

AN ABSTRACT OF THE THESIS OF

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A concept image is that collection of all images, pictures, symbols, definitions and properties associated with any given mathematical concept. One of the most important components in the mental representation of concepts in the concept image of advanced mathematical thinkers is *visualization*. This component, in turn, is indispensable in the intuition process which is essential to advanced mathematical thought. The visual aspect of intuitive reasoning in mathematics falls into three main categories - diagrammatic reasoning, which is predominantly though not exclusively graphical, analogic reasoning, relying heavily on non-mathematical experiences as models for abstract mathematical concepts, and the use of prototypes, the selection of one typical example as a representative of the concept.

This study was designed to examine and describe the nature of visual images used by advanced mathematical thinkers, as prototypical, analogic or diagrammatic images. We also sought to identify *hooks*, which provide initial access to the concept image, we looked for *links* among them, and for image schemas, which provide the mental 'scaffolding' for the concept image. We sought evidence of progress in

the construction process of concept images by looking for the *interiorization* and *condensation* stages, during which time the concept is internalized and all related information is condensed into a gestalt, and in particular, the *reification* stage, an event which produces a radical restructuring of the concept image.

Nine case studies are presented and analyzed, in which advanced mathematics undergraduates, mathematics graduate students, and mathematics faculty were extensively interviewed, and their responses audiotaped and transcribed. The interviews were reflective in nature, comprised of a series of questions, which were asked regarding twenty-one different mathematical concepts. A detailed analysis of each individual, in light of the above questions, is presented, summarizing the individual nature of the concept image in advanced mathematical thought.

Visualization and Reification  
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in  
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by  
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This work is dedicated to my parents,  
Helen and Ian,  
who came before and paved the way,  
and to my daughters,  
Vicky and Nina,  
who came after to follow in my footsteps.  
I am very proud of them all.

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# VISUALIZATION AND REIFICATION OF CONCEPTS IN ADVANCED MATHEMATICAL THINKING

## CHAPTER I INTRODUCTION

The most essential aspect of advanced mathematical thought is the facility to employ mental representations of abstract ideas in multiple and flexible ways. Mental images can include both verbal or linguistic symbols as well as pictorial images. Indeed, perhaps the most powerful images are those that combine both pictorial and linguistic elements or use pictures *as symbols*. Commonly shared mental images become part of the language of advanced mathematics—they allow the community of mathematicians to communicate with each other in ways that transcend the spoken word. But mental images are also a means of “internal” communication. That is, they are a tool for our own individual cognitive processes, and can guide that most elusively defined notion of mathematical thought—intuition.

Intuition seems to require and be the result of an amalgam of cognitive processes of which *visualization* is perhaps the most important. Some aspects of intuition which are also important in developing a visual interpretation of a mathematical concept are the processes of abstracting, generalizing and synthesizing, all requiring active engagement with the subject matter. These skills, so necessary to advanced mathematical thinking, seem to be interwoven with visualization. It is difficult to conceive of carrying out any one of these processes without some kind of visualization process to accompany and facilitate it. It was this centrality of the process of visualization that led me to want to investigate its use among mathematicians.

In considering the role of visualization in advanced mathematical thinking, two questions naturally arose: What types of visual imagery are important? How exactly are visual images used in advanced mathematical thinking? However, it

has proved a difficult task to identify the exact characteristics of visualization, they seem to vary widely, among psychologists, among mathematicians, and between both schools.

According to Gauss, once a mathematical argument has been constructed, none of the “scaffolding” used in the process should be in evidence—only the fine finished product should be visible, all the scaffolding and other evidence of construction should be hidden from view. It was exactly this scaffolding, and in particular the *visual* component of this scaffolding, that I wished to investigate.

Some research on advanced mathematical thinking has already been done, but the emphasis of these studies has been on beginning college students, predominantly pre-calculus and calculus students. Studies of students’ understanding of such concepts as function and limit are widespread. Studies on acquisition of the concept of derivative, on the understanding of  $\epsilon - \delta$  arguments, on the acquisition and understanding of proof techniques may also be found in the literature. There is, however a paucity of research on the truly advanced mathematical thinker - the mathematician, the mathematics specialist, the person who has chosen this area for her life’s work. This is where I have placed my focus.

I am convinced that no-one learns mathematics in a vacuum and that mathematical knowledge is best understood and retained by connecting it to previously existing knowledge. It is only when a new mathematical idea or concept is connected to an individual’s existing mathematical knowledge - either about that concept or about related concepts - that it begins to be understood. This process is facilitated by linking the mathematical idea to something in the experiential realm, as when mathematical induction is understood as analogous to a row of dominoes poised to fall, or as when infinity is understood as the north pole on a sphere or ball. These connections (or *scaffolding*), intuitive and most often visual, are of central importance to mathematical understanding and take time to build. They often need restructuring, expanding and remodeling. The connections between the experiential and the mathematical are provided by analogies and metaphors, both of which are

found as the *idealized cognitive models* of Lakoff(1987). The radical restructuring of the scaffolding is found in the *reification* event of Sfard(1991, 1994).

This study examines the advanced mathematical thinking of nine mathematicians - three advanced undergraduates selected to participate in a summer research program, four graduate students in their second year of graduate study or beyond, and two mathematics faculty members, both of whom are full professors.

By interviewing these mathematicians extensively I was able to gain some insight into their mathematical knowledge, the ways in which this knowledge is represented visually, how they connect their knowledge of concepts together, as well as some insight into where *they* feel they are in their mathematical journey.

Reflection is also an important part of the mathematical process, since it enables the individual to assess the progression of her mathematical thinking, and to monitor the development of mathematical understanding. It was this process that I employed in my study to gain access to each individual's mathematical thinking and understanding. I developed a set of reflective questions designed to 'pull out' from each individual some idea of her/his cognitive content and the scaffolding by which it was held in place. The idea of a mental scaffolding for constructing mathematical conceptions came to me from work of Sfard(1992). It provides a wonderful analogy and language for discussing this constructive process. In my analysis of these reflective interviews I lean heavily on this analogy for a language to use for discussing my findings.

There seem to be many ways in which an individual may carry out this construction process; all at once, quickly putting up a sometimes shaky building; gradually, perhaps beginning with a shaky foundation but slowly reinforcing the weak spots; building up then tearing down the structure, only to start all over again with some entirely new structure. These are some but by no means all of the possible representative construction projects.

In Chapter II, I elaborate on the theoretical framework for my study. Here, the emphasis is mainly on the psychological underpinnings for my study. In this

chapter, the reader will find in particular a historical development of visualization as a psychological construct. In Chapter III, the reader will find the mathematical framework for employing visualization to represent mathematical concepts, and as a major component in mathematical intuition. Here, one finds the intuitive reasoning with analogies, paradigms, and diagrams described by Fischbein(1987); the image schemas described by Lakoff(1987) which provide models for the types of scaffolding; and the process/object duality described by Sfard(1991) and the idea of a progression from operational to structural understanding of concepts, particularly through the process of reification, which suggests the tearing down and rebuilding of the scaffolding. In chapter IV, the reader will find an analysis of each case study. In chapter V, the reader will find an overall summary discussion of the findings.

## CHAPTER II VISUALIZATION AND ADVANCED MATHEMATICS

### MATHEMATICAL VISUALIZATION

There is no consensus in the mathematics and mathematics education literature as to the precise nature of mathematical visualization. Senechal(1990) makes the following remarks about visualization: "Like the weather, everyone talks about visualization, but no one does much about it. Visualization is not a simple matter: it is a deep subject, ... and still not well understood(p171) ... It is not true that we instinctively know how to 'see' any more than we instinctively know how to swim. Visualization is a tool that must be cultivated for careful and intelligent use."(p168).

Several authors have implied, through the context of their articles, that visualization is restricted to a mental activity and that any visualizing occurs in the mind. They have essentially borrowed the *visual imagery* idea from cognitive psychology and in fact several authors have used the terms interchangeably. For example, Goldin(1987) equates visualization with *feeling, imagining*, and therefore confines it to mental imagery. Presmeg(1986) defines a visual image as "a mental scheme depicting visual or spatial information"(p297) and notes that this "allows for the possibility that verbal, numerical or mathematical symbols may be arranged spatially to form the kind of numerical or algebraic imagery sometimes designated 'number forms'." (p297).

Recently, mathematics educators have directed much attention to the issue of technology in the classroom and in the curriculum, and to how that technology can aid visualization, particularly of graphs. This suggests that a description of mathematical visualization should include not only mental imagery but also pictorial imagery produced in more concrete form by pencil and paper, calculator, computer.

Lynn Steen(1990) offers this broader characterization of visualization, one in keeping with the proliferation of copmuterized visual displays:

Visualization ... is one of the most rapidly growing areas of mathematical and scientific research. The first step in data analysis is the visual display of data to search for hidden patterns. Graphs of various types provide visual display of relations and functions; they are widely used throughout science and industry to portray the behavior of one variable ... that is a function of another .... Now computer graphics automate these processes and let us explore as well the projections of shapes in higher-dimensional space. Learning to visualize mathematical patterns enlists the gift of sight as an invaluable ally in mathematics education.(p6).

Without actually defining visualization formally, it is clear from the context that Steen allows both mental imagery as well as more concrete and tangible visual imagery such as tables, diagrams and computer graphics. Zimmerman and Cunningham(1991) provide an even broader and all-encompassing description of mathematical visualization, which they emphasize must include the ability to produce and understand how to *use correctly* such a visual image:

From the perspective of mathematical visualization, the constraint that images must be manipulated mentally, without the aid of pencil and paper, seems artificial. In fact, in mathematical visualization what we are interested in is precisely the student's ability to draw an appropriate diagram (with pencil and paper, or in some cases, with a computer) to represent a mathematical concept or problem and to use the diagram to achieve understanding, and as an aid in problem solving. In mathematics, visualization is not an end in itself but a means toward an end, which is understanding. Notice that, typically, one does not speak about visualizing a *diagram* but visualizing a *concept* or *problem*. To visualize a diagram means simply to form a mental image of the diagram, but to visualize a problem means to understand the problem in terms of a diagram or visual image. Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding.(p3) (italics theirs).

## **Cognitive psychology and mathematical visualization**

The acceptance of the concept of imagery has fluctuated over time in the field of cognitive psychology. Psychologists of the behaviorist school believed that there was no place for any mental constructs in psychology, claiming that any such mental construct was of no value if it could not be observed and thus could not be effectively quantified. Behaviorism demanded the elimination of all mentalistic constructs such as thought, imagery and attention. Another attack came from psychologists in the 1960s who felt that the current (60's) depiction of imagery as synonymous with *pictures* - in the sense that an image had to be completely formed and static, much like a photograph, - was false, and consequently the idea of mental imagery was deeply flawed. Any further progress had to wait until a new way of describing the process was developed. The renewed interest in imagery in the late 60's and early 70's came first from various fields outside psychology, such as medicine, dream analysis and, later, from a renewed interest in the psychology of thinking. With this renewed interest there appeared many new theories purporting to explain the nature and function of imagery.

### **Flexible and static imagery in learning**

One of the most important and influential studies in the development of a theory of the nature of imagery in learning was produced by Paivio(1971). Paivio argued that there had been too much emphasis on verbal processes in the attempts to understand learning, memory and language, and that one should consider imagery as a variable. Paivio considered the process of imagery formation to be a dynamic system, allowing and encouraging flexibility and transformation in processing information. He contrasted this with verbal processes which he considered to be static in nature, simply providing a naming or describing function. He considered visual



imagery processes to be parallel in nature, in contrast to verbal processes which were sequential.

Piaget and Inhelder (1971) furthered the study of the development of imagery, claiming that it becomes increasingly abstract as a child cognitively develops. They distinguished two types of imagery - *reproductive* and *anticipatory*. Reproductive imagery is the representation of objects and/or events which are already known to the imager. Anticipatory images arise in the imagination to represent events which have not been previously perceived. After several experiments, Piaget and Inhelder further refined their idea of reproductive imagery to distinguish between *static reproductive images* and *kinetic images*, involving some motion.

They also refined their notion of anticipatory imagery to include two different stages. The first stage comprised images which could only be reproduced from a transformation which the subject had already experienced in some more concrete form (e.g. transforming an arc into a straight line when the subject had already experienced that done with a piece of wire). The second more advanced stage was one in which the subjects reported imagery of novel transformational products. These could be of two types, one where the new image was simply the kinetic transformation of an already known image, and also a more advanced type where the new image was truly *new* in that it was not a simple transformation of available images. Thus Piaget developed the theory of visual imagery to encompass the possibility of a dynamic process.

This revised position on imagery is reflected in Neisser (1972) where he asserts that the classical theory viewed images as mental pictures and imagery as seeing with the mind's eye, whereas the more modern view is that imagery is an active constructive process rather than a passive product. In fact, several psychologists, including Neisser and Paivio, assert that visual imagery is especially suited to the parallel processing of information, meaning that several different pieces of information could be processed simultaneously, in contrast with the verbal processing mode, which they claim is suited for sequential processing.

Werner and Kaplan(1963) also distinguish between different types of imagery, *imaginal* and *linear* visual representations, saying that imaginal representations are more concrete and primitive whereas linear representations are more abstract. They conclude that imagery is an earlier (genetical) level of representation.

Bruner(1966) describes a similar distinction, differentiating between analytic and intuitive thinking. Analytic thinking he characterizes as being a rational, sequential processing system, while intuitive thinking involves parallel processing of the information as a whole, a Gestalt view. He also considers more abstract, dynamic imagery to be symbolic rather than pictorial. He describes three different stages in the cognitive development of mental representations. The first he describes as *enactive representation*, i.e. knowledge through actions. This occurs through habit and is limited in scope. The second stage he calls *iconic* representation, in which a child is able to represent the world to himself through an image, abstract from, and independent of action. This representation is only slightly more flexible in scope, but still has limitations which do not allow the abstraction of invariant features. The third stage he describes as *symbolic* (verbal) representation, which he claims makes possible a highly abstract form of thinking. This is according to Bruner the most flexible, most desired form of representation. It subsumes both of the earlier types. Bruner regards imagery as a slow and rather unwieldy form of representation. He regards it as concrete and static, not lending itself to transformations. He claims that language is the most efficient way to perform transformations. He feels that imagery is suppressed as language develops. However, it should be emphasized that Bruner allows pictures to be a part of the language of symbols also.

This discussion points to the central question among cognitive psychologists when considering visual representations. What exactly is the nature of visual imagery? This will be discussed in the following section. We will see that there is no single, simple answer.

## The Nature of Visual Imagery

There is much debate among psychologists as to the nature of visual imagery. The two main opposing views differ on how visual imagery is encoded. One view holds that visual imagery is encoded in an abstract propositional format, similar to the way in which verbal information is stored. An account of the propositional theory of imagery may be found in Pylyshyn (1973). The other view is that there is a sharp distinction between the codes used for verbal and visual information, and that visual information is encoded in a pictorial format. For a detailed analysis of these two views see Anderson(1978).

Shepard(1978) notes that, historically, mental imagery has been ignored or at least downplayed by psychologists. He claims, however, that this should not be the case since "visual imagery seems to have played such a central role in the origin of the most creative of my own ideas"(p125) and also those of many scientists, inventors and writers. He writes,

... the fact that an internal representation is more abstract than a picture does not entail that such a representation is nonvisual. So-called 'imageless thought' may constitute just one end of a continuum of representational processes ranging from the most concrete and pictorial to the most abstract and conceptual. And the thinking of those who claim to experience little imagery may simply tend to be less concretely imagistic in this sense.(p130).

He appears to be saying that visual imagery is an all pervasive occurrence, but that an individual may not be aware of his or her use of visual imagery.

Kosslyn(1983) explores whether people use a pictorial or a propositional format to remember something. He concludes that "people can *choose* in many cases whether to use an image or a propositional format to remember something. It appears that whether we think of ourselves as 'mainly visual' or 'mainly verbal', most

of us have the capacity to shift our thinking in the other direction when it is useful to do so.”(p59).

He also addresses the question of how and when images are actually used in thinking and memory, and suggests two main applications. One is as a simulation of a real situation, the other is as a symbolization, which then does not actually depict an object or situation as it really appears, but represents it symbolically. He gives examples which back up his assertion that many problems may be solved using either form, and concludes from his experiments that imagery of either form is likely to be useful in solving problems as long as imagery is faster to use than propositions. Much of the debate seems to center on the question of which method is faster and more efficient.

Kosslyn lays out the four main components of visual thinking: generation, inspection, manipulation, and maintaining of images. He also notes that “much of the power of imagery comes from our ability to modify imaged objects and to see if the changes lead to anything.”(p191). Here, he is emphasizing the importance of the dynamic property of visual imagery. He also notes that “people do not differ simply in terms of ‘general imagery ability’; you are not just good or bad at imagery, but relatively good or bad at a host of separate imagery abilities,”(p200). Thus Kosslyn emphasizes that imagery is a complex set of processes, not a simple discrete process which an individual either possesses or does not possess.

Visualization follows an interesting pattern in cognitive development, according to Kosslyn. Young children are extremely adept at visualizing. In contrast, adults

depend on images mostly to call up facts about relatively unfamiliar properties; but to children, so many things are relatively unfamiliar that imagery may be their primary source of information. Over time, as they have occasion to draw on these depictive memories, they translate the facts they contain into propositional form. After this happens, they still have the option of using an image, but they no longer need it.(p201).

Another contributing factor he cites is that adults are adept at logical deduction, which uses previously stored propositional information, whereas children do not possess that facility.

### Visual representation - Arnheim's theory

The theory proposed by Arnheim(1969) offers several different types of visual representation rather than a simple dichotomy between static and dynamic images, sequential and holistic images, concrete and abstract images, pictorial and symbolic images.

Arnheim(1969) quotes John Locke (p99) regarding visual images. Locke interpreted any visual presence as an obstacle because he assumed that it must be specific, that it could not represent generalities (e.g. a generic triangle). Because of its inherent specificity, he claims that the mind will abandon it as being unhelpful, second best. The question becomes that of whether one could do without images. The idea of imageless thought was considered, and psychologists asked whether one could indeed think effectively without images. The distinction was drawn between conscious imagery and that which was not conscious. Only conscious occurrences could be labelled imagery. This idea fell out of favor, however, and it could no longer be concluded that the absence of conscious imagery meant the absence of imagery altogether.

Arnheim discusses the difference between generic and particular images. In Greek philosophy, mental images were assumed to be faithful replicas of that which they represented, with all the detail of the original objects. As Arnheim points out, however, these images "can serve as material *for* thought, but are unlikely to be a suitable instrument *of* thought."(p104). The mental images needed for thought are incomplete, rather hazy representations of objects and ideas. These memory images provide flexibility precisely by being fluid and dynamic, and by allowing the

thinker to bring into focus in the foreground the pertinent features required for the particular task at hand, while setting undersirable or unnecessary features in the background.

Arnheim distinguishes images on the basis of their relation to their referent, which he describes as their *function* rather than their type. His classification distinguishes among pictures, symbols and signs. He emphasizes that an image may often serve more than one function at the same time, and also that an image most often does not indicate its function(s) explicitly. The following is a short description of the basic differences in function as Arnheim describes them.

An image serves as a *sign*, he claims, when it stands for a particular content without reflecting its characteristics visually (e.g. numerals and verbal languages).

An image is a *picture* when it portrays things located at a *lower* level of abstractness than it is itself. It will render some relevant qualities e.g. shape, color, movement of the referent.

An image acts as a *symbol* when it portrays things which are at a *higher* level of abstractness than the symbol itself. As an example, he offers Holbein's portrait of Henry VIII. It is at the same time a picture of the king, but also a symbol of kingship and its associated abstract qualities such as brutality, strength, exuberance, power, all of which are at a higher level of abstraction than the painting itself. The painting, in contrast, is more abstract than the visual appearance of the king himself.

As regards abstraction in general, Arnheim says "Instead of relying on sensory experience, abstract thinking was supposed to take place in words." (p154). He claims that there exists an extremely harmful dichotomy between abstract and concrete, and the two are not mutually exclusive. He also points out that it is misleading to call concrete things 'physical' and abstract things 'mental'. Arnheim claims that, traditionally, abstraction was thought to be based on generalization. He disagrees, however, and makes the case that abstraction is a necessary precursor to generalization.

### Concluding remarks

This then provides an account of the history and development of visual (mental) imagery among psychologists. All of these accounts, however, suggest several distinct types of imagery. On the one hand we see such terms as linear, static, verbal, sentential, sequential, analytic. On the other hand we see such terms as dynamic, pictorial, holistic, Gestalt, visual, intuitive, kinetic, parallel. The idea of symbolic processing seems to capture aspects of both.

### ADVANCED MATHEMATICAL THOUGHT

In this section, we examine a framework for discussing conceptual understanding in mathematics, and look at how this framework applies to advanced mathematical thinking in particular. One such theoretical framework for discussing mathematical conceptual understanding is provided in the literature by Tall and Vinner(1981), Vinner(1983), and Vinner and Dreyfus(1989). They describe the development of this framework as follows:

The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure logic that gives us insight, nor is it chance that makes us make mistakes ... We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experience of all kinds, changing as the individual meets new stimuli and matures ... As the concept image develops it need not be coherent at all times. The brain does not work that way. Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole. (Tall and Vinner 1981).

For any given mathematical concept, Vinner(1983), and Tall and Vinner(1981) define the term *concept image* as the total cognitive structure associated with a given concept, and suggest that the concept image is a network of all these ideas, which are linked together in some way so that an individual may move between ideas and switch from one part of the concept image to any other. It comprises a mental network of properties, processes, examples, pictures, and any other information associated with the concept in the individual's mind. It is formulated over time as a synthesis of mathematical experiences, changing as new knowledge is assimilated. A concept image can have many different components, and as this image develops it may include conflicting parts. At any one time, however, only a part of the entire concept image may be evoked, and thus any inconsistencies may not be apparent unless conflicting parts of the whole are evoked *simultaneously*.

In this way, Tall and Vinner distinguish between the way an individual may think about a mathematical concept, via the *concept image*, and its formal mathematical definition, defined as the *concept definition*. The former is culturally developed and derives from the fact that mathematics is a human activity and is therefore culturally and contextually dependent. They argue that an individual will often (but not always) possess, concurrently with a concept image, a concept definition, which they describe as a verbal definition which explains the concept, a form of words used to specify the concept. They point out that, despite wishful thinking to the contrary, the concept image is not built from the concept definition in a logical linear fashion, but that, often the concept definition is not even present and yet a rich concept image, complete with examples, notation, visual images, may be present.

There may be two distinct types of concept definition - personal and formal. The personal definition is a mental construct, formulated informally by the individual to suit his/her personal needs. Note however that the personal concept definition may not exist, may be incomplete, or may be inconsistent with other parts of the concept image. On the other hand, a formal concept definition is the



formal definition accepted by the mathematical community at large, often learned in rote fashion by the student. This formal concept definition is relatively free of social influences, and is thus an attempt at structuring mathematics as a formal system. The concept definition, if it exists, may in turn generate its own concept image, referred to as the concept definition image, which may or may not be related in a coherent way to the other parts of the concept image.

Tall and Vinner assert that, in order to work successfully with concepts, an individual needs to have available both a concept image and a concept definition. However, they note that it is often the case that a concept definition remains forgotten and unused and that when one engages in thinking, other parts of the concept image are evoked. Vinner(1983) notes that many secondary and college teachers like to think that students form their concept images from the given formal concept definitions in a one-way process where the formal definition is encountered first, and then the concept image is built from that.

In actuality, this was found to be seldom the case. It is far more likely that the concept definition is developed (when needed) from the starting point of the already existing concept image, or that the concept definition is either missing entirely, or only fragments are present. Even if a concept definition is present it is entirely likely that it will be ignored in any application of the concept, and that only the concept image part of the concept will be evoked. Vinner claims that there is no way to *require* a cognitive structure to work from the concept definition instead of from the concept image, and that most often the flow of the cognitive process is through the concept image, bypassing the concept definition entirely. In this case, the concept definition is not connected to the rest of the network for this concept. They suggest that it is extremely important that, as educators and teachers, we attempt to reveal students' concept images, since this is the only way in which we may find incorrectly formed concept images. In the course of his research, Vinner discovered that students often had formed concept images that were inconsistent with and in conflict with textbook definitions.

It is important to distinguish between the *logical foundations* of a mathematical concept, and the *cognitive foundations* developed when learning about the concept. Mathematics is often taught utilizing a foundational approach in the form of a formal, logical development of a given concept, often beginning with a formal concept definition and proceeding from there with a logical, linear development in the form of definition-theorem-proof. In contrast, a cognitive structure may be developed in a much more non-linear, holistic, experiential fashion. Any student will try hard to link new concepts to existing cognitive structures, and to make some sense of the new ideas by applying her/his existing mathematical experience.

Tall(1992) points out that the most concise, logical development is often not the best way to first introduce and develop a new idea. For this purpose, he introduces the idea of a *cognitive root*, which he describes as being material that has “a dual role of being familiar to the students, and providing the basis for later mathematical development.(p.497)” He claims, however, that cognitive roots are very often hard to find. The archetypal example he provides is that of *local straightness* as being a cognitive root for the concept of derivative. It is intuitively understandable, connects readily to previous experience and thus provides a much better starting point than the traditional mathematical foundation, that of the *limit* concept. This latter concept was developed and honed by many great mathematicians over many years, and while it provides a good logical foundation for derivatives to a well trained mathematician it is not a good cognitive root to provide the cornerstone for the understanding of derivative for beginning calculus students.

It is important to note that throughout Tall and Vinner’s discussion of concept image, the existence of a network structure is alluded to, although the exact nature of such a network is not explained in detail. One’s *mathematical understanding* of a concept is determined as much or more by the structure of the concept image as by the sheer quantity of formal definitions, propositions and facts associated with the concept. That is, the concept image is more than just a *container* of all ideas, examples and pictures associated with a particular concept. It also consists of some

cognitive structure for these contents - links and relations between the elements. How one thinks about and uses mathematics is dictated by this structure. Learning can be defined in terms of changes in the cognitive structure - new contents, new links and relations, reorganization of the structure. Indeed, a mathematical *insight* can be thought of in terms of this structure - when a match is found suddenly between a problem situation or context and a relevant mathematical cognitive structure. Milestones in the development of one's mathematical understanding come from either radical new structures or a radical reorganization of existing structures.

### **Cognitive models and mathematics - Idealized cognitive models**

Students in the same class are exposed to the same content material and the same presentation, but they may structure the ideas presented to them in very different ways in their concept image. Large variations have been observed in the concept images of students in such similar circumstances by Presmeg(1992), who provides an explanation for these large variations by indicating that the visual pictorial component of mathematics plays a central role in learning, rather than a secondary, peripheral role. This is consistent with the views of proponents of the experientialist view of reason, such as Lakoff, and Johnson. Lakoff(1987) provides a possible structure for a concept image through his discussion of imaginative structures employed in reason which he refers to as *idealized cognitive models*, (ICM's), which are structured by image schemas, "recurring dynamic pattern(s) of our perceptual interactions and motor programs that give coherence and structure to our experience."(p xiv). We will examine this general viewpoint first, and then return to Presmeg's analysis.

Johnson(1987) makes a case for the centrality of imagination in any theory of meaning and rationality: "Without imagination, nothing in the world could be meaningful. Without imagination, we could never make sense of our experience.

Without imagination, we could never reason toward knowledge of reality.”(pix). One of the most widespread manifestations of imagination in reason is the occurrence of the use of metaphors. These are used to link abstract ideas and concepts to individual experiences in one’s physical world. The use of metaphor provides a *mental hook* on which to hang very abstract ideas and to endow them with some mental imagery that facilitates one’s recall and understanding of these otherwise very intangible concepts. According to Johnson: “There is a growing body of evidence that metaphor is a pervasive, irreducible, imaginative structure of human understanding that influences the nature of meaning and constrains our rational inferences.”(pxii).

