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In this work we study the following problem: *Given a topological space X, is there a map $F : R^n \to R^n$ such that X is an attractor for F?* R. F. Williams(1967), M. Misiureuicz(1985), W. Szczechla(1989), and M. Barge & J. Martin(1990) gave partial answers for this problem.

Barge and Martin showed that for any given continuous map $f : I \to I$, where I is a compact interval, there is an embedding of $\varprojlim(I, f)$ in R^2 and a homeomorphism $h : R^2 \to R^2$ such that $h(\varprojlim(I, f)) = \varprojlim(I, f)$, the restriction of h to $\varprojlim(I, f)$ is equal to $\hat{f}$, and $\varprojlim(I, f)$ is a global attractor for h. Here $\varprojlim(I, f)$ is the inverse limit of the sequence with bonding maps f and $\hat{f}$ is the induced homeomorphism on the inverse limit. Hence $\hat{f}$ on $\varprojlim(I, f)$ can be realized as the restriction of a homeomorphism h of the plane to its attractor.

In this work we extend these results to certain other compact subsets X of R^3. We show that X can be realized as local attractors for certain self homeomorphisms h of R^3 such that the restrictions of h to X are chaotic. These subsets X are cell-like sets arising as nested intersections of tori in a certain way. A typical example of these subsets is the Whitehead continuum, which is a non cellular embedding of the Knaster continuum in R^3. Technical difficulties arose in recognizing the self linking of certain subsets of R^3. This necessitated our working with inverse limits of pairs, and carefully analyzing a sequence of near homeomorphisms.

A Chaotic Embedding of the Whitehead Continuum

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1. Historical Background

1.1 *History of the Problem – Partial Answers.*

R. F. Williams [W] proved the following: Given a differentiable endomorphism f of a branched one-dimensional manifold K, the inverse limit $\varprojlim(K, f)$ can be embedded in S^4 and the shift map $\hat{f}$ extended to a diffeomorphism of S^4 possessing $\varprojlim(K, f)$ as an attractor.

M. Misiureuicz [M] proved the following: If $\tau : I \to I$ is the tent map ($x \to 1 - |2x - 1|$), then:

A. For every manifold M where $dim(M) \geq 3$, there exists a C^∞ diffeomorphism $h : M \to M$ such that h restricted to its attractor Λ is topologically conjugate to $\hat{\tau}$ (which is chaotic).

B. For every manifold M where $dim(M) \geq 2$, there exists a homeomorphism $h : M \to M$ such that h restricted to its attractor Λ is topologically conjugate to $\hat{\tau}$.

The results A and B hold for all maps conjugate to τ, for example the quadratic map $x \to 4x(1 - x)$.

W. Szczechla [Sz], in a paper entitled "*Inverse Limits of Certain Maps as Attractors in 2 Dimensions*" extended Miziureuicz's results.

Barge and Martin [BM4] proved that if $f : I \to I$ is a map of a closed interval. Then $\varprojlim(I, f)$ can be realized as a global attractor for a homeomorphism of R^2.

In this work we extend some of the results of Barge and Martin to certain other compact subsets X of R^3. These subsets are cell-like sets arising as nested intersections of tori in a certain way. A typical example of these subsets is the Whitehead continuum.

In the next few sections we define the Whitehead manifold and discuss some of its properties. We also define the Whitehead continuum and prove that it is a cell-like noncellular subset of R^3.

Definitions of some of the terms used here (for example, cell-like, cellular and UV^∞) can be found in *Section* 2.2.

1.2 *The Whitehead Manifold.* [H]

The Poincare' conjecture states that every homotopy 3-*sphere*, that is, every simply connected, compact 3-*manifold* without boundary, is a 3-*sphere*. This is still an open question. In 1935, J. H. C. Whitehead [Wh] showed that this conjecture cannot be generalized to *open* 3-*manifolds*. He constructed an open homotopy 3-*cell* M, that is, a noncompact simply connected, 3-*manifold* with trivial second homology group and without boundary, which is not homeomorphic to R^3. He constructed M as the union of an ascending sequence $T_1, T_2, \ldots$ of solid tori in R^3, $M = \bigcup_{i=0}^{\infty} T_i$, where T_i is embedded in T_{i+1} as shown in *Figure* 1.1.

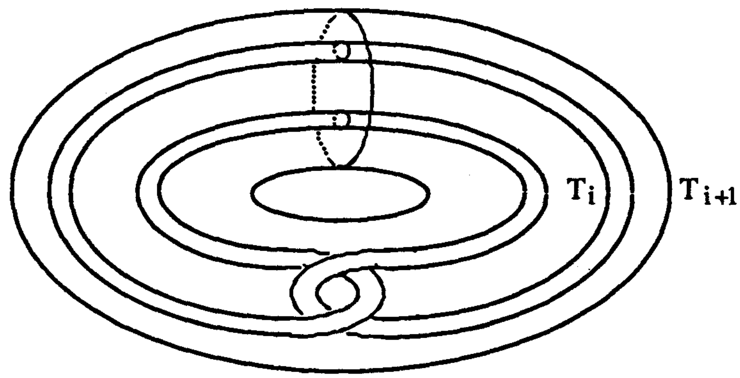

Figure 1.1

1.3 *Properties of the Whitehead Manifold.*

For completeness, we list some properties of the Whitehead manifold. Details can be found in [H].

(i) The space M is simply connected: every simple closed curve $C \subset M$ lies in a solid torus T_r since C is compact and therefore intersects at most finitely many tori T_i. But T_r is contractible in T_{r+1} and hence C is contractible in $T_{r+1} \subset M$.

(ii) The space M is not homeomorphic to R^3 since M contains a simple closed curve that does not lie in a 3-*cell* in M, for example the core curve C_1 of T_1. If C_1 lies in a 3-*cell* B^3 in M, it follows that $B^3 \subset Int(T_r)$, (for r sufficiently large), and that there exists a 3-*cell* D_1^3 in $Int(T_r)$ with $T_1 \subset Int(D_1^3)$ such that no connected component of $Bd(D_1^3) \cap Bd(T_i)$ could be a meridianal disk of T_i for $i = 2, 3, \ldots, r-1$. Hence $Bd(D_1^3)$ can be deformed out of T_2 obtaining a 3-*cell* D_2^3 in $Int(T_r)$ with $T_2 \subset Int(D_2^3)$. Continuing this way, one would finally obtain a 3-*cell* D_{r-1}^3 in $Int(T_r)$ with $T_{r-1} \subset Int(D_{r-1}^3)$, which is a contradiction (since this would imply that the Whitehead continuum, to be defined in *Section* 1.4 is cellular). For more details, see [H].

In [Bi3], Bing gives an alternate proof of the fact that M is not homeomorphic to R^3. He shows that a simple closed curve J on $Bd(T_1)$ that circles T_1 longitudinally does not lie on the interior of a topological cube in M. He does this by showing that each topological cube whose interior contains J, also contains a simple closed curve on $Bd(T_2)$ that circles T_2 longitudinally. It follows then that for every positive integer i, the cube contains a simple closed curve on $Bd(T_i)$ that circles T_i longitudinally. Hence the cube could not lie in M.

(iii) The space B can be embedded in R^3.

(iv) The product $B \times R^1 \cong R^4$ [Mc1]. The idea is to show that every product $T_{i+1} \times [a-\epsilon, b+\epsilon]$ contains a 4-*cell* which contains $T_i \times [a,b]$ where $[a,b] \subset R^1$ and $a-\epsilon < a < b < b+\epsilon$. Hence $T_1 \times [-1,1] \subset B_1^4 \subset T_2 \times [-2,2] \subset B_2^4 \subset T_3 \times [-3,3] \subset \cdots$, where B_i^4 is a 4-*cell* for all i. Hence $B \times R^1 = \bigcup_{i=1}^{\infty} T_i \times [-i,i] = \bigcup_{i=1}^{\infty} B_i^4$ can be represented as the union of an ascending sequence of 4-*cells*. Hence from a result of M. Brown's [Br2] stating that a space is homeomorphic to R^n if it is the union of an ascending sequence of open subsets each homeomorphic to R^n, it follows that $B \times R^1 \cong R^4$.

(v) The product $B \times B \cong R^6$ [Mc1].

1.4 *The Whitehead Continuum*.

Let T_0 be a solid torus in R^3. Let T_1 be a solid torus in $Int(T_0)$ as shown in *Figure* 1.2. Let T_2 be a solid torus embedded in $Int(T_1)$ as T_1 is embedded in $Int(T_0)$. Continue this construction. This results in a sequence $T_0, T_1, T_2 \ldots$ of solid tori in R^3 such that for all nonnegative integers n, $T_{n+1} \subset Int(T_n)$. Assume the tori $T_0, T_1, T_2, \ldots$ are constructed efficiently to force 1-*dimensionality* of their intersection. For example, each T_i can be required to retract to its core curve under a retraction r_i with $diam(r_i^{-1}(p)) < \frac{1}{i}$ for each p. Then $W = \bigcap_{i=0}^{\infty} T_i$ is called the Whitehead continuum.

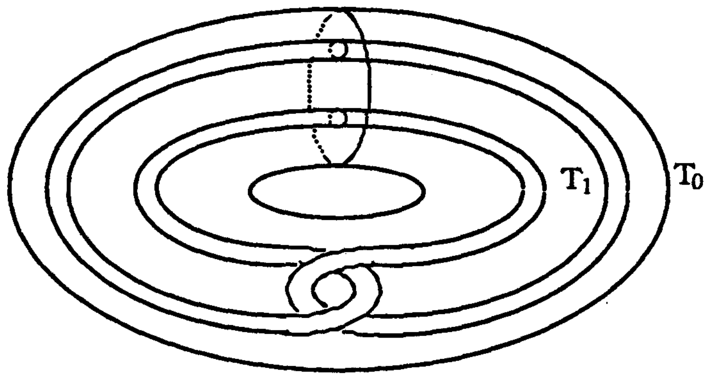

Figure 1.2

1.5 *Properties of the Whitehead Continuum.*

For completeness, we list some of the properties of the Whitehead continuum. For more details, see [D].

(i) The Whitehead continuum W is a noncellular subset of R^3. This will be proved in the next section.

(ii) The continuum W is a cell-like subset of R^3. This follows from the fact that if U is a neighborhood of W then for some integer $k \geq 0$, $T_k \subset U$. Hence $W \subset T_{k+1} \subset T_k \subset U$. Since T_{k+1} contracts to a point in T_k, W contracts to a point in U.

(iii) The continuum W is a UV^∞ continuum in R^3. This follows from the fact that W is cell-like and R^3 is an ANR (absolute neighborhood retract) [D, Prop.1, p.123].

(iv) The continuum W is cellular in R^4. This follows from the fact that W is UV^∞ in R^3 [Mc3].

1.6 *The Whitehead Continuum W is Noncellular in R^3.*

In this section we show that W is a noncellular subset of R^3. A few results from the literature are needed. These results and their proofs are included for completeness.

Notation. Let $T = h(S^1 \times D^2)$ be a solid torus in R^3, where $h : R^3 \to R^3$ is a homeomorphism. Assume $h(S^1 \times \{0\})$ lies in a plane P. Then $P - (P \cap T)$ has two components. By the *spanning 2-cell* D of T we mean the closure of the bounded component of $P - (P \cap T)$. The disk D is bounded by a "longitudinal loop" in $Bd(T)$.

If p is a loop, then by $p \simeq e$ we mean p is homotopically trivial.

Let $I^2 = [0,1] \times [0,1]$.

1.6.1 *Lemma.* [Mo, Th.5, p.113] *Let J_1, J_2 and J_3 be plane polygons, simply linked in a series, as shown in Figure 1.3. Let D be a plane 2-cell bounded by J_2, and suppose that D is simply punctured by J_1 and J_3, see Figure 1.3. Let p be a closed path in $U = D - (J_1 \cup J_2 \cup J_3)$. If $p \simeq e$ in $R^3 - (J_1 \cup J_3)$, then $p \simeq e$ in U.*

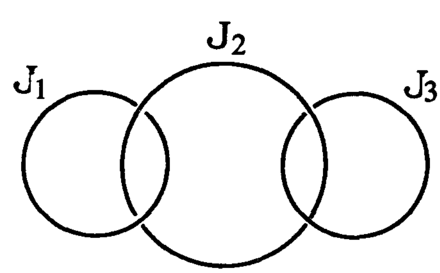

Figure 1.3

Proof. See [Mo, Th.5, p.113]. ∎

Let T_0 be a solid torus. In the interior of T_0 form a set T_1 which is the union of a finite collection of solid tori with planar cores, linked in cyclic order as shown in *Figure* 1.4.

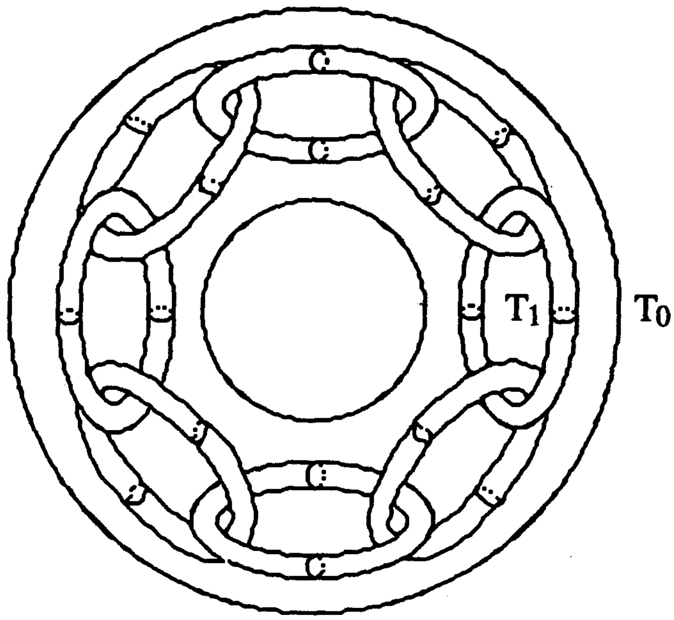

Figure 1.4

Suppose that the number of components C_i of T_1 is k, where $k \geq 4$. *Figure* 1.5 shows three successive components of T_1.

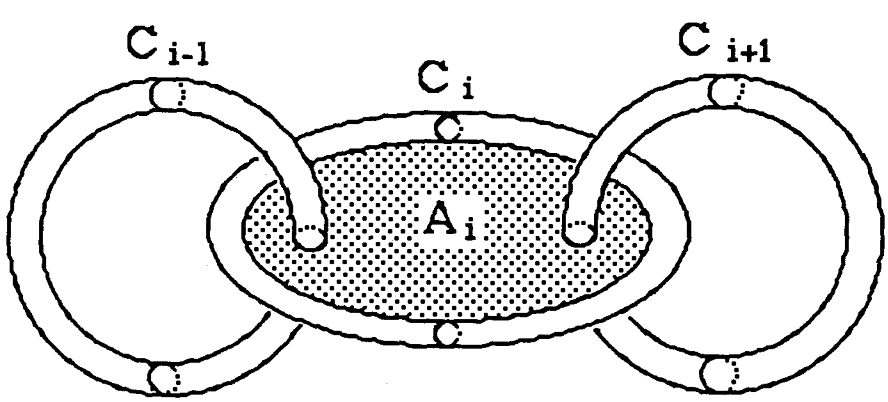

Figure 1.5

Let D_i be the *spanning* 2-*cell* of C_i. The set D_i is punctured by C_{i-1} and C_{i+1}, hence $A_i = Cl[D_i - (C_{i-1} \cup C_{i+1})]$ is a 2-*cell* with 2 holes.

<u>1.6.2 *Theorem*</u>. [Mo, Th.1, p.128] *Let the components C_i and the spanning 2-cells D_i, $i \leq k$ be as in the definition of T_1 above. Then $Bd(T_0)$ is a retract of the set $T_0 - [\bigcup_{i=1}^{k} C_i \cup \bigcup_{i=1}^{k} D_i]$.*

Proof. Note that the set $\bigcup_{i=1}^{k} C_i \cup \bigcup_{i=1}^{k} D_i$ contains a simple closed curve S_0 which is a core of T_0. Hence $T_0 - S_0$ retracts to $Bd(T_0)$, and $T_0 - (\bigcup_{i=1}^{k} C_i \cup \bigcup_{i=1}^{k} D_i) \subset T_0 - S_0$ retracts to $Bd(T_0)$ as well.

<u>1.6.3 *Theorem*</u>. [Mo, Th.2, p.129] *Let p be a closed path in $R^3 - T_0$. If $p \simeq e$ in $R^3 - T_1$, then $p \simeq e$ in $R^3 - T_0$.*

Proof. Let $A_i = Cl[D_i - (C_{i-1} \cup C_{i+1})]$, as in the definition of T_1. Suppose, without loss of generality, that p is a PL map, and let $\phi : I^2 \to R^3 - T_1$ be a PL contraction of p to e.

Choose p and $\phi(I^2)$ in general position relative to A_i, that is , there exists a triangulation K of I^2 such that if $\sigma^2 \in K$, and $\phi(\sigma^2)$ intersects A_i, then $\phi_{|_{\sigma^2}}$ is a simplicial homeomorphism, and A_i contains no vertex of $\phi(\sigma^2)$.

Let $J = \phi^{-1}(A_i \cap \phi(I^2))$. The set $\phi(J) = A_i \cap \phi(I^2)$ is a 1-*dimensional* polyhedron in A_i having no isolated points. $J \subset I^2$ is a finite union of disjoint polygons, since J contains no vertex of K. Let $J = \bigcup_{j=1}^{n} J_j$. Let J_j be a component of J which is innermost in I^2, that is, J_j is the boundary of a 2-*cell* d_j which contains no other components of J.

Consider the map $p_j = \phi_{|_{J_j}} : J_j \to A_j$. p_j is a closed path in A_j. Since $J_j = Bd(d_j)$ and $\phi(d_j) \subset R^3 - (C_{i-1} \cup C_{i+1})$ it follows that $p_j \simeq e$ in $R^3 - (C_{i-1} \cup C_{i+1})$. Hence by *Lemma* 1.6.1, $p_j \simeq e$ in $Int(A_i)$.

Extend p_j to a PL map $\phi_j : d_j \to A_i$. Define a new contraction $\phi' : I^2 \to R^3 - T_1$ by letting $\phi'_{|_{d_j}} = \phi_j$ and $\phi' = \phi$ elsewhere.

Now if N is a small connected neighborhood of d_j in I^2 then $\phi'(N)$ approches A_i from only one side, since $N - d_j$ is connected. Now define a new contraction $\phi'' : I^2 \to R^3 - T_1$ such that the intersection $\phi''(N) \cap A_i$ is empty and $\phi' = \phi$ elsewhere. Passing from ϕ to ϕ'' reduces the number of components of J by at least one. Hence after a finite number of steps, we get a contraction $\psi : I^2 \to R^3 - T_1$ such that $\phi(I^2) \cap A_i = \phi$.

We perform the procedure above for each $i = 1, 2, \ldots, k$. Note that A_i intersects A_{i-1} and A_{i+1} in linear intervals, see *Figure* 1.6.

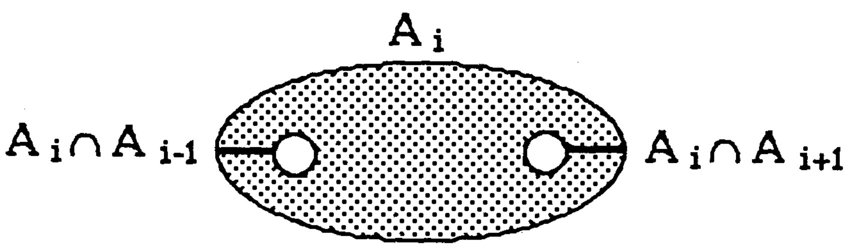

Figure 1.6

Thus if $\phi(I^2)$ is already disjoint from A_{i-1} (or A_{i+1}, or both), and $p_j : J_j \to A_i$ is a closed path in A_i, then p_j is contractible in $A_i - A_{i-1}$ (or $A_i - A_{i+1}$ or $A_i - (A_{i-1} \cup A_{i+1})$). Therefore we can *pull* $\phi(I^2)$ *off* the sets A_i, one at a time preserving the results of our earlier work. Thus after k steps we have a contraction $\psi_k : I^2 \to R^3 - T_1$ such that $\psi_k(I^2) \cap A_i = \phi$ for all i. Hence $\psi_k(I^2) \cap [\bigcup_{i=1}^{k} C_i \cup \bigcup_{i=1}^{k} D_i] = \phi$.

Let $r : T_0 - [\bigcup_{i=1}^{k} C_i \cup \bigcup_{i=1}^{k} D_i] \to Bd(T_0)$ be a retraction. Define $r_{|_{R^3 - T_0}}$ to be the identity map and let $\rho = r\psi_k : I^2 \to R^3 - Int(T_0)$. To get a contraction of p in $R^3 - T_0$, it suffices to pull $\rho(I^2)$ slightly off $Bd(T_0)$ into $R^3 - T_0$. ∎

<u>1.6.4 Theorem</u>. [Mo, Th.3, p.131] *Let p be a closed path in $R^3 - T_1$, and suppose that $p \simeq e$ in $R^3 - \bigcap_{i=0}^{\infty} T_i$. Then $p \simeq e$ in $R^3 - T_0$.*

Proof. Without loss of generality, assume that p is PL, and that there is a PL contraction $\phi : I^2 \to R^3 - \bigcap_{i=0}^{\infty} T_i$ of p. By compactness, for some integer n, the intersection $\phi(I^2) \cap T_n$ is empty. Let C be a component of T_{n-1}, and let $T_0' = C$ and $T_1' = C \cap T_n$.

Then $T_0' = C$ and $T_1' = C \cap T_n$ are related in the same way as T_0 and T_1, in fact, there is a homeomorphism of R^3 taking T_0 onto T_0' and T_1 onto T_1'. By *Theorem* 1.6.3, there is a contraction ϕ' of p onto e in $R^3 - C$ such that $\phi'(I^2) - \phi(I^2)$ lies in a small neighborhood of C, and hence intersects no other component of T_{n-1}.

Repeat the argument above for all components C of T_{n-1}. Hence in a finite number of steps we get a contraction of p in $R^3 - T_{n-1}$. By induction, p is contractible in $R^3 - T_0$. ∎

Let $\mathcal{C}_j$ be the union of the cores of the tori C_i making up T_j in *Figure* 1.4. The set $\mathcal{C}_1$ is a link of k unknotted circles arranged in a chain running around the solid torus T_0.

The following two theorems are generalizations of [Ro, Prop.G.1, p.70] and [Ro, Prop.G.4, p.72].

<u>1.6.5 Theorem</u>. *The meridian M of T_0 is not homotopically trivial in $R^3 - \mathcal{C}_i$ or in $T_0 - \mathcal{C}_i$ for all $i \geq 0$.*

Proof. Clearly, the theorem is true for $i = 0$. By *Theorem* 1.6.4, it suffices to prove it for $i = 1$. *Figure* 1.7 shows $\mathcal{C}_1 \subset T_0$.

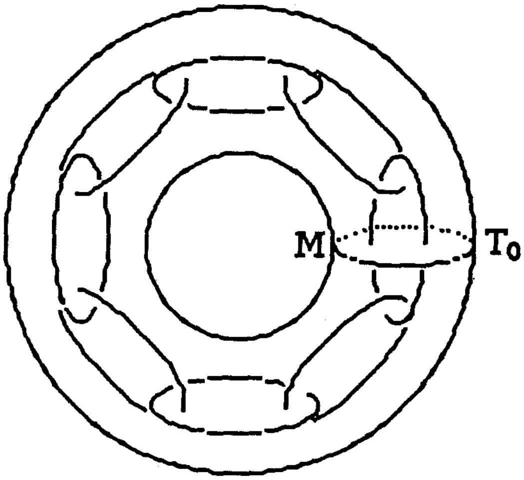

Figure 1.7

By *Theorem* 1.6.4, the loop $M \not\simeq e$ in $R^3 - T_1$ since $M \not\simeq e$ in $R^3 - T_0'$, where T_0' is a solid torus satisfying $\mathcal{C}_1 \subset Int(T_0') \subset Int(T_0)$. But T_1 retracts to $\mathcal{C}_1$. Hence $M \not\simeq e$ in $R^3 - \mathcal{C}_1$. ∎

Recall that the Whitehead continuum $W = \bigcap_{i=0}^{\infty} T_i$, where T_{i+1} is embedded in $Int(T_i)$ as shown in *Figure* 1.2. Let J_i denote the core of T_i for $i \geq 0$. *Figure* 1.8 shows $J_1 \subset T_0$.

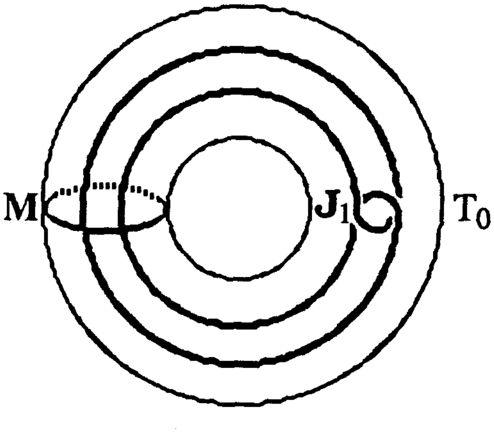

Figure 1.8

<u>1.6.6 Theorem</u>. *The meridian loop M of T_0 is not contractible in $T_0 - J_i$ for all $i \geq 0$.*

Proof. Clearly, the theorem is true for $i = 0$. By *Theorem* 1.6.4, it suffices to prove it for $i = 1$.

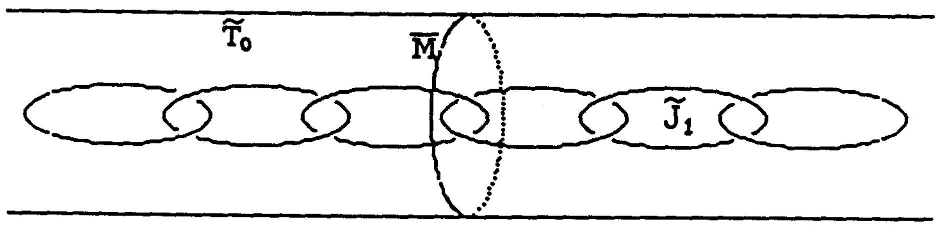

Figure 1.9

Let $p : \tilde{T} \to T_0$ be the universal cover of T_0. Let $\tilde{J}_1 = p^{-1}(J_1)$ and let $\overline{M}$ be one component of $p^{-1}(M)$. If H is a homotopy shrinking M in $T_0 - J_1$ to a point, then by the homotopy lifting property, H lifts to $\tilde{H}$ which shrinks $\overline{M}$ to a point in $\tilde{T}_0 - \tilde{J}_1$. Since $\tilde{H}(\overline{M} \times I)$ is compact, we may construct a finite circular chain C_1' missing $\tilde{H}(\overline{M} \times I)$ which contradicts *Theorem* 1.6.5 . (Appropriate twists may be needed for $\overline{M}$ and C_1' to be situated as M and C_1 shown in *Figure* 1.7). ∎

<u>1.6.7 Corollary</u>. *Every meridian disk of T_0, see Figure 1.8, intersects the Whitehead continuum.*

<u>1.6.8 Theorem</u>. *The Whitehead continuum W is noncellular in R^3.*

Proof. Consider the set $U = Int(T_0)$ as an open neighborhood of W. Assume that W is cellular in R^3. Hence there exists a 3-*cell* B^3 such that $W \subset Int(B^3) \subset B^3 \subset U$. Let M be a meridian loop of T_0. Let $f : B^2 \to T_0$ be a map such that f takes $Bd(B^2)$ homeomorphically onto M, where B^2 is a 2-*cell*. Choosing $f(B^2)$

in general position relative to B^3, we may assume that $f(B^2)$ misses a point p of $Int(B^3)$. Since there is a retraction $r : B^3 - \{p\} \to Bd(B^3)$, we may replace f by a map $g : B^2 \to T_0 - Int(B^3)$ with f and g agreeing on $Bd(B^2)$. Hence M is contractible in $T_0 - W$, which contradicts *Theorem* 1.6.6. ■

2. Definitions and Preliminary Theorems

2.1 *Chains and Chainable Continua.*

A *chain* $\mathcal{C}$ is a finite collection of open sets $\{C_1, C_2, \ldots, C_n\}$ such that $C_i \cap C_j \neq \phi$ if and only if $|i-j| \leq 1$. The sets C_i, $i = 1, 2, \ldots, n$ are called the *links* of the chain $\mathcal{C}$. Links are not assumed to be connected. If the links are of diameter less than ϵ, the chain is called an *ϵ-chain.* The links C_1 and C_n are called the *first* and *last* links of the chain, respectively. The chain $\mathcal{C}_2$ is a *refinement* of the chain $\mathcal{C}_1$ if each link of $\mathcal{C}_2$ is a subset of a link of $\mathcal{C}_1$.

