both expansions is uniform over compact subsets of $\mathbb{C}$.

Proof. For the first part, let $f \in \widetilde{\mathcal{B}}_{\pi,N}$ and let $p_N(z)$ be the Taylor polynomial of f of order N about zero. Then,

$$h(t) = \frac{f(t) - p_N(t)}{\eta(t)} \in L^2(\mathbb{R}),$$

where $\eta(t) = t^{N+1}$, since f is bounded by a polynomial of degree N. The function $h(z)$ is an entire function of exponential type π and thus it is in the space $\mathcal{B}_\pi^2$. We apply Theorem 2.5 and the fact that $|.| \leq \sqrt{2\pi}\, \|.\|_2\, e^{\pi|\mathrm{Im} z|}$ for functions in the space $\mathcal{B}_\pi^2$. Now, let $\mathcal{C} \neq \{0\}$ be a compact subset of $\mathbb{C}$. Then for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\left| \frac{f(z) - p_N(z)}{\eta(z)} - \sum_{k=-N}^{N} \frac{f(\lambda_k^*) - p_N(\lambda_k^*)}{\eta(\lambda_k^*)} \frac{\varphi^*(z)}{(z - \lambda_k^*)\varphi'(\lambda_k^*)} \right| < \frac{\epsilon}{\sup_{z \in \mathcal{C}} |\eta(z)|}.$$

for all $z \in \mathcal{C}$. It follows that

$$\begin{aligned} &\left| f(z) - p_N(z) - \sum_{k=-N}^{N} \left(f(\lambda_k^*) - p_N(\lambda_k^*)\right) \frac{\eta(z)}{\eta(\lambda_k^*)} \frac{\varphi^*(z)}{(z - \lambda_k^*)\varphi'(\lambda_k^*)} \right| \\ &= |\eta(z)| \left| \frac{f(z) - p_N(z)}{\eta(z)} - \sum_{k=-N}^{N} \frac{f(\lambda_k^*) - p_N(\lambda_k^*)}{\eta(\lambda_k^*)} \frac{\varphi^*(z)}{(z - \lambda_k^*)\varphi'(\lambda_k^*)} \right| \\ &< |\eta(z)| \frac{\epsilon}{\sup_{z \in \mathcal{C}} |\eta(z)|} \\ &\leq \epsilon \end{aligned}$$

which shows that the convergence is uniform over compact subsets of $\mathbb{C}$. For the second part, we replace p_N by q_N and we let $\eta(z) = \prod_{j=1}^{N+1} (z - \mu_j)$. □

Remark 4.1. *The Taylor polynomial of order N will require $N+1$ pieces of information. They are the value of the function and its derivative, up to order N, at zero.*

The result by Gilbert G Walter [44] was for a uniform sampling and uniform convergence over compact subsets of $\mathbb{R}$. Our result, Theorem 4.3, provides a sampling series with nonuniform sampling (perturbed points from the zeros of a π-sine-type function) and uniform convergence over compact subsets of $\mathbb{C}$.

The answer to Question 1 now follows as a corollary

Corollary 4.1. *Let* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *be a set of zeros of a* π*-sine-type function and let* $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ *be a sequence of complex numbers satisfying*

$$|\mathrm{Re}\lambda_k^* - \mathrm{Re}\lambda_k| \le dp, \sup_{k\in\mathbb{Z}} |\mathrm{Im}\lambda_k^* - \mathrm{Im}\lambda_k| < \infty$$

where $d < \frac{1}{4}$ *and* $p = \inf_k |\mathrm{Re}\lambda_k - \mathrm{Re}\lambda_{k+1}|$*. Then, for* $f \in \mathcal{B}_\pi^\infty$ *and* $\mu_1 \ne \lambda_k^*$ *for all* $k \in \mathbb{Z}$ *we have*

$$f(z) = f(\mu_1) + \sum_{k\in\mathbb{Z}} \left(f(\lambda_k^*) - f(\mu_1)\right) \left(\frac{z-\mu_1}{\lambda_k^* - \mu_1}\right) \frac{\varphi^*(z)}{\left(z-\lambda_k^*\right) \varphi^{*\prime}(\lambda_k^*)}. \tag{4.7}$$

The convergence is uniform over compact subsets of $\mathbb{C}$.

This shows that having one additional piece of information makes the reconstruction of functions in the space $\mathcal{B}_\pi^2$ works for the space $\mathcal{B}_\pi^\infty$ where the convergence is uniform over compact subsets of $\mathbb{C}$. Thus, it is an oversampling by one point.

4.2 Perturbation Beyond a Quarter

In this section we will discuss the perturbation that goes beyond a quarter from the integers. The problem will be addressed by oversampling by including one additional point.

The growth of the π-sine-type function was an important step to obtain the sampling series in Theorem 4.1. Hence, our strategy will be to study the growth of the canonical product φ with the perturbed integers (the integers is one example of zeros of a π-sine-type function) to deduce a sampling series of Lagrange-type. We do not have any result about the growth of the canonical product with a perturbation of the zeros of sine-type function in the sense of δ-perturbation (4.1).

In the following, we present an estimate for the canonical product by Hinsen, see [19].

Theorem 4.4. [19, Proposition 3.1] *Let* $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, *with* $\lambda_0 = 0$, *be such that* $|\lambda_k - k| \leq D < \frac{1}{2}$. *Then there are positive constants* C_1, C_2 *such that for all* $z = x + iy \in \mathbb{C}$ *with* $|z|$ *sufficiently large*

$$C_1 H_1(z) H_2(D;z) \leq |\varphi(z)| \leq C_2 H_1(z) H_2(-D;z), \tag{4.8}$$

where

$$H_1(z) = e^{\pi|y|} \begin{cases} 1, & |\mathrm{Im}(z)| > 1 \\ \prod_{k=M}^{M+2} |\lambda_k - z|, & |\mathrm{Im}(z)| \leq 1 \text{ and } \mathrm{Re}\,(z) > 0 \\ \prod_{k=-M-2}^{-M} |\lambda_k - z|, & |\mathrm{Im}\,(z)| \leq 1 \text{ and } \mathrm{Re}\,(z) < 0 \end{cases} \tag{4.9}$$

$$H_2\,(D;z) = \begin{cases} |z|^{-4D}\,, & 0 \leq |\sin\theta| \leq \sin\left(\frac{\pi}{2|z|}\right) \\ |z|^{-2D}\,|\sin\theta|^{2D}\,, & \sin\left(\frac{\pi}{2|z|}\right) < |\sin\theta| \leq 1 \end{cases} \tag{4.10}$$

$M = M\,(z)$ *is a suitable index to be defined below. In particular, for* $\theta = Arg(z) \in [0, \pi/2)$, *and for sufficiently large* $|z|$, M *is given by*

$$M := M(z) = max\{n \in \mathbb{N} : n + D \leq |z|\,/\cos\theta\} = \lfloor |z|\,/\cos\theta - D \rfloor\,. \tag{4.11}$$

The second case of $H_2\,(D;z)$ in (4.10) can be bounded below by $c\,|z|^{-4D}$ for some $c > 0$ and can be used in lower bound of (4.8). It is stated and proved in the following lemma.

Lemma 4.1. *Let* $0 < \theta < \pi/2$. *Then, for sufficiently large* $|z|$, $H_2\,(D;z) \geq \left(\frac{1}{2}\right)^{2D} |z|^{-4D}$.

Proof. We will study the second case in (4.10). Using Taylor expansion we can have the following inequality

$$\sin\theta \geq \theta - \frac{\theta^3}{6},\, 0 < \theta < \frac{\pi}{2}.$$

The right hand side is positive for $0 < \theta < \pi/2$. But, we have $\sin\,\theta > \sin\left(\frac{\pi}{2|z|}\right)$ and thus

$$\sin\,\theta > \frac{\pi}{2\,|z|} - \frac{\pi^3}{48\,|z|^3} > \frac{1}{2\,|z|}$$

for sufficiently large $|z|$. Therefore, for a sufficiently large $|z|$, we obtain

$$|z|^{-2D} |\sin\theta|^{2D} > \left(\frac{1}{2}\right)^{2D} |z|^{-4D}.$$

□

After this result, we can look at $H_2(D; z)$ as one case for simplicity. In the next result, we will show that M defined in (4.11) has one value for sufficiently large $|z|$ as $\mathrm{Im}z \in [0, 1]$.

Lemma 4.2. *Let $\{\lambda_n\}$ as in Theorem 4.4, $x_n = (\lambda_n + \lambda_{n+1})/2$, $y \in [0, 1]$, $z = x_n + iy$ and $M(z)$ as defined in (4.11). Then, for sufficiently large n we obtain $M(z) = n$.*

Proof. We have $0 \le \theta \le \tan^{-1}\left(\frac{1}{x_n}\right)$ and thus

$$\begin{aligned} \lfloor |z| - D \rfloor \le & M \le \left\lfloor \frac{|z|}{\cos\left(\tan^{-1}\left(\frac{1}{x_n}\right)\right)} - D \right\rfloor \\ \left\lfloor \sqrt{x_n^2 + y^2} - D \right\rfloor \le & M \le \left\lfloor \frac{\sqrt{x_n^2 + y^2}}{x_n / \sqrt{1 + x_n^2}} - D \right\rfloor \\ \lfloor x_n - D \rfloor \le & M \le \left\lfloor x_n + \frac{1}{x_n} - D \right\rfloor \end{aligned} \tag{4.12}$$

Now, we have

$$-D < x_n - \left(n + \frac{1}{2}\right) < D$$

which implies

$$\left(n + \frac{1}{2}\right) - 2D < x_n - D < \left(n + \frac{1}{2}\right).$$

But, $D < \frac{1}{2}$ and that leads to

$$n < x_n - D < \left(n + \frac{1}{2}\right). \tag{4.13}$$

For sufficiently large n we also have

$$n < x_n + \frac{1}{x_n} - D < \left(n + \frac{1}{2}\right). \tag{4.14}$$

It follows from (4.13) and (4.14) that $\lfloor x_n - D \rfloor = n = \left\lfloor x_n + \frac{1}{x_n} - D \right\rfloor$. Substituting in (4.12) we obtain that $M = n$. $\square$

For the purpose of perturbation that goes beyond a quarter we define $\widetilde{\psi}_N$ and Ψ_k^N as

$$\widetilde{\psi}_N(z) = \left(\prod_{i=1}^{N+2} (z - \mu_i)\right) \varphi(z), \tag{4.15}$$

$$\Psi_k^N(z) = \frac{\widetilde{\psi}_N(z)}{(z - \lambda_k)\, \widetilde{\psi}'_N(\lambda_k)}. \tag{4.16}$$

Those functions satisfy the condition $\Psi_k^N(\lambda_l) = \delta_{kl}$.

In (4.16) we have $\widetilde{\psi}'_N(\lambda_k)$ in the denominator. This can be a singularity. The next result treats this problem.

Theorem 4.5. *If $\{\lambda_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}$, is a set satisfying $|\lambda_k - k| \leq D < \frac{1}{2}$, and $\mu_j \neq \lambda_k$ for all $j = 1, \ldots, N+2$ and $k \in \mathbb{Z}$, then $\inf_{k\in\mathbb{Z}} \left|\widetilde{\psi}'_N(\lambda_k)\right| \neq 0$.*

Proof. We will show that the sequence $\left\{\left|\widetilde{\psi}'_N(\lambda_k)\right|\right\}_{\lambda_k\in\Lambda}$ is bounded away from zero. The zeros Λ are simple zeros for $\widetilde{\psi}_N(z)$ and so $\left|\widetilde{\psi}'_N(z)\right| > 0$ at those points. The concern is that $\left|\widetilde{\psi}'_N(\lambda_k)\right|$ becomes arbitrary small as λ_m becomes arbitrary large. Now, let λ_m be sufficiently large, $\underline{\delta} = \inf_k |\lambda_{k+1} - \lambda_k|$ and $|z - \lambda_m| = \delta$ where $\delta < \underline{\delta}/2$. Then,

$$\begin{aligned} |z - \mu_j| &= |z - \lambda_m + \lambda_m - \mu_j| \\ &\geq ||z - \lambda_m| - |\lambda_m - \mu_j|| \\ &= |\lambda_m - \mu_j| - \delta \\ &> \frac{1}{2} |\lambda_m - \mu_j| \end{aligned} \tag{4.17}$$

Now, we use Hinsen estimate, see Theorem 4.4, and the above inequality as follows

$$\left|\widetilde{\psi}_N(z)\right| = \left(\prod_{j=1}^{N+2} |z - \mu_j|\right) |\varphi(z)|$$

$$\geq C_1 \prod_{j=1}^{N+2} |z-\mu_j| \left(\prod_{k=M}^{M+2} |\lambda_k - z| \right) e^{\pi|y|} |z|^{-4D}$$
$$> C_1 \prod_{j=1}^{N+2} |z-\mu_j| \left(\prod_{k=M}^{M+2} |\lambda_k - z| \right) |z|^{-2}$$
$$\geq C_1 \prod_{j=1}^{N+2} |z-\mu_j| \left(\prod_{k=M}^{M+2} |\lambda_k - z| \right) |z|^{-N-2}. \tag{4.18}$$

The quantity $|z|$ can be bounded as $|z| = |z - \lambda_m + \lambda_m| \leq |z - \lambda_m| + |\lambda_m| = \delta + |\lambda_m| < 2|\lambda_m|$. Now, with use of Maximal Modulus Principle and (4.18), we obtain

$$\min_{|z-\lambda_m|=\delta_n} \left| \frac{\widetilde{\psi}_N(z)}{z-\lambda_m} \right| \geq \min_{|z-\lambda_m|=\delta} \left| \frac{\widetilde{\psi}_N(z)}{z-\lambda_m} \right|$$
$$> C_1 \min_{|z-\lambda_m|=\delta} \frac{\prod_{k=M}^{M+2} |\lambda_k - z|}{(4)^{N+2}\delta} \frac{\prod_{j=1}^{N+2} |\lambda_m - \mu_j|}{|\lambda_m|^{N+2}}$$
$$= C_1 \min_{|z-\lambda_m|=\delta} \frac{\prod_{k=M}^{M+2} |\lambda_k - z|}{(4)^{N+2}\delta} \prod_{j=1}^{N+2} \left| 1 - \left| \frac{\mu_j}{\lambda_m} \right| \right|$$

where $\left| \frac{\mu_j}{\lambda_m} \right| < \frac{1}{2}$ for sufficiently large λ_m. Also, for large λ_m we have $M = m$ by Lemma 4.2 and so

$$\frac{\prod_{k=M}^{M+2} |\lambda_k - z|}{\delta} = \prod_{k=m+1}^{m+2} |\lambda_k - z|. \tag{4.19}$$

Now using the fact that $\delta < \underline{\delta}/2$. We can bound the quantity in (4.19) as

$$\min_{|z-\lambda_m|=\delta} \prod_{k=m+1}^{m+2} |\lambda_k - z| = \min_{|z-\lambda_m|=\delta} |\lambda_{m+1} - z| \, |\lambda_{m+2} - z|$$
$$= \min_{|z-\lambda_m|=\delta} |\lambda_{m+1} - \lambda_m + \lambda_m - z| \, |\lambda_{m+2} - \lambda_m + \lambda_m - z|$$
$$\geq \min_{|z-\lambda_m|=\delta} ||\lambda_{m+1} - \lambda_m| - |z - \lambda_m|| \, ||\lambda_{m+2} - \lambda_m| - |z - \lambda_m||$$
$$> (\delta_m - \underline{\delta}/2)(\delta_m - \underline{\delta}/2) = (\delta_m - \underline{\delta}/2)^2 > (\underline{\delta}/2)^2 > 0,$$

where $\delta_m = |\lambda_{m+1} - \lambda_m|$. Now, by letting $\delta_n \to 0$, we obtain that

$$\left| \widetilde{\psi}'_N(\lambda_m) \right| > C_1 \left(\frac{1}{8} \right)^{N+2} (\underline{\delta}/2)^2 > 0$$

and therefore, $\inf_m \left|\widetilde{\psi}'_N(\lambda_m)\right| \neq 0.$ □

The following theorem explains the growth of functions of exponential type that has a polynomial growth on the real line. It is a corollary for the Theorem B.1.

