AN ABSTRACT OF THE DISSERTATION OF

Hamid R. Behmard for the degree of Doctor of Philosophy in Mathematics

presented on July 8. 1999.

Title: Nonperiodic Sampling Theorems and Filter Banks

Signature redacted for privacy.

Abstract approved: _____

Adel Faridani

Sampling theorems provide exact interpolation formulas for bandlimited functions. They play a fundamental role in signal processing. A function is called bandlimited if its Fourier transform vanishes outside a compact set. A generalized sampling theorem in the framework of locally compact Abelian groups is presented. Sampling sets are finite unions of cosets of closed discrete subgroups. Such sampling sets are not necessarily periodic and cannot be treated in that setting. An exact reconstruction formula is found for the case that the support of the Fourier transform of the function which needs to be reconstructed satisfies certain conditions.

The notion of a filter bank is generalized in the framework of locally compact Abelian groups. Conditions for perfect reconstruction are derived. It is shown that this theory includes some generalized sampling theorems and results in multisensor deconvolution problems as special cases.

Nonperiodic Sampling Theorems and Filter Banks

by

Hamid R. Behmard

A Dissertation

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed July 8, 1999
Commencement June 2000

Doctor of Philosophy dissertation of <u>Hamid R. Behmard</u> presented on <u>July 8, 1999</u>

APPROVED:

Signature redacted for privacy.

Major Professor, representing Mathematics

Signature redacted for privacy.

Chair of the Department of Mathematics

Signature redacted for privacy.

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Signature redacted for privacy.

Hamid R. Behmard, Author

ACKNOWLEDGMENT

TABLE OF CONTENTS

LIST OF FIGURES

NONPERIODIC SAMPLING THEOREMS AND FILTER BANKS

1. PRELIMINARIES

1.1. INTRODUCTION

In this section, we give a short history of the problem and the dissertation summary.

1.1.1. *A BRIEF HISTORY OF THE PROBLEM*

The majority of signals encountered in engineering and scientific environments can be viewed as analog signals, i.e., functions of a continuous variable. However, transmission, storage and manipulation of signals are easily permitted when signals are digital which can be presented by discrete functions. Thus, it was inevitable to find a means to convert analog signals to digital ones. In 1949 the classical sampling theorem was introduced to information theory by Shannon. Before Shannon, others have derived the theorem in different contexts, and the general idea has even been attributed to Cauchy; for a detailed review see [17]. In the literature on signal processing, the classical sampling theorem is roughly stated as follows.

Suppose $f \in L_2(\mathbb{R})$ and the Fourier transform $\widehat{f}(\xi)$ vanishes outside $K = (-W, W)$. Then $f(x)$ is completely determined by its values on $H = \{n/2W : n \in \mathbb{Z}\}$ by the formula

$$f(x) = \sum_{n \in \mathbb{Z}} f(n/2W) \frac{\sin \pi(2Wx - n)}{\pi(2Wx - n)}. \tag{1.1}$$

From above, one can see that the sampling theorem provides a reconstruction formula for a band-limited function of one variable from its sampled values. This theory has been extended to higher dimensions and to nonuniform sampling sets; see the reviews [2, 22, 23, 17, 15]. We are specifically interested in three branches of advancement.

The first branch of advancement is Kluvanek's important generalization which resulted from replacing $\mathbb{R}$ by an arbitrary locally compact Abelian (LCA) group G [19]. The sampling set is then a coset of a closed discrete subgroup H of G. The Fourier transform of the function f is assumed to vanish almost everywhere outside a set $K \subseteq R \subset \widehat{G}$, where R is a fundamental domain of $H^\perp$, i.e., a measurable subset R of $\widehat{G}$ such that every $\xi \in \widehat{G}$ can be uniquely written as $\xi = r + \eta$, where $r \in R$ and $\eta \in H^\perp$ (see (1.2.12) in § 1.2 below). One can view R as a complete set of coset representatives of $\widehat{G}/H^\perp$. Here $\widehat{G}$ and $H^\perp$ are the dual group of G and the annihilator subgroup of H respectively. The following is the sampling theorem proved in [19]:

Theorem 1.1.1 *Suppose $f \in L_2(G)$ and $\widehat{f}(\xi) = 0$ for almost all $\xi \notin R$. Then f is equal almost everywhere to a continuous function. If f itself is continuous, then*

$$f(x) = \frac{1}{c_{H^\perp}} \sum_{y \in H} f(y) \, \widehat{\chi}_R(-(x - y)) \tag{1.2}$$

uniformly on G and in the sense of the convergence in $L_2(G)$ where $\widehat{\chi}_R(-z)$ is the inverse Fourier transform of the characteristic function of R. Furthermore,

$$\| f \|^2 = \frac{1}{c_{H^\perp}} \sum_{y \in H} |f(y)|^2.$$

We note that in Kluvanek's paper, $c_{H^\perp} = 1$ and in our normalization $c_{H^\perp} = m_{\widehat{G}}(R)$, the Haar measure of R. If we choose $G = \mathbb{R}$, $K = (-W, W)$ and $H = (1/2W)\mathbb{Z}$, then

$$\widehat{\chi}_R(-(x - n/2W)) = \frac{\sin \pi(2Wx - n)}{\pi(2Wx - n)}.$$

Thus, if $f \in L_2(\mathbb{R})$ and $\widehat{f} = 0$ outside K, then we obtain (1.1).

We note that in Kluvanek's theorem, the support K of $\widehat{f}$ is contained in a fundamental domain R of $H^\perp$. This means that in the case of the classical sampling theorem, H and K must be chosen so that the translated sets $K + \eta$ with $\eta \in H^\perp$ are disjoint. But it is important to also consider the case where these translated sets are not disjoint. This case is part of the second branch of advancement which was first considered by Kohlenberg in [20] and its now known as periodic sampling. Consider $G = \mathbb{R}$, $H = (1/2W)\mathbb{Z}$, $H^\perp = 2W\mathbb{Z}$ and $K = (-W_0 - W, -W_0) \cup (W_0, W_0 + W)$. Since the (Lebesgue) measure of K is $2W$, we expect the optimal sampling rate to be $2W$. The classical sampling theorem allows exact reconstruction at this optimal rate only if W_0 is an integer multiple of W, since otherwise the translates $(K + 2lW)$, $l \in \mathbb{Z}$ are not disjoint. Kohlenberg overcame this restriction by considering sampling sets of optimal density which are no longer subgroups, but unions of cosets of a subgroup, e.g., by sampling the values of f on the subgroup $H_1 = W^{-1}\mathbb{Z}$ and on $x_1 + H_1$. Note that the set $H_1 \cup (x_1 + H_1)$ has the average density of $2W$. He gave the following reconstruction formula.

Theorem 1.1.2 *For any function $f(x)$ with $\widehat{f}(\xi) \equiv 0$ for all $\xi \notin K = (-W_0 - W, -W_0) \cup (W_0, W_0 + W)$, the exact interpolation formula is*

$$f(x) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{W}\right) k\left(x - \frac{n}{W}\right) + \sum_{n \in \mathbb{Z}} f\left(x_1 + \frac{n}{W}\right) k\left(\frac{n}{W} + x_1 - x\right) \qquad (1.3)$$

$$k(t) = \frac{cos[2\pi(W_0 + W)t - (r + 1)\pi W x_1] - cos[2\pi(rW - W_0)t - (r + 1)\pi W x_1]}{2\pi W t \; sin(r + 1)\pi W x_1}$$
$$+ \frac{cos[2\pi(rW - W_0)t - r\pi W x_1] - cos[2\pi W_0 t - r\pi W x_1]}{2\pi W t \; sinr\pi W x_1}.$$

(1.4)

We need to require that $x_1 \neq n/(Wr), x_1 \neq n/(W(r + 1))$ where r is an integer such that $(2W_0/W) \leq r < (2W_0/W) + 1$.

Since the sampling set is invariant under shifts by elements of the subgroup H_1, this area of sampling theory is known as periodic sampling. Further studies of nonequidistant but periodic sampling on $\mathbb{R}$ can be found in [18, 25, 30]. Generalizations to higher dimensions include [4, 11]. Further results for $\mathbb{R}^s \times [0, 2\pi)^r$ [6] with their applications to computed tomography can be found in [5, 6].

In his paper [7], Faridani gives a general and unified treatment of periodic sampling in the framework of LCA groups. He considered sampling sets which are unions of finitely many cosets of a closed subgroup H with K an arbitrary compact subset of $\widehat{G}$ to be the support of the Fourier transform of the function f. In case of periodic sampling the sets $K + \eta$, $\eta \in H^\perp$ are usually not disjoint. This induces a decomposition of the set K into disjoint subsets K_l, where each K_l consists of the points which are overlapped by the same translates, i.e., $K + \eta_1^{(l)}, \dots, K + \eta_{m_l - 1}^{(l)}$. Faridani uses the characteristic functions χ_{K_l} of these sets as the building blocks for his reconstruction formula. We will review the details of this decomposition at the beginning of the § 2.1.2 below. Another important difference to note is that the subgroup used by Faridani does not need to be discrete. We refer to the following theorem in [7] at various parts of the present work. It gives a reconstructed version of f from the knowledge of f on cosets $x_n + H$, $n = 0, \dots, N - 1$.

Theorem 1.1.3 *([7]) Assume that $f \in L_1(G)$ is continuous, every function $y \rightarrow f(x + y)$ belongs to $L_1(H)$, and that $\widehat{f}$ vanishes outside a compact set $K \subset \widehat{G}$.*

Let $M_l = \{\eta_1^{(l)}, \ldots, \eta_{m_l-1}^{(l)}\}$, $l = 1, \ldots, L$ be the values assumed by $M_\xi = \{\eta \in H^\perp \setminus \{0\} : \xi \in K + \eta\}$ as ξ runs through K. Let χ_{K_l} be the indicator function of $K_l = \{\xi \in K : M_\xi = M_l\}$. Assume $x_0 + H, \ldots, x_{N-1} + H \in G/H$ are such that for $l = 1, \ldots, L$ the systems of equations

$$\sum_{n=0}^{N-1} \beta_n^{(l)} = 1$$

$$\sum_{n=0}^{N-1} \beta_n^{(l)} e^{2\pi i <x_n, \eta_j^{(l)}>} = 0, \quad j = 1, \ldots, m_l - 1 \tag{1.5}$$

admit solutions $\beta_n^{(l)}$, $n = 0, \ldots, N-1$. Then

$$\widehat{f}(\xi) = \sum_{n=0}^{N-1} \sum_{l=1}^{L} \beta_n^{(l)} \chi_{K_l}(\xi) \int_H f(x_n + y) e^{-2\pi i <x_n + y, \xi>} dm_H(y) \tag{1.6}$$

and

$$f(x) = \sum_{n=0}^{N-1} \int_H f(x_n + y) k_n(x - x_n - y) dm_H(y) \tag{1.7}$$

with

$$k_n(z) = \sum_{l=1}^{L} \beta_n^{(l)} \check{\chi}_{K_l}(z). \tag{1.8}$$

We note that the case with $N = 1$ and $L = m_1 = 1$ yields the classical sampling theorem [19]. This requires that the sets $K + \eta$, $\eta \in H^\perp$ be mutually disjoint. In the case where $G = \mathbb{R}$, $H = \mathbb{Z}/W$ with $K = (-W_0 - W, -W_0) \cup (W_0, W_0 + W)$, Theorem 1.1.3 gives Kohlenberg's result (1.3).

The third branch of advancement is the case of sampling theory with nonperiodic sampled data but with partial structure. In this case, sampling sets are unions of cosets of possibly different subgroups and therefore no longer periodic. Walnut's paper on nonperiodic sampling of bandlimited functions [28] falls into this category. In [28], Walnut considered sampling sets in $\mathbb{R}$ to be the union of different subgroups,

i.e., $\Lambda = \cup_{i=1}^m \{n/2r_i\}_{n \in \mathbb{Z}}$ where r_i/r_j is irrational if $i \neq j$. As part of his work in [28], Walnut proves a uniqueness theorem for this case and higher dimensions. We quote the uniqueness theorem in one dimension.

Definition 1.1.4 : Given a finite set of positive numbers ρ, define $R_\rho = \sum_{r \in \rho} r$, $I_\rho = [-R_\rho, R_\rho]$, and $\Lambda_\rho = \{n/2r : n \in \mathbb{Z} \setminus \{0\}, r \in \rho\}$.

Theorem 1.1.5 ([28])Let $1 \leq p < \infty$, let $0 < r_1 < r_2 < \ldots < r_m$ be such that r_i/r_j is irrational if $i \neq j$, and let $\rho = \{r_i\}_{i=1}^m$. Suppose that $\widehat{f} \in L_p(I_\rho)$ vanishes almost everywhere outside I_ρ and f satisfies:

a. $f(\lambda) = 0$ for all $\lambda \in \Lambda_\rho$,

b. $f^{(j)}(0) = 0$ for $j = 0, \ldots, m - 1$.

Then $f(x) = 0$ for almost every $x \in \mathbb{R}$.

The condition **b** on the derivatives of f at zero is required since the sampling sets $\{n/2r_i : n \in \mathbb{Z}\}$ have zero as a common point. Walnut generalized these results to higher dimensions using tensor products. The Fourier transform of f is supported in the hypercube $I_\rho^d = \{x = (x_1, \ldots, x_d) : x_i \in I_\rho\}$ and the sampling sets are $\Lambda_\rho^d = \{\lambda = (\lambda_1, \ldots, \lambda_\rho) : \lambda_i \in \Lambda_\rho\}$, see Theorem 4.1 in [28, p. 445]. Walnut also constructed a collection of Lagrange-type interpolation functions which in principle would yield a reconstruction formula. However, Walnut also proved that this collection is unbounded in $L_2(\mathbb{R})$ (see Proposition 3.1 in [28, p. 442]). In a recent paper [29], Walnut used the results from nonperiodic sampling proved in [28] to compute explicit, compactly supported solutions, $\{\nu_{i,\Phi}\}_{i=1}^m$, to the deconvolution equation $\sum_{i=1}^m \mu_i * \nu_{i,\Phi} = \Phi$, where Φ is a given function and $\mu_i = \chi_{[-r_i, r_i]^d}$, $i = 1, \ldots, m$ with r_i as above.

The purpose of the present work is to consider sampling sets which are finite unions of cosets of closed discrete subgroups in the framework of locally compact

Abelian groups. In particular, we call H a lattice if G/H is compact. In our theory, we use admissible lattices which are lattices so that the annihilator subgroup $H^{\perp}$ of H is also a lattice with R as fundamental domain of $H^{\perp}$. We assume that $\widehat{f}$ vanishes outside a precompact subset K of $\widehat{G}$ which is constructed of the unions of fundamental domains of different H_is. By a precompact set, we mean a set whose closure is compact. Furthermore, K has a telescopic shape, i.e., for $R_1 \subset \ldots \subset R_N$, let $K_1 = R_1$ and for $j = 2, \ldots, N$

$$K_j = R_j \cup (\eta_j + R_{j-1}) \cup \ldots \cup ((\eta_j + \ldots + \eta_2) + R_1) \tag{1.9}$$

$$K_{j-1} \subset R_j$$

with $\eta_j \in H_j^{\perp}$. Then $K = K_N$. Using the classical sampling theorem, we first reconstruct a version of f, $S_{H_N} f$, with respect to H_N. Thus f and $S_{H_N} f$ coincide on $x_N + H_N$ and their difference vanishes on $x_N + H_N$ and using Theorem 2.1.4 can be represented by a function whose Fourier transform vanishes outside the set K_{N-1}. One can continue with this procedure till $K_1 = R_1$ is the only set left. This produces a recursive reconstruction formula to give f. To complete the introduction, we include the main result of the present work at this point. For definitions of admissible lattices and admissible subsets of $\widehat{G}$ with respect to H_i see Definition 1.2.12 and Definition 2.1.7 respectively.

Theorem 1.1.6 : *Suppose that H_i, $i = 1, \ldots, N$ are admissible lattices with R_i as a fundamental domain of $H_i^{\perp}$ for each i such that $R_1 \subset \ldots \subset R_N$. Assume that K is an admissible subset of $\widehat{G}$ with respect to $H_1, \ldots, H_N$ and K_j as in Definition 2.1.7. Let $f \in L_2(G)$ be continuous so that $\widehat{f}$ vanishes outside almost everywhere K. Assume that $\langle z - x_j, \eta_j \rangle \neq 0$ for $z \in \bigcup_{i=1}^{j-1}(x_i + H_i)$ and $1 < j \leq N$. Then there are continuous functions $f_j \in L_2(G)$ such that $supp(\widehat{f_j}) \subseteq K_j$, $j = 1, \ldots, N$ and for almost every $x \in G$*

$$f_1(x) = S_{H_1} f_1(x), \tag{1.10}$$

$$f_j(x) - S_{H_j} f_j(x) = f_{j-1}(x)(1 - e^{2\pi i \langle x - x_j, \eta_j \rangle}) \quad j = 2, \ldots, N, \tag{1.11}$$

$$f_N = f \tag{1.12}$$

with

$$S_{H_i} g(x) = \frac{1}{m_{\widehat{G}}(R_i)} \sum_{y \in H_i} g(x_i + y) \widehat{\chi}_{R_i}(-(x - x_i - y)), \quad x \in G. \tag{1.13}$$

This recursion provides an algorithm to compute f from the sampled values $f(z)$, $z \in \bigcup_{i=1}^{N}(x_i + H_i)$.