To describe these embodied, imaginal aspects of reasoning Johnson introduces two imaginative structures which he refers to as *image schemas* and *metaphoric projections*. He defines an image schema as “a recurring, dynamic pattern of our perceptual interactions and motor programs that gives coherence and structure to our experience”(pxiv). It is important to note the flexibility and non-static nature of this structure. This is what distinguishes an image schema from other more fixed, concrete images which are more readily rendered on paper. The same is true of metaphoric structures. Since any such structure is based on one’s experience of one’s world, which is dynamic, fluid, and constantly changing, it is clear that the metaphor structure is also constantly being adjusted to reflect one’s experiences. This related structure of metaphor he describes as “a pervasive mode of understanding by which we project patterns from one domain of experience in order to structure another domain of a different kind. . . . Through metaphor, we make use of patterns that obtain in our physical experience to organize our more abstract understanding.”(pp.xiv-xv).

Johnson describes image schemas as gestalt structures that consist of various parts that stand in relations to one another and make up a unified whole. These gestalt structures play a central role in the way in which we make sense of and find order in our experience. Mathematics is part of our experience, and these struc-

tures play a central role in our understanding of mathematics. They determine how we make sense of mathematical concepts, and how we understand abstract mathematical ideas. We map patterns in our everyday experience onto the mathematical domain in order to understand, to organize, to see patterns in, and to make connections among our mathematical ideas. These structures are different from any photograph-like mental picture of a more concrete type that we may form. Johnson makes an important distinction between image schemas and what he calls rich concrete images. "An image schema is a dynamic pattern that functions somewhat like the abstract structure of an image, and thereby connects up a vast range of different experiences that manifest this same recurring structure."(p2). Image schemas are more general than specific rich, concrete images or mental pictures, and have this very unique and important dynamic quality, and for these reasons are extremely useful and powerful in the reasoning process, especially in mathematics.

Image schemata exist at a level of generality and abstraction that allows them to serve repeatedly as identifying patterns in an indefinitely large number of experiences, perceptions, and image formations for objects or events that are similarly structured in the relevant ways. Their most important feature is that they have a few basic elements or components that are related by definite structures, and yet have a certain flexibility. As a result of this simple structure, they are a chief means for achieving order in our experience so that we can comprehend and reason about it.(p28).

Take, for example, an imaginal structure that one might use to represent a triangle. It has been argued that it is not possible to represent such an abstract concept in a specific image, since there are many different types of triangles, and that one specific image will not suffice to represent all triangles. It is exactly the dynamic, flexible nature of an image schema that makes it appropriate for this purpose. Kant(1965) had realized the existence of such schemas for representing the concept of triangle when he wrote "No image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the

concept which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere. The schema of the triangle can exist nowhere but in thought.”(p24). Johnson places the level of abstraction at which image schemas operate as a mental process on a continuum of abstractness between abstract propositional structures and particular concrete images:

On the one hand, [image schemas] are not propositions that specify abstract relations between symbols and objective reality . . . On the other hand, they do not have the specificity of rich mental images or mental pictures. They operate at one level of generality and abstraction above concrete, rich images. A schema consists of a small number of parts and relations, by virtue of which it can structure indefinitely many perceptions, images, and events. In sum, image schemata operate at a level of mental organization that falls between abstract propositional structures, on the one side, and particular concrete images, on the other.(pp 28-29).

Lakoff(1987) uses the term *idealized cognitive model* or ICM, to describe any of four major structuring principles used to form collections of objects. He sees the use of these cognitive structuring models as central to an experiential and imaginative interpretation of the way in which we reason. He contrasts his view of reason as embodied, experiential and imaginative, with the traditional, objectivist view of reason, which is to accept a propositional explanation of thought processes, as follows:

On the traditional [objectivist] view, reason is abstract and disembodied. On the new view, reason has a bodily basis. The traditional view sees reason as literal, as primarily about propositions that can be objectively either true or false. The new view takes imaginative aspects of reason - metaphor, metonymy, and mental imagery - as central to reason, rather than as a peripheral and inconsequential adjunct to the literal. (p. xi).

He goes on to point out that the traditional objectivist view “consists of the manipulation of abstract symbols ... and *all* rational thought involves the manipulation of abstract symbols which are given meaning only via conventional correspondences with things in the external world.”(p. xii). This traditional view is basically a *mind-as-machine* approach to reason and learning. In contrast, however, study of conceptual categories has shown that thought in general is embodied, imaginative, and has gestalt properties, as opposed to being merely the mechanical manipulation of abstract symbols which are internal representations of external reality. Lakoff explains that the concept of categories is essential to the understanding of thought processes since much of our mental activity involves the use of categories, by which he means collections or associations of objects having certain common properties. Lakoff discusses the way the mind uses ICM’s of different types to structure categories. He describes an ICM as “a complex structured whole, a gestalt, which uses four kinds of structuring principles”(p.68). He emphasizes that these four principles correspond to four types of cognitive models used to characterize categories:

- *Propositional models*
- *Image-schematic models*
- *Metaphoric models*
- *Metonymic models*

Categories are the result of the imposition of these structures on the mental space. We will describe each model in detail, and provide examples of how each of these structures may be found in advanced mathematical reasoning, but first we set the stage by considering the collection of different number systems used in mathematics. By examining the various ways in which one may understand different number systems - the complex numbers, Gaussian integers, real numbers, rational numbers, integers and natural numbers - we will be able to illustrate a variety of schemas, depending on the mathematical focus. It may be of interest to emphasize their topological properties, or their cardinalities, or their algebraic structures, or even to consider their different subsets and their relations to one another.

First, we illustrate the propositional model which describes the relationships holding among the various types of number systems used in mathematics.

### **Propositional models.**

Lakoff describes a propositional model as an ICM that does not use any imaginary devices of any kind. This is in contrast to the other three - metonymic models, metaphoric models or image schematic models - all of whose use entails the employment of mental imagery of some kind. A propositional model, at least in a mathematical setting, consists of elements which are words and/or mathematical shorthand symbols, together with a structure which consists of the properties of, or a hypothesis or set of hypotheses which characterize, the elements and the relations holding among them. The type of structure determines the type of propositional model: simple proposition, feature bundle structure, taxonomic structure, radial category structure, generative category structure. We will give examples of these in a mathematical setting.

#### • *Simple proposition.*

Any simple mathematical proposition of the form  $p \Rightarrow q$  is an example here. The structure comprises a typical logical propositional sentence, consisting of words and/or abstract symbols. For example, the statement 'any real number is a complex number' is a simple propositional model.

#### • *Feature bundle structure.*

A mathematical feature bundle structure would be perhaps a list of defining properties for a mathematical concept. This list would consist solely of words and/or symbols with no pictorial or visual imagery of any kind associated with it. For example, consider a collection of vector spaces. The hypotheses or properties of a vector space are what determine whether a given mathematical object is a vector space. This list of defining properties (features) are what determines whether or not



a given mathematical object is a vector space. This is a relatively simple *in-or-out* type of determination, achieved by merely checking off properties against the list. In this way, one can build a collection of vector space examples (and counter-examples) to hold in the concept image.

• *Taxonomic structure.*

In this structure there exists a hierarchy on the features (or hypotheses) characterizing the collection. Depending on how far up the check-off list one is able to advance when classifying an object, that object will be placed at a lower or a higher level. This is no longer a simple *in-or-out* categorical procedure, but more a checklist to determine the *level* at which the object belongs in the hierarchy. For example, take the collection of algebraic structures which are either groups, rings, integral domains or fields. We could first check off the group axioms, then for any objects qualifying as a group, we could check off the additional axioms necessary for characterization as a ring. For any such qualifying objects, we could next check off the additional axioms needed for characterization as an integral domain, and so on. In this way, we build up a hierarchical or taxonomic structure for the category of such algebraic structures. For example, note that

$$\{\text{groups}\} \supset \{\text{rings}\} \supset \{\text{integral domains}\} \supset \{\text{fields}\}.$$

• *Radial category structure.*

This structure is characterized by having a *central* or main list of axioms, against which one must initially check the object in order to establish a general membership in the category. Then one checks for additional defining features, which would serve to further classify the object as a specific type of element in the given category. There is no hierarchy on the types of objects, but rather a radial structure - each type of object is one additional step from the main object classification. For example, this type of structure may well categorize a student's collection of various types of groups. In this case group axioms form the initial defining properties. Once the determination has been made that the object is a group, then one checks for additional features, which in this case might be number of elements, commutativity,

cyclic property, symmetry and so on. In this way groups may be classified into different types, all of them radiating outward from the central group classification.

• *Generative category structure.*

This is a structure wherein there is a clear *best example*, from which all other objects in the category may be derived following a prescribed rule or set of rules. In this way, for example, the category of finite cyclic groups may be understood by understanding  $Z_n$ , the 'best example' of a finite cyclic group. The set of rules used to describe other examples in this case would be those of containing a generator  $x$  which satisfies the equation  $x^n = e$  for some positive integer  $n$ , and  $x^m \neq e$  for any  $0 < m < n$ .

In many cases, however, although these propositional structures are valid structures in and of themselves, it is important to note that such structures are often difficult to understand, and are devoid of any imagery that might aid in such understanding. Consider the simple proposition 'Every natural number is an integer, every integer is a rational number, every rational number is a real number, every real number is a complex number.' In this case, although this propositional structure is a valid structure in and of itself, it is important to note that such structures are often difficult to understand, and are devoid of any imagery that might aid in such understanding. They are given additional meaning by the use of the other three structuring principles, metaphoric models, metonymic models and image schemas, which serve as an aid in retaining a mental image of a propositional model. These models lend a gestalt nature to the structure of the category, and may provide additional understanding of the category.

### **Image schemas**

Lakoff(p283) maintains that, in general, categories that are structured by *image schemas* may be understood in terms of *container* schemas of various types:

- *Part-whole and up-down schemas* which provide hierarchical structure,
- *Link schemas* which provide relational structure,
- *Center-periphery schemas* which provide radial structure,
- *Front-back schemas* which provide foreground-background schemas, and
- *Linear order and up-down schemas* which provide linear quantity scales.

We will look at some examples of the structure that these different image schemas provide in a mathematical context, continuing with our example of the collection of number systems. Although a propositional model accurately describes any relationships that hold, for most people it does not provide a simple, convincing schematic of these relationships. This is where image schemas come into play. They provide a structure which facilitates the understanding of the propositional model, providing a wholistic model which captures the salient features diagrammatically. Some examples follow.

- *Part whole schema*

A part-whole schema may be used, in the following way, to describe the relationship among the sets of numbers.

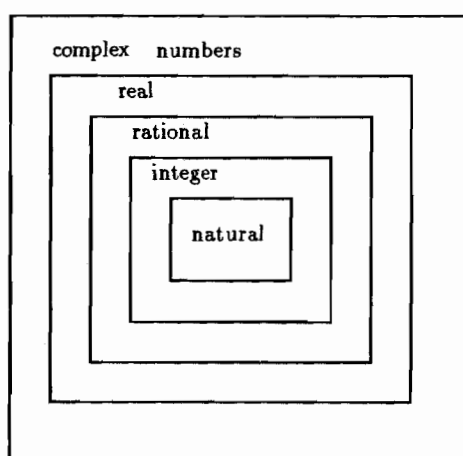


Figure 1. Part whole schema

Alternatively, a link schema or linear order schema may be used to indicate a succession of related items:

- *Link schema*, or *Linear order schema*

Such a schema is often used in mathematics to illustrate visually the linear ordering among the sets of numbers.

$$N \subset Z \subset Q \subset R \subset C$$

A linear order schema may also be used in a variety of ways to describe the relationship among the numbers of the number set  $N$  itself. For example:

$$N = \{1, 2, 3, \dots\}$$

Or,

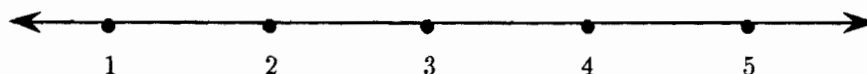


Figure 2. Linear order schema

Note, also, that this a linear order schema was actually employed earlier in my note following the taxonomically structured propositional model describing the classification of algebraic structures into groups, rings, integral domains and fields. I purposely left that illustration in place (even though it is an image schema) to illustrate the widespread and pervasive use of image schemas in any description of a mathematical concept or process. It is almost unavoidable.

- *Center-periphery schema*

Different types of subsets of the natural numbers with certain additional properties may be illustrated using a center-periphery image schema:

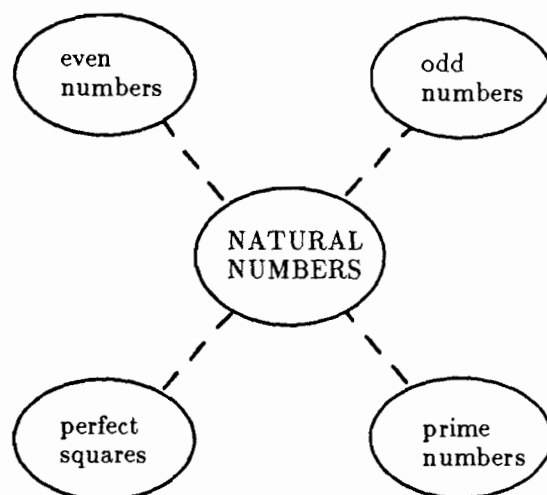


Figure 3. Center periphery schema

Or, if one wanted to emphasize the fact that all these number systems are subsets of the complex numbers, and their position relative to one another was unimportant, then perhaps a center-periphery model may be used with the complex numbers forming the central object.

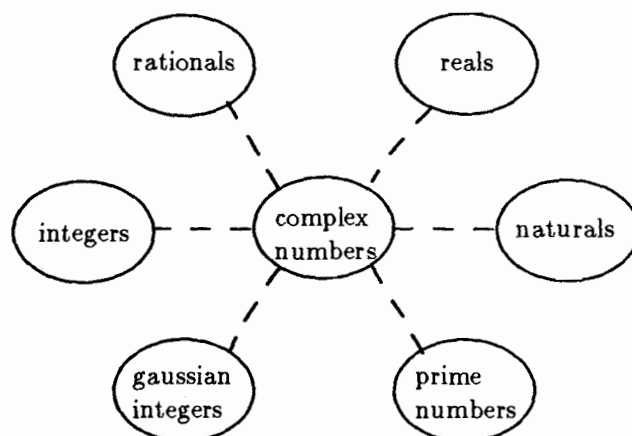


Figure 4. Another Center periphery schema

- *Front-back schema*

This type of schema may be used on the same collection of number systems to emphasize one particular system, or one property of a number system. For example, when considering the algebraic structure of the number systems, suppose that one wants to bring to the foreground an example of a ring - the Gaussian integers - while at the same time maintaining the relationship with other members of the collection. The front-back schema has the effect of pulling the desired object to the foreground, while keeping the others in the background, readily accessible if called upon, but out of the way so that they do not cloud the picture.

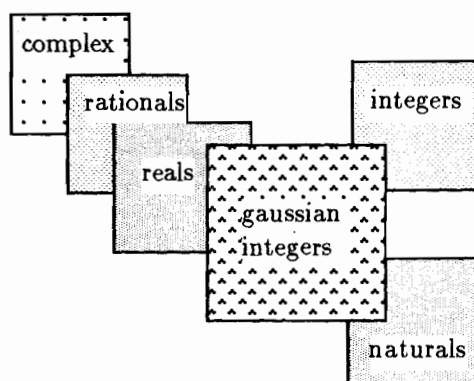


Figure 5. Front back schema

Thus, it is helpful to think of these sets of numbers in various ways, depending on the context. One needs not only the ability to decide on and evoke the most appropriate model, but one first must have available a variety of image schematic models of any given mathematical concept or structure, in order that a choice may be made.

## Metaphoric models

A metaphoric model involves a mapping from either a propositional or image schematic model in one domain to a corresponding structure in another domain. In other words, there is a function (*stands for*) associating two domains, the mathematical domain, and some other domain. This process allows for greater understanding of *new* ideas and concepts by relating them to existing ideas and concepts albeit in another domain of experience. A well-known example of a mathematical metaphoric model would be the mapping associating the model of falling dominoes with the principle of mathematical induction. These are two very different domains and yet the important mathematical principle of induction can be well understood by considering the properties of a series of upright dominoes, which continue to fall after the first domino falls. A model of this type provides a way in which new abstract mathematical principles may be understood in terms of some concept which is already very familiar and well-understood. It provides a method for connecting new mathematical knowledge with existing cognitive structures in other domains. It is therefore a powerful and effective cognitive model.

## Metonymic models

Lakoff defines a metonymic model as “a situation in which some subcategory or member or submodel is used (often for some limited and immediate purpose) to comprehend the category as a whole ... where some well-understood or easy-to-perceive aspect of something is used to stand for something else.”(pp77-79).

Prototype effects are the product of a metonymic model. In both propositional models and image schematic models, prototype effects may be found. For example, in a radial category structure, the center may be the *best example*, or metonymic model, whereas in a generative category structure, the generator acts in this way.

These prototypical elements occur frequently in advanced mathematical thinking, and play an important role in mathematical understanding. An example of this effect might be the tendency to think of the positive integers when considering the mathematical object defined as a well-ordered set. For many, the set of positive integers embodies all the salient features of a well-ordered set and thus may be thought of metonymically as *the* prototypical example of a well-ordered set.

Lakoff makes the following very important points about the effect this new understanding of reason will have on teaching and learning:

If we understand reason as embodied, then we will want to understand the relationship between the mind and the body and to find out how to cultivate the embodied aspects of reason. If we fully appreciate the role of the imaginative aspects of reason, we will give them full value, investigate them more thoroughly, and provide better education in using them. Our ideas about what people can learn and should be learning, as well as what they should be doing with what they learn, depend on our concept of learning itself. It is important that we have discovered that learning for the most part is neither rote learning nor the learning of mechanical procedures. It is important that we have discovered that rational thought goes well beyond the literal and the mechanical. (p. xvi).

### **Mathematics as a Cognitive Activity**

Lakoff devotes an entire chapter in his book *Women, Fire and Dangerous Things* (1991), to the discussion of mathematics as a cognitive activity, where it is very much the science of categorization, and that it is based on human rationality and comes from bodily experience, rather than transcending experience as in the Platonic view, where mathematics is perceived as “a unique body of absolute truths that hold of a timeless realm of mathematical objects, independent of the understandings of any beings” (p354). He claims rather that mathematics “is the study of structures that we use to understand and reason about our experience - structures



that are inherent in our preconceptual bodily experience and that we make abstract via metaphor”(p354-355). This agrees with the view of mathematics offered by the mathematician Saunders MacLane in his book *Form and Function*(1986), where MacLane points out that the set theoretic foundations of mathematics are not sufficient to explain the differences among the various branches of mathematics. He suggests that one must consider not only mathematics *form*, but also the *function* of mathematics in human activity. He proposes that the different branches of mathematics have arisen from different human activities and he makes the following connections between human activities and branches of mathematics(p463):

Table 1. Human activities and mathematics

Human Activities	Branches of Mathematics
counting	arithmetic and number theory
measuring	real numbers, calculus, analysis
shaping	geometry, topology
forming (eg. architecture)	symmetry, group theory
estimating	probability, measure theory, statistics
moving	mechanics, calculus, dynamics
calculating	algebra, numerical analysis
proving	logic
puzzling	combinatorics, number theory
grouping	set theory, combinatorics

“The real nature of these structures does not lie in their often artificial construction from set theory, but in their relation to simple mathematical ideas or to basic

human activities. . . mathematics is not the study of intangible Platonic worlds, but of tangible formal systems which have arisen from real human activities." (p.470, MacLane).

Lakoff suggests that what MacLane points to as *basic ideas* that recur across different branches of mathematics are similar to what he calls image schemas.

Presmeg(1992) borrows some of the terminology of Lakoff and Johnson, and applies their concept of categories and ICM's, and in particular, prototypes, metaphors and metonyms, to discuss her findings in her study of mathematical *visualizers*. She defines visualizers as students who prefer to use visualization to solve problems where visualization may be used, but where it is not essential to use it. She found that examples of all of these types of imagery abounded in her students. Presmeg also points out that Johnson's *image schematic structures* are different from what she refers to as *pattern imagery*.

The two constructs - that is, image schemata and pattern imagery - are not the same. Johnson was careful to distinguish image schemata from propositional formulations and from what he called rich concrete images or mental pictures: "*image schemata are not rich concrete images or mental pictures, either. They are structures that organize our mental representations at a level more general and abstract than that at which we form particular mental images*(Johnson, pp23-24, his emphasis)"(Presmeg, p.603).

She gives some examples of the use of prototypes, metaphors and metonymies in a mathematical setting. One student carried a powerful metonymic or prototypical model for a right triangle, where he found it necessary to redraw his picture to conform to his preferred orientation, namely with the right angle in the lower right corner of the triangle. He was unable to work on the problem unless this step was carried out. Another student related an elaborate metaphor about a ship sailing on the water that she used in order to remember the signs of trigonometric functions in different quadrants of the plane. A third student had difficulty with a problem where he had to find a parabola which opened downward but which had an axis of

symmetry other than the  $y$ -axis. His metonymic or prototypical example of such a downward opening parabola was of one that was symmetric about the  $y$ -axis. From these examples it is clear that while flexible visual imagery may be central to much mathematical reasoning, static or restrictive imagery may be a source of difficulty in mathematical reasoning. This is not to discourage the use of imagery, but rather to encourage its more frequent use, so that students may become more familiar with its flexible nature, and build a more varied concept image with a wider range of possible examples, prototypes and metonymies to choose from.

This approach to mathematical reasoning, in an embodied, experiential way makes a great deal of sense to many students of mathematics when they reflect on their mathematical experiences. Many individuals find it easy to retain knowledge if they can *own* it in the sense of being able to fit it into their existing experiential mathematical structure, whereas on the other hand, they find it almost impossible to retain any mathematics that they are not able to fit into their existing experience. Although many found rote learning easier at a younger age, it may now seem so pointless as they get older, that they are unable to do it. And, if they have no pre-existing cognitive models structured by image schemas within which to interpret the new material, it simply is not retained. If mathematics were truly a disembodied, transcendental truth then any new mathematics could be easily learned by rote addition of new propositional models to the existing mathematics category. This is not the case. To make sense of mathematics it is necessary to interpret it in an experiential way, personalizing it, individually tailoring it.

It therefore makes sense to investigate the mathematical experiences of mathematics students and professors, and to attempt to document their various and varying image schemas for understanding mathematical concepts. Although it is to be expected, from this viewpoint, that there will be many different experiences to recount, at the same time there may well emerge some striking similarities that are common experiences to all.

### CHAPTER III CHARACTERISTICS OF ADVANCED MATHEMATICAL THINKING

In my research, I am primarily interested in characterizing the cognitive processes of advanced mathematical thinking. By advanced mathematical thinking I mean the activity of someone who has come beyond mathematics as the science of numbers and algorithms, into the realm of definition, theorem and proof. Someone who has an understanding of the logical development of mathematical ideas, and an awareness of and appreciation of the need for theorems and proofs. Someone who, as Tall(1991) describes, is preparing to “participate in the full cycle of activity in advanced mathematical thinking, from the creative act of considering a problem context in mathematical research that leads to the creative formulation of conjectures and on to the final stage of refinement and proof.”(p3). Someone who is active in the process of mathematical thinking, rather than just passive in the acquisition of the product of mathematical thought.

As Tall points out, the processes of *generalization* and *abstraction* are important distinguishing processes in advanced mathematical thinking. These processes are usually not present at more elementary levels, or if present, are not essential. A generalization of a mathematical process entails an extension of a familiar process, whereas an abstraction requires a reorganization of existing mental structures. In short, the differences may be summarized in the following example: The definition of  $R^n$  is a generalization of  $R^2$ , whereas the process of defining a vector space  $V$  over a field  $F$  is an abstraction of the concept of the vector space  $R^2$ , or even, for that matter,  $R^n$ , which involves a reformulation and reorganization of ideas.

Tall also discusses the importance of intuition in advanced mathematical thinking, which is also emphasized by Fischbein (see later). He takes issue with claims that mathematicians often regard intuition and rigor as mutually exclusive. He sees intuition as

the product of the concept images of the individual. The more educated the individual in logical thinking, the more likely the individual's concept imagery will resonate with a logical response... [A]spects of logic too can be honed to become more 'intuitive' to the mathematical mind. The development of this refined logical intuition should be one of the major aims of more advanced mathematical thinking.(p14)

Dreyfus(1991) emphasizes the importance of *reflection* in advanced mathematical thinking. This is a process that is just not present in elementary mathematics: "We would not usually expect an elementary math student to stop, after having solved a problem, and think or recount how he went about solving this problem"(p25). There are, however, many processes interwoven to form the complex cognitive activities associated with advanced mathematical thinking. Dreyfus provides a (not exhaustive) list of such processes: representing, visualizing, generalizing, classifying, conjecturing, inducing, analyzing, synthesizing, abstracting and formalizing. He points out that although these are often not present in elementary mathematics,

there is no sharp distinction between many of the processes of elementary and advanced mathematical thinking... Many [of these processes] are not *exclusively* used in advanced mathematics, nor indeed are they exclusive to mathematics... However, we will describe the processes as they are relevant for advanced mathematical thinking, focussing on those processes whose characteristics make the mathematical thinking advanced.(p26).

The major distinction between elementary and advanced mathematical thinking is the complexity of these processes and the way in which this complexity is managed. Dreyfus claims that the main tools for such management are the processes of *abstracting* and *representing*. By such means one can move among different levels of complexity. What makes advanced mathematical thinking interesting and relevant for understanding learning and thinking are the close links between its mathematical

and psychological aspects. Dreyfus investigates the processes involved in representation at length, discussing the use of *symbolic representations*, which may or may not have any intrinsic meaning, or may simply abstractly represent the concept, without any resemblance to it. He also discusses *mental representations* such as examples and visual images.

As Dreyfus points out, mathematicians are often unaware of the extent of their use of different processes in their thinking, and therefore are not aware of the complexity involved in learning advanced mathematics. Little has been written about the cognitive aspects of advanced mathematical thinking, the exception in the past being Hadamard (1945), who emphasizes the importance of the use of informal reasoning, thinking in the absence of words, visual imagery, mental images and playing around with ideas. Few of these tools however are used by mathematicians when teaching mathematics.

Dreyfus points out that each individual has her/his own mental representation for any mathematical concept or process as it is being discussed. These mental representations are very personalized, individual, and widely varied. "Although most mathematicians can be expected to come up with roughly equivalent definitions, of say, a function, their respective mental representations of the notion may be vastly different. Have you ever asked mathematicians working in different areas what comes to their mind when they think about functions?"(p31). To make matters even more complex, in any given individual several mental representations of the same concept may be present and complement each other. They may eventually be integrated into a single representation of the concept. This process of integrating different representations into a cohesive whole is related to abstraction. To be a successful mathematician one must have a rich and varied set of mental representations for mathematical concepts. To be of use, these representations must be linked together and the individual must have the flexibility to move among these different representations. Thus a rich set of differing representations is not necessarily detrimental to advanced mathematical thinking. On the contrary, as long as one has the

facility to move among different representations one has a large toolkit with which to approach different mathematical situations.

One other extremely important process in advanced mathematical thinking discussed by Dreyfus is *abstracting*. Dreyfus states that the process of abstracting is first and foremost a constructive process involving building complex mental structures from (possibly complex) mathematical structures. This process of abstraction has two prerequisite processes in addition to representing: *generalizing* and *synthesizing*. In Dreyfus' words, "to generalize is to derive or induce from particulars, to identify commonalities, to expand domains of validity. . . . To synthesize means to combine or compose parts in such a way that they form a whole, an entity. This whole often amounts to more than the sum of its parts."(p35). These processes are both necessary in order for an individual to succeed at the complex task of abstracting, which in turn is a vital component of advanced mathematical thinking.

## VISUALIZATION IN ADVANCED MATHEMATICAL THINKING

We now look at the role that visualization plays in advanced mathematical thinking. It is however difficult to isolate those properties or processes that characterize visual processes in advanced mathematical thinking.

### **Visual reasoning and intuition in mathematics**

Schmalz (1988) developed what is described as a classical view of the way creative ideas in mathematics are attained by an individual. She describes how one might build up a mental representation of a problem and/or any relevant information. Schmalz is especially interested in intuition, defining it as being "... that faculty of the mind for which comprehension is spontaneous and immediate as op-

posed to rational and linear, and very often, though not always, sudden" (p34). She follows that with a disclaimer that something as fundamental as intuition is better understood by description rather than by definition, and she goes on to give accounts of the experiences of intuition by several famous mathematicians, including Henri Poincaré, Paul Halmos, and Philip J. Davis.