Theorem 4.6. (Phragmén-Lindelöf, [27, page 39]) *If $f(z)$ is an entire function of exponential type π, and*

$$|f\,(x)| \leq m\,(1+|x|)^N\,, x \in \mathbb{R}, N \in \mathbb{N}$$

then

$$|f(z)| \leq C\,(1+|z|)^N\, e^{\pi|y|}$$

for all z in the complex plane $\mathbb{C}$.

One method we will use in this section is the ***Residue Theorem***. The Residue Theorem is applicable for arbitrary finite number of singularities. We will be using a sequence of contours that contains arbitrary large number of sampling points. Accordingly, we should be concerned about the convergence of the infinite series. But, having the difference, between the function that we want to represent and the sequence of the partial sum, goes to zero shows the convergence of the infinite series.

The contour Γ_n that will be used in the next theorem is defined as $\Gamma_n = \eta_1 \cup \eta_2 \cup \gamma_1 \cup \gamma_2$ where

$$\eta_1 = \{x+iy_n \mid x_{-n} = \mathrm{Re}(\lambda_{-n}+\lambda_{-n-1})/2 \leq x \leq x_n = \mathrm{Re}(\lambda_n+\lambda_{n+1})/2,\, y_n = x_n\}\,,$$

$$\eta_2 = \{x-iy_n \mid x_{-n} = \mathrm{Re}(\lambda_{-n}+\lambda_{-n-1})/2 \leq x \leq x_n = \mathrm{Re}(\lambda_n+\lambda_{n+1})/2,\, y_n = x_n\}\,,$$

$$\gamma_1 = \{x_n+iy \mid x_n = \mathrm{Re}(\lambda_n+\lambda_{n+1})/2,\, \mid y \mid < x_n\}$$

and,

$$\gamma_2 = \{x_{-n}+iy \mid x_{-n} = \mathrm{Re}(\lambda_{-n}+\lambda_{-n-1})/2,\, \mid y \mid < x_n\} \tag{4.20}$$

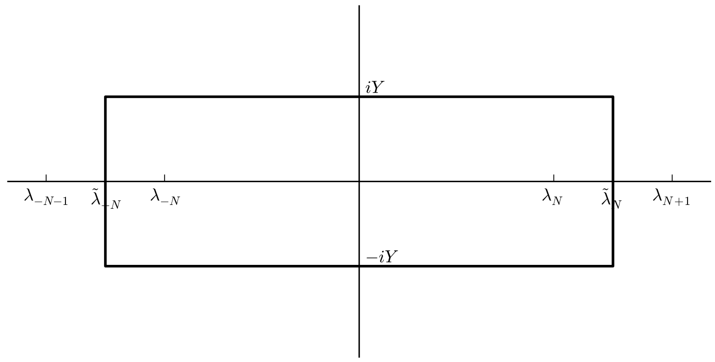

FIGURE 4.1: The contour for the contour integral method.

and can be depicted as in Figure 4.1.

The next theorem is the main result of this section.

Theorem 4.7. *Let* $f \in \widetilde{\mathcal{B}}_{\pi,N}$ *and* $\{\lambda_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}$ *be a set satisfying* $|\lambda_k - k| \leq D < \frac{1}{2}$, $\mu_j \neq \lambda_k$ *for all* $j = 1, \ldots, N+2$ *and* $k \in \mathbb{Z}$. *Then, we have the following expansion*

$$f(z) = \sum_{j=1}^{N+2} f(\mu_j) \frac{\widetilde{\psi}_N(z)}{(z-\mu_j)\,\widetilde{\psi}'_N(\mu_j)} + \sum_{k\in\mathbb{Z}} f(\lambda_k)\Psi_k^N(z) \tag{4.21}$$

where $\widetilde{\psi}_N$ *and* Ψ_k^N *as in (4.15) and (4.16). The convergence is uniform over compact subsets of* $\mathbb{C}$.

This theorem is another generalization of Higgins theorem in the sense that we reconstruct functions in the space $\widetilde{\mathcal{B}}_{\pi,N}$ and the λ_k's are allowed to go beyond the quarter from the integers. Gilbert G Walter did an extensive study for this class of functions. In [44], he obtained a sampling expansion that includes a correction by a polynomial where the sampling set is uniform. While in [45], he derived a sampling expansion with nonuniform sampling points that satisfy the three conditions: (i) $\lambda_k = -\lambda_{-k}, k = 0, \pm1, \pm2, \ldots$, (ii) $\sup_k |\text{Re}\lambda_k - k| < \frac{1}{4}$, (iii) $|\text{Im}\lambda_k| \leq C < \infty$.

Proof. Let $z \in \mathcal{C} \subset \mathbb{C}$ be a compact set and let n be sufficiently large such that $\mathcal{C}$ and

$\{\mu_j\}_{j=1}^{N+2}$ are inside the contour Γ_n defined in (4.20). Then, for any entire function we use Cauchy Integral Formula to obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{f(w)}{(w-z)}dw \\ &= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{[\widetilde{\psi}_N(w)-\widetilde{\psi}_N(z)+\widetilde{\psi}_N(z)]}{(w-z)}\frac{f(w)}{\widetilde{\psi}_N(w)}dw \\ &= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{[\widetilde{\psi}_N(w)-\widetilde{\psi}_N(z)]}{(w-z)}\frac{f(w)}{\widetilde{\psi}_N(w)}dw + \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widetilde{\psi}_N(z)}{(w-z)}\frac{f(w)}{\widetilde{\psi}_N(w)}dw \end{aligned} \tag{4.22}$$

For the first integral in (4.22) we will use residue theorem with the contour Γ_n. It follows that

$$\mathrm{Res}\left(\frac{[\widetilde{\psi}_N(w)-\widetilde{\psi}_N(z)]}{(w-z)}\frac{f(w)}{\widetilde{\psi}_N(w)},\lambda_k\right) = \frac{\widetilde{\psi}_N(z)f(\lambda_k)}{(z-\lambda_k)\,\widetilde{\psi}'_N(\lambda_k)},$$

for all λ_k's that are inside the contour Γ_n. Similarly,

$$\mathrm{Res}\left(\frac{[\widetilde{\psi}_N(w)-\widetilde{\psi}_N(z)]}{(w-z)}\frac{f(w)}{\widetilde{\psi}_N(w)},\mu_j\right) = \frac{\widetilde{\psi}_N(z)f(\mu_j)}{(z-\mu_j)\,\widetilde{\psi}'_N(\mu_j)},$$

where $j = 1,\ldots,N+2$. Thus,

$$f(z) = \sum_{j=1}^{N+2}\frac{\widetilde{\psi}_N(z)f(\mu_j)}{(z-\mu_j)\,\widetilde{\psi}'_N(\mu_j)} + \sum_{k=-n}^{n} f(\lambda_k)\frac{\widetilde{\psi}_N(z)}{(z-\lambda_k)\,\widetilde{\psi}'_N(\lambda_k)} + E_n(z) \tag{4.23}$$

where

$$E_n(z) = \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widetilde{\psi}_N(z)}{(w-z)}\frac{f(w)}{\widetilde{\psi}_N(w)}dw.$$

We will compute the error in (4.23) for the region $\mathrm{Re}(w)\geq 0$ and $\mathrm{Im}(w)\geq 0$ as n goes to infinity. The computation for the other regions follow the same way because of the relations (A.2) and (A.3). Now we estimate the integral over γ_1. It follows that $\prod_{j=1}^{N+2}|x_n-\mathrm{Re}\mu_j| = \prod_{j=1}^{N+2}|\mathrm{Re}\,(w-\mu)| \leq \prod_{j=1}^{N+2}|w-\mu_j|$ and $|x_n-\mathrm{Re}z| = |\mathrm{Re}\,(w-z)| \leq |w-z|$, where $w = x+iy$. By applying Phragmén-Lindelöf to f, see Theorem 4.6, the integral can be estimated as

$$\left|\oint_{\gamma_1}\frac{\widetilde{\psi}_N(z)}{(w-z)}\frac{f(w)}{\widetilde{\psi}_N(w)}dw\right| \leq C\max_{z\in\mathcal{C}}\left|\widetilde{\psi}_N(z)\right|\int_0^{x_n}\frac{(1+|x_n+iy|)^N e^{\pi|y|}}{|x_n-\mathrm{Re}z|\left|\widetilde{\psi}_N(x_n+iy)\right|}dy.$$

If we use the bounds (4.9) and (4.10) in Theorem 4.4 and Lemma 4.1, then for sufficiently large n, the function $\left|\widetilde{\psi}_N(w)\right|$ will be bounded below by

$$C_1 \left(\frac{1}{2}\right)^{2D} \prod_{j=1}^{N+2} |w-\mu_i| \, |w|^{-4D} \, e^{\pi|y|} \begin{cases} \prod_{k=M}^{M+2} |w-\lambda_k| \,, & |\text{Im}(w)| \le 1 \\ 1, & |\text{Im}(w)| > 1 \end{cases}$$

and thus the integral splits into two integrals as follows

$$\left| \oint_{\gamma_1} \frac{\widetilde{\psi}_N(z)}{(w-z)} \frac{f(w)}{\widetilde{\psi}_N(w)} dw \right|$$

$$\le (2)^{2D} C \max_{z\in\mathcal{C}} \left|\widetilde{\psi}_N(z)\right| \int_{[0,1]} \frac{(1+|x_n+iy|)^N \, |x_n+iy|^{4D} \, e^{\pi|y|}}{|x_n - \text{Re}z| \prod_{j=1}^{N+2} |x_n+iy-\mu_j| \prod_{k=M}^{M+2} |x_n+iy-\lambda_k| \, e^{\pi|y|}} dy$$
$$+ (2)^{2D} C \max_{z\in\mathcal{C}} \left|\widetilde{\psi}_N(z)\right| \int_{[1,x_n]} \frac{(1+|x_n+iy|)^N \, |x_n+iy|^{4D} \, e^{\pi|y|}}{|x_n - \text{Re}z| \prod_{j=1}^{N+2} |x_n+iy-\mu_j| \, e^{\pi|y|}} dy.$$

Again, we have $\prod_{k=N}^{N+2} |x_n-\lambda_k| \le \prod_{k=N}^{N+2} |x_n+iy-\lambda_k|$ and so

$$\left| \oint_{\gamma_1} \frac{\widetilde{\psi}_N(z)}{(w-z)} \frac{f(w)}{\widetilde{\psi}_N(w)} dw \right|$$

$$\le (2)^{2D} C \max_{z\in\mathcal{C}} \left|\widetilde{\psi}_N(z)\right| \int_{[0,1]} \frac{(1+|x_n+iy|)^N \, |x_n+iy|^{4D}}{|x_n - \text{Re}z| \prod_{j=1}^{N+2} |x_n-\text{Re}\mu_j| \prod_{k=M}^{M+2} |x_n-\text{Re}\lambda_k|} dy$$
$$+ (2)^{2D} C \max_{z\in\mathcal{C}} \left|\widetilde{\psi}_N(z)\right| \int_{[1,x_n]} \frac{(1+|x_n+iy|)^N \, |x_n+iy|^{4D}}{|x_n - \text{Re}z| \prod_{j=1}^{N+2} |x_n-\text{Re}\mu_j|} dy$$
$$\le \frac{(2)^{2D} C \, \max_{z\in\mathcal{C}} \left|\widetilde{\psi}_N(z)\right| \max_{y\in[0,1]} \left\{(1+|x_n+iy|)^N \, |x_n+iy|^{4D}\right\}}{|x_n - \text{Re}z| \prod_{j=1}^{N+2} |x_n-\text{Re}\mu_j| \prod_{k=M}^{M+2} |x_n-\text{Re}\lambda_k|}$$
$$+ \frac{(2)^{2D} C \, \max_{z\in\mathcal{C}} \left|\widetilde{\psi}_N(z)\right| \max_{y\in[1,x_n]} \left\{(1+|x_n+iy|)^N \, |x_n+iy|^{4D}\right\} |x_n-1|}{|x_n - \text{Re}z| \prod_{j=1}^{N+2} |x_n-\text{Re}\mu_j|}$$
$$\le \frac{3^N \, (2)^{4D} \, C \max_{z\in\mathcal{C}} \left|\widetilde{\psi}_N(z)\right| |x_n|^{N+4D}}{|x_n - \text{Re}z| \prod_{j=1}^{N+2} |x_n-\text{Re}\mu_j| \prod_{k=M}^{M+2} |x_n-\text{Re}\lambda_k|}$$
$$+ \frac{3^N \, (2)^{4D} \, C \max_{z\in\mathcal{C}} \left|\widetilde{\psi}_N(z)\right| |x_n|^{N+4D} \, |x_n-1|}{C_1 \, |x_n - \text{Re}z| \prod_{j=1}^{N+2} |x_n-\text{Re}\mu_j|}$$

$$= O\left(\frac{1}{|x_n|^{2-4D}}\right).$$

The right hand side goes to zero as n goes to infinity.

