AN ABSTRACT OF THE THESIS OF

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(Name) (Degree) (Major)

Date thesis is presented May 14, 1963

Title ON THE ENUMERATION OF CERTAIN EQUIVALENCE CLASSES OF EULER PATHS OF FULL GRAPHS

Abstract approved [redacted]

This thesis treats the problem of enumerating equivalence classes of Euler paths of full graphs. A full graph is a complete, unordered, graph with no loops or repeated edges. Two Euler paths are equivalent if and only if one can be transformed into the other by a finite sequence of rotations and reflections of the path and permutations of its vertices. We obtain the number of equivalence classes for full graphs on 3 and 5 vertices but obtain only partial results for full graphs on 7 vertices. We prove theorems which enable us to obtain representatives of all equivalence classes with relatively few repetitions for any full graph. Finally we prove a monotoneity theorem for the number of equivalence classes.

ON THE ENUMERATION OF CERTAIN EQUIVALENCE CLASSES OF EULER PATHS OF FULL GRAPHS

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF ARTS

June 1963

APPROVED:

[redacted]

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Date thesis is presented May 14, 1963

Typed by Carolyn Joan Whittle

Acknowledgement

The writer wishes to thank Dr. Robert Stalley for his help and guidance in the preparation of this thesis.

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ON THE ENUMERATION OF CERTAIN EQUIVALENCE CLASSES OF EULER PATHS OF FULL GRAPHS

Chapter I

Introduction

Graph theory has its origin in the works of Leonhard Euler in the early part of the eighteenth century [3]. The subject has experienced a strong revival in the past decade. Not only is graph theory being recognized as a powerful tool in applications but the theory is being extended at an increasing rate. The standard references in graph theory are those by Berge, König, and Ore found in the bibliography [2], [4], [5].

The statement of our main problem and its treatment are graph theoretical. However, we may alternatively state the problem as a problem in combinatorial analysis as follows: "Enumerate the non-equivalent cyclic arrangements of symbols taken from a set of n distinct symbols where each unordered pair of distinct symbols occurs in adjacent positions once and only once. Two arrangements are equivalent if and only if one may be obtained from the other by a finite sequence of rotations, reflections, and permutations of the distinct symbols."

This problem has not received attention in the literature. The problem has its origin in the design of computers [6], but no such application of the results of this thesis are envisioned.

Let n be the number of distinct symbols from which arrangements are formed. We prove that there exists an arrangement if and only if n is odd and $n \geq 3$. We enumerate the non-equivalent arrangements for $n = 3$ and $n = 5$, but fail to obtain the enumeration for arbitrary n. We do obtain partial results for $n = 7$. We develop several rules of construction which give us all arrangements with at most a small number of repetitions from each equivalence class. However, even when we take $n = 7$ this method is prohibitive without a high speed computer partly because of the necessity of eliminating the repetitions mentioned. Finally, we prove a monotoneity theorem for the number of pairwise non-equivalent arrangements.

Chapter II

Basic Definitions and Theorems

We are interested in full graphs in this thesis. A _full graph_ $G(V)$ is defined by the equation,

$$G(V) = \left\{\{a,b\} \;\middle|\; a\varepsilon V,\ b\varepsilon V,\ a\neq b \right\},$$

where $V = \{j \mid 1 \leq j \leq n\}$, $n \geq 2$, and the elements of $G(V)$ are unordered pairs. Full graphs are known also as _finite_, _unordered_, _complete graphs with no loops and with non-repeated edges_. The elements of V are called _vertices_ and the elements of $G(V)$ are called _edges_.

An Euler path on a full graph is a cyclic sequence of edges which contains each edge of the graph once and for which the intersection of two consecutive edges is a single vertex. Thus an Euler path is

(2.1) $$\Big\{\{a_1,a_2\},\ \{a_2,a_3\},\ \cdots,\ \{a_{i-1},a_i\},\ \{a_i,a_{i+1}\},\ \cdots,\ \{a_N,a_1\}\Big\},$$

where $a_i \varepsilon V$, $1 \leq i \leq N$, and where $N = \frac{1}{2}n(n-1)$ is the number of edges of $G(V)$.

A simpler notation for the Euler path in (2.1) is

(2.2) $$a_1\ a_2\ a_3\ a_4 \cdots a_{i-1}\ a_i\ a_{i+1} \cdots a_N\ a_1.$$

This arrangement is also referred to as the Euler path. It is now evident that the graph theoretical definition of an Euler path is equivalent to the combinatorial definition given in the introduction.

Example 1. If $V = \{1,2,3\}$, an Euler path is $\{\{1,2\},\{2,3\},\{3,1\}\}$, or in the simpler notation of (2.2), 1 2 3 1.

We will refer to an Euler path on a full graph with n vertices as simply an _Euler path on n vertices_. The first N symbols of (2.2) are the _corners_ of the Euler path on n vertices. The symbol a_i is the _ith corner_ where $1 \leq i \leq N$. Thus several corners may denote the same vertex.

The next step in the investigation of Euler paths is to determine necessary and sufficient conditions on the order of the vertex set of the full graph for the existence of an Euler path on that graph. The following theorems will present these conditions.

Theorem 1. If there exists an Euler path on a full graph $\mathcal{G}(V)$, then the order of V is odd and not less than three.

Proof. We first prove that the order, n, is not less than three. That n is at least two follows from the definition of a full graph. If $n = 2$, then the full graph contains one edge. An Euler path would contain one edge with a_1 as both the first and last vertex by (2.1), whereas by definition the vertices of an edge are distinct.

That n is odd also follows from a consideration of the Euler path as defined in (2.1). We have already noted that the vertex a_1 occurs in the first and last edges. Also, if a vertex occurs in any other edge it must occur in either the preceding edge or the following edge. Hence since a_1 does not occur in the second edge or the N-1 st edge, then a_1 must occur an even number of times in the remaining edges of (2.1). However, the number of edges containing vertex a_1 of a full graph on n vertices is $n-1$. Thus $n-1$ must be even which implies that n must be odd and the theorem is proved.

Theorem 2. If the order of the vertex set of a full graph $G(V)$ is odd and not less than three, then there exists an Euler path on $G(V)$.

The proof which will be given here is different from known proofs which may be found in any standard reference in graph theory and in some works of topology [1]. This proof is useful in the construction of Euler paths for study in later chapters. The proof is by induction on the order of the vertex set.

Proof. An Euler path on three vertices is given by Example 1. We assume that there is an Euler path on n vertices denoted by (2.2) where $a_1 = 1$. Then an Euler path on $n + 2$ vertices is given by

(2.3) $\quad b_1\ b_2\ b_3\ \cdots\ b_i\ \cdots\ b_N\ b_{N+1} \cdots\ b_{N+2n+1}\ b_1,$

where $b_1 = a_1 = 1,$

$$b_i = a_i \ (2 \le i \le N),$$

$$b_{N+2k-1} = k \ (1 \le k \le n),$$

$$b_{N+4k-2} = n+1 \ (1 \le k \le \tfrac{1}{2}(n+1)),$$

$$b_{N+4k} = n+2 \ (1 \le k \le \tfrac{1}{2}(n-1)),$$

$$b_{N+2n+1} = n+2.$$

Clearly consecutive edges contain a common vertex. We need to show that each edge occurs once and only once. We will begin by showing that each edge appears at least once. The edges may be divided into four classes.