If $\{(1,q_1),(2,q_2),\ldots,(n,q_n)\}$ is a collection of pairs of positive integers, the chain $\mathcal{C}_2$ *follows the pattern* $\{(1,q_1),(2,q_2),\ldots,(n,q_n)\}$ in the chain $\mathcal{C}_1$ if the ith link of $\mathcal{C}_2$ is a subset of the q_ith link of $\mathcal{C}_1$.

A *continuum* is a compact connected metric space. A continuum is called *chainable* (or *snakelike*) if for each positive number ϵ it can be covered by an ϵ-chain. A continuum is *decomposable* if it is the union of two proper subcontinua; otherwise it is *indecomposable.*

Let $A \subset X$. Then by $Bd(A)$, $Int(A)$ and $Cl(A)$ we mean the topological boundary, interior and closure of A in X respectively.

2.2 *Cellular Sets.* [D]

A subset X of R^n (of any *n-manifold*) is said to be *cellular* if there exists a sequence of *n-cells* B_i in R^n such that $B_{i+1} \subset Int(B_i)$, for $i = 1, 2, \ldots$ and $X = \bigcap_{i=1}^{\infty} B_i$. Alternatively, $X \subset R^n$ is *cellular* if and only if for every open set $U \supset X$ there exists an *n-cell* B such that $X \subset Int(B) \subset B \subset U$. As a second alternative definition, $X \subset R^n$ is *cellular* if and only if X is compact and has arbitrarily small neighborhoods homeomorphic to R^n.

A compact subset C of a space X is *cell-like* in X if for every neighborhood U of C in X, C can be contracted to a point in U.

A set $A \subset X$ has *Property n-UV* in X if for every neighborhood U of A in X there corresponds another neighborhood V of A in U such that every map of $Bd(B^{n+1})$ into V, where B^{n+1} is an $(n+1)$-*cell*, extends to a map of B^{n+1} into U. The set A has *Property* UV^n in X if it has Property k-UV in X for all $k \in \{0, 1, 2, \ldots, n\}$. The set A has *Property* UV^∞ in X if it has Property k-UV in X for all $k \geq 0$.

2.2.1 *Lemma*. [D, Prop.4, p.121] *Let C be a compact subset of an ANR X. Then C is cell-like in X if and only if, for each neighborhood U of C, some neighborhood V of C in U is contractible in U.*

2.2.2 *Lemma*. [D, Prop.1, p.123] *Every cell-like subset A of an ANR X has Property UV^∞ in X.*

Proof. Let $A \subset X$ be cell-like. Let U be a neighborhood of A in X. By *Lemma* 2.2.1, there exists a neighborhood V of A such that $A \subset V \subset U$ and V is contractible in U. Let $f : Bd(B^{n+1}) \to V$ be a map. Note that $Bd(B^{n+1})$ is closed in B^{n+1}. The set V is contractible in U; let $\phi_t : V \to U$ be that contraction. Then the map f is homotopic to the constant map $\phi_1 \circ f : Bd(B^{n+1}) \to U$. Since $\phi_1 \circ f$ extends over B^{n+1}, so does f. Let $F : B^{n+1} \to U$ be an extension of f. Hence A has Property n-UV in X for all $n \in \{0, 1, 2, \ldots\}$. Hence A has Property UV^∞ in X. ∎

2.3 *Inverse Limit Spaces.*

Motivation. Inverse limit spaces proved to be a valuable tool in the study of the dynamics of certain maps as evident from [BM1], [BM2], and [BM3]. In these papers Barge and Martin began investigating the relationship between the dynamics of an iterated interval map and the associated inverse limit space. They showed that complicated or "chaotic" dynamics of an interval map f is reflected in complicated topology in the inverse limit space, the existence of an indecomposable subcontinuum to be more specific. They also showed the following: Suppose $f : I \to I$ is continuous, onto and there is a finite set $\{a_0, a_1, \ldots, a_l\}$, $a = a_0 < a_1 < \cdots < a_l = b$ in $I = [a, b]$ such that f is monotone on $[a_{i-1}, a_i]$ for $i = 1, 2, \ldots$. Then if $\varprojlim(I, f)$ is indecomposable, f has a periodic point whose period is not a power of 2.

A motivation for their study was that certain strange attractors can be realized as the inverse limit spaces of certain interval maps.

Studying the dynamics of $f : X \to X$ by utilizing $\varprojlim(X, f)$ has two advantages [BM3]:

(1) Spaces of the type $\varprojlim(X, f)$ have been extensively studied, in particular in the cases where $X = I$ and $X = S^1$.

(2) The function $f : X \to X$ becomes a homeomorphism $\hat{f} : \varprojlim(X, f) \to \varprojlim(X, f)$ and this allows certain arguments to be "inverted".

For more on inverse limit spaces the reader is referred to [ES], [CV], [HY] or [Be], and for more on inverse limits and dynamical systems [Sc] serves as a good introduction.

An inverse sequence is a double sequence (X_n, f_n), $n = 1, 2, \ldots$ such that each coordinate space X_n is a topological space and each *bonding map* $f_n : X_{n+1} \to X_n$ is continuous. The inverse limit of the inverse sequence (X_n, f_n) is the set $\varprojlim(X_n, f_n) = \{(x_n) \in \prod_{n=1}^{\infty} X_n : \forall n \geq 1, f_n(x_{n+1}) = x_n\}$ topologized with the relativized product topology.

Let π_k denote the natural projection from both $\prod_{n=1}^{\infty} X_n$ and its subset $\varprojlim(X_n, f_n)$ onto X_k defined by $\pi_k((x_n)) = x_k$.

We now include some basic results about inverse limits needed for our constructions later on.

2.3.1 *Lemma*. *The collection* $B = \{\pi_k^{-1}(U_k) : k \geq 1$ *and* U_k *is open in* $X_k\}$ *is a basis for the topology on* $\varprojlim(X_n, f_n)$.

Proof. Suppose that U is open in $\varprojlim(X_n, f_n)$ and $(x_n) \in U$. Since $\varprojlim(X_n, f_n)$ has the relativized product topology, there is an open set $W = U_{n_1} \times U_{n_2} \times \cdots \times U_{n_m} \times \prod_{n \neq n_i} X_n$ such that $(x_n) \in W \cap \varprojlim(X_n, f_n) \subset U$.

Choose $n \geq n_i$ for $i = 1, 2, \ldots, m$, and let $V = \bigcap_{i=1}^{m} f_{n_i,n}^{-1}(U_{n_i})$. One can easily verify that $(x_n) \in \pi_n^{-1}(V) \subset W \cap \varprojlim(X_n, f_n)$. ■

Given (X_n, f_n) and (Y_n, g_n). For all $n \geq 1$, let $h_n : X_n \to Y_n$ be a function such that $h_n f_n = g_n h_{n+1}$. Then there is an induced function $\hat{h} : \varprojlim(X_i, f_i) \to \varprojlim(Y_i, g_i)$ defined by $\hat{h}((x_n)) = (h_n(x_n))$.

2.3.2 *Lemma*. *Consider the following commutative diagram:*

$$\begin{array}{ccccccccc}
X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & X_3 & \xleftarrow{f_3} & \cdots & \varprojlim(X_i, f_i) \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & & \downarrow{\scriptstyle \hat{h}} \\
Y_1 & \xleftarrow{g_1} & Y_2 & \xleftarrow{g_2} & Y_3 & \xleftarrow{g_3} & \cdots & \varprojlim(Y_i, g_i)
\end{array}$$

If each h_n *is continuous then so is* $\hat{h}$.

Proof. Let U be open in $\varprojlim(Y_i, g_i)$ and $\hat{h}(\underline{x}) \in U$. Since U is open, there exists a positive integer n and an open subset $V_n \subset Y_n$ such that $\hat{h}(\underline{x}) \in \pi_n^{-1}(V_n) \subset$

U. Let $W' = X_1 \times X_2 \times \cdots \times X_{n-1} \times h_n^{-1}(V_n) \times X_{n+1} \times \cdots$. Since h_n is continuous, $h_n^{-1}(V_n)$ is open in X_n. Hence the set $W = W' \cap \varprojlim(X_i, f_i)$ is open in $\varprojlim(X_i, f_i)$. Since $\hat{h}(\underline{x}) \in \pi_n^{-1}(V_n)$ then $h_n(x_n) \in V_n$, $x_n \in h_n^{-1}(V_n)$, and hence $\underline{x} \in W$. If $\underline{z} \in W$ then $z_n \in h_n^{-1}(V_n)$, $h_n(z_n) \in V_n$ and hence $\hat{h}(\underline{z}) \in \pi_n^{-1}(V_n) \subset U$. Hence $\hat{h}$ is continuous. ∎

Given an inverse sequence (X_n, f_n), $n = 1, 2, \ldots$ such that each coordinate space X_n is a metric space with metric d_n. Define a new metric d'_n on X_n by $d'_n(x, z) = \min\{1, d(x, z)\}$. The metrics d_n and d'_n are equivalent metrics, that is, they generate the same topology on X_n. The space $\prod_{n=0}^{\infty} X_n$ is metrizable with metric ϱ defined by $\varrho(\underline{x}, \underline{z}) = \sum_{i=1}^{\infty} \frac{d'_n(x_i, z_i)}{2^i}$. The topology induced by ϱ is the product topology. The subset $\varprojlim(X_n, f_n)$ inherits this metric [CV, Theorem 6.A.15].

Let X be a metric space and $f : X \to X$ be continuous. Let $\varprojlim(X, f)$ denote the inverse limit of the sequence

$$X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \cdots$$

Let $\hat{f}$ be the induced map by the diagram

$$\begin{array}{ccccccccc} X & \xleftarrow{f} & X & \xleftarrow{f} & X & \xleftarrow{f} & \cdots & \varprojlim(X, f) \\ \downarrow f & & \downarrow f & & \downarrow f & & & \downarrow \hat{f} \\ X & \xleftarrow{f} & X & \xleftarrow{f} & X & \xleftarrow{f} & \cdots & \varprojlim(X, f) \end{array}$$

The map $\hat{f}$ is defined by $\hat{f}(\underline{x}) = (f(x_1), f(x_2), f(x_3), \ldots) = (f(x_1), x_1, x_2, \ldots)$.

<u>2.3.3 Lemma</u>. *Let X be a metric space and $f : X \to X$ be continuous and onto. Then $\hat{f} : \varprojlim(X, f) \to \varprojlim(X, f)$ is a homeomorphism.*

Proof. Suppose $\hat{f}(\underline{x}) = \hat{f}(\underline{z})$. Hence $x_i = z_i$ for all $i \geq 1$. Hence $\underline{x} = \underline{z}$ and $\hat{f}$ is *one-to-one*.

Suppose $\underline{z} \in \varprojlim(X, f)$ and $\underline{z} = (z_1, z_2, z_3, \ldots)$. Let $\underline{x} = (z_2, z_3, \ldots)$. Clearly, $\hat{f}(\underline{x}) = (f(z_2), f(z_3), \ldots) = (z_1, z_2, \ldots) = \underline{z}$ and hence $\hat{f}$ is onto.

By *Lemma* 2.3.2, $\hat{f}$ is continuous.

Let U be open in $\varprojlim(X, f)$ and $\underline{x} = (x_1, x_2, \ldots) \in U$. Since U is open in $\varprojlim(X, f)$, there exists an integer n and an open subset $V_n \subset X_n$ such that $\underline{x} \in \pi_n^{-1}(V_n) \subset U$. Let $W' = X_1 \times X_2 \times \cdots \times X_n \times f^{-1}(V_n) \times X_{n+2} \times \cdots$. Let $W = W' \cap \varprojlim(X, f)$. Hence W is open in $\varprojlim(X, f)$. Since $\hat{f}(\underline{x}) = (f(x_1), x_1, x_2, \ldots)$ and $x_n \in f^{-1}(V_n)$, we have $\hat{f}(\underline{x}) \in W$. If $\underline{z} = (z_1, z_2, \ldots) \in W$, then $z_{n+1} \in f^{-1}(V_n)$. Hence $\underline{x} = (z_2, z_3, \ldots) \in U$ and $\hat{f}(\underline{x}) = \underline{z}$. It follows that $W \subset \hat{f}(U)$ and $\hat{f}$ is open.

The map $\hat{f}$ is *one-to-one*, onto, continuous and open, hence it is a homeomorphism. ■

2.3.4 Lemma. *If (X_n, f_n) is an inverse sequence and the bonding maps are inclusion maps, then* $\varprojlim(X_n, f_n) \cong \bigcap_{n=1}^{\infty} X_n$.

Proof. Given

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \cdots.$$

We prove that the map $h : \varprojlim(X_n, f_n) \to \bigcap_{n=1}^{\infty} X_n$ defined by $h(x, x, x, \ldots) = x$ is a homeomorphism.

Clearly, h is *one-to-one* and onto.

Let U be open in $\bigcap_{n=1}^{\infty} X_n$ and $h(\underline{x}) \in U$. Then there exists an open subset $U' \subset X_1$ such that $U = U' \cap \bigcap_{n=1}^{\infty} X_n$. Since U' is open in X_1 and $h(\underline{x}) \in U'$, there is an open subset $V' \subset X_1$ such that $h(\underline{x}) \in V' \subset U'$. Let $V = V' \cap \bigcap_{n=1}^{\infty} X_n$. V is an open subset of $\bigcap_{n=1}^{\infty} X_n$ such that $h(\underline{x}) \in V \subset U$. Let $W = \pi_n^{-1}(V)$ for some integer n. W is open in $\varprojlim(X_n, f_n)$ and $\underline{x} \in W$. If $\underline{z} = (z, z, \ldots) \in W$, then $z \in V$ and $h(\underline{z}) \in V \subset U$. Hence h is continuous.

Let U be open in $\varprojlim(X_n, f_n)$ and $\underline{x} \in U$. Then by *Lemma* 2.3.1 there exists an integer n and an open subset $V_n \subset X_n$ such that $\underline{x} \in \pi_n^{-1}(V_n) \subset U$. Let $V = X_1 \cap X_2 \cap \cdots \cap X_n \cap V_n \cap X_{n+1} \cap \cdots$. $V \subset V_n$ is open in $\bigcap_{n=1}^{\infty} X_n$ and $h(\underline{x}) = x \in V$. Since $\pi_n^{-1}(V) \subset \pi_n^{-1}(V_n) \subset U$, then $V \subset h(U)$ and h is open. Hence h is a homeomorphism. ∎

<u>2.3.5 Lemma</u>. *Given* (X_n, f_n). *If* $X_i \cong X_j$ *for all* i *and* j, *and* f_i *is a homeomorphism for all* i, *then* $\varprojlim(X_n, f_n) \cong X_i$ *for all* i.

Proof. The map $F_i : \varprojlim(X_n, f_n) \to X_i$ defined by $F_i(x_1, x_2, \ldots) = x_i$ is a homeomorphism for all i. ∎

<u>2.3.6 Theorem</u>. [Be, Th.7, p.8] *Given* (X_i, f_i) *and* $n_1, n_2, \ldots$ *an increasing sequence of positive integers. Then* $\varprojlim(Y_i, g_i) \cong \varprojlim(X_i, f_i)$ *where for each* i, $Y_i = X_{n_i}$ *and* $g_i = f_{n_i, n_{i+1}}$.

Proof. We prove that $F : \varprojlim(X_i, f_i) \to \varprojlim(Y_i, g_i)$ defined by $F(\underline{x}) = (x_{n_1}, x_{n_2}, \ldots)$ where $\underline{x} = (x_1, x_2, \ldots)$ is a homeomorphism.

Clearly, $F(\underline{x}) \in \varprojlim(Y_i, g_i)$.

Suppose $F(\underline{x}) = F(\underline{z})$. Then $x_{n_i} = x_{n_i}$ for all $i \geq 1$. Given k a positive integer, there exists an i such that $n_i > k$. Hence $x_k = f_{k,n_i}(x_{n_i}) = f_{k,n_i}(z_{n_i}) = z_k$. Hence $\underline{x} = \underline{z}$ and F is *one-to-one*.

Suppose $\underline{y} = (y_1, y_2, \ldots) \in \varprojlim(Y_i, g_i)$. Let $\underline{x} = \big(f_{1,n_1}(y_1), f_{2,n_1}(y_1), \ldots, y_1, f_{n_1+1,n_2}(y_2), f_{n_1+2,n_2}(y_2), \ldots, y_2, \ldots\big)$. Clearly, $\underline{x} \in \varprojlim(X_i, f_i)$ and $F(\underline{x}) = \underline{y}$. Hence F is onto.

Suppose U is open in $\varprojlim(Y_i, g_i)$ and $F(\underline{x}) \in U$. By *Lemma* 2.3.1, there exists a positive integer $k > 1$ and an open set $V \subset Y_k$ containing $\pi_k\big(F(\underline{x})\big)$ such that a point $\underline{z}$ of $\varprojlim(Y_i, g_i) \in U$ if $\pi_k(\underline{z}) \in V$. Define $W' = X_1 \times X_2 \times \cdots \times X_{n_k-1} \times V \times X_{n_k+1} \times \cdots$ and let $W = W' \cap \varprojlim(X_i, f_i)$. Clearly, W is open in $\varprojlim(X_i, f_i)$ and $\underline{x} \in W$. Note that $\pi_k\big(F(W)\big) \subset V$, hence $F(W) \subset V$ and F is continuous.

Suppose U is open in $\varprojlim(X_i, f_i)$ and $\underline{x} \in U$. By *Lemma* 2.3.1 there is a positive integer p such that for any integer $n > p$ there is an open set $V \subset X_n$ containing $\pi_n(\underline{x})$ such that $\pi_n^{-1}(V) \subset U$ and $\underline{x} \in \pi_n^{-1}(V)$. Choose n to be any $n_k > p$ and V as above. Clearly $F\big(\pi_{n_k}^{-1}(V)\big)$ contains $F(\underline{x})$ and is open in $\varprojlim(Y_i, g_i)$ since $F\big(\pi_{n_k}^{-1}(V)\big) = \pi_k^{-1}(V)$. The map F is continuous, *one-to-one*, onto and open, hence it is a homeomorphism. ∎

The following three corollaries follow from the previous lemma.

2.3.7 *Corollary*. *Given* (X_i, f_i) *and an integer* $n \geq 1$. *Then* $\varprojlim(Y_i, g_i) \cong \varprojlim(X_i, f_i)$ *where for each* i, $Y_i = X_{(n-1)+i}$ *and* $g_i = f_{(n-1)+i}$.

2.3.8 *Corollary*. *Given* (X, f) *and an integer* $n \geq 1$. *Then* $\varprojlim(X, f^n) \cong \varprojlim(X, f)$ *for all* n.

2.3.9 *Corollary*. *Given* (X_i, f_i) *and* $n_1, n_2, \ldots$ *a sequence of positive integers. Then* $\varprojlim(X_i, f_i) \cong \varprojlim(X, f)$ *where for each* i, $X_i = X$ *and* $f_i = f^{n_i}$.

2.3.10 *Lemma*. *If* $F : X \to X$ *is a one-to-one map. Then* $\Lambda = \bigcap_{n \geq 0} F^n(X)$ *is homeomorphic to* $\varprojlim(X, F)$.

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc}
X & \xleftarrow{F} & X & \xleftarrow{F} & X & \xleftarrow{F} & \cdots & \varprojlim(X,F) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F^2} & & & \downarrow{\scriptstyle F_\infty} \\
X & \xleftarrow{i_0} & F(X) & \xleftarrow{i_1} & F^2(X) & \xleftarrow{i_2} & \cdots & \Lambda
\end{array}$$

By *Lemma* 2.3.4, $\Lambda \cong \bigcap_{n\geq 0} F^n(X)$. This diagram induces a homeomorphism $F_\infty : \varprojlim(X,F) \to \bigcap_{n\geq 0} F^n(X)$ defined by $F_\infty(x_0, x_1, x_2, \ldots) = (x_0, F(x_1), \ldots) = (x_0, x_0, \ldots)$. ∎

<u>2.3.11 *Lemma*</u>. *If A is a closed subset of $\varprojlim(X_i, f_i)$, and for each n, $\pi_n(A) = X_n$, then $A = \varprojlim(X_i, f_i)$.*

Proof. Suppose $\underline{x} \in \varprojlim(X_i, f_i)$ and U is an open set containing $\underline{x}$. By *Lemma* 2.3.1, there exist an integer n and an open set $U_n \subset X_n$ such that $\underline{x} \in \pi_n^{-1}(U_n) \subset U$. But $\pi_n^{-1}(U_n) \cap A \neq \phi$ since $\pi_n(A) = X_n$. Hence $\underline{x}$ is a limit point of A. But A is closed, hence $\underline{x} \in A$. ∎

<u>2.3.12 *Theorem*</u>. [En, Theorem 1.13.2] *For every compact metric space X such that dim$X \leq n$ there exists an inverse sequence(K_i, f_i) consisting of polyhedra of dim $\leq n$ whose limit is homeomorphic to X; moreover, one can assume that for $i = 1, 2, \ldots$, K_i is the underlying polyhedron of a nerve $\mathcal{K}_i$ of a finite open cover of the space X and that for each i, the bonding map f_i is linear on each simplex in $\mathcal{K}_{i+1}$.*

<u>2.4 *Chaos and Chaotic Maps*.</u>

We begin with some definitions. We also state some results which will be used later on. Some proofs are included for completeness.

If $f : X \to X$ is a map, a point $x \in X$ has *period* n, where n is a positive integer, if $f^n(x) = x$, and if for all integers $1 \leq k < n$, $f^k(x) \neq x$. The *orbit* of x, $\mathcal{O}_x = \{f^n(x) : n = 0, 1, 2, \ldots\}$.

Let $F : X \to X$ and Λ be a closed subset of X. Then Λ is an *attractor* for F if there exists an open neighborhood U of Λ such that $Cl(F(U)) \subset U$ and $\Lambda = \bigcap_{n \geq 0} F^n(U)$.

Let X be a compact metric space. Then a map $f : X \to X$ is said to be *chaotic* if it satisfies the following conditions:

(1) The map f has sensitive dependence on initial conditions (SIC). That is, there exists a $\delta > 0$ such that for each $x \in X$ and for each $\epsilon > 0$ there exists an $x' \in X$, such that $d(x, x') < \epsilon$ and a positive integer n such that $d(f^n(x), f^n(x')) \geq \delta$.

(2) The map f has a dense orbit. That is, there exists an $x \in X$ whose orbit $\mathcal{O}_x$ is dense in X.

(3) The periodic points of f are dense in X.

In [BM1], Barge and Martin define *topological stability* as follows: Let X be a metric space and $f : X \to X$ be a map. Let $x \in X$, then x is *topologically stable if and only if* for every $\delta > 0$, there is an $\epsilon > 0$ such that if $z \in X$ and $d(z, x) < \epsilon$ then for each positive integer n, $d(f^n(x), f^n(z)) < \delta$. If x is not topologically stable, then x is *topologically unstable.*

Examining the definitions above, we see that $f : X \to X$ is *SIC if and only if* every point $x \in X$ is topologically unstable.

A map $f : X \to X$ is *topologically transitive* if and only if for every pair of nonempty open sets U, V in X, there exists an $n \geq 0$ such that $f^n(U) \cap V \neq \phi$.

2.4.1 *Lemma*. [Si] *Let X be a metric space with no isolated points. If $f : X \to X$ has a dense orbit, then f is topologically transitive. The converse is true if X is a complete separable metric space.*

Proof. We first prove the following claim:

Claim: In a metric space with no isolated points, every nonempty open subset is infinite.

To prove the claim, let $V \subset X$ be a nonempty open subset. Let $x \in V$. Then there exist $x_n \in V$, $n = 1, 2, \ldots$, such that $x_n \neq x$ and $d(x, x_n) < \frac{1}{n}$. The set $\{x_1, x_2, \ldots\}$ cannot be finite because $d(x, x_n) \to 0$. This proves the claim.

Let U and V be nonempty open subsets of X. Let $\mathcal{O}_x = \{x_0, x_1, \ldots\}$ be a dense orbit. Then there exist integers k and m such that $x_k \in U$ and $x_m \in V - \{x_0, x_1, \ldots, x_k\}$ which is open and nonempty. Since $m > k$, then $f^{m-k}(U) \cap V = \phi$.

To prove the converse, suppose that f has no dense orbit and $\{B_n\}_{n=1}^{\infty}$ is a countable basis for X. For each $x \in X$ there exists an integer $n(x)$ such that $f^k(x) \notin B_{n(x)}$ for all $k \geq 0$.

The union $\bigcup\limits_{k=0}^{\infty} f^{-k}(B_{n(x)})$ is open and is dense in X since f is topologically transitive. Let $A_{n(x)} = X - \bigcup\limits_{k=0}^{\infty} f^{-k}(B_{n(x)})$, then $x \in A_{n(x)}$ and $A_{n(x)}$ is closed and nowhere dense. Hence $X = \bigcup\limits_{x \in X} A_{n(x)}$ is a countable union of closed nowhere dense subsets of X, contradicting the fact that X is of second category. The union $\bigcup\limits_{x \in X} A_{n(x)}$ is countable because for every $x \in X$, $A_{n(x)} = X - \bigcup\limits_{k=0}^{\infty} f^{-k}(B_m)$ for some $m = 1, 2, 3, \ldots$. ∎

Let X and Y be topological spaces and let $f : X \to X$ and $g : Y \to Y$ be maps. f and g are said to be *topologically conjugate* if and only if there exists a homeomorphism $h : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

2.4.2 Lemma. *Let X be a compact metric space. If $f : X \to X$ is chaotic and is topologically conjugate to $g : Y \to Y$, then g is chaotic.*

Proof. Let $y \in Y$ and $\epsilon_1 > 0$. Then there exist $x \in X$ and $\epsilon_2 > 0$ such that $h(x) = y$ and for all $z \in X$ if $d(x, z) < \epsilon_2$ then $d(y, h(z)) < \epsilon_1$. Choose $\delta_f > 0$ for $f : X \to X$ as in the definition of SIC. Then there exist an $x' \in X$ such that $d(x, x') < \epsilon_2$ and an integer $n > 0$ such that $d\big(f^n(x), f^n(x')\big) \geq \delta_f$.

Let $y' = h(x')$. Now we have $d(g^n(y), g^n(y')) = d(g^n h(x), g^n h(x')) = d(hf^n(x), hf^n(x'))$.

Claim: The map $h : X \to Y$ satisfies the condition : for each $\delta > 0$ there exists an $\epsilon > 0$ such that if $d(x, x') > \delta$ then $d(h(x), h(x')) > \epsilon$ for all $x, x' \in X$.

Note that if h satisfies the previous condition then taking $\delta = \delta_f$ we get $\delta_g = \epsilon > 0$, we see that g is SIC.