The other integral will be over the path η_1, see (4.20). It follows that $\prod_{j=1}^{N}|y_n - \mathrm{Im}\mu_j| = \prod_{j=1}^{N}|\mathrm{Im}\,(w-\mu_j)| \le \prod_{j=1}^{N}|w-\mu_j|$ and $|y_n - \mathrm{Im}z| = |\mathrm{Im}\,(w-z)| \le |w-z|$. If we use Phragmén-Lindelöf, Theorem 4.6, Theorem 4.4 and Lemma 4.1, then for sufficiently large n we have the function $\left|\widetilde{\psi}_N(w)\right|$ bounded below by $C_1\left(\frac{1}{2}\right)^{2D}\prod_{j=1}^{N+2}|w-\mu_j|\,|w|^{-4D}\,e^{\pi|y|}$ and thus we obtain

$$\left|\oint_{\eta_1}\frac{\widetilde{\psi}_N(z)}{(w-z)}\frac{f(w)}{\widetilde{\psi}_N(w)}dw\right| \le C\max_{z\in\mathcal{C}}\left|\widetilde{\psi}_N(z)\right|\int_{[0,x_n]}\frac{(1+|x+iy_n|)^N\,e^{\pi|y_n|}}{|x+iy_n-z|\left|\widetilde{\psi}_N(x+iy_n)\right|}dx$$

$$\le (2)^{2D}\,C\max_{z\in\mathcal{C}}\left|\widetilde{\psi}_N(z)\right|\int_{[0,x_n]}\frac{(1+|x+iy_n|)^N\,|x+iy_n|^{4D}}{|y_n-\mathrm{Im}z|\prod_{j=1}^{N+2}|y_n-\mathrm{Im}\mu_j|}dx$$

$$\le \frac{(2)^{2D}\,C\max_{z\in\mathcal{C}}\left|\widetilde{\psi}_N(z)\right|\max_{x\in[0,x_n]}\left\{(1+|x+iy_n|)^N\,|x+iy_n|^{4D}\right\}|x_n|}{|x_n-\mathrm{Im}z|\prod_{j=1}^{N+2}|x_n-\mathrm{Im}\mu_j|}$$

$$\le \frac{3^N\,(2)^{4D}\,C\max_{z\in\mathcal{C}}\left|\widetilde{\psi}_N(z)\right|\,|x_n|^{4D+N+1}}{|x_n-\mathrm{Im}z|\prod_{j=1}^{N+2}|x_n-\mathrm{Im}\mu_j|}$$

$$= O\left(\frac{1}{|x_n|^{2-4D}}\right).$$

The right hand side goes to zero as n goes to infinity. This completes the proof. □

Remark 4.2. *(1) In Corollary 4.1, we can see that we obtain a reconstruction for functions in the space $\mathcal{B}_\pi^\infty$, with a perturbation within a quarter, by oversampling by one point. While, K. Seip [39] had a ratio-type oversampling for the same class of functions.*

(2) In Theorem 4.7, if we consider the class $\mathcal{B}_\pi^\infty$, then we can notice that the two points oversampling allows the perturbation to go beyond a quarter from the integers.

4.3 The Stability Over Compact Sets

The stability that is due to small error in the function values input should follow easily since the convergence is over compact sets. It can be shown as follows:

Let

$$\Psi_k^N(z) = \frac{\widetilde{\psi}(z)}{(z-\lambda_k)\,\widetilde{\psi}'(\lambda_k)}$$

where Λ is the sampling set after re-indexing and let $\mathcal{C}$ be a compact set. Let $\{\lambda_k\}_{k\in S}$ be such that λ_k's are in $\mathcal{C}\subset\mathbb{C}$. The set S can be chosen to be of symmetric index as $\{k\in\mathbb{Z}\mid \lambda_k \text{ or } \lambda_{-k} \text{ belongs to } \mathcal{C}\}$. Now,

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}}\alpha(\lambda_k)\,\Psi_k^N(x)\right| &\le \sum_{k\in\mathbb{Z}}|\alpha(\lambda_k)|\left|\Psi_k^N(x)\right| \\ &\le \sup_{k\in\mathbb{Z}}|\alpha(\lambda_k)|\sum_{k\in\mathbb{Z}}\left|\Psi_k^N(x)\right| \\ &\le \sup_{k\in\mathbb{Z}}|\alpha(\lambda_k)|\left(\sup_{x\in\mathcal{C}}\sum_{k\in S}\left|\Psi_k^N(x)\right| + \sum_{k\in\mathbb{Z}\setminus S}\left|\Psi_k^N(x)\right|\right)\end{aligned}$$

Now we use the fact that we are studying convergence over the compact set $\mathcal{C}$ and thus there exists ϵ_1 and ϵ_2 such that $|x-\lambda_k|\ge\epsilon_1|k|$ and $|\lambda_k-\mu_i|\ge\epsilon_2|k|$ for all $k\in\mathbb{Z}\setminus S$ and all $i=1,\ldots,N$. We obtain

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}}\alpha(\lambda_k)\,\Psi_k^N(x)\right| &\le \sup_{k\in\mathbb{Z}}|\alpha(\lambda_k)|\left(\sup\nolimits_{x\in\mathcal{C}}\sum_{k\in S}\left|\Psi_k^N(x)\right| + \sum_{k\in\mathbb{Z}\setminus S}\left|\Psi_k^N(x)\right|\right) \\ &\le \sup_{k\in\mathbb{Z}}|\alpha(\lambda_k)|\left(C_1+\frac{C_2}{\epsilon_1\epsilon_2}\sup_{x\in\mathcal{C}}\left|\widetilde{\psi}_k(x))\right|\sum_{k\in\mathbb{Z}\setminus S}\frac{1}{k^2}\right) \\ &\le M(\mathcal{C})\sup_{k\in\mathbb{Z}}|\alpha(\lambda_k)|\end{aligned}$$

which shows that $\left\|\sum_{k\in\mathbb{Z}}\alpha(\lambda_k)\,\Psi_k^N(.)\right\|_\infty$ is small whenever $\sup_{k\in\mathbb{Z}}|\alpha(\lambda_k)|$ is sufficiently small.

4.4 Numerical Example

In the following we have the plot of the function

$$f(x) = \cos(\pi x) - \left(\frac{3}{4}\right)\sin\left(\frac{\pi}{\sqrt{2}}x\right)$$

that is in the space $\mathcal{B}_\pi^\infty$. The sampling set is a perturbed set from the integers within $\frac{1}{2}$. The number of terms considered is $2N+2$ for 1-point oversampling and $2N+3$ for 2-point oversampling, see the expansion in Theorem 4.7. The additional points are $\mu_1 = 0.5$ and $\mu_2 = 1.5$ for the 2-points oversampling, and only μ_1 in case of 1-point oversampling. For 1-point oversampling we used the expansion (4.7) where the sampling set is the integers and no perturbation is considered. While the perturbation is beyond a quarter for the 2-point oversampling and follows the expansion (4.21). Also, we have a reconstruction with the sampling set $0.7\mathbb{Z}$ that is of a ratio type. The following table demonstrates the truncation error among the number of terms considered over different intervals.

	$N=20$	$N=200$	$N=400$
1-point, $\|e\|_{L^\infty[-10,10]}$	0.3281	0.0318	0.01602
$0.7\mathbb{Z}$, $\|e\|_{L^\infty[-10,10]}$	0.0261	0.0013	5.9527×10^{-4}
2-point, $\|e\|_{L^\infty[-10,10]}$	0.0258	2.444×10^{-5}	3.1543×10^{-6}
1-point, $\|e\|_{L^\infty[-20,20]}$	1.2984	0.0637	0.0318
$0.7\mathbb{Z}$, $\|e\|_{L^\infty[-20,20]}$	1.7181	0.0014	6.2729×10^{-4}
2-point, $\|e\|_{L^\infty[-20,20]}$	0.6008	1.907×10^{-4}	2.6021×10^{-5}

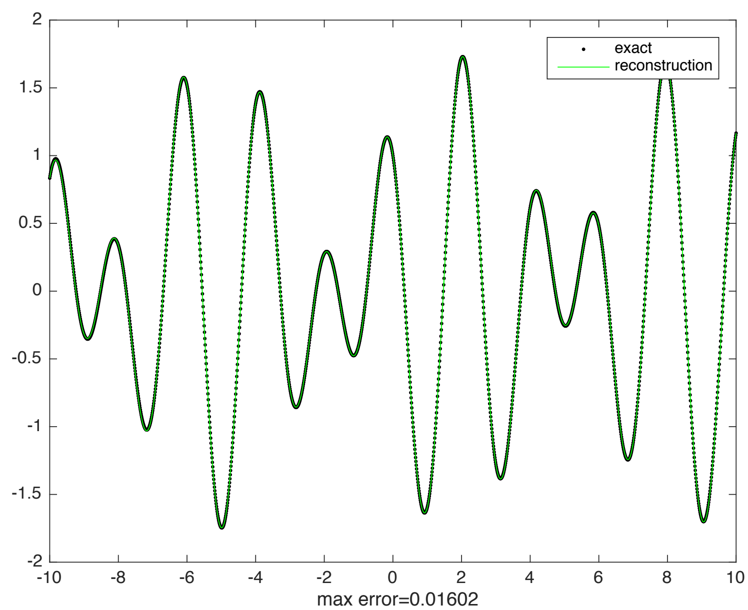

FIGURE 4.2: The graph of the function and its reconstruction with 1-point oversampling

5 QUANTIZATION

In this chapter, we will control the reconstruction error of function when the perturbation is not only in the sampling set, but in the function values as well. The perturbation in the function values occurs as round-off in the sampled function values. This kind of problem occurs in quantization processes and can be found in digital signal processing, see [2], [7], [15], [21] and [10].

The quantization operator Υ_δ is defined as

$$\Upsilon_\delta(f) = \left\lfloor \frac{\mathrm{Re} f}{2\delta} + \frac{1}{2} \right\rfloor 2\delta + \left\lfloor \frac{\mathrm{Im} f}{2\delta} + \frac{1}{2} \right\rfloor 2\delta i. \tag{5.1}$$

In [6, Corollary 2], Boche and Mönich showed that the error of the reconstruction can grow arbitrarily large for functions in the space $\mathcal{PW}_\pi^1$ where they considered the WKS sampling series together with the quantization operator of step size ($0 < \delta < \frac{1}{\pi}$), see [33, Corollary 6.12].

Throughout this chapter, we will consider a uniform discrete sampling set $\{\lambda_n\}_{n\in\mathbb{Z}}$. In the next result the sampling set is the integers.

Theorem 5.1. [33, Corollary 6.12] *Let* $0 < \delta < 1/\pi$ *and let*

$$\left(B_\delta f\right)(t) = \sum_{n=-\infty}^{\infty} \left(\Upsilon_\delta f\right)(n) \frac{\sin\left(\pi\left(t-n\right)\right)}{\pi\left(t-n\right)}.$$

Then,

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \le 1} \left\|f - B_\delta f\right\|_\infty = \infty.$$

The difficulty that arises in the quantization process is due to the fact that the quantization operator is nonlinear. That causes the operator B_δ to be nonlinear as well for given $\delta > 0$.

Butzer and Splettstößer [7] studied the functions f in $\mathcal{C}\left(\mathbb{R}\right) \cap L^1\left(\mathbb{R}\right)$ with Fourier transform $F\left(w\right) = 0$ for all $|w| > \pi W$, for some $W > 0$. For $0 < \delta \le 1/e$ and each $|t| < \lfloor 1/\delta \rfloor / W$, they used the WKS sampling series i.e.

$$\widetilde{f}(t) = \sum_{n=-\infty}^{\infty} \widetilde{f}\left(\frac{n}{W}\right) \frac{\sin\left(\pi\left(Wt-n\right)\right)}{\pi\left(Wt-n\right)},$$

where $\widetilde{f}\left(\frac{n}{W}\right) = f\left(\frac{n}{W}\right) + \delta_n$ with $|\delta_n| < \delta$ for some small $\delta > 0$. They proved that the error of the reconstruction using quantization is as follows

$$\left| f(t) - \sum_{k\in\mathbb{Z}} \widetilde{f}\left(\frac{n}{W}\right) \frac{\sin\left(\pi\left(Wt-n\right)\right)}{\pi\left(Wt-n\right)} \right| \leq \left(\frac{1+\pi}{\pi}\right) \left(2 + \sqrt{2\pi} W \left\| f \right\|_1\right) \delta \log\left(1/\delta\right).$$

This error is for the construction with quantization over compact subsets of $\mathbb{R}$. With an additional constraint, that is $|f(t)| \leq M\,|t|^{-\gamma}$ for some $M > 0$ and $0 < \gamma \leq 1$, they proved, for $\delta \leq \min\left\{\left(M/e^2\right)\left(\frac{1}{2}W\right)^{\gamma}, 1/\left(MW^{\gamma}\right)\right\}$, that

$$\left\| f(.) - \sum_{k\in\mathbb{Z}} \widetilde{f}\left(\frac{n}{W}\right) \frac{\sin\left(\pi\left(W\left(.\right)-n\right)\right)}{\pi\left(W\left(.\right)-n\right)} \right\|_{\infty} \leq \left(13/\gamma\right) \delta \log\left(1/\delta\right).$$

This error is uniform over $\mathbb{R}$ and it is independent of f but does depend on its decay factor γ.