Class 1. Edges of the form $\{h,j\}$ where $1 \le h < j \le n$. All edges of this class are found in the form $\{b_i, b_{i+1}\}$ where $1 \le i \le N$, since $b_1\ b_2 \cdots b_N\ b_{N+1}$ forms an Euler path on n vertices.

Class 2. Edges of the forms $\{h,n+1\}$ and $\{h,n+2\}$ where $2 \le h \le n$. These edges may also appear in the form $\{n+1,h\}$ and $\{n+2,h\}$. Consider the edges $\{b_{N+2h-2}, b_{N+2h-1}\}$ and $\{b_{N+2h-1}, b_{N+2h}\}$. If h is even they are $\{n+1,h\}$ and $\{h,n+2\}$ respectively, whereas if h is odd they are $\{n+2,h\}$ and $\{h,n+1\}$ respectively.

Class 3. Edges of the forms $\{1,n+1\}$ and $\{1,n+2\}$. These edges may also appear in the form $\{n+1,1\}$ and $\{n+2,1\}$ respectively. The edge $\{1,n+1\}$ is $\{b_{N+1}, b_{N+2}\}$ and the edge $\{n+2,1\}$ is $\{b_{N+2n+1}, b_1\}$.

Class 4. The edge $\{n+1, n+2\}$. This edge is $\{b_{N+2n}, b_{N+2n+1}\}$.

Finally, the number of edges of (2.3) is $N+2n+1$. Since

$$N+2n+1 = \tfrac{1}{2}n(n-1) + 2n + 1 = \tfrac{1}{2}(n^2 - n + 4n + 2)$$
$$= \tfrac{1}{2}(n^2 + 3n + 2) = \tfrac{1}{2}(n+2)(n+1)$$

is the order of $G(V)$ for a set V of order $n+2$, no edge can occur more than once. Thus each edge occurs exactly once and the proof is complete.

Let k be the number of corners of an Euler path on n vertices which denote a specific vertex. We find that $N = nk$ and $k = \tfrac{1}{2}(n-1)$.

Chapter III

Characteristic Sequences

In this chapter we define equivalent Euler paths and devise a method of determining whether or not two given Euler paths are equivalent under this definition.

Two Euler paths P and P' on the same full graph are defined to be <u>equivalent</u> if and only if P can be transformed into P' by a finite sequence of rotations and reflections of the path and permutations of the vertices of V. Here the terms rotation and reflection are taken in the geometric sense. More precisely, two Euler paths, $P = a_1\ a_2\ a_3\ a_4 \cdots a_N\ a_1$, and $P' = b_1\ b_2\ b_3\ b_4 \cdots b_N\ b_1$, of the same full graph are defined to be <u>equivalent</u> if and only if there exists a permutation $a_i \rightarrow b_i$ on the set $\{a_1, a_2, a_3, a_4, \cdots, a_N\}$ which is an element of the permutation group generated by elements of the following three types: <u>Vertex Permutation</u>, that is permutations for which corners which denote the same vertex have images which are corners which denote the same vertex; <u>Rotations</u>, that is permutations of the type, $b_i = a_{i+k}$ for $1 \leq i \leq N$; <u>Reflections</u>, that is permutations of the type, $b_1 = a_1$ and $b_i = a_{N+2-i}$ for $2 \leq i \leq N$. We define a_t for an arbitrary integer t by $a_t = a_j$ where $j \equiv t \pmod{N}$ and $1 \leq j \leq N$.

The above definition defines an equivalence relation on the set of all Euler paths on a given full graph. Each equivalence class of Euler paths contains paths which are sequences of rotations, reflections, and vertex permutations of a given path. Our problem is that of enumeration of equivalence classes for each odd n which is not less than three. This statement of our problem agrees with the statement given in the introduction.

We wish to devise a practical way of determining whether or not two given Euler paths belong to the same equivalence class. For this purpose we define the right representative sequence. For a given Euler path,

$$a_1\ a_2\ a_3\ \cdots\ a_i\ a_{i+1}\ \cdots\ a_N\ a_1,$$

the <u>right representative sequence</u>, or <u>RRS</u>, is

$$r_1\ r_2\ r_3\ \cdots\ r_i\ r_{i+1}\ \cdots\ r_N,$$

where r_i corresponds to the corner a_i and

(3.1) $$r_i = \min \{k \mid a_{i+k} = a_i,\ k \geq 1\}.$$

The element, r_i, of this sequence is called the <u>right characteristic number</u> associated with the corner a_i. Since each vertex is denoted by $k = \frac{1}{2}(n-1)$ corners, then if $n \geq 3$, each vertex is denoted by more than one corner and will have more than one right characteristic number corresponding to it.

A vertex permutation may be considered to be a relabelling of the elements of V. Thus, the RRS is invariant under a vertex permutation since it is dependent upon the order of the vertices and not upon the symbols used to represent those vertices. We always perform a vertex permutation upon each Euler path we are considering in order to bring it to a form such that the first corner denoting the vertex i precedes the first corner denoting the vertex j if $1 \leq i < j \leq n$.

If the Euler path is subjected to a rotation then the effect upon the RRS is to operate upon it with the permutation,

(3.2) $$r_i \rightarrow r_{i+k}.$$

We define r_t for any integer t by $r_t = r_j$ where $j \equiv t \pmod{N}$ and $1 \leq j \leq N$.

For the purpose of characterizing Euler paths to within a reflection, we also always perform a rotation upon each Euler path to transform it to a form $a_1\ a_2 \cdots a_N\ a_1$, such that the corresponding number,

$$\sum_{i=1}^{N} r_i\ N^{N-i},$$ is maximal.

The right representative sequence of an Euler path which has been transformed in the aforementioned way is called the <u>right characteristic sequence</u>, or <u>RCS</u>. The RCS is a characterization of an equivalence class corresponding to an equivalence relation where two Euler paths P and P' are equivalent if and only if P can be transformed into P' by a finite sequence of rotations and vertex permutations as defined above.

If the Euler path is subjected to a reflection, the effect upon the right representative sequence is to operate upon it with the permutation,

(3.3) $$r_i \rightarrow s_{i+r_i}.$$

Upon examination of this result we find that the sequence of s's may be defined in another way which is equivalent to the above. For a given Euler path,

$$a_1\ a_2\ a_3\ \cdots\ a_i\ a_{i+1}\ \cdots\ a_N\ a_1,$$

let the <u>left representative sequence</u>, or <u>LRS</u> be,

$$s_1\ s_2\ s_3\ \cdots\ s_i\ s_{i+1}\ \cdots\ s_N,$$

where s_i corresponds to the corner a_i and where

(3.4) $$s_i = \min \{k \mid a_{i-k} = a_i,\ k \geq 1\}.$$

We define s_t for any integer t by $s_t = s_j$ where $j \equiv t \pmod N$ and $1 \leq j \leq N$.

Thus, upon comparing (3.1) with (3.4) we see that if $a_j = a_{i+r_i}$, then $a_i = a_{j-s_j}$. Moreover, we find that the only difference between the right and left representative sequences is the sense of direction in which we must count from a corner to find the next corner denoting the same vertex. Thus, the effect of a reflection on an Euler path is to interchange its right representative sequence with its left representative sequence. This result will be used extensively in Chapter V.

The Euler path to which a given left representative sequence corresponds is sometimes transformed by a rotation such that the number,

$$\sum_{i=1}^{N} s_i N^{i-1}$$

is maximal.