We prove the claim by contradiction. So assume that there exists a $\delta_0 > 0$ and points $\{x_n\}$, $\{x'_n\}$, $n = 1, 2, \ldots$ such that $d(x_n, x'_n) > \delta_0$ and $d(h(x_n), h(x'_n)) \to 0$ as $n \to \infty$. Since X is compact, $\{x_n\}$ and $\{x'_n\}$ have convergent subsequences $\{x_{n_i}\}$ and $\{x'_{n_i}\}$ respectively. Assume $x_{n_i} \to x$ and $x'_{n_i} \to x'$ as $i \to \infty$. By the triangular inequality, $d(x, x') > \delta_0$, hence $h(x) \neq h(x')$ since h is *one-to-one*. By continuity of h, $d(h(x_{n_i}), h(x'_{n_i}) \to d(h(x), h(x'))$ as $i \to \infty$. But $d(h(x_{n_i}), h(x'_{n_i}) \to 0$ as $i \to \infty$, hence $d(h(x), h(x')) = 0$. Hence $h(x) = h(x')$, a contradiction.

If $\mathcal{O}_x = \{x, f(x), f^2(x), \ldots\}$ is dense in X, then $\mathcal{O}_{h(x)} = \{h(x), g(h(x)), g^2(h(x)), g^3(h(x)) \ldots\}$ is dense in Y. To show this, let U be open in Y. The set $h^{-1}(U)$ is open in X, hence $f^n(x) \in h^{-1}(U)$ for some positive integer n. Since $h \circ f = g \circ h$, we have $hf^n(x) = g^n h(x) \in U$.

Assume that $Per(f) = \{x \in X : f^n(x) = x,\ \textit{for some integer}\ n \geq 0\}$ is dense in X. Let $U \subset Y$ be open, then $h^{-1}(U)$ is open in X. Hence there exist an $x \in X$ and an integer $n \geq 0$ such that $x = f^n(x) \in h^{-1}(U)$. Hence $h(x) = h(f^n(x)) \in U$. But since $h(f^n(x)) = g^n(h(x))$, we have $h(x) = g^n(h(x)) \in U$. Therefore $Per(g) = \{y \in Y : g^n(x) = x,\ \textit{for some integer}\ n \geq 0\}$ is dense in Y. ■

The following lemma is needed for the proof of *Theorem* 2.4.4.

<u>2.4.3 *Lemma*</u>. [Sc, Lemma 32] *Let X be a metric space. Then for each $\underline{x} \in \varprojlim(X, f)$ and $\epsilon > 0$, there exists a positive integer k and an $\alpha > 0$ such that if $\underline{z} \in \varprojlim(X, f)$, where $d'(x_k, z_k) < \alpha$, then $\varrho(\underline{x}, \underline{z}) < \epsilon$.*

Proof. Given $\underline{x} \in \varprojlim(X, f)$ and $\epsilon > 0$, let k be such that $2^{-k} < \epsilon/2$. Using the continuity of the bonding maps, for each $i = 1, 2, \ldots, k-1$, there exists $\alpha_i > 0$ such that if $z_k \in X_k$, where $d'(z_k, x_k) < \alpha_i$, then $d'(f^i(z_k), f^i(x_k)) < \epsilon/2$. Let $\alpha = min\{\epsilon/2, \alpha_1, \ldots, \alpha_{k-1}\}$. Now, if $\underline{z} \in \varprojlim(X, f)$, where $d'(x_k, z_k) < \alpha$, then for each $i = 1, \ldots, k$, $d'(x_i, z_i) < \epsilon/2$ and

$$\varrho(\underline{x}, \underline{z}) = \sum_{i=1}^{\infty} \frac{d'(x_i, z_i)}{2^i} \leq \frac{\epsilon}{2}\left(\sum_{i=1}^{k} \frac{1}{2^i}\right) + \sum_{i=k+1}^{\infty} \frac{1}{2^i} \leq \frac{\epsilon}{2} + 2^{-k} < \epsilon. \qquad ■$$

<u>2.4.4 *Theorem*</u>. *Suppose that X is a metric space and $f : X \to X$ is onto. If f is chaotic, then $\hat{f} : \varprojlim(X, f) \to \varprojlim(X, f)$ is chaotic.*

Proof. Suppose that f is SIC. Let δ be given from the assumption that f is SIC. Let $\underline{x} \in \varprojlim(X, f)$ and $\epsilon > 0$. Assume that $\epsilon < \delta$. Apply *Lemma* 2.4.3 to obtain k and α such that if $\underline{z} \in \varprojlim(X, f)$, where $d'(x_k, z_k) < \alpha$, then $\varrho(\underline{x}, \underline{z}) < \epsilon$. Since f is SIC, there exist $w_k \in X_k$ such that $d'(w_k, x_k) < \alpha$ and a positive integer

m such that $d'(f^m(x_k), f^m(w_k)) \geq \delta$. Since f is onto, the set $\{\underline{z} \in \varprojlim(X,f) : \pi_k(\underline{z}) = w_k\}$ is nonempty; choose such a $\underline{z}$. Recall that $d'(f^m(x_k), f^m(z_k)) = d'(x_{k-m}, z_{k-m}) \geq \delta$. Since $d'(x_i, z_i) < \epsilon/2 < \delta$, for $i = 1, 2, \ldots, k$, we have $m > k$. Now, $\hat{f}^{m-k+1}(\underline{x}) = (f^{m-k+1}(x_1), f^{m-k}(x_1), \ldots, f(x_1), x_1, x_2, \ldots) = (x_{k-m}, x_{k-m+1}, \ldots, x_0, x_1, \ldots)$, and hence, if $n = m - k + 1$, then $\varrho(\hat{f}^n(\underline{x}), \hat{f}^n(\underline{z}) \geq \delta/2$. Therefore, $\hat{f}$ is SIC.

Assume that $\mathcal{O}_x = \{x, f(x), f^2(x), \ldots\}$ is dense in X. Choose $\underline{x} \in \varprojlim(X,f)$ such that $x_1 = x$ and consider $\mathcal{O}_{\underline{x}} = \{\underline{x}, \hat{f}(\underline{x}), \hat{f}^2(\underline{x}), \ldots\}$. Let U be open in $\varprojlim(X,f)$, then there exists an integer α and an open subset $U_\alpha \subset X_\alpha$ such that $\pi_\alpha^{-1}(U_\alpha) \subset U$. Since $\mathcal{O}_x = \{x, f(x), f^2(x), \ldots\}$ is dense in X, there exists an integer m such that $f^m(x) \in U_\alpha$. Hence $(f^{m+\alpha}(x), f^{m+\alpha-1}(x), \ldots, f^m(x), f^{m-1}(x), \ldots, f(x), x, \ldots) \in U$ and $\mathcal{O}_{\underline{x}}$ is dense in $\varprojlim(X,f)$.

Assume that $Per(f)$ is dense in X and let U be open in $\varprojlim(X,f)$, then there exists an integer α and an open subset $U_\alpha \subset X_\alpha$ such that $\pi_\alpha^{-1}(U_\alpha) \subset U$. Since $Per(f)$ is dense in X, there exists a periodic point x of period m in U_α. Let $\beta \equiv \alpha \pmod{m}$. Consider $\underline{x} = (f^\beta(x), f^{\beta-1}(x), \ldots, f(x), x, f^{m-1}(x), f^{m-2}(x), \ldots, f(x), x, f^{m-1}(x), \ldots)$. Clearly $\underline{x} \in \pi_\alpha^{-1}(U_\alpha)$ and $\hat{f}^m(\underline{x}) = \underline{x}$. Hence $Per(\hat{f})$ is dense in $\varprojlim(X,f)$. ∎

2.5 *Maps of the Compact Interval* $f : I \to I$.

In this section we include a few results on maps of the compact interval. These give alternative characterizations of chaos for maps $f : I \to I$.

2.5.1 Lemma. [MB1, Lemma 2] *Suppose f has a dense orbit $\mathcal{O}_x$. For $x \in I$ and s and k integers, $s \geq 1$, $k \geq 0$, let $A_{s,k}(x) = A_{s,k} = \{f^{sn+k}(x) : n \geq 0\}$. Then* *(i) If $A_{2,0}$ is dense in I, then $A_{s,k}$ is dense in I for all $s \geq 1$, $k \geq 0$.*

(ii) If $A_{2,0}$ is not dense in I, then $I = \overline{A}_{2,0} \cup \overline{A}_{2,1}$, $\overline{A}_{2,0}$, $\overline{A}_{2,1}$ are closed intervals which intersect in a point, and $f(\overline{A}_{2,0}) = \overline{A}_{2,1}$, $f(\overline{A}_{2,1}) = \overline{A}_{2,0}$. Moreover, for each $k \geq 1$, $A_{2k,0}$ is dense in $\overline{A}_{2,0}$ and $A_{2k,1}$ is dense in $\overline{A}_{2,1}$.

2.5.2 *Corollary*. [MB1, Cor., p.359] *Suppose f has a dense orbit $\mathcal{O}_x$. Then the set of periodic points of f is dense in I.*

Proof. Let $V \subset I$ be an open subinterval. Choose $x \in V$ such that $\mathcal{O}_x$ is dense in I. If $\{f^{2n}(x) : n \geq 0\}$ is not dense in I, we may assume, by *Lemma* 2.5.1, that $V \subset Cl\{f^{2n}(x) : n \geq 0\}$. Let j be an integer such that $f^j(x) \in V$. We may assume that $x < f^j(x)$. From *Lemma* 2.5.1, it follows that $\{g^k(x) : k \geq 0\}$ is dense in V. Now let l be the smallest positive integer such that $g^l(g(x)) < g(x)$. Then $g^l(x) = g^{l-1}(g(x)) \geq g(x) > x$ and $g^l(g(x)) < g(x)$. So $g^l(x) > x$ and $g^l(g(x)) < g(x)$. Hence g^l has a fixed point y, $x < y < g(x)$. Since $g^l(y) = y$, $f^{kl}(y) = y$ and since $y \in V$, V contains a periodic point of f. ∎

2.5.3 *Corollary*. [MB1, Cor., p.359] *Suppose f has a dense orbit $\mathcal{O}_x$. Then every point of I is topologically unstable.*

Proof. Suppose $y \in I$ and y is topologically stable. Let $x \in I$ have a dense orbit $\mathcal{O}_x$. We first show that $\mathcal{O}_y$ is dense.

Suppose $U \subset I$ is an open subinterval and for all $n \geq 0$, $f^n(x) \notin U$. Let V be an open interval which is the open middle third of U. Let $\epsilon = \frac{1}{3}diam(U)$. Then since y is topologically stable, there is a $\delta > 0$ such that if $|z - y| < \delta$ then for all n, $|f^n(y) - f^n(z)| < \epsilon$. In particular, is $|z - y| < \delta$ then, for each n, $f^n(z) \notin V$. Now since x has a dense orbit, there is an integer j such that $|f^j(x) - y| < \delta$. Then there exists an integer $k > j$ such that $f^k(x) \in V$. But then $f^{k-j}(f^j(x)) \in V$ and this is a contradiction. Hence $\mathcal{O}_y$ is dense.

By *Lemma* 2.5.1, there exists a positive number ϵ and a subinterval C of I such that $diam(C) > 3\epsilon$, and for each $n \geq 0$ $\{f^{kn}(y) : k \geq 0\}$ is dense in C. Now choose $\delta > 0$ such that if $|z - y| < \delta$ then for each j, $|f^j(z) - f^j(y)| < \epsilon$. Now by *Corollary* 2.5.2, let t be a periodic point such that $|t - y| < \delta$. Let n be the period of t. Then for each k, $|f^{kn}(t) - f^{kn}(y)| < \epsilon$ so $|t - f^{kn}(y)| < \epsilon$. But then $\{f^{nk}(y) : k \geq 0\}$ is dense in C. ∎

Hence for maps f of the interval $I \cong [0,1]$, f having a dense orbit is equivalent to f being chaotic.

In [BM1] and [BM2], Barge and Martin prove results, which yield the equivalence of (1)-(4) in the following theorem. In [CM], Coven and Mulvey prove that (5) is equivalent to the rest if f is piecewise monotone. They do so by proving that if $f : I \to I$ is piecewise monotone and if f^n is transitive for every $n > 0$, then for every subinterval $J \subset I$ there exists an n such that $f^n(J) = I$ [CM, Lemma 4.1].

2.5.4 Theorem. *Let $f : I \to I$ be continuous. Then the following statements are equivalent:*

(1) f is transitive and has a point of odd period greater than one.

(2) f^2 is transitive.

(3) f^n is transitive for every $n > 0$.

(4) For every pair U, V of nonempty open sets, there exists an N, such that $f^{-n}(U) \cap V \neq \phi$ for all $n \geq N$.

Furthermore, if f is piecewise monotone, then the following statement is equivalent to the rest:

(5) For every interval $J \subseteq I$, there exists an n such that $f^n(J) = I$.

2.6 *Maps of the Circle $f: S^1 \to S^1$.*

In this section we include a few results on maps of the circle. These give alternative characterizations of chaos for maps $f: S^1 \to S^1$.

2.6.1 *Theorem.* [CM, Theorem C] *Let $f: S^1 \to S^1$ be a continuous map of the circle to itself. Then the following statements are equivalent:*

(1) There is an m such that f^m is transitive and has a fixed point and a point of odd period greater than one.

(2) There is an m such that f^{2m} is transitive and f^m has a fixed point.

(3) f^n is transitive for every $n > 0$ and f has periodic points.

(4) For every pair U, V of nonempty open sets, there exists an N, such that $f^{-n}(U) \cap V \neq \phi$ for all $n \geq N$.

Furthermore, if f is piecewise monotone, then the following statement is equivalent to the rest:

(5) For every interval $J \subseteq S^1$, there exists an n such that $f^n(S^1) = S^1$.

2.6.2 *Theorem.* [Si, Theorem 7.1] *If $f: S^1 \to S^1$ has a dense orbit then any of the following are equivalent to f being chaotic:*

(1) f has a periodic point.

(2) f is not one-to-one.

(3) f has sensitive dependence on initial conditions.

(4) f has a non-dense orbit.

(5) f is not conjugate to an irrational rotation.

2.6.3 *Corollary*. [CM, Cor. 3.4] ***For transitive maps of the circle with periodic points, the periodic points are dense.***

A map $f : X \to X$ is called topologically transitive if any of the following equivalent conditions hold [CM]:

(1) For every pair U, V of nonempty open sets, there exists an n, such that $f^{-n}(U) \cap V \neq \phi$.

(2) The only closed invariant set K with $Int(K) \neq \phi$ is $K = X$

(3) If $Int(K) \neq \phi$, then $\overline{\cup_{n \geq 0} f^n(K)} = X$.

(4) f is onto and has a dense orbit.

3. The Whitehead Continuum

In this section we construct two spaces homeomorphic to the Whitehead continuum. One in R^3 which we refer to as the Whitehead continuum and one in R^2 which we refer to as the Knaster continuum. We chain the Knaster continuum in a specific way and then analogously we chain the Whitehead continuum. We use these chainings to prove that the Whitehead and the Knaster continua are homeomorphic. This result is stated without proof in [A].

Let $C = I$ or $C = S^1$. Let $f : B^2 \times C \to R^3$ be an embedding. Let $N(f(\{0\} \times C), r) = \{x \in R^3 : d(x, f(\{0\} \times C)) \leq r\}$. We say that $f(B^2 \times C)$ has "*cross sectional diameter* $\leq r$" if it is a subset of $N(f(\{0\} \times C), r)$ and if $diam(f(B^2 \times c)) < r$ for all $c \in C$.

Let $f : I \times I \to R^3$ be an embedding. Let $N(f(I \times \{\frac{1}{2}\}), r) = \{x \in R^3 : d(x, f(I \times \{\frac{1}{2}\})) \leq r\}$. We say that $f(I \times I)$ has "*width* $\leq r$" if it is a subset of $N(f(I \times \{\frac{1}{2}\}), r)$ and if $diam(f(t \times I)) < r$ for all $t \in I$.

3.1 *Construction of the Whitehead Continuum.*

Let T_0 be a solid torus in R^3. Let T_1 be a solid torus in $Int(T_0)$ as in *Figure* 3.1. Let T_2 be a solid torus embedded in $Int(T_1)$ as T_1 is embedded in T_0. Continue this construction. This results in a sequence $T_0, T_1, T_2, \ldots$ of solid tori in R^3 such that for each $n \in Z^+ \cup \{0\}$, $T_{n+1} \subset Int(T_n)$. Assume that the *cross sectional diameter* of $T_n \leq (\frac{1}{10})^n$ for all n. The Whitehead continuum W is defined by $W = \bigcap_{i=0}^{\infty} T_i$. Note that the conditions on the *cross sectional diameters* force W to *one-dimensional* and that W is homemorphic to the Whitehead continuum defined earlier in *Section* 1.4.

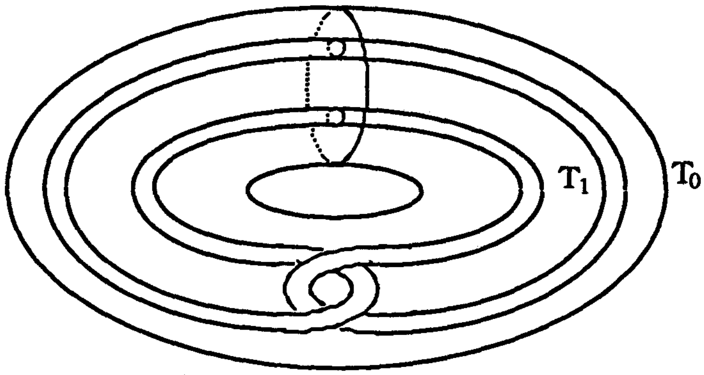

Figure 3.1

3.2 *Construction of the Knaster Continuum.*

let D_0 be a 2-*dimensional* disk in R^2 of *width* 1. Let D_1 be a 2-*dimensional* disk in $Int(D_0)$ as in *Figure* 3.2. Let D_2 be a 2-*dimensional* disk in $Int(D_1)$ as D_1 is embedded in D_0. Continue this construction. This results in a sequence $D_0, D_1, D_2, \ldots$ of 2-*dimensional* disks in R^2 such that for each $n \in Z^+ \cup \{0\}$, $D_{n+1} \subset Int(D_n)$. Assume that the *width* of $D_n \leq (\frac{1}{10})^n$ for all n. Then the Knaster continuum K is defined by $K = \bigcap_{i=0}^{\infty} D_i$. Note that the conditions on the *width* of D_n force K to be *one-dimensional*.

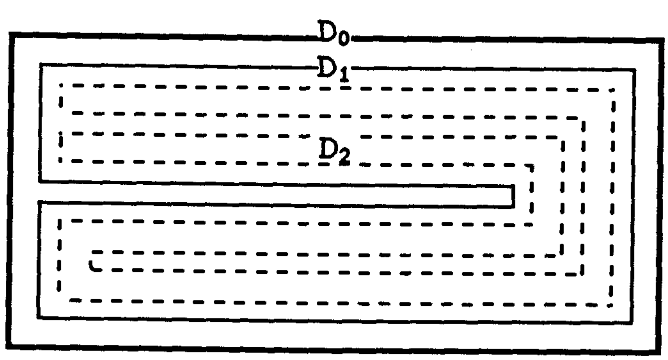

Figure 3.2

Next, will chain the Knaster continuum K in a specific way and then analogously chain the Whitehead continuum W. These chainings will be used to prove that W is homeomorphic to K.

3.3 Chaining the Knaster Continuum.

We will inductively define chains $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \ldots$ where $\mathcal{E}_i$ covers D_i.

Defining the Chain $\mathcal{E}_0$.

Consider the 2-*cell* D_0 shown in *Figure* 3.3.

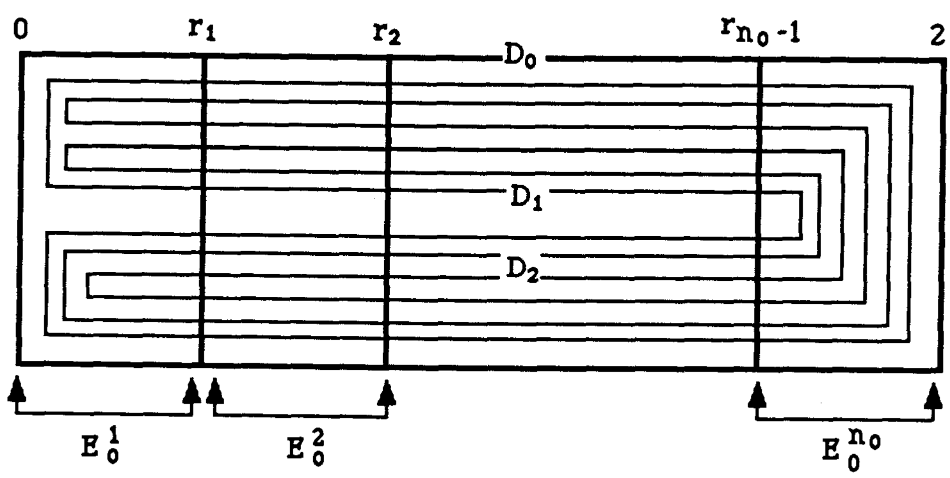

Figure 3.3

Let $D_0 = I_0 \times J_0$, where $I_0 = [0,2]$ and $J_0 = [0,1]$. Partition the interval I_0 into n_0 subintervals $[r_0, r_1], [r_1, r_2], \ldots, [r_{n_0-1}, r_{n_0}]$, where $r_0 = 0$ and $r_{n_0} = 2$. Require that $diam([r_i, r_{i+1}]) < \frac{2}{10}$ for all $0 \leq i \leq n_0 - 1$.

Let $E_0^i = [r_{i-1}, r_i] \times J_0$ for all $1 \leq i \leq n_0$. Choose the links E_0^i such that $E_0^i \cap D_1$ has exactly two components for $1 \leq i < n_0$ and $E_0^{n_0} \cap D_1$ has exactly one component. Now, slightly expanding each link E_0^i, as shown in *Figure* 3.4, we produce the open links (still denoted by $E_0^1, E_0^2, \ldots, E_0^{n_0}$) making up the chain

$\mathcal{E}_0$. Hence $\mathcal{E}_0 = \{E_0^1, E_0^2, \ldots, E_0^{n_0}\}$ is a chain made up of n_0 open links such that $E_0^i \cap E_0^j \neq \phi$ *if and only if* $|i-j| \leq 1$.

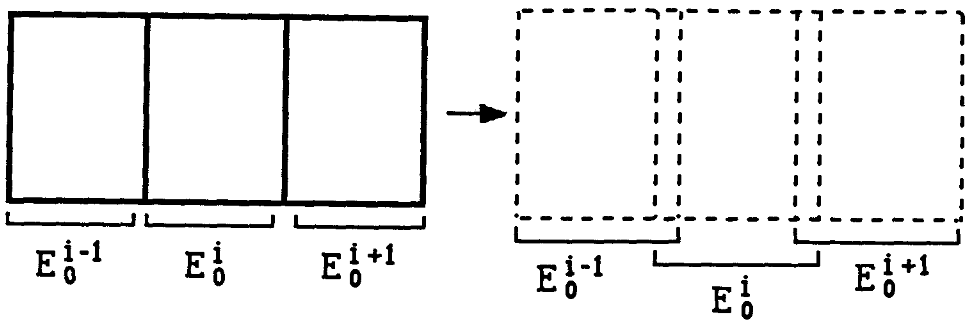

Figure 3.4

Defining the Chain $\mathcal{E}_1$.

Let $E_0^i \cap D_1 = {}^uS_0^i \cup {}^lS_0^i$ for all $1 \leq i < n_0$, where ${}^uS_0^i = \left([r_{i-1}, r_i] \times [\frac{1}{2}, 1]\right) \cap D_1$ and ${}^lS_0^i = \left([r_{i-1}, r_i] \times [0, \frac{1}{2}]\right) \cap D_1$.

Let ${}^lS_0^1 = I_1 \times J_1$. Partition the interval I_1 into m subinterval $[r_0, r_1]$, $[r_1, r_2], \ldots, [r_{m-1}, r_m]$. Require that :

(1) $Diam\left([r_i, r_{i+1}]\right) < \frac{2}{100}$.

(2) The intersection $D_2 \cap \left([t_0, t_1] \times J_1\right)$ has one component.

(3) The intersection $D_2 \cap \left([t_i, t_{i+1}] \times J_1\right)$ has two components for all $1 < i \leq m-1$.

Let $E_1^{n_1} = [t_0, t_1] \times J_1$, $E_1^{n_1-1} = [t_1, t_2] \times J_1$, $E_1^{n_1-2} = [t_2, t_3] \times J_1, \ldots$, $E_1^{n_1-m} = [t_{m-1}, t_m] \times J_1$. Expand these links slightly to produce open links $E_1^{n_1}, E_1^{n_1-1}, E_1^{n_1-2}, \ldots, E_1^{n_1-m}$. Hence we have defined the last m links of the chain $\mathcal{E}_1$. See *Figure* 3.5.

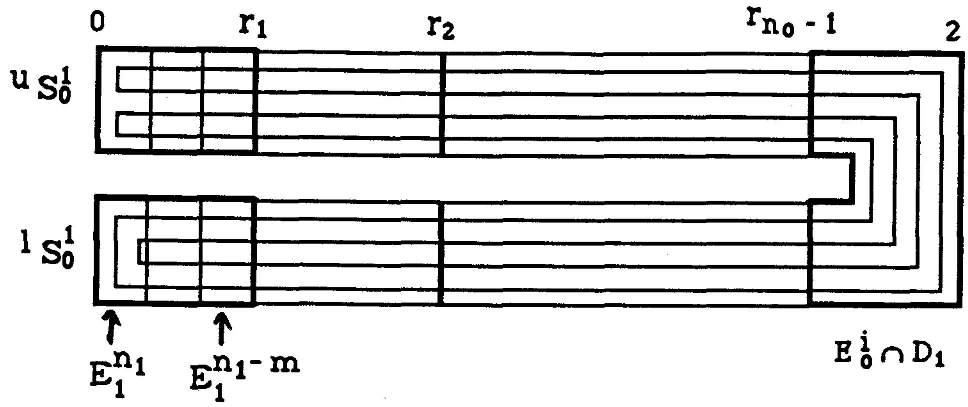

Figure 3.5

Similarly, partition ${}^{l}S_0^i$ into m sublinks $E_1^{i_1}$, $E_1^{i_2}, \ldots,$ $E_1^{i_m}$ such that ${}^{l}S_0^i =$ $E_1^{i_1} \cup E_1^{i_2} \cup \cdots \cup E_1^{i_m}$ for all $1 \leq i < m$. Also partition ${}^{u}S_0^i$ into m sublinks $E_1^{i_1}$, $E_1^{i_2}, \ldots,$ $E_1^{i_m}$ such that ${}^{l}S_0^i = E_1^{i_1} \cup E_1^{i_2} \cup \cdots \cup E_1^{i_m}$ for all $1 \leq i < m$. Here each sublink intersects D_2 in two components.

Now consider $E_0^{n_0} \cap D_1$. Let $h : R^3 \to R^3$ be a homeomorphism taking $E_0^{n_0} \cap D_1$ onto the *2-cell* $I_2 \times J_2$ such that $diam\big(h^{-1}(r \times J_2)\big) < \frac{1}{10}$. See *Figure* 3.6.