We can differentiate this work from periodic sampling by the following points.

- Using nonperiodic sampling sets.

- Under certain conditions, it is advantageous to treat a periodic sampling set as union of cosets of different subgroups. Example 2.1.21 shows one such case.

The set K which is considered here needs to satisfy certain conditions which are not necessary in periodic sampling. Furthermore, in periodic sampling sampling sets need not to be discrete. We also differentiate the work here from Walnut's nonperiodic sampling in the framework of $G = \mathbb{R}^d$ by the following points.

- Using the general setting of locally compact Abelian groups.

- Using the cosets of the subgroups instead of the condition on the derivatives of f at zero.

- Sampling sets are not built by using tensor products of lower dimensional sets.

- The support of the Fourier transform of the function that needs to be reconstructed does not need to be a hypercube.

- A recursive formula is provided that can reconstruct a bandlimited function under certain conditions.

On the other hand, the theory developed here requires a compatibility condition involving the support K of $\widehat{f}$ and the sampling sets which are not necessary in the nonperiodic theory presented by Walnut. We also make a note that the theory provided here is not general enough to treat the general case of nonperiodic irregular sampling. This is the case where the sampling set only needs to meet some density requirements. For a treatment of this case see, e.g., [2, 9, 10, 12, 21] and the references given there.

The classical sampling theorem is only one method in which data can be extracted from a function and still contain sufficient information to reconstruct the function. Shannon also noted that one could reconstruct the function from the knowledge of the function and its first derivative at every other sample point. This idea was extended to higher derivatives. These and other reconstruction ideas were remarkably unified in a generalization of the sampling theorem by Papoulis [25]. The generalization concerns reconstruction of a signal with data sampled at $1/N$th the optimal rate from the output of N filters through which the signal has been sent. The result is a generalization of the reconstruction from the filtered signal's samples. On $\mathbb{R}$, this is an analogue of what is called filter bank in a discrete setting. Since the area of filter banks is vast, we won't be able to give a comprehensive introduction. However, the following should suffice to serve our purpose.

A digital signal is a function defined on the integers, denoted by a doubly infinite vector. The delay operator S is given by $S\,x(m) = x(m-1)$ where $m \in \mathbb{Z}$.

The inverse of the delay operator is called advance. A digital filter is a linear combination of delays and advances. The N-fold decimator (N being an integer) is presented by the relation $(\downarrow N)\, x(m) = x(Nm)$, which indicates that only the input samples occurring at times that are multiples of N are saved. The symbol $(\downarrow N)$ indicates *downsampling* or decimation. The expander is described by the relation

$$(\uparrow N)\, x(m) = \begin{cases} x(m/N), & \text{if } m \text{ is a multiple of } N; \\ 0, & \text{otherwise.} \end{cases}$$

That means, the expanders insert $N - 1$ zeros between adjacent components of $x(m)$. The symbol $(\uparrow N)$ indicates *upsampling* or expansion. Upsampling and downsampling are not shift-invariant operators. A filter bank is a set of filters, which are linked by sampling operators and sometimes by delays. An analysis bank consists of two steps, *filtering* and *downsampling*. A synthesis bank which follows an analysis bank has also two steps, *upsampling* and *filtering*. When the reconstructed output $\widehat{x}(n)$ from the synthesis bank is identical to the original input x of the analysis bank(with only a time delay), the reconstruction is called "perfect reconstruction".

We see that a filter bank is the analogue of the generalization of the sampling theory defined on $\mathbb{R}$ given by Papoulis. Our purpose is to give a general definition of the concept of a filter bank which can be used as a unifying tool in areas such as periodic sampling theory.
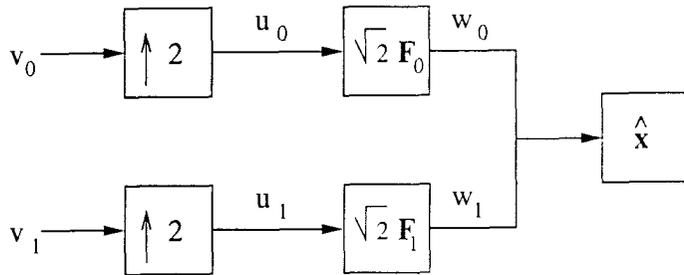
FIGURE 4.2. Synthesis Bank
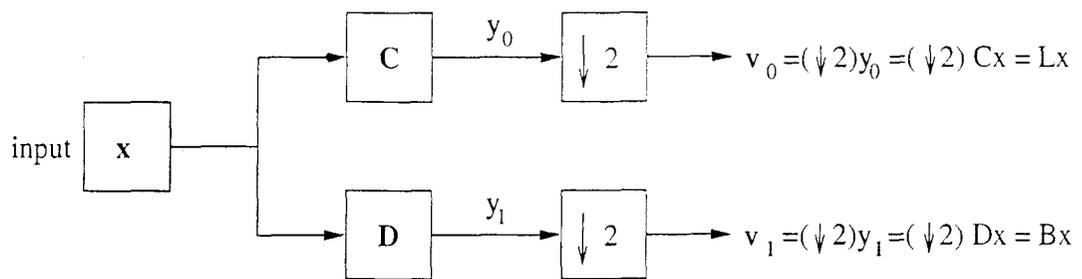
FIGURE 4.1. Analysis Bank

1.1.2. *DISSERTATION SUMMARY*

A brief outline of the dissertation is as follows.

In § 1.2, we introduce the necessary facts about Fourier analysis on locally compact Abelian groups (LCA groups). We will also provide lemmas and conventions such as normalization of the Haar measures which will be used in the next two chapters.

The main results are proved in chapters 2 and 3.

In chapter 2, we prove a uniqueness theorem and introduce a recursive method as a reconstruction formula for a class of nonperiodic sampling sets. In § 2.1, we prove theorems and lemmas which facilitate our work for proving uniqueness and reconstruction. Of those, Theorem 2.1.4 contributes results which are significant on their own. Consider a function f which vanishes on a closed discrete subgroup H and its Fourier transform vanishes outside a precompact subset $K \subseteq \widehat{G}$. If $K + \eta$, $\eta \in H^\perp$ are disjoint, the classical sampling theorem implies that f must vanish almost everywhere. Theorem 2.1.4 tells us what can be said about f, if the translated sets are not disjoint.

A uniqueness theorem for the case of nonperiodic sampling sets is proved in § 2.1.1, i.e., if f vanishes on a union of cosets of finite number of different admissible lattices and $\widehat{f}$ vanishes outside a precompact set $K \subseteq \widehat{G}$, then f vanishes everywhere on G. An example in 2 dimensions is given.

In § 2.1.2, we assume the same hypotheses as uniqueness with the exception that f does not vanish on H. A recursive formula is found which can reconstruct f. In case of $H_1 = H_2 = H$, it is shown that this recursive formula has consistent results with Theorem 1.1.3 by Faridani (cf. [7]). Further examples in one dimension are furnished.

In chapter 3, we consider how to generalize the notion of filter banks to use it as a unifying tool for periodic sampling theory and multi-sensor deconvolution. In § 3.1.1, upsampling and downsampling are generalized in the framework of LCA groups. By appropriate choices of LCA groups, we showed that the results of our generalization are consistent with the results found in the area of filter bank theory.

These results are used in § 3.2 to prove a generalized form of what is known as perfect reconstruction in signal processing. We applied these results to two different areas, periodic sampling and multi-sensor deconvolution problem.

1.2. STANDARD DEFINITIONS AND THEOREMS

We present some useful notation, definitions, and theorems which will be used throughout the course of the next two chapters. Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{C}$, denote the reals, integers, and complex numbers respectively. If S is any set, we define the characteristic function (indicator function) χ_S of the set S to be the function given by

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S; \\ 0, & \text{if } x \notin S. \end{cases}$$

We use $\bar{S}$, ∂S, and $|S|$ to denote the closure, the boundary, and the number of elements in S respectively. In the next two chapters, the setting of our work is based on locally compact Abelian groups and elementary locally compact Abelian groups. We use the abbreviations LCA and ELCA for them respectively. We recall that a topological space is locally compact if every point has a closed compact neighborhood. We follow the definitions given in [13]. Recall that ELCA groups are defined through their topological factors as:

$$G \cong \mathbb{R}^d \times \mathbb{Z}^p \times \mathbb{T}^q \times \Gamma_m \qquad (d \geq 0, p \geq 0, q \geq 0, m \geq 1), \qquad (1.14)$$

where $\mathbb{R}^d$ is the Euclidean d-space, $\mathbb{Z}^p$ is the p-dimensional lattice of integers, $\mathbb{T}^q$ is the q-dimensional torus group and Γ_m is a finite Abelian group with cardinality m. The product topology inherited from the factors makes G a LCA group. If G is an ELCA group, then $L_2(G)$ is separable. Unless the contrary is explicitly stated, throughout the next two chapters G represents a LCA or an ELCA group with binary operation of addition. A character of a group G is defined to be a homomorphism of G into the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We use the notation $\langle x, \xi \rangle$ to denote the character $\xi \in \widehat{G}$ at the point $x \in G$. The group of all continuous characters of G is called the character group or the dual group of G and is denoted by $\widehat{G}$. It is also an LCA group if it is equipped with a natural addition and a natural topology, see [26, p. 10]. This can be used to prove the famous Pontryagin duality theorem.

Theorem 1.2.1 *(Pontryagin) If G is a LCA group, then the dual group of $\widehat{G}$ is G.*

Proof. For a proof, see [26, p. 28]. □

Classic examples are: $G = \mathbb{R}$, $\widehat{G} = \mathbb{R}$; $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\widehat{G} = \mathbb{Z}$; $G = \mathbb{Z}$, $\widehat{G} = \mathbb{T}$.

With every LCA group G, there is associated a non-negative regular measure m_G, the so-called Haar measure of G. This measure is not identically zero and is translation invariant. That means $m_G(E + x) = m_G(E)$ for all $x \in G$ and every Borel set $E \subset G$. The Haar measure is unique up to multiplication by a positive constant. $m_G(G)$ is finite if and only if G is compact. In this case we normalize m_G so that $m_G(G) = 1$. If G is discrete, m_G will be a multiple of the counting measure. If G is discrete but not compact, m_G can be normalized so that it equals the counting measure. For a more detailed discussion, see [26, pp.1-2].

Theorem 1.2.2 *If G is discrete, then $\widehat{G}$ is compact. If G is compact, then $\widehat{G}$ is discrete.*

Proof. For a proof, see [26, p. 10]. □

As part of the proof for Theorem 1.2.2, we have the following useful fact which gives an orthogonality relation.

Lemma 1.2.3 *If G is compact and its Haar measure is normalized so that $m_G(G) = 1$, then*

$$\int_G e^{2\pi i \langle x, \xi \rangle} dm_G(x) = \begin{cases} 1, & \text{if } \xi = 0; \\ 0, & \text{if } \xi \neq 0. \end{cases}$$

Proof. For a proof see e.g., [14, Sec. 23.19] or [26, p. 10]. □

Definition 1.2.4 : Suppose H is a closed subgroup of an LCA group G and $H^\perp$ is the set of all $\eta \in \widehat{G}$ such that $\langle y, \eta \rangle = 0$ for all $y \in H$. $H^\perp$ is called the annihilator of H.

Theorem 1.2.5 *With the above notation $H^\perp$ and $\widehat{G}/H^\perp$ are isomorphically homeomorphic to $(G/H)^\wedge$ and $\widehat{H}$ respectively, i.e.,*

$$H^\perp = (G/H)^\wedge \text{ and } \widehat{H} = \widehat{G}/H^\perp.$$

Proof. For a proof see [26, p. 35]. □

Lemma 1.2.6 *If $H^\perp$ is the annihilator of H, then H is the annihilator of $H^\perp$, i.e.*

$$(H^\perp)^\perp = H.$$

Proof. For a proof see [26, p. 36]. □

If m_G is the Haar measure associated with G and $0 < p < \infty$, $L_p(G)$ denotes the set of all Borel functions f on G such that the norm

$$\|f\|_p = \left(\int_G |f(x)|^p \, dm_G(x) \right)^{1/p}$$

is finite. If $p = \infty$, the L_∞ is the space of all bounded Borel functions on G, normed by

$$\|f\|_\infty = \text{ess sup}_{x \in G} |f(x)|;$$

where the essential supremum of $|f|$ is defined to be the smallest number λ such that $m(\{x : f(x) > \lambda\}) = 0$.

For all $f \in L_1(G)$, the continuous function $\hat{f}$ defined on $\widehat{G}$ by

$$\hat{f}(\xi) = \int_G f(x)e^{-2\pi i \langle x,\xi \rangle} \, dm_G(x) \tag{1.15}$$

is called the Fourier transform of f. From now on, we will always assume that the Haar measures of G and $\widehat{G}$ are normalized so that the following inversion theorem holds.

Theorem 1.2.7 *If $f \in L_1(G)$ is continuous and $\hat{f} \in L_1(\widehat{G})$, then*

$$f(x) = \int_{\widehat{G}} \hat{f}(\xi)e^{2\pi i \langle x,\xi \rangle} \, dm_{\widehat{G}}(\xi) = (\widehat{\hat{f}})(-x). \tag{1.16}$$

The Fourier transform can be extended to a linear isomorphism of $L_2(G)$ onto $L_2(\widehat{G})$ by means of the Plancherel Theorem.

Theorem 1.2.8 *The Fourier transform is a linear isometry of $L_2(G)$ onto $L_2(\widehat{G})$, and the inverse Fourier transform is a linear isometry $L_2(\widehat{G})$ onto $L_2(G)$. These two transformations are inverse of each other.*

Proof. For a proof see [14, Sec. 31.18]. □

Lemma 1.2.9 *If $f \in L_2(G)$ and $\widehat{f}$ has compact support, then $\widehat{f} \in L_2 \cap L_1(\widehat{G})$ and f is equal almost everywhere to a continuous function.*

Proof. $\widehat{f} \in L_2(\widehat{G})$ by the Plancherel Theorem. Thus $\widehat{f} \in L_1(\widehat{G})$ since $\widehat{f}$ has compact support. Since $\widehat{f} \in L_1(\widehat{G})$, f is also the inverse Fourier transform of $\widehat{f}$ in the L_1 sense, i.e.,

$$f(x) = \int_{\widehat{G}} \widehat{f}(\xi)e^{2\pi i \langle x,\xi \rangle} dm_{\widehat{G}}(\xi) \qquad \text{a.e.}$$

Since the right hand side is continuous, the proof is complete. □

Remark 1.2.10 : If $f \in L_2(G)$ and $\widehat{f}$ has compact support, then by Lemma 1.2.9 f is equivalent to a continuous function $\widetilde{f}$. If we define the restriction of f to $H \subset G$ to be the restriction of $\widetilde{f}$ to H, then this restriction is well defined.

For the remainder of this section and the next two sections, we will assume that G is an ELCA group. This choice is broad enough to cover almost all applications.

Definition 1.2.11 : A closed discrete subgroup H of an ELCA group G is called a lattice, if the quotient G/H is a compact group.

Definition 1.2.12 : An admissible lattice is a lattice which satisfies the following two conditions:

i) $H^\perp$ is also a lattice, and

ii) there exists a measurable precompact subset R of $\widehat{G}$ such that ∂R has measure zero and every $\xi \in \widehat{G}$ can be uniquely written as $\xi = r + \eta$, where $r \in R$ and $\eta \in H^\perp$. We call such a set R a fundamental domain of $H^\perp$.

Remark 1.2.13 : Suppose that H is an admissible lattice and R a fundamental domain of $H^\perp$. Then m_H and $m_{H^\perp}$ equal non-zero constants c_H and $c_{H^\perp}$ times the counting measure respectively. For $\widehat{f} \in L_1(\widehat{G})$ define the function $R_{H^\perp}\widehat{f} : \widehat{G}/H^\perp \to \mathbb{C}$ by

$$R_{H^\perp}\widehat{f}(\xi + H^\perp) = c_{H^\perp} \sum_{\eta \in H^\perp} \widehat{f}(\xi + \eta).$$

We notice that the sum on the right-hand side does not change when ξ is replaced by $\xi + \eta$ with $\eta \in H^\perp$, so $R_{H^\perp}\widehat{f}$ is indeed a function of the coset $\xi + H^\perp$. According to Theorem 28.54 in [14], $R_{H^\perp}\widehat{f}$ belongs to $L_1(\widehat{G}/H^\perp)$. Throughout this thesis, we use the following convention.