Fischbein, Tirosh and Melamed(1981) point out the importance of understanding and acknowledging one's own natural intuitive biases, since these affect the formation of one's concepts, one's interpretations, one's ability to understand. For example, the authors claim that "it is intuitively natural to affirm that the points of a line segment and those of an (infinite ) line can be put into one-to-one correspondence *because both are infinite sets*. It is also natural to affirm that the sets of points of a line segment and those of an (infinite) line *cannot be put into one-to-one correspondence because the segment is limited and the line is infinite.*"(pp 503-504). They claim that students of mathematics find it much easier to accept and retain interpretations that do not conflict with these natural intuitive biases, and to reject or change those that do.

Fischbein(1987) discusses intuition in great detail. He lists several aspects characteristic of intuitive understanding, and asserts that the main factor contributing to the immediacy of intuition is visualization. He goes so far as to say that intuitive knowledge and visual representation are often taken to mean the same thing. He claims that one tends to think naturally in terms of visual images and "that what one cannot imagine visually is difficult to realize mentally"(p103). He notes that Poincaré referred to intuitive mathematicians as *geometers*, and that Hilbert noted the importance of images in mathematical thought. However, Fischbein asserts that visual representations may not be equated with intuitive knowledge, since immediacy is not a sufficient condition for producing an intuitive cognition. However, "visualization embedded *in an adequate cognitive activity* remains an essential factor contributing to an intuitive understanding"(p103). He also notes that visualization is an important factor in globalization, and in their concreteness these visual im-



ages help create a feeling of self-evidence and immediacy. "A visual image not only organizes the data at hand . . . but it is also an important factor guiding the *analytical* development of a solution; visual representations are an essential anticipatory device"(p104). He emphasizes the fact that visualization is more than just *seeing* mentally, more than just a static image, but that it involves motion and change. It involves a dynamic constructive representation, and provides something that may be mentally manipulated. "In the . . . example [of the scientist Kekulé discovering the structure of the benzene molecule by experiencing a visual image of a snake taking hold of its own tail] the scientist was playing with the image, with the chain of atoms"(p105). In classifying types of models for shaping acceptable cognition, Fischbein makes a preliminary distinction between abstract and intuitive models. Under this classification he states that mathematical relations (e.g. formulas, functions) are usually abstract models of concrete reality (e.g.  $s = \frac{1}{2}at^2$ , is an abstract model for accelerated motion). On the other hand, an intuitive model is of a sensory nature (e.g. representations of vectorial magnitudes by directed line segments, tree diagrams used in combinatorial problem solving, and graphs of functions).

The progression for the formulation of an intuitive model is captured in the following process, as one might proceed from concrete reality, to abstract model, to intuitive model. For example, take the concrete reality of a falling body. One may well construct from this the abstract model of a quadratic function, and then progress to the intuitive model of the graph of the function, which Fischbein describes as a visual behavioral representation of the dynamic relation between the variables involved.

Fischbein refines this classification of intuitive models into three different categories - *analogic*, *paradigmatic*, and *diagrammatic* models. (Note that the graph constructed in the above example would be classified as a diagrammatic model.)

Fischbein defines an analogy and an analogic model as follows: "two entities are considered to be in a relation of *analogy* if there are some systematic similarities between them, which would entitle a person to assume the existence of other

similarities as well. In the case of analogic models, the model and the original belong to two distinct conceptual systems" (p122). He provides examples of analogic models: the description of electric current by analogy with that of the flow of a liquid through a very fine tube, and also the description of the structure of the atom by Rutherford by analogy with the model of planetary motion. Notice that in each case, the systems are two conceptually distinct classes: in the first case, electric current and fluid flow, and in the second case, atomic structure and planetary motion.

He further refines the idea of an analogic model by subdividing into three distinct categories:

1. Both model and original do not use explicit intuitive means, but only a numerical algebraic symbolism. e.g. operations with imaginary numbers, defined by analogy with the reals, operations with transfinite numbers, defined by analogy with finite cardinals.
2. One term in the analogy is an intuitive, usually geometric representation, whereas the second term is a symbolic expression. e.g. the geometric interpretation of functions based on an isomorphism of numbers and figures, as between the algebraic expression for a function and its graphical representation in the Cartesian plane.
3. The model is extramathematical, often a concrete material representation of mathematical concepts, e.g. Dienes or Cuisenaire rods as analogies for numbers and operations with them, spots for points, sets of spots for numbers.

Fischbein describes a paradigmatic model, on the other hand, as being more than just an example, but rather a representative of a whole class, a prototype, through which one could see the entire concept. All the salient features and properties of any member of the entire class are represented by this prototypical example. He refers to a paradigmatic or prototypical model as an exemplar. As an example he employs the prototypical example of the tangent to a circle to represent *any* tangent line. This prototype is used by calculus students in forming the concept

of derivative as slope of the tangent line. This often leads to misconceptions since this particular prototype is too restrictive in that it allows for the tangent line to intersect the curve at one point only.

The third category of intuitive model is that of a diagrammatic model, which, in mathematics, is most often some sort of graphical representation. As Fischbein describes it:

diagrammatic models are graphical representations of phenomena and relationships amongst them. . . . A diagram possesses important intuitive features. Firstly, it offers a synoptic, global representation of a structure or a process and this contributes to the globality and the immediacy of the understanding. Secondly, a diagram is an ideal tool for bridging between a conceptual interpretation and the practical expression of a certain reality. A diagram is a synthesis between these two apparently opposed types of representation - the symbolic and the iconic.(p154).

To distinguish diagrammatic models from other models, Fischbein points out that the original system in the model exists in its own right, while the other system, the diagram, is an artificial construct, intentionally created to model the original system. The symbolism used in creating a diagram is not intuitively understood, and does not stand alone. The language and conventions needed for interpreting diagrams need to be learned. They are not self-evident since diagrams are not generally the direct image of a certain reality, but contain layers of implicit information that require experience to extract. Graphs are the most important examples of diagrammatic models in mathematics, and as such possess certain inherent properties, but they are not good intuitive devices unless one has the necessary interpretive skills to extract the information. Consider the impossibility of finding the roots of a polynomial from its graph without knowing that these occur precisely at those points where the graph intersects the  $x$ -axis, or of finding the solutions to a system of equations or inequalities without knowing how to interpret these from the graph.

A graph with its figural properties very often has the properties of a Gestalt. It imposes itself on the learner as a *figure*, in the Gestalt sense, as a structured, directly interpretable reality. For that reason it should represent an excellent intuitive device. In fact, a graph is not, by itself, generally an intuitive device. Like other types of diagrams the graph is neither an example nor an analogy in respect to the phenomenon to be represented. . . . the relation between the graph and the original is indirect, it takes place through an intervening conceptual structure. A graph may become an intuitive device only *after* the system of conventions relating the original reality, the intervening conceptual system (the function) and the graphical representation have been internalized and automatized.(p160).

This discussion of diagrammatic models helps to explain why students often have such trouble with graphs, and often find them to be such complicated constructs, while at the same time those who have internalized and automatized the interpreting conventions wonder what all the fuss is about.

These three types of intuitive models compare loosely with the types of idealized cognitive models (ICM's) suggested by Lakoff and Johnson. Fischbein's analogic model compares with Lakoff's metaphoric model, and Fischbein's paradigmatic model compares with Lakoff's metonymic model. However, Fischbein's diagrammatic model does not match Lakoff's propositional model, in part because the propositional model does not use any imaginary devices. It is different in the sense that a diagram may be considered to be a stripped down, uncluttered version of the actual situation, with unnecessary features removed, and only the most salient, pertinent features retained. Thus a diagram may be construed as being representative of a more complicated situation. We should keep in mind that Lakoff discusses ICM's in a more general setting. This diagrammatic model may be unique to mathematics (and physics), in that it is a formal structure designed for the purpose of providing a representation other than by analogy. We will consider this diagrammatic model to be a separate model, special to mathematics. These three types of intuitive models are all highly visual in nature. They provide us with a convenient

framework for classification of the types of visual information processing used by the advanced mathematical thinkers in our study.

Dreyfus discusses the use of *process* and *object* in formulating a framework for the acquisition of mathematical knowledge. Mathematics deals with many things, numbers, variables, functions and so forth, which may be considered as objects, related to each other by various structures. Processes are operations on objects which transform these objects into different objects. At the same time, the associated underlying structure may or may not be preserved. For example, a function from  $R$  to  $R$  may be considered as a process which transforms any real number into another, possibly different, real number. The structural relationship among the real numbers, in this case  $<$ , may or may not be preserved depending on the function used, whether it is increasing, decreasing, or neither. The complex nature of mathematics is due in part to the fact that what is a process in one context may be considered as an object in a different context.

For example, when considering the process of differentiation, this process acts on a function, transforming it into another function. In this instance, the function serves as the object upon which the process acts, whereas in other contexts, the function itself may be viewed as the process. This *objectification* of function, as Dreyfus called it, making an object out of a process, is "one of the most essential steps in learning" (p118). As Dreyfus points out, many of these objects and processes can be associated with visual models that help to identify and clarify the underlying mathematical structure, and that allow for ease and speed of information processing.

Visualization, from the point of view of mathematics education, includes two directions: the interpretation and understanding of visual models and the ability to translate into visual images information that is given in symbolic form. In addition to this aspect of coding and decoding, the direct processing of information in visual form may take on central importance in learning mathematics. (p119).

Tall(1992) discusses the difficulties students encounter when making the transition to advanced mathematical thinking. His focus is on the conflicts which arise between concept image and concept definition. The transition, from accepting concepts intuitively and experientially (and from this, experience in developing an appropriate concept image) to requiring all concepts to be formally defined and their properties logically deduced (i.e. forming concept definitions), gives rise to conflict.

One way to ease this transition is through the use of a cognitive root, which we discussed earlier. Tall also claims that, in developing the transition to mathematical proof, it could be more important to nurture and encourage mathematical insight than mathematical precision. His examples indicate that this mathematical intuition is often of a visual nature. He provides as an example a visual proof of the mean value theorem, and suggests that a visual understanding of such a theorem may be far more valuable than a possibly unsuccessful, or partially successful, attempt at memorizing and/or understanding a formal logical proof. He also claims that in order to be successful in advanced mathematical thinking one needs to have good intuition which comes from having had appropriate experiences to enable one to build a rich set of concept images. This points to the importance of developing mathematical intuition through the use of visualization in order to make the transition to advanced mathematical thinking.

Sfard(1991) discusses the dual nature of the process/object interpretation of mathematical concepts, claiming that they are best thought of as different sides of the same coin. She makes the distinction between *concept* or *notion* as the official form of a mathematical idea, and *conception*, the personal, internal understanding of such an idea. Since these internal advanced mathematical constructs are inaccessible to our senses, we have to *see* them with our mind's eye. "Being capable of somehow *seeing* these invisible objects appears to be an essential component of mathematical ability."(p3). But this objectification is not the only way in which a conception may be structured. Instead of this *structural* approach, sometimes an

*operational* approach is used, wherein a conception may be perceived as a process. This is the object/process duality which Sfard considers. She makes the important point that the process (or operational) approach to learning abstract notions is not incompatible with the object (or structural) approach, but instead the two approaches are complementary, with each approach requiring the other in order to be accomplished successfully. This interpretation explains, at least in part, why mathematics learning is a difficult business for many. The objectification (or structural conception) of mathematical ideas is the most difficult step. Much of the blame for this difficulty is put on the abstract nature of mathematics. The question is however, what is the nature of this abstractness, and how does it differ from that required in other scientific disciplines.

The central distinction between a structural conception of a notion and an operational conception of a notion is that while the former is a static structure existing somewhere in space and time, the latter is dynamic. Whereas with a structural approach one has the ability to recognize the idea at a glance without details, the operational approach means a detailed, sequential grasp. As Sfard points out, many mathematical notions are understood both structurally and operationally, for example function, symmetry, natural numbers, rational numbers, circle. Sfard emphasizes that the difference in approach lies in the basic (usually implicit) beliefs about the nature of mathematical entities, and that there is a deep ontological gap between the two conceptions. However, she points out that they are not mutually exclusive, but rather complementary, similar to the dual nature of light, which may be considered as both particle and wave. Similarly, the ability to see a function as both a process and an object is "indispensable for a deep understanding of mathematics." (p5). Most mathematical concepts may be defined in both ways.

This dual nature is apparent in symbolic representations, as well as verbal. Consider both a graphical representation for a function, and an algorithmic representation similar to a computer program. The symbolic algebraic representation lies somewhere in between these two extremes. These structures show up during pro-

cessing knowledge mentally; mathematical concepts are sometimes envisioned with the help of mental pictures, at other times a verbal representation may be more useful. In general, however, mental images support a structural conception. The distinction between these two conceptions - operational and structural - is made apparent in the following comparison table from Sfard(p33):

Table 2. Operational and Structural Conceptions

	Operational conception	Structural conception
General characteristic	a mathematical entity is conceived as a product of a certain process or is identified with the process	a mathematical entity conceived as a static structure - as if it was a real object
Internal representations	is supported by verbal representations	is supported by visual imagery
Its place in concept development	develops at the first stages of concept formation	evolves from the operational conception
Its role in cognitive processes	is necessary, but not sufficient, for effective problem-solving & learning)	facilitates all the cognitive processes (learning, problem-solving

Sfard claims that of the two conceptions, objects are the more abstract. In concept formation, the operational precedes the structural, the structural is more abstract; to talk about *objects* we need the ability to deal with *products* of processes without bothering about the processes themselves. Sfard claims that, in general, (but not exclusively), the majority of mathematical ideas originated in processes rather than in objects. If this is true, and she gives convincing evidence that it is,



then forcing the objects first - via rote learning of definitions - before developing an operational understanding, does not work. It merely leaves the student with empty definitions of objects about which they have no clear understanding. For successful mathematical learning then, there would appear to be a process/object/process ... spiral.

In arguing that the procedural precedes the structural, and that the development of the structural conception occurs in three stages: *interiorization*, *condensation*, and *reification*, Sfard says: "First there must be a process performed on the already familiar objects, then the idea of turning this process into an autonomous entity should emerge, and finally the ability to see this new entity as an integrated, object-like whole must be acquired." (p18). The first two processes are gradual, the third is a qualitative quantum leap. Process solidifies into object, and becomes a static structure.

"Various representations of the concept become semantically unified by this abstract, purely imaginary construct. The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category. At some point, this category rather than any kind of concrete construction becomes the ultimate base for claims on the new object's existence. A person can investigate general properties of such category and various relations between its representatives... Process can be performed in which the new-born object is an input. New mathematical objects may now be constructed out of the present one." (p 20).

Sfard discusses how structural process aids in our mental storing of mathematical ideas. These objects

function like simplified pictures or symbols which can be seized at one glance and may be used instead of 'the real thing' (the corresponding process) at certain stages of problem-solving ... These compact abstract entities serve as pointers to more detailed information. Thus almost any mathematical activity may be seen as an intricate interplay between the operational and the structural versions of the same mathematical ideas."(pp26-28). Many mathematical entities are too complex to be grasped fully all at once, and the gap between the operational conception and the structural one may be too large to bridge easily. Sfard explains that "we overcome this difficulty by creating intervening abstract objects which serve us as a kind of way-station in our intellectual journey.(p 29.)

In the first stage, interiorization, one becomes acquainted with the processes which will ultimately give rise to the new concept. As skill is acquired, interiorization results when the processes can be carried out through mental representations. "A process has been interiorized if it can be carried out through [mental] representations, and in order to be considered, analyzed and compared it needs no longer to be actually performed."(p 20). The next stage, condensation, is "a period of 'squeezing' lengthy sequences of operations into more manageable units. At this stage a person becomes more and more capable of thinking about a given process as a whole, without feeling an urge to go into details. It is like turning a recurrent part of a computer program into an autonomous procedure."(p19). At this point, combining with other processes, generalizing, making comparisons, moving among different representations, all become easier. Although a new concept may emerge in this stage, it is not fully developed until the final, reification stage. Progress in condensation is made when it becomes easier to alternate between different representations.

Sfard explains that the third step, reification, is so difficult because it involves an ontological shift, a quantum jump, a radical restructuring of the existing knowledge structure, "a sudden ability to see something familiar in a totally new light." (p 19). In (1994) Sfard claims that "although reification itself may be difficult to achieve, once it happens, its benefits become immediately obvious. The decrease in difficulty and the increase in manipulability is immense. What happens in such a transition may be compared to what takes place when a person who is carrying many different objects loose in her hands decides to put all the load in a bag." (p198). It is a significant event in one's mathematical experience that is not easily forgotten.

Sfard compares the reification step to that of an earthquake. "The transition from purely operational to a dual process-object outlook is probably not a gradual smooth movement toward a higher level. Like any reification it is likely to be a quantum leap toward a higher vantage point." (p212). At the same time, she points out that reification, or seeing a process in a new light as an object, requires manipulating it as a whole, considering it as a static object. As such, it must serve as the lower level input to a higher level process. But in order to do this one must be able to conceive of it as being acted upon as part of a higher level process. Thus, in order to achieve lower level reification one needs higher level interiorization, and vice versa. From this it follows that the process of advanced mathematical thinking involves a constant back and forth between structural and operational conceptions for any given notion, thus requiring much time and patience to reach a full understanding, and often being a slow and painful process.

Sfard is careful to emphasize that, although structural and operational approaches are dual, each one dependent upon the other, nevertheless there is a prevailing tendency for the operational approach to most often occur first. Only in the exceptional case of an advanced mathematician is it possible that a structural approach is successful first. Sfard points out that textbooks, in contrast, often present mathematical material structurally first.

Sfard contrasts the cognitive schemas that are employed in an operational approach with those employed in a structural approach. In the former approach, operationally conceived information can only be stored in unstructured, linear sequential cognitive schemas. In this type of schema it is not necessary to assimilate new knowledge. In contrast, with a structural approach, information is stored in a "static, object-like representation which squeezes the operational information into a compact whole and turns the cognitive schema into a more convenient structure." (p 26). Sfard employs a diagrammatic illustration to make her point. While not claiming to have faithful images of mental structures in which information is stored, she nevertheless employs the diagram to illustrate a schema storing structural information.

For a cognizing person, the [parts of the diagram] function like simplified pictures or symbols, which can be seized at one glance and may be used instead of 'the real thing' (the corresponding process) at certain stages of problem-solving. Naturally, more often than not, these abstract constructs can only be seen with our mind's eyes. (p 26).

This last paragraph describes the intent and purpose of my thesis: to investigate and describe the nature of these various abstract visual constructs used by advanced mathematical thinkers, and seen only (or mainly) in the mind's eye.

### DESIGN AND IMPLEMENTATION OF STUDY

My study was designed to answer the following questions:

1. What is the nature of visual images used by advanced mathematical thinkers?  
How may they be classified as to the type of visual image? Are they prototypical? Analogic? Diagrammatic?
2. Can we identify any structure - hooks, which provide access to information, links, which forge connections among different pieces of information, or schema, which provide the scaffolding - within an individual's concept image?
3. Can we identify different stages in the learning processes of an advanced mathematical thinker by looking at the associated visual images and structures?  
In particular, can we distinguish whether a given mathematical concept is in the interiorization, condensation, or reification stage of understanding? More particularly, is there evidence of an earthquake-like event indicative of the reification process?

The study was conducted on individuals at Oregon State University during the academic year 1991-1992, and during the summer of 1992. Nine individuals were involved: two faculty members, four graduate students, and three advanced under-

graduate students in the REU (Research Experience for Undergraduates) program. All nine were mathematicians, seven were men, two were women. One woman was an undergraduate, the other was a graduate student. The study was conducted as a series of individual audiotaped interviews - usually held in two or three sessions altogether, each session being approximately two hours in duration. All the tape recordings were transcribed to provide a hard copy of the interviews. The format of the interviews was reflective in nature, and designed so as to encourage the respondents to reflect upon various aspects of their mathematical experience. The reflective interviews were designed to examine each individual's concept image associated with each of twenty-one mathematical concepts.

The mathematical concepts were chosen in large part from a careful examination of the book *Form and Function* by MacLane. The author describes the connections between everyday processes such as measuring, counting, sorting, collecting, and the corresponding mathematical concepts. He explains how he feels mathematics arose in a very experiential fashion, grounded in and organized by practical problems. From these concepts, I selected a subset of important and widely known concepts. Some of the concepts - such as ring, field - were chosen because there were non-mathematical interpretations for them. Others - such as infinity, set - were chosen because individuals would be familiar with them since early in their childhood. Yet others were chosen to be central concepts in any mathematics major's learning.

The following protocol was used during the interview, concerning each concept:

- What first comes to mind when you think of the concept?

- What specific examples come to mind?
- Do you have any visual images associated with the concept?
- Are there specific theorems that come to mind that are associated with this concept?
- When were you first introduced to the concept?
- How has your understanding of this concept changed/evolved over time? Is your understanding different now than it once was?
- Can you point to any particular milestone that changed your understanding?

These questions were designed to provide only what each individual evoked in connection with each concept. No probing or follow-up questions were asked if nothing was voluntarily evoked. It is important to note that the responses are not intended to indicate the totality of the knowledge that the individual may possess associated with the concept, only that material which was easily and voluntarily evoked, without any prompting beyond the above seven questions. It is also important to understand that the interview was designed to uncover each individual's *own* hooks to information, not to see what was evoked, *if hooks were provided*.. In many instances it is quite possible that far more information may have been available, once some essential hook was provided, but that was not the purpose of this study.

For each of the concepts, the questions were asked initially in the same order. However, the order in which the concepts themselves were asked was random. Also, the respondents quickly became familiar with the questions and began to anticipate them, leading to answers that were sometimes out of sequence. I chose not to inter-

fere with the free flow of their ideas, resulting in their sometimes anticipating the questions, answering them out of order, and sometimes responding to two questions at the one time, perhaps discussing examples and theorems simultaneously. In the presentation of the transcripts I have used standardized subheadings to preface the responses, so that for example, when an individual is discussing the visual images associated with the given concept, that part of the transcript will be prefaced with 'visual images'. As much as possible, I have separated the responses to agree with each standard subheading. The concepts that were discussed during the interview are shown in Table 3.

Table 3. Concepts used in the Study

Well ordered set	Group	Metric Space
Partially ordered set	Ring	Infinity
Zorn's Lemma	Field	Limit
Induction	Function	Convergence
Set	Vector	Continuity
Cardinality	Vector Space	Derivative
Transformation	Basis	Integral

The analysis was carried out on an individual basis, with a separate analysis for each individual. Each person's responses were analyzed individually. The



concepts were arranged in groupings with common threads, depending on that individual's responses. Consequently, different individual responses may be grouped quite differently.

Complete transcripts of responses are presented, unless the responses were insignificant, followed by a summary of the individual's responses for each of the concepts. Discussion of each individual's interview, and interpretation in light of the theoretical framework follows. An overall comparison and contrasting of the nine case studies concludes the study.

## CHAPTER IV CASE STUDIES AND INTERPRETIVE ANALYSES

### CASE STUDY 1 - ADAM (undergraduate)

#### • Partially Ordered Set, Zorn's Lemma, Group, Ring, Field, Metric Space

Adam is not familiar with any of these concepts, with the exception that he knows that the real numbers are an example of a field. However, he does not evoke any further information about fields.

#### • Transformation, Vector, Vector Space, Basis - linear algebra concepts

Adam first encountered these concepts in a linear algebra class. Consequently, there are links among them.

#### • Transformation

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* I don't know much about transformations. I know you can define some transformations by matrices and things like that.

Specific examples. [No response]

Visual images.

*Adam:* Just matrices. That's what comes to mind. I see a matrix because it was an operation to me, the idea of a transformation doesn't make a lot of sense to me, I just deal with it, or did deal with it at that time. It's just an operation, they're calling it a transformation, but this is what I have to do to do the problem.

Specific images. [No response]

### • Vector

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* A line with an arrow at one end and a dot at the other. And something that for me is difficult to deal with. I don't deal well with vectors and I don't know why because vectors are very useful, very, very powerful mathematical entities I guess. We were introduced to vector calculus, vector valued functions, and defining functions by vectors, regular polynomials by vectors, and then doing integration and differentiation and things like that, and I didn't do so well in that term at all. And then we used them a lot in physics too, and I didn't really like physics either. And they're hard. They're hard for me because the way our physics professor used vectors to describe natural occurring phenomena, in particular I'm thinking of torque right now, and when you apply a torque to a rod, the torque vector goes perpendicular to the way that you apply the force, and that to me makes absolutely no sense and not just for that reason, but that was one of the many reasons that I didn't like vectors in physics, 'cause they were tough. Physics was hard for me, and physics used vectors so that made vectors hard for me.

Specific examples.

*Adam:* From physics, vectors are used all the time in physics. We have

velocity vectors, displacement vectors, acceleration vectors, and all those things, they seemed to me a harder way to represent what was happening, because I already had a pretty good feel for how to use differentiation to get these things and to apply these concepts, but of course in physics you have to do what the teacher says and so you use vectors, and the torque vector bothers me to this day. We use vectors for fields. I guess I was lying when I was talking about fields, but I don't think it was the same fields you were talking about, we did a lot of things in calculus, or in physics with magnetic fields, and uh, uh, electric, electric fields, we used vectors on all kind of things, but now that I think about it we did use a different definition of fields

Visual images.

*Adam:* A line that, not a line, a line segment that starts at the origin and goes out somewhere and has an arrow on the other end. That's what I think of.

Specific images.

*Adam:* Just the one about torque, because it really bothers me that we can apply a force in one direction and the way we represent that force graphically is perpendicular to the actual way that the force is applied and that's just notation I guess but yeah, it bothers me, so I remembered it.

First introduction.

*Adam:* I don't know, probably in high school calculus, my first. I don't think we rigorously used them or anything, but I think we were first introduced to them in high school in calculus.

Changes in understanding.

*Adam:* A little bit, a little bit. I haven't done much with them since calcu-

lus and physics, since the vector valued part of calculus, and since physics, but we did do a little bit with them. I remember one problem in particular, I had a problem solving seminar this year as a 2 credit, pass-fail class, we had a problem that I was working on and it was a geometrical problem where I was trying to figure out where this location would have to be, and I was trying to do it geometrically and I was using trig and all this and I could not get this problem. I just worked and worked and I couldn't get it and I finally broke down and I asked my professor how to do it and he said well did you ever try and use vectors? And I said no, I never even thought about. And he whipped off like a four line proof of what I'd been dying trying to get, using properties of vectors, and if nothing else, I'm at least a little more aware of their use in everyday math, rather than their specific uses that I had encountered before that. And I do recognize them today as useful, I mean outside of the realm that I saw them previously.

Milestones. [No response]

### • Vector Space

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* I don't even know what a vector space is.

### • Basis

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* Basis. Linear algebra comes to mind. I do know this definition too, a basis (pause) it spans and, (pause) oh it's linearly independent I think. A group, not a group by definition but to me it's a group, but a collection of vectors, a basis would span that which means

you can get a linear combination of every vector from that basis, and then they're also linearly independent because, yeah, linear dependence was a bad thing, a nondesirable attribute.

Specific examples.

*Adam:* Well, in  $\mathbb{R}^3$ , (1,0,0), (0,1,0) and (0,0,1) that comes to mind, the trivial cases do. But that's probably all the ones that come to mind because we encountered them on I think one or two tests you know in linear algebra and then on an exam. I had a really interesting professor I guess for that, the tests were, the whole class was quite easy, very easy, so we didn't really have to learn the material very well. And I wasn't particularly interested in it and strictly by my own doing I didn't learn it very well because I didn't have to and I didn't really want to and now I regret it because I know it's something that I should know, but I don't.

Visual images.

*Adam:* I remember trying to prove that they would span the given vector space, or whatever you want to call it. Not a visual image per se, I guess not, I just remember how hard sometimes it was to show whether or not the given set of vectors was a basis.

Specific images.

*Adam:* Just the definition of a basis, what you have to do to show that a collection of vectors is a basis. There were other theorems, and I can't remember what any of them were, but I know there were some, of course there were some. No, I don't recall.