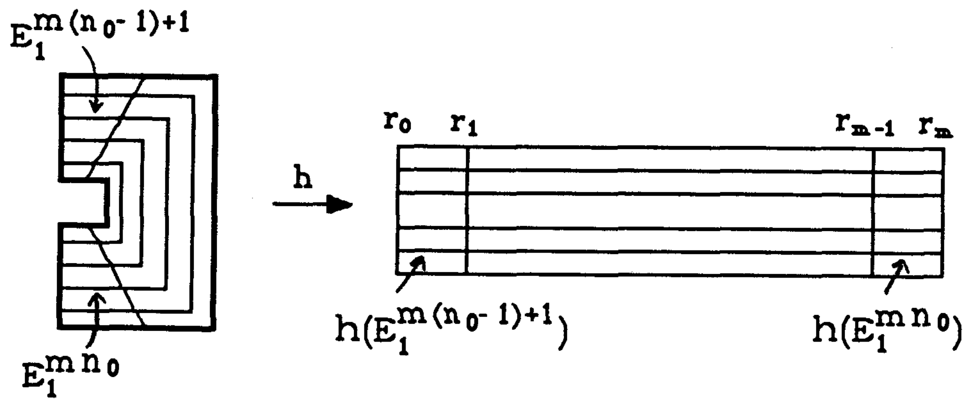

Figure 3.6

Partition I_2 into m subintervals $[r_0, r_1]$, $[r_1, r_2], \ldots,$ $[r_{m-1}, r_m]$ of equal diameters. Let $E_1^{m(n_0-1)+1} = h^{-1}\big([r_0, r_1] \times J_2\big)$, $E_1^{m(n_0-1)+2} = h^{-1}\big([r_1, r_2] \times J_2\big), \ldots,$ $E_1^{mn_0} = h^{-1}\big([r_{m-1}, r_m] \times J_2\big)$. This defines the middle m links of $\mathcal{E}_1$.

Hence the definition of the chain $\mathcal{E}_1$ is complete. Note that $\mathcal{E}_1$ has $n_1 = 2m(n_0 - 1) + m$ open links such that $E_1^i \cap E_1^j \neq \phi$ *if and only if* $|i - j| \leq 1$. Note also that each link of the chain $\mathcal{E}_0$ is the union of exactly m links of the chain $\mathcal{E}_1$.

Defining the Chain $\mathcal{E}_k$.

In general, having defined the chain $\mathcal{E}_{k-1}$, define the chain $\mathcal{E}_k$ to be a chain with $n_k = 2m(n_{k-1} - 1) + m$ open links such that $E_k^i \cap E_k^j \neq \phi$ *if and only if* $|i - j| \leq 1$ and each link of the chain $\mathcal{E}_k$ is the union of exactly m links of the chain $\mathcal{E}_{k-1}$. We call $\mathcal{E}_k$ a *U-chain* in $\mathcal{E}_{k-1}$ since the first link of $\mathcal{E}_k$ is a subset of the first link of $\mathcal{E}_{k-1}$ and then $\mathcal{E}_k$ goes straight through $\mathcal{E}_{k-1}$, turns around and comes back through $\mathcal{E}_{k-1}$. Note that $diam(E_i^j) \to 0$ as $i \to \infty$.

3.4 *Chaining the Whitehead Continuum*.

Having chained the Knaster continuum, we analogously chain the Whitehead continuum. We inductively define chains $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ where $\mathcal{C}_i$ covers T_i.

Consider the cylinder $B_0^2 \times I_0$, where $I_0 = [0, 2]$ shown in *Figure* 3.7.

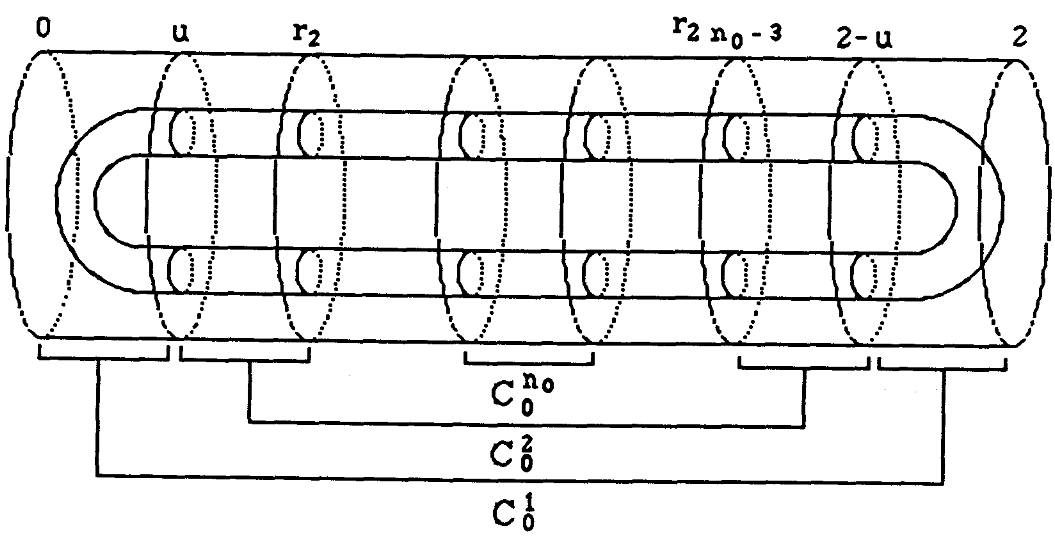

Figure 3.7

Let $\frac{2}{10} < u < \frac{1}{2}$, $u = \frac{3}{10}$ is a good choice. Consider the following identification: Given a disk $B_0^2 \times \{r\}$ where $2-u \leq r \leq 2$. Rotate $B_0^2 \times \{r\}$ about the axis $\{0\} \times I_0$ through an angle $\theta = \frac{\pi}{2}$ is the counter-clockwise direction and identify it with the disk $B_0^2 \times \{r + u - 2\}$. The quotient space of this identification yields the first two stages T_0 and T_1 in the construction of the Whitehead continuum shown in *Figure* 3.1.

Defining the Chain C_0.

Consider the cylinder $B_0^2 \times [u, 2-u]$ shown above. Partition the interval $[u, 2-u]$ into $2n_0-3$ subintervals $[r_1, r_2], [r_2, r_3], \ldots, [r_{n_0-1}, r_{n_0}], \ldots, [r_{2n_0-3}, r_{2n_0-2}]$ where $r_1 = u$ and $r_{2n_0-2} = 2-u$. Require that $diam([r_i, r_{i+1}]) < \frac{2}{10}$ for all $1 < i \leq 2n_0 - 3$.

Let $C_0^1 = B_0^2 \times [0, r_1] \cup B_0^2 \times [r_{2n_0-2}, 2]$, $C_0^2 = B_0^2 \times [r_1, r_2] \cup B_0^2 \times [r_{2n_0-3}, r_{2n_0-2}]$, $\ldots$, $C_0^{n_0-1} = B_0^2 \times [r_{n_0-2}, r_{n_0-1}] \cup B_0^2 \times [r_{n_0+1}, r_{n_0+2}]$, and $C_0^{n_0} = B_0^2 \times [r_{n_0-1}, r_{n_0}]$.

Note that the link C_0^i has exactly two components for all $1 < i < n_0$ and C_0^1 and $C_0^{n_0}$ have one component each; recall that $B_0^2 \times [0, r_1]$ is identified with $B_0^2 \times [r_{2n_0-2}, 2]$. Note also that $C_0^i \cap T_1$ has exactly two components for $1 \leq i \leq n_0$. Expand the links defined above slightly to produce the open links (still denoted by $C_0^1, C_0^2, \ldots, C_0^{n_0}$) making up the chain $\mathcal{C}_0$. Hence $\mathcal{C}_0 = \{C_0^1, C_0^2, \ldots, C_0^{n_0}\}$ is a chain made up of n_0 open links such that $C_0^i \cap C_0^j \neq \phi$ *if and only if* $|i-j| \leq 1$. See *Figure* 3.7.

Defining the Chain C_1.

Let $C_0^1 \cap T_1 = {}^lS_0^1 \cup {}^rS_0^1$ and consider ${}^lS_0^1$. Let $h: R^3 \to R^3$ be a homeomorphism taking ${}^lS_0^1$ onto the cylinder $B_1 \times I_1$ as shown in *Figure* 3.8.

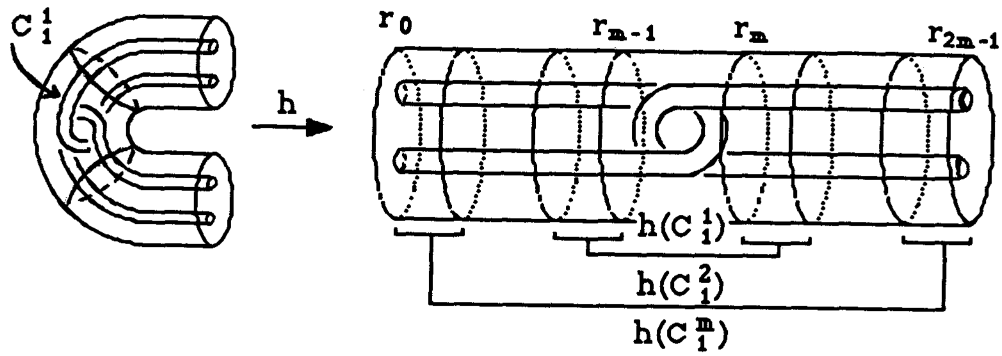

Figure 3.8

Partition the interval I_1 into $2m-1$ subintervals $[r_0, r_1]$, $[r_1, r_2]$, $\ldots$, $[r_{m-1}, r_m]$, $\ldots$, $[r_{2m-2}, r_{2m-1}]$. Choose the subintervals such that $\frac{2}{100} < diam([r_{m-1}, r_m]) < \frac{3}{100}$ and $diam([r_i, r_{i+1}]) < \frac{2}{100}$ for $i \in \{0, 1, \ldots, m-2, m, \ldots, 2m-2\}$.

Let $C_1^1 = h^{-1}(B_1^2 \times [r_{m-1}, r_m])$, $C_1^2 = h^{-1}(B_1^2 \times [r_{m-2}, r_{m-1}]) \cup h^{-1}(B_1^2 \times [r_m, r_{m+1}]), \ldots, C_1^{m-1} = h^{-1}(B_1^2 \times [r_1, r_2]) \cup h^{-1}(B_1^2 \times [r_{2m-3}, r_{2m-2}])$, and $C_1^m = h^{-1}(B_1^2 \times [r_0, r_1]) \cup h^{-1}(B_1^2 \times [r_{2m-2}, r_{2m-1}])$.

Require that $C_1^i \cap T_2$ has four components for $1 < i \leq m$ and C_1^1 has two components.

Expand the links $C_1^1, C_1^2, \ldots, C_1^m$ slightly to produce open links; use $C_1^1, C_1^2, \ldots, C_1^m$ to denote these open links also. This defines the first m links in the chain $\mathcal{C}_1$. Similarly partition ${}^rS_0^1$ into m sublinks $C_1^{i_1}$, $C_1^{i_2}, \ldots, C_1^{i_m}$ such that ${}^rS_0^1 = C_1^{i_1} \cup C_1^{i_2} \cup \cdots \cup C_1^{i_m}$. This defines the last m links in the chain $\mathcal{C}_1$.

Recall that, $C_0^2 = B_0^2 \times [r_1, r_2] \cup B_0^2 \times [r_{2n_0-3}, r_{2n_0-2}]$. Now consider ${}^lC_0^1 = B_0^1 \times [r_1, r_2]$. Partition the interval $[r_1, r_2]$ into m subintervals $[t_0, t_1]$, $[t_1, t_2], \ldots$, $[t_{m-1}, t_m]$ of equal diameters where $t_0 = r_1$ and $t_m = r_2$.

Let $C_1^{m+1} = T_2 \cap (B_0^2 \times [t_0, t_1])$, $C_1^{m+2} = T_2 \cap (B_0^2 \times [t_1, t_2]), \ldots, C_1^{2m} = T_2 \cap (B_0^2 \times [t_{m-1}, t_m])$, This defines the second m links of $\mathcal{C}_1$. Note that C_1^i has two components for all $m+1 \leq i \leq 2m$.

Similarly, define the rest of the links of $\mathcal{C}_1$. Hence the definition of the chain $\mathcal{C}_1$ is complete. Note that $\mathcal{C}_1$ has $n_1 = 2m(n_0 - 1) + m$ open links such that $C_1^i \cap C_1^j \neq \phi$ *if and only if* $|i - j| \leq 1$. Note also that each link of the chain $\mathcal{C}_0$ is the union of exactly m links of the chain $\mathcal{C}_1$.

Defining the Chain $\mathcal{C}_k$.

In general, having defined the chain $\mathcal{C}_{k-1}$, define the chain $\mathcal{C}_k$ to be a chain with $n_k = 2m(n_{k-1} - 1) + m$ open links such that $C_k^i \cap C_k^j \neq \phi$ *if and only if* $|i - j| \leq 1$ and each link of the chain $\mathcal{C}_k$ is the union of exactly m links of the chain $\mathcal{C}_{k-1}$. Note that $\mathcal{C}_k$ is a *U-chain* in $\mathcal{C}_{k-1}$. Note that $diam(C_i^j) \to 0$ as $i \to \infty$.

The proof of the following theorem parallels that of [Bi2,Theorem 11].

<u>3.5 Theorem</u>. *The Whitehead continuum W is homeomorphic to the Knaster continuum K.*

Proof. Given $x \in W$, let $C_i^*(x) = \bigcup_{x \in C_i^j} C_i^j$. Note that $C_i^*(x) \subset C_{i+1}^*(x)$ for all i, hence $x = \bigcap_{i=1}^{\infty} C_i^*(x)$. let $E_i^*(x) = \bigcup_{x \in C_i^j} E_i^j$. Define $h : W \to K$ by $h(x) = \bigcap_{i=0}^{\infty} E_i^*(x)$.

Clearly, h is *well-defined* and onto. To show that h is continuous, we show that if $x \in W$ and U is an open subset of K containing $h(x)$, then there exists an open set $V \subset W$ containing x such that $h(V) \subseteq U$. So let U be an open set in K such that $h(x) \in U$. There exists j such that any link of $\mathcal{E}_j$ containing $h(x)$ is a subset of U. Now if $x \in C_j^r$, then $h(C_j^r \cap W) \subseteq E_j^r$. But $h(x) \in E_j^r \subseteq U$, hence h is continuous.

We will argue by contradiction to show that h is *one-to-one*. So let $x_1, x_2 \in W$ such that $h(x_1) = h(x_2)$. Then there exists k such that no element of $\mathcal{C}_k$ contains both x_1 and x_2. Let E_k^r be such that $h(x_1) = h(x_2) \in E_k^r$. Then there exists a $j > k$ such that every element of $\mathcal{E}_j$ containing $h(x_1) = h(x_2)$ is a subset of E_k^r. Let C_j^μ , C_j^ν contain x_1, x_2 respectively. Since E_j^μ, E_j^ν contain $h(x_1) = h(x_2)$ then E_j^μ, $E_j^\nu \subset E_k^r$. Then C_j^μ and C_j^ν are subsets of C_k^r. But no elements of $\mathcal{C}_k$ contains both x_1 and x_2. A contradiction, hence h is *one-to-one*.

The map $h : W \to K$ is *one-to-one*, onto, and continuous; but W is compact and K is Hausdorff, so h is a homeomorphism. ∎

4. The Whitehead Continuum Viewed as a Nontransitive Attractor

In this section we view a specific Whitehead continuum as an attractor of a map $F : T \to T$. A projection map $P : T \to S^1$ is defined. The maps F and P induce a map $f : S^1 \to S^1$. We prove that the attractor of the map F, that is, $W = \bigcap_{k=0}^{\infty} F^k(T)$ is homeomorphic to $\varprojlim(S^1, f)$. Hence $F_{|W}$ is topologically conjugate to $\hat{f} : \varprojlim(S^1, f) \to \varprojlim(S^1, f)$ which is not topologically transitive. Finally, we discuss the dynamics of F.

Let $C = I$ or $C = S^1$. Let $f : B^2 \times C \to R^3$ be an embedding. Let $N\big(f(\{0\} \times C), r\big) = \big\{x \in R^3 : d\big(x, f(\{0\} \times C)\big) \leq r\big\}$. We say that $f(B^2 \times C)$ has "*cross sectional diameter* $\leq r$" if it is a subset of $N\big(f(\{0\} \times C), r\big)$ and if $diam\big(f(B^2 \times c)\big) < r$ for all $c \in C$.

By a *Whitehead map* f we mean an embedding $f : T \to T$ such that the attractor of f is homeomorphic to the Whitehead continuum constructed in *Section* 3.1 and is embedded in R^3 just as the Whitehead continuum is.

By a *Whitehead continuum* we mean an attractor of a Whitehead map.

The results in this chapter parallel those in [Ba2]. In [Ba2], Barge considers horseshoe maps and realizes their attracting sets as inverse limits of maps of the interval. Here we consider Whitehead maps and realize their attracting sets as inverse limits of maps of the circle.

4.1 *Construction.*

Suppose that T is a solid torus, $T = S^1 \times B^2$. Given an angle $0 < \theta_0 < \frac{\pi}{2}$, let $\theta_1 = \frac{3\pi}{2} - \theta_0$, $\theta_2 = \frac{3\pi}{2} + \theta_0$, $\theta_3 = \frac{\pi}{2} - \theta_0$, and $\theta_4 = \frac{\pi}{2} + \theta_0$. Let $C_1 = \{\theta : \theta_1 \leq \theta \leq \theta_2\} \times B^2$, $C_2 = \{\theta : \theta_2 \leq \theta \leq \theta_3\} \times B^2$, $C_3 = \{\theta : \theta_3 \leq \theta \leq \theta_4\} \times B^2$, and $C_4 = \{\theta : \theta_4 \leq \theta \leq \theta_1\} \times B^2$.

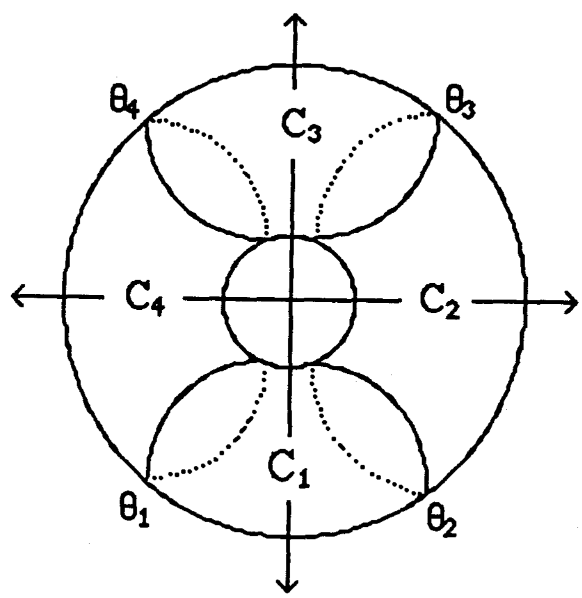

Figure 4.1

Consider $B^2 \times I$ shown in *Figure* 4.2. Identify each disk $B^2 \times \theta$ (located to the right of C_3) with the disk $B^2 \times \theta$ (located to the left of C_3) for $\theta_1 \leq \theta \leq \theta_2$. That is identify the cylindrical sections (labeled C_1) at the ends of $B^2 \times I$. The quotient space corresponding to this identification is a solid torus T. Hence T can be thought of as depicted *Figure* 4.2.

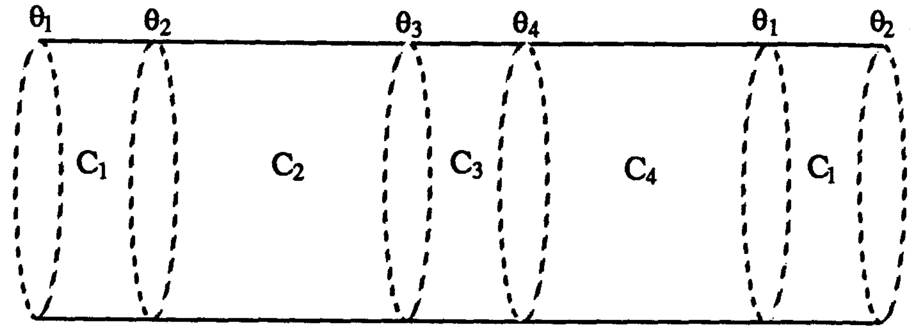

Figure 4.2

Let S^1 be the quotient space of the interval $[0,1]$ resulting from identifying the end points $\{0\}$ and $\{1\}$.

Define $P : T \to S^1$ as follows:

$P(C_1) = \{0\}$, $P(C_3) = \{1\}$, and

$$P(\theta, B^2) = \begin{cases} \frac{\theta-\theta_2}{\theta_3-\theta_2}, & \text{if } (\theta, B^2) \subseteq C_2; \\ \frac{\theta_1-\theta}{\theta_1-\theta_4}, & \text{if } (\theta, B^2) \subseteq C_4. \end{cases}$$

Define $F : T \to T$ as follows:

(i) Radially contract C_2 and C_4 by a factor of $\delta = \frac{1}{10}$ and linearly stretch them by a factor of μ to get two cylinders $F(C_2)$ and $F(C_1)$ of cross-sectional diameter 2δ and of length $\mu(\theta_3 - \theta_2)$.

(ii) Radially contract C_1 and C_3 by a factor of $\delta = \frac{1}{10}$ and horizontally shrink them by a factor of $\lambda < 1$ to get two cylinders $F(C_1)$ and $F(C_3)$ of cross-sectional diameter δ and of length $\lambda(\theta_2 - \theta_1)$.

(iii) Embed $F(C_1)$, $F(C_2)$,$F(C_3)$, and $F(C_4)$ in T as shown in *Figure* 4.3.

This defines an embedding $F : T \to T$. We want $F(T)$ to be embedded in T just as T_1 is embedded in T_0 in *Figure* 1.2. That is, we want $F(T)$ to self-link in T. One way to realize this self-linking, is to obtain T as a quotient space of the cylinder shown in *Figure* 4.3 with the following identification: Given a disk B^2, $\theta_1 \le \theta \le \theta_2$, located to the right of $B^2 \times \theta_4$ in the figure below. Rotate $B^2 \times \theta$ about the axis $\{0\} \times I$ through an angle $\theta = \frac{\pi}{2}$ in the counter-clockwise direction and identify it with $B^2 \times \theta$ located to the left of $B^2 \times \theta_4$.

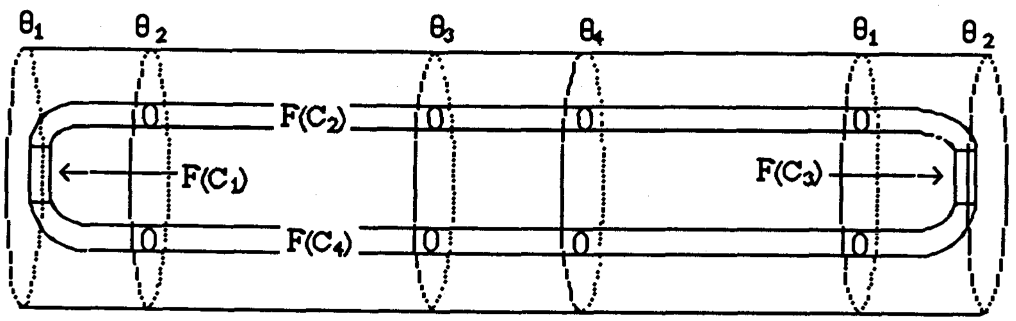

Figure 4.3

By choosing λ and μ appropriately, we can require that the embedding F have the following properties:

(i) $F\big(P^{-1}(P(z))\big) \subseteq P^{-1}\big(P(F(z))\big)$ for all $z \in T$.

(ii) $F(C_1) \subseteq IntC_1$, and $F(C_3) \subseteq IntC_1$.

(iii) For all $x \in S^1$, $P^{-1}(x) \cap F(T)$ has exactly four components; and

(iv) $Diam\big(F^k(P^{-1}(P(z))\big) \to 0$ uniformly in $z \in T$ as $k \to \infty$.

The attracting set for F is $W = \bigcap_{k=0}^{\infty} F^k(T)$. This means that for $z \in T$, $d(F^k(z), W) \to 0$ as $k \to \infty$. The set W is a Whitehead continuum.

So we have the following diagram:

$$\begin{array}{ccc} T & \xrightarrow{F} & T \\ \downarrow P & & \downarrow P \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

F induces a continuous map $f : S^1 \to S^1$ defined by $f(x) = P\Big(F\big(P^{-1}(x)\big)\Big)$.

The embedding F is required to satisfy property (i) above to insure that the induced map f is *well-defined* and the commutativity of the diagram above.

To show that $f : S^1 \to S^1$ is *well-defined*, let $x = P(z_1) = P(z_2)$ be in S^1 such that $z_1 \neq z_2 \in T$. Then $f(x) = PFP^{-1}(x) = PFP^{-1}(P(z)) \subseteq P(P^{-1}PF(z_1))$; the inclusion follows from property (i). Then $f(x) = PF(z_1)$. Similarly, $f(x) = PF(z_2)$. But since $P(z_1) = P(z_2)$, then $z_1 \in P^{-1}P(z_2)$. Hence $PF(z_1) \in PF(P^{-1}P(z_2)) = P(FP^{-1}P(z_2)) \subseteq P(P^{-1}PF(z_2)) = PF(z_2)$. Hence $PF(z_1) = PF(z_2)$ and f is *well-defined.*

To show the commutativity of the diagram above, assume F satisfies property (i). That is, assume $FP^{-1}P(z) \subseteq P^{-1}PF(z)$ for all $z \in T$. Then $P(FP^{-1}P(z)) \subseteq P(P^{-1}PF(z)) = PF(z)$. Hence $fP(z) \subseteq PF(z)$ for all $z \in T$. To show that $PF(z) \in fP(z)$, let $z \in T$, then $PF(z) \subseteq PF(P^{-1}P(z)) = fp(z)$. Hence $f \circ P = P \circ F$.

The map f has the following properties:

(i) $f(0) = 0$, $f(1) = 0$, and

(ii) For $i = 1, \ldots, 4$, there exists $a_i \in S^1$, $0 = a_0 < a_1 < a_2 < a_3 < a_4 < 1$ such that f is strictly monotone on $[a_{2i-1}, a_{2i}]$ for $i = 1, 2$ and for $i = 0, 1, 2$

$$f([a_{2i}, a_{2i+1}]) = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even.} \end{cases}$$

See *Figure* 4.4.

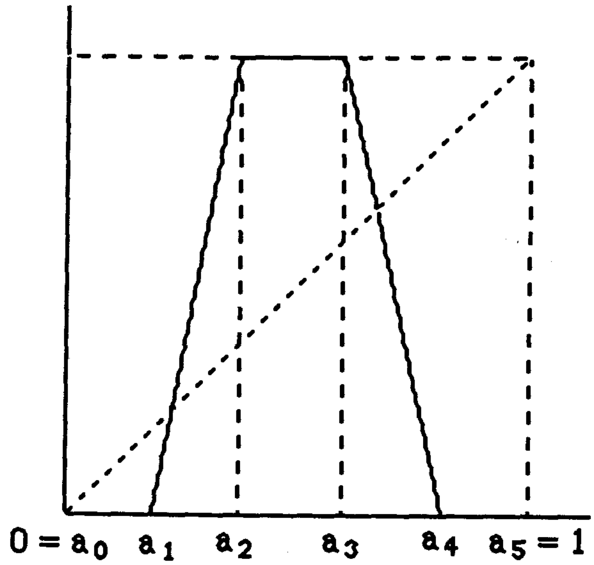

Figure 4.4

The proof of the following theorem is a modification of the proof of Theorem 1 of [Ba2], tailored to our needs.