The left representative sequence of an Euler path which has been transformed in the aforementioned way is called the <u>left characteristic sequence</u>, or <u>LCS</u>. When working with the left characteristic sequence but not directly with the Euler path to which it corresponds, we will write the LCS in the following form:

$$s_N \; s_{N-1} \; s_{N-2} \cdots s_{N-i} \cdots s_2 \; s_1 .$$

By doing this we sacrifice the correspondence between the left characteristic numbers and the corners of the Euler path to which they correspond for the purpose of being able to compare characteristic sequences more easily.

Now to each Euler path correspond two sequences, the right and left characteristic sequences, and these sequences completely characterize an equivalence class of Euler paths on a full graph. Since the effect of a rotation of an Euler path upon its RRS and LRS is to interchange them such a rotation also interchanges the RCS and LCS of the path. Thus the RCS of an Euler path is identical to the LCS of that path after that path has been reflected once. Two Euler paths are equivalent if and only if the RCS and LCS of one are identical to the RCS and LCS of the other in either order. More simply, two Euler paths are equivalent if and only if the RCS of one equals the RCS or LCS of the other.

Example 2. If n = 5, the following Euler path is given by (2.3) using Example 1 as an Euler path on three vertices. We also give the RRS and LRS for this path.

Euler path:	1 2 3 1 4 2 5 3 4 5 1,
RRS:	3 4 5 7 4 6 3 5 6 7,
LRS:	7 6 5 3 6 4 7 5 4 3.

The right and left characteristic sequences of the Euler path are,

RCS:	7 4 6 3 5 6 7 3 4 5,
LCS:	7 4 6 3 5 6 7 3 4 5.

The above procedure of writing the RRS and LRS for a path by writing their characteristic numbers directly below the corners with which they are associated is used throughout this thesis. Note that in Example 2 the RCS is identical to the LCS. This is found later to be a property of the RCS and LCS of any Euler path on five vertices. It is not, however, a property of the RCS and LCS of Euler paths in general.

Chapter IV

Construction Theorems and Some Applications

The purpose of this chapter is first to developtheorems which aid in the construction of the Euler paths which represent the equivalence classes described in Chapter III. These theorems are then to be used to construct representative Euler paths from each equivalence class for n equal to three and five. We also prove a theorem which will give some information about the number of equivalence classes of Euler paths on seven vertices.

To construct the Euler paths we assign a vertex from the vertex set as a value to each of the N corners of the path, subject to the definition of an Euler path. We use the convention, described in the preceding chapter, of constructing Euler paths $a_1 a_2 \cdots a_N a_1$ such that the number, $\sum_{i=1}^{N} r_i N^{N-i}$, is maximal and such that the first corner denoting the vertex i precedes the first forner denoting the vertex j if $1 \leq i < j \leq n$.

Let the <u>segment</u> <u>associated</u> <u>with</u> <u>the</u> <u>corner</u> a_i which denotes the vertex h be defined as the set $S(i,h) = \{a_j \mid 1 < j \leq s,\ a_j \neq a_i \text{ if } 1 < j < s,\ a_s = h\}$. The segment associated with the first corner of an Euler path constructed according to the convention mentioned above is called the <u>principal segment</u> of the Euler path. The length of a segment associated with a corner is its order. Thus, the length of the segment associated with a_i is r_i. We denote the length of the principal segment by L. Clearly, the length of the principal segment is greater than or equal to the lengths of the segments associated with each of the other N-1 corners. Thus, in the Euler path on three vertices, 1 2 3 1, the length of the principal segment, $\{2,3,1\}$, is three.

Lemma 1. The sum of the right characteristic numbers associated with the corners denoting a specific vertex is N.

Proof. Let $S(i,h)$ be the empty set if the corner i does not denote the vertex h. Thus the Euler path is just, $\bigcup_{1\leq i\leq N} S(i,h)$ for any fixed h. Let $[A]$ be the order of the set A. The sum of the characteristic numbers associated with a specific vertex is just

$$\sum_{i=1}^{N} [S(i,h)] = \left[\bigcup_{1\leq i\leq N} S(i,h)\right] = N.$$

This completes the proof of Lemma 1.

Theorem 3. If $n \geq 5$, the principal segment of an Euler path on n vertices has length greater than n.

Proof. First we prove that $L \geq n$. Assume $L < n$. Consider the vertex 1. This vertex is denoted by $\frac{1}{2}(n-1)$

corners in an Euler path on n vertices. Since L is the length of the principal segment, the right characteristic numbers of the other $\frac{1}{2}(n-1) - 1$ corners denoting the vertex 1 must be less than or equal to L. Thus the sum, W, of the characteristic numbers associated with corners denoting the vertex 1 satisfies $W \leq \frac{1}{2}L(n-1) < \frac{1}{2}n(n-1)=N$ contrary to Lemma 1 which states that $W = N$.

Now we prove $L > n$. Assume $L = n$. By Lemma 1 the sum of the characteristic numbers associated with corners denoting a specific vertex must equal N. Each vertex is denoted by $\frac{1}{2}n(n-1)$ corners throughout the Euler path. All characteristic numbers of the RCS must be less than or equal to n. If any of the characteristic numbers were less than n, the sum of the characteristic numbers of the corners denoting some vertex would be less than N. Thus all characteristic numbers must equal n. But if the RCS consists only of n's then the edge $\{a_1, a_2\}$ equals the edge $\{a_{n+1}, a_{n+2}\}$. If $n \geq 5$ these edges are distinct since $n < N = \frac{1}{2}n(n-1)$, and we have a contradiction; so Theorem 3 is proved.

Theorem 4. Every vertex of an Euler path is denoted by some corner in the principal segment.

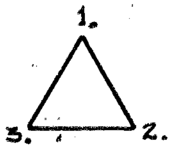
Proof. Assume the vertex i is not denoted by a corner in the principal segment. Recall that the length of the principal segment is denoted by L. Let the first corner denoting the vertex i be corner m where $m > L$.

Assume that the $\frac{1}{2}(n-1)$st corner denoting the vertex i is corner p. Certainly p is less than or equal to N. Consider the segment associated with corner p. This segment will contain all corners of the principal segment. It will also contain corner one. Thus the length of this segment is at least $L+1$ which contradicts the definition of the principal segment and Theorem 4 is proved.

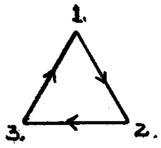
Theorem 5. There is exactly one equivalence class of Euler paths on three vertices.

Proof. Since each vertex is denoted by $\frac{1}{2}(n-1)$ corners in an Euler path on n vertices, each vertex is denoted by $\frac{1}{2}(3-1) = 1$ corner in an Euler path on three vertices. The first corner denoting the vertex i precedes the corner denoting the vertex j if $1 \leq i < j \leq 3$ in this case. Hence since every corner denoting a vertex in an Euler path on three vertices is the first corner denoting that vertex, the only Euler path given by our construction is 1 2 3 1, and the theorem is proved.

Let us examine the geometric significance of the results of Theorem 5. The full graph on three vertices may be represented by,



The Euler path which we have found as a representative of all Euler paths on three vertices may be represented by,



In the above geometric interpretations we represent vertices by points, edges by line segments, and Euler paths by cyclic sequences of directed line segments.