5.1 Quantization Error Over $\mathbb{R}$.

In [33] Mönich studied the behavior of a sampling series with equidistant sampling points (=uniform) and the quantization operator (5.1). In general, he deduced that the error of the sampling series can be controlled over all the functions in the space $\mathcal{PW}_{\pi}^{1} \supset \mathcal{PW}_{\pi}^{2}$ if the resolution of the quantization operator is made sufficiently small, provided that the kernel (the sampling functions) in the sampling series is in $L^1(\mathbb{R})$, see [33, Corollary 6.28]. In [33, Theorem 6.15], he used oversampling together with a kernel from the class $\mathcal{M}(a)$, see [33, Definition 3.23].

In this chapter, we will study the quantization problem with nonuniform sampling set. Mönich's work was for uniform sampling points and he pointed out an interest in the nonuniform sampling points in the following passage (see [33, page 159, section 6.4]):

"All the results in this chapter were obtained for equidistant sampling. It would be interesting to extend the results to non-equidistant sampling. Ordinary non-equidistant sampling series without quantization or thresholding were analyzed in Section 3.3. However, the more general problem which treats the convergence behavior of non-equidistant sampling series with sample values that are disturbed by the threshold operator or the quantization operator is difficult to analyze and still open."

His first result towards solution of this problem was obtained in [6] where oversampling is considered. Also, he discussed the nonuniform sampling for $\mathcal{PW}_\pi^2$ with stable linear time-invariant (LTI) system.

Accordingly, we state the main question as follows:

Question: Is there a sampling series, with a nonuniform sampling set, that uses the quantization operator (5.1) such that the quantity

$$\sup_{f\in\mathcal{PW}_\pi^1} \|f - B_\delta f\|_\infty$$

can be made small for sufficiently small δ ?

To answer the question, we will use the following scheme: First, we find a sampling series for functions in the space $\mathcal{PW}_\pi^2$ with a sufficiently fast decay. Then, we use this sampling series to obtain a reconstruction for functions in the space $\mathcal{PW}_\pi^1$ using the density property of $\mathcal{PW}_\pi^2$ in $\mathcal{PW}_\pi^1$. The scheme will be used for lifting the space $\mathcal{PW}_\pi^2$ to $\mathcal{PW}_\pi^1$. It is a scheme that solves the problem over $\mathbb{R}$ uniformly.

A sampling series found by Cvetković, Daubechies and Logan [9] can be used to answer the question. Cvetković, Daubechies and Logan [9, Theorem 3.1] found a series

$$f(t) = \sum_{n\in\mathbb{Z}} f(\lambda_n)\psi_n(t-\lambda_n) \tag{5.2}$$

for $f \in \mathcal{PW}_\pi^2$ with a uniformly discrete sampling set $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ that satisfies $\sup_n \left|\lambda_n - \frac{n}{\lambda}\right| < \infty$, $\lambda > 1$. The convergence holds pointwise, absolutely, in L^2 and uniformly on compact subsets of $\mathbb{R}$. The functions ψ_n are complicated but have rapid decay.

In the following, we give an explicit definition of the function ψ_n. First we consider C^∞ function θ that is defined as $\theta : \mathbb{R} \to [0,1]$ such that $\theta(x) = 0$ for $x < -1/2$, $\theta(x) = 1$ for $x > 1/2$, and $\theta(x) + \theta(-x) = 1$ for all x. For $\nu > 1$ we define the function $\hat{g}$ as

$$\hat{g}(w) = \frac{1}{\sqrt{\pi(1+\nu)}} \sin\left[\frac{\pi}{2}\theta\left(\frac{1+\nu-2|w|/\pi}{2(\nu-1)}\right)\right]$$

Then $\hat{g}(w) = 1/\sqrt{\pi(1+\nu)}$ for $|w| < \pi$, $\hat{g}(w) = 0$ for $|w| > \nu\pi$ and

$$\sum_m \left|\hat{g}\left(w + \frac{(\nu+1)m}{2}\right)\right|^2 = \frac{1}{\pi(\nu+1)}.$$

It follows that the functions $g_k(t) := g\left(t - \frac{2k}{\nu+1}\right)$, $k \in \mathbb{Z}$ are orthonormal. The function $\psi_n(t)$ will be defined as

$$\psi_n(t) = \sum_k \left(B^{-1}\right)_{k,n} g(t + \lambda_n - \frac{2k}{\nu+1}) \tag{5.3}$$

where $B = [B_{n,k}] = [g_k(\lambda_n)]$. It is proved that for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$ and $N \geq 1$ the function ψ_n satisfies

$$|\psi_n(t)| \leq C_N (1+|t|)^{-N}.$$

The sampling series (5.2) has L^2 convergence and thus it has the uniform convergence over $\mathbb{R}$ since $|f(t)| \leq \sqrt{2\pi}\,\|f\|_2$ for $f \in \mathcal{PW}_\pi^2$.

Cvetković, Daubechies and Logan with use of (5.2), estimated the error of the reconstruction with quantization for the space $\mathcal{PW}_\pi^2$, see [9, Corollary 3.3]. The claim is for all $t \in \mathbb{R}$ one has

$$\left|f(t) - \sum_{n\in\mathbb{Z}} \alpha_n \psi_n(t - \lambda_n)\right| \leq A\delta,$$

where $|f(\lambda_n) - \alpha_n| < \delta$ for all $n \in \mathbb{Z}$ and α_n is a finite precision of $f(\lambda_n)$. The constant A is independent of the sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$. The functions ψ_n appear difficult to calculate.

We need a few additional definitions and results before answering the question.

Let $B^{\delta}_{CD}f$ denote a sampling series with the quantization operator using the sampling functions by Cvetković and Daubechies. The sampling series $B^{\delta}_{CD}f$ is defined as follows:

$$\left(B^{\delta}_{CD}f\right)(t) = \sum_{n=-\infty}^{\infty} \left(\Upsilon_{\delta}f\right)(\lambda_n)\psi_n(t-\lambda_n), \tag{5.4}$$

where $\psi_n(t)$ as in (5.3).

Also, we define the operator $(S_N f)$ as follows:

$$\left(S_N f\right)(t) = \sum_{n=-N}^{N} f(\lambda_n)\psi_n(t-\lambda_n).$$

where $\psi_n(t)$ as in (5.3).

For completeness, the following known result.

Lemma 5.1. *$\mathcal{PW}^2_{\pi}$ is dense in $\mathcal{PW}^1_{\pi}$.*

Proof. We use the fact that $\mathcal{D}(\pi B)$ is dense in $L^2(\pi B)$. If $f \in \mathcal{PW}^1_{\pi}$, then $f(t) = \int_{\pi B}\phi(w)e^{iwt}dw$ where $\phi \in L^1[-\pi,\pi]$ and $\|f\|_{\mathcal{PW}^1_{\pi}} = \int_{\pi B}|\phi(w)|\,dw < \infty$. Now, let $\epsilon > 0$. Then there exist $\phi_{\delta} \in \mathcal{D}(\pi B)$ with $\int|\phi_{\delta}|^2\,dw < \infty$ such that $\int|\phi-\phi_{\delta}|\,dw < \epsilon$. We define $g_{\delta}(t) = \int_{\pi B}\phi_{\delta}(w)\,e^{iwt}dw$. It follows that $\|f-g_{\delta}\|_{\mathcal{PW}^1_{\pi}} = \int|\phi-\phi_{\delta}|\,dw < \epsilon$ which completes the proof. □

Lemma 5.2. *Let $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a uniformly discrete sampling set and let*

$$\left(S_N f\right)(t) = \sum_{n=-N}^{N} f(\lambda_n)\,\varphi_n(t) \tag{5.5}$$

be an approximation for functions in $\mathcal{PW}^2_{\pi}$ that converges to f uniformly on $\mathbb{R}$ as $N \to \infty$. If the sampling functions satisfy

$$\sum_{n\in\mathbb{Z}} |\varphi_n(t)| < M < \infty, \ \textit{for all } t \in \mathbb{R}, \tag{5.6}$$

then

$$\lim_{N\to\infty} |f(t) - (S_N f)(t)| = 0$$

for all $f \in \mathcal{PW}_\pi^1$. *The convergence is uniform over* $\mathbb{R}$.

Proof. Let t be arbitrary but fixed in $\mathbb{R}$. The space $\mathcal{PW}_\pi^2$ is dense in $\mathcal{PW}_\pi^1$ and so for a given ϵ, there exist $g \in \mathcal{PW}_\pi^2$ such that $\|f-g\|_{\mathcal{PW}_\pi^1} < \epsilon/2$. Now, we compute the following:

$$\begin{aligned} |f(t) - (S_N f)(t)| &= |f(t) - g(t) + g(t) - (S_N g)(t) + (S_N g)(t) - (S_N f)(t)| \\ &\leq |f(t) - g(t)| + |g(t) - (S_N g)(t)| + |S_N(f-g)(t)|. \end{aligned}$$

The first quantity can be made less than $\epsilon/2$ by using the density property and the fact that $|f(t) - g(t)| \leq \|f-g\|_\infty \leq \|f-g\|_{\mathcal{PW}_\pi^1}$. While for the second quantity, we choose N sufficiently large in the sampling series (5.5). It remains to show the analysis of the last quantity

$$\begin{aligned} |S_N(f-g)(t)| &= \left| \sum_{k=-N}^{N} [f(\lambda_k) - g(\lambda_k)]\, \varphi_k(t) \right| \\ &= \left| \sum_{k=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{f}(\xi) - \hat{g}(\xi)\right) e^{i\xi\lambda_k} d\xi \varphi_k(t) \right| \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\hat{f}(\xi) - \hat{g}(\xi)\right) \sum_{k=-N}^{N} e^{i\xi\lambda_k} \varphi_k(t) d\xi \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\hat{f}(\xi) - \hat{g}(\xi)\right| \left| \sum_{k=-N}^{N} e^{i\xi\lambda_k} \varphi_k(t) \right| d\xi \\ &\leq \max_{\xi\in[-\pi,\pi]} \left| \sum_{k=-N}^{N} e^{i\xi\lambda_k} \varphi_k(t) \right| \|f-g\|_{\mathcal{PW}_\pi^1} \\ &\leq \sum_{k=-N}^{N} |\varphi_k(t)|\, \|f-g\|_{\mathcal{PW}_\pi^1} \\ &\leq \sum_{k\in\mathbb{Z}} |\varphi_k(t)|\, \|f-g\|_{\mathcal{PW}_\pi^1}. \end{aligned}$$

By using (5.6) we obtain

$$|S_N(f-g)(t)| \leq M\epsilon$$

uniformly over $\mathbb{R}$. Thus, for a given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|f(t) - (S_N f)(t)| \leq (1+M)\,\epsilon \text{ for all } t \in \mathbb{R}$$

The proof is complete. □

The following result is the answer to the main question.

Theorem 5.2. *Let the sampling set* $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ *satisfy* $\sup_{n\in\mathbb{Z}} \left|\lambda_n - \frac{n}{\lambda}\right| < \infty$, $\lambda > 1$. *Then, for the reconstruction defined in (5.4), we obtain*

$$\lim_{\delta\to 0}\left(\sup_{f\in\mathcal{PW}_\pi^1}\left\|f - B_{CD}^{\delta} f\right\|_\infty\right) = 0$$

Proof. The function reconstruction (5.5) can be used for the functions in the space $\mathcal{PW}_\pi^1$ by Lemma 5.2 since the sampling functions $\{\psi_n\}$ satisfy (5.6). Thus, for arbitrary fixed t in $\mathbb{R}$ and all $f \in \mathcal{PW}_\pi^1$ we obtain

$$\begin{aligned}
\left|f(t) - \left(B_{CD}^{\delta} f\right)(t)\right| &= \left|\sum_{n\in\mathbb{Z}} f(\lambda_n)\psi_n(t-\lambda_n) - \sum_{n\in\mathbb{Z}} \left(\Upsilon_\delta f\right)(\lambda_n)\psi_n(t-\lambda_n)\right| \\
&= \left|\sum_{n\in\mathbb{Z},} \left[f(\lambda_n) - \left(\Upsilon_\delta f\right)(\lambda_n)\right]\psi_n(t-\lambda_n)\right| \\
&\leq \sum_{n\in\mathbb{Z}} \left|f(\lambda_n) - \left(\Upsilon_\delta f\right)(\lambda_n)\right| \left|\psi_n(t-\lambda_n)\right| \\
&\leq 2\delta \sum_{n\in\mathbb{Z}} \left|\psi_n\left(t-\lambda_n\right)\right| \leq 2\delta M
\end{aligned}$$

by (5.4). The right hand side is independent of t and f. Therefore, the conclusion follows. □

5.2 Quantization Using Finite Points Oversampling Sampling Functions

In this section, we will use the sampling functions of finite points oversampling to control the error from quantization, see (4.15) and (4.16). We need to check the condition (5.6) over compact subsets of $\mathbb{C}$. Let $\mathcal{C}$ be a compact subset in $\mathbb{C}$, $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ and $\{\mu_i\}_{i=1}^{m+2}$ be the set of oversampling. The sampling functions that we consider here are

$$\Psi_k^m(t) = \frac{\prod_{i=1}^{m+2}(t-\mu_i)\,\varphi(t)}{(t-\lambda_k)\prod_{i=1}^{m+2}(\lambda_k-\mu_i)\,\varphi'(\lambda_k)}.$$

Let $S = \{k \in \mathbb{Z}\,|\lambda_k \text{or} \lambda_{-k} \in \mathcal{C}\}$ and also consider λ_k's that satisfy

$$|\lambda_k| > 2\,|\mu_{m+2}| \tag{5.7}$$

Then, let $\widetilde{S}$ be the symmetric set of indices such that it contains S and makes the condition (5.7) satisfied. For some $\epsilon > 0$ we have $|t-\lambda_k| \geq \epsilon$ for all $k \in \mathbb{Z}\backslash\widetilde{S}$. Also, $|\varphi'(\lambda_k)| > c > 0$ for all $k \in \mathbb{Z}$ and $\prod_{i=1}^{m+2}|\lambda_k-\mu_i| > \varepsilon\,|k|^{m+2}$ for some $\varepsilon > 0$. Now for $t \in \mathcal{C}$ we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}|\Psi_k^m(t)| &= \sum_{k\in\mathbb{Z}}\left|\frac{\prod_{i=1}^{m+2}(t-\mu_i)\,\varphi(t)}{(t-\lambda_k)\prod_{i=1}^{m+2}(\lambda_k-\mu_i)\,\varphi'(\lambda_k)}\right| \\
&\leq \sup_{t\in\mathcal{C}}\left(\sum_{k\in\widetilde{S}}|\Psi_k^m(t)|\right) + \sum_{k\in\mathbb{Z}\backslash\widetilde{S}}\left|\frac{\prod_{i=1}^{m+2}(t-\mu_i)\,\varphi(t)}{(t-\lambda_k)\prod_{i=1}^{m+2}(\lambda_k-\mu_i)\,\varphi'(\lambda_k)}\right| \\
&\leq \sup_{t\in\mathcal{C}}\left(\sum_{k\in\widetilde{S}}|\Psi_k^m(t)|\right) + \sum_{k\in\mathbb{Z}\backslash\widetilde{S}}\left|\frac{\prod_{i=1}^{m+2}(t-\mu_i)\,\varphi(t)}{(t-\lambda_k)\prod_{i=1}^{m+2}(\lambda_k-\mu_i)\,\varphi'(\lambda_k)}\right| \\
&\leq A(\mathcal{C}) + \frac{\sup_{t\in\mathcal{C}}\left|\prod_{i=1}^{m+2}(t-\mu_i)\,\varphi(t)\right|}{\varepsilon\epsilon c}\sum_{k\in\mathbb{Z}\backslash\widetilde{S}}\frac{1}{|k|^{m+2}} \\
&< \infty
\end{aligned}$$

for all $t \in \mathcal{C}$.