Convention 1.2.14 : Let H be an admissible lattice. We normalize the Haar measures on G, $\widehat{G}$, H, $H^\perp$, G/H and $\widehat{G}/H^\perp$ so that

i) The Fourier inversion formula (1.16) holds.

ii) An analogue inversion formula holds for the Fourier transform on H, i.e., for $F \in L_1(H)$ with $\widehat{F} \in L_1(\widehat{G}/H^\perp)$ so that

$$F(y) = \int_{\widehat{G}/H^\perp} \widehat{F}(\zeta) e^{2\pi i \langle y, \zeta \rangle} dm_{\widehat{G}/H^\perp}(\zeta).$$

iii) $m_{\widehat{G}/H^\perp}(\widehat{G}/H^\perp) = 1.$

iv) $\int_{\widehat{G}/H^\perp} R_{H^\perp} \widehat{f}(\xi + H^\perp) \, dm_{\widehat{G}/H^\perp}(\xi + H^\perp)$

$$= \int_{\widehat{G}/H^\perp} c_{H^\perp} \sum_{\eta \in H^\perp} \widehat{f}(\xi + \eta) \, dm_{\widehat{G}/H^\perp}(\xi + H^\perp) = \int_{\widehat{G}} \widehat{f}(\xi) \, dm_{\widehat{G}}(\xi). \tag{1.17}$$

We note that such a normalization is possible [26, section 2.7.3]. Below, we will show that $c_H = 1$ and $c_{H^\perp} = m_{\widehat{G}}(R)$.

The next lemma establishes an identification between $\widehat{G}/H^\perp, m_{\widehat{G}/H^\perp}$ and $R, m_{\widehat{G}}$ respectively, where R is a fundamental domain of $H^\perp$. This identification is used throughout the next chapter.

Lemma 1.2.15 *Assume that H is an admissible lattice and R a fundamental domain of $H^\perp$. For $g \in L_1(\widehat{G}/H^\perp)$, define the function $\widetilde{g} \in L_1(\widehat{G})$ by $\widetilde{g}(\xi) = \chi_R(\xi) g(\xi + H^\perp)$. Then the following equality holds:*

$$\int_{\widehat{G}/H^\perp} g(\xi + H^\perp) \, dm_{\widehat{G}/H^\perp}(\xi + H^\perp) = \frac{1}{c_{H^\perp}} \int_R \widetilde{g}(\xi) \, dm_{\widehat{G}}(\xi). \tag{1.18}$$

Proof. Define $\widehat{f}$ in Remark 1.2.13 to be $\chi_R(\xi) \widetilde{g}(\xi)$. Since for $\xi \in R$ and $\eta \in H^\perp$

$$\chi_R(\xi + \eta) = \begin{cases} 1, & \text{if } \eta = 0; \\ 0, & \text{if } \eta \neq 0. \end{cases}$$

We have

$$R_{H^\perp}(\chi_R \widetilde{g})(\xi + H^\perp) = c_{H^\perp} \sum_{\eta \in H^\perp} \chi_R(\xi + \eta)\widetilde{g}(\xi + \eta) = c_{H^\perp}\widetilde{g}(\xi) \qquad (1.19)$$

Thus, for $\xi \in R$

$$\int_R \widetilde{g}(\xi) dm_{\widehat{G}}(\xi) = \int_{\widehat{G}} \chi_R(\xi)\widetilde{g}(\xi) \, dm_{\widehat{G}}(\xi) \qquad (1.20)$$

$$= \int_{\widehat{G}/H^\perp} R_{H^\perp}(\chi_R \widetilde{g})(\xi + H^\perp) \, dm_{\widehat{G}/H^\perp}(\xi + H^\perp) \qquad (1.21)$$

$$= c_{H^\perp} \int_{\widehat{G}/H^\perp} g(\xi + H^\perp) \, dm_{\widehat{G}/H^\perp}(\xi + H^\perp) \qquad (1.22)$$

where (1.21) and (1.22) follow by (1.17) and (1.19) respectively. $\square$

We are ready now to determine the two constants c_H and $c_{H^\perp}$.

Note 1.2.16 : To find c_H, consider the function $f \in L_2(H)$ whose Fourier transform satisfies $\widehat{f}(\zeta) = 1$. Then by the inverse Fourier transform, we have

$$f(y) = \int_{\widehat{G}/H^\perp} e^{2\pi i \langle \zeta, y \rangle} dm_{\widehat{G}/H^\perp}(\zeta)$$
$$= \begin{cases} 1, & \text{if } y = 0; \\ 0, & \text{if } y \neq 0. \end{cases} \qquad (1.23)$$

Note that (1.23) follows from Lemma (1.2.3) and the fact that $\widehat{G}/H^\perp$ is compact. Now by the Fourier transform, we have

$$1 = \widehat{f}(\zeta) = c_H \sum_{y \in H} f(y)e^{-2\pi i \langle \zeta, y \rangle}$$
$$= c_H.$$

Furthermore, if in Lemma 1.2.15, we let $g \equiv 1$, then $c_{H^\perp}$ is equal to $m_{\widehat{G}}(R) \neq 0$.

The following version of the Poisson summation formula is given by Gröchenig in [13, p. 217].

Theorem 1.2.17 *Suppose H is an admissible lattice with R as a fundamental domain of $H^\perp$. If $\widehat{f} \in L_1(\widehat{G})$, then the periodization $R_{H^\perp}\widehat{f}(\xi + H^\perp) = m_{\widehat{G}}(R) \sum_{\eta \in H^\perp} \widehat{f}(\xi + \eta)$ is in $L_1(\widehat{G}/H^\perp)$ and for $y \in H$*

$$(R_{H^\perp}\widehat{f})^\wedge(y) = f(y). \tag{1.24}$$

If furthermore $\sum_{y \in H}|f(y)|^2 < \infty$, then $R_{H^\perp}\widehat{f} \in L_2(\widehat{G}/H^\perp)$ and

$$R_{H^\perp}\widehat{f}(\xi + H^\perp) = m_{\widehat{G}}(R)\sum_{\eta \in H^\perp}\widehat{f}(\xi + \eta) = \sum_{y \in H}f(y)e^{-2\pi i\langle y,\xi\rangle} \tag{1.25}$$

where the equality (1.25) holds almost everywhere and the right hand side converges in $L_2(\widehat{G}/H^\perp)$.

Proof. For a proof, see [13, p. 217]. $\square$

2. NONPERIODIC SAMPLING THEOREMS

2.1. SAMPLING THEOREMS

In this chapter, we prove a uniqueness theorem for nonperiodic sampling sets. We present a reconstruction formula for this case as well. We will assume throughout the next two chapters that H is an admissible lattice and R a fundamental domain of $H^\perp$. We normalize the required Haar measures according to Convention 1.2.14. With this normalization, we have found $c_H = 1$ and $c_{H^\perp} = m_{\widehat{G}}(R)$. Furthermore, we assume that $f \in L_2(G)$ is continuous, and that $\widehat{f}$ vanishes almost everywhere outside a precompact subset of $\widehat{G}$. In earlier sections, we mentioned a decomposition of K into disjoint subsets K_l. This decomposition plays an important role throughout this work. To motivate the abstract theory, we will give an example and then present the decomposition as it is introduced in [7].

Example 2.1.1 : Let $G = \mathbb{R}$, $H = \mathbb{Z}$. Then $\widehat{G} = \mathbb{R}$, $H^\perp = \mathbb{Z}$, and $\widehat{H} = \mathbb{R}/\mathbb{Z} = \mathbb{T}$. Hence fundamental domain R of $H^\perp$ can be chosen to be $[0,1)$. Thus $m_{\widehat{G}}(R) = 1$. Assume that $K = [0, 3/2]$, and $f \in L_2(\mathbb{R})$ such that the hypothesis of Theorem 1.2.17 holds. Define $F(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n \xi}$. Following Poisson summation formula (1.25), we have

$$F(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n \xi} = \sum_{n \in \mathbb{Z}} \widehat{f}(\xi + n) \qquad (2.1)$$

where $\eta \in H^\perp$. Since $\widehat{f}$ vanishes outside K, (2.1) becomes

$$F(\xi) = \begin{cases} \widehat{f}(\xi) + \widehat{f}(\xi + 1), & \text{if } \xi \in [0, 1/2] = K_2, \\ \widehat{f}(\xi), & \text{if } \xi \in (1/2, 1) = K_1, \\ \widehat{f}(\xi - 1) + \widehat{f}(\xi), & \text{if } \xi \in [1, 3/2] = K_3. \end{cases}$$

For $\xi \in K_1 = (1/2, 1)$, $\widehat{f}(\xi)$ can be recovered from $f(n)$ with $n \in \mathbb{Z}$. However, for $\xi \in K_2 \cup K_3$, $\widehat{f}(\xi)$ can not be recovered uniquely from $f(n)$. The Poisson

summation formula gives an equation which relates points in K_2 with points in K_3. Thus in general, given K, the Poisson summation formula gives an equation of the form $F(\xi) = \sum_{i=0}^{m-1} \widehat{f}(\xi + \eta_i)$ where the only nonvanishing terms have η_i for which $\xi + \eta_i \in K$ and there are m such η_is. Let $\widetilde{M}_\xi$ be the set of all such η_i, i.e., $\widetilde{M}_\xi = \{\eta \in H^\perp : \xi + \eta \in K\}$. For a given ξ the set K_ξ consists of all points $\xi' \in K$ for which $\widetilde{M}_{\xi'} = \widetilde{M}_\xi$. Furthermore, for $\eta \in \widetilde{M}_\xi$, we have $\eta \in K - \xi \subseteq K - K$, i.e., $\eta \in H^\perp \cap (K - K)$. Keeping this example in mind, next we go through the formal theory.

Lemma 2.1.2 *Suppose H is an admissible lattice and K a precompact subset of $\widehat{G}$. Then there is a decomposition of K into finitely many subsets $K_{j,k}$ with the following properties:*

i) The sets $K_{j,k}$ are mutually disjoint for $j = 1, \ldots, J$, $k = 0, \ldots, n_j - 1$ and

$$K = \bigcup_{j=1}^{J} \bigcup_{k=0}^{n_j - 1} K_{j,k}.$$

ii) There are $\eta_n^{(j,k)} \in H^\perp$, $j = 1, \ldots, J$; $k, n = 0, \ldots, n_j - 1$ such that for $\eta \in H^\perp$, $(K_{j,k} + \eta) \cap K \neq \emptyset$ if and only if $\eta \in \{0 = \eta_0^{(j,k)}, \ldots, \eta_{n_j-1}^{(j,k)}\} = \widetilde{M}_{(j,k)}$. If $(j,k) \neq (j',k')$, then $\widetilde{M}_{(j,k)} \neq \widetilde{M}_{(j',k')}$.

iii) With the $\eta_0^{(j,k)}$ as in ii), one has $K_{j,k} = K_{j,0} + \eta_k^{(j,0)}$, i.e., for fixed j, the sets $K_{j,k}$ are translates of $K_{j,0}$.

Proof. For $\xi \in K$, define M_ξ to be the set $(H^\perp \backslash \{0\}) \cap (K - \xi)$ which is contained in $H^\perp \cap (K - K)$ where $K - K = \bigcup_{\xi \in K} (K - \xi)$. Since $K - K$ is contained in the compact set $\bar{K} - \bar{K}$, by Lemma 2.3 in [7] $H^\perp \cap (K - K)$ is a finite set. Thus, we let $M_\xi = \{\eta_1, \ldots, \eta_{m-1}\}$ where $\xi \in K$ and $m - 1$ represents the cardinality of the set M_ξ. Since the finite set $H^\perp \cap (K - K)$ has only finitely many subsets, as

ξ runs through K, M_ξ will assume only finitely many values, say $M_1, \ldots, M_L$. The relation $\xi \equiv \xi' \Leftrightarrow M_\xi = M_{\xi'}$ gives an equivalence relation on K induced by the subgroup H. The equivalence classes are

$$K_l = \{\xi \in K : M_\xi = M_l\}, \ l = 1, \ldots, L. \tag{2.2}$$

The sets K_l are mutually disjoint and $K = \bigcup_{l=1}^L K_l$. Furthermore, each K_l consists of the points ξ for which $\xi + \eta \in K$ if $\eta \in M_l \cup \{0\}$, and $\xi + \eta \notin K$ if $\eta \in H^\perp \backslash (M_l \cup \{0\})$. To facilitate the notation, we let

$$M_l = \{\eta_1^{(l)}, \ldots, \eta_{m_l-1}^{(l)}\}, \quad l = 1, \ldots, L \tag{2.3}$$

be the values assumed by $(H^\perp \backslash \{0\}) \cap (K - \xi)$ as ξ runs through K, and let $\widetilde{M_l} = M_l \cup \{0\} = \{0 = \eta_0^{(l)}, \ldots, \eta_{m_l-1}^{(l)}\}$. We have that $(K_l + \eta) \subset K$ if $\eta \in \widetilde{M_l}$, and $(K_l + \eta) \cap K = \emptyset$ for $\eta \in H^\perp \backslash \widetilde{M_l}$. Hence, for each $\eta \in \widetilde{M_l}$ there must be an $l' \in \{1, \ldots, L\}$ such that $K_{l'} = K_l + \eta$, and $\widetilde{M_{l'}} = \widetilde{M_l} - \eta$. It follows that we can define an equivalence relation on the set of indices $\{1, \ldots, L\}$ by letting $l \equiv l'$ if and only if there is $\eta \in \widetilde{M_l}$ such that $K_{l'} = K_l + \eta$. Let J be the number of equivalence classes for the set $S = \{1, \ldots, L\}$ where $1 \leq J \leq L$. Assume that each equivalence class S_j for $j = 1, \ldots, J$ contains n_j elements. We rearrange the sets K_l so that the sets which belong to the same equivalence class are grouped together. This gives

$$K_1, \ldots, K_L = \underbrace{K_1, \ldots, K_{n_1}}_{j=1}, \underbrace{K_{n_1+1}, \ldots, K_{n_1+n_2}}_{j=2}, \ldots, \underbrace{K_{n_1+\cdots+n_{J-1}+1}, \ldots, K_{n_1+\cdots+n_J}}_{j=J}.$$

Note that for each $j \in \{1, \ldots, J\}$,

$$l = \left(\sum_{j'=1}^{j-1} n_{j'}\right) + 1 + k, \qquad k = 0, \ldots, n_j - 1. \tag{2.4}$$

We can describe this arrangement by a double index (j, k), where j identifies the equivalence class and k is as in (2.4). Thus using this correspondence between l

and (j,k), we write $K_{j,k}$ for K_l, $\eta_n^{(j,k)}$ for $\eta_n^{(l)}$ and $\widetilde{M}_{(j,k)} = \{0 = \eta_0^{(j,k)}, \ldots, \eta_{n_j-1}^{(j,k)}\}$ for $\widetilde{M}_l = \{0 = \eta_0^{(l)}, \ldots, \eta_{m_l-1}^{(l)}\}$. Furthermore, one can order the sets K_l in each equivalence class in such a way that $K_{j,k} = K_{j,0} + \eta_k^{(j,0)}$ and $\widetilde{M}_{j,k} = \widetilde{M}_{j,0} - \eta_k^{(j,0)}$, $j = 1, \ldots, J$, $k = 0, \ldots, n_j - 1$. $\square$

To clarify the abstract notation, we continue by giving an example in one dimension.

Example 2.1.3 : Let $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}, \widehat{G} = \mathbb{Z}, H = \{j/N : j = 0, \ldots, N - 1\}$ with addition modulo 1 where $N \geq 5$ is a positive integer. It follows that $H^\perp = N\mathbb{Z}$. Let $K = \{-P, \ldots, N + P\}$ where $P < \frac{N}{2} - 1$ is a positive integer. Then for $\xi \in K$, we have

$$(H^\perp \setminus \{0\}) \cap (K - K) = \{-N, N\}.$$

It follows that K_1, K_2, and K_3 are the sets $\{P + 1, \ldots, N - P - 1\}, \{-P, \ldots, P\}$, and $\{N - P, \ldots, N + P\}$ respectively. Hence, $L = 3$ and M_l for $l = 1, 2$ and 3 is respectively $\{\emptyset\}$, $\{N\}$, and $\{-N\}$. The equivalence relation on the set of indices gives the double indices as follows:

$$S_1 = \{1\} \quad \text{with} \quad \eta_0^{(1,0)} = 0$$

$$S_2 = \{2, 3\} \quad \text{with} \quad \eta_1^{(2,0)} = N \quad \text{and} \quad \eta_1^{(2,1)} = -N.$$

Furthermore, the compact sets can be reindexed and defined as

$$K_1 = K_{1,0}, \ K_2 = K_{2,0} \quad \text{and} \quad K_3 = K_{2,1}$$

where

$$K_{2,1} = K_{2,0} + \eta_1^{(2,0)}.$$

The following theorem tells what can be said about f if the function vanishes on an admissible lattice H but the translated sets $K + \eta$, $\eta \in H^\perp$ are not necessarily disjoint.