First introduction.

*Adam:* Linear algebra, this year, my junior year.

Changes in understanding.

*Adam:* Probably not. I just had it last semester so.

Milestones. [No response]

### • Interpretive Analysis

Adam's visual image of a vector is the algebraic symbol of a line segment with a dot at one end and an arrow at the other. He has this concept linked with physics in his mind, and as a result has a conflict associated with it. He learned about torque vectors in physics, but still does not understand this application, thus making him feel very uneasy about vectors in general. On a personal level, Adam is very uncomfortable with this concept, allowing that he does not deal well with vectors. He has a strong self-awareness of cognitive conflict with this concept that is causing a weakness in the 'scaffolding' of his concept image. His understanding has broadened to include different applications of vectors and he has come to appreciate their power and diversity, while at the same time he has not been able to eradicate the uncomfortable feelings associated with this concept.

Although Adam claims to be unfamiliar with the concept of a vector space, he has in fact worked with them, and mentions them in connection with a basis. Adam first describes the context of linear algebra for 'basis'. Immediately, this allows him to pull out his concept definition of a basis, as a linearly independent, spanning set of vectors. He evokes no particular visual image of this concept, other than the algebraic symbols  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ , a prototypical example of the standard basis in  $R^3$ . Adam is not really familiar with transformation, although he has encountered this concept in an operational way. He has a tentative link with matrices and linear algebra.

## • Continuity, Derivative, Integral, Limit, Convergence, Function

Adam was initially exposed to these concepts in the general context of calculus, or in some cases, before that in high school. He has revisited most of the concepts in a real analysis class, and this is causing him to re-examine and perhaps reconstruct his concept images.

### • Continuity

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* Continuity, well what first comes to mind would be a curve with no holes in it, because that's the first intuitive idea of continuity that we got back in calculus, back in high school. Continuous functions and things like that, so yeah, probably the graph.

Specific examples.

*Adam:* Nothing in particular, any graph really to me is what comes to mind, but it has changed also since I've had real analysis again, because of epsilon and deltas and things like that, and so we kind of redefined continuity in real analysis, and so I guess my perspective has changed a little bit. I sometimes think of epsilons and deltas but normally I still see a continuous, a smooth curve, I think that's what we called them back in high school, a smooth curve would be what I think of as continuous.

Visual images.

*Adam:* Of continuity? Yeah, a graph, a solid line of a graph, probably, that's the first thing that comes to mind when I think about it



Specific images. [No response]

First introduction.

*Adam:* I think in high school calculus, I believe, when we did what was it? Continuity implies differentiability [sic] I think, but differentiability [sic] doesn't necessarily imply continuity or maybe it's backwards but anyway, in the beginning of high school calculus, somewhere around there.

Changes in understanding. [No response]

Milestones. [No response]

## • Derivative

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* A calculus operation, very neat, I loved it, it was really a neat thing for me, because it was very simple, and I also think of velocity equations, versus distance, or acceleration, and rate of change equations. It's very fascinating to me when I first encountered it that it could be a useful operation, because nothing in math up to that point, other than geometry had had any use to me, it was just something that you just learned. But now we take derivatives and I can figure out if I'm pulling a boat at such an angle with such a speed on my winch, you know what's the rate of change with respect to the given axis, or something like that, and that was really neat, that was a great moment.

Specific examples.

*Adam:* Yeah, the rate of change problems. Although it's funny because

today I don't like to do those because they're applied, and I have a harder time with applied mathematics than I do with theoretical, but those still come to mind because I tutor a lot and I get a lot of students in calculus doing derivatives and applications of derivatives which are mostly applied problems and I still, sometimes enjoy doing them, I enjoy doing them when I get them, but when I don't get them then I don't enjoy doing them.

Visual images.

*Adam:*  $dy/dx$ ,  $dy$  over  $dx$ , the sym

Specific images. [No response]

First introduction.

*Adam:* My junior year of high school, in calculus.

Changes in understanding.

*Adam:* Again, in real analysis we redefined the derivative by, well actually we didn't redefine it, we re-introduced the old definition that I had learned, the limit-taking process of derivatives way back in high school, but I instantly forgot it as soon as we were taught the shortcut method. And so quite honestly until I took real analysis this year I forgot even how you defined it, that delta  $x$  delta  $y$  process it's called, or something like that, and then now I think I understand the process a little more, how the derivative evolved. I also had a history class where we discussed a little bit, not rigorously, how Isaac Newton and Leibniz first defined their 'derivatives' and how they first came upon it and first looked at it, and it gives me a better feel for the whole concept. Rather than just taking  $x^2$  and bringing the exponent down and dropping the exponent by the power of one, I kind of understand what's happening and that would probably be responsible to real analysis and that history of math class.

Milestones. [No response]

## • Integral

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* Integrals. Area. Integrals to me were the reason that I'm here sitting here today, that I'm a math major, that I even like math.

Specific examples.

*Adam:* The rotating sphere, or the rotating curve problems, as a whole, to me, were phenomenal, and also surface area, that I can find the surface area of a curve rotated around an axis would generate a solid and I could find the volume of that solid and I could find the surface area of that solid and those types of problems to me were just great. I mean, god, they were so neat. Because I remember the very first time we ran into a problem like that, we looked at a curve, we rotated it around the x-axis and it had a finite volume and infinite surface area and I'll never forget that. I don't remember what the function, or the polynomial is that generates that, but that was a great example. But I must say that my understanding of integrals has changed specifically and absolutely as a result of tutoring because today I can stand up and I could walk anybody through how to do the shell method for finding the volume of revolution and I know exactly what I'm talking about. And the only reason is because I've had to explain it to people when they come in. And when I was in high school, although it was fascinating to me that I could do that, I didn't really understand what I was doing. I just knew the process, you know, I'm given a problem, I'll do this this and this and I'll get the answer. But today I really understand why it works and so I can stand up and I can teach somebody how to do the shell method or the disc method or find an area under a curve, why it works, and that is directly from tutoring over the last two years that I've tutored at this school.

Visual images.

*Adam:* The notation of course, the little squiggly s sign and, and I always, always, always think of just a curve, a continuous curve in Euclidean two space with the inscribed rectangles under it when they're first showing you what was that, it's like Simpson's approximation, or one of those approximation rules where they show you how to inscribe or circumscribe the rectangles. That's, that's what I see.

Specific images. [No response]

First introduction.

*Adam:* That was the first time encountered it, high school calculus. When he told me that I could find the area under a curve I could not believe him. And he showed me and it worked, and then he told me that I could take that curve and rotate it about an axis and find the volume of the solid that was generated, I didn't believe it, and then he showed me and it worked. And the integral to me is the most brilliant application of mathematics that I have ever seen. It's so, I don't want to say basic in concept, it's just brilliant, I just love it. I think it's so great, I love it when I'm tutoring and people come in and they're on applications of integration section in their calculus class, I get so excited I can't hardly contain myself because I know a lot of little tricks about integration that they don't know yet because they're taking the class and so I get to look smart for one thing 'cause I know a lot of tricks that they don't. And I just love to do it, I think it is so neat that I can apply as simple a concept as inscribing rectangles under a curve to approximate area and then let it go to infinity, take the limit and there you go, you have an exact representation of what is there, and it's exact, it's not close, it's not within a certain percentage of error, it is exact, and it works, and I love it.

Changes in understanding. [No response]

Milestones. [No response]

## • Limit

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* Infinity. Not only infinity but very smallness too. As I approach something and get infinitesimally close then where really am I? That's what a limit is to me. Very interesting. Whoever it was that first started using those I think it was just a brilliant way to represent what happens at infinity or as I approach zero, or as I approach anything, as I get infinitesimally close to it, where am I or what happens?

Specific examples.

*Adam:* Well, we did some work in real analysis with limits, a little bit before we got to derivatives and used limits and in the definition for derivatives.

Visual images.

*Adam:* Just the notation, the little  $\lim$ , and then as  $x$  approaches whatever  $x$  is approaching.

Specific images.

*Adam:* Probably not. Well ok, no there is one. There is one that I remember from high school trig. And it's what everybody uses and that is if you're looking at the limit of a polynomial and it's approaching a finite number, say one, you're looking at the limit as  $x$  approaches one you can just take one and plug it in to  $x$  everywhere in the polynomial and evaluate it and that's really what it is. And that's a very useful theorem to me and it makes perfect sense that that's what you would do, but then you also get into problems when you are approaching infinity, because you can't just take infinity and plug it in. But there are ways that you can manipulate it, and that

was very neat to me. But then it also raised problems, because then you had indiscriminate [sic] forms when you're dealing with limits and you have a hard time with that too, thinking that one to the infinity isn't necessarily one, but one to every power is itself, until you get to infinity and then it's not necessarily itself. And I think we ran into that in high school calculus.

First introduction.

*Adam:* In high school trig was where we used limits.

Changes in understanding.

*Adam:* I can't really put it in a class but probably since I've tutored, I guess that would be the best answer. When you try and explain a limit to a person, I had to understand it before I could explain it, and that made me understand indiscriminate [sic] forms a little better in that I had always thought that when I take the limit and I plug in numbers, if I'm approaching one and I plug in one to a polynomial it's a shortcut, it's not an actual defined method of doing things, so when I plug in infinity I should expect that it's not going to obey natural laws. And so in that sense I understand a limit today better, in that I feel more comfortable with the indiscriminate [sic] forms that it creates, or that arise from using the limit. And they make more sense to me and that's probably due, I would say not to any one class or anything like that, over the last two years just trying to explain it to other people.

Milestones. [No response]

## • Convergence

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* To me I think of it as infinitesimally close, smallness, it's very close in concept to me to a limit. To being as absolutely as close as possible, but not getting there. And although I know that's not what convergence of a limit is, because you can have a constant sequence and it's converging to one number because it's constant, but that's not what I normally think of when I think of convergence, I think of  $\frac{1}{n}$ , where  $n$  goes to infinity, and so I converge to zero, but I don't quite get there, real close, but not quite.

Specific examples. [No response]

Visual images.

*Adam:* Yes, doing real analysis I can see it now, and again I don't normally think of constant series. I know they are convergent but that's not what comes to mind when I think of one. I see a neighborhood around the convergence point and I see for any epsilon, for any size neighborhood that there's infinitely many points within that neighborhood. So when I see it I see a real line and my convergence point and then a little neighborhood around that convergence point.

Specific images.

*Adam:* There were a lot of convergence theorems, and divergence theorems. And I remember some of them. I remember  $p$  series, where one over  $k$  to the  $p$  where  $k$  is the constant, and if  $p$  is greater than one the series is converging I think, where  $p$  is one or less, then it's diverging I think. I don't know, something similar to that though. And geometric series, I can never remember what the sum of the geometric series is, but I know that there's a formula and I can look it up real quick. Arithmetic series and progressions, but those aren't necessarily converging. But they're all related. There's a lot more of them that I don't remember. I remember there were some integral tests. I remember my favorite convergence test was a comparison test, where if I wanted to show that a given series was convergent, I had to find a 'bigger' series that was also convergent. Or if I wanted to show that one was divergent I had to find one that was a 'smaller' series that was also, that was a really great theorem, 'cause that was powerful to me, and I knew how to use

that one pretty well.

First introduction. [No response]

Changes in understanding.

*Adam:* Yes, since real analysis. Calculus first allowed me to work with convergence much more and to prove that a given series or sequence was convergent or divergent, but then when I got to real analysis we did the epsilon delta proofs and that gave me more of an intuitive feel for what it meant to be convergent or divergent.

Milestones. [No response]

Evoked images. (What first comes to mind when you think of concept?)

*Adam:*  $f(x)$ .  $f(x)$  comes to mind, functions and polynomials also.  $f(x)$  equals and then whatever polynomial follows, and I don't normally think of trig functions or hyperbolic functions or inverses or anything like that, it's  $x^3 + 2x^2$  something like that, of that nature. And that's probably what comes to mind. There's a lot more to them than that though. That's what I see.

Specific examples. [No response]

Visual images.

*Adam:* Now that I think about that machine I do. I hadn't thought about that in quite a while, but on an average day, probably just a polynomial,  $f(x)$  equals . . . , a polynomial equation, rather than a mapping or anything like that because those to me don't feel as comfortable as other things, but I've just recently been introduced to them in the last year or so I guess.



Specific images.

*Adam:* One of the Cantor theorems that says that two sets are equivalent if you can find a one to one function from one set to the other and then an inverse. And also bijective functions, I forgot about those, I like the theorem that every bijective function has an inverse that is also bijective. Once you understand the concept behind bijective it makes perfect sense, but when I first encountered it I thought that it was a little bit abstract, a little bit strange, but it makes a lot more sense to me now. I like them, I like functions.

First introduction.

*Adam:* I don't know, probably, well we worked with polynomials of course in algebra, I don't think we called them functions then, probably weren't introduced to a function, well I don't know, we used to have function machines, I remember those. I'm going to say somewhere between eighth grade and my junior year of high school, I don't know, I'm sure by calculus I was, and I remember function machines, where you plug in a number and it spits out the answer, and I can't remember when we used those, it may have been eighth grade algebra but I'm not sure.

Changes in understanding.

*Adam:* I do have a different understanding of a function now because of real analysis and that discrete math class, because we used functions as mappings from one set to another, and I'd never encountered that before and they're far more useful things than I ever thought they were before. I'm thinking of topology of the real number line and finding equivalences between sets. Functions are, to me now they're a lot more useful, rather than just making neat curves with polynomials, now I can take a function and define it any way I want and get it to do a job for me, to be one to one from this set to this set under a given restriction that I make. Very powerful tools to me now and that would all be probably due to discrete math and real analysis.

Milestones. [No response]

• **Interpretive Analysis**

Adam first evokes a visual image of continuity from high school calculus, a diagrammatic one of the graph of a continuous smooth function. He describes revisiting continuity in real analysis class, and this causes a significant change in his understanding, when he re-learned the  $\epsilon - \delta$  definition of continuity. However, he says that although he sometimes thinks of  $\epsilon - \delta$  in association with continuity, what is evoked is the smooth continuous curve. Thus he apparently continues to rely heavily on the ideas and images first learned in calculus.

Adam first evokes his self-awareness of derivative as a 'neat operation' that on top of being fun to apply is actually useful! His visual image of derivative is of the algebraic symbols,  $\frac{dy}{dx}$ , commonly used to denote this concept, and he understands derivative as a rate of change, a calculus operation. Revisiting the formal definition of derivative in analysis class has added depth to Adam's predominantly algebraic and operational understanding of derivative, but is not considered by him to be a significant event in his understanding. He describes the link he has between derivative and physics.

Adam immediately evokes this self-awareness of the importance and aesthetic appeal of this concept. His visual image of integral is a diagrammatic image of the area under a curve, approximated by rectangles. He also evokes the algebraic symbol for integral,  $\int$ . He describes an enthusiasm for the power of this concept because of its rich and varied applications, particularly volumes of solids of revolution. He describes a significant deepening of his understanding which occurred as a result of tutoring. His understanding underwent a significant change when he made the transition from understanding an integral as a rote application of method - an operational understanding of an integral - to a richer, deeper more conceptual

understanding of an integral. This may be setting the stage for a reification event. However, there seems to be little evidence here that Adam's understanding of integral passed through the reification stage. Certainly Adam has been positively and deeply affected by his tutoring experience. He is certainly very self-aware of its effect.

Adam has the concept of limit linked with infinity, and infinitesimals, as a way of discussing the behavior of a function when  $x$  approaches infinity, or zero, or any other value. His visual image is of the algebraic symbol for limit,  $\lim_{x \rightarrow}$ . Adam links convergence with limit. Again this is a concept he has revisited since his first introduction, this time in his college tutoring. This has given him a better understanding of limit, because as he points out, to explain it, you must first understand it.

He describes a diagrammatic visual image of the real line and a neighborhood around the point of convergence. His prototypical example for a convergent sequence is the sequence  $\{\frac{1}{n}\}$ , whose terms never equal the limit point. It is unclear whether he is aware of this limitation of his example, although he does make reference to the convergence of a constant sequence, but nevertheless prefers to use  $\{\frac{1}{n}\}$  to represent convergence. He describes his real analysis class as being significant in changing his understanding of convergence by providing a rigorous  $\epsilon - \delta$  definition for convergence. He says, however, that this gives him a more intuitive feel for what convergence is.

Adam first evokes the algebraic symbols for a function  $f(x)$ . His prototypical example for a function is a polynomial function. Even though he notes that there are many others, he prefers to think of polynomials. He recalls a visual image of a function machine from high school, but does not use that as an image any more. He describes a significant event in his understanding as being real analysis class, where he learned that functions may be defined so as to get them to do whatever job you want done. He realizes at this point that functions are a powerful mathematical tool. This was a big paradigm shift from thinking of a function as a polynomial, prescribed by a fixed, predetermined formula,  $f(x) = \dots$

For each of these concepts, the same pattern emerges. Adam first evokes that part of his concept image developed earliest, often in high school mathematics. Once he has hooked into his concept image scaffold in this way he is then easily able to access the more recent construction and/or remodeling which resulted from his real analysis class. It is important to note that the original access hook is not discarded or left unused, but is still the predominant access hook to the concept image, even in the presence of newer construction.

- **Well Ordered Set, Induction - natural numbers as prototype**

Adam has these concepts linked through the prototype of the natural numbers, although the connection with induction is implicit.

- **Well Ordered Set**

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* Well, I don't have a real good grasp, but what I do know is that the naturals are well-ordered, I think of that in relation to the axiom of choice, I don't know if I should or not, but I do. A well-ordered set, I think and I could be wrong, 'cause I don't think we've ever talked about them a whole lot, but they're sets that, for lack of a better term, can be put in order, can be put in a given sequence or something like that.

Specific examples.

*Adam:* I know that the open interval of real numbers from zero to one is not a well-ordered set, because you can't say well ok here's my first

element, you can't do that, and so for that reason it's not well-ordered, but that the naturals of course are 'cause you can just choose the smallest element and then proceed up the line and then there you go, they're well-ordered. And I think it does have to do with axiom of choice in certain situations, 'cause the axiom of choice says that I can choose a function that will, I think that will partition any set into well-ordered sets, or something like that, I don't know.

Visual images.

*Adam:* Natural numbers, 'cause that is easy for me to grasp. I see one, comma two, comma three, a set of natural numbers or any subset thereof.

Specific images. [No response]

First introduction.

*Adam:* This year in discrete math. The way I remember it was I was working on a proof that wasn't related to the class. I was working on a proof that the power set of the natural numbers has cardinality of the continuum, cardinality  $c$ , and I went in and I was asking my professor about something, I don't even remember what, and he showed me some things with the axiom of choice and there's one theorem, the well-ordering principle, and of course I don't remember what it was but I remember seeing it, and so I did have a brief introduction to it but I never studied it per se.

Changes in understanding. [No response]

Milestones. [No response]

## • Induction

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* A very, very nice way of proving things related to natural numbers, not necessarily, but normally to the natural numbers. I like induction proofs, and it's a brilliant way to prove too. Again I have to credit whoever first thought of it.

Specific examples.

*Adam:* I was trying to think of one, but I was failing. I can't think of one right now. I've done it a lot I should be able to, but I'm drawing a blank.

Visual images.

*Adam:*  $n$  and  $n + 1$ , I see the actual heart of the induction proof, assuming it's true for  $n$  and showing it's true for  $n + 1$ .

Specific images.

*Adam:* With induction? No, not that relate to induction. The method of induction.

First introduction.

*Adam:* I was first introduced to that my sophomore year in college, I took a class that was just how to do proofs, it was like a transition from calculus to higher mathematics, where we had to be able to prove things. I'd never seen induction before and I thought it was pretty fascinating. It might be easier to make an example. If I want to prove that a given relationship holds, to assume that it's true for base  $n$  and show that from base  $n$  that  $n + 1$  is true, that to me doesn't prove it. You know, when I first encountered it that's what I'm thinking, I'm thinking well that shows nothing, that shows that

from one step to the next that it works. Then he says 'you also have to prove the base case,' and I said, 'ok, fine, I have a base case and then way out somewhere in the natural numbers I have two cases that I know, but there's an infinite number that I don't know.' And then it finally dawned on me that if you know  $n$  and that implies  $n + 1$ , and you know one, then one implies two, two implies three and so on and so forth, but the concept of induction finally made sense and it's really a nice way to prove things I think.

Changes in understanding.

*Adam:* Not since that class. I think that class my sophomore year did very well. I understand, well it's a fairly easy concept of how it works, the proofs aren't always easy, but the concept of why induction works is really easy, I think I understood that to begin with.

Milestones. [No response]

### • Interpretive Analysis

Adam's prototypical example of a well ordered set is the set of natural numbers, and just as important is his non-example of the open interval  $(0,1)$ . He has this concept hooked to the Axiom of Choice in his concept image, but the connection is very shaky, though he is able to describe the least element principle. He also recalls the words 'well ordering principle', but is not able to say what it actually is, just that it is connected in some way to the concept.

Adam has the concept of induction linked with the natural numbers in his concept image. His visual image is of the algebraic symbols  $n$  and  $n + 1$  associated with a proof by induction.

Most of what he has to say about induction concerns his self-awareness of his feelings about this concept. He really likes the technique of proof by induction. Although he was not immediately convinced of its validity when he first encountered

it, he has struggled with the proof technique and has come to see it as a clever way to prove things. He is aware of the struggle that he had. Having grappled and won, he now has a fondness for this concept.

## • Cardinality Conflict

### • Set

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* A finite collection. To me, that would be the first thing that comes to mind. And probably I would again see a symbol of the little brackets with numbers in them, but not necessarily a set, that's just anything, any collection but probably numbers come to mind first and a finite set comes to mind, although I know very few of those are actually.

Specific examples.

*Adam:* Well we just learned about the Cantor set, I just learned about it, a lot of people already knew about it but that was a really neat set. There's a lot of infinite sets that come to mind, just because of their paradoxes to me, open-interval between 0 and 1, closed interval, from 0 to 1. The set of rationals, the set of irrationals and those all come to mind because of the cardinality conflict I have with them, that there can actually be different sizes of infinity.

Visual images.

*Adam:* I see notation. When I think of a set I see brackets, and if it's an interval I would just see a closed bracket for a closed interval and



just parentheses for an open-ended interval, but for a finite set I just see brackets with elements.

Specific images.

*Adam:* That the real numbers has cardinality  $c$ . Versus the rationals, or the naturals for that matter having cardinality aleph null. I don't remember the name, but the one theorem that states that you can show that two sets are equivalent by showing a one to one function in both directions, one from  $A$  to  $B$  and one function from  $B$  to  $A$ , that that would show that  $A$  and  $B$  are equivalent. That was a really neat theorem. That an infinite set always has another infinite subset of the same cardinality, has a proper subset that has the same cardinality as itself. That blew my mind, that I could take some away and still be as big. But those are probably the big ones.

First introduction.

*Adam:* I really don't know. I'm sure it was in high school or earlier, probably not earlier, because I took algebra in eighth grade and I don't think you cover sets in eighth grade. Sometime in high school I would imagine, but I'm not positive.

Changes in understanding.

*Adam:* Real analysis, the almighty class. Yeah, it sure has. It evolved from strictly finite sets, or maybe I should say, it evolved from the only sets I understood were finite sets to now I think that I have a fairly good understanding of some infinite sets and their relationship to other infinite sets mainly rationals to the reals, or the irrationals to the reals, or the irrationals to the rationals, or things like that. And that began in real analysis and then I had another class after that that was called discrete math but I don't really think it was what everybody thinks of when they think of discrete math and we did a little bit with cardinality in there and at that time I was more interested in it, rather than learning it, and I actually did a few proofs on my own, using cardinality and functions and things

like that. And now I think I have a better understanding of how those sets are given different sizes of different cardinalities. That's all just in the recent past, 6 months or so.

Milestones. [No response]

## • Cardinality

Evoked images. (What first comes to mind when you think of concept?)

*Adam:* Cantor is what I think of. I should say cardinality of finite sets to me was very intuitive of course, I mean it just is very intuitive, it was just a different name for size at first and that's still how I think of it, it's just a different name for size, because size doesn't really fit anymore when you go to an infinite set. And when I think of cardinality, I think of aleph null, I think of  $c$ , I think of the power set of the reals whatever that has as cardinality I don't know.

Specific examples. [No response]

Visual images.

*Adam:* Cardinality? Aleph null, because it's a neat symbol, I think it's cool. That probably comes to mind, and that little  $c$  that they always use for cardinality of the reals.

Specific images.

*Adam:* There's a lot of neat anti-intuitive things that come to mind when I think of cardinality. Every infinite set has a proper set of the same cardinality. The open interval zero to one has the same cardinality as the whole euclidean two space, that bothers me, but I can see it. I'm not going to say it makes sense, but it does to some extent.

There's all kinds of things. The natural numbers has the same cardinality as the rationals yet the rationals are dense in the real number line, but the naturals aren't, not even close, but they have the same 'size' as a set. That the cross product of the rationals with themselves, any finite number of times, million, trillion, a billion times still is the same 'size' as the rational numbers. And it's still 'smaller' than the open interval zero to one. You know, those kinds of things, they're just, I could go on, 'cause they don't seem intuitive but when you think about the definitions that you're dealing with, when you think about the one to one correspondence that must hold in order for two sets to have the same cardinality, then it will make sense to me. But when I stand back and look at the big picture it doesn't make sense.

First introduction.

*Adam:* I think of real analysis, interesting, very interesting, but difficult to me, it was hard at first, but today I understand Cantor's definitions, how he related cardinality to functions, you know, how a one to one correspondence is a very natural way of defining 'size'. And if you can't find a bijection between two sets then one of them has to be 'bigger', and to me that makes sense now, and that that has just come I don't know, over time I guess, just thinking about it. I didn't really understand it when I was in real analysis. I had that first semester this year. But as I thought about it more and more and did some proofs on my own then it would make a little more sense to me that, even though to me infinity is still one 'size' it's all huge, there are still sets that you can't put in one to one correspondence with each other. So therefore one of them has to be bigger, sort of.

Changes in understanding. [No response]

Milestones. [No response]

## • Infinity

Evoked images. (What first comes to mind when you think of con-

cept?)

*Adam:* Huge. Infinity, there's something that when I first encountered it was very difficult because it was so big to me. It just was beyond reality, but you just get used to it I guess.

Specific examples.

*Adam:* Probably the coordinate system, that it can go on forever in all directions, or three space for that matter and any physical object that I can see that will go on forever in my mind. Like when I look outside I don't see infinity, I look at the sky I don't see infinity, but when I look at a line on a chalkboard and it has two arrows on each end I see infinity, because that goes to infinity in my mind.

Visual images.

*Adam:* The infinity symbol. The figure eight. Other than that no, because I can't really grasp something going on forever, in such a way. So probably just the infinity symbol I guess.

Specific images. [No response]

First introduction.

*Adam:* I don't know. I'm going to say high school calculus, but it may have been before that. Yeah, I think it was high school trigonometry because we first used the idea of limits at that time.

Changes in understanding.

*Adam:* Real analysis was where I was first introduced to cardinality, and different sizes of infinity and that really blew my mind. I had a real hard time at first grasping that, I'd finally accepted that there was

infinity, and that it was a very real thing, and then they tell me that there's different sizes and that probably there's an infinite number of sizes of infinity. According to cardinality there is no upper bound on the cardinality of sets so maybe there's sets that have infinite cardinality, if that makes any sense, it doesn't make much sense to me. It has changed because at first you're introduced to limits and it was kind of shocking to think first that there *was* infinity, but I've recently taken real analysis and that's probably the milestone. But it's changed because, I can deal with a lot more because I have to think of it now as an abstract concept rather than a concrete entity, like figures, a figure's concrete but cardinality is not it's just, to me it's more like a definition that I just have to abide by and according to these rules what happens. Anyways, that's how I think a lot.

Milestones. [No response]

### • Interpretive Analysis

The word set evokes for Adam a finite collection, even though he is aware that many sets are not finite. He provides several examples, all of infinite cardinality: the Cantor set, the rational numbers, the irrational numbers, open and closed unit intervals. This provides a link with the cardinality conflict (his words), and perhaps provides an explanation as to why he chooses to stay away from infinite sets. He describes a visual image of the algebraic symbols, brackets with numbers inside, usually used to denote finite sets. Alternatively, he visualizes interval notation. Adam realizes that he will have reached a true milestone when he feels comfortable in his understanding of infinite sets, though he is clearly aware of his internal conflicts.