Since these sampling functions can be used for the space $\mathcal{B}_\pi^\infty$, we do not need the scheme suggested earlier for the space $\mathcal{PW}_\pi^1$. The convergence in this case is studied over compact subsets of $\mathbb{R}$.

We define the sampling series $(B_F f)(t)$ that uses the sampling functions $\{\Psi_k^m\}_{k\in\mathbb{Z}}$ and the quantization operator (5.1) as

$$(B_F f)(t) = \sum_{n\in\mathbb{Z}} (\Upsilon_\delta f)(\lambda_n)\, \Psi_k^m(t)$$

where $\Upsilon_\delta f$ is as defined in (5.1).

In the following the quantization error is controlled over compact subset of $\mathbb{R}$.

Theorem 5.3. *Let the sampling set $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ satisfy $|\lambda_k - k| \leq D < \frac{1}{4}$. Then, we obtain*

$$\lim_{\delta\to 0}\left(\sup_{f\in\mathcal{B}_\pi^\infty} \|f - B_F f\|_\infty\right) = 0$$

The uniform norm is over compact subsets of $\mathbb{R}$.

Proof. We have

$$\begin{aligned}
|f(t) - (B_F f)(t)| &= \left|\sum_{n\in\mathbb{Z}} f(\lambda_n)\, \Psi_k^m(t) - \sum_{n\in\mathbb{Z}} (\Upsilon_\delta f)(\lambda_n)\, \Psi_k^m(t)\right| \\
&\leq \sum_{n\in\mathbb{Z}} |f(\lambda_n)\, \Psi_k^m(t) - (\Upsilon_\delta f)(\lambda_n)\, \Psi_k^m(t)| \\
&= \sum_{n\in\mathbb{Z}} |f(\lambda_n) - (\Upsilon_\delta f)(\lambda_n)|\, |\Psi_k^m(t)| \\
&\leq 2\delta \sum_{n\in\mathbb{Z}} |\Psi_k^m(t)| \leq 2\delta M
\end{aligned}$$

by the above computation. Therefore, the conclusion follows. □

5.3 Speed of Convergence

For computing the rate of convergence we consider a compact set $\mathcal{C} \subset \mathbb{R}$. Choose N and $\epsilon > 0$ such that $|\lambda_k| \geq 2\,|\mu_{m+2}|$ and $|t - \lambda_k| > \epsilon$ for all $t \in \mathcal{C}$ and all $|k| \geq N$. Also we have $|\varphi'(\lambda_k)| \geq c$ for all $k \in \mathbb{Z}$ and $\prod_{i=1}^{m+2} |\lambda_k - \mu_i| > \varepsilon |k|^{m+2}$ for some $\varepsilon > 0$. Now,

$$\begin{aligned}
&\left| \sum_{|k|\geq N} f(\lambda_k) \frac{\prod_{i=1}^{m+2} (t-\mu_i)\varphi(t)}{(t-\lambda_k)\prod_{i=1}^{m+2} (\lambda_k-\mu_i)\varphi'(\lambda_k)} \right| \\
&\leq \sum_{|k|\geq N} \left| f(\lambda_k) \frac{\prod_{i=1}^{m+2} (t-\mu_i)\varphi(t)}{(t-\lambda_k)\prod_{i=1}^{m+2} (\lambda_k-\mu_i)\varphi'(\lambda_k)} \right| \\
&\leq \frac{\sup_{t\in\mathcal{C}} \left| \prod_{i=1}^{m+2} (t-\mu_i)\varphi(t) \right| \sup_{k\in\mathbb{Z}} |f(\lambda_k)|}{\epsilon\varepsilon c} \sum_{|k|\geq N} \frac{1}{|k|^{m+2}} \\
&\leq A(\mathcal{C}) \sup_{k\in\mathbb{Z}} |f(\lambda_k)| \int_{N-1}^{\infty} \frac{dx}{x^{m+2}} \\
&= \tilde{A}(\mathcal{C}) \frac{1}{(m+1)(N-1)^{m+1}}
\end{aligned}$$

which shows that the error $= O\left(\frac{1}{N^{m+1}}\right)$ if f is a bounded function.

We proved some results on controlling the error caused by the quantization operator (5.1) for the space $\mathcal{PW}_\pi^1$. The main idea was finding a sampling series with sampling functions that have sufficiently fast decay for the space $\mathcal{PW}_\pi^2$ and applying it for the space $\mathcal{PW}_\pi^1$ by using the density property of $\mathcal{PW}_\pi^2$ in $\mathcal{PW}_\pi^1$. Also, we treated the problem over compact subsets by using the sampling functions from finite points oversampling where in this case we considered the space $\mathcal{B}_\pi^\infty \supset \mathcal{PW}_\pi^1$.

6 PEAK VALUE PROBLEM

The problem of finding an upper bound for the infinity norm of signals from their samples is called *Peak Value Problem.* The Peak Value Problem is a significant problem related to e.g. Orthogonal Frequency Division Multiplexing (OFDM) which has application in wireless networks, digital television, audio broadcasting and mobile communications, see [48]. In this chapter, we will first answer a question posed by Boche and Mönich where the sampling set is the zeros of a π-sine-type function. Then we will address a related a question where the sampling set only requires a bound on the maximum distance between two consecutive sampling points. The norm that we consider in this section is the infinity norm unless otherwise specified.

The next result is about the infinity norm estimate of the subclass of $\mathcal{B}_\pi^\infty$ that contains all trigonometric polynomials of degree $\leq n$ with real coefficients, that is

$$\mathcal{T}_n = \left\{ f \mid f(t) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \sin kt + b_k \cos kt, t, a_k, b_k \in \mathbb{R} \right\}.$$

Theorem 6.1 (Ehlich and Zeller, [12])**.** *Let $m > n$, $m, n \in \mathbb{N}$ and $f \in \mathcal{T}_n$. Then*

$$\|f\|_\infty \leq \frac{1}{\cos \frac{\pi n}{2m}} \max_{0 \leq k < 2m} \left| f\left(\frac{k}{m}\pi\right)\right|. \tag{6.1}$$

The estimate in (6.1) is sharp if and only if $n \mid m$.

Building on an earlier result by Wunder and Boche [47], Jetter, Pfander, and Zimmermann [22] deduced an estimate for oversampling. It is as follows

$$\|f\|_\infty \leq \sqrt{\frac{m+n}{m-n}} \max_{0 \leq k \leq N-1} \left| f\left(k\frac{2\pi}{N}\right)\right| \tag{6.2}$$

for $f \in \mathcal{T}_n$ where $m \geq n+1$ and $N \geq m+n$. In particular for $N \geq 2n+1$ the choice $m = N-n$ gives

$$\sqrt{\frac{n+m}{n-m}} = \sqrt{\frac{N}{N-2n}}.$$

For a background of the problem in signal processing, see [42], [35] and [31].

6.1 Solvability Question For Stable Sampling

In [5], Wunder and Boche obtained a result for the class of functions $\mathcal{B}^{\infty}_{\beta\pi,0}$, $0 < \beta < 1$ that are real when restricted to the real line. The result reads as

Theorem 6.2 (Wunder and Boche, [5])**.** *If* $0 < \beta < 1$ *and* $f \in \mathcal{B}^{\infty}_{\beta\pi,0}$, *then*

$$\| f \|_{\infty} \leq \frac{1}{\cos(\beta\pi/2)} \sup_{k\in\mathbb{Z}} | f(k) | . \tag{6.3}$$

In (6.3) the sampling set is $\mathbb{Z}$. Here $\mathcal{S}$ will denote the set of all sampling patterns $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ that are made of the zeros of a π-sine-type function. Accordingly, an interesting question is the following:

Question 1: (Boche, Mönich, [32, page 2218])

Let $0 < \beta < 1$ and $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \in \mathcal{S}$. Does there exist a constant $C = C(\beta)$ such that

$$\| f \|_{\infty} \leq C(\beta) \sup_{k\in\mathbb{Z}} | f(\lambda_k) |$$

for all $f \in \mathcal{B}^{\infty}_{\beta\pi}$?

There is a result by Beurling that shows an existence of such a constant. Before we state the result, we define the following density conditions

$$\text{u.u.d}\,(\Lambda) = \lim_{r\to\infty} \max_{x\in\mathbb{R}} \frac{\#\left(\Lambda \cap [x, x+r]\right)}{r}$$

and

$$\text{l.u.d}\,(\Lambda) = \lim_{r\to\infty} \min_{x\in\mathbb{R}} \frac{\#\,(\Lambda \cap [x, x+r])}{r}.$$

Beurling's result is as follows

Theorem 6.3. (Beurling, [3, page 346]) *Let $f \in \mathcal{B}^{\infty}_{\beta\pi}$ with $0 < \beta < 1$. Then, we obtain*

$$\|f\|_{\infty} \le K\,(\Lambda, \beta) \sup_{k\in\mathbb{Z}} |f\,(\lambda_k)| \tag{6.4}$$

where $K\,(\Lambda, \beta) < \infty$ if and only if $\text{l.u.d}\,(\Lambda) > \beta$.

To answer Question 1, we need the following two theorems, see [49, page 144] and [1].

Theorem 6.4. (Levin-Golovin, [49, page 144]) *If $\{\lambda_k\}_{n\in\mathbb{Z}}$ is a set of zeros of a function of π-sine-type, then the system of $\left\{e^{i\lambda_k t}\right\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2\,[-\pi, \pi]$.*

Theorem 6.5 (see Avdonin-Joo, [1])**.** *If $\left\{e^{i\lambda_k t}\right\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2\,[-\pi, \pi]$, where λ_k's are ordered with respect to their real parts, then $\lim_{k\to\pm\infty} \frac{\lambda_k}{k} = 1$.*

We answer the question as follows:

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a set of zeros of a π-sine-type function. Then, for $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\left|\frac{\lambda_k}{k} - 1\right| < \epsilon$$

for all $|k| \ge N$. Thus, we obtain $|k|\,(1-\epsilon) < |\lambda_k| < |k|\,(1+\epsilon)$ for all $|k| \ge N$. It follows that

$$\text{l.u.d} = \lim_{r\to\infty} \min_{x\in\mathbb{R}} \frac{\#\,(\{\lambda_k\} \cap [x, x+r])}{r} \ge \lim_{r\to\infty} \min_{x\in\mathbb{R}} \frac{\#\,(\{|k|\,(1+\epsilon)\} \cap [x, x+r])}{r} = \frac{1}{1+\epsilon}$$

for arbitrary $\epsilon > 0$. Thus, for $\beta < 1$, if we pick $\epsilon < \frac{1}{\beta} - 1$, then we obtain $\text{l.u.d} > \frac{1}{1+\epsilon} > \beta$. Therefore, we conclude that

$$\|f\|_{\infty} < K\,(\Lambda, \beta) \sup_{n\in\mathbb{Z}} |f(\lambda_n)|$$

exists for all $f \in \mathcal{B}^{\infty}_{\beta\pi}$ by Theorem 6.3.

6.2 Generalized Valiron-Tschakaloff Sampling Series

In this section we will generalize the Valiron-Tschakaloff sampling series. The generalization appears in Theorem 6.7 below. We will need to estimate a certain contour integral where the contour is the one given in (4.20). For this we will use the following estimate of the canonical product.

Theorem 6.6. (Levinson, [29, page 56]) *Let φ be defined as (1.5) with*

$$|\lambda_k - k| \leq D < \frac{p-1}{2p},\ 1 < p \leq 2$$

Then,

$$A_p\, |\mathrm{Im} z|\, (|z|+1)^{-4D-1} e^{\pi|\mathrm{Im} z|} < |\varphi(z)| < B_p(|z|+1)^{4D} e^{\pi|\mathrm{Im} z|} \tag{6.5}$$

We need the following result for the proof of the main result. The proof of this result follows the same way as the proof of Theorem 4.7. Here, for the residue part computation, we will deal with singularity of order 2.

Theorem 6.7. *Let $f \in \mathcal{B}_\pi^\infty$ and let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ satisfy*

$$|\lambda_k - k| \leq D < 1/4 \tag{6.6}$$

Then, for any $m \in \mathbb{Z}$ we obtain the following expansion

$$f(z) = L(z) + \sum_{k\in\mathbb{Z}\setminus\{m\}} f(\lambda_k) \frac{(z-\lambda_m)\,\varphi(z)}{(\lambda_k - \lambda_m)\,(z-\lambda_k)\,\varphi'(\lambda_k)} \tag{6.7}$$

where φ denotes the canonical product (1.5) and $L(z)$ is

$$L(z) = \left\{ \frac{\varphi(z)}{\varphi'(\lambda_m)} \right\} f'(\lambda_m) + \left\{ \left[\frac{1}{(z-\lambda_m)\,\varphi'(\lambda_m)} - \frac{\frac{1}{2}\varphi''(\lambda_m)}{(\varphi'(\lambda_m))^2} \right] \varphi(z) \right\} f(\lambda_m).$$

The convergence is uniform over compact subsets $\mathcal{C}$ of $\mathbb{C}$.