<u>4.2 *Theorem*</u>. *Consider the following diagram:*

$$\begin{array}{ccccccccc}
T & \overset{i}{\longleftarrow} & F(T) & \overset{i}{\longleftarrow} & F^2(T) & \overset{i}{\longleftarrow} & \cdots & \bigcap_{n=0}^{\infty} F^n(T) = W \\
\uparrow{\scriptstyle id} & & \uparrow{\scriptstyle F} & & \uparrow{\scriptstyle F^2} & & & \\
T & \overset{F}{\longleftarrow} & T & \overset{F}{\longleftarrow} & T & \overset{F}{\longleftarrow} & \cdots & \varprojlim(T, F) \\
\downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} & & & \\
S^1 & \overset{f}{\longleftarrow} & S^1 & \overset{f}{\longleftarrow} & S^1 & \overset{f}{\longleftarrow} & \cdots & \varprojlim(S^1, f)
\end{array}$$

The map $\hat{P} : W \to \varprojlim(S^1, f)$ *given by*

$$\hat{P}(z) = \Big(P(z), P(F^{-1}(z)), P(F^{-2}(z)), \ldots\Big)$$

is a homeomorphism and the following diagram of homeomorphisms commutes:

$$\begin{array}{ccc} W & \xrightarrow{F} & W \\ \downarrow \hat{P} & & \downarrow \hat{P} \\ \varprojlim(S^1, f) & \xrightarrow{\hat{f}} & \varprojlim(S^1, f) \end{array}$$

That is , $F_{|W}$ is topologically conjugate to $\hat{f}$.

Proof. Since $P \circ F = f \circ P$, then $f\Big(P(F^{-(i+1)}(z))\Big) = P\Big(F(F^{-(i+1)}(z))\Big) = P(F^{-i}(z))$ and $\hat{P}(z) \in \varprojlim(S^1, f)$.

The map $\hat{P}$ is clearly continuous. To see that $\hat{P}$ is *one-to-one* and onto, let $\underline{x} = (x_1, x_2, \ldots) \in \varprojlim(S^1, f)$ and let $C_k = F^k(P^{-1}(x_k))$ for $k = 1, 2, \ldots$. Then C_k is a closed, nonempty subset of T for each $k \geq 1$, and since $F(P^{-1}(x_{k+1})) \subseteq P^{-1}(f(x_{k+1})) = P^{-1}(x_k)$, we have $C_{k+1} \subseteq C_k$ for $k = 1, 2, \ldots$. Thus $\bigcap_{k=1}^{\infty} C_k$ is a nonempty set and if $z \in \bigcap_{k=1}^{\infty} C_k$, then $P(z) = x_1$, $P(F^{-1}(z)) = x_2, \ldots$. That is, $\hat{P}(z) = \underline{x}$. Moreover, if $\hat{P}(z) = \underline{x}$ then z must be in $\bigcap_{k=1}^{\infty} C_k$. But since $diam\Big(F^k(P^{-1}(x_k))\Big) \to 0$ as $k \to \infty$, we have $\bigcap_{k=1}^{\infty} C_k = \{z\}$ and $\hat{P}$ is *one-to-one* and onto. ■

<u>4.3 lemma</u>. *The map $\hat{f} : \varprojlim(S^1, f) \to \varprojlim(S^1, f)$ is not topologically transitive.*

Proof. let $U = \pi_1^{-1}(a_0, a_1)$ and $V = \pi_1^{-1}(a_2, a_3)$.

claim: $\hat{f}^n(U) \cap V = \phi$ for all $n \geq 0$.

$\hat{f}^n(U) = \hat{f}^n(\pi_1^{-1}(a_0, a_1)) = \pi_1^{-1}(f^n(a_0, a_1)) = \pi_1^{-1}\{a_0\}$.

Clearly, $\pi_1^{-1}\{a_0\} \cap \pi_1^{-1}(a_2, a_3) = \phi$. Hence the claim is proved and $\hat{f}$ is not transitive. ■

Consider the following inverse sequence:

$$T \xleftarrow{i} F(T) \xleftarrow{i} F^2(T) \xleftarrow{i} \cdots \varprojlim(F^j(T), i)$$

From *Lemma* 2.3.4, it follows that $\varprojlim(F^j(T), i)$ is homeomorphic to $\bigcap_{n=0}^{\infty} F^n(T) = W$.

Consider the following diagram:

$$\begin{array}{ccccccccc} T & \xleftarrow{F} & T & \xleftarrow{F} & T & \xleftarrow{F} & \cdots & \varprojlim(T, F) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F^2} & & & \\ T & \xleftarrow{i} & F(T) & \xleftarrow{i} & F^2(T) & \xleftarrow{i} & \cdots & W \end{array}$$

Since $F^j : T \to F^j(T)$ is a homeomorphism for all j we have $\varprojlim(T, F) \cong W$. Hence by *Theorem* 4.2, $\varprojlim(T, F)$ is homeomorphic to $\varprojlim(S^1, f)$.

From *Theorem* 1 of [Ba2] we conclude that the Knaster continuum is homeomorphic to $\varprojlim(I, h)$ where h has the following properties:

(i) $h(0) = 0$, $h(1) = 0$, and

(2) For $i = 1, \ldots, 4$, there exists $a_i \in S^1$, $0 = a_0 < a_1 < a_2 < a_3 < a_4 < 1$ such that f is strictly monotone on $[a_{2i-1}, a_{2i}]$ for $i = 1, 2$ and for $i = 0, 1, 2$,

$$h([a_{2i}, a_{2i+1}]) = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even.} \end{cases}$$

From *Theorem* 5 of [Ba2], we conclude that $\varprojlim(I, h)$ is homeomorphic to $\varprojlim(I, g)$ where $g : I \to I$ is defined by

$$g(\frac{i}{2}) = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even.} \end{cases}$$

for $i = 0, 1, 2$ and g is linear on $[\frac{i}{2}, \frac{i+1}{2}]$ for $i = 0, 1$.

Consider the following diagram

$$\begin{array}{ccccccc} I & \overset{h}{\longleftarrow} & I & \overset{h}{\longleftarrow} & I & \overset{h}{\longleftarrow} \cdots & \underleftarrow{\lim}(I,h) \\ \downarrow G & & \downarrow G & & \downarrow G & & \downarrow \hat{G} \\ S^1 & \overset{f}{\longleftarrow} & S^1 & \overset{f}{\longleftarrow} & S^1 & \overset{f}{\longleftarrow} \cdots & \underleftarrow{\lim}(S^1,f) \end{array}$$

where $G : I \to S^1$ is defined by $G(0) = G(1) = 0$ and $G(x) = x$ for $x \notin \{0,1\}$.

<u>4.4 Lemma</u>. *The map* $\hat{G} : \underleftarrow{\lim}(I,h) \to \underleftarrow{\lim}(S^1,f)$ *is a homeomorphism.*

Proof. We only need to show that $\hat{G}$ is *one-to-one*, the rest is obvious. So assume $\underline{x} = (x_1, x_2, \ldots)$ and $\underline{y} = (y_1, y_2, \ldots)$ are in $\underleftarrow{\lim}(I,f)$ such that $\hat{G}(\underline{x}) = \hat{G}(\underline{y})$. Then $(G(x_1), G(x_2), \ldots) = (G(y_1), G(y_2), \ldots)$. Assume without loss of generality that $x_i \neq y_i$, $x_i = 0$ and $y_i = 1$. Then $x_{i+1} \in [a_0, a_1] \cup [a_4, a_5]$ and $y_{i+1} \in [a_2, a_3]$. This contradicts the fact that $G(x_{i+1}) = G(y_{i+1})$. Hence $\underline{x} = \underline{y}$. ■

<u>4.5 *The Dynamics Of The Whitehead Map*</u>.

The discussion given here parallels that given in [De, Sec. 2.3] for the horseshoe map.

The Whitehead map F embeds T into itself as described earlier. Note that $F(T) \subset T$ and F is *one-to-one*. Now we study the dynamics of F in T.

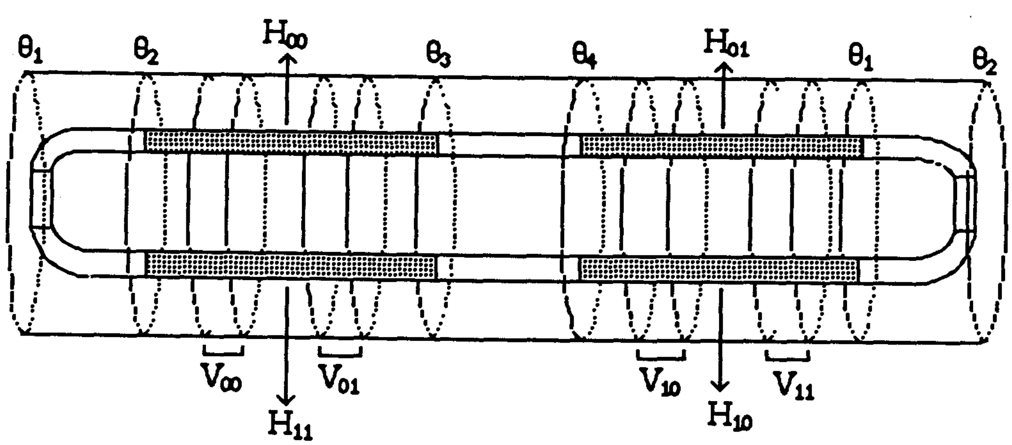

Figure 4.5

The set $F^{-1}(C_2 \cup C_4) = V_{00} \cup V_{01} \cup V_{10} \cup V_{11}$, is a union of four vertical cylinders which are mapped linearly onto the four horizontal components H_{00}, H_{01}, H_{10} and H_{11} of $F(C_2 \cup C_4) \cap C_2 \cup C_4$. The height of V_{ij} is $\frac{1}{\mu}$. The cross sectional diameter of H_{ij} is δ. By linearity of F on C_2 and C_4, F preserves cross sections in C_2 and C_4. If C is a cylinder in $C_2 \cup C_4$ whose image lies in $C_2 \cup C_4$ then the length of $F(C)$ is expanded by a factor of μ times the length of C and the cross sectional diameter of C is shrunk by a factor of δ.

The map F is a contraction on C_1. By the Contraction Mapping Theorem, F has a unique fixed point $p \in C_1$ and $\lim_{n \to \infty} F^n(q) = p$ for all $q \in C_1$. Since $F(C_3) \subset C_1$, all forward orbits in C_3 behave likewise. Similarly, if $q \in C_2 \cup C_4$ but $F^k(q) \in C_1 \cup C_3$ for some $k > 0$, then we have $F^n(q) \in C_1 \cup C_3$ for $n \geq 2$, so $F^n(q) \to p$ as $n \to \infty$. Hence, to understand the forward orbits of F, it suffices to consider the set of points whose forward orbits lie entirely in $C_2 \cup C_4$.

Now, if the forward orbit of q lies in $C_2 \cup C_4$ then $q \in V_{00} \cup V_{01} \cup V_{10} \cup V_{11}$, for all other points in $C_2 \cup C_4$ are mapped into $C_1 \cup C_3$. Also $F(q) \in C_2 \cup C_4$, then $F(q) \in V_{00} \cup V_{01} \cup V_{10} \cup V_{11}$, that is, $q \in F^{-1}(V_{00}) \cup F^{-1}(V_{01}) \cup F^{-1}(V_{10}) \cup F^{-1}(V_{11})$.

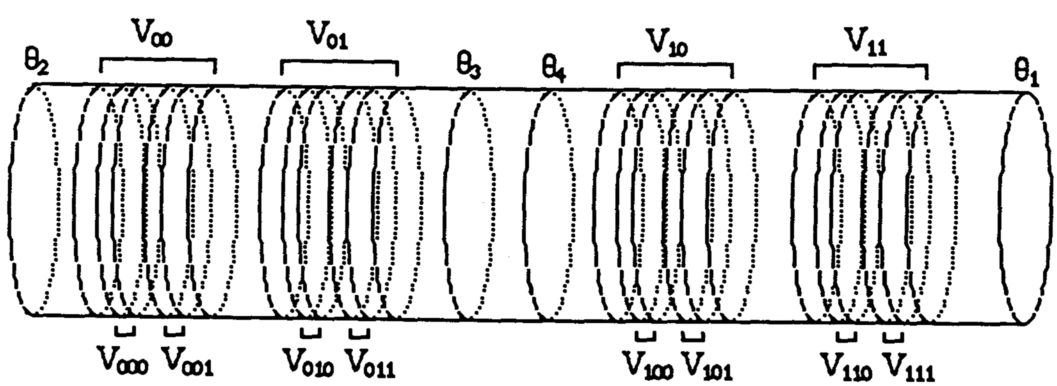

Figure 4.6

Inductively, if V is any vertical cylinder in $C_2 \cup C_4$ of cross sectional diameter c, then $F^{-1}(V)$ is a pair of cylinders of cross sectional diameter δc, one in each V_i. Hence $F^{-1}(F^{-1}(V_i)) = F^{-2}(V_i)$ consists of four cylinders each of cross sectional diameter δ^2c, $F^{-3}(V_i)$ consists of eight cylinders of cross sectional diameter δ^3c.

Hence $\Lambda_+ = \{q : F^k(q) \in C_2 \cup C_4 \; for \; k = 0, 1, 2, \ldots\}$ is the product of a Cantor set with a vertical disk. Analogously, $\Lambda_- = \{q : F^{-k}(q) \in C_2 \cup C_4 \; for \; k = 0, 1, 2, \ldots\}$ is the product of a Cantor set with a horizontal disk. Let $\Lambda = \{q \in C_2 \cup C_4 : F^k(q) \in C_2 \cup C_4 \; for \; all \; k \in Z\}$. Note that $\Lambda = \Lambda_+ \cap \Lambda_-$.

Now we introduce symbolic dynamics into the system. Given a cylinder $C \subset \Lambda_+$, $F^k(C)$ is a cylinder of length μ^k in either V_0 or V_1. Attach an infinite sequence $s_0 s_1 s_2 \ldots$ of 0's and 1's to any point in C according to the rule $s_j = \alpha$ if and only if $F^j(C) \subset V_\alpha$. s_0 tells us which cylinder C lies in, s_1 tells where its image is located, etc. Similarly, attach a sequence of 0's and 1's to any horizontal cylinder H. Write this sequence $\ldots s_{-3} s_{-2} s_{-1}$, where $s_{-j} = \alpha$ if and only if $F^{-j}(H) \subset V_\alpha$ for $j = 1, 2, 3, \ldots$. Note that $F^{-1}(H), F^{-2}(H), \ldots$ are horizontal line segments of decreasing lengths.

Hence, if p is a point in Λ, we may associate a pair of sequences of 0's and 1's to p. One sequence gives the itinerary of the forward orbit of p; the other describes the backward orbit. Let us amalgamate both of these sequences into one, doubly infinite sequence of 0's and 1's. That is, we define the itinerary $S(p)$ by the rule

$$S(p) = (\ldots s_{-2} s_{-1} . s_0 s_1 s_2 \ldots)$$

where $s_j = k$ if and only if $F^j(p) \in V_k$. This then gives the symbolic dynamics on Λ. Let $\sum_2$ denote the set of all doubly infinite sequences of 0's and 1's:

$$\sum\nolimits_2 = \{(s) = (\ldots s_{-2} s_{-1} . s_0 s_1 s_2 \ldots) : s_j = 0 \; or \; 1\}$$

Define a metric on $\sum_2$ by

$$d((s),(t)) = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}$$

Define the shift map σ by

$$\sigma(\ldots s_{-2} s_{-1} . s_0 s_1 s_2 \ldots) = (\ldots s_{-2} s_{-1} s_0 . s_1 s_2 \ldots)$$

σ is a homeomorphism on $\sum_2$.

Hence we have the following commutative diagram:

$$\begin{array}{ccc} \Lambda & \xrightarrow{F} & \Lambda \\ \downarrow S & & \downarrow S \\ \sum_2 & \xrightarrow{\sigma} & \sum_2 \end{array}$$

$S : \Lambda \to \sum_2$ is a homeomorphism. Hence $F_{|\Lambda}$ is topologically conjugate to σ. But σ has a dense orbit, hence F is chaotic on Λ.

In summary, we have shown that $F : W \to W$ is topologically conjugate to $\hat{f} : \varprojlim(S^1, f) \to \varprojlim(S^1, f)$. Also $\hat{f}$ is not chaotic, by *Lemma* 3.3. Hence $F : W \to W$ is not chaotic, whereas $F : \Lambda \to \Lambda$ is chaotic.

Our goal now is to define an embedding $F' : T \to T$ such that the attractor W' for F' is a Whitehead continuum and $F' : W' \to W'$ is chaotic. This is done in the next two chapters.

5. Partial Results

In this chapter we focus our attention on functions $f : I \to I$ which are continuous, onto and satisfy Conditions (1) or (2) stated below. We show that for a function f satisfying Condition (1), if H is a proper nondegenerate subcontinuum of $\varprojlim(I, f)$ then H is homeomorphic to I; if in addition f satisfies Condition (2) then $\varprojlim(I, f) - H$ is dense in $\varprojlim(I, f)$ which implies that $\varprojlim(I, f)$ is an indecomposable continuum.

Condition 1: If $(J_n), n = 1, 2, 3, \ldots$ is a sequence of nondegenerate proper closed subintervals of I such that $f(J_{i+1}) = J_i$ for $i \geq 1$, then there exists an integer $N \geq 1$ such that $f : J_{i+1} \to J_i$ is a homeomorphism for $i \geq N$.

Condition 2: If J is a proper nondegenerate closed subinterval of I then there exists an integer $M \geq 1$ and a collection of pairwise disjoint proper closed subintervals $J_1, J_2, \ldots, J_n$ where $n \geq 2$ such that $f^{-M}(J) = J_1 \cup J_2 \cup \cdots \cup J_n$.

5.1 *Examples*. It can be easily verified that the following functions satisfy Conditions 1 and 2 above. We will prove that g_2 satisfies Conditions 1 and 2 in *Lemma* 4.5.

Let $g_n : I \to I$, $n \geq 2$ be defined by

$$g_n(\frac{i}{n}) = \begin{cases} 0, & \text{if } i \text{ is even;} \\ 1, & \text{if } i \text{ is odd.} \end{cases}$$

$i = 0, 1, \ldots, n$ and g_n is linear on $[\frac{i}{n}, \frac{i+1}{n}]$, $i = 0, 1, \ldots, n-1$.

5.2 *Theorem*. *Suppose that $f : I \to I$ satisfies Condition 1 above. If H is a proper nondegenerate subcontinuum of $\varprojlim(I, f)$ then H is homeomorphic to I.*

Proof. Let H be a proper nondegenerate subcontinuum of $\varprojlim(I, f)$. Let H_i denote $\pi_i(H)$ for all i. From *Lemma* 2.3.11 it follows that there exists a $j \geq 1$ such that $H_j \neq I$, and hence $H_i \neq I$ for all $i > j$. Let $\alpha = \min\{i : H_i \neq I\}$. Since H_i is connected it follows that H_i is a proper closed subinterval of I for all $i \geq \alpha$.

The set $\{H_i : i \geq \alpha\}$ is a collection of proper closed subintervals of I and $f(H_{i+1}) = H_i$ for all $i \geq \alpha$. Since f satisfies Condition 1, there exists an $N \geq 1$ such that $f : H_{i+1} \to H_i$ is a homeomorphism for all $i \geq \alpha + N$. Let $\beta = \alpha + N$.

Now, $H_i \cong I$ for all $i \geq \beta$ and $f : H_{i+1} \to H_i$ is a homeomorphism for all $i \geq \beta$ imply that the inverse limit X of the inverse sequence

$$H_\beta \xleftarrow{f} H_{\beta+1} \xleftarrow{f} H_{\beta+2} \xleftarrow{f} \cdots$$

is homeomorphic to the closed interval I. It follows from *Theorem* 2.3.6 that $X \cong H$. Hence $H \cong I$. ■

5.3 *Theorem*. *Suppose that $f : I \to I$ satisfies Conditions 1 and 2 above. If H is a proper nondegenerate subcontinuum of $\varprojlim(I, f)$ then $\varprojlim(I, f) - H$ is dense in $\varprojlim(I, f)$.*

Proof. Let H be a proper nondegenerate subcontinuum of $\varprojlim(I, f)$. We need to show that if $U \subseteq \varprojlim(I, f)$ is open then $U \cap \left(\varprojlim(I, f) - H\right) \neq \phi$.

Assume that $U \cap \left(\varprojlim(I, f) - H\right) = \phi$ for some open subset U of $\varprojlim(I, f)$. It follows that $U \subseteq H$. The map f satisfies Condition 1, hence there exists an $N \geq 1$ such that $f : H_{i+1} \to H_i$ is a homeomorphism for all $i \geq N$.

The set $U_N = \pi_N(U)$ is open in I; choose a proper closed subinterval $J \subset U_N$. Since f satisfies Condition 2, there exists an $M \geq 1$ and a collection of pairwise disjoint proper closed subintervals $J_1, J_2, \ldots, J_n$, $n \geq 2$, such that $f^{-M}(J) = J_1 \cup J_2 \cup \cdots \cup J_n$. Since $f^M : H_{N+M} \to H_N$ is a homeomorphism, at most one of the intervals $J_1, J_2, \ldots, J_n$, without loss of generality, J_1 is a subset of H_{N+M}. Hence if $\underline{x} \in \varprojlim(I, f)$ such that $\pi_N(\underline{x}) \in U_N$ and $\pi_{N+M}(\underline{x}) \notin J_1$, then $\underline{x} \in U$ and $\underline{x} \notin H$. A contradiction to the assumption that $U \subset H$. Hence $\varprojlim(I, f) - H$ is dense in $\varprojlim(I, f)$. ∎

5.4 *Theorem*. *Suppose $f : I \to I$ satisfies Conditions 1 and 2 above. Then $\varprojlim(I, f)$ is an indecomposable continuum.*

Proof. By the previous theorem, if H is a proper nondegenerate subcontinuum of $\varprojlim(I, f)$, then $\varprojlim(I, f) - H$ is dense in $\varprojlim(I, f)$. Hence by *Theorem* 2 of [JK], $\varprojlim(I, f)$ is an indecomposable continuum . ∎

5.5 *Lemma*. *Let $f : I \to I$ be defined by*

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ 2 - 2x, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then f satisfies Conditions 1 and 2.

Proof. Let $J_1, J_2, J_3, \ldots$ be a sequence of proper nondegenerate closed subintervals of I such that $f(J_{i+1}) = J_i$ for all $i \geq 1$. We need to consider the following three cases :

(1) $J_i = [0, \epsilon]$, $0 < \epsilon < 1$.

(2) $J_i = [\epsilon_1, \epsilon_2]$, $\epsilon_1 < \epsilon_2$, $0 < \epsilon_1, \epsilon_2 < 1$.

(3) $J_i = [1 - \epsilon, 1]$, $0 < \epsilon < 1$.

Case 1: If $J_i = [0, \epsilon]$, then $J_{i+1} = [0, \frac{\epsilon}{2}]$ or $J_{i+1} = [1 - \frac{\epsilon}{2}, 1]$. It suffices to consider $J_{i+1} = [1-\frac{\epsilon}{2}, 1]$. It follows that $J_{i+2} = [\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}+\frac{\epsilon}{4}]$ and $J_{i+3} = [\frac{1}{4}-\frac{\epsilon}{8}, \frac{1}{4}+\frac{\epsilon}{8}]$ or $J_{i+3} = [\frac{3}{4} - \frac{\epsilon}{8}, \frac{3}{4} + \frac{\epsilon}{8}]$. The maps $f_{|J_{i+3}} : [\frac{1}{4} - \frac{\epsilon}{8}, \frac{1}{4} + \frac{\epsilon}{8}] \to [\frac{1}{2} - \frac{\epsilon}{4}, \frac{1}{2} + \frac{\epsilon}{4}]$ and $f_{|J_{i+3}} : [\frac{3}{4} - \frac{\epsilon}{8}, \frac{3}{4} + \frac{\epsilon}{8}] \to [\frac{1}{2} - \frac{\epsilon}{4}, \frac{1}{2} + \frac{\epsilon}{4}]$ are homeomorphisms. Hence if $N = i + 2$ then $f : J_{k+1} \to J_k$ is a homeomorphism for all $k \geq N$.

Case 2: if $J_i = [\epsilon_1, \epsilon_2]$ then $J_{i+1} = [\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}]$ or $J_{i+1} = [1-\frac{\epsilon_2}{2}, 1-\frac{\epsilon_1}{2}]$. The maps $f_{|J_{i+1}} : [\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}] \to [\epsilon_1, \epsilon_2]$ and $f_{|J_{i+1}} : [1 - \frac{\epsilon_2}{2}, \frac{1-\epsilon_1}{2}] \to [\epsilon_1, \epsilon_2]$ are homeomorphisms. Hence if $N = i$ then $f : J_{k+1} \to J_k$ is a homeomorphism for all $k \geq N$.

Case 3: If $J_i = [1 - \epsilon, 1]$ then $J_{i+1} = [\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}]$ and $J_{i+2} = [\frac{1}{4} - \frac{\epsilon}{4}, \frac{1}{4} + \frac{\epsilon}{4}]$ or $J_{i+2} = [\frac{3}{4} - \frac{\epsilon}{4}, \frac{3}{4} + \frac{\epsilon}{4}]$. The maps $f_{|J_{i+2}} : [\frac{1}{4} - \frac{\epsilon}{4}, \frac{1}{4} + \frac{\epsilon}{4}] \to [\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}]$ and $f_{|J_{i+2}} : [\frac{3}{4} - \frac{\epsilon}{4}, \frac{3}{4} + \frac{\epsilon}{4}] \to [\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}]$ are homeomorphisms. Hence if $N = i + 1$ then $f : J_{k+1} \to J_k$ is a homeomorphism for all $k \geq N$.

Hence f satisfies Condition 1. Moreover, It follows from the proof that f satisfies Condition 2. ■

Define $\tau : I \to I$ by

$$\tau(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ 2 - 2x, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

5.6 *Corollary*. *If H is a proper nondegenerate subcontinuum of $\varprojlim(I, \tau)$ then H is homeomorphic to I and $\varprojlim(I, \tau) - H$ is dense in $\varprojlim(I, \tau)$.*

Proof. This corollary follows from *Theorem* 5.2, *Lemma* 5.5 and *Theorem* 5.3. ■

It can be shown that $\varprojlim(I, \tau)$ is homeomorphic to the Knaster continuum defined in *Section* 3.2.

5.7 *Corollary*. *The Knaster continuum is indecomposable.*

Proof. This corollary follows from *Lemma* 5.5 and *Corollary* 5.4. ■

In the following theorem we give an elementary proof that the Knaster continuum is indecomposable. The proof parallels that of *Theorem* 9.*D*.14 in [CV].

5.8 *Theorem*. *The Knaster continuum K is indecomposable.*

Proof. Let A and B be subcontinua of K such that $K = A \cup B$. We show that $A \subset B$ or $B \subset A$.

Claim: For all i, $\pi_i(A) \subset \pi_i(B)$ or $\pi_i(B) \subset \pi_i(A)$.

Assume the claim is false for some k, and let $a \in \pi_k(A) - \pi_k(B)$ and $b \in \pi_k(B) - \pi_k(A)$. The set $K = \underleftarrow{\lim}(I, f)$ and f is onto, hence π_i is onto for all i. Thus $\frac{a}{2} \in \pi_{k+1}(A) \cup \pi_{k+1}(B)$. If $\frac{a}{2} \in \pi_{k+1}(B)$, then $a = f(\frac{a}{2}) \in f\pi_{k+1}(B) = \pi_k(B)$, which is impossible. Hence $\frac{a}{2} \in \pi_{k+1}(A) - \pi_{k+1}(B)$. Similarly we have $1 - \frac{a}{2} \in \pi_{k+1}(A) - \pi_{k+1}(B)$ and $\frac{b}{2}, 1 - \frac{b}{2} \in \pi_{k+1}(B) - \pi_{k+1}(A)$.