He still prefers to think of a set as finite, since this allows him to avoid the conflict.

Adam first describes how intuitive cardinality is when applied to finite sets. It is merely a renaming of 'size'. His visual images are of the algebraic symbol,  $\aleph_0$ , representing the cardinality of a countable set, and  $c$ , representing the cardinality

of an uncountable set. He quickly goes on to talk about the conflict he has with the cardinality of infinite sets. It seems that rationally he has resolved the conflict that he perceives among sets with countable cardinality, and also between those sets and others with uncountable cardinality. He claims this distinction makes sense, in the sense that he believes the proofs to be mathematically correct, yet on another level he has not resolved this conflict, he still 'feels' that there is only one 'size' for infinity. This self-awareness is causing Adam cognitive conflict.

He also has a link with the topological property of discreteness in his concept image. He is bothered by the fact that the rationals and the natural numbers have the same cardinality, but the rationals are dense in the real numbers whereas the natural numbers are not. A source of cognitive conflict is his apparent desire to connect this topological property with a set's cardinality.

Adam first evokes the adjective 'huge' rather than any particular prototypical model or visual image associated with infinity. He evokes the algebraic symbol,  $\infty$ , as a visual image, and explains that he has no others because it is not possible for him to visualize something that goes on forever. However, he later describes prototypical examples of infinity - the real line, and three space. He makes a distinction between being able to see infinity in his mind's eye - a line, drawn on a chalk board, with two arrows at the ends 'goes to infinity in my mind' - but not being able to see infinity when he looks at the sky. He seems not to see the sky going on forever in his mind's eye, and yet mathematical three dimensional space somehow represents infinity to him.

He has this concept intimately linked with the concept of cardinality in his understanding. We have seen evidence of the major trouble spot that cardinality poses in his mathematical experience. In this particular instance, he points to two major events in his understanding of infinity. He first describes his initial difficulty in accepting the existence and validity of such an abstract, non-concrete idea, only to be confronted with the fact that, not only was infinity an accepted and rigorously defined mathematical concept, but that there were different 'sizes' of infinity, in the

form of countable and uncountable cardinalities. He has reconciled this somewhat in his concept image, by accepting a formal concept definition of infinity (although he does not actually define it in the interview setting), and accepting that it is not a concrete entity, but a purely abstract concept. It seems that he has accepted this concept formally via the formal concept definition, but his intuitive understanding is still in turmoil, suffering under conflicting ideas. This in turn means that his concept image is also in a confused state. These two events just described are significant milestones in his understanding of infinity, both of them having caused major shifts in his understanding. There is still progress to be made.

Intuitive understanding, according to Fischbein, is largely dependent on visual images. Adam seems to have mostly symbols for images. Other visual images such as denseness of the rational numbers and 'gaps' in the natural numbers do *not* help him to develop the intuition needed to overcome his conflict. In fact, it may be the major factor contributing to it.

### CASE STUDY 2 - BETH (undergraduate)

• **Well Ordered Set, Partially Ordered Set, Zorn's Lemma, Transformation, Vector, Vector Space, Basis, Ring, Field and Induction.**

• **Interpretive Analysis**

Beth evoked very little of interest related to these mathematical concepts, and so the transcripts have been omitted. To her, rings and fields are algebraic objects with more structure than groups, with the rational numbers and the real numbers being prototypical examples for each concept. She understands induction as a proof

technique, and recalls the classic false induction proof that all cows are purple. This serves as a metaphorical example of what can go wrong with an induction proof if one does not take care. She is unfamiliar with the concepts of a well ordered set, a partially ordered set, Zorn's Lemma and a transformation. She is familiar with vectors from physics class, but is not comfortable with the concepts of vector space and basis, although she has encountered the terms.

### • Group and Function - hooks and links

Beth makes an interesting connection between these two concepts, and the concept of function became objectified as a result.

#### • Group

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* Oh gosh, about something that's closed, and something abstract.

Specific examples.

*Beth:* Specific examples that come to mind are dihedral groups again and that's the visual one I think of.

Visual images. [No response]

Specific images.



*Beth:* That it's not necessarily commutative, non-abelian groups are what they're called. Let's see, maybe I just haven't gotten enough food or something.

First introduction.

*Beth:* In Algebraic Structures I, first semester of this past year.

Changes in understanding.

*Beth:* I think the dihedral groups changed it a lot because the fact that something wasn't commutative blew my mind. I had a hard time thinking of that. Then I finally got down to the shoe-sock principle. Putting on your shoes first and then your socks is not the same as putting on your socks and then your shoes. And so I suppose that was the milestone that changed my understanding.

Milestones.

*Beth:* [The idea of a dihedral group] was basically the milestone that changed my understanding a bit, when I started learning all sorts of things about groups of things other than numbers, and started getting away from the concept of numbers.

### • Function

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* A means of taking a number and finding another number from it.

Specific examples. [No response]

Visual images.

*Beth:* Oh gosh, right now I'm thinking of different functions. Specific examples that come to mind are like I think of dihedral groups and the rotating of the square comes to mind right off, the rotating and the flipping of squares. That's a fairly visual image.

Specific images.

*Beth:* That you can do functions in different orders and come up with different answers.

First introduction.

*Beth:* I was first introduced to the concept of a function in high school, senior year, calculus class.

Changes in understanding.

*Beth:* I know it's not just numbers now it's not always plug and chug. A lot of the work sometimes comes in finding the function. So it's a little different.

Milestones.

*Beth:* A milestone that changed my understanding was algebra class probably. It changed to getting rid of the numbers and thinking of it more generally than just oh  $x + 3$ , something like that and that it isn't always something that would normally be graphed. We used the dihedral groups and functions to get from one element of the dihedral group to another element. So I was thinking of those examples.

## • Interpretive Analysis

Beth's prototypical model for a group is the dihedral group. This group seems to have made a strong and lasting impression on her, because it was the first example of a non-commutative group that she encountered, and she had difficulty with that idea initially, until she learned the 'socks and shoes' analogy. This linking of an abstract mathematical concept with a common, readily understandable life experience makes it easier for Beth to understand the concept of commutativity, and to remember its properties. She has a mathematical visual image associated with the dihedral group, of the group acting on a square, (which she describes also when talking about function), again linking the mathematical concept with a simple model. She visualizes the square being rotated and flipped. Her understanding changed significantly with the introduction of non-commutativity, and also with the idea that a group could consist of elements other than numbers. Again, the dihedral group is the prototype here, and Beth may choose to think of its elements geometrically, as rotations and reflections, or as functions (or permutations of the vertices of the square).

Beth first evokes her understanding of a function as a process which has a number as input and a number as output. Apparently the 'function as process' idea is the hook that provides the initial access to her concept image. Once accessed, she is able to link up with the 'function as object' part of the scaffold. She talks of the prototypical example of the dihedral groups, and the accompanying rotations and reflections of a square. She describes a significant change in her understanding which occurred when she learned, in abstract algebra, that a function may be something other than a formula and its graph, something other than a process which takes a number and spits out another number. This occurred when she learned about permutation groups, whose elements are functions. Beth seems to be describing a reification event in her understanding of function, where she began to understand a function in a dual operational-structural way, as both a process and an object.

It is important to note that the understanding of function as object did not occur until it was *necessary* for her to understand function in that way, as an element in a higher level process - as an element in the dihedral group of symmetries of the square.

### • Set and Cardinality

Although both of these concepts are often linked with infinity, Beth does not make the connection here.

#### • Set

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* Venn diagrams.

Specific examples.

*Beth:* I just think of a triangle.

Visual images.

*Beth:* I would think the most visual images I have associated with the concept is the Venn diagrams. Getting the intersections and stuff.

Specific images.

*Beth:* Mostly with intersections and stuff. Specific theorems again aren't in my mind. I never get specific theorems in my mind, unless I'm in the middle of a class.

First introduction.

*Beth:* I was first introduced to the concept of a set in probably discrete math, which was first semester my sophomore year in college.

Changes in understanding.

*Beth:* It's changed into looking at different kinds of sets. Once you get into algebra it changes, knowing that sets can be classified and that sort of thing is a little different, not too much though. And I guess algebra changed my understanding a little bit, by classifying different kinds of sets into groups and stuff. Stuff like that.

Milestones. [No response]

## • Cardinality

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* The number of elements in a set.

Specific examples.

*Beth:* Trying to find the number of elements in an intersection, Venn diagrams.

Visual images.

*Beth:* Visual images I have come from back in discrete math when sets were nice and small when we first started working with them, listing out all the elements in the set and counting them. You actually see the elements if you count them.

Specific images. [No response]

First introduction.

*Beth:* I was first introduced to the concept of cardinality in discrete math.

Changes in understanding.

*Beth:* My understanding of this concept hasn't changed a heck of a lot. It's still the same. The number of elements in a set.

Milestones. [No response]

### • Interpretive Analysis

Beth's visual image of a set is a diagrammatic one of Venn diagrams, intersecting circles arranged in a triangular pattern. Her understanding has broadened due to her algebraic structures class, where she encountered sets of elements other than numbers, endowed with additional properties, but it has not undergone a radical change.

Beth's understanding of cardinality is of the number of elements in a set. There is no evidence of a link with infinite (countable and uncountable) cardinality, although she is aware of these, since she talks about them when discussing infinity. Even though Beth understands that 'countable' and 'uncountable' are adjectives that may be used to describe different 'sizes' of infinity, she does not extend the

term 'cardinality' to infinite sets. Her visual image is of Venn diagrams, and of writing out the (small, finite) number of elements in any given set.

### • Metric Space

Beth encountered a great deal of difficulty with this concept, apparently because she was unable to connect it to any other mathematical (or non-mathematical) concepts (or experiences).

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* Oh gosh, real analysis and sheer hell! That was a concept I had a huge amount of trouble with and still do. I don't think the professor ever was quite able to get it into my head what a metric space really was for some reason. None of us in that class were able to figure out what a metric space was and we spent a long time on it.

Specific examples.

*Beth:* Specific examples, trying to figure out whether certain spaces were compact and trying to figure out exactly what compactness meant, anyway, and open and closed and getting the neighborhood. The only visual images I ever saw

Visual images.

*Beth:* with metric spaces were blobs. I couldn't figure much out from blobs.

Specific images.

*Beth:* If a metric space is closed and bounded it's compact comes to mind.

First introduction.

*Beth:* I was first introduced to the concept in real analysis. We were all very scared of metric spaces and had to figure out whether it was closed or not.

Changes in understanding.

*Beth:* My understanding of it hasn't changed much because I still don't understand it. Partially because the way it was explained to us was very abstract and there was nothing to visualize, and there was not even that much for me to be able to think of in terms of playing around with it at all and the professor was never very specific about what they were. He just said 'blob this is a metric space.' 'Ok, but what about this and what about this?' 'Oh, those don't matter.' 'Ok.' And so since he was never really clear about what a metric space was in the first place, without having really gotten the definition down he started going into theorems and things like that and we were still 'wait, wait, we don't know what this is' and he never did manage to explain the concept to us. So I remember trying to prove theorems dealing with metric spaces when I had no idea what a metric space was. So, there wasn't really any milestone that changed it.

Milestones. [No response]

### • Interpretive Analysis

Beth had a rather unfortunate experience in a real analysis class where metric spaces were introduced and worked with. She did not learn much about metric spaces in that class, and still has difficulty with this concept. It appears that the topic of metric spaces was rather quickly introduced, at least too quickly for most of



the students to really grasp the concept, and then the professor quickly moved on to theorems involving metric spaces. It seems that the students were not given enough time to absorb the formal definition and form a conception of a metric space. In any event, Beth seems not to have been able to connect this new concept to any existing cognitive structure. This has resulted in her not understanding any of the theorems that followed, and in her not having any clear conception of a metric space other than 'a circular blob'. This seems to indicate that the first step - interiorization - in the process-object progression was not effectively begun and that during the class only a (possibly incomplete or even erroneous) concept definition formed in her concept image, which was quickly lost at the conclusion of the course because it was not linked to any other cognitive structure. It is difficult to retain a formal concept definition when there is no context within which to place it. This emphasizes the importance of connecting new mathematics to old and existing mathematical beliefs and structures.

### • Infinity, Limit, Convergence

These three concepts all have graphical interpretations for Beth.

#### • Infinity

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* Infinity? Very large.

Specific examples.

*Beth:* In calculus, getting infinitely close to a specific line.

Visual images.

*Beth:* One visual image associated with the concept of infinity is something that increases exponentially and just heads off to infinity, a function. A graph of a function.

Specific images.

*Beth:* Oh gosh! I'm short on specific theorems recently, or much of anything.

First introduction.

*Beth:* Probably I had heard of it in middle school. I hadn't really gotten an idea of how insane infinity was until then.

Changes in understanding.

*Beth:* My understanding of the concept has changed a lot in terms of I think of it as being a bit more spatial and also I started learning about things like countable infinity, uncountable infinity. So, as I've gone through I've learned a bit more about infinity and it's a lot different than 'think of the biggest number you can possibly think of.' Whenever I think of infinity I generally think of graphing a function, at this point the line is heading toward infinity and when I first heard of the concept I just thought, well, think of the biggest number I can think of off the top of my head and it's bigger than that.

Milestones.

*Beth:* I guess my real analysis class. First, I learned about countable

infinity and second, it really pounded it in that infinity wasn't quite a number or anything, it was more of a general concept.

### • Limit

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* I think of getting really close to something but never quite getting there.

Specific examples.

*Beth:* Specific examples coming to mind are as  $x$  approaches infinity the limit of  $1$  over  $x$  approaches zero.

Visual images.

*Beth:* The graph that's associated with that. No matter how close up you get to the graph, you won't quite get there, you'll always be getting closer.

Specific images.

*Beth:* Epsilon-delta, not much else.

First introduction.

*Beth:* First introduced to the concept in high school.

Changes in understanding.

*Beth:* My understanding of this concept has changed a little bit because the first time I took calculus I had a really bad problem with the idea of a limit. I couldn't figure out that the function would never quite hit the limit and it never really hit home until I got to college and took it again and figured out that it never did hit. Eventually it gets there but that's a long ways off so you can't see it yet. That's basically how it's changed, taking Calculus I again in college.

Milestones. [No response]

### • Convergence

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* I think of getting closer and closer to the asymptote. About the same as a limit.

Specific examples.

*Beth:* Getting a point on a line and then going from different directions towards that point and getting closer and closer to it

Visual images. [No response]

Specific images. [No response]

First introduction.

*Beth:* I was first introduced to the concept of convergence in calculus in high school.

Changes in understanding.

*Beth:* My understanding of this hasn't changed much either.

Milestones.

*Beth:* I still haven't found the milestone for it.

### • Interpretive Analysis

In all three of these cases, Beth's concept image seems to be tied to the graphical or (diagrammatical) image of the limiting behavior of a function. In all three cases, there is the idea of getting closer and closer, or larger and larger, but never quite getting there. Infinity, limit points and points of convergence are all unattainable points, somehow out of reach. A significant change occurred in Beth's understanding of infinity, from infinity being just a number that was bigger than any other number imaginable, to a visual, graphical understanding of infinity as something approached but never reached. Learning about countable and uncountable infinity helped to change her understanding also.

### • Calculus Concepts

Beth's understanding of these concepts appears to be that of a typical calculus student.

### • Continuity

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* I think of a line that just has no breaks in it.

Specific examples.

*Beth:* Not off the top of my head, but again continuous but nowhere differentiable functions.

Visual images.

*Beth:* Particular visual images again a line on a graph with no breaks.

Specific images.

*Beth:* I'm trying to drag out my calculus. Epsilon-delta stuff comes into my head.

First introduction.

*Beth:* I was introduced to the concept of continuity in my high school calculus class.

Changes in understanding.

*Beth:* It hasn't really changed and evolved that much over time. My understanding isn't that much different than it was in calculus so I can't think of any particular milestone that changed that.

Milestones. [No response]

**• Derivative**

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* Nasty high school stuff. Some fun.

Specific examples.

*Beth:* Derivative of  $x$  to the  $x$  to the  $x$ .

Visual images.

*Beth:* Any visual images associated with the concept of derivatives would be the tangent of the line.

Specific images.

*Beth:* The only thing I can think of when I think of derivatives is methods of finding the derivative.

First introduction.

*Beth:* I was first introduced to the concept of derivatives in high school in calculus.

Changes in understanding. [No response]

Milestones. [No response]

**• Integral**

Evoked images. (What first comes to mind when you think of concept?)

*Beth:* Finding the area under a curve.

Specific examples.

*Beth:* The area under a bell curve. I can't remember how to do it right off but I can remember looking for it.

Visual images.

*Beth:* All those little bars underneath the curves that approximate. As you make them smaller and smaller they come closer and closer to the integral.

Specific images. [No response]

First introduction.

*Beth:* I was first introduced to the concept in high school my calculus course there.

Changes in understanding.

*Beth:* My understanding changed a little because I wasn't really familiar with the concept of the bars under the curve and how you approximate it with the bars until I got to college and didn't really quite understand it although I could do it until Calculus II, second semester freshman year. I thought of it more in terms of approximation. I guess Calculus II was the milestone that changed my



understanding by realizing that just those bars getting smaller and smaller and closer to the curve.

Milestones. [No response]

- **Interpretive Analysis**

Beth's visual images of these three concepts are all graphical - smooth continuous graphs, slopes of tangent lines, and area under a curve. Continuity is tenuously linked with its formal concept definition, while derivatives and integrals are little more than computational tools. She describes a change in her understanding occurring when she began to understand Riemann sum approximations to integrals, using areas of rectangles. However, her understanding of derivative and integral appear to be predominantly operational.

### **CASE STUDY 3 - CALVIN (undergraduate)**

- **Well Ordered Set, Zorn's Lemma, Induction, Infinity, Cardinality and Set**

Calvin has these concept images linked to one another and switches readily back and forth among them.

- **Well Ordered Set**

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Well-ordered? I don't know, well I think of the natural numbers.

Specific examples. [No response]

Visual images.

*Calvin:* As sets, as continuous, just define that 0 is the empty set, that one is the set that contains zero, you just go down the line. That's what I think about, that's an example of them anyway so, it's a visual image.

Specific images.

*Calvin:* Well when I think about it, I think every set can be well-ordered, the well ordering principle. Um, no big theorems come to mind.

First introduction.

*Calvin:* It was my junior year in high school, fall semester, no sophomore year, fall semester. That was in our introduction to higher mathematics. Using that in induction to prove what you're doing, to show that they're equivalent. Did I? Maybe I didn't, something along those lines.

Changes in understanding.

*Calvin:* Just how to use it. Just the well-ordering principle. Not even, no, not even using that just you know, the equivalence with Zorn's lemma, whichever one's easier to use, is what you use.

Milestones. [No response]

• **Zorn's Lemma**

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Infinity. I don't know, when I think, usually I think of a descending chain of inclusions, that's bounded, or an increasing, whichever way I want to go. Actually it's increasing, I usually think of increasing, and just there's something out there beyond it. When you get to the last point there's something, infinity plus one, there we go. Take an intuitive approach.

Specific examples.

*Calvin:* Yeah, non-measurable sets, stuff to do with prime ideals.

Visual images.

*Calvin:* The first thing that just popped into my head when you said that was a long line. Don't ask me why, I just thought of a long line. There are other long lines, just a long line, it has nothing to do with Zorn's lemma, but that's like infinity plus one. I like using analogies to think about things, usually even non-related topics and just twisting them, and getting an idea out of that.

Specific images.

*Calvin:* Well not theorems, just the Axiom of Choice and the Well Ordering theorem, I associate them with it. The existence of non measurable sets. I haven't done much with Zorn's lemma, very little.

First introduction.

*Calvin:* Topology, when were doing set theory. Well, like into the first part of topology, the review of set theory and stuff that's where I heard it. Also I think of it as, like a super induction, like an induction plus, you know.

Changes in understanding.

*Calvin:* I'm just in the beginning stages, I think. I just really learned it fairly recently.

Milestones. [No response]

## • Induction

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Induction?  $n + 1$ . It really does. Another problem just popped into my head, it was a Putnam problem, an example that I used induction two years ago, the first question, I proved it right. Even examples pop into my head about it. Proving arithmetic progression is one of them, that's crazy.

Specific examples. [No response]

Visual images.

*Calvin:* Going out to a point and being able to reach the next point. Infinity plus one. A little thing of Zorn's lemma.

Specific images.

*Calvin:* The well-ordering principle, I think I can prove it. Well I'm not totally convinced.

First introduction.

*Calvin:* Really the first time I'd seen it, and I didn't know how to do it, was in high school, there were two different types of math classes, and I was in the one that didn't do induction, and some girl asked me how do I do this, and now I realize what it was and how to prove it, I didn't know and she didn't know it. And when I first really learned about it was my sophomore year at college, very late to learn, but that's when I first learned it.

Changes in understanding.

*Calvin:* Just with the introduction of Zorn's lemma. You see, when you're asking how has it changed, it's so hard to think about what I think about now and what I thought about then. It's so hard for me to say. Most of the time I'm going to say it really hasn't changed, because I don't think it has. My perception is almost the same.

Milestones. [No response]

### • Infinity

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Probably the symbol.

Specific examples.

*Calvin:* The circle, yeah, that's my first example. Sequences and numbers, that's just what I relate infinity to - space, time.

Visual images.

*Calvin:* A line. Just the real number line.

Specific images. [No response]

First introduction.

*Calvin:* In the third grade, with the symmetries of a circle. That's first time I remember.

Changes in understanding.

*Calvin:* It was a theorem that responsible for that, Zorn's lemma, of what infinity is, that changed my mind, that you can actually do an infinite number of steps in a finite amount of time, sort of a Zorn's lemma and infinite chain. Oh yea, and chains, infinite chains, when I first learned that I realized infinity is a solid concept. It's not just boom, if you don't understand it it's infinity.

Milestones. [No response]

### • Cardinality

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Aleph nought, I don't know, just size, how big things are.

Specific examples.

*Calvin:* Cardinality of the Cantor set, just cardinality of all types of sets. I don't know if that's an example.

Visual images.

*Calvin:* The only thing I can think of is aleph nought, the closed interval from 0 to 1. Natural numbers, real numbers, just something. The most amazing result I thought of cardinality is that the rationals have the same cardinality as the natural numbers. That's like, wow, that still blows me away. I can't, I can't visualize that, but I know it's true, I can prove it, and I trust, I have confidence in the math. But what if it's all wrong? The biggest thing to me is the rational numbers. It's weird. The rationals are dense, it doesn't make sense. I mean it makes sense but wow, it's hard to visualize. It's like two contradicting ideas, not ideas, you can't relate them, denseness and cardinality don't go, 'cause there's no logic to their relationship at all, but when you first learned about it they go hand in hand.

Specific images.

*Calvin:* The continuum hypothesis.

First introduction.

*Calvin:* Probably topology during set theory, or it could have been the introduction to higher mathematics.

Changes in understanding.

*Calvin:* I haven't done much with it so it hasn't changed at all. Still the same.

Milestones. [No response]

• Set

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* The Cantor set. I don't want to say a collection of objects because that's a class. Um, a class with a certain structure. I think that's right. Um, when I think of sets I think of topology, group theory, even analysis with sequences, doing sequences, series, a group of people, that's an example.

Specific examples. [No response]

Visual images.

*Calvin:* Just a group of things. If it's abstract I just use mathematical properties to think about it. I would say it's tied up with visualization. Not a pictorial visualization, just an abstract visualization, just combine those two words. I visualize the tertiary representation of the Cantor set, just how it's formed using little trees down to it. That's probably the easiest way for me to say it, is the trees, because it's visual, well you can't visualize the whole thing, or maybe you can, I can't.

Specific images. [No response]

First introduction.

*Calvin:* Oh, some time in high school, or even before that most likely, I don't know when. The first time I actually remember it was in high school when we started doing intersection, stuff like that. That was the first time I probably remember.



Changes in understanding.

*Calvin:* Yeah. When I first thought of it I just thought of it as a collection of things, I still do. When I learned how the power of what a set is, not the power set, the uses, like all the properties of sets, the first time I actually saw it, when I read the proof of the Axiom of Choice implies Zorn's Lemma implies the Well Ordering Principle. Wow! Then I realized that you don't even need sets for that, just classes, I think you can just use classes, but with sets it's easy. That's when I found the power, it still is the three driving theorems in set theory.

Milestones. [No response]

### • Interpretive Analysis

For all of these concepts, Calvin evokes algebraic symbols as visual images. His underlying prototypical model for well ordered set and induction is the natural numbers. He describes a conflict he has between the cardinality of the rationals and its denseness. He cannot reconcile the fact that the rationals and the natural numbers have the same cardinality and yet the former is dense while the latter is not. He is aware of his internal conflict because he is unable to visualize this phenomenon. Calvin's paradigmatic model for a set is of the Cantor set. This set looms larger in Calvin's interview than it might otherwise have done, probably because the REU students have recently been discussing this (and also Zorn's lemma, and also Fibonacci numbers) in their lectures. He describes his visual image of a set as an 'abstract visualization' not a 'pictorial visualization.' He visualizes the Cantor set with a diagrammatic image of trees.

It is important to note that for each concept in this group, Calvin consistently points to some other members of the group as an agent of change in his understanding. In particular, the equivalence of the Axiom of Choice, Zorn's Lemma and the Well Ordering Principle seems to have played a role in the development of his

concept image of set and well ordered set. These concept images are all linked to one another and accessible one from the other.

## • Group, Ring, Field

### • Group

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Well, the actions of the group first come to mind.

Specific examples.

*Calvin:* I think of quaternions, p-adics, rationals, a square, symmetries of a square. The first time I saw group theory and modular arithmetic was when I was in tenth grade, I remember this, when I was in tenth grade, I remember where I was sitting, looking at the clock and doing the arithmetic. The first time I'd seen it. Symmetries of a square. I had no idea what it was until I looked at the book and I saw a square sitting there and I'm like, oh I can't believe this. I still remember exactly, I can picture myself sitting there and looking at the board in tenth grade, I was like what does this all mean. We didn't know properties about it we just didn't mention group theory at all. Maybe he did and it went shoo, over my head, but.

Visual images.

*Calvin:* A square. I think of the permutations as well. Just sets, well, elements of a set. I think of them, and an abstract group, mathematical.

Specific images.

*Calvin:* Cayley's theorem, isomorphic to a subgroup of  $S_n$ , Lagrange's theorem, Sylow's theorem. Those three theorems, any other theorems I can't name, those are the big ones, off the top of my head. Don't ask me to prove them, but I know what they are.

First introduction.

*Calvin:* In my sophomore year, and then my junior year, last year, I did independent study on commutative groups, then we hit moduluses, and all that, that just blew me right out the window. When I first got into class I wasn't ready for it, because I was taking introduction classes to induction and all that stuff, just the higher mathematics, and I first went in there and they were doing simple induction proofs, one to one and onto stuff, and I was lost, I had no idea what they were doing, and after that year, after I took the introduction class, and I did the topics that we did in algebra, that was so simple, I was kicking myself for not understanding it the first time. It was like, this is so simple and it happens like that, it's getting easier and easier, to get things out of there.

Changes in understanding.

*Calvin:* I don't know. It's harder than I thought. When I first did group theory it was not that hard. The big part of the beginning of algebra was just introduction to induction, well I had no clue what that was, no clue, one to one functions how to prove them all that, he didn't explain it to us and I had to figure it out myself, that was probably when we first learned it. That doesn't really apply to the group theory stuff, that was just in my mind all the time, I just couldn't, see if I have a problem and I'm, I can't solve it, just don't talk to me. I'll just think about it constantly, everything just kind of flies by me while I'm thinking about it. It took me so long to catch up and when eventually I did I was able to keep up with the rest of the class, and it got easier from there on, not easier as in material wise, but to comprehend what was going on, I was able to comprehend it after that and I got the foundations.

Milestones. [No response]

• **Ring**

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Marriage. No, um, a sloppy field, I don't know just all the axioms. Integral domains, UFD's. I think of Herstein because his picture was in the book.