Proof. Let $m \in \mathbb{Z}$ be arbitrary, $\mathcal{C} \subset \mathbb{C}$ compact, and $z \in \mathbb{C}$. Let n be sufficiently large such that both λ_m and $\mathcal{C}$ lie in the interior of the contour Γ_n defined in (4.20). Then, by the Cauchy integral formula we obtain that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{\Gamma_n} \frac{f(w)}{(w-z)}dw = \\
&\frac{1}{2\pi i}\oint_{\Gamma_n} \frac{[(w-\lambda_m)\,\varphi(w) - (z-\lambda_m)\,\varphi(z) + (z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)}dw \\
&= \frac{1}{2\pi i}\oint_{\Gamma_n} \frac{[(w-\lambda_m)\,\varphi(w) - (z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)}dw \\
&+ \frac{1}{2\pi i}\oint_{\Gamma_n} \frac{[(z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)}dw
\end{aligned} \tag{6.8}$$

For the first integral in (6.8) we obtain by the residue theorem for $k \neq m$, $|k| \leq n$,

$$\text{Res}\left(\frac{[(w-\lambda_m)\,\varphi(w) - (z-\lambda_m)\,\varphi(z)]f(w)}{(w-\lambda_m)\,(w-z)\,\varphi(w)}, \lambda_k \right) = \frac{(z-\lambda_m)\,\varphi(z)f(\lambda_k)}{(\lambda_k-\lambda_m)\,(z-\lambda_k)\,\varphi'(\lambda_k)}.$$

We have that λ_m is a singularity of order 2. After computing and simplifying, we obtain

$$\text{Res}\left(\frac{[(w-\lambda_m)\,\varphi(w) - (z-\lambda_m)\,\varphi(z)]f(w)}{(w-\lambda_m)\,(w-z)\,\varphi(w)}, \lambda_m \right) =$$

$$\left\{ \frac{\varphi(z)}{\varphi'(\lambda_m)} \right\} f'(\lambda_m) + \left\{ \left[\frac{1}{(z-\lambda_m)\,\varphi'(\lambda_m)} - \frac{\frac{1}{2}\varphi''(\lambda_m)}{(\varphi'(\lambda_m))^2} \right] \varphi(z) \right\} f(\lambda_m).$$

Thus, we obtain

$$\begin{aligned}
f(z) = &\left\{ \frac{\varphi(z)}{\varphi'(\lambda_m)} \right\} f'(\lambda_m) + \left\{ \left[\frac{1}{(z-\lambda_m)\,\varphi'(\lambda_m)} - \frac{\frac{1}{2}\varphi''(\lambda_m)}{(\varphi'(\lambda_m))^2} \right] \varphi(z) \right\} f(\lambda_m) \\
&+ \sum_{|k|\leq n, k\neq m} f(\lambda_k) \frac{(z-\lambda_m)\,\varphi(z)}{(\lambda_k-\lambda_m)\,(z-\lambda_k)\,\varphi'(\lambda_k)} + E_n
\end{aligned}$$

where

$$E_n = \frac{1}{2\pi i}\oint_{\Gamma_n} \frac{[(z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)}dw.$$

We now show that E_n goes to zero as $n \to \infty$. For n sufficiently large we will have $|\, y_n \,| > \eta$ (η for (2.2)) and $|\, y_n \,| > |\, \bar{y} \,|$ for all $z = \bar{x} + i\bar{y} \in \mathcal{C}$. Thus, we have $|\, f(x+iy) \,| \leq$

$\|f\|_\infty e^{\beta\pi|y|}$ by Phragmén-Lindelöf, see Theorem 4.6, and also we have $\mid \varphi(x+iy) \mid \geq A\,|y|\,(|z|+1)^{-4D-1}\,e^{\pi|y|}$ as (6.5). The error over the horizontal segment η_1 will be

$$\begin{aligned}
&\mid \frac{1}{2\pi i}\oint_{\eta_1} \frac{[(z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)} dw \mid \leq \\
&\qquad \frac{M\,\|f\|_\infty}{2\pi A} \int_{x_{-n}}^{x_n} \frac{(|x+iy_n|+1)^{4D+1}}{|y_n| \mid x+iy_n-\bar{x}-i\bar{y} \mid\mid x+iy_n-\lambda_m \mid} dx \\
&\qquad = O\left(\frac{1}{|y_n|^{1-4D}}\right),
\end{aligned} \tag{6.9}$$

where $M = \sup_{z\in\mathcal{C}} \mid (z-\lambda_m)\,\varphi(z) \mid$. The quantity in (6.9) goes to zero as n goes to infinity. For the integral over the path γ_1, we only estimate the integral over the segment γ_1^+ in the first quadrant. The computation of the remaining segment γ_1^- in the fourth quadrant can be estimated in the same way using the symmetry relations (A.2) and (A.3). Also, the computation over the path γ_2 is similar to γ_1. We will use the estimate in [29, page 57, eq 16.08] for the canonical product that is

$$|\varphi\,(x+iy)| \geq \frac{(1+|\lambda_N-(x+iy)|)\,e^{\pi|y|}}{(1+|x+iy-N|)\,(1+|x+iy|)^{4D}},$$

where N is determined by

$$N-\frac{1}{2} \leq |w|\sec\theta < N+\frac{1}{2},$$

$w = x+iy$ and $\theta = \mathrm{Arg}\,(w)$. Now,

$$\begin{aligned}
&\mid \frac{1}{2\pi i}\oint_{\gamma_1^+} \frac{[(z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)} dw \mid \leq \\
&\qquad \frac{M\,\|f\|_\infty}{2\pi} \int_0^{x_n} \frac{(1+|x_n+iy-N|)\,(|x_n+iy|+1)^{4D}}{\mid x_n+iy-\lambda_m \mid\mid x_n+iy-\bar{x}-i\bar{y} \mid |\lambda_N-x_n-iy|} dy.
\end{aligned}$$

The quantity

$$\frac{1+|x_n+iy-N|}{|\lambda_N-x_n-iy|}$$

can be shown to be bounded for $y \in [0, x_n]$ and thus

$$\left|\frac{1}{2\pi i}\oint_{\gamma_1} \frac{[(z-\lambda_m)\,\varphi\,(z)]}{(w-z)} \frac{f\,(w)}{(w-\lambda_m)\,\varphi\,(w)} dw\right| = O\left(\frac{1}{|x_n|^{1-4D}}\right). \tag{6.10}$$

The right hand side of (6.10) goes to zero as n goes to infinity. This completes the proof. $\square$

The previous result has again been obtained by adding one additional piece of information. The additional piece of information required is the value of the derivative of the function f at λ_m, while the additional piece information in Corollary 4.1 is the value of the function f at μ_1.

We point out here that if $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is chosen to be the integers, then we obtain the interpolation formula by Valiron-Tschakaloff, see (F.1).

6.3 Estimating The Infinity Norm

In this section we will estimate the infinity norm of function in the space $\mathcal{B}^{\infty}_{\beta\pi}, 0 < \beta < 1$, by knowing the information of the function over a set that is not necessarily uniform with a certain gap. Accordingly, we state the following question:

Question 2: Let $0 < \beta < 1$ and $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ with $|\lambda_k - k| < \delta$ for all $k \in \mathbb{Z}$ for some $\delta < \frac{1}{4}$. Does there exist a constant $C = C(\Lambda, \beta)$ such that

$$\|f\|_{\infty} \leq C\left(\Lambda, \beta\right) \sup_{k\in\mathbb{Z}} \left|f\left(\lambda_k\right)\right| \tag{6.11}$$

for all $f \in \mathcal{B}^{\infty}_{\beta\pi}$? If so, then can $C(\Lambda, \beta)$ be estimated?

For the first part of the question, we have l.u.d (Λ) for $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ that satisfies $|\lambda_k - k| \leq D < 1/4$ as

$$\text{l.u.d}\left(\Lambda\right) = 1 > \beta.$$

Thus, for all $f \in \mathcal{B}^{\infty}_{\beta\pi}$ with $0 < \beta < 1$ with the condition, $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, $|\lambda_k - k| \leq D < 1/4$ an estimate such as (6.4) exists. In order to bound $C\left(\Lambda, \beta\right)$ in (6.11), we will use Theorem 6.7. Before stating the main result of this section, we need some definitions that are given below

Let $\widetilde{\mathcal{B}}_{\beta\pi}^{\infty} = \left\{ f \in \mathcal{B}_{\beta\pi}^{\infty} \mid f(t) \in \mathbb{R} \text{ for } t \in \mathbb{R} \right\}$. Then, we define $\|.\|_{\Lambda,\infty}$ as

$$\|f\|_{\Lambda,\infty} = \sup_{\lambda_k \in \Lambda} |f(\lambda_k)|,$$

and we define the quantity $C_\Lambda(\beta)$ as

$$C_\Lambda(\beta) = \sup_{f \in \widetilde{\mathcal{B}}_{\beta\pi}^{\infty}, \|f\|_{\Lambda,\infty} \leq 1} \|f\|_\infty . \tag{6.12}$$

Following the condition (6.6), we show that $C_\Lambda(\beta) > 1$.

Example 6.1. *We consider the function $g(z) = c\,\mathrm{sinc}(\beta\pi z)$, $0 < \beta < 1$, where $c = \frac{1}{\mathrm{sinc}(\beta\pi/4)} > 1$. Then, $|g(0)| > 1$ and $\left|g\left(-\frac{1}{4}\right)\right| = \left|g\left(\frac{1}{4}\right)\right| = 1$. Now, if $f(z) = g(z - t_{k_0})$, $t_{k_0} = (\lambda_{k_0} + \lambda_{k_0+1})/2$ for some fixed $k_0 \in \mathbb{Z}$, then $\|f\|_{\Lambda,\infty} \leq 1$ since $\lambda_{k+1} - \lambda_k > \frac{1}{2}$ for all $k \in \mathbb{Z}$, while $\|f\|_\infty \geq |f(t_{k_0})| > 1$. We conclude that $C_\Lambda(\beta) \geq \frac{1}{\mathrm{sinc}(\beta\pi/4)} > 1$.*

For a more general set up, we let $\Gamma \subset \mathbb{R}$, $\|f\|_\Gamma = \sup_{\gamma\in\Gamma} |f(\gamma)|$ and $d_\Gamma = \sup_{x\in\mathbb{R}} \mathrm{dist}\,(x, \Gamma)$, where $\mathrm{dist}\,(x,\Gamma)$ denotes the distance of the point x to the set Γ. That means the largest gap between two successive points in Γ has width $2d_\Gamma$. Then, we define $C_\Gamma(\beta)$ as

$$C_\Gamma(\beta) = \sup_{f \in \widetilde{\mathcal{B}}_{\beta\pi}^{\infty}, \|f\|_\Gamma \leq 1} \|f\|_\infty$$

Now, we state and prove the main result.

Theorem 6.8. *Let $f \in \widetilde{\mathcal{B}}_{\beta\pi}^{\infty}$, $0 < \beta < 1$, $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ be such that it satisfies $|\lambda_k - k| \leq D < \frac{1}{4}$, $\lambda_0 = 0$, $\varphi'(0) > 0, |\varphi'(0)| = \inf_{\lambda_k\in\Lambda} |\varphi'(\lambda_k)|$. Let $E(t) = \varphi'(0)\varphi'(t) - \varphi''(0)\varphi(t)$ and $J = [\beta' a, \beta' b]$, $\lambda_{-1} \leq a < 0 < b \leq \lambda_1$, such that $d_0(\beta') = \min_{t\in J} E(t) > 0$ for $0 < \beta' < 1$. If $\Gamma \subset \mathbb{R}$ with $d_\Gamma \leq (b-a)/2$, then*

$$\|f\|_\infty \leq \frac{(\varphi'(0))^2}{d_0(\beta)} \sup_{\gamma\in\Gamma} |f(\gamma)|.$$

Proof. Let $\epsilon > 0$ be arbitrary. Then there is $f_\epsilon \in \widetilde{\mathcal{B}}^\infty_{\beta\pi}$ with $\|f_\epsilon\|_\Gamma \leq 1$ and $\|f_\epsilon\|_\infty > C_\Gamma(\beta) - \epsilon$. We now construct a related function $f_{\epsilon,\delta}$ such that $|f_{\epsilon,\delta}|$ assumes its maximum at a certain point. For this purpose let $h \in C_0^\infty(\mathbb{R})$ be an even non-negative function such that $h(t) = 0$ for $|t| \geq 1$ and $\int_{\mathbb{R}} h(t)\,dt = 1$. For $\delta > 0$ let $h_\delta(t) = h(t/\delta)/\delta$. Then $\widehat{h_\delta}(t) \in \mathbb{R}$ for $t \in \mathbb{R}$, and $\left|\widehat{h_\delta}(t)\right| \leq \int_{\mathbb{R}} |h_\delta(t)|\,dt = \int_{\mathbb{R}} |h(t)|\,dt = \int_{\mathbb{R}} h(t)\,dt = 1$. Let $f_{\epsilon,\delta}(t) = f_\epsilon(t)\,\widehat{h_\delta}(t)$. Then $|f_{\epsilon,\delta}(t)| \leq |f_\epsilon(t)|$, $\lim_{|t|\to\infty} f_{\epsilon,\delta}(t) = 0$, $f_{\epsilon,\delta}$ converges pointwise to f_ϵ as $\delta \to 0$, $f_{\epsilon,\delta} \in \widetilde{\mathcal{B}}^\infty_{\beta'\pi}$ for $\beta' = \beta + \delta/\pi$, and $|f_{\epsilon,\delta}(t)|$ assumes its maximum at some point $t_{\epsilon,\delta}$. Now let δ be sufficiently small such that $\beta' = \beta + \delta/\pi < 1$ as well as

$$|f_{\epsilon,\delta}(t_{\epsilon,\delta})| = \|f_{\epsilon,\delta}\|_\infty \geq \|f_\epsilon\|_\infty - \epsilon \geq C_\Gamma(\beta) - 2\epsilon.$$

The inequality $\|f_{\epsilon,\delta}\|_\infty \geq \|f_\epsilon\|_\infty - \epsilon$ holds because there is some t_ϵ with $|f_\epsilon(t_\epsilon)| \geq \|f_\epsilon\|_\infty - \epsilon/2$ and since $f_{\epsilon,\delta}$ converges pointwise, one has for sufficiently small δ that $\|f_\epsilon\|_\infty \geq \|f_{\epsilon,\delta}\|_\infty \geq |f_{\epsilon,\delta}(t_\epsilon)| \geq |f_\epsilon(t_\epsilon)| - \epsilon/2 \geq \|f_\epsilon\|_\infty - \epsilon$. Since $-f_{\epsilon,\delta}$ has the same desired properties as $f_{\epsilon,\delta}$, we may assume that $f_{\epsilon,\delta}(t_{\epsilon,\delta}) > 0$. Furthermore, since $f_{\epsilon,\delta}(t)$ is maximal at $t = t_{\epsilon,\delta}$ we have $f'_{\epsilon,\delta}(t_{\epsilon,\delta}) = 0$.