Theorem 6. Suppose an Euler path on n vertices, where $n \geq 5$, is operated upon by a rotation permutation such that its RRS is its RCS. Suppose further that a vertex permutation is performed so that the first corner denoting the vertex i precedes the first corner denoting the vertex j if $1 \leq i < j \leq n$. Then the first four corners denote the vertices 1, 2, 3, and 4 in that order.

Proof. By Theorem 3 the principal segment must be of length greater than five. Thus the vertex 1 which occurs in the first corner cannot occur in corners two, three, or four. Consider the second corner. The only possible vertex for the second corner to denote is a new vertex, the vertex 2. The third corner can denote either the vertex 2 or a new vertex, namely 3. But if the vertex 2 is denoted by corner three we would have $\{2,2\}$ occurring in the Euler path and this is not an edge. Thus, the third corner must denote the vertex 3. The fourth corner may denote any of the vertices 2, 3, or the new vertex 4. The vertex 3 cannot be denoted by corner four because $\{3,3\}$ is not an edge. The vertex 2 if denoted by corner four would cause the edge $\{2,3\}$ to be repeated which violates the definition of an Euler path. Thus, the only possible vertex which corner four can denote is the new vertex 4, and the theorem is proved.

A loop is a pair of consecutive corners of (2.2), both of which denote the same vertex. Because a loop is not an edge, it is easily seen that a full graph, and therefore an Euler path of a full graph, cannot contain a loop.

Theorem 7. The principal segment of an Euler path on five vertices cannot be of length six.

Proof. Assume there is such an Euler path. We construct the representative Euler path of the equivalence class to which it belongs. By Theorem 6 the first four corners must denote the vertices 1 2 3 4. Corner five can possibly denote only two vertices, the vertex 2 or the new vertex 5. Thus the vertex 1 cannot be denoted by a corner preceding the seventh corner whereas the vertices 4 and 3 would give rise to a loop or a repeated edge, respectively. The vertex 2 is impossible also because the length of the segment associated with corner five would then be seven, a number greater than the length of the principal segment which is assumed to be six. This follows from Lemma 1 and the fact that each vertex is denoted by $\frac{1}{2}(5-1) = 2$ corners in an Euler path on five vertices. The partial Euler path is now 1 2 3 4 5 X X X X X 1. By a partial path we mean a path partially determined. Each X denotes a corner with an undetermined value. Since the principal segment is assumed to have length six, the seventh corner of the Euler path must denote the vertex 1. Leaving the sixth corner undetermined, the partial path now becomes 1 2 3 4 5 X 1 X X X 1 . We now consider the possible vertices which can be denoted by corner six. Vertices 1, 2, 4, and 5 are impossible because they would give rise to loops or repeated edges. The vertex 3 is impossible also because the length of the segment associated with the sixth corner would then be seven which would contradict the assumption that the principal segment is of length six. Thus there is no vertex which satisfies the conditions required for a vertex to be denoted by corner six. The theorem is now proved.

Similar, though more complex, proofs can be devised to show that the length of the principal segment of an Euler path is greater than $n + 1$ if n is seven or nine. The lack of an inductive scheme keeps us from obtaining this result for more general values of n.

Theorem 8. The number of equivalence classes of Euler paths on five vertices is three.

Proof. We use our construction procedure to construct all representatives of equivalence classes of Euler paths on five vertices. We will then consider the RCS and LCS of each of these representative paths to see if any are equivalent. Those Euler paths which remain will then be the representatives of the various equivalence classes of Euler paths on five vertices.

By Theorem 6, the first four corners of all representative Euler paths must denote the vertices 1 2 3 4. The eighth corner denotes the vertex 1 by Theorems 3 and 7, for otherwise the ninth or tenth corner would denote the vertex 1 and we would have a repeated edge or loop. The fifth corner denotes either the vertex 2 or the new vertex 5, for otherwise we would have a repeated edge or loop. Thus we have two partial paths,
A = 1 2 3 4 2 X X 1 X X 1 and B = 1 2 3 4 5 X X 1 X X 1.

Consider the partial path A. Corner six of A must denote the vertex 5 to avoid loops or repeated edges, so we now have A = 1 2 3 4 2 5 X 1 X X 1. Corner seven denotes either of two vertices, the vertex 3 or the vertex 4, because vertices 2 and 5 would give rise to a repeated edge and a loop, respectively. Thus partial path A is replaced by the two partial paths,
C = 1 2 3 4 2 5 3 1 X X 1 and D = 1 2 3 4 2 5 4 1 X X 1.

Consider the partial path D with its partially determined RRS,

D = 1 2 3 4 2 5 4 1 X X 1

RRS = 7 3 3 7 7 3 .

Note that for RRS to be the RCS, (that is, for $\sum_{i=1}^{N} r_i N^{N-i}$ to be maximal) the characteristic number for corner six must be three. This requires that the vertex denoted by corner nine is the vertex 5, which in turn requires that corner ten denotes the vertex 3 to avoid loops and repeated edges. Thus we have as an equivalence class representative and corresponding RCS,

(4.1) D = 1 2 3 4 2 5 4 1 5 3 1,

RCS = 7 3 7 3 7 3 7 3 7 3 .

Now let us consider the partial path C together with its partial RRS,

C = 1 2 3 4 2 5 3 1 X X 1,

RRS = 7 3 4 7 6 .

Note that the right characteristic number for corner six is at least three and the triple of right characteristic numbers for corners five, six, and seven is at least 7 3 6. The triple of characteristic numbers for corners one, two, and three is 7 3 4. This fact contradicts the assumption that the right representative sequence of the path we are constructing is to be the RCS of the path. Thus C is not the representative of an equivalence class.

Next to be considered is the partial path B = 1 2 3 4 5 X X 1 X X 1. The vertices 4 and 5 are not denoted by corner six otherwise we would have a repeated edge or a loop. Since the vertices 2 and 3 are acceptible,

partial path B is replaced by two partial paths,
E = 1 2 3 4 5 2 X 1 X X 1 and F = 1 2 3 4 5 3 X 1 X X 1. Path F is not the representative of an equivalence class since if corner seven denotes any vertex, we have a repeated edge or loop. To avoid repeated edges and loops corner seven of partial path E must denote the vertex 4 and so we have E = 1 2 3 4 5 2 4 1 X X 1. Corner nine may denote either of the vertices 3 or 5. Vertices 1, 2, and 4 would give loops or repeated edges. So this partial path is replaced by the two partial paths,
G = 1 2 3 4 5 2 4 1 3 X 1 and H = 1 2 3 4 5 2 4 1 5 X 1. Each of these partial paths may be continued by letting corner ten denote the only vertex which will give no loops or repeated edges, and we obtain the partial paths
G = 1 2 3 4 5 2 4 1 3 5 and H = 1 2 3 4 5 2 4 1 5 3. Thus we have as equivalence class representatives and their corresponding RCS's,

(4.2) G = 1 2 3 4 5 2 4 1 3 5 1,

RCS = 7 4 6 3 5 6 7 3 4 5 .