Theorem 2.1.4 : *Suppose H is an admissible lattice and R a fundamental domain of $H^{\perp}$. Assume that $f \in L_2(G)$ is continuous, f vanishes on H, and that $\widehat{f}$ vanishes almost everywhere outside a precompact set $K \subset \widehat{G}$. Let $K_{j,k}$ be the sets of the decomposition of K according to Lemma 2.1.2. Then there exist continuous functions $h_{j,k} \in L_2(G)$ with $\widehat{h}_{j,k}$ vanishing almost everywhere outside $K_{j,k}$ for $j = 1, \ldots, J$, $k = 1, \ldots, n_j - 1$ such that*

$$\widehat{f}(\xi) = \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} \widehat{h}_{j,k}(\xi) - \widehat{h}_{j,k}(\xi + \eta_k^{(j,0)}) \tag{2.5}$$

for almost every $\xi \in \widehat{G}$ and

$$f(x) = \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} h_{j,k}(x)\big(1 - e^{-2\pi i \langle x, \eta_k^{(j,0)} \rangle}\big) \tag{2.6}$$

for almost every $x \in G$.

Note that if $K + \eta$, $\eta \in H^{\perp}$ are disjoint, then $J = 1$, $n_1 = 1$ so (2.6) reads $f(x) = 0$ which is consistent with the classical sampling theorem.

Proof. Let $h_{j,k} \in L_2(G)$ be a continuous function such that $\widehat{h}_{j,k}(\xi) = \chi_{K_{j,k}}(\xi) \widehat{f}(\xi)$ for $j = 1, \ldots, J$ and $k = 0, \ldots, n_j - 1$. Then

$$\widehat{f}(\xi) = \sum_{j=1}^{J} \sum_{k=0}^{n_j-1} \widehat{h}_{j,k}(\xi) \qquad a.e. \tag{2.7}$$

Since $\widehat{f} \in L_1(\widehat{G})$ and $\int_H |f(y)|^2 \, dm_H(y) = 0$ by hypothesis, the Poisson summation formula (1.25) can be applied. Thus, we have

$$\sum_{\eta \in H^{\perp}} \widehat{f}(\xi + \eta) = \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} f(y) e^{-2\pi i \langle y, \xi \rangle} \qquad a.e. \tag{2.8}$$

where (2.8) converges in $L_2(\widehat{G}/H^{\perp})$. Since f vanishes on H, the right hand side of (2.8) is equal to zero and hence for almost every $\xi \in K_{j,0}$

$$\sum_{\eta \in H^{\perp}} \widehat{f}(\xi + \eta) = \sum_{k=0}^{n_j - 1} \widehat{f}(\xi + \eta_k^{(j,0)}) = 0 \tag{2.9}$$

where $\eta_0^{(j,0)} = 0$ and $j = 1, \ldots, J$. Combining (2.7) and (2.9) gives

$$\widehat{h}_{j,0}(\xi) + \sum_{k=1}^{n_j - 1} \widehat{h}_{j,k}(\xi + \eta_k^{(j,0)}) = 0 \tag{2.10}$$

for almost every $\xi \in K_{j,0}$. If $\xi \notin K_{j,0}$, then $\xi + \eta_k^{(j,0)} \notin K_{j,k}$ for fixed j and $k = 1, \ldots, n_j - 1$. Hence, (2.10) holds for almost every $\xi \in \widehat{G}$. We solve for $\widehat{h}_{j,0}$ in (2.10) and substitute in (2.7) to get

$$\widehat{f}(\xi) = \sum_{j=1}^{J} \Big(- \sum_{k=1}^{n_j - 1} \widehat{h}_{j,k}(\xi + \eta_k^{(j,0)}) + \sum_{k=1}^{n_j - 1} \widehat{h}_{j,k}(\xi) \Big)$$

$$= \sum_{j=1}^{J} \sum_{k=1}^{n_j - 1} \widehat{h}_{j,k}(\xi) - \widehat{h}_{j,k}(\xi + \eta_k^{(j,0)}) \tag{2.11}$$

for almost every $\xi \in \widehat{G}$. The inverse Fourier transform of (2.11) gives (2.6). $\square$

Corollary 2.1.5 : *Assume that the hypotheses of the Theorem 2.1.4 holds except the condition that f vanishes on H. Assume instead that f vanishes on $\alpha + H$, where $\alpha \in G$.*

Then there exist functions $h_{j,k} \in L_2(G)$ with $\widehat{h}_{j,k}$ vanishing almost everywhere outside $K_{j,k}$ for $j = 1, \ldots, J$, $k = 1, \ldots, n_j - 1$ such that

$$f(x) = \sum_{j=1}^{J} \sum_{k=1}^{n_j - 1} h_{j,k}(x)(1 - e^{-2\pi i \langle x - \alpha, \eta_k^{(j,0)} \rangle}) \tag{2.12}$$

for almost every $x \in G$.

Proof. Define $g(x) = f(x + \alpha)$. Note that $\widehat{g}$ vanishes almost everywhere outside K and g vanishes on H. Thus Theorem 2.1.4 can be applied to g to give

$$g(x) = \sum_{j=1}^{J} \sum_{k=1}^{n_j - 1} \widetilde{h}_{j,k}(x)(1 - e^{-2\pi i \langle x, \eta_k^{(j,0)} \rangle}).$$

Define $h_{j,k}(x) = \widetilde{h}_{j,k}(x - \alpha)$. Then

$$
\begin{aligned}
f(x) &= g(x - \alpha) \\
&= \sum_{j=1}^{J} \sum_{k=1}^{n_j - 1} \widetilde{h}_{j,k}(x - \alpha)\left(1 - e^{-2\pi i \langle x - \alpha, \eta_k^{(j,0)} \rangle}\right) \\
&= \sum_{j=1}^{J} \sum_{k=1}^{n_j - 1} h_{j,k}(x)\left(1 - e^{-2\pi i \langle x - \alpha, \eta_k^{(j,0)} \rangle}\right).
\end{aligned}
$$

This completes the proof. $\square$

The following example treats a specific case which will be important for the remainder of the chapter.

Example 2.1.6 : Assume H is an admissible lattice with R a fundamental domain of $H^{\perp}$. Suppose K' is a nonempty precompact subset of R. Let $K = R \cup (\eta_1 + K')$ where $\eta \in H^{\perp} \setminus \{0\}$. Then the decomposition of K yields

$$
K_1 = K' + \eta, \quad K_2 = K' \quad \text{and} \quad K_3 = R \setminus K'.
$$

We obtain two equivalence classes:

$$
K_{1,0} = \eta + K', \quad K_{1,1} = K' \quad \text{and} \quad K_{2,0} = R \setminus K'.
$$

Hence $n_1 = 2$, $n_2 = 1$ and $\eta_1^{(1,0)} = -\eta$. Equation (2.12) implies that

$$
f(x) = h(x)\left(1 - e^{2\pi i \langle x - \alpha, \eta \rangle}\right) \tag{2.13}
$$

with $\widehat{h}$ vanishing almost everywhere outside $K_{1,1} = K'$.

In Theorem 2.1.4, K was an arbitrary precompact subset of $\widehat{G}$. For the remainder of this section, we consider K to be a union of shifted copies of fundamental domain of different $H^{\perp}$s. We will treat this formulation here in detail and use it thereafter.

Definition 2.1.7 : Suppose that $H_1, \ldots, H_N$ are admissible lattices and R_i a fundamental domain of $H_i^\perp$ for each i such that $R_1 \subset \ldots \subset R_N$. If $K_1 = R_1$, and for $j = 1, \ldots, N-1$

i) $K_{j+1} = R_{j+1} \cup (\eta_{j+1} + K_j)$ with $\eta_{j+1} \in H_{j+1}^\perp \setminus \{0\}$, and

ii) $K_j \subset R_{j+1}$,

then $K = K_N$ is called an admissible subset of $\widehat{G}$ with respect to the subgroups $H_1, \ldots, H_N$.

Note that this gives

$$K_j = R_j \cup (\eta_j + R_{j-1}) \cup ((\eta_j + \eta_{j-1}) + R_{j-2}) \cup \ldots \cup ((\eta_j + \ldots + \eta_2) + R_1) \quad (2.14)$$

where $j = 1, \ldots, N$.

2.1.1. *UNIQUENESS*

In the next theorem, we prove the uniqueness for generalized sampling theorem with nonperiodic sampling sets. We also give an example in $\mathbb{R}^2$.

Theorem 2.1.8 : *Suppose that $H_1, \ldots, H_N$ are admissible lattices and R_i a fundamental domain of $H_i^\perp$ for each i such that $R_1 \subset \ldots \subset R_N$. Assume that $f \in L_2(G)$ is continuous and vanishes on $S_i = x_i + H_i$ for $i = 1, \ldots, N$, where $x_i \in G$. Assume that $\widehat{f}$ vanishes almost everywhere outside an admissible subset K of $\widehat{G}$ with respect to the subgroups $H_1, \ldots, H_N$. If $\langle z - x_j, \eta_j \rangle \neq 0$ for $z \in \bigcup_{i=1}^{j-1} S_i$ where $1 < j \leq N$, then f vanishes almost everywhere on G.*

Proof. The proof is by induction on N. If $N = 1$, then $K = R_1 = K_1$. Since f vanishes on a coset of H_1 and $supp(\widehat{f}) \subset \bar{R}_1$, by the standard sampling theorem f vanishes on G. For $N > 1$, assume that the theorem holds with N replaced by $N - 1$. Now H_N is an admissible lattice and R_N a fundamental domain of $H_N^\perp$.

Furthermore, K_{N-1} is a nonempty subset of $\widehat{G}$ such that $K = R_N \cup (\eta_N + K_{N-1})$. Since $K_{N-1} \subset R_N$ by Example 2.1.6 and Corollary 2.1.5, there exists a function $h_N \in L_2(G)$ with $\widehat{h}_N$ vanishing almost everywhere outside K_{N-1} such that

$$f(x) = h_N(x)(1 - e^{2\pi i \langle x - x_N, \eta_N \rangle}). \tag{2.15}$$

Let $S'_N = \bigcup_{i=1}^{N-1} S_i$. If $z \in S'_N$, then $z \in S_i$ for some $i = 1, \ldots, N-1$. Thus, by the hypothesis, $0 = f(z) = h_N(z)(1 - e^{2\pi i \langle z - x_N, \eta_N \rangle})$. Since $\langle z - x_N, \eta_N \rangle \neq 0$, $h_N(z) = 0$ $\forall z \in S_i$ $i = 1, \ldots, N-1$. Now $h_N \in L_2(G)$ vanishes on $S_i = x_i + H_i$ for $i = 1, \ldots, N-1$ where $x_i \in G$ and $\langle z - x_j, \eta_j \rangle \neq 0$ for $z \in \bigcup_{i=1}^{j-1} S_i$ where $1 < j \leq N-1$. Furthermore, $\widehat{h}_N$ vanishes almost everywhere outside K_{N-1}. Thus, h_N satisfies the hypotheses of the theorem with N replaced by $N-1$. Hence, by the induction hypothesis $h_N(x)$ vanishes almost everywhere on G. Following (2.15), f vanishes almost everywhere on G. $\square$

We continue by giving an example on the group $G = \mathbb{R}^2$.

Example 2.1.9 : Let $G = \mathbb{R}^2$, and let H_1, H_2 and H_3 be closed subgroups of G generated by the matrices $W_i,, i = 1, 2, 3$ where

$$W_i = \begin{pmatrix} r_i & 0 \\ 0 & d_i \end{pmatrix}$$

with $r_i, d_i \in \mathbb{R}^+$. Furthermore, assume that $d_3 < d_2 < d_1$ and $r_3 < r_2 < r_1$ such that $\frac{1}{d_1} < \frac{1}{d_3} - \frac{1}{d_2}$. Let R_i to be a fundamental domain of $H_i^\perp$ for $i = 1, 2, 3$. Hence

$$R_i = \{(\nu_i, \sigma_i) \in \mathbb{R} \times \mathbb{R} : 0 \leq \sigma_i < 1/d_i, \quad 0 \leq \nu_i < 1/r_i\} \tag{2.16}$$

for each i. It follows that $R_1 \subset R_2 \subset R_3$. Let

$$K = R_3 \cup \left(\begin{pmatrix} 1/r_3 \\ 0 \end{pmatrix} + R_2 \right) \cup \left(\begin{pmatrix} 1/r_3 \\ 1/d_2 \end{pmatrix} + R_1 \right). \tag{2.17}$$

Suppose $f \in L_2(\mathbb{R}^2)$ is continuous and vanishes on $S_i = \left\{ \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} + H_i \right\}$ $i = 1, 2$ and H_3. Assume that $\hat{f}$ vanishes almost everywhere outside K. Suppose $\left\langle z, \begin{pmatrix} 0 \\ 1/d_2 \end{pmatrix} \right\rangle \neq 0$ for $z \in S_i, i = 1, 2$. and $\left\langle z - \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/d_2 \end{pmatrix} \right\rangle \neq 0$ for $z \in S_1$. Note that here $\eta_3 = \begin{pmatrix} 1/r_3 \\ 0 \end{pmatrix}$ and $\eta_2 = \begin{pmatrix} 0 \\ 1/d_2 \end{pmatrix}$. Following Theorem 2.1.8, $K_2 = R_2 \cup \left(\begin{pmatrix} 0 \\ 1/d_2 \end{pmatrix} + R_1 \right)$ and $K_1 = R_1$. Since $K_2 \subset R_3$, f vanishes almost everywhere on G by Theorem 2.1.8. Note that $K_1 = R_1 \subset R_2$.
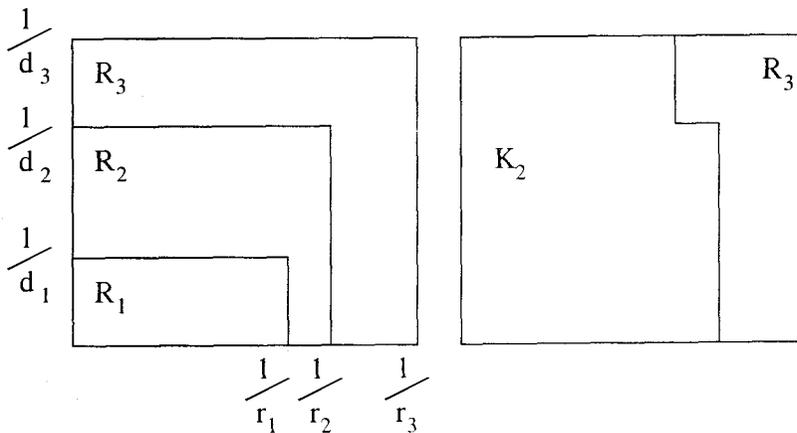
FIGURE 2.1. $R_1 \subset R_2 \subset R_3$ with $K_2 \subset R_3$
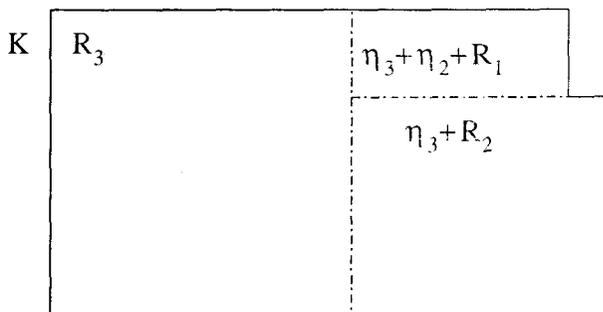
FIGURE 2.2. $K = R_3 \cup (\eta_3 + R_2) \cup ((\eta_3 + \eta_2) + R_1)$

2.1.2. *RECONSTRUCTION*

In the next Theorem, we consider the following: Let f be defined on an ELCA group G such that $\widehat{f}$ vanishes almost everywhere outside an admissible subset K of $\widehat{G}$ with respect to $H^\perp$. Using the classical sampling theorem, we find a reconstructed version of f, say $S_H f$, such that f and $S_H f$ are equal on the coset of H. Now applying the Theorem 2.1.4 to the difference $g = f - S_H f$, we can find a function h with $\widehat{h}$ vanishing outside K_{N-1}. By repeating this procedure with the knowledge of f on a finite union of cosets of different subgroups of G, one can reconstruct f recursively.

Remark 2.1.10 : Assume that H is an admissible lattice with R a fundamental domain of $H^\perp$. Then $R + \eta$ are disjoint for $\eta \in H^\perp$ since R is a fundamental domain of $H^\perp$. If $f \in L_2(G)$ is continuous and $supp(\widehat{f}) \subseteq R$, then

$$R_{H^\perp}\widehat{f}(\xi + H^\perp) = m_{\widehat{G}}(R) \sum_{\eta \in H^\perp} \widehat{f}(\xi + \eta)$$
$$= m_{\widehat{G}}(R)\,\widehat{f}(\xi). \tag{2.18}$$

To be more specific on $S_H f$ and its properties, we need the following technical lemmas.