Let $\varphi(t)$ be the canonical product of the sampling set Λ and let $\varphi_{\beta'}(t) = \varphi(\beta' t)$ and $\tilde{g}(\tau) = g(\tau/\beta')$ where

$$g(t) = f_{\epsilon,\delta}(t + t_{\epsilon,\delta}) - \|f_{\epsilon,\delta}\|_\infty \frac{\varphi'_{\beta'}(t)}{\varphi'_{\beta'}(0)}.$$

Now we substitute the function $\tilde{g}$ in the sampling series (6.7) to obtain

$$\tilde{g}(\tau) - \left\{\frac{\varphi(\tau)}{\varphi'(0)}\right\}\tilde{g}'(0) = \tau\varphi(\tau)\left\{\sum_{k\in\mathbb{Z}\setminus\{0\}} \tilde{g}(\lambda_k)\frac{1}{\lambda_k(\tau - \lambda_k)\,\varphi'(\lambda_k)}\right\}. \tag{6.13}$$

We have $\varphi'(\lambda_k) = (-1)^k c_k$, $c_k > 0$ and $\lambda_k(\tau - \lambda_k) < 0$ for $\tau \in (\lambda_{-1}, \lambda_1)$. But, $\tilde{g}(\lambda_k) = d_k(-1)^{k+1}$, $d_k > 0$ and if we substitute in (6.13) we obtain

$$\tilde{g}(\tau) - \left\{\frac{\varphi(\tau)}{\varphi'(0)}\right\}\tilde{g}'(0) = \tau\varphi(\tau)\left\{\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^{k+1} d_k}{\lambda_k(\tau - \lambda_k)\,(-1)^k c_k}\right\}. \tag{6.14}$$

It follows that

$$\tilde{g}(\tau) - \left\{ \frac{\varphi(\tau)}{\varphi'(0)} \right\} \tilde{g}'(0) \geq 0 \text{ for } \tau \in (\lambda_{-1}, \lambda_1).$$

Now, let $\tau = \beta' t$. Then $\tilde{g}(\tau) = g(t)$ and thus

$$g(t) - \left\{ \frac{\varphi(\beta' t)}{\varphi'(0)} \right\} \frac{g'(0)}{\beta'} \geq 0 \text{ for } \beta' t \in (\lambda_{-1}, \lambda_1). \tag{6.15}$$

We compute $g'(0)$ and it is

$$g'(0) = -\|f_{\epsilon,\delta}\|_\infty \frac{\beta' \varphi''(0)}{\varphi'(0)}.$$

Substitute back in (6.15) we obtain

$$g(t) + \left\{ \frac{\varphi(\beta' t)}{\varphi'(0)} \right\} \|f_{\epsilon,\delta}\|_\infty \frac{\varphi''(0)}{\varphi'(0)} \geq 0 \text{ for } \beta' t \in (\lambda_{-1}, \lambda_1).$$

Replacing g by its definition we obtain

$$f_{\epsilon,\delta}(t + t_{\epsilon,\delta}) - \|f_{\epsilon,\delta}\|_\infty \frac{\varphi'(\beta' t)}{\varphi'(0)} + \|f_{\epsilon,\delta}\|_\infty \frac{\varphi''(0)\,\varphi(\beta' t)}{(\varphi'(0))^2} \geq 0 \text{ for } \beta' t \in (\lambda_{-1}, \lambda_1).$$

Then,

$$f_{\epsilon,\delta}(t + t_{\epsilon,\delta}) \geq \|f_{\epsilon,\delta}\|_\infty \frac{\varphi'(0)\,\varphi'(\beta' t) - \varphi''(0)\,\varphi(\beta' t)}{(\varphi'(0))^2},$$

for $\beta' t \in (\lambda_{-1}, \lambda_1)$. We have $E(\beta' t) = \varphi'(0)\,\varphi'(\beta' t) - \varphi''(0)\,\varphi(\beta' t) > d_0(\beta')$ for $t \in [a, b]$. Also, we have $2d_\Gamma \leq (b - a)$ and thus there is $t_\gamma \in [a, b]$ such that $t_\gamma + t_{\epsilon,\delta} = \gamma \in \Gamma$ and so $|f_{\epsilon,\delta}(t_\gamma + t_{\epsilon,\delta})| \leq 1$. By the Intermediate Value Theorem there is t^* between 0 and t_γ such that $f_{\epsilon,\delta}(t^* + t_{\epsilon,\delta}) = 1$. Therefore,

$$\|f_{\epsilon,\delta}\|_\infty \leq \frac{f_{\epsilon,\delta}(t^* + t_{\epsilon,\delta})\,(\varphi'(0))^2}{E(\beta' t^*)} \leq \frac{(\varphi'(0))^2}{d_0(\beta')}.$$

That means

$$C_\Gamma(\beta) \leq \|f_{\epsilon,\delta}\|_\infty + 2\epsilon \leq \frac{(\varphi'(0))^2}{d_0(\beta')} + 2\epsilon.$$

Letting both ϵ and δ go to zero completes the proof. □

The earlier result by Wunder and Boche, Theorem 6.2, now follows as a corollary if we choose $\Gamma = \Lambda = \mathbb{Z}$.

7 CONCLUSIONS

Several sampling theorems that are often used in signal processing are expansions with uniform sampling points. The Poisson Summation Formula (PSF) is a central concept for the uniform sampling set up. Both the PSF and the sampling theorems with uniform sampling are built upon having a sampling set that forms a coset of a discrete subgroup of $\mathbb{R}$. Beyond that there are cases where the sampling set is the union of finitely many cosets of a discrete subgroup. In that case, the sampling set is called nonuniform periodic. Yet the PSF still plays a role in the derivations of sampling theorems for this case, see [13, Section 3]. In this thesis, we generalize a sampling series for the space $\mathcal{PW}_\pi^2$ by considering a sampling set that is a perturbation of the zero set of a π-sine-type function. The perturbation of each point is within a quarter of the minimum distance between two consecutive zeros. Such a set can be nonuniform and nonperiodic and still maintains the stability in the sense of a Riesz basis.

The other type of generalization in this thesis is considering bigger spaces than $\mathcal{PW}_\pi^2$. It is known that the functions in the space $\mathcal{PW}_\pi^2$ are functions in $L^2(\mathbb{R})$ when restricted to the real line. Additionally, those functions go to zero at infinity. We derived a sampling series for $\widetilde{\mathcal{B}}_{\pi,N}$, the space of band-limited functions that have polynomial growth when restricted to the real line. We developed a method using a smooth cut-off function. The purpose of this function is to overcome the growth of the function $f \in \widetilde{\mathcal{B}}_{\pi,N}$ to be reconstructed. The new function is called f_δ and has a rapid decay. The function f_δ can be made arbitrary close, over compact sets, to the function f by choosing smaller values of δ.

Another way of controlling the growth of the band-limited functions in the class $\widetilde{\mathcal{B}}_{\pi,N}$ is by using successive divisions to create an auxiliary function Q_{N+1}, see (4.5), that is in $\mathcal{PW}_\pi^2$. This method has some advantages such as

(1) For the auxiliary function Q_{N+1}, we can use any sampling series for the space $\mathcal{PW}_{\pi}^{2}$.

(2) The sampling set need not be of a ratio-type. The oversampling is a finite oversampling determined by the degree of the growth of the function to be reconstructed.

A direct consequence is that one additional piece of information (1-point oversampling) is sufficient to obtain a reconstruction for functions in the space $\mathcal{B}_{\pi}^{\infty}$ using a sampling series for the space $\mathcal{B}_{\pi}^{2} = \mathcal{PW}_{\pi}^{2}$, see Corollary 4.1.

An important result in this thesis is Theorem 4.7 where the perturbation can be beyond a quarter. The oversampling is by a finite number of points and not of ratio-type. The sampling series is for band-limited functions that have polynomial growth when restricted to the real line. The oversampling by adding additional points can be used to generate a faster decay of the sampling functions which causes more rapid convergence in the sampling series. The example in Section 4.4 demonstrates the higher reconstruction accuracy by adding one additional sampling point.

Sometimes the sampled function values are only available in quantized form. The key requirement to control the resulting error in the reconstruction is a sufficiently rapid decay of the sampling functions. This we achieved for band-limited functions of polynomial growth by using oversampling by finitely many points. The convergence is uniform over compact sets. We also showed that a sampling theorem by Daubechies et al. [9] can be used for functions in $\mathcal{PW}_{\pi}^{1}$ with uniform convergence over the whole real line. On the other hand, their sampling functions may be more difficult to evaluate.

The problem of estimating the infinity norm of a function from its supremum over a subset of $\mathbb{R}$ is called a peak value problem. Boche and Mönich asked whether an estimate of the form

$$\|f\|_{\infty} \leq C(\beta) \sup_{\gamma \in \Gamma} |f(\gamma)|, \quad f \in \widetilde{\mathcal{B}}_{\beta\pi}^{\infty}, \quad 0 < \beta < 1, \tag{7.1}$$

exists if Γ is the zero set of a π-sine-type function. We provide an affirmative answer using a result of Beurling as well as some properties of sine-type functions.

We proved a generalized Valiron-Tschakaloff sampling theorem for sampling sets that are perturbations of the integers by up to a quarter. This sampling theorem is then used to derive a general method to find upper bounds for $C(\beta)$ in the inequality (7.1) if Γ is a subset of $\mathbb{R}$ with maximum gap of $2d_\Gamma$. For $\Gamma = \mathbb{Z}$, i.e., $d_\Gamma = 1/2$, we reproduce an earlier estimate by Wunder and Boche [5, 48].

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APPENDICES

A APPENDIX Sine-type Function

One example of a sine-type function with non-equidistant zeros is the the function

$$\varphi_{\alpha,\beta}(z) = \cos(\pi z) - \beta \sin(\alpha \pi z) \text{ where } 0 < \alpha < 1 \text{ and } 0 \leq \beta \leq 1.$$

The function has a non-equidistant set of real zeros, see [11, Theorem 1]. Setting $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{\sqrt{3}}$ we obtain nonuniform, nonperiodic set of zeros.

The Canonical Product

Lemma A.1. *The infinite product*

$$\prod_{k=1}^{\infty} (1+ \mid a_k \mid)$$

converges to a non-zero limit if and only if

$$\sum_{k=1}^{\infty} \mid a_k \mid < \infty.$$

Throughout this work we will be considering the infinite product that is expressed below

$$\varphi(z) = \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{\lambda_k}\right). \tag{A.1}$$

The convergence can be conditional. The infinite product on the right hand side of (A.1) is understood in the sense of

$$\lim_{N \to \infty} \prod_{|k|<N} \left(1 - \frac{z}{\lambda_k}\right).$$

The factor $\left(1 - \frac{z}{\lambda_k}\right)$ will be replaced by z if $\lambda_k = 0$.

Let $G(\{\lambda_k\}; z)$ denote the canonical product $\varphi(z)$ for the set of zeros $\{\lambda_k\}_{k\in\mathbb{Z}}$. If $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, one has the relations

$$G(\{\lambda_k\}; \overline{z}) = \overline{G(\{\lambda_k\}; z)} \tag{A.2}$$

and

$$G(\{\lambda_k\}; -z) = s(\lambda_0) G(\{-\lambda_{-k}\}; z), \quad \text{with } s(\lambda_0) = \begin{cases} 1 & \text{if} \quad \lambda_0 \neq 0 \\ -1 & \text{if} \quad \lambda_0 = 0 \end{cases} \tag{A.3}$$

B APPENDIX Phragmén-Lindelöf

We define $M_f(r)$ as

$$M_f(r) = \sup\left\{\left|f\left(re^{i\theta}\right)\right| : \theta_1 < \theta < \theta_2\right\}.$$

In the next result we set $\theta_1 = 0$ and $\theta_2 = \pi$.

Theorem B.1. (Phragmén-Lindelöf, [27, page 38]) *If $f(z)$, $z = x + iy$, is an analytic function in the upper half-plane $\{z \mid Im\, z > 0\}$ such that for all $\epsilon > 0$,*

$$M_f(r) \overset{as}{<} e^{(\sigma+\epsilon)r}$$

for sufficiently large $r = \mid z \mid$ and $\mid f(x) \mid \leq M$ on the real axis, then

$$|f(x + iy)| \leq Me^{\sigma y}$$

in the upper half-plane.

We also have the following useful corollary.

Corollary B.1. (Phragmén-Lindelöf, [27, page 39]) *If $f(z)$ is an entire function of exponential type π, and*

$$|f(x)| \leq m\ (1 + |x|)^N,\ x \in \mathbb{R}$$

then

$$|f(z)| \leq C\,(1 + |z|)^N\, e^{\pi|y|}$$

for all z in the complex plane $\mathbb{C}$.