The set A is connected implies that $\pi_{k+1}(A)$ includes at least the interval $[\frac{a}{2}, 1 - \frac{a}{2}]$. Similarly, $\pi_{k+1}(B)$ includes at least the interval $[\frac{b}{2}, 1 - \frac{b}{2}]$. Clearly $[\frac{a}{2}, 1 - \frac{a}{2}] \subset [\frac{b}{2}, 1 - \frac{b}{2}]$ or $[\frac{b}{2}, 1 - \frac{b}{2}] \subset [\frac{a}{2}, 1 - \frac{a}{2}]$ contradicting the fact that $\frac{a}{2}, 1 - \frac{a}{2} \in \pi_{k+1}(A) - \pi_{k+1}(B)$ and $\frac{b}{2}, 1 - \frac{b}{2} \in \pi_{k+1}(B) - \pi_{k+1}(A)$. Hence the claim is proved. We now have two possibilities: $\pi_i(A) \subset \pi_i(B)$ for infinitely many i, or $\pi_i(B) \subset \pi_i(A)$ for infinitely many i.

Now if $\pi_i(A) \subset \pi_i(B)$ for infinitely many i, then for every i there exists a j, $j > i$, such that $\pi_j(A) \subset \pi_j(B)$. Hence $\pi_k(A) \subset \pi_k(B)$ for all $k \leq j$, since $\pi_{k-1}(A) = f\pi_k(A) \subset f\pi_k(B)$. Hence $\pi_i(A) \subset \pi_i(B)$ for all i. Similarly, if $\pi_i(B) \subset \pi_i(A)$ for infinitely many i then $\pi_i(B) \subset \pi_i(A)$ for all i.

Now observe that A is the inverse limit of the sequence $(\pi_i(A), f|_{\pi_i(A)})$ and B is the inverse limit of the sequence $(\pi_i(B), f|_{\pi_i(B)})$. Hence $A \subset B$ or $B \subset A$ follow from the following diagrams:

$$\begin{array}{ccccccccc}
\pi_1(A) & \xleftarrow{f} & \pi_2(A) & \xleftarrow{f} & \pi_3(A) & \xleftarrow{f} & \cdots & A \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & & \downarrow{\scriptstyle \cap} \\
\pi_1(B) & \xleftarrow{f} & \pi_2(B) & \xleftarrow{f} & \pi_3(B) & \xleftarrow{f} & \cdots & B
\end{array}$$

or

$$\begin{array}{ccccccccc}
\pi_1(B) & \xleftarrow{f} & \pi_2(B) & \xleftarrow{f} & \pi_3(B) & \xleftarrow{f} & \cdots & B \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & & \downarrow{\scriptstyle \cap} \\
\pi_1(A) & \xleftarrow{f} & \pi_2(A) & \xleftarrow{f} & \pi_3(A) & \xleftarrow{f} & \cdots & A
\end{array}$$

Hence the Knaster continuum is indecomposable. ∎

6. Main Results

Consider a solid torus $T_1 = S^1 \times D_1$ such that $T_1 \subset B_3$ where D_1 is a 2-*cell* and B_3 is a 3-*cell*. Our objective is to construct a *near homeomorphism* $H : B_3 \to B_3$ satisfying:

(1) There is a sequence of homeomorphisms $H_{t_i} : B_3 \to B_3$ converging uniformly to H such that each H_{t_i} is a Whitehead map.

(2) There exists a homeomorphism $F : \varprojlim(B_3, H) \to \varprojlim(B_3, H_{t_i})$ such that $F(\varprojlim(T_1, H)) = \varprojlim(T_1, H_{t_i})$.

(3) Taking S^1 to be the quotient space of $[0,1]$ generated by identifying the endpoints $\{0\}$ and $\{1\}$, then the restriction of H to S^1 is the the function $\tau : S^1 \to S^1$ defined by

$$\tau(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2}; \\ 2 - 2x, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

which is chaotic.

(4) The set $\varprojlim(T_1, H)$ is a local attractor for $\hat{H} : \varprojlim(B_3, H) \to \varprojlim(B_3, H)$.

Note that (2) implies that $\varprojlim(T_1, H)$ is embedded in $\varprojlim(B_3, H)$ just as the standard Whitehead continuum is embedded in B_3. Note alse that (3) implies $\hat{H}$ restricted to $\varprojlim(T_1, H)$ is chaotic.

While [Br1, Theorem 3], stated below, supplies us with a homeomorphism $F : \varprojlim(B_3, H) \to \varprojlim(B_3, H_{t_i})$, it does not guarantee that $F(\varprojlim(T_1, H)) = \varprojlim(T_1, H_{t_i})$. This is rectified by proving a generalization of [Br1, Theorem 3] for inverse sequences of pairs.

[Br1, Theorem 3]. *Let* $X^f_\infty = \varprojlim(X_i, f_i)$ *where the* X_i *are compact metric spaces. For* $2 \leq i$*, let* G_i *be a nonempty collection of maps from* X_i *into* X_{i-1}*. Suppose that for each* $i \geq 2$ *and* $\epsilon > 0$ *there exists a* $g \in G_i$ *such that* $\|f_i - g\| < \epsilon$*. Then there is a sequence* (g_i) *where* $g_i \in G_i$ *and* X^f_∞ *is homeomorphic to* $\varprojlim(X_i, g_i) = X^g_\infty$.

The homeomorphism in [Br1, Theorem 3] is defined in [Br1, Theorem 1] and [Br1, Theorem 2]. For completeness we will state these theorems. The following technical definitions are needed first:

(1) Let $f : X \to Y$ be a map, where X and Y are compact metric spaces. Then for $\epsilon > 0$ define $L(\epsilon, f)$ by $L(\epsilon, f) = Sup\{\delta < diam(X) : x, y \in X \text{ and } d_X(x, y) < \delta \text{ implies } d_Y(f(x), f(y)) < \epsilon\}$. Since X is compact $0 < L(\epsilon, f) \leq diam(X)$.

(2) Given the inverse sequence (X_i, f_i). A sequence (a_i) of positive real numbers is a *Lebesgue sequence* for (X_i, f_i) if there is a sequence (b_i) of positive real numbers such that

   (a) $\sum_{i=1}^{\infty} b_i < \infty$, and

   (b) Whenever $x, y \in X_j$, $i < j$ and $d_j(x, y) < a_j$, then $d_i(f_{ij}(x), f_{ij}(y)) < b_j$.

(3) A sequence (c_i) of positive real numbers is a *measure* for (X_i, f_i) if

   (a) $\sum_{i=n+1}^{\infty} c_i < \frac{1}{2} c_n$ for $n = 1, 2, \ldots$, and

   (b) For any two distinct points $\underline{x}$, $\underline{x}' \in \varprojlim(X_i, f_i)$ there is an integer n such that $d_{n+1}(x_{n+1}, x'_{n+1}) > c_n$.

We now state [Br1, Theorem 1] and [Br1, Theorem 2].

[Br1, Theorem 1]. *Let* $X^f_\infty = \varprojlim(X_i, f_i)$ *and* $X^g_\infty = \varprojlim(X_i, g_i)$ *where the* X_i *are compact metric spaces. Suppose* $\|f_{i+1} - g_{i+1}\| < a_i$, $i = 1, 2, \ldots$, *where* (a_i) *is a Lebesgue sequence for* (X_i, g_i). *Then the function* $F_N : X^f_\infty \to X_N$ *defined by* $F_N = \lim_{n\to\infty} g_{Nn}\pi_n$ *is well-defined and continuous. Moreover the function* $F : X^f_\infty \to X^g_\infty$ *defined by* $F(\underline{x}) = \big(F_1(\underline{x}), F_2(\underline{x}), \ldots\big)$ *is well-defined, continuous, and onto.*

[Br1, Theorem 2]. *Let* $X^f_\infty = \varprojlim(X_i, f_i)$ *and* $X^g_\infty = \varprojlim(X_i, g_i)$ *where the* X_i *are compact metric spaces. Suppose* $\|f_i - g_i\| < \min\big[c_{i-1}; \min_{k<i-1} L(c_{i-1}, g_{k,i-1})\big]$ *where* (c_i) *is a measure for* (X_i, f_i). *Then the map* $F : X^f_\infty \to X^g_\infty$ *described in [Br1, Theorem 1] is a homeomorphism .*

<u>*Notation*</u>. By the pair (X_i, Y_i) we mean a metric space X_i, equipped with a metric d_i, and a closed subset $Y_i \subseteq X_i$. By a map $f_i : (X_i, Y_i) \to (X_{i-1}, Y_{i-1})$ we mean a map $f_i : X_i \to X_{i-1}$ satisfying $f_i(Y_i) \subseteq Y_{i-1}$.

Let $\big((X_i, Y_i), f_i\big)$ denote the inverse sequence

$$(X_1, Y_1) \xleftarrow{f_2} (X_2, Y_2) \xleftarrow{f_3} (X_3, Y_3) \xleftarrow{f_4} \cdots.$$

Let (X^f_∞, Y^f_∞) denote the inverse limit of the sequence $\big((X_i, Y_i), f_i\big)$. That is, let X^f_∞ and Y^f_∞ be the inverse limits of the sequences (X_i, f_i) and $(Y_i, f_{i|_{Y_i}})$ respectively. Similarly, define $\big((X_i, Y_i), g_i\big)$ and (X^g_∞, Y^g_∞).

By *Lemma* 1 and *Lemma* 2 of [Br1], If the X_i are compact metric spaces then (X_i, g_i) has a Lebesgue sequence (a_i) and a measure (c_i).

The following theorem is a generalization of [Br1, Theorem 1] .

<u>6.1 *Theorem*</u>. *Let* $(X^f_\infty, Y^f_\infty) = \varprojlim\big((X_i, Y_i), f_i\big)$ *and* $(X^g_\infty, Y^g_\infty) = \varprojlim\big((X_i, Y_i), g_i\big)$ *where the* X_i *are compact metric spaces and for all* i, Y_i *is a*

closed subset of X_i. *Suppose* $\|f_{i+1} - g_{i+1}\| < a_i$, $i = 1,2,3,\ldots$; *where* a_i *is a Lebesgue sequence for* (X_i, g_i). *Then the function* $F_N : (X^f_\infty, Y^f_\infty) \to X_N$ *defined by* $F_N = \lim_{n\to\infty} g_{Nn}\pi_n$ *is well-defined and continuous. Moreover the function* $F : (X^f_\infty, Y^f_\infty) \to (X^g_\infty, Y^g_\infty)$ *defined by* $F(\underline{x}) = \big(F_1(\underline{x}), F_2(\underline{x}), \ldots\big)$ *is well-defined, continuous, onto and* $F(Y^f_\infty) = Y^g_\infty$.

Proof. We only need to show that $F(Y^f_\infty) = Y^g_\infty$, since the rest of the proof is identical to that of [Br1, Theorem 1] . Assume $\underline{z} \in (X^f_\infty, Y^f_\infty)$ and $z_i \in Y_i$ for all i. Clearly $F(\underline{z}) \in (X^g_\infty, Y^g_\infty)$. Since $g_i(Y_i) \subseteq Y_{i-1}$ for all $i \geq 2$, it follows that $F_i(\underline{z}) = \lim_{n\to\infty} g_{in}\pi_n(\underline{z}) \in Y_i$ for all i.

Let $\underline{w} = (w_1, w_2, w_3, \ldots) \in Y^g_\infty$. Fix a positive integer N. We first show that there exists $\underline{x}^N \in (X^f_\infty, Y^f_\infty)$ such that $F_N(\underline{x}^N) = w_N$. Let $\epsilon > 0$. From the proof of [Br1, Theorem 1] we have:

(1) $\lim_{\substack{i\to\infty \\ N<i<j}} \|g_{Ni}f_{ij} - g_{Ni}g_{ij}\| = 0$ and

(2) $g_{Ni}\pi_i$ converges uniformly to F_N as $i \to \infty$

(1), (2) and the fact that $\sum_{i=1}^{\infty} b_i < \infty$ imply that there exists an $i > N$ such that $\|F_N - g_{Ni}\pi_i\| < \frac{\epsilon}{3}$, $\|g_{Ni}f_{ij} - g_{Ni}g_{ij}\| < \frac{\epsilon}{3}$ for all $j > i$ and $b_i < \frac{\epsilon}{3}$. Fix this i.

Now, $\bigcap_{j=i}^{\infty} f_{ij}(Y_j) = \pi_i(Y^f_\infty)$. Since Y_j is compact, there exists a $j > i$ such that if $y_i \in f_{ij}(Y_j)$ then there exists $x_i \in \pi_i(Y^f_\infty)$ such that $d_i(y_i, x_i) < a_i$. Hence there exists $\underline{x} \in (X^f_\infty, Y^f_\infty)$ where $x_i \in \pi_i(Y^f_\infty)$ such that $d_i\big(f_{ij}(w_j), \pi_i(\underline{x})\big) < a_i$. Hence $d_N\big(g_{Ni}f_{ij}(w_j), g_{Ni}\pi_i(\underline{x})\big) < b_i < \frac{\epsilon}{3}$. Then $d_N\big(F_N(\underline{x}), w_N\big) \leq d_N\big(F_N(\underline{x}), g_{Ni}\pi_i(\underline{x})\big)$ $+d_N\big(g_{Ni}\pi_i(\underline{x}), g_{Ni}f_{ij}(w_j)\big)$ $+d_N\big(g_{Ni}f_{ij}(w_j), g_{Ni}g_{ij}(w_j)\big) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

The function F_N is continuous and (X^f_∞, Y^f_∞) is compact, hence there exists $\underline{x}^N \in (X^f_\infty, Y^f_\infty)$ where $x^N_i \in Y_i$ for all $i \geq 1$ such that $F_N(\underline{x}^N) = w_N$. For all N, $F_N(\underline{x}^N) = w_N$ implies that $F_i(\underline{x}^N) = w_i$ for all $i < N$.

Since (X^f_∞, Y^f_∞) is compact, $\{\underline{x}^N\}$ has a convergent subsequence. If $\underline{y}$ is a limit point of this subsequence then $F(\underline{y}) = \underline{w}$. Hence $F(Y^f_\infty) = Y^g_\infty$. ∎

The following three theorems are generalizations of [Br1, Theorem 2], [Br1, Theorem 3] and [Br1, Theorem 1] respectively. The proofs are identical to those found in [Br1], hence they are omitted.

6.2 Theorem. *Let* $(X^f_\infty, Y^f_\infty) = \varprojlim((X_i, Y_i), f_i)$ *and* $(X^g_\infty, Y^g_\infty) = \varprojlim((X_i, Y_i), g_i)$ *where the* X_i *are compact metric spaces and for all* i, Y_i *is a closed subset of* X_i. *Suppose* $\|f_i - g_i\| < \min\left[c_{i-1}; \min_{k<i-1} L(c_{i-1}, g_{k,i-1})\right]$ *where* (c_i) *is a measure for* (X_i, f_i). *Then the map* $F : (X^f_\infty, Y^f_\infty) \to (X^g_\infty, Y^g_\infty)$ *described in Theorem* 6.1 *is a homeomorphism satisfying* $F(Y^f_\infty) = Y^g_\infty$.

6.3 Theorem. *Let* $(X^f_\infty, Y^f_\infty) = \varprojlim((X_i, Y_i), f_i)$ *where the* X_i *are compact metric spaces and for all* i, Y_i *is a closed subset of* X_i. *For* $i \geq 2$, *let* G_i *be a nonempty collection of maps from* (X_i, Y_i) *into* (X_{i-1}, Y_{i-1}). *Suppose that for each* $i \geq 2$ *and* $\epsilon > 0$ *there exists a* $g \in G_i$ *such that* $\|f_i - g\| < \epsilon$. *Then there is a sequence* (g_i) *where* $g_i \in G_i$ *and a homeomorphism* $F : (X^f_\infty, Y^f_\infty) \to (X^g_\infty, Y^g_\infty)$ *satisfying* $F(Y^f_\infty) = Y^g_\infty$.

6.4 Theorem. *Let* $(X^f_\infty, Y^f_\infty) = \varprojlim((X_i, Y_i), f_i)$ *where:*

(1) For all i, *there exists a homeomorphism* $h_i : (X_i, Y_i) \to (X, Y)$, *where* X *is a compact metric space and* $Y \subset X$ *is closed such that* $h_i(Y_i) = Y$, *and*

(2) For all i, f_i *is a near homeomorphism.*

Then there exists a homeomorphism $\phi : (X^f_\infty, Y^f_\infty) \to (X, Y)$ *satisfying* $\phi(X^f_\infty) \subseteq Y$.

A point p in the *xy-plane* is coordinatized using the familiar polar coordinate system (r, θ). Let R_ϕ be a rotation of the *xy-plane* through an angle ϕ measured counterclockwise from the positive *x-axis* about the line $x = -1$. For any point $p = (r, \theta)$ in the *xy-plane* let $R_\phi(p)$ be the image of p under the rotation R_ϕ. To be more specific, let $R_\phi((x, y, 0)) = ((x+1)\cos\phi - 1, y, -(x+1)\sin\phi)$. Consider *Figure* 6.1.

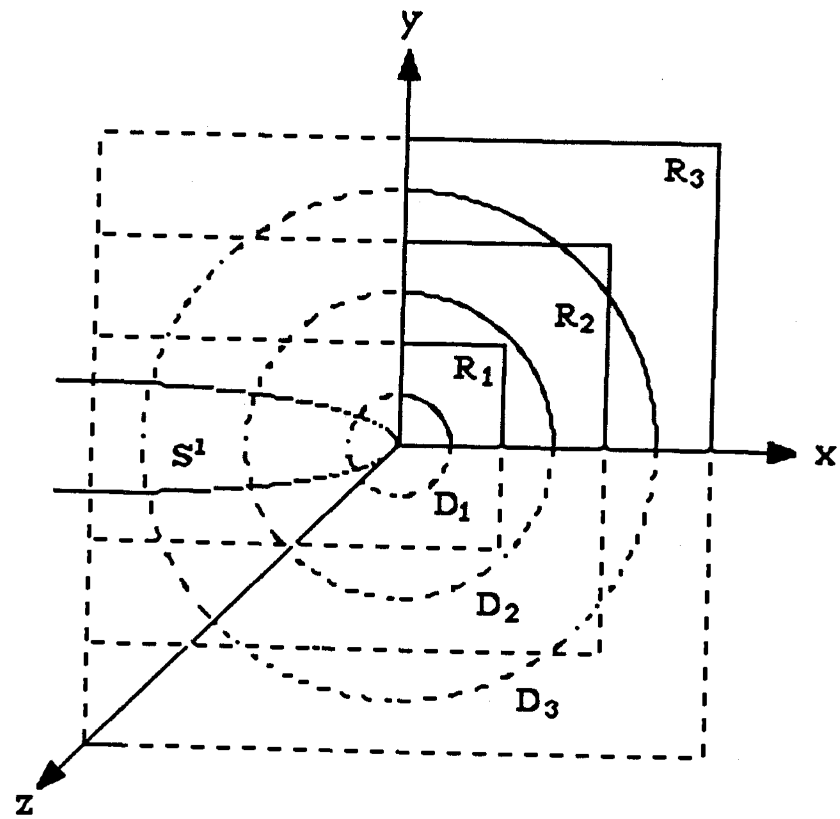

Figure 6.1

Let $D_1 = \{(r, \theta) : 0 \leq r \leq r_1 \text{ and } 0 \leq \theta \leq 2\pi\}$, $D_2 = \{(r, \theta) : 0 \leq r \leq r_3 \text{ and } 0 \leq \theta \leq 2\pi\}$, and $D_3 = \{(r, \theta) : 0 \leq r \leq r_5 \text{ and } 0 \leq \theta \leq 2\pi\}$. For $i = 1, 2$ and 3 let $T_i = \{R_\phi(D_i) : 0 \leq \phi \leq 2\pi\}$. For $i = 1, 2$ and 3, the T_i are solid tori with diameters $2r_1$, $2r_3$, and $2r_5$ respectively satisfying $T_1 \subset Int(T_2) \subset Int(T_3)$.

Let $R_1 = \{(x, y) : -1 \leq x \leq r_2 \text{ and } -r_2 \leq y \leq r_2\}$, $R_2 = \{(x, y) : -1 \leq x \leq r_4 \text{ and } -r_4 \leq y \leq r_4\}$, and $R_3 = \{(x, y) : -1 \leq x \leq r_6 \text{ and } -r_6 \leq y \leq r_6\}$.

For $i = 1, 2$ and 3 let $B_i = \{R_\phi(R_i) : 0 \leq \phi \leq 2\pi\}$. For $i = 1, 2$ and 3, the B_i are 3-*cells* satisfying: $T_1 \subset Int(B_1)$, $T_2 \subset Int(B_2)$, $T_3 \subset Int(B_3)$ and $B_1 \subset Int(B_2) \subset Int(B_3)$.

Let $S^1 = \{R_\phi((0,0)) : 0 \leq \phi \leq 2\pi\}$. To simplify notation, we will denote a point $p = R_\phi((0,0)) \in S^1$ by ϕ^*. For example, $R_0((0,0))$ will be denoted by 0^* and $R_\pi((0,0))$ will be denoted by π^*. The set S^1 is a circle of radius 1 centered at the point $(-1,0,0)$. Let $\phi_i = \frac{2\pi i}{n}$ for $i = 0, 1, 2, \ldots, n$ where n is an even positive integer.

We now describe a typical Whitehead type of embedding $g : S^1 \to T_1$ where the image of S^1 has a self-linking in T_1.

Let $g : S^1 \to T_1$ be the embedding shown in *Figure* 6.2 and satisfying:

(1) $g(\phi_0^*) = \phi_{n-2}^*$ and $g(\phi_{\frac{n}{2}}^*) = \phi_2^*$.

(2) For $\phi_2 \leq \phi \leq \phi_{\frac{n}{2}-2}$ let $g(\phi^*)$ be in $R_{2\phi}(D_1)$ and for $\phi_{\frac{n}{2}+2} \leq \phi \leq \phi_{n-2}$ let $g(\phi^*)$ be in $R_{2\phi_n - 2\phi}(D_1)$. Note that if $\phi_0 \leq \phi_j \leq \phi_{\frac{n}{2}}$ and $\psi_j = \phi_n - \phi_j$ then $\phi_{\frac{n}{2}} \leq \psi_j \leq \phi_n$ and $R_{2\phi_j}(D_1) = R_{2\phi_n - 2\psi_j}(D_1)$.

(3) The set $\{g(\phi^*) : \phi_{n-2} \leq \phi \leq \phi_n$ *or* $\phi_0 \leq \phi \leq \phi_2\}$ is a subset of the plane P determined by the *y-axis* and the straight line passing through the points $(0,0,0)$ and ϕ_{n-2}^*.

(4) The set $\{g(\phi^*) : \phi_{\frac{n}{2}-2} \leq \phi \leq \phi_{\frac{n}{2}+2}\}$ is a subset of the *xz-plane*

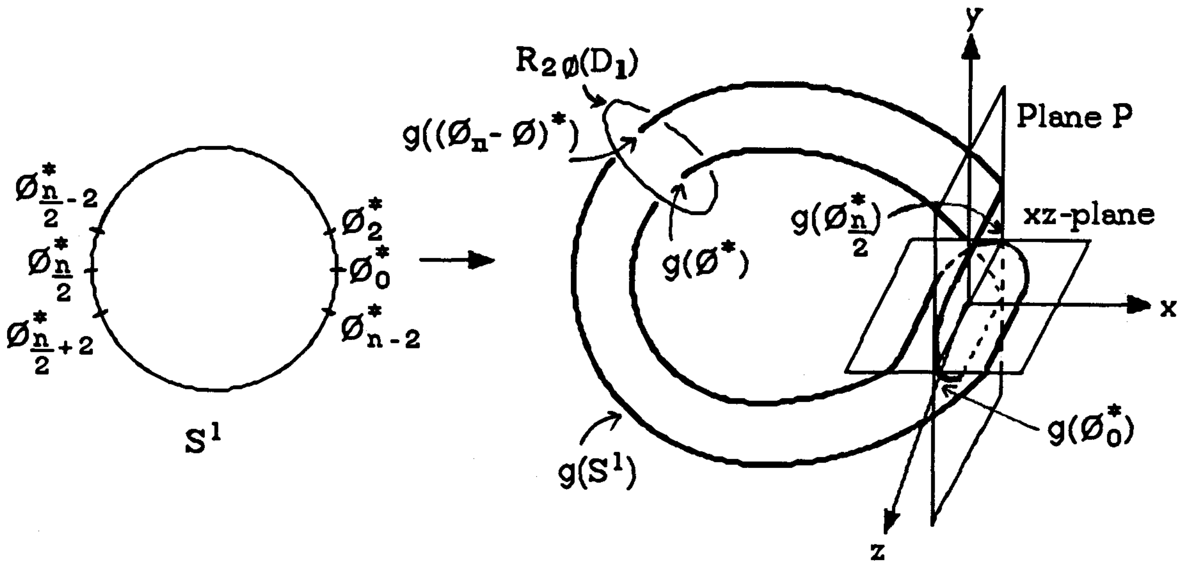

Figure 6.2

Extend g to a homeomorphism $G : B_3 \to B_3$ such that:

(1) The set $G(T_1)$ is a solid torus contained in $N(g(S^1), \frac{1}{n}) \subset Int(T_1)$.

(2) $G(T_2) \subset Int(T_2)$.

(3) $G_{|B_3 - B_2} = \text{id}$.

The homeomorphism G can be visualized by the sequence of pictures in *Figure* 6.3. Imagine twisting a "*flexible*" 3-*cell* B_2 in such a way that the boundary stays fixed and the interior is twisted so that a top view of $S^1 \subset Int(B_2)$ goes through the following stages:

(1) A half twist is introduced.

(2) Another half twist is introduced.

(3) The top loop is folded down over the bottom loop which produces the desired self-linking.

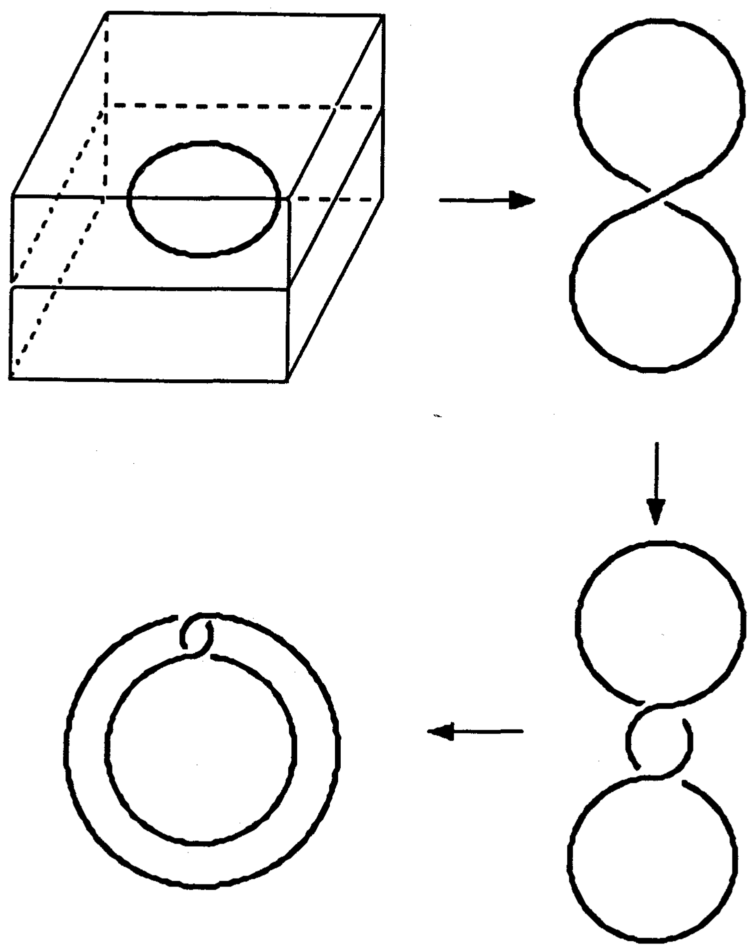

Figure 6.3

We now define three *pseudo-isotopies* P_t^1,P_t^2 and P_t^4 and an *isotopy* P_t^3 of B_3 onto itself. The effects of these maps are represented graphically in *Figure* 6.4. The map P_1^1 shrinks the solid torus $G(T_1)$ to $G(S^1)$ leaving $G(S^1)$ fixed. The map P_1^2 "eliminates" the self-linking of $G(S^1)$. Note that the number of components of $R_\phi(D_1) \cap G(S^1)$ is equal to

$$\begin{cases} 2, & \text{if } \phi_2 < \phi < \phi_{n-2}; \\ 3, & \text{if } \phi = \phi_0 \text{ or } \phi = \phi_n; \\ 4, & \text{if } \phi_0 < \phi < \phi_2 \text{ or } \phi_{n-2} < \phi < \phi_n. \end{cases}$$

Note also that the number of components of $R_\phi(D_1) \cap P_1^2 \circ P_1^1 \circ G(S^1)$ is equal to

$$\begin{cases} 2, & \text{if } \phi_0 < \phi < \phi_n; \\ 1, & \text{if } \phi = \phi_0 \text{ or } \phi = \phi_n. \end{cases}$$

The map P_1^4 shrinks the torus T_1 to its core S^1. The map P_1^3 is defined such that for $\phi_0 \le \phi \le \phi_{\frac{n}{2}}$, $P_1^3 \circ P_1^2 \circ P_1^1 \circ G(\phi^*)$ is in $R_{2\phi}(D_1)$ and for $\phi_{\frac{n}{2}} \le \phi \le \phi_n$, $P_1^3 \circ P_1^2 \circ P_1^1 \circ G(\phi^*)$ is in $R_{2\phi_n - 2\phi}(D_1)$.