Lemma 2.1.11 *Assume that H is an admissible lattice and R a fundamental domain of $H^\perp$. Let $f \in L_2(G)$ be continuous so that $\widehat{f}$ vanishes almost everywhere outside K where K is a precompact subset of $\widehat{G}$. Assume that there exist $P \in \mathbb{N}$ such that $K \subseteq \bigcup_{i=0}^{P-1} (R + \eta_i)$ where η_i are distinct elements of $H^\perp$. If F is the restriction of f to the subgroup H, then $F \in L_2(H)$ and as an operator, this restriction is continuous.*

Proof. If $P = 1$, then K is contained in R. Thus, it suffices to show that F is the inverse Fourier transform of a function in $L_2(\widehat{G}/H^\perp)$. Then by the Plancherel

theorem the result will follow. By (1.24), F is the inverse Fourier transform of $R_{H^\perp}\widehat{f}$. Furthermore,

$$\int_{\widehat{G}/H^\perp} |R_{H^\perp}\widehat{f}(\xi + H^\perp)|^2 \, dm_{\widehat{G}/H^\perp}(\xi + H^\perp)$$

$$= (m_{\widehat{G}}(R))^2 \int_{\widehat{G}/H^\perp} |\widehat{f}(\xi)|^2 \, dm_{\widehat{G}/H^\perp}(\xi + H^\perp) = m_{\widehat{G}}(R) \int_{\widehat{G}} \chi_R(\xi) |\widehat{f}(\xi)|^2 \, dm_{\widehat{G}}(\xi)$$

where the first equality follows from Remark 2.1.10 and the second one follows from the Lemma 1.2.15. The right hand side is finite since $\widehat{f} \in L_2(\widehat{G})$. The continuity follows from Lemma 1.2.9.

If $P > 1$, then we let $f_i \in L_2(G)$ be continuous functions such that $\widehat{f_i}(\xi) = \chi_{R+\eta_i}(\xi)\widehat{f}(\xi)$, $i = 0, \ldots, P - 1$ where $K = \bigcup_{i=0}^{P-1} R + \eta_i$. Then

$$\widehat{f}(\xi) = \sum_{i=0}^{P-1} \widehat{f_i}(\xi). \tag{2.19}$$

But for each function f_i, $P = 1$ and the above argument applies to show that the restriction of each f_i to the subgroup H is in $L_2(H)$. Thus, $F \in L_2(H)$.

To show the continuity of the restriction as an operator, we will show its boundedness. Suppose

$$\|f_j|_H\|_2 = \max_{i=0,\ldots,P-1} \|f_i|_H\|_2$$

where $0 \le j \le P - 1$. Since the functions f_i are mutually orthogonal, we have

$$\|f_j\|_2^2 \le \sum_{i=0}^{P-1} \|f_i\|_2^2 = \|\sum_{i=0}^{P-1} f_i\|_2^2 = \|f\|_2^2 \tag{2.20}$$

where Pythagoras lemma is used for the first equality. Hence

$$\|f|_H\|_2^2 = \|\sum_{i=0}^{P-1}(f_i|_H)\|_2^2 \le (\sum_{i=0}^{P-1} \|(f_i|_H)\|_2)^2 \le P^2\|f_j|_H\|_2^2 = P^2 m_{\widehat{G}}(R)\|f_j\|_2^2 \tag{2.21}$$

$$\le P^2 m_{\widehat{G}}(R)\|f\|_2^2 \tag{2.22}$$

where the last equality in (2.21) follows from Kluvanek's Theorem 1.1.1 and (2.22) follows from (2.20). □

Lemma 2.1.12 *Assume that H is an admissible lattice and R a fundamental domain of $H^\perp$. Then for $y \in H$*

$$\widehat{\chi}_R(y) = \int_R e^{-2\pi i \langle y, \xi \rangle} \, dm_{\widehat{G}}(\xi) = \begin{cases} m_{\widehat{G}}(R), & \text{if } y = 0; \\ 0, & \text{if } y \neq 0. \end{cases} \tag{2.23}$$

Proof. Since H is a lattice, $\widehat{G}/H^\perp$ is compact. Thus, Lemma 1.2.3 can be applied. The proof is now complete by Lemma 1.2.15. □

To give a description of $S_H f$, assume that the hypotheses of Lemma 2.1.11 hold. Let F be the restriction of f to the subgroup H. It follows from Lemma 2.1.11 that $\widehat{F} \in L_2(\widehat{H}) = L_2(\widehat{G}/H^\perp)$. Thus, for $\xi \in \widehat{G}/H^\perp$,

$$\begin{aligned} \widehat{F}(\xi) &= \sum_{y \in H} F(y) e^{-2\pi i \langle y, \xi \rangle} \\ &= \sum_{y \in H} f(y) e^{-2\pi i \langle y, \xi \rangle}. \end{aligned} \tag{2.24}$$

Hence,

$$\begin{aligned} f(y) = F(y) &= \int_{\widehat{G}/H^\perp} \widehat{F}(\xi) e^{2\pi i \langle y, \xi \rangle} \, dm_{\widehat{G}/H^\perp}(\xi) \\ &= \frac{1}{m_{\widehat{G}}(R)} \int_{\widehat{G}} \chi_R(\xi) \widehat{F}(\xi) e^{2\pi i \langle y, \xi \rangle} \, dm_{\widehat{G}}(\xi) \end{aligned} \tag{2.25}$$

where $\xi \in \widehat{G}/H^\perp$ and (2.25) is true by Lemma 1.2.15.

Remark 2.1.13 : Define

$$S_H f(x) = \frac{1}{m_{\widehat{G}}(R)} \int_{\widehat{G}} \chi_R(\xi) \, \widehat{F}(\xi) \, e^{2\pi i \langle x, \xi \rangle} dm_{\widehat{G}}(\xi). \tag{2.26}$$

Following (2.25), for all $y \in H$, $S_H f(y) = F(y) = f(y)$. Furthermore,

$$(S_H f)^{\wedge}(\xi) = \frac{1}{m_{\widehat{G}}(R)} \chi_R(\xi) \widehat{F}(\xi)$$

where $\widehat{F} \in L_2(\widehat{H})$. Hence, $(S_H f)^{\wedge} \in L_2(\widehat{G})$.

Upon substitution of $\widehat{F}$ from (2.24) into (2.26), we get

$$
\begin{aligned}
S_H f(x) &= \frac{1}{m_{\widehat{G}}(R)} \int_{\widehat{G}} \chi_R(\xi) \left(\sum_{y \in H} f(y) e^{-2\pi i \langle y, \xi \rangle} \right) e^{2\pi i \langle x, \xi \rangle} dm_{\widehat{G}}(\xi) \\
&= \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} f(y) \left(\int_{\widehat{G}} \chi_R(\xi)\, e^{2\pi i \langle x-y, \xi \rangle} dm_{\widehat{G}}(\xi) \right) \qquad (2.27) \\
&= \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} f(y) \widehat{\chi}_R(-(x-y)). \qquad (2.28)
\end{aligned}
$$

We note that $\widehat{F}$ has compact support and hence is in $L_1(\widehat{H})$. The interchange of integration and summation in (2.27) is based on convergence in $L_1(\widehat{H})$ of the sum in (2.24) and on the fact that the bounded function $e^{2\pi i \langle x, \xi \rangle}$ does not violate this convergence.

The following theorem gives a reconstruction formula for the case of nonperiodic sampling with two different subgroups.

Theorem 2.1.14 : *Assume that H_1 and H_2 are two admissible lattices with R_i a fundamental domain of $H_i^\perp$ for each i such that $R_1 \subset R_2$. Assume that K is an admissible subset of $\widehat{G}$ with respect to H_1 and H_2, i.e., $K = R_2 \cup (\eta_2 + R_1)$, where $\eta_2 \in H_2^\perp \setminus \{0\}$. Let $f \in L_2(G)$ be continuous so that $\widehat{f}$ vanishes almost everywhere outside K. Let $S_1 = x_1 + H_1$ with $x_1 \in G$ be so that for all $z \in S_1$, $\langle z, \eta_2 \rangle \neq 0$. Then for almost every $x \in G$*

$$
\begin{aligned}
f(x) = {} &\frac{1 - e^{2\pi i \langle x, \eta_2 \rangle}}{m_{\widehat{G}}(R_1)} \sum_{y \in H_1} \frac{f(x_1 + y) - S_{H_2} f(x_1 + y)}{1 - e^{2\pi i \langle x_1 + y, \eta_2 \rangle}} \widehat{\chi}_{R_1}(-(x - x_1 - y)) \\
&+ S_{H_2} f(x) \qquad (2.29)
\end{aligned}
$$

where

$$
S_{H_2} f(x) = \frac{1}{m_{\widehat{G}}(R_2)} \sum_{y \in H_2} f(y) \widehat{\chi}_{R_2}(-(x-y)).
$$

Proof. Let $g \equiv f - S_{H_2}f$ where $S_{H_2}f$ is defined by (2.28). Then $supp(\widehat{g}) \subset supp(\widehat{f}) \cup supp(S_{H_2}f)^\wedge = K \cup R_2 = K$ and g vanishes on H_2 following Remark 2.1.13. Note that $g \in L_2(G)$ since $(S_{H_2}f)^\wedge \in L_2(\widehat{G})$. Since $K = R_2 \cup (\eta_2 + R_1)$ and $R_1 \subset R_2$ following Theorem 2.1.4 and Example 2.1.6, there exists a continuous function $h \in L_2(G)$ with $\widehat{h}$ vanishing almost everywhere outside R_1 such that

$$g(x) = h(x)(1 - e^{2\pi i \langle x, \eta_2 \rangle}) \qquad a.e. \tag{2.30}$$

It follows from (2.30) that for all $z \in S_1$

$$h(x_1 + y) = \frac{g(x_1 + y)}{1 - e^{2\pi i \langle x_1 + y, \eta_2 \rangle}}.$$

Using the classical sampling theorem, Theorem 1.1.1, we have

$$h(x) = \frac{1}{m_{\widehat{G}}(R_1)} \sum_{y \in H_1} h(x_1 + y) \widehat{\chi}_{R_1}(-(x - x_1 - y)) \tag{2.31}$$

$$= \frac{1}{m_{\widehat{G}}(R_1)} \sum_{y \in H_1} \frac{g(x_1 + y)}{1 - e^{2\pi i \langle x_1 + y, \eta_2 \rangle}} \widehat{\chi}_{R_1}(-(x - x_1 - y)) \tag{2.32}$$

$$= \frac{1}{m_{\widehat{G}}(R_1)} \sum_{y \in H_1} \frac{f(x_1 + y) - S_{H_2}f(x_1 + y)}{1 - e^{2\pi i \langle x_1 + y, \eta_2 \rangle}} \widehat{\chi}_{R_1}(-(x - x_1 - y)). \tag{2.33}$$

It follows that

$$\begin{aligned}
f(x) &= g(x) + S_{H_2}f(x) \\
&= h(x)(1 - e^{2\pi i \langle x, \eta_2 \rangle}) + S_{H_2}f(x) \\
&= \frac{1 - e^{2\pi i \langle x, \eta_2 \rangle}}{m_{\widehat{G}}(R_1)} \sum_{y \in H_1} \frac{f(x_1 + y) - S_{H_2}f(x_1 + y)}{1 - e^{2\pi i \langle x_1 + y, \eta_2 \rangle}} \widehat{\chi}_{R_1}(-(x - x_1 - y)) \\
&\quad + S_{H_2}f(x) \quad \square. \tag{2.34}
\end{aligned}$$

To compare the result of Theorem 2.1.14 with Theorem 1.1.3 by Faridani (see [7]), we assume that the subgroups in the hypotheses of Theorem 2.1.14 are the same, i.e., $H = H_1 = H_2$. We assert that in this case the two theorems give consistent results. We need the following lemma.

Lemma 2.1.15 *Suppose H is an admissible lattice with R a fundamental domain of $H^\perp$. For $x, \alpha \in G$ and $y, z \in H$*

$$\frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} \widehat{\chi}_R(-(\alpha + y - z)) \widehat{\chi}_R(-(x - \alpha - y)) = \widehat{\chi}_R(-(x - z)). \tag{2.35}$$

Proof. Let $f(t) = \widehat{\chi}_R(-(\alpha + t))$. Then $f \in L_2(G)$ and $supp(\widehat{f}) = R$. Thus, for $y' = y - z$ Theorem 1.1.1 applies to give

$$\frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} \widehat{\chi}_R(-(\alpha + y - z)) \widehat{\chi}_R(-(x - \alpha - y))$$

$$= \frac{1}{m_{\widehat{G}}(R)} \sum_{y' \in H} f(y') \widehat{\chi}_R(-(x - \alpha - z - y'))$$

$$= f(x - \alpha - z)$$

$$= \widehat{\chi}_R(-(\alpha + x - \alpha - z))$$

$$= \widehat{\chi}_R(-(x - z)). \quad \square$$

Corollary 2.1.16 *Assume that the hypotheses of Theorem 2.1.14 hold with the additional condition that $H_1 = H_2 = H$. Then*

$$f(x) = \frac{1}{m_{\widehat{G}}(R)} \sum_{n=0}^{1} \sum_{y \in H} f(x_n + y) \, k_n(x - x_n - y) \tag{2.36}$$

with

$$x_0 = 0, \ and \ x_1 = \alpha.$$

Here

$$k_0(x - y) = \beta_0^{(1)} \widehat{\chi}_R(-(x - y)) + \beta_0^{(2)} \widehat{\chi}_{\eta_2 + R}(-(x - y))$$

$$k_1(x - \alpha - y) = \beta_1^{(1)} \widehat{\chi}_R(-(x - \alpha - y)) + \beta_1^{(2)} \widehat{\chi}_{\eta_2 + R}(-(x - \alpha - y))$$

where $\beta_0^{(1)} = \beta_1^{(2)} = \frac{1}{1 - e^{-2\pi i \langle \alpha, \eta_2 \rangle}}$ and $\beta_0^{(l)} = \overline{\beta_1^{(l)}}$, $l = 1, 2$.

Proof. Following Theorem 2.1.14, we have

$$
\begin{aligned}
f(x) &= \frac{1 - e^{2\pi i \langle x, \eta_2 \rangle}}{m_{\widehat{G}}(R)} \sum_{y \in H} \frac{f(\alpha + y) - S_H f(\alpha + y)}{1 - e^{2\pi i \langle \alpha + y, \eta_2 \rangle}} \widehat{\chi}_R(-(x - \alpha - y)) + S_H f(x) \\
&= \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} f(\alpha + y) \left(\frac{\widehat{\chi}_R(-(x - \alpha - y))}{(1 - e^{2\pi i \langle \alpha, \eta_2 \rangle})} + \frac{\widehat{\chi}_{\eta_2 + R}(-(x - \alpha - y))}{(1 - e^{-2\pi i \langle \alpha, \eta_2 \rangle})} \right) \\
&\quad - \frac{(1 - e^{2\pi i \langle x, \eta_2 \rangle})}{(1 - e^{2\pi i \langle \alpha, \eta_2 \rangle})} \left[\frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} \left(\frac{1}{m_{\widehat{G}}(R)} \sum_{z \in H} f(z) \widehat{\chi}_R(-(\alpha + y - z)) \right) \right] \widehat{\chi}_R(-(x - \alpha - y)) \\
&\quad + S_H f(x)
\end{aligned}
$$

(2.37)

where (2.37) follows because

$$
\frac{e^{2\pi i \langle x, \eta_2 \rangle}}{(1 - e^{2\pi i \langle \alpha, \eta_2 \rangle})} \widehat{\chi}_R(-(x - \alpha - y)) = \frac{\widehat{\chi}_{\eta_2 + R}(-(x - \alpha - y))}{e^{-2\pi i \langle \alpha, \eta_2 \rangle} - 1}
$$

and the definition of $S_H f(\alpha + y)$ respectively. Using Lemma 2.1.15, (2.37) can be written as

$$
\begin{aligned}
&-\frac{(1 - e^{2\pi i \langle x, \eta_2 \rangle})}{(1 - e^{2\pi i \langle \alpha, \eta_2 \rangle})} \frac{1}{m_{\widehat{G}}(R)} \sum_{z \in H} f(z) \widehat{\chi}_R(-(x - z)) + S_H f(x) \\
&= \frac{1}{m_{\widehat{G}}(R)} \sum_{z \in H} f(z) \left(\frac{\widehat{\chi}_{\eta_2 + R}(-(x - z))}{(1 - e^{2\pi i \langle \alpha, \eta_2 \rangle})} - \frac{\widehat{\chi}_R(-(x - z))}{(1 - e^{2\pi i \langle \alpha, \eta_2 \rangle})} \right) + S_H f(x) \\
&= \frac{1}{m_{\widehat{G}}(R)} \sum_{z \in H} f(z) \left(\frac{\widehat{\chi}_{\eta_2 + R}(-(x - z))}{(1 - e^{2\pi i \langle \alpha, \eta_2 \rangle})} + \frac{\widehat{\chi}_R(-(x - z))}{(1 - e^{-2\pi i \langle \alpha, \eta_2 \rangle})} \right).
\end{aligned}
$$

(2.38)

Hence

$$
\begin{aligned}
f(x) &= \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} f(y) \left(\beta_0^{(1)} \widehat{\chi}_R(-(x - y)) + \beta_0^{(2)} \widehat{\chi}_{\eta_2 + R}(-(x - y)) \right) \\
&\quad + \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} f(\alpha + y) \left(\beta_1^{(1)} \widehat{\chi}_R(-(x - \alpha - y)) + \beta_1^{(2)} \widehat{\chi}_{\eta_2 + R}(-(x - \alpha - y)) \right)
\end{aligned}
$$

where

$$
\beta_0^{(1)} = \frac{1}{1 - e^{-2\pi i \langle \alpha, \eta_2 \rangle}} \qquad \beta_0^{(2)} = \frac{1}{1 - e^{2\pi i \langle \alpha, \eta_2 \rangle}}
$$

and

$$
\beta_1^{(1)} = \frac{1}{1 - e^{2\pi i \langle \alpha, \eta_2 \rangle}} \qquad \beta_1^{(2)} = \frac{1}{1 - e^{-2\pi i \langle \alpha, \eta_2 \rangle}}.
$$

This is consistent with the results of Theorem 1.1.3 by Faridani (see [7]) with $N = 2$ and K as above with the exception of the extra constant $\frac{1}{m_{\widehat{G}}(R)}$ in the above. This constant is resulted from the fact that H is an admissible lattice and with the normalization of the measures given in [7]. $\square$

The following example illustrates a case where nonperiodic sampling is preferred over periodic sampling because of higher efficiency.