C APPENDIX The Indicator Diagram and The Sine-type Function

The definition of the sine-type function in Katsnel'son [23] and (2.2) are equivalent. The condition of having a function of exponential type satisfying

$$0 < m \leq |f(x + iH)| \leq M < \infty,$$

for some m and M for all $x \in \mathbb{R}$ in some horizontal line $y = H$ together with the indicator diagram width condition is equivalent to:

There exist A, B and η such that

$$Ae^{\sigma|y|} \leq\mid f(x + iy) \mid\leq Be^{\sigma|y|}$$

for all $x, y \in \mathbb{R}$, such that $|y| \geq \eta$. The computation is as follows:

Let us assume that the indicator diagram is 2π. i.e. $h_f\left(\frac{\pi}{2}\right) = h_f\left(-\frac{\pi}{2}\right) = 2\pi$, where

$$h_f(\theta) = \limsup_{|r|\to\infty} \frac{\log\left|f\left(re^{i\theta}\right)\right|}{r^\rho}$$

for some order ρ. The function h_f is called the indicator function. Now, If $f(x + iy)$ is an entire function of exponential type and it is bounded on some horizontal line $y = H$, then by Phragmén-Lindelöf, Theorem B.1, we obtain

$$|f(x + iy)| \leq Be^{\sigma|y|}.$$

Moreover, if the zeros are in a horizontal strip $|y| \leq H$, then

$$Ae^{\sigma|y|} \leq |f(x + iy)| \leq Be^{\sigma|y|}$$

for $|y| \geq H$ and thus we have

$$\widetilde{A} + \sigma|y| \leq \log|f(y)| \leq \widetilde{B} + \sigma|y|.$$

It follows that

$$\sigma \leq \limsup_{|y|\to\infty} \frac{\log|f(y)|}{|y|} \leq \sigma$$

and therefore, $\sigma = \pi$. Now, we show the equivalence in the two different statements that are used as a definition. Let

$$m \leq |f(x+iH)| \leq M \tag{C.1}$$

then by using (C.1) and Phragmén-Lindelöf we have

$$|f(x+iy)| \leq \widetilde{M} e^{\pi|y|}. \tag{C.2}$$

By using (C.2), it can be shown that

$$|f(x+iy)| \geq m e^{\pi|y|}$$

whenever $\operatorname{dis}(z, \{\lambda_k\}) > \eta$, see [27, page 163]. Hence,

$$m e^{\pi|y|} \leq |f(x+iy)| \leq M e^{\pi|y|} \tag{C.3}$$

for all $x, y \in \mathbb{R}$ with $|y| \geq \eta$. On the other hand, if (C.3) satisfied then

$$0 < \alpha \leq |f(x+iH)| \leq \beta < \infty$$

where $-\infty < x < \infty$ for $|H| \geq \eta$.

D APPENDIX Riesz Basis

Theorem D.1. ([49, page 27]) *Let $\mathcal{H}$ be a separable Hilbert space. Then, the following statements are equivalent.*

(1) The sequence $\{f_n\}$ forms a Riesz basis for $\mathcal{H}$, that is the sequence $\{f_n\}$ is complete in $\mathcal{H}$, and there exist positive constants A and B such that for any arbitrary positive integer n and arbitrary scalars $c_1, \ldots, c_n$ one has

$$A \sum_{i=1}^{n} |c_i|^2 \leq \left\| \sum_{i=1}^{n} c_i f_i \right\|^2 \leq B \sum_{i=1}^{n} |c_i|^2 .$$

(2) The sequence $\{f_n\}$ is obtained from an orthonormal basis by means of a bounded invertible operator.

(3) There is an equivalent inner product on $\mathcal{H}$, with respect to which the sequence $\{f_n\}$ becomes an orthonormal basis for $\mathcal{H}$.

(4) The sequence $\{f_n\}$ is complete in $\mathcal{H}$, and its Gram matrix

$$((f_i, f_j))_{i=1,j=1}^{\infty}$$

generates a bounded invertible operator on l^2.

(5) The sequence $\{f_n\}$ is complete in $\mathcal{H}$ and possesses a complete biorthogonal sequence $\{g_n\}$ such that

$$\sum_{i=1}^{\infty} |(f, f_n)|^2 < \infty \text{ and } \sum_{n=1}^{\infty} |(f, g_n)|^2 < \infty$$

for every f in $\mathcal{H}$

Theorem D.2. *A sequence that is biorthogonal to a Riesz basis is also a Riesz basis.*

E APPENDIX The Paley-Wiener Spaces and Theorems

Theorem E.1 (The Plancherel-Polya Inequality)**.** *Let* $f \in \mathcal{B}^p_\sigma$, $p > 0$, *and let* $\Lambda = (\lambda_n)$, $n \in \mathbb{Z}$, *be an increasing sequence such that* $\lambda_{n+1} - \lambda_n \geq 2\delta$. *Then*

$$\sum_{n\in\mathbb{Z}} \mid f(\lambda_n) \mid^p \leq \frac{2e^{p\sigma\delta}}{\pi\delta} \parallel f \parallel_p^p .$$

Theorem E.2. *The Bernstein and the Paley-Wiener spaces are ordered as follows*

$$\mathcal{B}^2_\sigma \subseteq \mathcal{B}^3_\sigma \subseteq \ldots \subseteq \mathcal{B}^\infty_\sigma$$

and

$$\ldots \subset \mathcal{PW}^2_\sigma \subset \mathcal{PW}^1_\sigma.$$

Proof. If f is in the space $\mathcal{B}^p_\sigma$, $p \geq 2$ and $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1} - \lambda_k \geq 2\delta$, then by Plancherel-Polya the function satisfies

$$\sum_{n\in\mathbb{Z}} |f(\lambda_k)|^p \leq \frac{2e^{p\sigma\delta}}{\pi\delta} \|f\|_p^p .$$

We claim that the function is bounded on the real line. If not then $|f(x)|$ has a sequence $\{|f(\lambda_k)|\}_{k\in\mathbb{Z}}$ that is unbounded. Now we pick let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$and then

$$\|f\|_p^p \geq \frac{\sigma\delta}{2e^{p\sigma\delta}} \sum_{k\in\mathbb{Z}} |f(\lambda_k)|^p = \infty$$

which is a contradiction. Thus,

$$\int_{\mathbb{R}} |f|^{p+1} \, d\mu \leq \max_{x\in\mathbb{R}} |f(x)| \int_{\mathbb{R}} |f|^p \, d\mu < \infty,$$

and therefore $f \in \mathcal{B}^{p+1}_\sigma$ whenever $f \in \mathcal{B}^p_\sigma$. Also, since the function in any of these spaces is bounded in the real line then all these spaces contained in $\mathcal{B}^\infty_\sigma$. However, the fact that

$$\|f\|_{\mathcal{PW}^p_\sigma} = \left(\frac{1}{2\sigma} \int_{[-\sigma,\sigma]} |g(t)|^p \, dt\right)^{1/p}$$

$$\leq \left(\frac{1}{2\sigma} \int_{[-\sigma,\sigma]} |g(t)|^{p+1} \, dt \right)^{1/(p+1)}$$

$$= \|f\|_{\mathcal{PW}_\sigma^{p+1}}$$

shows the inclusions for the Paley-Wiener spaces. □

E_σ denotes the space of entire function of exponential type $\leq \sigma$.

Theorem E.1 (Paley-Wiener)**.** *Let $f \in L^2(\mathbb{R})$. Then, f has an analytic extension to $\mathbb{C}$ which belongs to E_σ if and only if $\widehat{f} \subseteq [-\sigma, \sigma]$.*

Theorem E.2. (Paley-Wiener-Schwartz, [38, page 198]) *(a) If $\phi \in \mathcal{D}(\mathbb{R}^n)$ has its support in rB, B is the closed unit ball of $\mathbb{R}^n$, and if*

$$f(z) = \int_{\mathbb{R}^n} \phi(t) \, e^{-iz.t} dm_n(t), \ (z \in \mathbb{C}^n) \tag{E.1}$$

then f is entire, and there is a constant $\gamma_N < \infty$ such that

$$|f(z)| \leq \gamma_N (1 + |z|)^{-N} e^{r|Imz|}, (z \in \mathbb{C}^n, N = 0, 1, 2, \ldots) \tag{E.2}$$

(b) Conversely, if f is an entire function in $\mathbb{C}^n$ which satisfies (E.2), then there exists $\phi \in \mathcal{D}(\mathbb{R}^n)$, with support in rB, such that (E.1) holds.

Theorem E.3. (Paley-Wiener-Schwartz, [38, page 199]) *(a) If $u \in \mathcal{D}'(\mathbb{R}^n)$ has its support in rB, if u has order N, and if*

$$f(z) = u(e_{-z}), \ (z \in \mathbb{C}^n) \tag{E.3}$$

then f is entire, the restriction of f to $\mathbb{R}^n$ is the Fourier transform of u, and there is a constant $\gamma < \infty$ such that

$$|f(z)| \leq \gamma (1 + |z|)^N e^{r|Imz|}. \tag{E.4}$$

(b) Conversely, if f is an entire function in $\mathbb{C}^n$ which satisfies (E.4) for some N and some γ, then there exists $u \in \mathcal{D}'(\mathbb{R}^n)$, with support in rB, such that (E.3) holds.

Theorem E.4. *The spaces $\mathcal{B}_\pi^2$ and $\mathcal{PW}_\pi^2$ represent the same class of functions.*

Proof. To show that we let $f \in \mathcal{B}_\pi^2$. Then, by Fourier transform

$$f(z) = \int_{\mathbb{R}} \hat{f}(z) e^{izw} dw$$

where $\hat{f} \in L^2[-\pi, \pi]$ since supp$\hat{f} \in [-\pi, \pi]$. Thus, $f \in \mathcal{PW}_\pi^2$. The other inclusion, let $f \in \mathcal{PW}_\pi^2$. Then,

$$f(z) = \int_{-\pi}^{\pi} g(w) e^{izw} dw$$

where $g \in L^2[-\pi, \pi]$. We can see that

$$|f(z)| \leq \int_{-\pi}^{\pi} |g(w)| e^{-w\mathrm{Im}z} dw \leq \|g\|_1 e^{\pi|y|} \leq C \|g\|_2 e^{\pi|z|}$$

Also, $\|f\|_{L^2(\mathbb{R})} < \infty$ either by Plancheral or by Hausdorff-Young Inequality. $\square$

In fact, the space $\mathcal{B}_\pi^2$ is a Hilbert space since it is a closed subspace of the Hilbert space $(L^2(\mathbb{R}), < f, g >= \int_{\mathbb{R}} f(x)\overline{g(x)}dx)$. It is closed subspace since $\mathcal{F}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\mathcal{F}^{-1}\left\{L^2[-\pi, \pi]\right\} = \mathcal{B}_\pi^2$ where $L^2[-\pi, \pi]$ is a closed subspace of $L^2(\mathbb{R})$. The Fourier transform is a unitary operator that carries closed space to a closed space. It is unitary since it is an isomorphism on $L^2(\mathbb{R})$ and preserves inner product by Parseval's Identity.

F APPENDIX Interpolation In Bernstein Spaces

In this section we give a short summary of interpolation formulae in Bernstein spaces.

- 1925 Valiron-Tschakaloff, an interpolation formula for $f \in \mathcal{B}_\pi^\infty$, see [16, page 60].

$$f(t) = \frac{\sin\pi t}{\pi}\left\{ f'(0) + \frac{f(0)}{t} + \sum_{n\in\mathbb{Z}\setminus\{0\}} f(n)\frac{\pi t(-1)^n}{n(t-n)} \right\} \tag{F.1}$$

- 1976 J. R. Higgins, an interpolation formula for $f \in \mathcal{B}_\pi^2$, $|\lambda_n - n| \le D < \frac{1}{4}$, see [17]. Biorthogonality argument.

$$f(t) = \sum_{n\in\mathbb{Z}} f(\lambda_n)\frac{\varphi(t)}{(t-\lambda_n)\varphi'(\lambda_n)} \tag{F.2}$$

  uniformly for $t \in \mathbb{R}$.

- 1987 Kristian Seip, an interpolation formula for $f \in \mathcal{B}_{\pi-\delta}^\infty$ $(0 < \delta < \pi)$,$|\lambda_n - n| \le D < \frac{1}{4}$, see [39]. Oversampling type.

$$f(z) = \sum_{n\in\mathbb{Z}} f(\lambda_n)\frac{\varphi(z)}{(z-\lambda_n)\varphi'(\lambda_n)} \tag{F.3}$$

- 1993 G. Hinsen, an interpolation formula for $f \in \mathcal{B}_\pi^p$, $|\lambda_n - n| \le D$, $D < \frac{1}{4}$ for $1 \le p \le 2$, and $D < \frac{1}{2p}$ for $2 \le p < \infty$, see [19]. More precise growth for $\varphi(t)$ and distribution property for $\{\lambda_n\}_{n\in\mathbb{Z}}$.

$$f(z) = \sum_{n\in\mathbb{Z}} f(\lambda_n)\frac{\varphi(z)}{(z-\lambda_n)\varphi'(\lambda_n)} \tag{F.4}$$

- 2010 Boche, Mönich, an interpolation formula for $f \in \mathcal{B}^{\infty}_{\pi,0}$, $\{\lambda_n\}_{n\in\mathbb{Z}}$ zeros of π-sine-type function, see [32].

$$f(z) = \sum_{n\in\mathbb{Z}} f(\lambda_n) \frac{\varphi(z)}{(z-\lambda_n)\varphi'(\lambda_n)} \tag{F.5}$$

- 2010 Boche, Mönich, an interpolation formula for $f \in \mathcal{B}^{\infty}_{\beta\pi}$, $\{\lambda_n\}_{n\in\mathbb{Z}}$ zeros of π-sine-type function, see [32].

$$f(z) = \sum_{n\in\mathbb{Z}} f(\lambda_n) \frac{\varphi(z)}{(z-\lambda_n)\varphi'(\lambda_n)} \tag{F.6}$$

  and if we consider $\mathcal{B}^{\infty}_{\beta\pi,0}$ the convergence will be uniform over $\mathbb{R}$.

All convergence are over compact subsets of $\mathbb{C}$, unless otherwise stated.