Specific examples.

*Calvin:* Well, any time I think of a ring I think of a field. I like fields better than rings. I can't come up with a ring off the top of my head which is not in a field. (pause) Oh yea I can. I can come up with just the integers, that will do. Took me a while to think of it, but I did.

Visual images.

*Calvin:* Same thing with a field, just two tables, finite stuff. I mean first off I just go what the properties are.

Specific images.

*Calvin:* Just about unique factorization. That's really fields.

First introduction.

*Calvin:* In abstract algebra, two terms of it. First was group theory, the second was rings and fields.

Changes in understanding.

*Calvin:* I want to say the uses of them, off the top of my head I can't think where I use them. I know there are uses out there for them, I just don't know any specific examples. I mean I learned - maybe I didn't learn - that's why I can't come up with any. No, I would say I haven't, my understanding hasn't changed about rings.

Milestones. [No response]

### • Field

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Galois. I don't know.

Specific examples.

*Calvin:* Well the rationals, the complexes, Gaussian integers, I think of polynomials.

Visual images.

*Calvin:* A table, addition and multiplication tables, because that's how I was introduced to it and that sort of stuck in my head. Actually, it works when it's finite too. Infinite things I guess, there's no visualization, actually yeah there is, with the complexes I think it's easier, 'cause I visualize that by vectors, field of vectors, a vector space. That's how I think of it, in that way.

Specific images.

*Calvin:* Extensions, I don't know. I think it's Eisenstein's criterion isn't it? Popped into my head. I'm doing pretty good.

First introduction.

*Calvin:* Sophomore year, both in linear algebra and abstract algebra. No time before.

Changes in understanding.

*Calvin:* I'm still contemplating.

Milestones. [No response]

### • Interpretive Analysis

Calvin's conceptions of the concepts of a group, a ring, and a field are, by his own admission, still being formulated. He is able to evoke several prototypical examples of each concept - quaternions for a group, the integers for a ring, and the rationals for a field - and has a visual image of group tables to accompany them. He is however, somewhat confused about the distinction among the three concepts and interchanges his examples. He is familiar with the geometrical interpretation of group actions, such as the symmetries of a square, and also at one point mentions the infinite symmetry group of a circle. He is also aware of the hierarchical relationship of these algebraic structures, with a group at the bottom of the hierarchy, then a ring and finally a field. In the case of a ring, but not of a field, he has a link with the non-mathematical image of marriage. He has had a basic introduction to these concepts in an algebra course, but has not had much chance for developing that understanding.

• **Partially Ordered Set**

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* I'd have to go way back and think what a partially ordered set was. Confusion, um, I don't even know, I know what it is but nothing comes to mind but the definition. That's really bad, I grade this stuff in school.

Specific examples. [No response]

Visual images.

*Calvin:* A tree. I don't know. I also think of groups and subgroups.

Specific images. [No response]

First introduction.

*Calvin:* It was in the introduction to higher mathematics in my sophomore year, first semester.

Changes in understanding.

*Calvin:* No. Since I've graded this stuff I've seen the confusion from other people about the definition. So I'm glad that I understand, but maybe I'm not glad because I have to explain it. Just the properties of them, all the general properties. Just from 'given a set of objects prove conditions.' Strict ordering, show that. An example that pops into my head are the words in the dictionary, that's not a partially ordered set. I'd have to go through things in my head to figure out what that set was. I remember doing that, with simple two letter words, ordering, that's it.

Milestones. [No response]

• **Interpretive Analysis**

Despite his feeling that he should know about this concept, Calvin relates mostly confusion. He does not have things sorted out in his mind. He has an incomplete concept image, although he is able to evoke a visual image of a tree, and also thinks of groups and subgroups. Perhaps he has the subgroup lattice of a group in mind.

• **Transformation, Function, Limit, Convergence**

• **Transformation**

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Transformation? Linear. Hundreds of examples. Such as, contractions, retractions, I can't think of the name of it, but quadratic forms, I think of vector spaces when I think of vectors, affine transformations, that's all.

Specific examples. [No response]

Visual images.

*Calvin:* When I think of a transformation I usually think of rotations, that's another example, I think of rotations and just how things happen



graphically, stretching, twisting, flipping, rotating. Symbolically, I know how to manipulate, like multiplying which is easier, diagonalize things, get them in the right form and to relate those together, what's the mathematical properties happening to it and inside my mind to figure out visually what's going on. And back too. It depends on, if I don't understand it then I'll start drawing pictures until I understand it. And if I understand it I'll draw a picture just to see what it's like.

Specific images.

*Calvin:* Well in calculus, proving just different, like the inverse function theorem, because you can do all that and the implicit function, because I just took real analysis so it's fresh, well I think it's fresh in my mind, and just when I did quadratic forms, Fibonacci numbers. I consider, when I think of transformation I think of a two by two matrix, because that's what you really heard about two dimensions in math, it's easy, so two by two, and Fibonacci's because you can define Fibonacci's by using a two by two.

First introduction.

*Calvin:* In calculus, proving all the time.

Changes in understanding. [No response]

Milestones. [No response]

## • Function

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* A mapping from one set to another, that's what comes to mind, a

very plain definition. I just think of  $e^x$  when I think of a function. Examples, millions of them, an uncountable number.

Specific examples. [No response]

Visual images.

*Calvin:*  $e^x$ ,  $x^2$  circles, well circles aren't functions but I consider them. The graph of  $e^x$ . But I also consider it even without the graph, but as soon as I think of  $e^x$  the graph is in my mind, it's almost instantaneous. I can't think of  $e^x$  without the graph. That's what I was using before when I said when I use the picture to help me to mathematically define a picture. Usually they just come both at the same time, and if it's really difficult they don't, then it takes me a while, but once I've done it it's there, the correlation is one to one, once I understand it, but I don't know many of those that come instantaneously like  $e^x$  with the graph. Usually I have to work at it. Once I know it then I'll sit down maybe take a few minutes to think about it, what's going on there, and then out pops an idea.

Specific images.

*Calvin:* Just in topology, open sets. Functions, that's such a broad topic. That's like, you could sit here for hours trying to think of something, think of everything, just I don't know, I've never really sat down and thought about what they really are, although I'm doing all this stuff, it's just too jumbled right now.

First introduction.

*Calvin:* Eighth grade maybe, I was still in junior high school. Introduction to algebra, not abstract algebra, and learning how to graph things. I don't know if that was the first time but I remember doing it then. Ok. That might not be the first time, but that's an early time.

Changes in understanding.

*Calvin:* Well when I took abstract algebra, introductions to higher mathematics, with the one to one bijective and all the properties of the functions, that's what I think about it.

Milestones. [No response]

### • Limit

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Infinity. That's another one with infinity. I think of the limit. Well now I think of inverse limits. 'Cause I need that for my proof, for my project I'm doing here. I think of sequences, I think of continuous functions because you can use those. Fibonacci numbers, the limit process, not representing these, the golden ratio represents the Fibonacci numbers, those are examples.

Specific examples. [No response]

Visual images.

*Calvin:* The epsilon delta proofs. Just getting real close but never touching, you know, if anybody, a non-math major asked me what's a limit, it's when you get really really close, and that's what a limit is. That's the most intuitive sentence I can come up with.

Specific images.

*Calvin:* Like with double limits and the limit of the sum is the sum of the limits, I've got to say this right, and all those properties. Just constants you can pull out. I'm just stating the properties.

First introduction.

*Calvin:* 12th grade in calculus. Just basically when things are continuous we use limits, and how to show what limits are, different values, numerical values of limits.

Changes in understanding.

*Calvin:* Yea, when I first learned what the definition of what a limit was, that's cleared up a lot. I learned about limits very intuitively before that. Then I saw the definition in a later calculus class. I don't know when, I don't know when I definitely saw it. What I know is, in the first year of college, or even high school, end of high school. Instead of that intuitive sense of it getting really really close, thinking about it mathematically, just epsilons, if you have an epsilon then you can do any limit, you can take any limit, oh a circular argument! I can't say a definition without the word in it, off the top of my head.

Milestones. [No response]

### • Convergence

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Limits.

Specific examples.

*Calvin:* Closed sets, I don't know.  $1/n$  goes to zero, there's a classic example, that's it.

Visual images.

*Calvin:* They get really, really close. Um, that's basically it.

Specific images.

*Calvin:* Any bounded monotone sequence converges, Cauchy sequences.

First introduction.

*Calvin:* Junior year, fall semester, in topology, in set theory.

Changes in understanding.

*Calvin:* When I first learned about them I couldn't prove anything. And now I can prove simple things about them, I understand them better, know how to use them, how to apply them.

Milestones. [No response]

### • Interpretive Analysis

Calvin links the concept of transformation with linear, so that his prototypical model for a transformation is of a linear transformation. His visual images are geometric - rotations, reflections. He also visualizes a transformation symbolically through the use of matrices. He says that his understanding has changed, but it is not clear how.

He first evokes the definition of a function as a mapping, and his prototypical models of functions are of the exponential function and polynomial functions. He visualizes these graphically, saying that the graphs are evoked simultaneously with the algebraic symbols. A significant change in his understanding occurred

when he took abstract algebra, but this experience seems to have broadened his understanding rather than having caused a radical restructuring.

He links the concept of a limit with the concept of infinity. His visual image is of the algebraic symbols  $\epsilon$  and  $\delta$  associated with the formal concept definition. He describes a more intuitive diagrammatic visual image of something getting close but not touching. His understanding underwent a significant change when he learned the formal concept definition. Prior to that time, he had an intuitive, visual understanding of limit as 'getting close', but the formal definition gave him a clearer, more precise visual understanding.

Calvin links the concept of convergence with that of a limit, using the same diagrammatic visual image of something getting really close. His prototypical model is of the sequence  $\frac{1}{n}$  as  $n \rightarrow \infty$ . His understanding changed in that he has a certain facility with using and applying properties and theorems associated with convergence. It would seem that perhaps this facility comes once the concept has been condensed, and all the varying facts, definitions, images, properties have been condensed into a compact, unified whole. With this wholistic understanding, one can then begin to *do* things, prove things more easily.

### • Metric Space

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Topology. Metric spaces, distance functions. And all the examples of metric spaces such as Riemann surfaces, Lebesgue measure.

Specific examples. [No response]

Visual images.

*Calvin:* All my examples are in two or three dimensions. I just think of subspaces of  $R^n$  or  $E^n$ . I learned  $E^n$  as the Euclidean space with the metric,  $R^n$  is without the metric. That's what I remember, that's what I was told. So when I said  $E^n$  before I meant  $R^n$  with the Euclidean metric, right,  $E^n$ . Spheres, Moebius bands, things like that.

Specific images. [No response]

First introduction. [No response]

Changes in understanding.

*Calvin:* When I first started topology it seemed pretty easy, it didn't get hard, and then we just faded away and then I started reading out of a book and oh my gosh, I didn't know anything about manifolds. Once I read that I was like, alright this is getting hard. I thought it was easy at first, just a little topic, everything's been done in that. By the way, I like the word '*misunderstanding*' a lot better than the word '*understanding*'. I mean, there's a lot more I don't know than do. When I think of things I perceive, in my mind it's, well, not right but that's the way I think of it, it's a misinterpretation. Stuff that I've never seen but I think about, come up with theorems yourself that have been proven, you haven't seen them before, and you just think about them wrong. That's what I think a misunderstanding would be. It gets harder and harder, but I understand more and more of it, but one is growing exponentially, and the other in polynomial fashion. That's the way it is, when I think about things I come up with so many questions.

### • Interpretive Analysis

Calvin links this concept with topology and distance functions, and he visualizes a metric space as the familiar two or three space. He describes visual images of

a sphere, and a moebius band. His understanding seems to have undergone a rather gradual evolution from misunderstandings to understanding. At the same time, he realizes that there is a lot more to know than he currently does the concept. As his understanding increases so do the number of questions he has related to the concept. But he does not talk about metric spaces metaphorically.

## • Vector, Vector Space, Basis

### • Vector

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* A line with an arrow on one end and a dot on the other. I was a Physics major.

Specific examples.

*Calvin:* That's pretty generic. I think almost everyone thinks, well, in my mind I mean, examples of vectors I think of tensors, like stresses tensors and other tensors. I also think of curves, like tangent curves, tangent lines, just curvature, curves. Examples from vector calculus.

Visual images.

*Calvin:* Ok, and also just like when I deal with rotation and inertia, vectors that way, rules for electricity, that's probably the way I visualize it.

Specific images. [No response]



First introduction.

*Calvin:* I was in high school, I was taking physics, even introduction to geology, just stresses and stuff.

Changes in understanding.

*Calvin:* Yes, in what way. Again how hard they can get, I thought I was understanding them and I'm not understanding them. Um, just the uses, on how you can simplify mathematically. I also think of vectors with transformations. They go pretty much hand in hand, you know, they should represent the vector in matrix form.

Milestones. [No response]

### • Vector Space

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* I think of field theory. I think of Galois theory. Algebra proofs. Physics.

Specific examples.

*Calvin:*  $E^n$ , euclidean space. I don't know, just, oh, rotations in physics.

Visual images.

*Calvin:* I'm just thinking of, I've got objects going through my mind. A cube, circles, and everything else like, when I think of vector spaces, I think of objects, I can characterize the space, that way stick to

three dimensions of course, and if it's over three dimensions then it's all mathematical, I can't, I just don't like doing that, I can't visualize that.

Specific images.

*Calvin:* The only thing I think of is the extension fields of vector spaces. Oh maybe I'm even thinking of that wrong right now, I'm not sure. Not really.

First introduction.

*Calvin:* Well, without knowing what it was, in an introduction to physics class, we used vectors, vector stuff, often we didn't know any of the properties and it wasn't, I didn't even know algebra at the time, so I had no clue what we were actually doing. When I first really saw it I saw it in abstract algebra, we did field theory. No wait, I saw vector spaces before that, saw it in linear algebra.

Changes in understanding.

*Calvin:* I'm still trying to understand it. It hasn't changed much.

Milestones. [No response]

### • Basis

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Linear algebra, topology. (pause) You know with linear algebra just vector spaces, stuff like that.

Specific examples. [No response]

Visual images.

*Calvin:* Just what the basis is, the objects, the defining parameters, of the space that you have. Because I consider basis, and then I know there is group theory stuff that I know nothing about. Groebner basis, or something like that. I read it somewhere. I don't have a visualization, mathematical visual image.

Specific images.

*Calvin:* Like  $n$ -dimensional space, you know, and  $n$  linearly independent basis elements that define it.

First introduction.

*Calvin:* Linear algebra.

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

As with the group, ring and field, Calvin has had a basic introduction to these concepts, in this case in linear algebra. By his own admission, Calvin feels his understanding is not yet complete. His visual image of a vector is a diagrammatic one of an arrow. His prototypical models are applications from physics and mathematics, tensors, tangent lines. He links this concept with transformations and matrices.

He links the concept of vector space with field theory in physics and mathematics. His prototypical model of a vector space is of Euclidean  $n$  space. He visualizes a vector space as a cube or circles.

The concept of basis is linked with the general areas of linear algebra and topology. He visualizes a basis symbolically, evoking the algebraic symbols for the basis elements.

## • Continuity, Derivative, Integral

### • Continuity

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Continuity? A curve. A smooth curve, that's continuous. It's one of those intuitive type things. Functions, so many examples, we'll get to use those.

Specific examples. [No response]

Visual images.

*Calvin:* That line, or I think of discontinuous functions, I think that's the easy analogy, something that's discontinuous.

Specific images.

*Calvin:* Just what continuity means, the definition of it, just stuff in topology with open sets, continuous images of continuous functions,

probably topology, all continuous stuff is from topology, and analysis.

First introduction.

*Calvin:* Probably, it was fifth grade. Doing an example the teacher used to do on the board, put a line of little tiny dots and if it looked like a line from far away, but once you get close it's discontinuous, she didn't say discontinuous, but now I know what she put on the board, and that's when I think the first time it was introduced to me, without me knowing.

Changes in understanding.

*Calvin:* Well the first time I started doing continuous stuff was in calc 1, we did it without the definition, just with visualization, when I first learned the definition I was in calculus three, I mean like really used it. I knew the definition before, the analysis definition. Um, that's when I visualized what it actually was. Because an epsilon delta proof, I think, is perfectly clear, whether something is continuous or not continuous. So I can visualize just a little neighborhood of points. That's what I think of about it.

Milestones. [No response]

## • Derivative

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Derivative? A rate of change. I was a physics major too, ok, so it's a rate of change. When I think of derivatives I think of all the problems with derivatives and integrals. I think also continuous, continuous derivatives, functions, noncontinuous derivatives, almost continuous everywhere derivatives. Partial derivatives, dif-

ferent kinds of derivatives.

Specific examples. [No response]

Visual images.

*Calvin:* When I just graph a two dimensional figure. Just the function, a one variable function, concave up, concave down, that's what I visualize as far as derivatives, how fast it's changing, not how fast the derivative's changing but the value of the derivative, slope, also transformations come to my mind when I think about it.

Specific images.

*Calvin:* Well the definition, the 'if it's differentiable it's continuous.' All the little theorems about, they're not little theorems, because I have them all floating around in there and if I need them I'll think about them, not theorems that pop in my head all of a sudden, it's when I need them, they might come to me, but I just can't sit there and think of them.

First introduction.

*Calvin:* In Calc I.

Changes in understanding.

*Calvin:* Well, like not understanding it would be part of that. Again examples, not just strictly derivatives but the partial derivatives and using those. Every time I think of derivative, integration comes up.

Milestones. [No response]

• Integral

Evoked images. (What first comes to mind when you think of concept?)

*Calvin:* Derivatives. When I first think of integrals I, the first thing that came into my head is a Lebesgue. 'Cause I just did it. Um, and then I've seen, I always have questions in my head about these, and I know there's a Darboux integral. I think I've seen that, that's why it comes to my head, I think I've seen that. I've asked everyone, no one knows what it is. That's the first thing that comes to mind. Um, all the transformations, come to mind, about how to solve them, It's like when you integrate the integral  $e$  to  $x^2$  from negative infinity to infinity, that exists but it's not integrable. I mean yea, the negative  $x^2$  right. It has to be negative. That integral exists but it's not integrable, I forget what that's called, I always forget.

Specific examples. [No response]

Visual images.

*Calvin:* Area, volume. When I think area, I think of surface area, I picture what the object is or at least try to picture what the object is, and look at the surface, volume you just look at the volume, the structure.

Specific images.

*Calvin:* The fundamental theorem of calculus, that's derivatives and integrals combined in one shot, that's the big one, that's the big one.

First introduction.

*Calvin:* High school. In 12th grade probably.

Changes in understanding.

*Calvin:* These I understand better now. I know how to solve the harder ones, at least I think I do, I can relate physics to integral equations now I hope. Just the relationship between other mathematics and calculus.

Milestones.

*Calvin:* There's a few. First is, all the transformations you use, that was pretty big to me, and then the applications to physics, geology.

### • Interpretive Analysis

Calvin's prototypical models for continuity are both an example, of a smooth curve, and a non-example, of a discontinuous function, which he visualizes graphically. His understanding underwent a significant change when he learned the formal concept definition, at which time his understanding became less intuitive. The formal concept definition gave him a clearer visual image of continuity, enabling him to visualize  $\epsilon$  and  $\delta$  neighborhoods clearly. This was a major event in his understanding.

Calvin links the concept of derivative with physics applications - he initially was a physics major - and with the concept of continuity. He visualizes this concept graphically, as both the concavity of a function, and the slope of its tangent line. He also links derivative closely with integration.

Calvin links the concept of integral with derivatives, and evokes several different types of integrals as prototypical models. In particular he evokes an example of a non-elementary function,  $e^{-x^2}$ . He visualizes integrals diagrammatically as both surface area of a solid, and as volume of a solid. He describes his understanding



as having changed for the better in that operationally he can compute more integrals. He is more aware of applications of integration in other fields. However, his understanding does not seem to have undergone a radical restructuring.

#### CASE STUDY 4 - ANDY (graduate)

Andy is unfamiliar with the concepts of a well ordered set and of Zorn's Lemma. His only connection is a tentative link between Zorn's Lemma and the Axiom of Choice. Any link between Zorn's Lemma and a well ordered set is not present.

##### • Natural number or integers as prototype

Andy has similar prototypical images for these two concepts.

##### • Partially ordered set

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* I think of one of those funny little symbols, one of those curly less than signs. I think of the integers, the less than, I just think of a bunch of objects and some relation that goes one way and not the other. Partially ordered set, I'm trying to remember what they are. That's where you just compare certain elements. A poset - yeah I guess that's it, I think of more of a tree structure ... objects on one branch of the tree might be comparable to one another but not necessarily ... a different branch may not be comparable to other branches.

Specific examples. [No response]

Visual images.

*Andy:* Family trees, binary trees, and plenty of other trees, that's sort of a loose description ... somehow you have a set of objects that can be compared that you have grouped together somehow, maybe like the leaves ... the branches.

Specific images. [No response]

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

## • Induction

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* All I guess, stuff going on, and all goes back to the base case. Induction occurs. When you say induction I'm thinking of proof by induction, base case ...

Specific examples.

*Andy:* Well the basic one, summation formulas, proof by induction, sum of the first  $n$  squares, once you see something like that you automatically think induction, any formula involving  $n$ .

Visual images.

*Andy:* I don't know if it's visual. It's sort of visual in the sense that I think of it as steps of moving up and down and you start at the  $k^{\text{th}}$  step and you know that's true because  $(k - 1)^{\text{st}}$  step is true and you can drop back down to that one and you recurse all the way back down to the bottom level and that one you've shown to be true and it's like boom!

Specific images.

*Andy:* The principle of mathematical induction. There's usually a very well known theorem that people state and I automatically think, oh yeah that's true by induction, but I don't think of theorems as, just theorems about induction but theorems that are true by induction.

First introduction.

*Andy:* Probably discrete math.

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

Andy's visual images of a partially ordered set include an algebraic symbol associated with this concept, namely the  $\prec$  symbol, often used in mathematical writings to represent a partial ordering relation. Tree diagrams act also as a concrete analogy to help him reconstruct the salient properties of a partially ordered set. He

elaborates on this analogy by describing the elements as leaves of the tree, and those elements which are comparable must appear on the same branch of the tree. This analogy (or metaphor) with a tree provides him with an extensive vocabulary, associated with a very familiar non-mathematical concept, with which to describe this mathematical concept.

At one point in his reflection he recalls the familiar mathematical contraction for a partially ordered set, namely the word '*poset*'. This provides the verbal, symbolic link to more information in his concept image. Indeed, this recollection of the term '*poset*' may have been a necessary trigger for Andy to recall anything more about a partially ordered set. This term evokes the tree-like structure that allows him to describe the salient properties of partially ordered set.

The integers form his prototype for modeling the features of a partially ordered set, thus any partially ordered set should have the same salient features as the integers. But, in order for this set to be an effective paradigm for a partially ordered set, Andy must be aware of the particular features of the integers shared by any partially ordered set. Initially, he seems to have trouble with this. Perhaps, since the integers are a *totally* ordered set, that set alone, without the tree-like diagram, would not be sufficient to provide an effective paradigm for a partially ordered set. However, with the tree-like diagram clearly in his mind, he is able to describe the property that some pairs of elements are directly comparable, but not necessarily all.

It seems that both the word '*poset*' and the tree-like diagram serve as links which connect the concept of partially ordered set with all the information that Andy has in his own personal concept image regarding partially ordered sets.

Andy's description of his visual image associated with induction includes words like 'steps' and 'moving up and down' that hint at an underlying ladder metaphor, though he does not make this explicit in any way. The implicit underlying mathematical model for this metaphor is the natural numbers. It is clear, however that he actually thinks of these steps, that they do form a part of his concept image of

induction, and that they provide a way for him to accurately describe the process of induction. Thus, these words seem to play the role of a link between Andy's 'life experiences' and his 'world of mathematics', again providing him with a language for his mathematical environment.

### • Group, Ring, Field - a Linear Order Schema

Andy uses a linear order schema to organize his concept images of algebraic structures in a hierarchical arrangement. He knows that the higher an element's position in the schema, the more structure it will possess.

#### • Group

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* I think of a collection of objects where one stands out. A circle with dot, and in a group you have the identity. A bunch of stuff, a bunch of objects, one of them is the only one that stands out, one important object. I think of it in a hierarchy of semi-groups, semi-group or monoid, I don't know which, I'd have to play around ... and then you have a group, they don't have that much structure, rings, I know you don't have nice operations, I usually think of examples. That reminded me again of a little bit why ... I used to I guess ... I used to think of groups sort of as discrete sets, very discrete and you don't have that division, whereas fields I thought of as more continuous but that's changed in the last year or so since I learned about topological groups ... that's something that's changed, thinking of groups as discrete sets.

Specific examples. [No response]

Visual images. [No response]

Specific images.

*Andy:* Sylow theorems. Group homomorphism theorems, mod out by a normal subgroup. Mod out by the kernel. Vague recollections about all these awful theorems about the order, which sub-group divides the order into prime powers and all these number theoretical relations.

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

## • Ring

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* Something better than a group but not quite as good as a field. I always have to stop and think ... a ring, you get yourself your second operation, you can add things, you can multiply things, but you still can't divide. So examples, integers. I guess it's a group so I can multiply.

Specific examples.

*Andy:* [No response]

Visual images.

*Andy:* Sometimes I sort of see a collection with an identity element.

Specific images.

*Andy:* The theorems when you get rings, when you take a field and you mod out by an ideal and you get rings. So I guess theorems with rings.

First introduction.

*Andy:* In algebra.

Changes in understanding. [No response]

Milestones. [No response]

## • Field

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* Like on the quad out there? 'Field of Dreams'? Mathematical field? Usually when people say field, I struggle a little bit because algebra I never enjoyed that much. So I think of real numbers and I think of being able to ... When someone says field I always have to remind myself 'Ok, that means I should divide,' I sort of say that to myself to remind myself 'Ok, field, that means I can divide.' So, I might think of just the real line.

Specific examples. [No response]

Visual images.

*Andy:* Um, not really except I see symbols like  $x$  and maybe also the symbol  $\frac{1}{x}$ , and I know that I have those around.

Specific images. [No response]

First introduction.

*Andy:* Probably senior year in college.

Changes in understanding.

*Andy:* Not too terribly much. I may be able to see it sort of at the top in the hierarchy a little bit better than I used to when I learned about monoids and groups and rings and fields. Sort of at the top of the hierarchy. One of the best things in math, if you're going to do any work, you need coefficients, a set of scalars. If you have scalars in this field then certainly a lot more power then, in my understanding it's a sort of a powerful thing I guess.

Milestones. [No response]

### • Interpretive Analysis

Andy's visual images of a group include a diagrammatic model of a circular blob containing the group elements, with one special element distinguished, this being the identity. This simple model helps Andy to remember the existence of an identity element as one important property of groups. To recall other properties of groups, he visualizes a hierarchical image schema comprising many abstract structures, in which a group sits in relation to other abstract algebraic structures. Thus,



Andy's concept image of a group is intimately linked with his concept images of other familiar abstract algebraic structures. Not only does he understand a group as a set itself (with additional properties), but also as an element in an ordered set of algebraic structures, including groups, rings and fields.

A milestone he mentions is that at one time he considered groups as being somehow discrete, with spaces or gaps between elements, as opposed to fields which he considered as being more 'continuous'. This was changed by his introduction to topological groups. This may indicate that his prototypical example of a group is the set of integers under addition, whereas in contrast, his prototypical example of a field is the real numbers. This would account for his remark of "spaces in between" with respect to groups. If this is the case, the realization that the non-essential topological differences of these two number systems do not suggest general differences between the algebraic structures of group and field was a milestone in his understanding. While this was a significant event in sharpening Andy's concept image of groups, we point out that it is not an earthquake-like *reification* event (in the sense of Sfard) of the concept.

Andy's visual image of a ring is diagrammatic, essentially the same as that of a group. Thus it is not this visual image that links him to other parts of his concept image of a ring. However, he is also aware of a ring's place in the hierarchy of abstract algebraic structures, and it is this position in the hierarchy - above groups - that provides the hook to additional properties of rings. Once he positions a ring in the correct place, he knows that it must have additional structure, in the form of a second operation, which he immediately recalls. He describes the integers as his prototypical model.