Example 2.1.17 : Suppose that $G = \mathbb{R}$, $H_1 = N\mathbb{Z}$, and $H_2 = (N/(N-1))\mathbb{Z}$ for $1 < N \in \mathbb{Z}$. Then $\widehat{G} = \mathbb{R}$, $H_1^{\perp} = 1/N\mathbb{Z}$, and $H_2^{\perp} = ((N-1)/N)\mathbb{Z}$. Furthermore, R_1 and R_2 can be chosen to be $[0, 1/N)$ and $[0, ((N-1)/N))$ respectively. Hence $R_1 \subset R_2$. We have $K_1 = R_1 \subset R_2$ and $K_2 = K = R_2 \cup (\eta_2 + R_1) = [0, (N-1)/N)) \cup [(N-1)/N, 1) = [0, 1)$ where $\eta_2 = (N-1)/N$. Let $S_1 = \alpha + H_1$ with $\alpha \in \mathbb{R}$ so that $(\alpha + H_1) \cap H_2 = \emptyset$. Now if $f \in L_2(\mathbb{R})$ is continuous and $\widehat{f}$ vanishing almost everywhere outside K, then we can find an explicit reconstruction formula.

$$
\begin{aligned}
f(x) = N(1 - e^{2\pi i x/N}) \sum_{l \in \mathbb{Z}} \frac{f(\alpha + Nl) - S_{H_2}f(\alpha + Nl)}{1 - e^{2\pi i(\alpha + Nl)/N}} \widehat{\chi}_{[0,1/N)}(-(x - \alpha - l)) \\
+ S_{H_2}f(x)
\end{aligned}
\tag{2.39}
$$

where

$$
S_{H_2}f(x) = \frac{N}{N-1} \sum_{k \in \mathbb{Z}} f(Nk/(N-1))(e^{2\pi i x/N}) \frac{\sin((2\pi/N)(x - (Nk/(N-1))))}{\pi(x - (Nk/(N-1)))}
\tag{2.40}
$$

and

$$
\widehat{\chi}_{[0,1/N)}(-(x - \alpha - l)) = (e^{(\pi i/N)(x - \alpha - Nl)}) \frac{\sin((2\pi/N)(x - \alpha - Nl))}{\pi(x - \alpha - Nl)}.
$$

Note that Example 2.1.21 can use the theory for periodic sampling with $H = N\mathbb{Z}$. However, as N gets larger, the number of sampling cosets increases which in turn creates a larger systems of equations in (1.5) to be solved.

The result of Theorem 2.1.14 can be generalized to N subgroups. The proof is by induction.

Theorem 2.1.18 : *Suppose that H_i, $i = 1, \ldots, N$ are admissible lattices with R_i as a fundamental domain of $H_i^{\perp}$ for each i such that $R_1 \subset \ldots \subset R_N$. Assume that K is an admissible subset of $\widehat{G}$ with respect to $H_1, \ldots, H_N$ and K_j as in Definition 2.1.7. Let $f \in L_2(G)$ be continuous so that $\widehat{f}$ vanishes outside almost everywhere K. Assume that $\langle z - x_j, \eta_j \rangle \neq 0$ for $z \in \bigcup_{i=1}^{j-1}(x_i + H_i)$ and $1 < j \leq N$. Then there are continuous functions $f_j \in L_2(G)$ such that $supp(\widehat{f_j}) \subseteq K_j$, $j = 1, \ldots, N$ and for almost every $x \in G$*

$$f_1(x) = S_{H_1} f_1(x), \tag{2.41}$$

$$f_j(x) - S_{H_j} f_j(x) = f_{j-1}(x)(1 - e^{2\pi i \langle x - x_j, \eta_j \rangle}) \quad j = 2, \ldots, N, \tag{2.42}$$

$$f_N = f \tag{2.43}$$

with

$$S_{H_i} g(x) = \frac{1}{m_{\widehat{G}}(R_i)} \sum_{y \in H_i} g(x_i + y)\widehat{\chi}_{R_i}(-(x - x_i - y)), \quad x \in G. \tag{2.44}$$

This recursion provides an algorithm to compute f from the sampled values $f(z)$, $z \in \bigcup_{i=1}^{N}(x_i + H_i)$.

Proof. The proof is by induction on N. If $N = 1$, then $K = R_1 = K_1$. Since $supp(\widehat{f}) \subset K = R_1$, by Theorem 1.1.1, $f = S_{H_1} f$. For $N > 1$, assume that the theorem holds with N replaced by $N - 1$. Let $g = f - S_{H_N} f$. Then $supp(\widehat{g}) \subset supp(\widehat{f}) \cup supp(S_{H_N} f)^{\wedge} = K \cup R_N = K$ and g vanishes on $S_N = x_N + H_N$ following Remark 2.1.13. Furthermore, $g \in L_2(G)$ since $(S_{H_N} f)^{\wedge} \in L_2(\widehat{G})$. Since

$K = R_N \cup (\eta_N + K_{N-1})$ and $K_{N-1} \subset R_N$ following Example 2.1.6, there exists a continuous function $f_{N-1} \in L_2(G)$ with $supp(\widehat{f}_{N-1}) \subseteq K_{N-1}$ such that

$$g(x) = f(x) - S_{H_N} f(x) = f_{N-1}(x)(1 - e^{2\pi i \langle x - x_N, \eta_N \rangle}). \qquad (2.45)$$

With $f_N = f$, this gives the recursion formula for $j = N$. Now $f_{N-1} \in L_2(G)$ satisfies the hypothesis of the theorem with K and N replaced by K_{N-1} and $N - 1$ respectively. By the induction hypothesis, we assumed that the theorem holds for $N-1$. Hence, we can apply the theorem to f_{N-1} to find a recursion for $j = 1, \ldots, N - 1$. We note that $f_{N-1}(z)$, $z \in \bigcup_{i-1}^{N-1}(x_i + H_i)$ can be found from the sampled values since $f_{N-1}(z) = \frac{f(z) - S_{H_N} f(z)}{1 - e^{2\pi i \langle z - x_N, \eta_N \rangle}}$ and $\langle z - x_N, \eta_N \rangle \neq 0$ for $z \in \bigcup_{i=1}^{N-1}(x_i + H_i)$. $\square$

This is a description of the recursive algorithm.

Algorithm 2.1.19 *If $N = 1$ then $f = S_{H_N} f$;*

    else do;

    compute sampled values of $g(z) = \frac{f(z) - S_{H_N} f(z)}{1 - e^{2\pi i \langle z - x_N, \eta_N \rangle}}$;

        for $z \in \bigcup_{j=1}^{N-1}(x_j + H_j)$;

    Call algorithm with N, f replaced by $N - 1, g$, to compute $g(x)$, $x \in G$;

    $f = 1 - e^{2\pi i \langle x - x_N, \eta_N \rangle} g + S_{H_N} f$;

    end;

end;

We consider the case $N = 3$ and give an example for this case as well.

Corollary 2.1.20 *Suppose that H_1, H_2, H_3 are admissible lattices with R_i as a fundamental domain of $H_i^\perp$ for each i such that $R_1 \subset R_2 \subset R_3$. Let K be an admissible subset of $\widehat{G}$ with respect to H_i, $i = 1, 2, 3$. Let $f \in L_2(G)$ be so that $\widehat{f}$ vanishes almost everywhere outside K. Assume that $\langle z - x_j, \eta_j \rangle \neq 0$ for $z \in \bigcup_{i=1}^{j-1}(x_i + H_i)$ and $j = 2, 3$. Then for every $x \in G$*

$$f(x) = (1 - e^{2\pi i \langle x - x_3, \eta_3 \rangle}) \left[(1 - e^{-2\pi i \langle x - x_2, \eta_2 \rangle}) f_1(x) + S_{H_2} f_2(x) \right] + S_{H_3} f(x) \quad (2.46)$$

where f_2, f_1 and $S_{H_i} f_i$, $i = 2, 3$ are given respectively as follows:

$$f_2(x) = \frac{f(x) - S_{H_3} f(x)}{1 - e^{2\pi i \langle x - x_3, \eta_3 \rangle}} \quad (2.47)$$

$$f_1(x) = \frac{1}{m_{\widehat{G}}(R_1)} \sum_{y \in H_1} \frac{f_2(x_1 + y) - S_{H_2} f_2(x_1 + y)}{1 - e^{2\pi i \langle x_1 + y - x_2, \eta_2 \rangle}} \widehat{\chi}_{R_1}(-(x - x_1 - y)) \quad (2.48)$$

$$S_{H_i} f_i(x) = \frac{1}{m_{\widehat{G}}(R_i)} \sum_{y \in H_i} f_i(x_i + y) \widehat{\chi}_{R_i}(-(x - x_i - y)) \quad (2.49)$$

where $x \in (x_1 + H_1) \cup (x_2 + H_2)$ and $f_3 = f$.

We note that in above $K = R_3 \cup (\eta_3 + R_2) \cup ((\eta_3 + \eta_2) + R_1)$ with $K_1 = R_1$ and $K_2 = R_2 \cup (\eta_2 + R_1)$ where $K_1 \subseteq R_2$ and $K_2 \subseteq R_3$.

Example 2.1.21 : Let $G = \mathbb{Z}_{18}$, $H_1 = \langle 9 \rangle$, $H_2 = \langle 6 \rangle$, $H_3 = \langle 3 \rangle$. Then $\widehat{G} = \{j/18 : j = 0, \ldots, 17\}$, $H_1^\perp = \langle 1/9 \rangle$, $H_2^\perp = \langle 1/6 \rangle$, $H_3^\perp = \langle 1/3 \rangle$ with addition modulo 1. Thus, we can let $R_1 = (1/18)\{0, 1\}$, $R_2 = (1/18)\{0, 1, 2\}$, $R_3 = (1/18)\{0, 1, 2, 3, 4, 5\}$ where $\eta_1 = 1/9$, $\eta_2 = 1/6$ and $\eta_3 = 1/3$. It follows that $R_1 \subset R_2 \subset R_3$. Furthermore, $K_1 = R_1$, $K_2 = (1/18)\{0, 1, 2, 3, 4\}$ and $K = K_3 = (1/18)\{0, \ldots, 10\}$. Hence, $K_1 = R_1 \subset R_2$ and $K_2 \subset R_3$. In order to satisfy the hypotheses of the Corollary 2.1.20, we choose $x_1 = 1$, $x_2 = 5$ and $x_3 = 0$. Now the support of the Fourier transform of the function $f(l) = 2 + 2(-1)^l \cos(\pi l/9)$ is in K. Thus, Corollary 2.1.20 applies. A program written in Matlab shows that the relative error between f and the reconstructed version of f is 5.4886e-15.
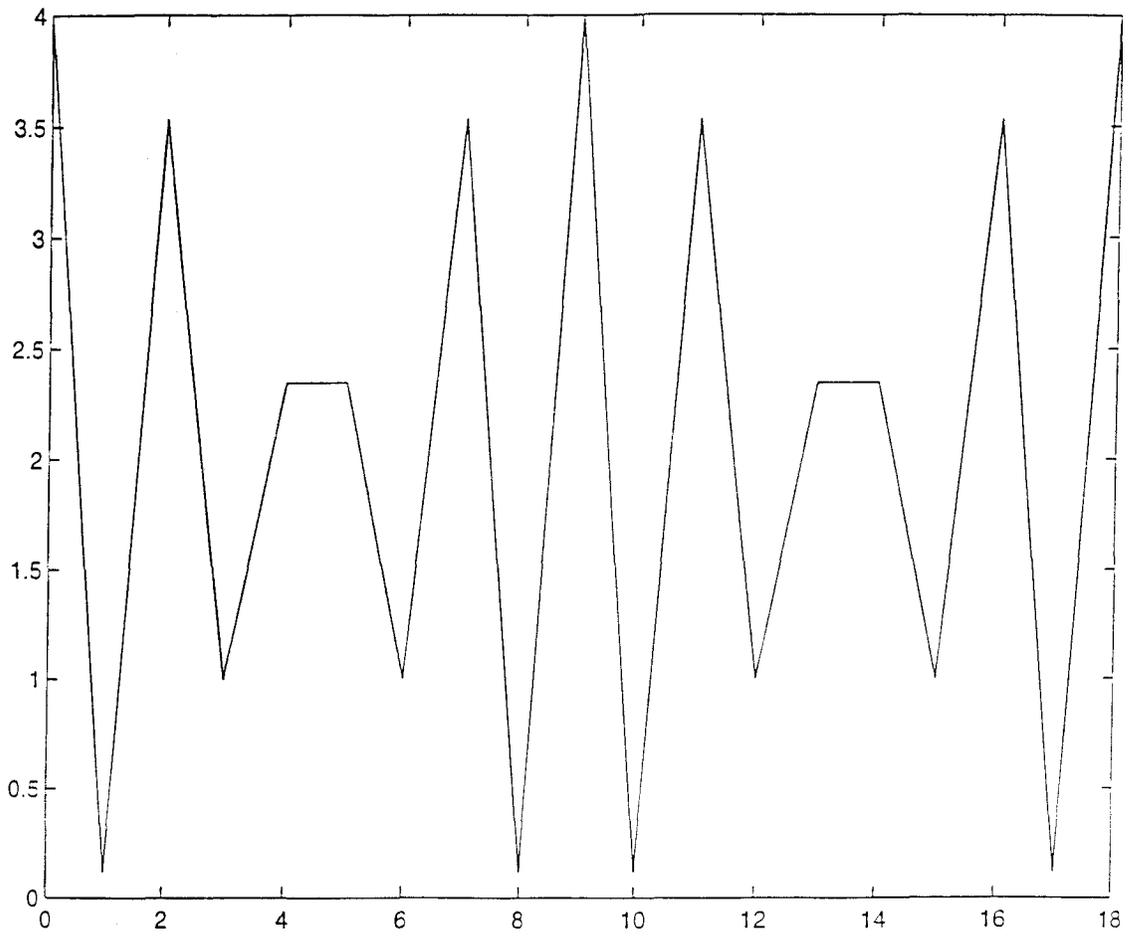
FIGURE 5.3. $f(x) = 2 + 2(-1)^x \cos(\pi/9x)$

3. GENERALIZED FILTER BANKS

In this chapter, we generalize the notion of filters, upsampling and downsampling in the framework of LCA groups. We derive their properties. With appropriate choice of G, we show consistency between our generality and their counterparts in signal processing context. In § 3.2, we derive the conditions for perfect reconstruction in a generalized form. These conditions are used to show that they have consistent results with nonperiodic sampling and multisensor deconvolutuion.

Holschneider in [16] has a similar argument in Wavelet theory where he defines a sampling space $V \subset L_2(G)$ over Abelian groups and what he calls a perfect sampling operator (cf. [16, pp. 373-377]).

3.1. FILTERS, UPSAMPLING AND DOWNSAMPLING FOR LCA GROUPS

In the following, let G be an LCA group, and H an admissible lattice with R a fundamental domain of $H^\perp$. For $a \in G$, let τ_a be the translation operator defined by $\tau_a f = f(x - a)$. Furthermore, the Fourier transform of the delay is

$$(\tau_a f)^\wedge(\xi) = \widehat{f}(\xi)e^{-2\pi i \langle a, \xi \rangle} \tag{3.1}$$

with $a \in G$ and $\xi \in \widehat{G}$.