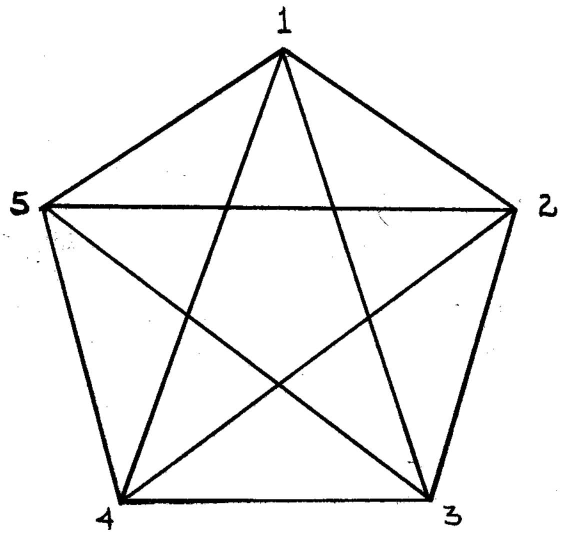
(4.3) H = 1 2 3 4 5 2 4 1 5 3 1,

RCS = 7 4 7 3 4 6 7 3 6 3 .

Thus we have (4.1), (4.2), and (4.3) as equivalence classes. We must now examine their RCS's and LCS's to see if some pair represents the same equivalence class. The following is a list of the RCS and LCS of each of the three Euler paths:

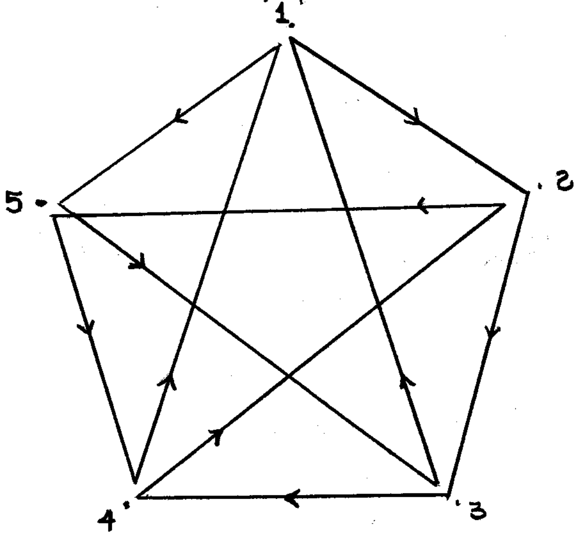
D - RCS = 7 3 7 3 7 3 7 3 7 3 and LCS = 7 3 7 3 7 3 7 3 7 3.
G - RCS = 7 4 6 3 5 6 7 3 4 5 and LCS = 7 4 6 3 5 6 7 3 4 5.
H - RCS = 7 4 7 3 4 6 7 3 6 3 and LCS = 7 4 7 3 4 6 7 3 6 3.
Thus we find that the three Euler paths D, G, and H must represent different equivalence classes of Euler paths on five vertices. The three equivalence classes represented by these three Euler paths are all that are possible on five vertices and the theorem is proved.

Let us examine the geometric significance of this result. As for the case $n = 3$, we represent vertices by points, edges by line segments, and Euler paths by cyclic sequences of directed line segments. The full graph on five vertices may be represented by

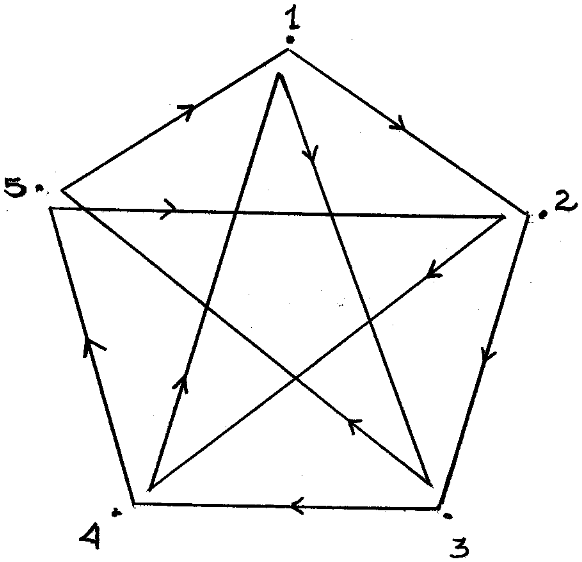


The three Euler paths which we have found as representatives of each of the three equivalence classes of Euler paths on five vertices, each followed by a representation of that path, are as follows,

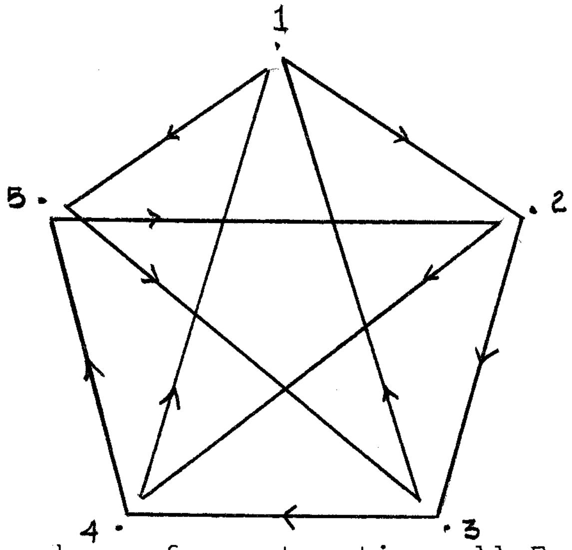
Equivalence Class One, path D = 1 2 3 4 2 5 4 1 5 3 1:



Equivalence Class Two, path G = 1 2 3 4 5 2 4 1 3 5 1:



Equivalence Class Three, path H = 1 2 3 4 5 2 4 1 5 3 1:



This procedure of constructing all Euler paths which are representatives of equivalence classes for a given value of n becomes prohibitive for values of n which are greater than or equal to seven. The number of representatives for n = 7 which would be found using the theorems of this chapter is estimated to be of the order of 10,000. This estimate was made on the basis of the results of a partial enumeration of the representatives. The following gives a lower bound on the number of equivalence classes of Euler paths on seven vertices.

Theorem 9. The number of equivalence classes of Euler paths on seven vertices is greater than or equal to three.

Proof. To prove this theorem we will display three non-equivalent Euler paths on seven vertices. The three Euler paths we display are generated by using (2.3) and the representative Euler paths of each of the three equivalence classes of Euler paths on five vertices as found above. The Euler path on seven vertices obtained by using (2.3) and path D as the Euler path on five vertices will be called D_7 and similarly for G and H. The three Euler paths on seven vertices, together with their respective right and left representative sequence (RRS and LRS) are,

Path D_7: 1 2 3 4 2 5 4 1 5 3 1 6 2 7 3 6 4 7 5 6 7 1,

RRS: 7 3 7 3 8 3(10) 3(10) 5(11) 4(10) 4 9 4 8 3 8(13)(14),

LRS: (11)(10)9 8 3 8 3 7 3 7 3(13) 8(14) 5 4(10) 4(10) 4 3.

Path G_7: 1 2 3 4 5 2 4 1 3 5 1 6 2 7 3 6 4 7 5 6 7 1,

RRS: 7 4 6 3 5 7(10) 3 6 9(11) 4(10) 4 9 4 8 3 7(13)(14),

LRS: (11)(10)9 8 7 4 3 7 6 5 3(13) 7(14) 6 4(10) 4 9 4 3.

Path H_7: 1 2 3 4 5 2 4 1 5 3 1 6 2 7 3 6 4 7 5 6 7 1,

RRS: 7 4 7 3 4 7(10) 3(10) 5(11) 4(10) 4 9 4 8 3 7(13)(14),

LRS: (11)(10)9 8 7 4 3 7 4 7 3(13) 7(14) 5 4(10) 4(10) 4 3.