It is interesting to note that Andy's evoked visual images of fields include non-mathematical images, though he is well aware of the mathematical context of the interview. He switches easily to the mathematical images, including algebraic symbols,  $x$  and  $\frac{1}{x}$ , for field elements, and also a field's position at the top of the hierarchy, providing the link to remind him of the division property. His prototypical

model for a field is the real numbers, which he visualizes as the real number line. It seems that this model provides an image of a field as a set which has no spaces between its elements, since the real numbers lie on a continuum, thus giving rise to his understanding that fields are continuous, whereas groups are discrete.

This hierarchical schema of algebraic structures is of central importance for Andy in his organization of and access to his linked concept images of groups, rings and fields.

### • Metric Space, Convergence - similar diagrammatic images

Andy evokes similar diagrammatic images associated with each of these concepts.

#### • Metric Space

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* How far. I think of, easy topology, a topology, the fun part about topology is when you do all this stuff about a metric ... once you're given a metric, everything you want happens. Metric space, I usually think of, what I think of is a topological space, a very powerful topological space where nothing bad happens.

Specific examples.

*Andy:* If I'm thinking of a topological space and once I'm given a metric I automatically drop back to Euclidean space and, you know, draw balls with radius. That was my first exposure to metric spaces.

Visual images.

*Andy:* I see these little dots and circles around them, the Hausdorff separation axiom. Usually when I think of metric I think of Hausdorff and visual image is two dots that are separated with circles around them.

Specific images.

*Andy:* The Urysohn metrization theorem, when you can, when, ... existence of a metric, when you take a space and begin building a metric, that's the big one I guess, of the topology theorems. You have a category that are all metric spaces.

First introduction.

*Andy:* Maybe in advanced calculus as a junior in college, we did a little topology of  $R^n$ . Maybe not. I did a little in undergraduate topology course. If they had the definition for it probably it didn't make much sense. All the spaces you were working in were metric spaces. Once you have topology then you get introduced to topological spaces that aren't metric spaces.

Changes in understanding. [No response]

Milestones. [No response]

### • Convergence

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* I think of a curve ...  $\frac{1}{x}$  curve and all that stuff.

Specific examples.

*Andy:* Well another thing I think of is, more in a topological sense, where if you have a convergent sequence, pick any neighborhood of the point you're converging to, and that neighborhood will contain other points of the sequence no matter how small you go.

Visual images. [No response]

Specific images.

*Andy:* I guess all the little tests for convergence.

First introduction. [No response]

Changes in understanding.

*Andy:* Well, maybe in the sense that, when people write sequences up, people talk about sequence, thoughts that you might have in your head does it converge or does it not converge and before you might have to stop and count to myself and think does this sequence converge, but now it's completely different when someone writes a sequence on the board I know what to do.

Milestones. [No response]

### • Interpretive Analysis

Andy's visual image associated with a metric space is a diagrammatic model - triggered by the link between metric space and Hausdorff space - of the Hausdorff separation property, one of the important topological features of a metric space: two

points separated by disjoint open sets. His prototypical model for a metric space is Euclidean space, which is accompanied in his concept image by a diagrammatic image of balls with radius. He describes Euclidean space and the image of open balls as the models he relies on whenever he works with a metric space.

Andy describes two important visual images for convergence - a graphical model, and also a more general topological diagrammatic model of a neighborhood around the point of convergence. It seems that although Andy has been introduced to the more general topological model of convergence, he is not yet ready to let go of his earlier graphical model. In fact, he evokes the models in the order in which they were learned, with the graphical one being evoked *first*, suggesting that this model still occupies the foreground in his concept image of convergence.

### • Infinity, Cardinality - linked concepts causing change in understanding

Andy's understanding of countable and uncountable cardinalities is associated with his understanding of infinity. These two concepts developed simultaneously for Andy, each affecting the other.

#### • Infinity

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* I think of the very end of the real line. In either direction, or a lot of times I'll think of the North Pole on the sphere.

Specific examples. [No response]

Visual images. [No response]

Specific images. [No response]

First introduction.

*Andy:* I guess it was probably in calculus. Maybe before.

Changes in understanding.

*Andy:* I'm sure at one point it was pretty confused, infinity was something kind of bizarre . . . I'm sure at first it wasn't something very concrete, didn't seem very rigorous, just out there, but now I understand that when mathematicians use the symbol they actually mean something by it, it's defined.

Milestones.

*Andy:* Thinking of it as the north pole of a sphere. You think of it as some place you can actually see and get hold of and draw a picture of rather than always something that's off the board. Try to draw the real line and say infinity is at the end of the real line, well of course you can never draw it, but you can draw the point at infinity when it's up there as the north pole of the sphere.

### • Cardinality

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* Measure, size of something.

Specific examples. [No response]

Visual images.

*Andy:* As you get into the different cardinalities, is the cardinality of the integers larger or smaller than cardinality of the reals? When I think of the rationals, I know it's a countable set. In some sense there's room in between.

Specific images.

*Andy:* I was thinking of the theorem that tells you how many functions there are from one set to another, and the cardinality of that set is greater than the cardinality of the set.

First introduction.

*Andy:* In discrete math, junior year of college.

Changes in understanding.

*Andy:* Not of finite cardinality, that's just the size of the set. Maybe being used to the idea of different notions of infinity.

Milestones. [No response]

### • Interpretive Analysis

For Andy, the word 'infinity' evokes contrasting visual images. These include images developed earlier in his mathematics education, where infinity represents

the imagined 'ends' of the real line, and a more tangible analogic image of the north pole of the sphere. This analogic model of the sphere to represent the complex numbers (or the points in the plane) is extremely useful and important to Andy since it allows him to '*concretize the referent*' (in the sense of Presmeg), and he emphasizes the importance of being able to get his hands on infinity. It provides a link between an abstract mathematical object and an experiential object - a sphere or ball. Indeed, this seems to have been a reification event in Andy's understanding that radically changed his conception of infinity.

The rational numbers form Andy's prototypical model of a set with countable cardinality. He describes his associated visual image as the idea of there being, in some sense (which he does not make clear), space between rational numbers. It seems he is attempting to link the mathematical concepts of cardinality and denseness, similar to the way in which he attempts to link the topological and algebraic properties of the abstract structures of groups, rings and fields.

He describes a change in his understanding that occurred as a result of learning about different notions of infinity, of countable versus uncountable cardinality. It seems unlikely that this change was of earthquake-like proportions, but rather that it contributed to a gradual broadening of his understanding of both infinity and cardinality.

### • Transformation, Function - interchangeable concepts

Andy considers transformation and function to be interchangeable. He evokes one generic diagrammatic image that serves both concepts. However, he also evokes different images associated with transformation than with function.

### • Transformation



Evoked images. (What first comes to mind when you think of concept?)

*Andy:* Matrix.

Specific examples.

*Andy:* The obvious ones are the rotations, reflections, various rotation matrices with sines and cosines. At a more advanced level I think of the geometry of coordinate vectors, transformations, locally given by matrices.

Visual images.

*Andy:* I just thought of it. Square brackets and stuff. I think of just rotations, reflections, things shrinking and stretching. Other than that, the more abstract visualization, I guess you could call it a visualization, is where you draw one picture over on the left hand side and draw what that picture looks like on the right hand side after it's been transformed according to the rules ... sort of a before and after type thing.

Specific images. [No response]

First introduction.

*Andy:* In that language, probably in linear algebra, my sophomore year.

Changes in understanding.

*Andy:* It gets more complicated, I think, integration,  $u$ -substitutions, changing coordinates with transformations which is something I didn't think of in that way.

Milestones. [No response]

• **Function**

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* Sort of a black box with a hopper that you can put something in, or transformation, the letter  $f$ . I guess, whenever anyone mentions function, if it's not obvious or it's not written down explicitly, I almost always have to write down the picture, like  $f$  and actually write down the domain, the range, what sort of objects going in and what sort of objects coming out. Especially reading papers in geometry the objects going in and out are usually not functions in terms of it acts on fifteen vectors and four co-vectors and out pops maybe a scalar function, it's really confusing what sort of things it's acting on and what sort of things are coming out, usually ... the domain and range.

Specific examples. [No response]

Visual images.

*Andy:* The graph of the function. Very similar I guess to tranformation, before and after ... what the domain looks like, what the range looks like.

Specific images.

*Andy:* Implicit function theorem comes to mind. I always have to stop and think what does the implicit function theorem say.

First introduction. [No response]

Changes in understanding.

*Andy:* It's probably changed from wanting to know the domain and the range. How it's changed? ... technical definitions of functions. Now I recognize when things are or are not functions.

Milestones. [No response]

### • Interpretive Analysis

Andy visualizes a transformation as a matrix including the square brackets and the elements inside. Like the set braces, this algebraic symbol for matrix essentially doubles as a diagrammatic representation, since it consists of a two dimensional array of elements enclosed in brackets. He also describes a very generic and non-specific diagrammatic '*before and after*' model of two blobs representing the domain and the range, and the transformation acting between them.

An important family of examples consists of geometric transformations such as rotations and reflections, and their representative matrices whose elements are sines and cosines.

Andy describes several different types of visual images associated with this concept, the first one being the algebraic symbol  $f$  used to represent a function symbolically. He also describes graphical images of functions - diagrammatic images - and a black box with a hopper - an analogic model for function, which emphasizes the operational nature of a function. He also describes the same '*before and after*' diagram that he associates with transformation, which applies because he considers function and transformation to be equivalent. Nevertheless, he evokes the image of a matrix associated with transformation but not with function. Conversely, one image associated with function but not with transformation is the graphical one. It is important to note that Andy describes a total of four different types of images

associated with function - algebraic symbol, graphical model, diagrammatic model, and analogic model. However, his understanding actually changed as a result of his learning the formal 'technical' concept definition and applying it to recognize a function.

• **Vector, Vector Space, Basis - linked linear algebra concepts**

These concepts are all ones that Andy learned in linear algebra, and he has their concept images linked: A vector space is a collection of vectors, and a basis is a special collection of vectors which may be used to represent any vector in the space.

• **Vector**

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* An arrow, a line segment with an arrow at one end and a dot at the other. Just the notation, I think of an arrow where the top looks like that. Vector fields ... anywhere from the little line in the plane to the differential operators depending on what context I'm in.

Specific examples. [No response]

Visual images.

*Andy:* I see arrows, or collections of arrows like vector fields, ... tangent vectors.

Specific images.

*Andy:* I thought of one, but it's not directly related to vectors. I'm thinking of the theorem of when a manifold admits a metric, a metric is a bilinear form that acts on vectors. Another one, non-vanishing vector fields.

First introduction.

*Andy:* Probably linear algebra. Vectors, that's all, an expression, a change, a certain angle, going from a vector as an arrow in a plane to a differential operator or even more abstract, vector space, anything that belongs to a vector space is a vector.

Changes in understanding. [No response]

Milestones. [No response]

## • Vector Space

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* A collection of vectors.

Specific examples. [No response]

Visual images.

*Andy:* I usually think of  $R^n$  as the vector space, as I think through the theorem or the statement, I think of the points in  $R^n$  as vectors, not necessarily the line parallel ... each point. I think of that

or just as a collection of abstract objects that I know satisfy the theorems that I have. If I have to try to visualize the addition of vectors again I would visualize in  $R^n$ . I guess I'm more seeing  $R^3$ . I don't claim to be able to see higher dimensions! Not me. As far as vectors go I just think of points in  $R^3$ , and addition, where you picture your own position vectors. Although at this stage when I think of adding vectors in  $R^n$  I don't really think of drawing one arrow and a different arrow and drawing the diagonal and parallelogram, I don't go through all that. I'm more comfortable thinking of the vector just as points.

Specific images. [No response]

First introduction.

*Andy:* In linear algebra,  $R^3$ .

Changes in understanding.

*Andy:* It's changed. I guess the first vector space you see is a little bit like  $R^2$  and then they abstract the idea and say well vector space is anything that's a collection of objects satisfying some hypotheses. Now the problem is now I have the idea I visualize everything as  $R^n$ , I think of it abstractly.

Milestones.

*Andy:* It must have been gradual.

### • Basis

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* The  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , in  $R^3$ . A little prejudice there.

Specific examples. [No response]

Visual images.

*Andy:* I see them like toothpicks with ... a basis in  $R^3$ .

Specific images.

*Andy:* Change of basis theorems.

First introduction.

*Andy:* As far as bases, vector spaces, in linear algebra, as a sophomore.

Changes in understanding.

*Andy:* Probably not that I can remember. I think of it as the minimum number of objects you need to identify a point in space in space.

Milestones. [No response]

### • Interpretive Analysis

Andy's visual image of a vector is either the standard diagrammatic one of an arrow, or the algebraic symbol,  $\vec{x}$ , a letter with an arrow above, used to denote a vector, although he also describes vectors as merely points in  $R^n$ .

Andy's visual image of a vector space is of  $R^3$ . This, or more generally  $R^n$ ,

serves also as his prototypical model for a vector space. He describes using this model whenever he has vector operations, such as addition of vectors, to perform. However, he stresses that he no longer uses arrow diagrams, but rather he now thinks of vectors as points. He describes some gradual changes to his understanding attributable to the idea of generalizing to arbitrary  $n$ -space.

Consistent with the fact that  $\mathcal{R}^3$  serves as his prototypical model for a vector space, Andy's prototypical model for a basis is the standard basis for  $\mathcal{R}^3$ , represented using the algebraic symbols  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ . He visualizes this basis diagrammatically, as toothpicks in three space. He summarizes his general intuitive understanding of the concept of basis in the phrase "the minimum number of objects you need to identify a point in space."

- Limit, Continuity, Derivative, Integral - linked calculus concepts

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* Something like a symbol is all I get really. Graphs in calculus. A graph zipping up to infinity as it approaches an asymptote. Actual pictures of functions that behave strangely at certain points where you're probably interested in their behavior.

Specific examples.

*Andy:* Well the problems in calculus where you draw step functions and the left-hand limit is different from the right-hand limit, that's an example.



Visual images. [No response]

Specific images.

*Andy:* Definition of the derivative, that isn't really a theorem. Theorems? Lebesgue dominated convergence theorem, when you can pass the limit through an integral sign.

First introduction.

*Andy:* Probably after being in calculus, high school. That's how the definition grew in terms of a limit. I imagine that was the first time I saw it.

Changes in understanding.

*Andy:* Well, I understand what a limit is now when I probably at first didn't. I'm more comfortable with the idea of a limit now.

Milestones.

*Andy:* Nothing specific, unless it's just my applying something else, as far as milestone, to understanding something better.

### • Continuity

Evoked images. (What first comes to mind when you think of concept?) Oh, a nice smooth graph, one with no breaks.

Specific examples. [No response]

Visual images.

*Andy:* The one picture I drew [of a nice smooth graph with no breaks], and then to sort of complement my pictures, graphs with points missing, and step functions. I think of those when I think of continuity.

Specific images.

*Andy:* Any theorem in calculus, any theorem of differential geometry, requires continuity, especially in relativity and differential geometry, in topology.

First introduction.

*Andy:* Probably in calculus, as a freshman, no actually in high school I guess, in calculus. I'm sure I was introduced to continuity then. I honestly don't remember what I thought about it in high school.

Changes in understanding.

*Andy:* Of the things I described already, I can certainly see the  $\epsilon - \delta$  definitions of continuity and the relationships between the definitions alot better now I'm sure. I can see now that they're the same thing. Shrinking down. Shrinking epsilons over here, and deltas ...

Milestones. [No response]

## • Derivative

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* The first thing that came to mind, I saw the little prime symbol,  $f'(x)$ ,  $\frac{dy}{dx}$ , and rates of change, slopes of tangent lines to curves.

Specific examples.

*Andy:* The one that comes to mind is the slope of the tangent line.

Visual images.

*Andy:* Certainly the tangent line. The tangent lines to curves. Rates of change I don't see visually. The limit definition of derivative. Visually you usually think of the slope of the tangent line.

Specific images. [No response]

First introduction.

*Andy:* High school, senior year.

Changes in understanding.

*Andy:* It's changed alot. To be able to, when someone says derivative, still think of all the different things associated with derivatives, rates of change so many things can be interpreted as a derivative, the fact that I understand now what the definition means whereas before you know I could still crank the formula out.

Milestones. [No response]

• Integral

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* Limits of Riemann sums, area under a curve. Usually I think more of the summation notation, If I'm writing an integral on the board I'm not exactly sure what context I'm thinking of and I'm trying to figure out what it is that I'm trying to get a handle on, I just think of it as I'm adding up a bunch of rectangles.

Specific examples.

*Andy:* The area under the curve.

Visual images. [No response]

Specific images. [No response]

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

The word 'limit' initially evokes for Andy the algebraic symbol for limit, which is immediately followed by an entire family of both graphical examples and graphical non-examples of the different limiting behaviors of functions. It appears that non-examples are at least as important as examples in providing a scaffolding for his understanding of limit.

He uses the same type of diagrammatic model for continuity as he does for limit, namely graphs of both examples and non-examples, of both continuous and discontinuous functions, emphasizing the importance and the usefulness of non-examples as well as examples in solidifying his understanding of this mathematical concept.

He describes the link he has formed between the graphical model for a continuous function and the formal (analytic) concept definition of continuity using an  $\epsilon - \delta$  argument. It seems that, at one time, he held both the graphical model and the formal concept definition in his concept image but they were not linked. This linking may have been a significant event in his understanding of continuity, but he is so far removed from it that he is now only dimly aware that he once knew the two models only separately, without being conscious of how they might be linked. Thus, if there was a reification event here, he does not remember it.

Andy first describes algebraic symbols associated with derivative, namely the notation  $f'(x)$  and also  $\frac{dy}{dx}$ , followed by a diagrammatic visual image of derivative as the tangent line to a curve, linked to the formal concept definition of derivative as a limit.

He describes a significant milestone in his understanding occurring when he made the transition from a formulaic or algorithmic understanding of a derivative as something to be calculated by applying a formula, to a much broader understanding of a derivative in terms of the many interpretations and applications of derivatives one encounters in mathematics and physics, together with an understanding of the formal concept definition of a derivative as a limit of a difference quotient. It seems plausible from his language that what he is really describing is his passage from an operational understanding of a derivative to a dual operational-structural one.

The concept of integral first evokes the idea of a Riemann sum, in fact Andy thinks of an integral *as* a sum, as opposed to having an algorithmic understanding of an integral as a computational tool. This indicates that he has a predominantly structural rather than operational understanding of an integral. His visual image

of integral is of the area under a curve. He also describes the algebraic symbol of summation notation associated with integral, but does not mention the integral sign itself. Although the occurrence of any reification event is not discernable, it is significant that, while Andy considers both derivative and integral to be computational tools as well as objects, there is no evidence that the two concepts are linked as inverse operations in any way.

• Set

Evoked images. (What first comes to mind when you think of concept?)

*Andy:* A group of stuff. Braces and stuff inside.

Specific examples.

*Andy:* Well, pretty much the ones I taught in discrete math and people are always confused as to what sets are. I always have to tell them, just collections of objects: cars and trucks, people or numbers, and stuff like that.

Visual images.

*Andy:* I see those curly braces holding the stuff inside, or Venn diagrams I think of a lot, some boundary and then junk inside it and that's a set.

Specific images. [No response]

First introduction.

*Andy:* I don't know. Well, you deal with sets of real numbers a lot in calculus all the time, but you don't think abstractly you just call them sets and here's something else you do with them. I guess my junior year in college I had a discrete math course. But I don't know my reaction when they said this is a definition of a set. I'm sure I had seen the word before and probably even used the word before but never had it clear in my mind how I would use it mathematically.

Changes in understanding.

*Andy:* Not really too much, it's always been a bunch of stuff. I don't think it's changed much. I guess, no let me change that, it has changed a little bit... if you look at... you really get interested in what a set is when you start reading things like Russell's paradox in set theory. You realize that a set, even though it seems so easy to define, is actually very hard to define. And if you talk of the set of all sets, well it's not even in the set, supposedly that's the paradox, so I guess it has changed in the sense that even though I know it's a very basic concept you have to be really careful, especially when you're talking about sets of uncountable numbers and things like that. It's still a mystery.

Milestones. [No response]

### • Interpretive Analysis

Simple visual images - algebraic symbols of braces which act as the container for the elements, and the elements themselves - appear to serve as much as a diagrammatic representation of a set as Venn diagrams for Andy. These collections of objects are first evoked by the mention of 'set'. Later on, Andy describes two significant events - an awareness of Russell's paradox, and also the mysteries he associates with uncountable sets - which deepened his understanding and awareness of the complexity of sets.

CASE STUDY 5 - BILL (graduate)

• **Well Ordered Set, Partially Ordered Set, Zorn's Lemma, Induction**

These concepts are all linked together for Bill.

• **Well Ordered set**

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I think of induction again. A well ordered set I think of as being just totally ordered, that there's a relationship between all the elements.

Specific examples.

*Bill:* The integers are my example. I really don't have an image of a well ordered set though. Doesn't seem that it's something that you really use very much, directly or have cause to think about. The first time I was introduced to it though was in number theory as an undergrad. But I think it was just mentioned in the passing. In fact I think normally it's always just always mentioned in passing.

Visual images. [No response]

Specific images. [No response]

First introduction. [No response]



Changes in understanding. [No response]

Milestones. [No response]

### • Partially Ordered Set

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I think of Zorn's lemma because that's the first time that we ever used it. I remember the first time I was introduced to it was in my first year here in linear algebra. We did this little exercise about partially ordered sets, and I remember just thinking that in order to really think of these things correctly you just need to not think so hard. Not everything has to be comparable to the other thing, that's what it means to be partially ordered, you can't compare everything to everything else, so that's the image. Bob had drawn this picture of these things that kind of branched off so once you've branched off there was no comparing and that was alright because you were only partially ordered.

Specific examples. [No response]

Visual images.

*Bill:* Yeah, just Bob's tree.

Specific images. [No response]

First introduction.

*Bill:* Yea, in graduate linear algebra. My ideas have changed a little bit because now it's easier to recognize when things are partially

ordered and when you need to talk about the totally ordered subset and all that kind of stuff.

Changes in understanding. [No response]

Milestones. [No response]

### • Zorn's Lemma

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I love Zorn's lemma now. Just yesterday in the seminar Bent said something and I said 'I smell Zorn's lemma' and I was right, because two seconds later he says 'we order by inclusion and now we use Zorn's principle' and I'm like yes, I got it. I only really began to recognize when you would use it this year. I think of it as being a lot like induction. I don't know why. It's that same process, the same idea behind it basically. With induction, it's not necessarily a maximal element, but with Zorn's lemma, you have your set, you have something you can say about it, you have a way you can order it and then you just need to show something about it. You know there is a maximal element and then usually you want to show that it's a certain thing that you suspected it was going to be. I always think of it just in the same light as induction.

Specific examples.

*Bill:* Just yesterday [in the seminar], and examples of when I used it on the qualifier. That's very near and dear to my heart because I used it correctly. As for a Zorn's lemma proof that I would just pull out of a bag, I think there are a lot all the same so I really don't have a specific one. There's nothing that really sticks in my head when I think of a Zorn's lemma proof. I think of it also being equivalent to the axiom of choice, use it to prove that every vector space has a basis.

Visual images.

*Bill:* All I usually think of is big, because usually it's really a big set, or a big something. It's never something puny that you use Zorn's lemma on. It's kind of a strange question. In a way I get, it's not really an idea or mental image, it's just a feeling. It's sometimes more of a feeling than it is a visual image.

Specific images. [No response]

First introduction.

*Bill:* The first year of graduate school was the first time I had heard about it and I didn't really know what exactly it was about.

Changes in understanding.

*Bill:* When I first heard it I didn't get it at all. Then we learned about all the partial ordering stuff, and it's like ok, I still didn't get this, and then he'd say oh and by Zorn's lemma and I'd go 'pardon me', 'cause it didn't make much sense. But now I understand just how the process works, how you would apply Zorn's lemma. I really understand it a lot better in that I can recognize it as a Zorn's lemma, and I can also recognize when to use it in a problem whereas before I would never have thought to use Zorn's lemma.

Milestones. [No response]

## • Induction

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* Induction. The next one. I always think of being able to get to the next one from where you're at. And I usually don't think of having to start off on the ladder, that just always seems like a nuisance. One way it was explained to me was that induction is like a ladder and you have to first be able to get on the ladder so you have to get on some rung, and then you always have to know that from whatever rung you're on, you can get to the next one. And so that's kind of the idea, and so I guess I always think of getting to the next rung, but actually getting on the ladder doesn't seem like you need it in the earlier problems. A little later, I did an example of one where you couldn't get on the ladder. So I think of just being able to get to the next step or the next rung.

Specific examples. [No response]

Visual images.

*Bill:* There's the ladder image. But there's also the image of  $k+1$  because that's always the variable you use. And I guess it depends on how I'm using it, what I think of it. If I know that induction is going to get me where I want to go then I just basically run through the motions, because, if I know that it is going to be true for all those integers then I just basically do the whole  $n+1$  thing, and I just kind of whip through it. In that way it just becomes a tool and it's just a manipulation thing. If it's just part of a deeper proof, it seems like it's pretty simple. If I can see my way to the end of the induction then I just turn on autopilot and just go through it. But if that's really what it's going to be, then I think of how does the previous one relate, what's the relationship that will bring me up there to the next one. The other thing that I think of when I think of induction, that always made me mad, is the idea of weak induction, because I never really worked this out, so maybe I should work this out sometime, weak induction, when you assume that they all work below or whatever, and I thought to myself that's just the same. So that's the major example that pops into my head just because maybe that's the question I haven't answered for myself, is, if it's really always ok to just, I mean, it seems like strong induction is the same, in some cases, as weak induction.

Specific images.

*Bill:* Well, besides the principal of mathematical induction, I'm trying to think of a good induction proof. It reminds me of Zorn's Lemma, which isn't a theorem, just a Lemma! It reminds me of that mainly because it's related to the well ordering principle. I just think it's the same idea. The proofs have the same feeling, the ideas are the same. In Zorn's Lemma you are looking for a maximal element, but for some reason they are always related in my head, induction and Zorn's Lemma are a lot the same. It's just that you use them in different places, but they're basically the same type of tool.

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

Bill describes his prototypical model of a well ordered set as the integers, which has the property that every element is comparable to every other. It does not however, satisfy the least element principle. He does not have any visual images. He has this concept linked with induction.

He describes a diagrammatic image of a tree-like structure to represent a partially ordered set. He uses the language provided by the analogy with a tree to talk about the properties of such a set. He links Zorn's lemma with this concept.

He describes a visual image as 'big', a feeling rather than any clear, sharp image. He links the Axiom of Choice and also induction with Zorn's lemma. He describes a gradual change in his understanding rather than a sudden one. Gradually he has come to understand this lemma more, and can recognize and anticipate its use. He indicates that he likes this feeling of accomplishment that comes from recognizing when and how to use it correctly.

His analogic model for induction is the idea of getting onto and climbing up a ladder. This provides him with a language that he can use, and an effective image to use to help him understand the process of induction. He also describes an algebraic symbolic image of  $k+1$ , symbolizing the next rung on the ladder. He links induction with Zorn's lemma, since he sees them as being the same type of tool, providing an algorithmic 'turn on autopilot' way of solving certain problems. He does not feel that there was any significant event that changed his understanding.

### • Group, Ring, Field

Bill links these algebraic structures, and orders them hierarchically, and connects them with the underlying set structure.

#### • Group

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* It's hard to say. Initially what comes into my head is the same idea as a set. I know there's an operation, I'm just not very concerned with it usually. When I first look at a problem I'm usually concerned that it's a group, that it's a set of objects. So I usually think of it in terms of a set. So you've got this sort of circular blob? [At this point in the interview, the concept of set had already been discussed so I was reiterating the image he associated with a set.] Yeah, same type of circular blob, it seems to be working for me.

Specific examples.