Below we define a filter to be the bounded linear transformation where we abuse the notation by letting the letter M denote both the operator $M : L_2(G) \to L_2(G)$ and the corresponding function $\widetilde{M}(\gamma)$.

Definition 3.1.1 : Let $f \in L_2(G)$. A *filter* M is a linear translation-invariant operator such that its action on f in the Fourier domain takes the following form.

$$\widehat{(Mf)}(\xi) = M(\xi)\,\widehat{f}(\xi) \tag{3.2}$$

where M is an element of $L_\infty(\widehat{G})$ and $\xi \in \widehat{G}$.

Throughout this chapter, we assume that K is a precompact subset of $\widehat{G}$ such that $K \subseteq \bigcup_{i=0}^{P-1} (R + \eta_i)$ where η_is are distinct elements of $H^\perp$ and $P \in \mathbb{N}$.

Definition 3.1.2 *Downsampling and upsampling:* Suppose H is an admissible lattice and R a fundamental domain of $H^\perp$. Let the space of admissible functions be denoted by

$$\mathcal{A} = \{f \in L_2(G) : f \text{ is continuous and } \widehat{f}(\xi) \text{ vanishes almost everywhere outside } K\}$$

$$(3.3)$$

If $f \in \mathcal{A}$, then by Lemma 2.1.11 the restriction of f to H is in $L_2(H)$. We define the *downsampling* operator $(\downarrow H)f : L_2(G) \to L_2(H)$ by the restriction of f to H. That is

$$(\downarrow H)f = f|_H. \tag{3.4}$$

By Lemma 2.1.11, $(\downarrow H)$ is a continuous linear operator from $L_2(G) \to L_2(H)$. For $g \in \mathcal{A}$, define the *upsampling* operator $(\uparrow H)g : L_2(H) \to L_2(G)$ to be the adjoint of the *downsampling* operator. That is, for $g \in L_2(H)$, $(\uparrow H)g = (\downarrow H)^*g$. Hence, for $x \in G$, we have

$$(\uparrow H)g(x) = \sum_{y \in H} g(y)\, \widehat{\chi}_K(-(x-y)). \tag{3.5}$$

In the following the unit impulse δ is defined as follows:

$$\delta(n) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & n \neq 0. \end{cases}$$

Example 3.1.3 *:* Let $G = \mathbb{Z}$, $H = 2\mathbb{Z}$. Then $\widehat{G} = \mathbb{T} \simeq [0,1)$, and $H^\perp = \{0, 1/2\}$. It follows that H is an admissible lattice with $R = [0, 1/2)$. Let $f, g \in L_2(\mathbb{Z})$ be so that $\widehat{g}, \widehat{f}$ vanish outside $K = \widehat{G}$. We write $(\downarrow 2)$, $(\uparrow 2)$ for $(\downarrow H)$ and $(\uparrow H)$

respectively. Then $(\downarrow 2)f(n) = f(2n)$ for $n \in \mathbb{Z}$. This result is identical with (3.2) of [24, pp. 88-89]. Furthermore,

$$
\begin{aligned}
(\uparrow 2)g(n) &= \sum_{y \in H} g(y)\, \widehat{\chi}_K(-(n-y)) \\
&= \sum_{l \in \mathbb{Z}} g(2l)\, \widehat{\chi}_K(-(n-2l)) \\
&= \begin{cases} g(2l), & \text{if } n = 2l; \\ 0, & \text{if } n = 2l+1. \end{cases}
\end{aligned}
$$
(3.6)

(3.7)

If we define a function $\widetilde{g}$ on $\mathbb{Z}$ by $\widetilde{g}(l) = g(2l)$, we see that this result is identical with (3.5) of [24, pp. 88-89]. In the Example 3.1.3, H could be chosen to be $M\mathbb{Z}$ where M is a positive integer. This gives the upsampling and downsampling as it is defined in the signal processing literature. Thus, the general definition of upsampling and downsampling used here contains the one found in the literature.

In this chapter, we use $\mathcal{F}_G$ and $\mathcal{F}_H$ to represent the Fourier transform with respect to G and H respectively. However, to simplify the notation, we will use $\wedge$ for both $\mathcal{F}_G$ and $\mathcal{F}_H$ as long as the distinction is possible from the context.

Lemma 3.1.4 : *Suppose H is an admissible lattice and R a fundamental domain of $H^\perp$. If $g \in \mathcal{A}$, then the Fourier transform of $(\downarrow H)g$ with respect to H is given by*

$$
((\downarrow H)g)^\wedge(\xi + H^\perp) = m_{\widehat{G}}(R) \sum_{\eta \in H^\perp} \widehat{g}(\xi + \eta)
$$
(3.8)

with $\xi \in \widehat{G}$.

Proof. Since Theorem 1.2.17 holds, we have

$$
\begin{aligned}
\mathcal{F}_H\Big((\downarrow H)g\Big)(\xi + H^\perp) &= \sum_{y \in H} g(y)\, e^{-2\pi i \langle y, \xi \rangle} \\
&= m_{\widehat{G}}(R) \sum_{\eta \in H^\perp} \mathcal{F}_G\, g(\xi + \eta). \quad \square
\end{aligned}
$$

Lemma 3.1.5 : *Suppose H is an admissible lattice. If $g \in \mathcal{A}$, then the Fourier transform of $(\uparrow H)g$ with respect to G is given by*

$$((\uparrow H)g)^{\wedge}(\xi) = \chi_K(\xi)\,\widehat{g}(\xi + H^{\perp}) \tag{3.9}$$

where $\xi \in \widehat{G}$.

Proof. For $\xi \in \widehat{G}$, we have

$$\mathcal{F}_G\Big((\uparrow H)g\Big)(\xi) = \mathcal{F}_G \sum_{y \in H} g(y)\,\widehat{\chi}_K(-(x - y))$$

$$= \chi_K(\xi) \sum_{y \in H} g(y)\,e^{-2\pi i \langle y, \xi \rangle} \tag{3.10}$$

$$= \chi_K(\xi) \sum_{y \in H} g(y)\,e^{-2\pi i \langle y, \xi + H^{\perp} \rangle} \tag{3.11}$$

$$= \chi_K(\xi)\,\mathcal{F}_H g(\xi + H^{\perp})$$

where the transition from (3.10) to (3.11) uses $\langle y, \xi \rangle = \langle y, \xi + H^{\perp} \rangle$ with $y \in G$, $\xi \in \widehat{G}$ on the left-hand side, and $y \in H$, $\xi + H^{\perp} \in \widehat{H}$ on the right-hand side. $\square$

The combined operation of downsampling followed by upsampling is used frequently.

Lemma 3.1.6 : *Suppose H is an admissible lattice and R a fundamental domain of $H^{\perp}$. If $g \in \mathcal{A}$, then the Fourier transform of $(\uparrow H)(\downarrow H)g$ with respect to G is given by*

$$((\uparrow H)(\downarrow H)g)^{\wedge}(\xi) = m_{\widehat{G}}(R)\,\chi_K(\xi) \sum_{\eta \in H^{\perp}} \widehat{g}(\xi + \eta) \tag{3.12}$$

with $\xi \in \widehat{G}$.

Proof. Substitute (3.8) into (3.9). $\square$

Example 3.1.7 : Let $G = \mathbb{Z}$, $H = N\mathbb{Z}$ where $N > 1$ is a positive integer. Then $\widehat{G} = \mathbb{T}$, $H^{\perp} = \{j/N : j = 0, \ldots, N - 1\}$ with addition modulo 1. Hence H is an admissible lattice with $R = [0, 1/N)$. Let $g \in L_2(\mathbb{Z})$. Then by (3.8), we have

$$((\downarrow H)g)^{\wedge}(\xi + H^{\perp}) = \frac{1}{N} \sum_{j=0}^{N-1} \widehat{g}(\xi + j/N). \tag{3.13}$$

Let $v = (\downarrow H)g$. Then by (3.9), we have

$$
\begin{aligned}
((\uparrow H)v)^{\wedge}(\xi) &= \widehat{v}(\xi + H^{\perp}) \\
&= \sum_{y \in N\mathbb{Z}} v(y)\, e^{-2\pi i \langle y, \xi + H^{\perp} \rangle} \\
&= \sum_{y \in N\mathbb{Z}} v(y)\, e^{-2\pi i (Nl\xi)}
\end{aligned}
\tag{3.14}
$$

where (3.14) follows since $\langle y, \xi + H^{\perp} \rangle = \langle y, \xi \rangle = Nl\xi$. Now, let $\tilde{v}(l) = v(Nl), l \in \mathbb{Z}$. Then becomes (3.14)

$$\sum_{l \in \mathbb{Z}} \tilde{v}(l)\, e^{-2\pi i l (N\xi)} = \mathcal{F}_G\, \tilde{v}(N\xi).$$

This agrees with formula (3.24) in "Wavelets and Filter Banks" by G. Strang and T. Nguyen [24].

3.2. RECONSTRUCTION

In this section, we present the reconstruction theorem. This theorem agrees with the perfect reconstruction theorem found in signal analysis literature provided G and H are chosen appropriately. Throughout the following section, we assume that H is an admissible lattice with R a fundamental domain of $H^{\perp}$. We define a generalized analysis-synthesis filter bank as follows.
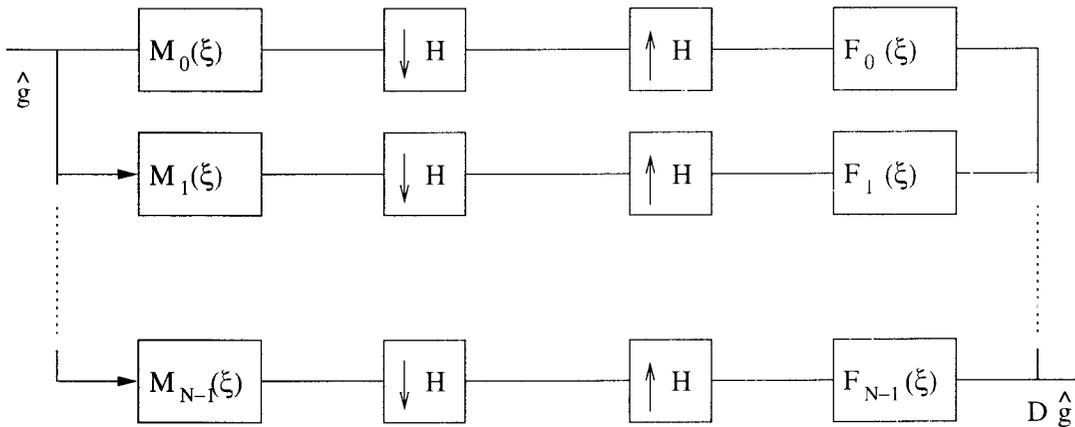
FIGURE 3.1. Generalized form of an N-channel filter bank

Definition 3.2.1 *In our case $H_0 = \ldots = H_{N-1} = H$. Assume that $g \in \mathcal{A}$. Assume that M_j and F_j for $j = 0, \ldots, N-1$ are filters. The output of a generalized analysis-synthesis filter banks pair is given by*

$$\widetilde{S}g(x) = \sum_{j=0}^{N-1} (F_j(\uparrow H)(\downarrow H)M_j g)(x). \tag{3.15}$$

In the following $\delta_{H^\perp}(\eta) = 1$ if $\eta = 0$ and $\delta_{H^\perp}(\eta) = 0$ otherwise.

Theorem 3.2.2 *: Suppose H is an admissible lattice and R a fundamental domain of $H^\perp$. Let $g \in \mathcal{A}$. Suppose D is a filter and M_j, F_j for $j = 0, \ldots, N-1$ are filters in analysis and synthesis banks respectively. If*

$$m_{\widehat{G}}(R)\,\chi_K(\xi) \sum_{j=0}^{N-1} F_j(\xi)\, M_j\,(\xi + \eta) = \delta_{H^\perp}(\eta)\, D(\xi) \tag{3.16}$$

with $\xi \in \widehat{G}$ and $\eta \in \widetilde{M_\xi} = \{\eta \in H^\perp : \xi + \eta \in K\}$, then

$$(Sg)(x) = (Dg)(x) \tag{3.17}$$

for $x \in G$.

Proof. For $\xi \in \widehat{G}$ and $\eta \in H^\perp$, the Fourier transform of the output of the jth-channel with respect to G is

$$\mathcal{F}_G \left(F_j (\uparrow H)(\downarrow H) M_j g \right)(\xi) = F_j(\xi) \, \mathcal{F}_G \left((\uparrow H)(\downarrow H) M_j g \right)(\xi) \tag{3.18}$$

$$= F_j(\xi) \left(\chi_K(\xi) \, m_{\widehat{G}}(R) \sum_{\eta \in H^\perp} \mathcal{F}_G (M_j g)(\xi + \eta) \right) \tag{3.19}$$

$$= F_j(\xi) \left(\chi_K(\xi) \, m_{\widehat{G}}(R) \sum_{\eta \in H^\perp} M_j(\xi + \eta) \, \widehat{g}(\xi + \eta) \right) \tag{3.20}$$

where (3.19) follows from Lemma 3.1.6, (3.18) and (3.20) resulted from Definition (3.1.1). For an N-channel filter bank, total output is summed over $j = 0, ..., N - 1$. Thus using (3.20), we have

$$(\widetilde{S} g)^\wedge(\xi) = \sum_{j=0}^{N-1} \left[F_j(\xi) \, \chi_K(\xi) \, m_{\widehat{G}}(R) \sum_{\eta \in H^\perp} M_j(\xi + \eta) \, \widehat{g}(\xi + \eta) \right]$$

$$= \sum_{\eta \in H^\perp} \left(m_{\widehat{G}}(R) \, \widehat{\chi}_K(\xi) \sum_{j=0}^{N-1} \left[F_j(\xi) \, M_j(\xi + \eta) \right] \right) \widehat{g}(\xi + \eta)$$

$$= \delta_{H^\perp}(\eta) \, D(\xi) \, \widehat{g}(\xi + \eta) \tag{3.21}$$

$$= \mathcal{F}_G (Dg)(\xi)$$

where (3.21) follows from (3.16) and the fact that $\widehat{g}(\xi + \eta) = 0$ for $\eta \notin \widetilde{M_\xi}$. $\square$

To see if the condition in Theorem 3.2.2 agrees with the perfect reconstruction condition found in literature, we recall that perfect reconstruction of an input signal is the same signal possibly with a shift of the input signal. Thus, D in the above is the shift of the function g. Thus, we compare the conditions in Theorem 3.2.2 with the perfect reconstruction conditions found in [24]. We need the following definition.

Definition 3.2.3 : The z-transform of a sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}. \tag{3.22}$$

If the summation does not converge for any $z \in \mathbb{C}$, the z-transform does not exist.

Example 3.2.4 : Let $G = \mathbb{Z}$, $H = N\mathbb{Z}$ where $N > 1$ is a positive integer. Then $\widehat{G} = \mathbb{T}$, $H^\perp = \{j/N : j = 0, \ldots, N-1\}$ with addition modulo 1. Hence H is an admissible lattice with $R = [0, 1/N)$. Let $K = \widehat{G}$ and hence $\widetilde{M_\xi} = H^\perp$. Let D be the shift of the function g by l, i.e., $(Dg)^\wedge(\xi) = \widehat{g}(\xi)e^{-2\pi i \xi l}$. Using (3.16), we have

$$\frac{1}{N} \sum_{j=0}^{N-1} F_j(\xi) \, M_j(\xi + \eta) = \delta_{H^\perp}(\eta) \, e^{-2\pi i \xi, l}. \tag{3.23}$$

Let $\eta = k/N$, $k \in \{0, \ldots, N-1\}$ where $\delta_{H^\perp}(\eta) = 1$ for $k = 0$ and $\delta_{H^\perp}(\eta) = 0$ for $k = 1, \ldots, N-1$. Then replacing $e^{2\pi i \xi}$ with z, (3.23) becomes

$$\frac{1}{N} \sum_{j=0}^{N-1} F_j(z) \, M_j(z \, e^{2\pi i k/N}) = \begin{cases} z^{-l}, & \text{if } k = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3.24}$$

Let $W = e^{(2\pi i)/N}$, then (3.24) becomes

$$\frac{1}{N} \sum_{j=0}^{N-1} F_j(z) \, M_j(z \, W^k) = \begin{cases} z^{-l}, & \text{if } k = 0; \\ 0, & \text{otherwise.} \end{cases}$$

That is

$$\frac{1}{N} \sum_{j=0}^{N-1} F_j(z) \, M_j(z) = z^{-l}$$

$$\frac{1}{N} \sum_{j=0}^{N-1} F_j(z) \, M_j(z \, W^k) = 0, \quad k = 1, \ldots, N-1.$$

These are the perfect reconstruction conditions for N channels given in [24, p. 112].