Now if we find that the RCS of each of these three paths does not equal the RCS or LCS of any other of these three paths, we can say that these three paths are pairwise non-equivalent and therefore that there are at least three equivalence classes of Euler paths on seven vertices.

The six characteristic sequences are

RCS of D_7: (14) 7 3 7 3 8 3 (10) 3 (10) 5 (11) 4 (10) 4 9 4 8 3 8 (13),

LCS of D_7: (14) 8(13) 3 7 3 7 3 8 3 8 9 (10)(11) 3 4 (10) 4 (10)4 5.

RCS of G_7: (14) 7 4 6 3 5 7 (10) 3 6 9 (11) 4 (10) 4 9 4 8 3 7 (13),

LCS of G_7: (14) 7(13) 3 5 6 7 3 4 7 8 9 (10)(11) 3 4 9 4 (10)4 6.

RCS of H_7: (14) 7 4 7 3 4 7 (10) 3 (10) 5 (11) 4 (10) 4 9 4 8 3 7 (13),

LCS of H_7: (14) 7(13) 3 7 4 7 3 4 7 8 9 (10)(11) 3 4 (10) 4 (10)4 5.

Since these six characteristic sequences are distinct, there exist at least three equivalence classes of Euler paths on seven vertices and the theorem is proved.

Chapter V

A Monotoneity Theorem

The purpose of this chapter is to establish the following general theorem.

Theorem 10. The number of equivalence classes of Euler paths on full graphs on n vertices, where n is an arbitrary odd integer not less than 3, is a monotonically increasing function of k, where $k = \frac{1}{2}(n-1)$.

Preliminary part of the proof. Let $r(n)$ be the number of equivalence classes of Euler paths on n vertices. We must prove

$$r(n+2) \geq r(n) \tag{5.1}$$

where n is odd and $n \geq 3$. The validity of (5.1) for $n = 3$ and for $n = 5$ follows from Theorems 5, 8, and 9. Hence we assume that $n \geq 7$. The proof of Theorem 2 is constructive where an Euler path on $n + 2$ vertices is constructed from an Euler path on n vertices. We will prove that Euler paths on n vertices which are from different equivalence classes yield Euler paths on $n + 2$ vertices which are from different equivalence classes under this construction. Our proof will be indirect. We will assume that two Euler paths on $n + 2$ vertices constructed from non-equivalent Euler paths on n vertices are equivalent and will then show that this assumption leads to a contradiction. We will use the notation of Chapter 2.

As the representative of a given equivalence class of Euler paths on n vertices we pick the path whose right representative sequence is identical to its RCS and for which the first corner denoting the vertex i precedes the first corner denoting the vertex j if $1 \leq i < j \leq n$. By Theorem 6 the path begins with 1 2 3 4. Hence, our construction yields the following path on $n + 2$ vertices:

$$(5.2) \quad \{1\ 2\ 3\ 4\ \cdots\}\ 1\ (n+1)\ 2\ (n+2)\ 3\ (n+1)\ 4\ (n+2)$$
$$5\ \cdots\ (n-2)\ (n+1)\ (n-1)\ (n+2)\ n\ (n+1)\ (n+2)\ 1.$$

All but the last corner of the path on n vertices from which (5.2) is constructed are enclosed in braces. This is called the <u>path part</u> or <u>P-part</u> of (5.2). The remaining corners of (5.2) form what is called the <u>induction part</u> or <u>I-part</u>. Note that the vertices $n + 1$ and $n + 2$ each are denoted by $k + 1$ corners of the I-part of (5.2), where $k = \frac{1}{2}(n-1)$. Note also that each of the vertices i, where $2 \leq i \leq n$ is denoted by exactly one corner of the I-part of (5.2). The vertex 1 is denoted by two corners of the I-part of (5.2), namely the $N + 1$ st corner and the first corner. The first corner of an Euler path appears twice. The second appearance here is needed to indicate that the edge $\{(n+2), 1\}$ is contained in the Euler path since these vertices are denoted by the $N + 2n + 1$ st corner and the first corner, respectively. For the remainder of this chapter we will assume that the I-part contains only one corner denoting the vertex 1, the N + 1 st corner of (5.2). Hereafter we will enclose the path part of such a path in braces. A path constructed in this way is called an induction path and is denoted P_{n+2}.

The right representative sequence of path (5.2) is of the form

(5.3) [X X X ··· X] (2n+1) 4 (2n) 4 (2n-1) 4 (2n-2) 4
X 4 X 4 X ··· 4 X 3 X (N+3) (N+4),

where each X denotes a number which can be determined only with additional information about the form of path (5.2). The part of (5.3) which corresponds to the path part of (5.2) is enclosed in brackets. This is called the path part or P-part and the rest of (5.3) is called the induction part or I-part. Hereafter we will enclose the path part of such RRS in brackets. Such an RRS is called an induction RRS and is denoted R_{n+2}.

The left representative sequence of (5.2) is of the form

(5.4) [(2n+1) (2n) (2n-1) (2n-2) X X ··· X] X
(N+3) X (N+4) X 4 X 4 X ··· 4 X 4 X 4 3,

where each X denotes a number which can be determined only with additional information about the form of path (5.2). The part of (5.4) which corresponds to the path part of (5.2) is enclosed in brackets and is called the path part or P-part. The rest of (5.4) is called the induction

<u>part</u> or <u>I-part</u>.

Preliminary lemmas. We now prove four lemmas.

Lemma 2. If n is odd, $n \geq 7$, then $N > 2n + 1$.

Proof. $N = \frac{1}{2}n(n-1) \geq 3n > 2n + 1$.

Lemma 3. If n is odd, $n \geq 7$, then $N + 3 > 4$.

Proof. Since $n \geq 7$, then by Lemma 2 we have

$$N + 3 > 2n + 4 > 4.$$

Lemma 4. If n is odd, $n \geq 7$, then the unordered pair $(N+3,N+4)$, occurs in adjacent positions of the path part or in the last two positions of the induction part of an induction RRS R_{n+2}.

Proof. We examine all other pairs of consecutive positions and find that always one of them is occupied by an integer smaller than $N + 3$. The integer 3 or the integer 4 occurs in every consecutive pair of positions except the last pair of the induction part of R_{n+2}. The first integer of the path part of R_{n+2} is not greater than $\frac{1}{2}n(n-1) = N$. The first integer of the induction part of R_{n+2} is $2n + 1$. The integer preceding the pair $(N+3,N+4)$ in the induction part of R_{n+2} is not greater than $\frac{1}{2}n(n-1) + 2 = N + 2$. Hence, by Lemmas 2 and 3 each of these other possible pairs of consecutive positions is occupied by at least one integer less than $N + 3$ and so cannot be occupied by the pair $(N+3,N+4)$.

Lemma 5. There is no permutation on the set $\{i \mid 1 \leq i \leq n + 2\}$ so that a sequence of consecutive corners of the induction part of a P_{n+2}, $k + 1$ of which denote the vertex $N + 1$ or which denote the vertex $n + 2$, is transformed into a sequence of corners with the same order which are successive corners in the path part of a P_{n+2}.

Proof. By hypothesis at least one of the vertices $n + 1$, $n + 2$, is denoted by $k + 1$ corners of the sequence of consecutive corners of the induction part of a P_{n+2}. Hence, if such a permutation does exist, the common image of the corners that denote that vertex is denoted by $k + 1$ corners of the path part of a P_{n+2}. However, each vertex of the path part of a P_{n+2} is denoted by $k = \frac{1}{2}(n-1)$ corners, which is a contradiction, and Lemma 5 is proved.