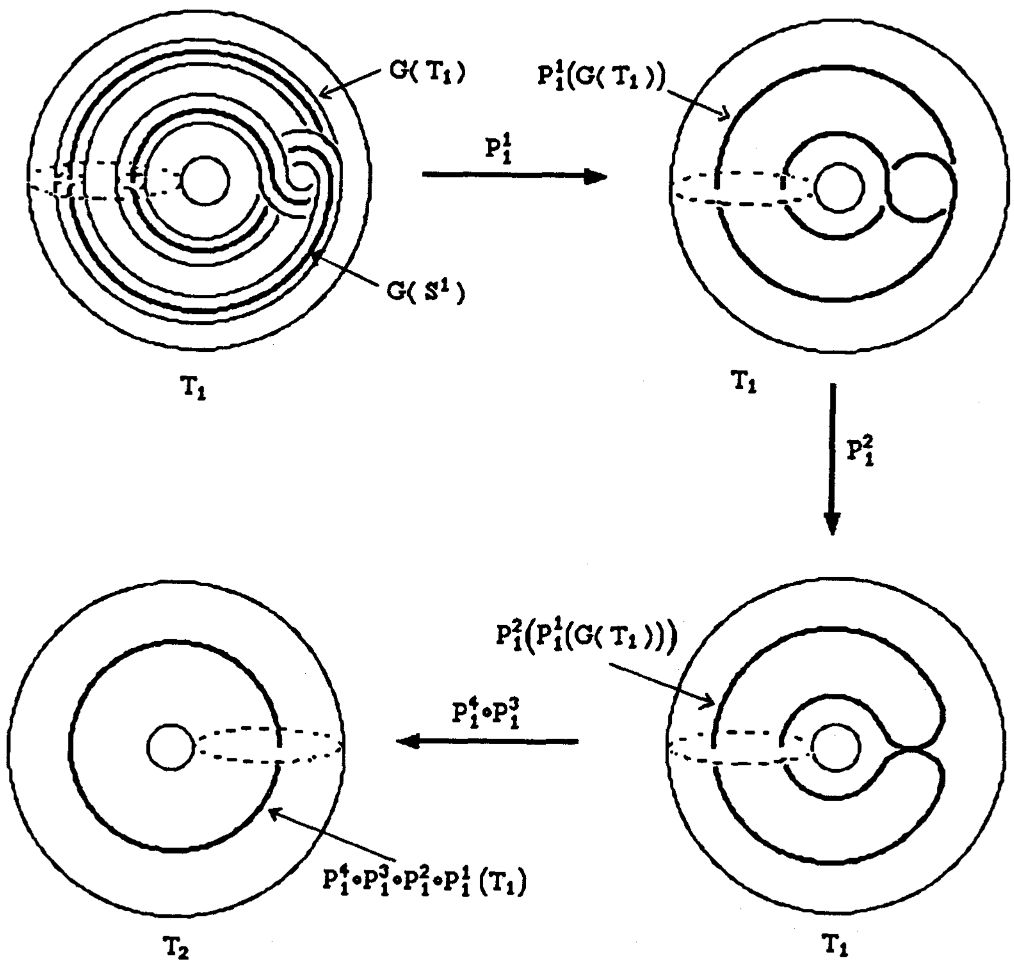

Figure 6.4

Defining $P_1^1 : B_3 \to B_3$.

Let $D_1' = \{(r,\theta) : 0 \le r \le r_1'\}$ where $r_1 < r' < r_2$. Consider the solid torus $T_1' = \{R_\phi(D_1') : 0 \le \phi \le 2\pi\}$ satisfying $T_1' \subset Int(T_2)$ and $G(T_1') \subset Int(T_2)$. Define the pseudo-isotopy P_t^1 such that P_0^1 is the identity map and P_1^1 collapses $G(T_1)$ to the linked circle $G(S^1)$ and is the identity map on $B^3 - G(T_1')$

To be more precise, consider D_1' shown in *Figure* 6.5 and define a pseudo-isotopy ${}^0P_t^1 : B_3 \to B_3$ as follows:

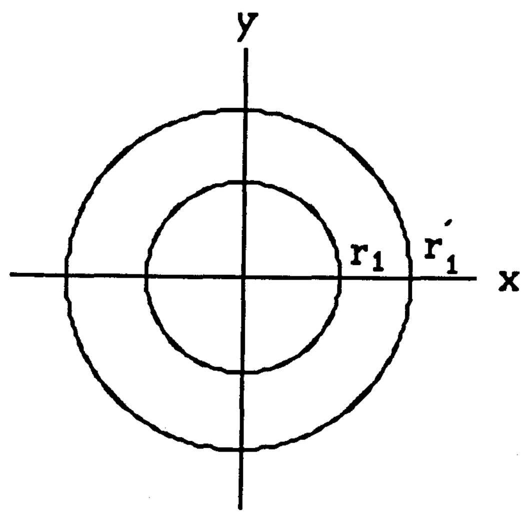

Figure 6.5

For any point $x = (r,\theta,0) \in R_3$ let ${}^0P_t^1(r,\theta,0) = (\mathcal{R}_t(r),\theta,0)$ where $\mathcal{R}_t(r)$ is defined as follows:

$$\mathcal{R}_0(r) = r, \text{ and}$$

$$\mathcal{R}_1(r) = \begin{cases} 0, & \text{if } 0 \le r \le r_1; \\ \frac{r_1'}{r_1'-r_1}(r - r_1), & \text{if } r_1 \le r \le r_1'; \\ r, & \text{if } r \ge r_1'. \end{cases}$$

Hence $\mathcal{R}_t(r) = (1-t)r + t\mathcal{R}_1(r)$. See *Figure* 6.6.

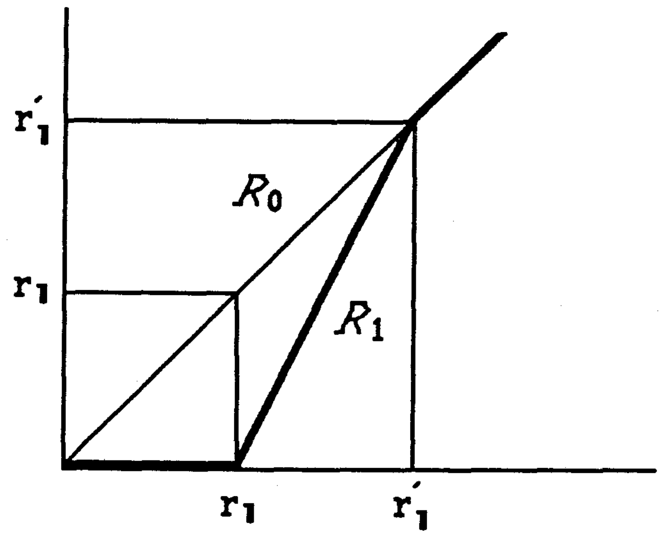

Figure 6.6

More precisely,

$$\mathcal{R}_t(r) = \begin{cases} (1-t)r, & \text{if } 0 \le r \le r_1 \\ (1-t)r + t\left[\frac{r_1'}{r_1'-r_1}(r-r_1)\right], & \text{if } r_1 \le r \le r_1'; \\ r, & \text{if } 0 \le r \le r_1. \end{cases}$$

Let ${}^0P_t^1(r,\theta,\phi) = R_\phi\big({}^0P_t^1(r,\theta,0)\big)$ for $\phi_0 \le \phi \le \phi_n$.

For $0 \le t < 1$, ${}^0P_t^1$ is a homeomorphism of B^3 onto itself under which $R_\phi(D_1)$ goes to $\{(r,\theta,\phi) : r = (1-t)r_1\}$ and $R_\phi(D_1' - Int(D_1))$ goes to $\{(r,\theta,\phi) : (1-t)r_1 \le r \le r_1'\}$.

The desired pseudo-isotopy P_t^1 is defined by $P_t^1 = G \circ {}^0P_t^1 \circ G^{-1}$.

Defining $P_1^2 : B_3 \to B_3$.

The objective is to pull the linked parts of $G(S^1)$ together via a pseudo-isotopy P_t^2 so that $P_1^2(G(\phi_0^*)) = P_1^2((G(\phi_{\frac{n}{2}}^*)) = (0,0,0)$.

Consider the wedge $\Delta_1 \subset Int(T_1)$, shown in *Figure* 6.7, whose central plane P_c is a subset of the plane P. Recall that $\{g(\phi^*) : \phi_{n-2} \leq \phi \leq \phi_n \text{ or } \phi_0 \leq \phi \leq \phi_2\}$ is a subset of the plane P. Choose Δ_1 such that $G(\phi^*) \notin \Delta_1$ for $\phi_1 < \phi < \phi_{n-1}$ and $G(\phi^*) \in P_c$ for $\phi_{n-1} \leq \phi \leq \phi_n$ or $\phi_0 \leq \phi \leq \phi_1$.

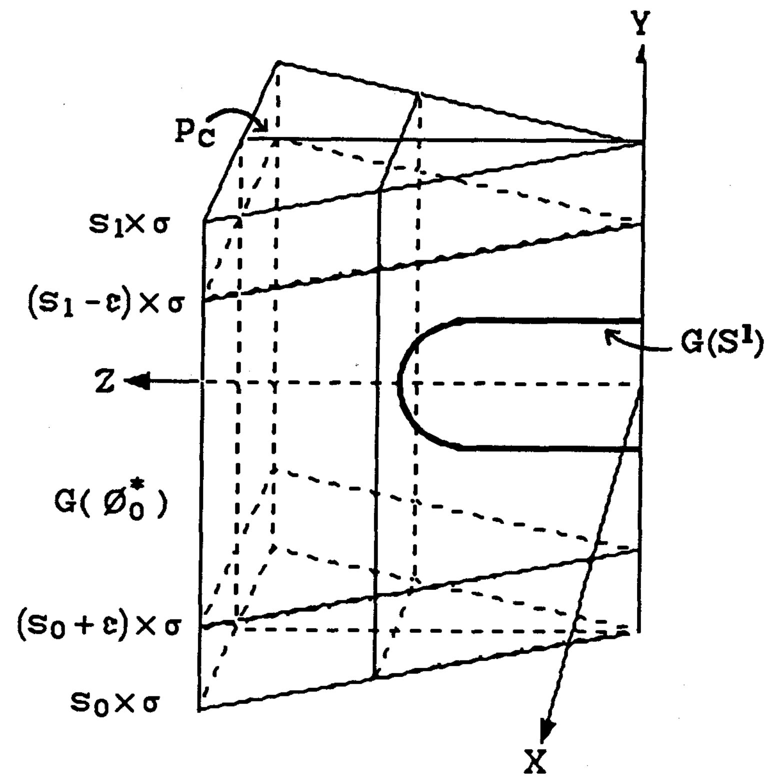

Figure 6.7

Consider σ shown in *Figure* 6.8 and view Δ_1 as $\sigma \times J_1$, where $J_1 = [s_0, s_1]$ as seen in *Figure* 6.7. Note that if $x' \in G(S^1) \cap \Delta_1$ then x' lies in $[e, d] \times J_1$. We define a pseudo-isotopy P_t of σ as follows: Let P_0 be the identity map and P_1 be the simplicial map which leaves the vertices a, b, c, d fixed and sends e to d.

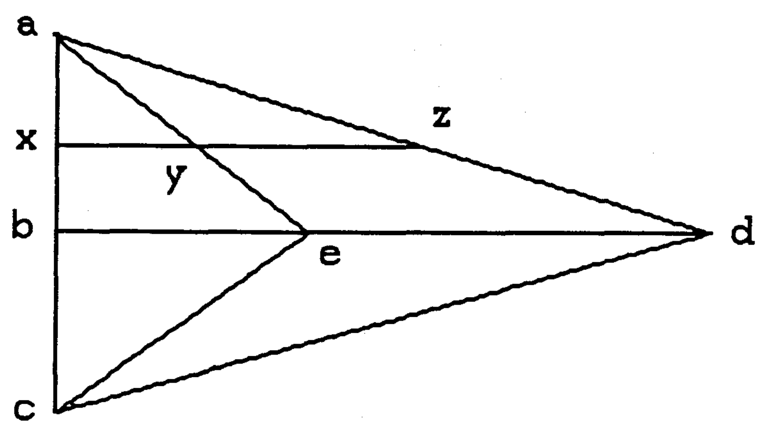

Figure 6.8

To be more precise, for $0 \le \alpha \le 1$ and $0 \le t \le 1$ define P_t on the line segment xz by:

$$P_t\big[(1-\alpha)x + \alpha y\big] = (1-\alpha)x + \alpha\big[(1-t)y + tz\big], \text{ and}$$

$$P_t\big[\alpha y + (1-\alpha)z\big] = \alpha\big[(1-t)y + tz\big] + (1-\alpha)z.$$

Note that P_t fixes the boundary of σ for all t. Now P_t on σ induces a pseudo-isotopy ${}^1P_t^2$ on Δ_1 where ${}^1P_1^2\big(G(\phi_0^*)\big) = (0,0,0)$ and the P_t action is phased out near the bottom, $\sigma \times s_0$, and the top, $\sigma \times s_1$, of Δ_1 so that ${}^1P_t^2$ fixes $Bd(\Delta_1)$.

The pseudo-isotopy ${}^1P_t^2 : \Delta_1 \to \Delta_1$ can be defined as follows:

$${}^1P_t^2(\sigma \times s) = \begin{cases} P_t(\sigma) \times s, & \text{if } s_0 + \epsilon \le s \le s_1 - \epsilon; \\ P_{\frac{(s-s_0)}{\epsilon}t}(\sigma) \times s, & \text{if } s_0 \le s \le s_0 + \epsilon; \\ P_{\frac{(s_1-s)}{\epsilon}t}(\sigma) \times s, & \text{if } s_1 - \epsilon \le s \le s_1. \end{cases}$$

Extend ${}^1P_t^2$ to B_3 by setting ${}^1P_t^2(x) = x$ for all $x \in B_3 - \Delta_1$.

We next consider the wedge $\Delta_2 \subset Int(T_1)$, shown in *Figure* 6.9, whose central plane P_c is a subset of the *yz-plane*. Note that $\Delta_2 \cap \Delta_1 = (0,0,0)$. Recall that $\{g(\phi^*) : \phi_{\frac{n}{2}-2} \leq \phi \leq \phi_{\frac{n}{2}+2}\}$ is a subset of the *yz-plane*. The wedge Δ_2 is chosen such that $G(\phi^*) \in P_c$ for $\phi_{\frac{n}{2}-1} \leq \phi \leq \phi_{\frac{n}{2}+1}$ and $G(\phi^*) \notin \Delta_2$ for $\phi_{\frac{n}{2}+1} < \phi \leq \phi_n$ or $\phi_0 \leq \phi < \phi_{\frac{n}{2}} - 1$.

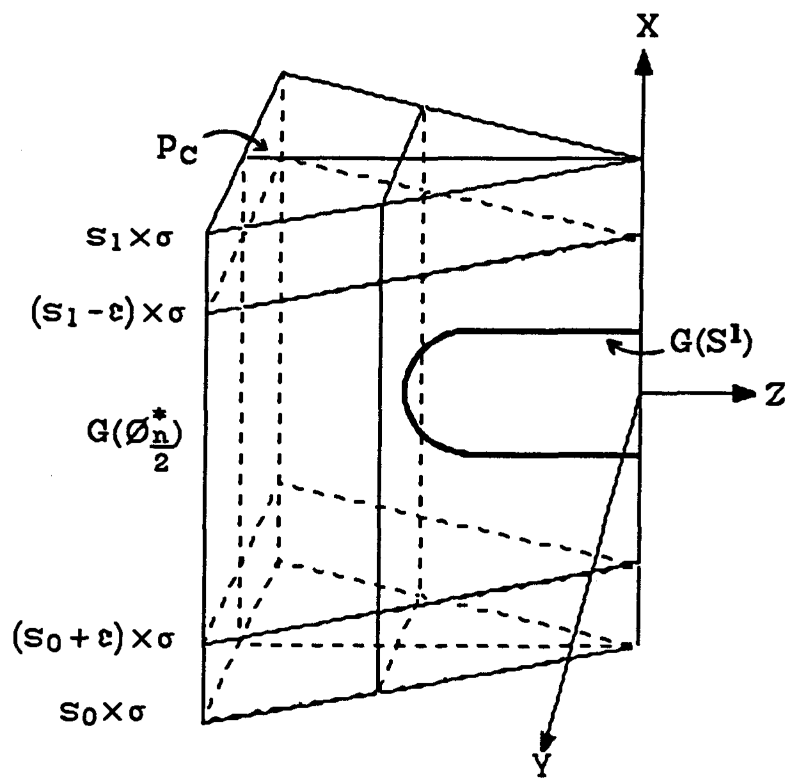

Figure 6.9

In a similar way, construct a pseudo-isotopy ${}^2P_t^2$ on B_3 that is the identity on $B_3 - Int(\Delta_2)$ and ${}^2P_1^2(G(\phi^*_{\frac{n}{2}})) = (0,0,0)$.

Define $P_t^2 : B_3 \to B_3$ by

$$P_t^2(x) = \begin{cases} {}^1P_t^2(x), & \text{if } x \in \Delta_1; \\ {}^2P_t^2(x), & \text{if } x \in \Delta_2; \\ x, & \text{if } x \in B_3 - (\Delta_1 \cup \Delta_2). \end{cases}$$

Defining P_1^3.

The objective is to position $P_1^2G(S^1)$ in $Int(T_1)$, via the isotopy P_t^3, in such a way that for $\phi_0 \leq \phi \leq \phi_{\frac{n}{2}}$, $P_1^3P_1^2G(\phi^*) \in R_{2\phi}(D_1)$ and for $\phi_{\frac{n}{2}} \leq \phi \leq \phi_n$, $P_1^3P_1^2G(\phi^*) \in R_{2\phi_n-2\phi}(D_1)$.

Consider $C = \{R_\phi(D_1) : \phi_0 \leq \phi \leq \phi_4 \text{ or } \phi_{n-4} \leq \phi \leq \phi_n\}$. The set $C \subset T_1$ is shown below. Let $S^r = \{\phi^* : \phi_0 \leq \phi \leq \phi_2 \text{ or } \phi_{n-2} \leq \phi \leq \phi_n\}$ and $S^l = \{\phi^* : \phi_{\frac{n}{2}-2} \leq \phi \leq \phi_{\frac{n}{2}} \text{ or } \phi_{\frac{n}{2}} \leq \phi \leq \phi_{\frac{n}{2}+2}\}$. Note that $P_1^2G(S^r)$ is a subset of the plane P and $P_1^2G(S^l)$ is a subset of the *xz-plane*.

Let $S_1 = \{\phi^* : \phi_0 \leq \phi \leq \phi_2\}$, $S_2 = \{\phi^* : \phi_{n-2} \leq \phi \leq \phi_n\}$, $S_3 = \{\phi^* : \phi_{\frac{n}{2}-2} \leq \phi \leq \phi_{\frac{n}{2}}\}$ and $S_4 = \{\phi^* : \phi_{\frac{n}{2}} \leq \phi \leq \phi_{\frac{n}{2}+2}\}$.

Let $g_1 = P_1^2G_{|S_1}$, $g_2 = P_1^2G_{|S_2}$, $g_3 = P_1^2G_{|S_3}$ and $g_4 = P_1^2G_{|S_4}$.

Let f_1 be an embedding of S_1 into T_1 satisfying:

(1) $f_1(S_1)$ is a subset of the plane P.

(2) $f_1(\phi_0^*) = g_1(\phi_0^*)$ and $f_1(\phi_2^*) = g_1(\phi_2^*)$.

(3) $f_1(\phi^*) \subset R_{2\phi}(D_1)$ for $\phi_0 \leq \phi \leq \phi_2$.

Let f_2 be an embedding of S_2 into T_1 satisfying:

(1) $f_2(S_2)$ is a subset of the plane P.

(2) $f_2(\phi_n^*) = g_2(\phi_n^*)$ and $f_2(\phi_{n-2}^*) = g_2(\phi_{n-2}^*)$.

(3) $f_2(\phi^*) \subset R_{2\phi_n-2\phi}(D_1)$ for $\phi_{n-2} \leq \phi \leq \phi_n$.

Let f_3 be an embedding of S_3 into T_1 satisfying:

(1) $f_3(S_3)$ is a subset of the *xz-plane*.

(2) $f_3(\phi^*_{\frac{n}{2}-2}) = g_3(\phi^*_{\frac{n}{2}-2})$ and $f_3(\phi^*_{\frac{n}{2}}) = g_3(\phi^*_{\frac{n}{2}})$.

(3) $f_3(\phi^*) \subset R_{2\phi}(D_1)$ for $\phi_{\frac{n}{2}-2} \le \phi \le \phi_{\frac{n}{2}}$.

Let f_4 be an embedding of S_4 into T_1 satisfying:

(1) $f_4(S_4)$ is a subset of the *xz-plane*.

(2) $f_4(\phi^*_{\frac{n}{2}+2}) = g_4(\phi^*_{\frac{n}{2}+2})$ and $f_4(\phi^*_{\frac{n}{2}}) = g_4(\phi^*_{\frac{n}{2}})$.

(3) $f_4(\phi^*) \subset R_{2\phi_n - 2\phi}(D_1)$ for $\phi_n \le \phi \le \phi_{\frac{n}{2}+2}$.

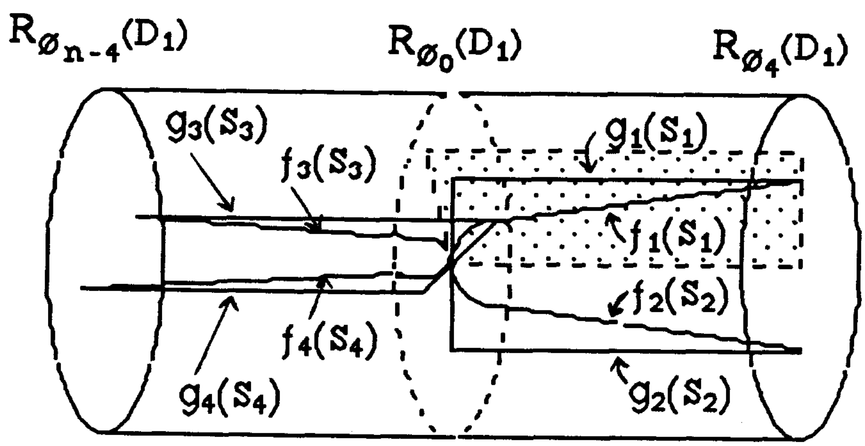

Figure 6.10

We now define four isotopies Q^1_t, Q^2_t, Q^3_t and Q^4_t from B_3 onto itself such that $Q^i_{0|S_i} = g_i$ and $Q^i_{1|S_i} = f_i$ for $i = 1, 2, 3$, and 4. Also, $Q^i_{t|_{S^1-S_i}} = P^2_1 G(S^1 - S_i)$ and $Q^i_{t|_{B_3-T_1}}$ is the identity map for all t.

To define Q^1_t, consider a 2-*cell* C^2 subset of the plane P and containing $g_1(S_1)$ and $f_1(S_1)$. Choose C^2 such that $C^2 \cap f_2(S_2) = C^2 \cap g_2(S_2) = C^2 \cap f_3(S_3) = C^2 \cap g_3(S_3) = C^2 \cap f_4(S_4) = C^2 \cap g_4(S_4) = G(\phi^*_0)$. The 2-*cell* C^2 is the shaded region in *Figure* 6.10.

Consider the square $I \times I$ where $I = [-1, 1]$ shown in *Figure* 6.11. We will make use of a theorem of *Schönflies* stating that any two closed simply connected regions whose boundaries are simple closed curves can be homeomorphically mapped onto each other so that the correspondence so determined between their boundaries is a preassigned *one-to-one*, continuous one [Sm].

Define the maps g_{11},$f_{11} : I \to I$ by

$$g_{11}(x) = \begin{cases} \frac{x+1}{2}, & \text{if } -1 \leq x \leq 0; \\ \frac{1-x}{2}, & \text{if } 0 \leq x \leq 1. \end{cases}$$

$$f_{11}(x) = \begin{cases} \frac{-(x+1)}{2}, & \text{if } -1 \leq x \leq 0; \\ \frac{x-1}{2}, & \text{if } 0 \leq x \leq 1. \end{cases}$$

Now, it follows from the *Schönflies* theorem, stated above, that there exists a homeomorphism h' from the plane P onto itself taking the 2-*cell* C^2 onto the square $I \times I$ such that:

(1) $h' \circ g_1(\phi^*) = g_{11}\left(\frac{2\phi}{\phi_2} - 1\right)$ and $h' \circ f_1(\phi^*) = f_{11}\left(\frac{2\phi}{\phi_2} - 1\right)$ for all $\phi_0 \leq \phi \leq \phi_2$

Note that (1) implies that:

(i) $h \circ g_1(\phi_0^*) = g_{11}(-1) = h \circ f_1(\phi_0^*) = f_{11}(-1) = (-1, 0)$,

(ii) $h \circ g_1(\phi_2^*) = g_{11}(1) = h \circ f_1(\phi_2^*) = f_{11}(1) = (1, 0)$,

(iii) $h \circ g_1(\phi_1^*) = g_{11}(0) = (0, \frac{1}{2})$, $h \circ f_1(\phi_1^*) = f_{11}(0) = (0, -\frac{1}{2})$, and

(iv) For all $\phi_0 \leq \phi \leq \phi_2$, the points $h \circ g_1(\phi^*)$ and $h \circ f_1(\phi^*)$ have the same *x-coordinate*. See *Figure* 6.11.

Note that the homeomorphism $h' : P \to P$ can be extended to a homeomorphism $h : R^3 = P \times R^1 \to R^3 = P \times R^1$ by letting $h(x, r) = \left(h'(x), r\right)$ for all $x \in P$ and $r \in R^1$.