In the setting of LCA groups, the conditions in Theorem 3.2.2 produce consistent results with Theorem 1.1.3 by Faridani (cf. [7]). We gave a simplified version of the sampling theorem for nonperiodic sampling by Faridani in the introduction. However, we give the full version of this theorem here.

Theorem 3.2.5 *Assume that $f \in L_1(G)$ is continuous, every function $y \to f(x+y)$ belongs to $L_1(H)$, and that $\widehat{f}$ vanishes outside a compact set $K \subset \widehat{G}$. Let $M_l = \{\eta_1^{(l)}, \ldots, \eta_{m_l-1}^{(l)}\}$, $l = 1, \ldots, L$ be the values assumed by $M_\xi = \{\eta \in H^\perp \setminus \{0\} : \xi \in$*

$\eta + K\}$ *as* ξ *runs through* K. *Let* χ_{K_l} *be the indicator function of* $K_l = \{\xi \in K :$
$M_\xi = M_l\}$. *Assume* $x_0 + H, \ldots, x_{N-1} + H \in G/H$ *are such that for* $l = 1, \ldots, L$ *the*
systems of equations

$$\sum_{n=0}^{N-1} \beta_n^{(l)} = 1$$

$$\sum_{n=0}^{N-1} \beta_n^{(l)} e^{2\pi i < x_n, \eta_j^{(l)} >} = 0, \quad j = 1, \ldots, m_l - 1 \tag{3.25}$$

admit solutions $\beta_n^{(l)}$, $n = 0, \ldots, N - 1$. *Let* $F \in L_2(G)$ *with* $supp(\hat{F}) \subseteq K$, *and*
define

$$(Sf)^\wedge(\xi) = \hat{F}(\xi) \sum_{n=0}^{N-1} \sum_{l=1}^{L} \beta_n^{(l)} \chi_{K_l}(\xi) \int_H f(x_n + y) e^{-2\pi i < x_n + y, \xi >} dm_H(y) \tag{3.26}$$

and $Sf(x) = \int_{\hat{G}} (Sf)^\wedge(\xi) e^{2\pi i < x, \xi >} dm_{\hat{G}}(\xi)$. *Then*

$$\hat{F}(\xi)\hat{f}(\xi) = (Sf)^\wedge(\xi) \quad for \ \xi \in \hat{G} \tag{3.27}$$

and

$$F * f(x) = Sf(x) = \sum_{n=0}^{N-1} \int_H f(x_n + y) k_n(x - x_n - y) dm_H(y) \tag{3.28}$$

with

$$k_n(z) = \sum_{l=1}^{L} \beta_n^{(l)} F * \bar{\chi}_{K_l}(z). \tag{3.29}$$

Remark 3.2.6 : Following (3.15) and applying the Poisson summation formula
(1.25), a formulation of the output of a generalized filter bank in the Fourier domain
can be written as

$$(\widetilde{S}g)^\wedge(\xi) = \sum_{n=0}^{N-1} \left[F_n(\xi) \chi_K(\xi) \sum_{y \in H} M_n g(y) e^{-2\pi i \langle y, \xi \rangle} \right]. \tag{3.30}$$

In periodic sampling the input is $g(x_n + y)$, $y \in H$. This can be viewed as $(\downarrow H)M_n g(y)$ if $M_n g(z) = g(x_n + z)$, i.e., $(M_n g)^\wedge(\xi) = \widehat{g}(\xi)e^{2\pi i \langle x_n, \xi \rangle}$ following (3.1). The reconstruction condition for a generalized filter bank in Theorem 3.2.2 then requires that the filters F_N to satisfy

$$m_{\widehat{G}}(R)\chi_K(\xi)\sum_{n=0}^{N-1}[F_n(\xi)e^{2\pi i \langle x_n, \xi+\eta \rangle}] = \delta_{H^\perp}(\eta)\widehat{F}(\xi) \tag{3.31}$$

for all η such that $\xi + \eta \in K$, i.e., $\eta \in \widetilde{M_\xi}$. Then the reconstruction function is $F * g$. Since $supp(\widehat{F}) \subseteq K$, 3.31 is equivalent to

$$m_{\widehat{G}}(R)\sum_{n=0}^{N-1}[F_n(\xi)e^{2\pi i \langle x_n, \xi+\eta \rangle}] = \delta_{H^\perp}(\eta)\widehat{F}(\xi) \tag{3.32}$$

for all $\eta \in \widetilde{M_\xi}$. Using (3.26) of Theorem 3.2.5 by A. Faridani, the reconstructed function Sg satisfies

$$(Sg)^\wedge(\xi) = \widehat{F}(\xi)\sum_{n=0}^{N-1}\sum_{l=1}^{L}\beta_n^{(l)}\chi_{K_l}(\xi)\frac{1}{m_{\widehat{G}}(R)}\sum_{y\in H}g(x_n + y)e^{-2\pi i \langle x_n + y \rangle}$$

$$= \frac{\widehat{F}(\xi)}{m_{\widehat{G}}(R)}\sum_{n=0}^{N-1}\sum_{l=1}^{L}\beta_n^{(l)}\chi_{K_l}(\xi)e^{-2\pi i \langle x_n, \xi \rangle}\sum_{y\in H}g(x_n + y)e^{-2\pi i \langle y, \xi \rangle}. \tag{3.33}$$

Comparison of (3.32) and (3.33), reveals that Sg can be viewed as output of a synthesis filter bank with the choice of filters

$$F_n(\xi) = \frac{\widehat{F}(\xi)}{m_{\widehat{G}}(R)}\sum_{l=1}^{L}\beta_n^{(l)}\chi_{k_l}(\xi)e^{-2\pi i \langle x_n, \xi \rangle}. \tag{3.34}$$

Now upon replacing (3.34) into (3.32), We get

$$\sum_{n=0}^{N-1}\sum_{l=1}^{L}\beta_n^{(l)}\chi_{k_l}(\xi)e^{-2\pi i \langle x_n, \xi \rangle} = \delta_{H^\perp}(xi)$$

for $\xi \in K$ and η such that $\xi + \eta \in K$. If $\xi in K_l$, then $\eta = 0$ or $\eta = \eta_k^{(l)}$. This gives the equations

$$\sum_{n=0}^{N-1}\beta_n^{(l)} = 1$$

$$\sum_{n=0}^{N-1} \beta_n^{(l)} e^{2\pi i \langle x_n, \eta_k^{(l)} \rangle} = 0$$

for $k = 1, ..., m_l - 1$. These are the system of equations (3.25) given in Theorem 3.2.5 by A. Faridani (cf. [7]).

The theory presented here is abstract and will ultimately be useful if it leads to a better understanding of concrete applications. Thus, one goal is to see that our results are sufficiently general to cover different areas related to signal processing. One such area is known as "multi-sensor deconvolution problem". We will give a short description here; see "System of Convolution Equations, Deconvolution, Shannon Sampling, and the Wavelet and Gabor Transforms" by S. Casey and D. Walnut [3] and references given there. Suppose that a signal or image, g, is detected via a bank of k linear, translation invariant sensors with impulse responses $\{\mu_i\}_{i=0}^{k-1}$, i.e., $s(t) = g * \mu(t)$ where the vector μ contains μ_i, $i = 0, \ldots, k-1$. Hence, the signal g is changed into a vector of data $s = (s_0, ..., s_{k-1})$ where $s_i = g * \mu_i$ for each i. The multi-sensor deconvolution problem (MDP) states that given a collection of compactly supported distributions such as $\{\mu_i\}_{i=0}^{k-1}$, find a collection of compactly supported distributions $\{\nu_i\}_{i=0}^{k-1}$ such that

$$\sum_{i=0}^{k-1} \mu_i * \nu_i = \delta. \tag{3.35}$$

Taking Fourier transforms in (3.35) gives,

$$\sum_{i=0}^{k-1} \widehat{\mu_i}(\xi) \widehat{\nu_i}(\xi) = 1 \tag{3.36}$$

where $\widehat{\mu_i}$ and $\widehat{\nu_i}$ can be viewed as band-limited functions. If (3.35) can be solved, then perfect reconstruction is possible and hence,

$$\sum_{i=0}^{k-1} s_i * \nu_i = \sum_{i=0}^{k-1} (g * \mu_i) * \nu_i = \sum_{i=0}^{k-1} g * (\mu_i * \nu_i) = g * \sum_{i=0}^{k-1} \mu_i * \nu_i = g * \delta = g.$$

The following theorem is given by S. Casey and D. Walnut in [3, p. 560]. In the following, let $\delta_{nT}(t) = \delta(t - nT)$ and let $sinc_T(t) = \frac{\sin(\frac{\pi}{T}t)}{\pi t}$.

Theorem 3.2.7 *Let $g \in L_2(\mathbb{R})$ be so that $\widehat{g}$ vanishes outside $K = [\Omega, \Omega]$ and*

$$s_i(t) = g * \mu_i(t), \qquad i = 0, \ldots, k - 1.$$

Assume that associated with the set of convolvers $\{\mu_i\}_{i=0}^{k-1}$, there exists a set of deconvolvers $\{\nu_{i,\psi}\}_{i=0}^{k-1}$ such that

$$\sum_{i=0}^{k-1} \mu_i * \nu_{i,\psi} = \psi \tag{3.37}$$

where ψ is an arbitrary close approximation of the Dirac δ. If $T \leq 1/2\Omega$, then the function $g * \psi$ may be reconstructed from the sampled functions $s_i = \mu_i * g$,

$$\sum_{n=-\infty}^{\infty} s_i \delta_{nT},$$

by

$$g * \psi(t) = T \sum_{i=1}^{k} \left(\sum_{n=-\infty}^{\infty} s_i(nT)\delta_{nT} \right) * \left(\frac{\sin(\frac{\pi}{T}t)}{\pi t} * \nu_{i,\psi} \right)(t). \tag{3.38}$$

With suitable choices of filters in analysis and synthesis banks, we will show that Theorem 3.2.2 and Theorem 3.2.7 have consistent results.

Remark 3.2.8 *:* Let $G = \mathbb{R}$ and $H = T\mathbb{Z}$ for some $T \in \mathbb{R}$. Then $\widehat{G} = \mathbb{R}$, $H^{\perp} = (1/T)\mathbb{Z}$ and the a fundamental domain is $R = [0, 1/T)$. Let $g \in L_2(\mathbb{R})$ be so that $\widehat{g}$ vanishes outside $K = [-\Omega, \Omega]$. In MDP, the input is $\mu_i * g(y)$, $y \in T\mathbb{Z}$, which can viewed as output of an analysis filter banks with $(\downarrow H)H_i g(y)$ if $H_i g(z) = \mu_i * g(z)$. The reconstruction condition for filters banks requires that the filters F_i to satisfy the condition

$$\frac{1}{T} \chi_{[-\frac{1}{2T}, \frac{1}{2T}]}(\xi) \sum_{i=0}^{k-1} [F_i(\xi)\widehat{\mu}_i(\xi + j/T)] = \delta_{H^{\perp}}(\eta)\widehat{\psi}(\xi) \tag{3.39}$$

for all $j \in \mathbb{Z}$ such that $\xi + (j/T) \in [-\Omega, \Omega]$. Then the reconstructed function is $g * \psi$.
The reconstruction formula for MDP can be viewed as an output of an synthesis
filter banks with the choice of filters

$$F_i(\xi) = T\widehat{\nu}_{i,\psi}(\xi). \tag{3.40}$$

Substitution of (3.40) into (3.39) yields

$$\frac{1}{T}\chi_{[-\frac{1}{2T}, \frac{1}{2T}]}(\xi) \sum_{i=0}^{k-1} [T\widehat{\nu}_{i,\psi}(\xi)\widehat{\mu}_i(\xi + j/T)] = \delta_{H^\perp}(\eta)\widehat{\psi}(\xi) \tag{3.41}$$

for $\xi \in [-\Omega, \Omega]$ and $j \in \mathbb{Z}$ such that $\xi + (j/T) \in [-\Omega, \Omega]$. This gives the equation

$$\sum_{i=0}^{k-1} \widehat{\nu}_{i,\psi}(\xi)\widehat{\mu}_i(\xi) = \widehat{\psi}(\xi).$$

This is the condition (3.37) given in Theorem 3.2.7 by S. Casey and D. Walnut [3].

BIBLIOGRAPHY

[1] Anant Agarwal, "Limits on Interconnection Network Performance", *IEEE Transactions on Parallel and Distributed Systems*, vol. 2, no. 4, pp. 398–412, October 1991.

[2] J. J. Benedetto, *Irregular sampling and frames,* Wavelets-A Tutorial in Theory and Applications (C. K. Chui, ed.), Academic Press, New York, 1992, pp. 445-507.

[3] S. Casey and D. Walnut, System of convolution equations, deconvolution, Shannon sampling and the wavelet and Gabor transforms, *SIAM Rev. 36(4),* pp. 537-577, December 1994.

[4] K.F. Cheung, *A multidimensional extension of Papoulis' generalized sampling expansion with the application in minimum density sampling.* In R.J. Marks II (ed.), Advanced Topics in Shannon Sampling and Interpolation Theory, Springer, New York, 1993, pp. 85-119.

[5] L. Desbat, *Efficient sampling on coarse grids in tomography,* Inverse Problems, 9(1993), pp. 251-269.

[6] A. Faridani, *An application of a multidimensional sampling theorem to computed tomography,* Contemp. Math., vol. 113, Amer. Math Soc., Providence, RI, 1990, pp. 65-80.

[7] A. Faridani, "A Generalized Sampling Theorem", *Mathematics of Computation,* vol 63, no. 207, pp. 307-327, July 1994.

[8] A. Faridani, *Sampling in Parallel-Beam Tomography,* Inverse Problems, Tomography, and Image Processing (A. G. Ramm, ed.), Plenum Press, New York, 1998, pp. 33-53.

[9] H. G. Feichtinger and K. Gröchenig, *Irregular sampling theorems and series expansions of band-limited functions,* J. Math. Anal. Appl. 167 (1992), 530-556.

[10] ———-, *Theory and practice of irregular sampling,* Wavelets: Mathematics and Applications (J. J. Benedetto and M. W. Frazier, eds.), CRC Press, 1993, pp. 305-363.

[11] N. T. Gaarder, *A note on the multidimensional sampling theorem,* Proc. IEEE, 60(1972), pp. 247-248.

[12] K. Gröchenig, *Reconstruction algorithms in irregular sampling,* Math. Comp. 59 (1992), 181-194.

[13] K. Gröchenig, *Gabor Analysis on LCA Groups,* Gabor Analysis and Algorithms (H. G. Feichtinger and T. Strohmer, ed.), Birkhäuser, Boston, 1998, pp. 211-229.

[14] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis,* Volume I,II, Springer, 1979, 1970.

[15] J.R. Higgins, *Five short stories about the cardinal series,* Bull. Amer. Math. Soc., 12(1985), pp. 45–89.

[16] M. Holschneider, *Wavelets, an analysis Tool,* Clarendon Press, New York, 1995.

[17] A. J. Jerri, *The Shannon sampling theorem-its various extensions and applications: a tutorial review,* Proc. IEEE 65 (1977), 339-347.

[18] R.E. Kahn and B. Liu, *Sampling representations and the optimum reconstruction of signals,* IEEE Trans. Informat. Theory, IT-11(1965), pp. 339-347.

[19] I. Kluvanek, *Sampling theorem in abstract harmonic analysis,* Mat. Casopis Sloven. Akad. Vied, 15(1965), pp. 43-48.

[20] A. Kohlenberg, *Exact interpolation of bandlimited functions,* J. Appl. Phys. 24 (1953), 1432-1436.

[21] F. Marvasti. *Nonuniform sampling,* Advanced Topics in Shannon Sampling and Interpolation Theory (R. J. Marks II, ed.), Springer, New York, 1993, pp. 121-156.

[22] R. J. Marks II, *Introduction to Shannon sampling and interpolation theory,* Springer, New York, 1991.

[23] R. J. Marks II, *Advanced topics in Shannon sampling and interpolation theory,* Springer, New York, 1993.

[24] G. Strang and T. Nguyen, *Wavelets and Filter Banks,* Wellesley-Cambridge Press 1996.

[25] A. Papoulis, *Signal Analysis,* McGraw-Hill, New York, 1977.

[26] W. Rudin, *Fourier Analysis on groups,* Wiley, New York, 1962.

[27] C. E. Shannon, *Communications in the presence of noise,* Proc. IRE, vol. 37, pp. 10-21, jan. 1949.

[28] D. Walnut, *Nonperiodic Sampling of Bandlimited Functions on Unions of Rectangular Lattices,* J. Fourier Anal. Appl., vol. 2, Num. 5, pp. 435-452, 1996.

[29] ———, *Solutions to Deconvolution Equations Using Nonperiodic Sampling*, J. Fourier Anal. Appl., vol. 4, Num. 6, pp. 669-709, 1998.

[30] J. L. Yen, *On nonuniform sampling of bandwidth-limited signals*, IRE Trans. Circuit Theory, CT-3(1956), pp. 251-257.