Our proof will be divided into two parts. In Part A we show that two induction paths which are constructed from non-equivalent Euler paths on n vertices are not related through rotations and vertex permutations. In part B we show that two induction paths which are constructed from non-equivalent Euler paths on n vertices are not related through a reflection. The theorem will then follow since we will have shown that two induction paths which are constructed from non-equivalent Euler paths on n vertices are not related through a finite sequence of rotations, reflections, or vertex permutations. This shows that representative Euler paths from different equivalence classes on n vertices generate representatives of different equivalence classes on $n + 2$ vertices and thus $r(n+2) \geq r(n)$.

Proof, Part A. The induction paths, P_{n+2}, obtained by the aforementioned process must fall into three mutually exclusive types. Each type is determined by the value of m such that the right characteristic number denoted by r_1 in the RCS of the Euler path corresponds to the corner a_m. The three mutually exclusive types are as follows.

Type A: $m = N + 2n + 1$. For a type A P_{n+2}, $r_1 = N + 4$.

Type B: $N < m < N + 2n + 1$. For a type B P_{n+2}, r_1 is dependent upon the Euler path on n vertices which is used to construct the induction path.

Type C: $1 \leq m \leq N$. For a type C P_{n+2}, r_1 is dependent upon the Euler path on n vertices which is used to construct the induction path.

Thus after a rotation we obtain each type of induction path with its corresponding RCS as follows:

Type A

path: (n+2) {1 2 3 4 ··· } 1 (n+1) 2 (n+2) 3 (n+1)
RCS: (N+4) [X X X X ··· X] (2n+1) 4 (2n) 4 (2n-1) 4

path (cont): 4 (n+2) 5 ··· (n+2) n (n+1) (n+2)
RCS (cont): (2n-2) 4 X ··· 3 X (N+3).(N+4).

Note that for Type A Euler path there is one vertex of the I-part occurring before the P-part, followed by the N vertices of the P-part, followed by the remaining 2n vertices of the I-part.

Type B

path: u ··· (n+2) n (n+1) (n+2) {1 2 3 4 ··· }

RCS: d ··· 3 X (N+3) (N+4) [X X X X ··· X]

path (cont): 1 (n+1) 2 (n+2) 3 (n+1) 4 (n+2) ··· u

RCS (cont): (2n+1) 4 (2n) 4 (2n-1) 4 (2n-2) 4 ··· .

Note that d is the right characteristic number of a corner denoting the vertex u and $d \geq N + 4$. We will assume that the vertex u is denoted by the jth corner of the I-part of (5.2), that is, we will assume that j-1 corners of the I-part follow the P-part of a Type B path.

Type C

path: v ··· } 1 (n+1) 2 (n+2) 3 (n+1) 4 (n+2) 5 ···

RCS: e ···X](2n+1) 4 (2n) 4 (2n-1) 4 (2n-2)4 X ···

path (cont): ··· (n+2) n (n+1) (n+2) {1 2 3 4 ··· v

RCS (cont): ··· 3 X (N+3) (N+4) [X X X X ··· .

Note that e is the right characteristic number of a corner denoting the vertex v and that $e \geq N + 4$. We will assume that the vertex v is denoted by the ith corner of the P-part of (5.2), that is, we will assume that there are i-1 corners of the P-part which follow the I-part of a type C path.

We now consider a set of mutually exclusive cases which will exhaust all induction paths which arise from the aforementioned construction process. In each case we assume that the two induction paths, which have been constructed from non-equivalent Euler paths on n vertices, are related through a finite sequence of rotations and vertex permutations. In other words, we assume that the RCS's of the two induction paths are identical. We obtain a contradiction by showing that the RCS's are not identical, and thus that if the induction paths are equivalent they are related through a reflection only.

Case 1. The two induction paths are of the same type. We are assuming that the two Euler paths on n vertices are non-equivalent. Therefore, there is some vertex a, where $1 \leq a \leq n$, which is denoted by different sets of corners for the two Euler paths on n vertices. Thus, the vertex a is denoted by different sets of corners for the P-parts of the two induction paths P_{n+2}, and so the sets of right characteristic numbers which correspond to the corners denoting the vertex a must be different. This is because the vertex a must be denoted by the same corner of the I-part of the two induction paths. Now assume that the corner to which the right characteristic number denoted by r_1 corresponds is the same for both P_{n+2}'s. Then the two RCS's cannot be identical because the set of corners which correspond to the right characteristic numbers of corners denoting the vertex a must be different for the two P_{n+2}'s. This means that the two P_{n+2}'s cannot be of type A since for type A paths r_1 corresponds to corner $N + 2n + 1$.

Assume the P_{n+2}'s are both of type B. The corners to which the right characteristic numbers equal to r_1 correspond cannot denote either the vertex $n + 1$ or the vertex $n + 2$. This is because all right characteristic numbers corresponding to corners denoting these vertices are known, and if the greatest of them, $N + 4$, were r_1 we would have a type A path.

Thus r_1 must correspond to different corners in each of the type B P_{n+2}'s. This is because each vertex different from the vertex $n + 1$ and the vertex $n + 2$ is denoted by exactly one corner of the I-part of a type B path. These vertices are denoted by corners occupying different, actually consecutive odd, positions of the I-part of the type B path. Thus the pair of right characteristic numbers $(N+3,N+4)$ which corresponds to the consecutive pair of corners denoting the pair of vertices $(n+1,n+2)$ must be identical with some other pair of consecutive right characteristic numbers from the I-part of the RCS of a type B path. But this is impossible by Lemma 4, and so the two paths cannot be of type B.

Assume now that the P_{n+2}'s are both of type C. Let the corner to which the right characteristic number denoted by r_1 corresponds be a_i for one of the paths and a_{i+t} for the other path. If t is greater than or equal to 2n, then the $k + 1$ corners which denote the vertex $n + 1$ which occur in the I-part of one of the P_{n+2}'s can be transformed by a vertex permutation into 2n corners of the P-part of the other P_{n+2}. But this is impossible by Lemma 5, and so t must be less than 2n. If t is less than 2n, the pair of right characteristic numbers $(N+3, N+4)$ which corresponds to the consecutive pair of corners which denote the vertex $n + 1$ and the vertex $n + 2$ which occurs in the I-part of one of the P_{n+2}'s must be identical with some pair consisting of right characteristic numbers of a consecutive pair of corners of the first 2n positions of the I-part of the other P_{n+2}. But this is impossible by Lemma 4, and so both paths cannot be of type C. Thus, Case 1 has been proved to be impossible.

Case 2. The two induction paths are of types A and B. The vertex u which is denoted by the first corner of the type B path is neither the vertex $n + 1$ nor the vertex $n + 2$. This is seen by considering the right characteristic numbers of the corners which denote these vertices. All are less than $N + 4$ except that of the last corner of the I-part which denotes the vertex $n + 2$. But if this corner denoted the vertex u, we would have a type A path instead of a type B path. Thus, u is one of the vertices of the set $\{1, 2, \cdots, n\}$. This means that d is one of the unknown right characteristic numbers of the I-part of an R_{n+2}. Consider the last element of the RCS of the type B path. This number must be either 4 or 3. Now consider the last element of the RCS of the type A path. This number is N + 3. If the two RCS's are to be identical these two right characteristic numbers must be equal. This is impossible by Lemma 3 and so this case is impossible.