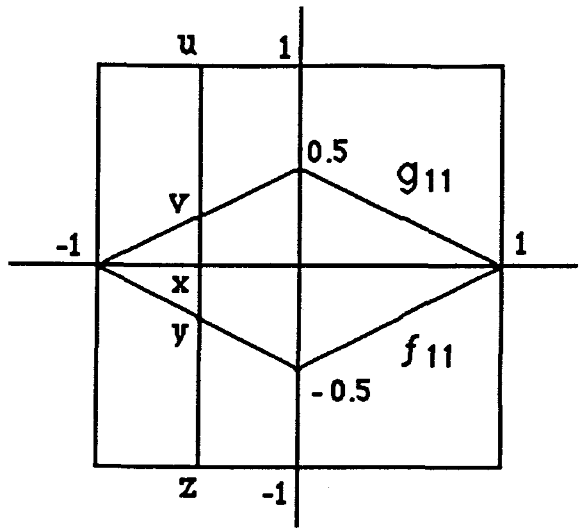

Figure 6.11

Now define a pseudo-isotopy ${}^0Q_t^1$ from $I \times I$ onto itself such that ${}^0Q_0^1$ is the identity map and ${}^0Q_1^1$ takes $g_{11}(x)$ onto $f_{11}(x)$ for all $x \in I$ and ${}^0Q_t^1$ fixes $Bd(I \times I)$ for all t. To be more precise, ${}^0Q_t^1$ can be defined as follows: Consider the line segment uz in *Figure* 6.11. For $0 \le \alpha \le 1$ and $0 \le t \le 1$ define ${}^0Q_t^1$ by:

$${}^0Q_t^1\left[(1-\alpha)u + \alpha v\right] = (1-\alpha)u + \alpha\left[(1-t)v + ty\right]$$

$${}^0Q_t^1\left[\alpha v + (1-\alpha)z\right] = \alpha\left[(1-t)v + ty\right] + (1-\alpha)z$$

Note that ${}^0Q_t^1$ fixes the boundary of $I \times I$ for all t. Let C^3 be a 3-*cell* containing $I \times I$ such that $h^{-1}(C^3) \subset Int(T_1)$ and $h^{-1}(C^3) \cap f_2(S_2) = h^{-1}(C^3) \cap g_2(S_2) = h^{-1}(C^3) \cap f_3(S_3) = h^{-1}(C^3) \cap g_3(S_3) = h^{-1}(C^3) \cap f_4(S_4) = h^{-1}(C^3) \cap g_4(S_4) = (0,0,0)$.

Let ${}^1Q_t^1$ be an extension of the isotopy ${}^0Q_t^1$ to C^3 fixing the boundary of C^3.

Now define Q_t^1 by $Q_t^1 = h^{-1} \circ {}^1Q_t^1 \circ h$. Note that Q_t^1 is the identity map on $B_3 - T_1$ and $Q_1^1\left(g_1(\phi^*)\right) = f_1(\phi^*)$ for all $\phi^* \in S_1$.

In a similar fashion, define the isotopies Q_t^2, Q_t^3, and Q_t^4. Now define the isotopy $P_1^3 : B_3 \to B_3$ by $P_1^3 = Q_t^4 \circ Q_t^3 \circ Q_t^2 \circ Q_t^1$.

Defining $P_1^4 : B_3 \to B_3$.

The objective is to define a pseudo-isotopy P_t^4 such that P_0^4 is the identity map and P_1^4 collapses T_1 onto its core S^1 and at the same time collapses T_2 onto T_1 fixing the boundary of T_3.

Consider D_3 shown in *Figure* 6.12.

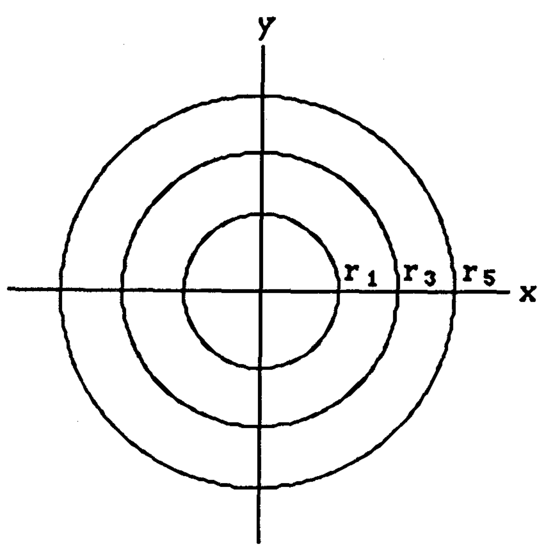

Figure 6.12

Define $P_t^4 : B_3 \to B_3$ as follows: For any point $x = (r, \theta, 0) \in R_3$ let $P_t^4(r,\theta,0) = (\mathcal{R}_t(r), \theta, 0)$ where $\mathcal{R}_t(r)$ is defined by

$$\mathcal{R}_t(r) = \begin{cases} (1-t)r, & \text{if } 0 \le r \le r_1; \\ r - t\big[r - (\frac{r-r_1}{r_3-r_1})r_1\big], & \text{if } r_1 \le r \le r_3; \\ r - t\big[\frac{r-r_5}{r_5-r_3}(r_1 - r_3)\big], & \text{if } r_3 \le r \le r_5; \\ r, & \text{if } r \ge r_5. \end{cases}$$

Let $P_t^4(r,\theta,\phi) = R_\phi\big(P_t^4(r,\theta,0)\big)$ for $\phi_0 \le \phi \le \phi_n$.

For $0 \leq t < 1$, P_t^4 is a homeomorphism of B^3 onto itself under which $R_\phi(D_1)$ goes to $\{(r,\theta,\phi) : r = (1-t)r_1\}$ and $\{(r,\theta,\phi) : r_1 \leq r \leq r_5\}$ goes to $\{(r,\theta,\phi) : (1-t)r_1 \leq r \leq r_5\}$. In addition, P_1^4 satisfies the conditions:

(1) $P_1^4(T_2) = T_1$.

(2) $P_1^4|_{B_3 - Int(T_3)} = id$.

(3) If $\phi^* \in S^1$ then $R_\phi(D_1) \subset (P_1^4)^{-1}(\phi^*)$.

(4) For every point $(r,\theta,\phi) \in Int(T_3)$ there exists an integer $n \geq 0$ such that $(P_1^4)^n(r,\theta,\phi) \in S^1$.

Define $H : B_3 \to B_3$ by $H = P_1^4 \circ P_1^3 \circ P_1^2 \circ P_1^1 \circ G$. The map H satisfies the following properties:

(1) The homeomorphisms $H_t : B_3 \to B_3$ where $H_t = P_t^4 \circ P_t^3 \circ P_t^2 \circ P_t^1 \circ G$ and $t \in [0,1)$ converge uniformly to H as $t \to 1$. Hence H is a near homeomorphism.

(2) $H(T_2) = T_1$.

(3) $H(T_1) = S^1$

(4) For every $(r,\theta,\phi) \in Int(T_3)$ there exists an integer $n \geq 0$ such that $H^n((r,\theta,\phi)) \in S^1$. Hence $\bigcap_{n\geq 0} H^n(Int(T_3)) = S^1$.

(5) $H|_{Bd(B_3)} = id$.

Note that T_1 is a closed subset of B_3, $H(T_1) \subset T_1$ and $H_t(T_1) \subset T_1$ for all $t \in [0,1]$. It follows from *Theorem* 6.3 that there is a sequence H_{t_i}, $i = 1,2,\ldots$, $H_{t_i} \in \{H_t : t = \frac{n}{n+1},\ and\ n \in \{1,2,\ldots\}\}$ and a homeomorphism $F : \varprojlim((B_3,T_1),H) \to \varprojlim((B_3,T_1),H)$ such that $F(\varprojlim(T_1,H)) = \varprojlim(T_1,H_{t_i})$.

Let $K = \varprojlim(T_1, H)$ and $W = \varprojlim(T_1, H_{t_i})$. By *Theorem* 6.4, there is a homeomorphism $\Phi : \varprojlim((B_3, T_1), H) \to (B_3, T_1)$.

Now, consider the following diagram:

$$\begin{array}{ccccccccc} T_1 & \xleftarrow{H_{t_1}} & T_1 & \xleftarrow{H_{t_2}} & T_1 & \xleftarrow{H_{t_3}} & \cdots & & W \\ \downarrow i & & \downarrow H_{t_1} & & \downarrow H_{t_1}H_{t_2} & & & & \\ T_1 & \xleftarrow{i} & H_{t_1}(T_1) & \xleftarrow{i} & H_{t_1}H_{t_2}(T_1) & \xleftarrow{i} & \cdots & & \bigcap_{i=1}^{\infty} H_{t_1}H_{t_2}\ldots H_{t_i}(T_1) \end{array}$$

This diagram defines a homeomorphism $h : W \to \bigcap_{i=1}^{\infty} H_{t_1}H_{t_2}\ldots H_{t_i}(T_1)$. Hence W is a standard Whitehead continuum (one with self-linking). Since $F : \varprojlim((B_3, T_1), H) \to \varprojlim((B_3, T_1), H_{t_i})$ takes $K = \varprojlim(T_1, H)$ onto $W = \varprojlim(T_1, H_{t_i})$, K is embedded in B_3 just as W is.

let h be the restriction of H to S^1 where S^1 is the core of T_1. Note that h is just the tent map on S^1. That is, considering S^1 as the quotient space of $[0, 1]$ resulting from identifying the end points $\{0\}$ and $\{1\}$ then

$$h(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2}; \\ 2 - 2x, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Now, consider the following diagram:

$$\begin{array}{ccccc} \varprojlim(S^1, h) & \xrightarrow{i} & \varprojlim((B_3, T_1), H) & \xrightarrow{\Phi} & B_3 \\ \downarrow \hat{h} & & \downarrow \hat{H} & & \downarrow \Psi = \Phi \hat{H} \Phi^{-1} \\ \varprojlim(S^1, h) & \xrightarrow{i} & \varprojlim((B_3, T_1), H) & \xrightarrow{\Phi} & B_3 \end{array}$$

Claim: $K = \varprojlim(T_1, H)$ is a local attractor for $\hat{H} : \varprojlim((B_3, T_1), H) \to \varprojlim((B_3, T_1), H)$.

To prove the claim, first note that since $H(T_1) = S^1$, it follows from the following diagram that $K = \varprojlim(T_1, H) = \varprojlim(S^1, h)$.

$$\begin{array}{ccccccccc}
T_1 & \xleftarrow{H} & T_1 & \xleftarrow{H} & T_1 & \xleftarrow{H} & \cdots & & K \\
\uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & & & \\
S^1 & \xleftarrow{h} & S^1 & \xleftarrow{h} & S^1 & \xleftarrow{h} & \cdots & & \varprojlim(S^1, H)
\end{array}$$

Since $H(S^1) = S^1$, then $\hat{H}(K) = K$.

Let $U = \{(x_1, x_2, \ldots) \in \varprojlim((B_3, T_1), H) : x_1 \in Int(T_2)\} = \pi^{-1}(Int(T_2))$. Clearly, U is open in $\varprojlim((B_3, T_1), H)$ and $K \subset U$. Now if $\underline{x} = (x_1, x_2, \ldots) \in U$, then $\hat{H}^n(\underline{x}) = (H^n(x_1), H^n(x_2), \ldots) \to K$ as $n \to \infty$.

Since $H(T_2) = T_1$, we have $\hat{H}(U) \subseteq \pi_1^{-1}(T_1)$ and hence $\overline{\hat{H}(U)} \subseteq \overline{\pi_1^{-1}(T_1)}$ $= \pi_1^{-1}(T_1) \subset \pi_1^{-1}(T_2) = U$. Therefore $Cl(\hat{H}(U)) \subset U$.

It follows from *Theorems* 2.6.1,2.6.2 and 2.4.4 that $\hat{h} = \varprojlim(S^1, h) \to (S^1, h)$ is chaotic. Hence $K = \bigcap_{n \geq 0} \hat{H}^n(U)$ is a local chaotic attractor for $\hat{H} : \varprojlim((B_3, T_1), H) \to \varprojlim((B_3, T_1), H)$.

Let $\Lambda = \Phi(K) = \bigcap_{n \geq 0} \Psi^n(\Phi(U))$. Since $\hat{H}_{|K}$ is *topologically conjugate* to $\Psi_{|\Phi(K)}$, then $\Phi(K)$ is a local chaotic attractor for Ψ.

7. Generalizations

Recall that we are studying the following problem: *Given a topological space X, is there a map $F : R^3 \to R^3$ such that X is an attractor for F?*

In *Chapter* 6, we showed that the Whitehead continuum can be embedded in R^3 as a local chaotic attractor. In this chapter, we define two infinite classes of continua, $\mathcal{W} = \{W(n,m) : n \geq 1, m \geq 1\}$ and $\mathcal{K} = \{K_n : n \geq 2\}$ to which the construction in *Chapter* 6 generalizes. Each of these continua is defined as the intersection of a nested sequence of solid tori. These continua have an important feature in common with the Whitehead continuum, namely the self-linking.

Defining $\mathcal{W}$.

Let T_0 be a solid torus in the interior of a 3-*cell* B_3. For all integers $n \geq 1$, $m \geq 1$, let $G_{nm} : B_3 \to B_3$ be a homeomorphism such that $T_1 = G_{nm}(T_0) \subset Int(T_0)$ is a solid torus which wraps around T_0 *n-times* in clockwise direction, then it self-links, and finally it wraps around T_0 *m-times* in counterclockwise direction as shown in *Figure* 7.1.

For all integers, $n \geq 1$ and $m \geq 1$, let $W(n,m) = \bigcap_{k \geq 0} G^k_{nm}(T_0)$. The continua $W(n,m)$ can be embedded in R^3 as local chaotic attractors.

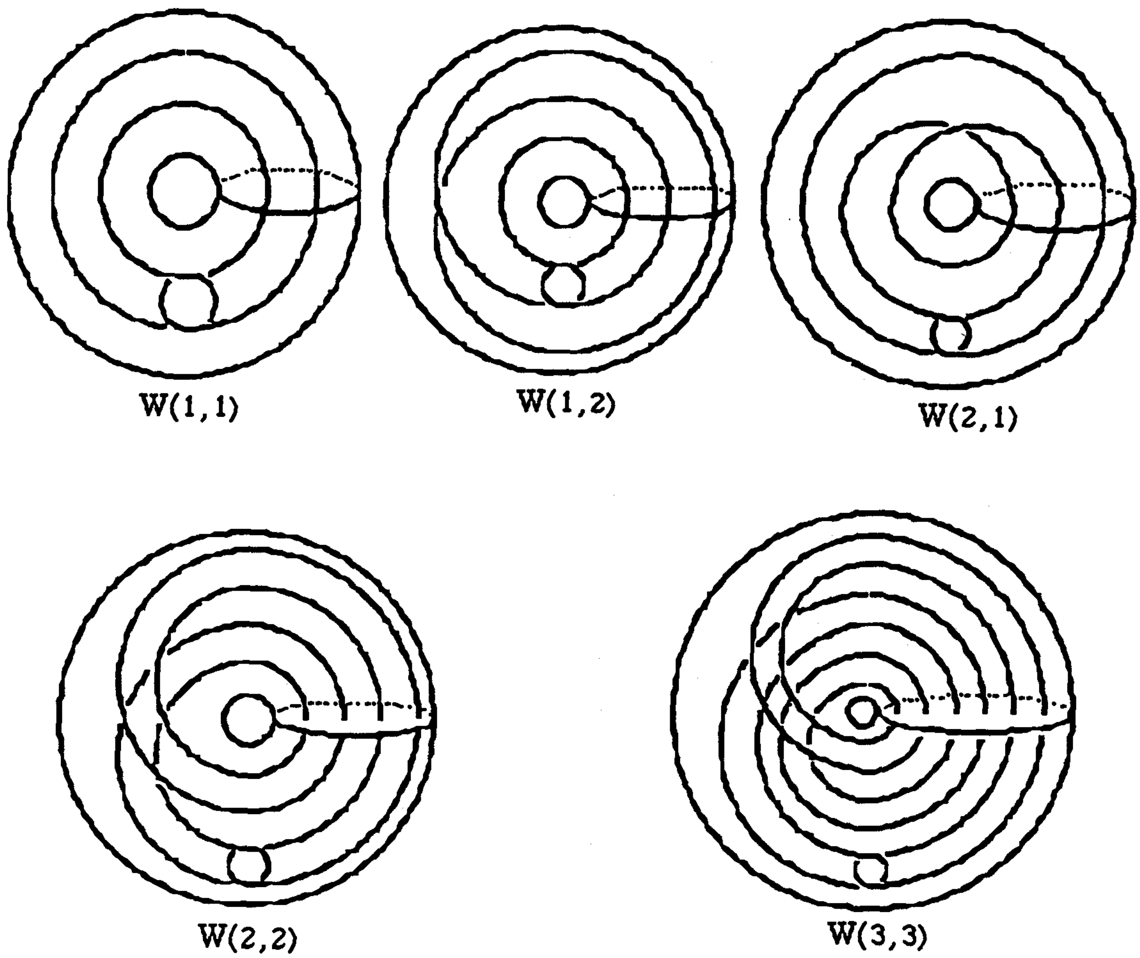

Figure 7.1

Shown in *Figure* 7.1 are the first stages in the construction of $W(1,1), W(1,2), W(2,1), W(2,2)$, and $W(3,3)$. The solid torus T_1 is not shown in its entirety, only its core is shown.

As we have done in *Chapter* 6, after a few pseudo-isotopies (eliminating the self-intersection), the homeomorphism G_{nm} is transformed into a near homeomorphism $H_{nm} : B_3 \to B_3$ such that the restriction of H_{nm} to S^1, the core of T_0, is the map $f_{nm} : S^1 \to S^1$ such that $W(n,m) = \varprojlim(S^1, f_{nm})$.

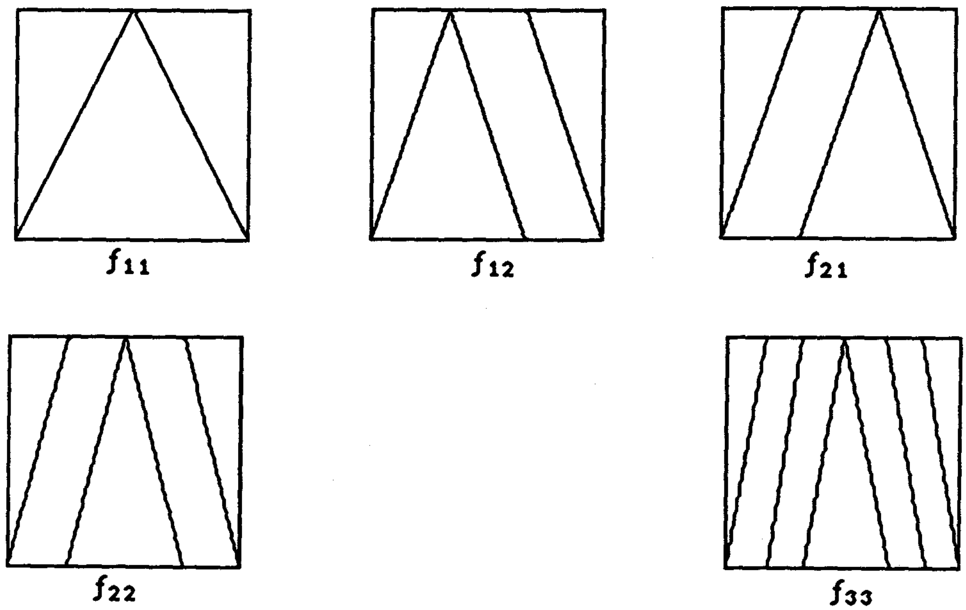

Figure 7.2

Shown in *Figure* 7.2 are the maps $f_{11}, f_{12}, f_{21}, f_{22}$, and f_{33}. Here S^1 is viewed as the quotient space of the interval $[0, 1]$ resulting from identifying the end points $\{0\}$ and $\{1\}$.

For $n \geq 1$ and $m \geq 1$, the map $f_{nm} : S^1 \to S^1$ has the following property: If $J \subset S^1$ with nonempty interior, there exists an integer N such that $f^k_{nm}(J) = S^1$ for all integers $k \geq N$. Hence by *Theorem* 2.6.1, f^k_{nm} is transitive for every $k > 0$. Clearly, f_{nm} has periodic points, hence *Theorem* 2.6.2 implies that f_{nm} is chaotic.

Defining $\mathcal{K}$.

For all integers $n \geq 2$, let $Q_n : B_3 \to B_3$ be a homeomorphism such that $T_1 = Q_n(T_0)$ is embedded in $Int(T_0)$ as shown in *Figure* 7.3. Shown in *Figure* 7.3 are the cores of $Q_i(T_0)$ for $i = 2, 3, \ldots, 7$. The images of T_0 under Q_n

for $n > 7$ are not shown, but can be drawn by noticing the pattern developing in $Q_2(T_0), Q_3(T_0), \ldots, Q_7(T_0)$.

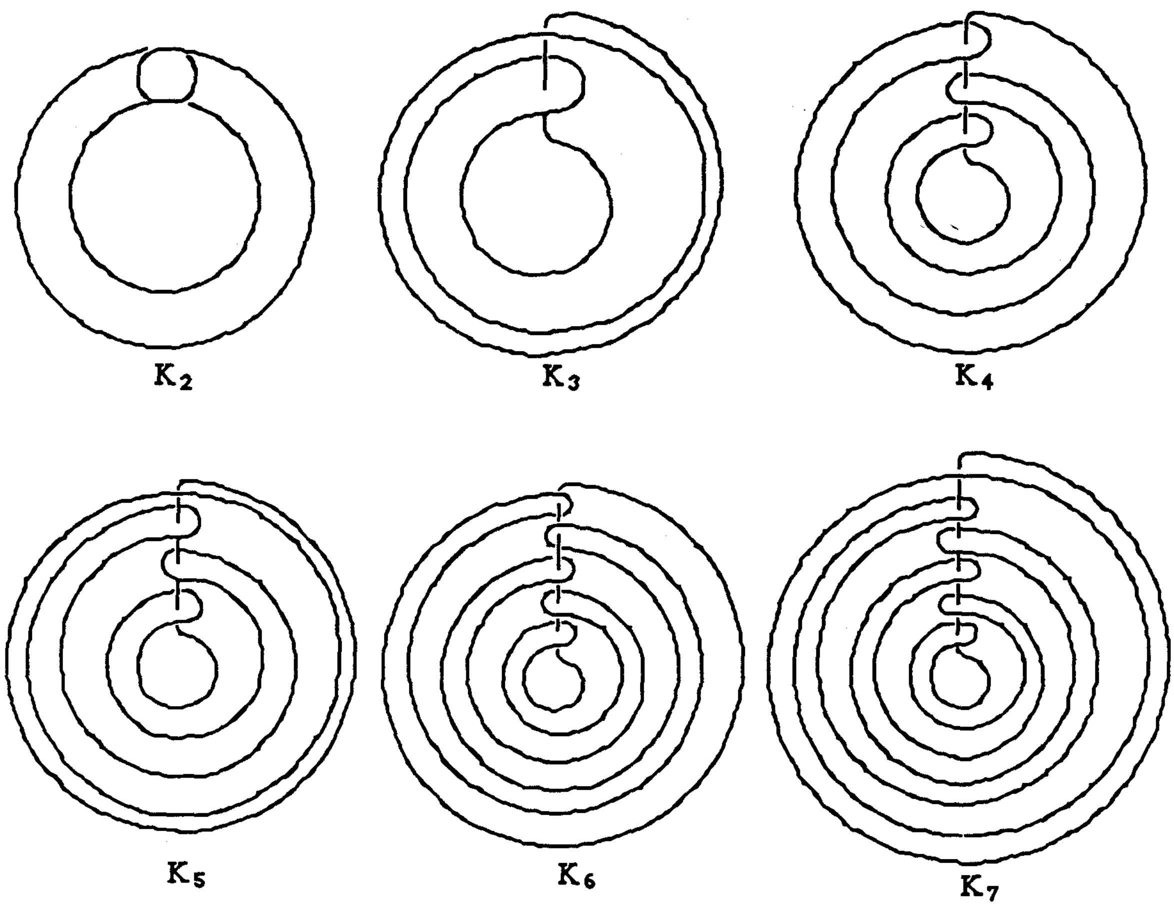

Figure 7.3

Let $K_n = \bigcap_{k \geq 0} Q_n^k(T_0)$ for $n \geq 2$. The continua K_n can be embedded in R^3 as chaotic local attractors.

Again, as we have done in *Chapter* 6, after a few pseudo-isotopies (eliminating the self-intersection), the homeomorphism Q_n is transformed into a near homeomorphism $H_n : B_3 \to B_3$ such that the restriction of H_n to S^1, the core of T_0, is the map $h_n : S^1 \to S^1$ such that $K_n = \varprojlim(S^1, h_n)$.

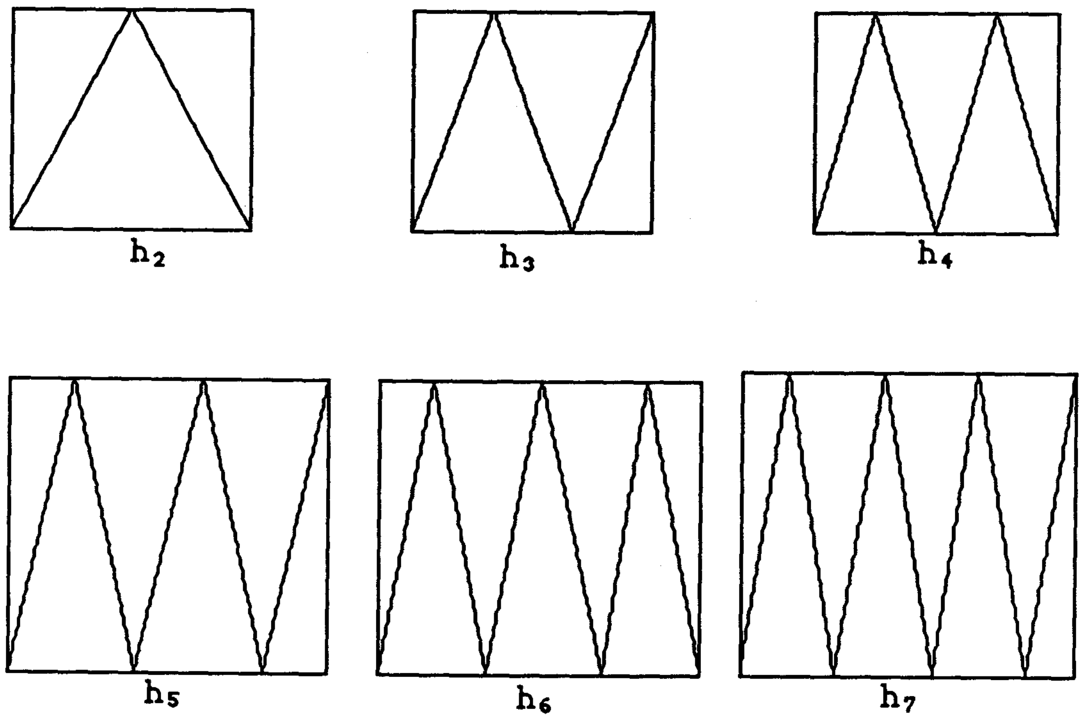

Figure 7.4

Shown in *Figure* 7.4 are the maps h_i for $i = 1, 2, \ldots, 7$. Here S^1 is viewed as the quotient space of the interval $[0, 1]$ resulting from identifying the end points $\{0\}$ and $\{1\}$. The maps $h_n : S^1 \to S^1$ are chaotic by *Theorems* 2.6.1 and 2.6.2.

The continua $K_n \simeq \varprojlim(S^1, h_n) \simeq \varprojlim(I, h_n)$. It follows from [W] that K_n is homeomorphic to K_m *if and only if* n and m have the same prime factors.

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