Case 3. The two induction paths are of types A and C. Remembering that the vertex v is denoted by the ith corner of the P-part of (5.2), we consider three subcases which exhause this case.

Subcase 3A. Assume $i < 2n$. This requires that the pair $r_{N+2n+1-i}, r_{N+2n+2-i}$ of the RCS of the type C path must equal some consecutive pair form the first 2n elements of the RCS of the I-part of the type A path. This is impossible by Lemma 4.

Subcase 3B. Assume $i = 2n$. This requires that r_{N+2} of the RCS of the type A path equals $r_{N+2n+2-i}$ of the RCS of the type C path. This leads to the equation, $2n + 1 = N + 4$, which is impossible by Lemma 2.

Subcase 3C. Assume $i > 2n$. This means that the I-part of the type A path may be transformed by a vertex permutation into a sequence which occupies consecutive corners in the P-part of the type C path, which is impossible by Lemma 5. Thus Case 3 has been proved to be impossible.

Case 4. The two induction paths are of types B and C. Remembering that the vertex v is denoted by the jth corner of the I-part of (5.2), we consider three subcases which exhaust this case.

Subcase 4A. Assume $N-i < 2n + 1 - j$. We now have two possibilities, either r_{2n+2-j} of the RCS of the type B path must equal r_{N+2-i} of the type C path, or the pair r_{2n+1-j}, r_{2n+2-j} of the RCS of the type B path must equal some consecutive pair from the first $j-1$ elements of the RCS of the I-part of the type C path. The first of these possibilities leads to the equation, $N + 4 = 2n + 1$, which is impossible by Lemma 2. The second possibility is impossible by Lemma 4, because the pair r_{2n+1-j}, r_{2n+2-j} is the pair $(N+3, N+4)$.

Subcase 4B. Assume $N - i = 2n + 1 - j$. This means that the I-part of the type C path can be transformed by a vertex permutation into a sequence which occupies successive corners in the P-part of the type B path which is impossible by Lemma 5.

Subcase 4C. Assume $N - i > 2n + 1 - j$. We show that the I-part of the type B path can be transformed into a sequence which occupies successive corners of the P-part of the type C path, which is impossible by Lemma 5. Consider the first $2n + 2 - j$ corners of the type B path. These corners are from the I-part of the type B path. Since $N - i > 2n + 1 - j$ these $2n + 1 - j$ corners can be transformed by a vertex permutation into a sequence occupying consecutive corners in the P-part of the type C path. Consider now the last $j-1$ corners of the type B path. These corners can be transformed by a vertex permutation into a sequence occupying consecutive corners in the P-part of the type C path also, because if they were not, one of the following would have to be true. Either $r_{N+2n+2-i}$ of the RCS of the type C path equals $r_{N+2n+2-j}$ of the RCS of the type B path, or the pair $r_{N+2n+1-i}, r_{N+2n+2-i}$ of the RCS of the type C path must equal some consecutive pair from the first $j-1$ elements of the RCS of the RCS of the I-part of the type B path. The first possibility leads to the equation, $2n + 1 = N + 4$, which is impossible by Lemma 2. The second possibility is impossible by Lemma 4, because the pair $r_{N+2n+1-i}, r_{N+2n+2-i}$ is the pair $(N+3, N+4)$. Thus, Subcase 4C is proved to be impossible which completes Part A of the proof.

Proof, Part B. In this part we show that two induction paths formed from non-equivalent Euler paths on n vertices cannot be related through a reflection. To do this we will consider an induction path, P_{N+2}, which has been reflected once. We will call this reflected induction path Q_{n+2}. As was seen in Chapter 3, the RCS of Q_{n+2} is identical to the LCS of P_{n+2}. If we compare the RCS of the P_{n+2} with the RCS of the Q_{n+2} and show that they cannot be identical, we will have shown that two P_{n+2}'s formed from non-equivalent Euler paths on n vertices cannot be related through a reflection. We will show that the RRS's of P_{n+2} and Q_{n+2} can never be identical and since the RCS is just a specific element of the set of all RRS's we will have shown that the RCS's cannot be identical. Consider the RRS's of P_{n+2} and Q_{n+2}.

The RRS of P_{n+2} is of the form

[X X X ··· X](2n+1) 4 (2n) 4 (2n-1) 4 (2n-2) 4 X
4 X 4 ··· X 4 X 3 X (N+3) (N+4).

The RRS of Q_{n+2} is of the form

[X X X ··· X (2n-2) (2n-1) (2n) (2n+1)] 3 4 X 4 X
4 X ··· X 4 X (N+4) X (N+3) X.

All the unknown right characteristic numbers of the I-part of P_{n+2} must be greater than or equal to n+2. This follows from the fact that the vertex i cannot occur before the ith position of the P-part because we are using only the representative Euler path of each equivalence class on n vertices to form P_{n+2}. Now consider the known right characteristic number pair (3,4) of the I-part of Q_{n+2}. All right characteristic numbers of the I-part of P_{n+2} are greater than or equal to 4 except the known right characteristic number 3, which precedes the known right characteristic number N+3. The pair (3,N+3) cannot equal the pair (3,4) by Lemma 3.

Since the first and last right characteristic numbers of the I-part of P_{n+2} are $2n + 1$ and $N + 4$, both of which are greater than 4 by Lemma 3, we find the following. No pair of consecutive right characteristic numbers of the RRS of P_{n+2} which contains at least one right characteristic number from the I-part of the RRS of that path can equal the pair (3,4). Thus if the RCS of Q_{n+2} is to be identical to the RCS of P_{n+2} the pair (3,4) of the RRS of the I-part of Q_{n+2} must equal some pair of right characteristic numbers from the RRS of the P-part of P_{n+2}.

By Lemma 5, the first $2n-2$ vertices of the I-part of Q_{n+2} cannot all have their images in the P-part of P_{n+2}. But unless the first $2n-4$ vertices of the I-part of Q_{n+2} all have their images in the P-part of P_{n+2}, there will be a vertex which occurs in the P-part of an induction path which occurs more than once in its I-part. This is impossible because of the way we constructed these induction paths. Thus we have that the first $2n-4$ vertices of the I-part of Q_{n+2} must have their images in the P-part of P_{n+2}. This means that the right characteristic number of the $2n-2$ nd corner of the I-part of Q_{n+2} must equal the right characteristic number of either the first or the second corner of the I-part of P_{n+2}. This last statement leads to the equations, $N + 4 = 2n + 1$, and $N + 4 = 4$, respectively, which are impossible by Lemmas 2 and 3, respectively. Thus the RRS of P_{n+2} cannot be identical to the RRS of Q_{n+2} and the proof of Part B is complete. This also completes the proof of Theorem 10.

INDEX OF NOTATION

n - The order of the vertex set of a full graph.

N - The number of edges in a full graph on n vertices. Also the number of edges in an Euler path on n vertices.

k - The number of corners of an Euler path on n vertices which denote a specific vertex.

a_i - The ith corner of an Euler path on n vertices.

r_i - The right characteristic number associated with the corner a_i of an Euler path.

s_i - The left characteristic number associated with the corner a_i of an Euler path.

RRS - Right representative sequence.

RCS - Right characteristic sequence.

LRS - Left representative sequence.

LCS - Left characteristic sequence.

L - The length of the principal segment of an Euler path.

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