*Bill:* Some, I guess, I think of  $Z_n$ , but I don't really think of that in

terms of group, I think of that more as modular arithmetic. I think of a group of units that was maybe my first interesting group. I also think of the example when I first really thought I got this idea of modding down  $\mathbb{Z}$  because Bob was saying something about you mod out by the ideal, the functions all have a zero at a certain point and you mod out by the ideal, so yes, when you mod out what do you get, the function is from 0 to 1, so if you mod out you get the real numbers and you know I was right. Well, then he says think of it in terms of the unit circle, it just keeps going around and around itself. But so it's the same idea. I thought to myself yes I really think I get this, so that's another example that comes to my head. Just because it's a really good memory for me. It's strange how it's not really the examples that you can use that are what come into your head, it's more associated with memories you have. Then I think of group actions and how I did not get group actions so very well. It took me a while to get, the idea of a group wasn't so bad, I remember finally getting the homomorphism theorems and then came group actions last fall quarter. I was not the group action wonder, but, looking at things like normal subgroups, when I had to know it to do all this group action stuff, then all of a sudden I started understanding the normal subgroups. I usually think of all the things that are attached, all the characteristics. When I think of a group then I also think of other things that go with  $\mathbb{Z}$  average group.

Visual images. [No response]

Specific images. [No response]

First introduction.

*Bill:* I was first introduced to the idea of a group as an undergrad again, and I think it was in an abstract algebra class. I remember, this year I looked back at an old exam, I don't know how I ran across it but I was looking through this stuff and we were using the homomorphism theorems and I thought, that doesn't even ring a bell. When Bob did that last year, that didn't even sound familiar. I didn't have any idea what, I just couldn't, and I did pretty well on that test. I'm pretty proud of myself, I don't have any idea how I did those.

Changes in understanding.

*Bill:* Yea, it's just gotten so that there's just been little chunks where I absorb how groups work and structures of groups and I just absorb it bit by bit, chunks, all of a sudden it just hits me, oh, and all of a sudden it's a familiarity, and I can use that idea. Also it's changed in that I've started thinking of it in terms of a permutation group. If it's a finite group then you can think of it in terms of permutations, or that it looks just like it but with the idea of isomorphism you can turn it into something that you know more about. Also, the idea of generators and relations, that makes more sense now that's another point of view of looking at groups. So I think that I have seen it from different angles.

Milestones.

*Bill:* As for a little moment of revelation when I first started thinking, I think the idea of modding out by a subgroup. I always think of it as sucking in or pulling in part of that group, just condenses or having it all fall into itself. So that idea of the kernel of the homomorphism, I think of it as taking all the things that are alike and sticking them together and then all of a sudden it's just one group. It pulls those things together or condenses, it just folds in on itself, there it is condensed.

## • Ring

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I think of a thing that goes around your finger, but that's not what you're after. I think of almost a field but not good enough. I never think of it as being better than a group, I always think of it as not being as good as a field, kind of a little moralistic judgement there. I guess I think that there's less structure on a ring. Actually I think I find rings annoying because groups are fine to work with, and then fields are fine too, but rings just aren't fine to work with.



Specific examples.

*Bill:* I think of PID, UFD, integral domain, the things that you prove for all that kind of business. Specific examples would be not rings but fields. I would think of the rationals, or the integers, and you think of just Noetherian, Artinian and that kind of thing. But those also turn out to be fields, no not the integers, but you know what I mean. So I guess the integers are a good ring for me.

Visual images.

*Bill:* I really don't, and also you don't use it, you usually jump from groups to fields so I really don't. So what do I think of when I think of rings? I guess I just think of a field that's not so strict or something. I never think of it as an older brother to a group, I think of it as a younger brother to a field.

Specific images. [No response]

First introduction.

*Bill:* Undergraduate algebra. The way it was first introduced was in terms of the little 4 by 4 grid. So you did your multiplication and addition tables and I don't think we got very far past that, and that was stupid. So that was my first introduction, nothing came out of that. I saw it again a year later and I think I understood a little better just what it was to be a ring in terms of what it was to not quite be a field. It's a little bit stronger having had algebra this year again because we talked about modules and things like that, vector spaces over a ring.

Changes in understanding. [No response]

Milestones. [No response]

• Field

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I'd say that was an open field. I always think of something that's a little more structured. You have these two operations that are to be used so I guess that's the first thing I would think of - more structured. There's really not a vision that pops into my head. It's just the feeling more or less. Just something more rigid.

Specific examples.

*Bill:* I guess the one that just popped into my head, I was thinking of  $Z_p$ . You know that reals are a field, but when you actually have a finite field then you can actually check that everything works. You know that everything works in the other cases but it seems a little more tangible and something that you can work with. And plus my algebra experience, I do a lot with prime power fields and their different applications, but I guess, and maybe it's you don't have to work hard at all but I think of first prime power fields. Usually in a group you'll know if it's either finite or if it's infinite. If it's infinite most likely I'll think of the rationals, just because I think they're easier to work with, with rationals you can do fractions. Reals you can't really. But if the problem lends itself to finite then I usually would pick  $Z_p$ , and I wouldn't think of the Galois field of higher order. I would normally think right away of straight  $Z_p$ . Then I would see if it needed something else. If I needed to look at it that is what I would think of first.

Visual images.

*Bill:* Nothing really comes to mind. I think that, I don't know if you're going to ask this about groups but I'll just go ahead anyway. For groups I do, especially when I'm thinking about the Sylow theorems I can envision things sort of running around. I remember when you did these Sylow theorem problems and it was using counterarguments and I just pictured in the background who belongs and you

get an image in your mind of this diagram if nothing else. But with fields I don't think so because there's two operations. With one operation you really don't worry about it, you know these things are just associated with each other, you know there's some operation. But with two operations then you worry about how they're associated, you know they have to associate in two different ways so I don't think, maybe it's just I don't understand it well enough to do that.

Specific images. [No response]

First introduction.

*Bill:* It would be when I took linear algebra as an undergrad, but when I first really understood it is a different story. I think that would most likely be here, because I had a couple of courses in algebra as an undergrad but really, we talked about fields but now I think it has sunk in a little more, what fields are, and that's when I feel I understood, here, as opposed to before it was more, more manipulation, when you just do it that way. But now it is involved more so I would say it developed here. It's introduced as an undergrad in algebra classes, but for actually getting what is going on I think would be here.

Changes in understanding.

*Bill:* Yea, it's definitely different, it's more intuitive. Before if I had to check if something, if you work a proof and you know you need to check a field, until you really have a good feeling you would actually check and make sure and otherwise sometimes you can just go oh and just know how it's going to work. So it's become a little more intuitive. And I think actually when it first really changed to be a little more intuitive was when I had linear algebra here with Bob. Because he put on the tests 'show that  $Z_p$  ...,' or 'show that  $Z_6$  is not a field,' and I went oh, cheap answer. So, I would say here is when that changed.

Milestones.

*Bill:* I don't think so. I think mostly because it's related more to the idea of groups and I think my understanding just transferred over to fields. Because I remember with groups just all of a sudden understanding how this modding out works. When you modify something, all those steps and how it works. And then I think that idea just kind of flowed right into being able to do prime ideals to get fields. I wouldn't say there was any time I mean one instance that I can throw out.

### • Interpretive Analysis

Bill describes a very rich and varied collection of examples, images, and different types of groups and their applications. His prototypical model for a group is the finite group  $Z_n$ . He links the idea of a group with that of a set, and this linking provides him with a visual image of a group in the same way he visualizes a set, using a circular blob idea. This linking of group with other types of groups, applications, and theorems, extends in many different directions. He mentions normal subgroups, permutation groups, homomorphism theorems, group actions, isomorphisms, generators and relations, and modding out by a normal subgroup. This last connection seems to be a particularly important one for Bill. It seems to be one area that, once understood, provided him with a significant change in his understanding of groups. At the same time, he describes a phenomenon that he himself has observed, which he refers to as 'chunking'. He seems to be describing the condensation phase in his understanding of a group, when several different aspects of a group seemed to combine together to form a coherent whole. He absorbed it, it becomes familiar to him, and he can use it. He also describes his experience of suddenly understanding normal subgroups, which he had encountered at a lower level, once he began to learn about group actions, at a higher level. This is clearly a description of one of the reification process. Sfard claims that to understand a concept structurally, rather than operationally, one must begin to use it as an object in a higher level process.

Bill first evokes a non-mathematical image of a ring. Mathematically, he evokes a feeling rather than any visual image, a feeling that places a ring as a little bit less than a field. A ring is somehow a deficient field. He links the concept of a ring to many types of rings, PID's, UFD's, integral domains, and gives two examples of rings, the rational numbers, and the integers. With reluctance he recalls a visual image of addition and multiplication tables which he learned in his first abstract algebra class, but which he found to be less than helpful in providing any insight about rings.

Here again, Bill first evokes a non-mathematical image of a field. Again, he describes a feeling, of more structure, associated with the concept of a field, rather than any clear visual image. His prototypical model for a field is of the finite field  $Z_p$ , rather than an infinite field such as the real numbers, although he is well aware of both. He describes a significant shift in his understanding which occurred at about the same time that he suddenly understood modding out by a normal subgroup. Once he understood that, he was able to transfer that understanding to the analogous construction of a field from a ring by modding out by a prime ideal. This led to a more intuitive understanding of a field.

## • Set, Cardinality

### • Set

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I usually think of a circle or something and then I just think of what's inside. Maybe there's points or objects, something like that.

Specific examples.

*Bill:* That's what always come into my head whenever I think of a set. And it's not really like the Venn diagram type circles, when they intersect, it's more like a circle, or not even, it doesn't need to be a circle, it just needs to be a little blob, and there's just these little parts, but I don't really even see the little parts I just know that there's all these things that are running around inside the circle. So I think of it as, maybe again I'm thinking of a space, I don't really see the objects in the set I just see the set as a whole. I don't really think of it as made up of all the little parts, I just see this little blob. I could draw it for you on the board. That's all I really see though, that's what visually comes to mind as an example.

Visual images.

*Bill:* Actually the image that always comes to mind is that, you take the power set of the power set of the power set of the power set and then you have the set of the set of the set of the set. So actually I envision how you would write a set with a set bracket and a set bracket, set bracket, set bracket, and then, just the nesting of all those things. I just thought that was how I was first able to understand how is this power set thing working? Take the power set on three elements and then there's eight elements in there and so I had to work it out, just do all of them and just see how it is. Now I just think of it as a set of objects and the objects inside might be a set, the objects inside that might be a set, but I just still think of it now in terms of it, the power set is just a set and what's inside is just, the elements are just sets and then what's inside that and I don't really go any deeper than that. It may or may not be sets depending on what the first set was. So I just think of it as a new type of set.

Specific images.

*Bill:* Oh, the one about the cardinality theorems. The idea of the power set. I remember when I first learned about that, the cardinality of the power set of the rationals, or of the integers, the reals, the Riemann continuum hypothesis. The one where is there any cardinality between the two of them. So that comes to mind just

because I remember being interested in just thinking of that a little bit, could it or could it not. In terms of sets though, I think when I think of a generic set I think of an infinite set. I don't usually think of a finite set. It may be just how it was introduced to me.

First introduction.

*Bill:* I first really got the notion of a set in a class called foundations of mathematics, which was basically here's how you do a proof. You get, in elementary school, here's a set of three objects, you put them in a box and so I guess that's when you first get that idea. You know how in calculus some people can't do composition of functions, and I think that is a lot like the idea of a set because if  $g(x)$  is  $x^2 + 1$ , and if you ask them what is  $g(x + 1)$  they just can't, and I think I've always been able to do that because I've always thought of that  $x$  as you just sort of plug it in, it's like you're taking something and putting it in wherever you see an  $x$ . It's just like taking a set, so your set might be  $x + 1$ , it's not really a set but it's still the idea of just,  $x$  is an object, you put it in there,  $x + 1$  is an object you plug in  $x + 1$ , so a set really is also just an object not just the parts. You really don't break it down, you think of it as a set. I think that I've always been ok with, or pretty comfortable with that idea, not since birth or anything but in elementary school and they ask you, is this a set, it always seemed a little silly. I didn't know why they were asking that.

Changes in understanding.

*Bill:* Well, then I had this set theory class and that's when we really started talking about the Zermelo-Fraenkel axioms of set theory. We started looking at some of the paradoxes of set theory, like the library, can you put in the book that has the name of all the books, and so I think that's when I first really got interested or really thought about it in that way. That's sort of a non-analytic point of view, not a real analytical point of view. Out here [graduate school], is when we started talking about compact, and open and closed or connected, so that changed my idea, I mean, not really changed it but it was something that was added on top of it. Before, I thought of sets as pretty generic. That's what I liked about sets, that they were so generic. They really are though. And then I got here and

they started slapping these labels on them like open and connected, and I thought, oh, you're ruining set theory for me. So I guess it changed when I got to graduate school and we started talking about real analysis type things. And we started concerning ourselves with well what kind of set is it in terms of open and all that. But before I would just think of it in terms of just these objects. And I mostly thought of it in terms of countable sets. Not necessarily finite, but just countable. The idea of uncountable set seemed not as warm and inviting as maybe a countable set. You know what I mean? Just there's an order in it, there is in the reals too but, so if you ask for specific examples I normally would think of either, I think of a countable set, or just again the idea that pops into my head is just this blob, this space, but I don't really see what's going on inside. I really don't look inside, if I'm thinking of a set then I think of the set, I don't need to look inside.

Milestones.

*Bill:* Right now I think of power set. That's the first thing I think of. That was the first really interesting idea that I think I was introduced to in cardinality.

### • Cardinality

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* Right now I think of power set. That's the first thing I think of. That was the first really interesting idea that I think I was introduced to in cardinality.

Specific examples. [No response]

Visual images.



*Bill:* Actually the image that always comes to mind is that, you take the power set of the power set of the power set of the power set and then you have the set of the set of the set of the set. So actually I envision how you would write a set with a set bracket and a set bracket, set bracket, set bracket, and then, just the nesting of all those things. I just thought that was how I was first able to understand how is this power set thing working? Take the power set on three elements and then there's eight elements in there and so I had to work it out, just do all of them and just see how it is. Now I just think of it as a set of objects and the objects inside might be a set, the objects inside that might be a set, but I just still think of it now in terms of it, the power set is just a set and what's inside is just, the elements are just sets and then what's inside that and I don't really go any deeper than that. It may or may not be sets depending on what the first set was. So I just think of it as a new type of set.

Specific images.

*Bill:* Well, I think we talked about some of these, like the Riemann hypothesis, but obviously I do not know the answer to that one, otherwise I wouldn't be talking to you! The cardinality of the power set of the real numbers, or the integers and the same with the rational power set, but I always think of the power set of the real numbers. I could give you the rationals, complexes, whatever, but that's the example I would think of. Other theorems? I guess another theorem that pops into my mind is when you talk about the arbitrary union of open sets, and then we have just the countable union of open sets is open. And then for closed sets, an arbitrary intersection of closed sets is closed and the countable union of closed sets is closed. I don't know if I have those right or not, but that's that's another thing that pops into my head. I guess the only reason that would fall into cardinality is because you talk about countable and uncountable. You know both are infinite but before it kind of seemed like well, what's the difference that it's infinite and it goes on forever. I guess cardinality I always associated with infinite. That was the first time when I think I really distinguished between countable and uncountable, that was really it.

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

Bill describes a visual image of a circle or a blob, representing the set, and an awareness of, rather than a clearly defined image of, the elements inside. This would seem to indicate that he understands a set as an object itself, even though it contains elements which are objects. At one point he explains that these elements may even be sets themselves, but that is not really important, he does not look inside the set, he does not need a clear picture of *what* is inside, just that there *are* elements inside. Although he is aware that there are elements in the set, he is not visualizing them in any detail, just acknowledging their existence, while saying at the same time that it is not necessary for him to look inside. He describes a second diagrammatic visual image of a set as a Venn diagram. He suggests that his understanding has been enhanced by learning about different types of sets, such as open, closed, countable, uncountable. This does not represent a radical restructuring of his understanding. his prototypical model of a set is of a countable set since this seems to be the most aesthetically pleasing to him.

Bill first evokes the linked concept of the power set of a set when asked about cardinality. He visualizes this via the usual algebraic symbols for a set, set brackets, with set brackets inside, since the elements of the power set of a set are themselves sets. As he is talking about this image, he begins talking about his image of a set, and describes how he thinks of the power set as a set, and, while acknowledging that the elements inside are sets, he does not really imagine them in any detail. Bill seems to have cardinality and set closely linked. His understanding of cardinality deepened when he learned about countable versus uncountable cardinality.

## • Infinity, Limit, Convergence

Bill links these concepts, and offers some interesting analogies in his understanding of different types of infinity. There is also evidence of a reification of infinity.

### • Infinity

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* The first thing that popped into my mind is the infinity symbol, and then the next thing that comes into my head is a plane, and that infinity is just out there, that it's just some point out there, and that maybe that evolved from just needing to think of it that way. Also, it's kind of strange. I just remember when I was little and I was thinking about not necessarily infinity as numbers going on forever, but it was the same idea and I was thinking of states and how they go on forever and ever and the same idea with numbers and how you can always get one more and it just seemed so strange. And I just remember the feeling of nausea and oh this is just too deep. But even so that's what it reminds me of too is just always going and never stopping.

Specific examples.

*Bill:* Dividing by zero is definitely one. Sequences. You just think of it getting a little bit further. If it's converging then there's some point but you keep going. Or, another idea of infinity is if I walk halfway to the end of the room and then I walk another half, and another half, and another half I will never get there. But in the limiting process you do. Also in induction I can remember thinking how strange that was, it just seemed a little bit odd that induction really doesn't seem to do infinity. I always think of it as a little paradox because it works for all integers so you can get any integer

that you want but it doesn't really go all the way out to infinity so I always thought induction was a little odd that way.

Visual images.

*Bill:* Sometimes I think of just spinning around and really I guess it breaks down to two types of infinity. Infinity that never gets anywhere, like if it converges, if it's a sine and you just keep moving up and down. And then there's infinity that just keeps going forever. But with that, what I envision is a long road. You know you can just keep going on this road, and that's how that infinity works. That's for the more finite one, the one that hangs around and spins so to speak. Things go on forever but a result is always in your grasp, as opposed to things that go on forever and they get out of your grasp. So the first one is just sort of hanging around. I guess the second one is what I use when I just don't picture anything spinning around, I just picture it going on forever.

Specific images.

*Bill:* Uh, I'm thinking here, (pause) nothing.

First introduction.

*Bill:* I remember when I was little I think I asked my mom about the universe. I acted like I was a really deep child but I really wasn't. I just said something about but there's a wall, but there's something beyond the wall in the universe. You know that's how I envisioned how the universe was. That there was a wall and there was a wall beyond. So that's when I first started thinking about it, of course not in the mathematical sense. Mathematically I suppose when I first was dividing by zero, and that might have been when I starting learning long division and that you can't divide by zero. I remember trying ones, I wrote it out, zero divided into 159. And I just thought you can never get there. So mathematically that's when it was introduced, but in a more formal sense I would suppose I really didn't get that until calculus when you do the limit process. That's when you work with it.

## Changes in understanding.

*Bill:* Yea, I remember, oh this is mass flashbacks. I remember when I first had complex analysis and we talked about the Riemann sphere and I thought it was so neat, because we talked about how you could project all the places on the sphere this way. Also the north pole is the same as infinity so you could actually put a label, you can lay your hand on infinity. And I thought that was great. So that changed a little bit my understanding of infinity. I think I thought of it as maybe a little bit of handwaving by mathematicians before that point. They just think it goes on for ever and ever, and then to actually fix it or put your hand on it or to actually know where you can find infinity then it made it seem like it was something you could more use. So it has changed in that, partly you know it changes every time you introduce a new concept of infinity, the limiting process, and then you talk about series and sequences, but now I think it has changed in that now I think of it as something that is more usable. Even in calculus when they always tell you you can never write the definite integral from 6 to infinity, and people think as  $v$  goes to infinity you always think to yourself what's really the difference. But now that helps to kind of understand the difference of when you really need to consider a limiting process and when you can really include the point of infinity. I think that you become more comfortable with how to use it. Before it was ok what do I do to avoid it, because if I just put infinity in there it works anyway. You know, you always think that though.

Milestones. [No response]

• **Limit**

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* Infinity. Because when you think of a limit you don't think of it as  $n$  goes to 6 you think of it, I mean usually you go to infinity. I mean, even if it's the limit as  $n$  goes to 6, the reason that you're going to six is because it's going to go to infinity. And so, that's what I've always

associated with it. The limit is usually something that I think is used if you're going to get in trouble if you just plug it in. Or the other way I think about a limit is if it's approximating something, if you think of, in analysis with the Lebesgue dominated convergence theorem, you approximate, you take the limit, the sequence of  $f_n$ 's, you know, and they go to  $f$ , and so you're approximating your function with all these other functions that you know something about. And so, in that way, I see the limit as turning these step functions into this curve, as it gets finer and finer until all of a sudden you can't really tell the difference or the error is negligible.

Specific examples. [No response]

Visual images.

*Bill:* That is [referring to response to evoked images]. When I think of it in terms of a limit. That's a visual image when I think of that. When I think of it in terms of a sequence, the idea of a limit there is, as the sequence  $a_n$  as  $n$  goes from 1 to infinity. Then I think of it as these points that, if it converges, then it gets closer and closer and if it's not then it's just a sequence. In terms of the limit as  $n$  goes to infinity of the integral, then I don't really think of it, I think of it more in terms of as a tool for seeing that oh will this all cancel out, does what I need to have cancel out cancel out? And then, I guess, is it going to make a difference? So I don't really visualize anything there.

Specific images. [No response]

First introduction. [No response]

Changes in understanding.

*Bill:* Yea, just because you look at the different types of convergence. So the idea of the limit is still the same but there's an aura that has to stick in your head. I guess what I associate with it as the limit is this little thing moving and so you fix an  $n$  so that for all

$n$  greater than whatever, you think of all that, and so you think of this limit as moving along and just riding along and you try and associate with where everything else is. So if you fix an  $n$ , if that's the kind of convergence you need, so that for all  $n$  greater than  $n$  etc. So then I picture  $n$  being fixed and then just moving along and passing, going by. So in that way it's changed somewhat, from just saying the limit of. Before when I was first introduced I remember going oh I don't get this limit stuff at all, and someone said just plug it in and see. And I thought well, that's dumb, it's not really what it is. So that is what my first initial idea of what the limit was, really just plug it in and see if it works. Of course that didn't work so well for the old  $\sin x$  over  $x$  business but that was my first idea, so it has changed quite a bit.

Milestones. [No response]

### • Convergence

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I think I answered this one because I remember saying that I think it's the points just going closer and closer. I have the same idea though [as limit]. Convergence I still think of it as these points and they're getting closer and closer and they never really get there.

Specific examples. [No response]

Visual images. [No response]

Specific images. [No response]

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

Bill first evokes the algebraic symbol,  $\infty$  for infinity, and then also describes infinity as a point 'out there' beyond the plane. When asked about visual images, however, he describes what to him are two distinct types of infinity, both of which he describes using language involving motion. The first type involves a motion of spinning around in place, never getting out of reach, but continuing for ever. The other type involves getting further and further out of reach. For the first type of infinity he uses the example of a sine curve moving up and down, and, especially if you think of this as a circular function, you get the idea of spinning in place. Although Bill did not describe the sine function this way, it is possible that he has this in mind. For the second one he does not give an explicit example but describes this type as getting out of one's grasp. It would seem that the distinction he is making, without actually saying it, is that the first type of infinity is represented by a bounded function, with domain all reals, whereas the second type is represented by an unbounded function on the same domain. He is making heavy use of non-mathematical ideas of time and motion to describe these two types of infinity metaphorically.

He describes a change in his understanding of infinity that occurred when he learned about the Riemann sphere, with the north pole representing infinity. This metaphorical and diagrammatic image gave him the ability to put his hand on infinity, as he describes it, providing him with an important concrete realization of infinity. Again the link with a useful non-mathematical image aids the understanding of infinity. It made infinity seem a more tangible and therefore more usable



and useful concept to him, perhaps because this representation of infinity helps to objectify the concept, a significant reification event for Bill.

Bill links limit with infinity, and with convergence. His visual image is a diagrammatic, graphical one of a sequence of step functions approximating a continuous curve. He describes a significant event in his understanding of limit which occurred when he began to understand what the limit process really meant, rather than just understanding limit and convergence as algorithmic processes.

### • Transformation, Vector, Vector Space, Basis

These concepts have the common link with the area of linear algebra.

#### • Transformation

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I think about being able to pull that thing right out of the transformation.  $T(a\vec{v}) = aT(\vec{v})$ . I guess that's really symbolic. I also think of a shift. I think of it more as just a shift as opposed to you know like function graphing. It's just a shift.

Specific examples.

*Bill:* Just the idea of shifting, normally the ones that I usually deal with are linear transformations, and so that's the idea that I'm comfortable with and so I would say that. But that's all just linear transformation, I guess I automatically think of linear transformation. Uh, and just that shift.

Visual images.

*Bill:* I just see something being shifted I don't really know if I'd think of it as a vector or anything. I just have this image of something being shifted, just a transformation, just being moved. That might just be because I think linear transformation stays to itself. I see something turning into something else and it's just kind of being stretched or pulled. So that may be where it comes from but I just do see a whole shift.

Specific images.

*Bill:* The one where you can turn it into a matrix.

First introduction.

*Bill:* I was first introduced I would suppose in linear algebra. I guess when we learned about linear transformation.

Changes in understanding.

*Bill:* I'm glad you asked. Yes, actually I remember, when I first learned it I kept thinking of the whole vector space, and now it is a lot more intuitive, I guess it has just matured some so that I can envision what is going on and I have a little bit of understanding as to what happens when there's a linear transformation. I also know that there are things that I don't get, but I can force myself to think of like geometric interpretations but that's not what pops into my head. So it's definitely changed, it's definitely matured in some ways. I usually again think in terms of linear transformations and of matrices and Jordan canonical forms. That I picture more. I had to give this talk on Jordan canonical form in linear algebra. And I remember having to stop and just let them know about invariant subspace. So I just stopped and stood in one spot and said, 'Alright here I am. I'm a subspace and I just never leave me. I just come back right here, you know, I'm a little function and I always come right back to my home.' And so these are broken up and those are

the blocks and so in that way that's a little bit more visual but that's also because I've become a little more familiar with it and understand it a little bit better. But that's still how I think of it when I think of transformation. Sometimes I think back, I think of the space that it's actually working on and what is happening and what's going on in the space and where do these things go and so if it is invariant under a certain subspace then I actually picture this little blob, and things actually stay and don't churn out.

Milestones.

*Bill:* Yea, I think when I thought of what it means to be an invariant subspace. A subspace that falls in on itself, like if you apply the transformation again and again it gets smaller until you finally get to what it really was, a really small part of it.

## • Vector

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I guess the arrow business is what first comes to mind. That's the way you see it in calculus. I also think of a vector as pulling, usually it never pushes, it always pulls for me. Because we did all these force problems. I didn't think of it in terms of vectors, I thought of it as pulling this box up an incline. As a sophomore in college we talked about vectors, and I always thought of it as just this pull, usually it was on the end of something, like it was a fixed structure. The problem that I always remember, and I hated this problem, I don't know why, you always had these window cleaner problems where the guy is standing on the platform, and how much weight, what is the pull, and how do you decide, and all this stuff. But there was one where it was a little different and the pull was, I don't know, it was like this thing that was sticking out from the wall and you had to figure out how much was pushing in and how much was pulling out. So, I think of it like, the vectors pulling, almost like a pipe sticking out of the wall and how it pushes, in a more applied sense, and that's how I think of it.

Specific examples. [No response]

Visual images. [No response]

Specific images.

*Bill:* Well, in an unrelated sort of way I guess, the dimension of vector spaces over each other. How you figure out the dimensions and how if it's algebraic you combine the bases, and then that's why if the dimension is 6 and the next extension is 3 then the extension is 18. So I do think of that in terms of a vector space, or in terms of the vectors, and so now your vector has to be 18 long because you have to have each one of these components and then you have to be able to combine them in every way. Um, so I guess that's one thing, I think maybe that was in my head because that's something I developed this year, that idea. I remember thinking of it in terms of  $n$ -tuples, but this year was when I really started, and it became a lot more natural to do that.

First introduction. [No response]

Changes in understanding.

*Bill:* My idea of vectors changed very much because, from there it went to thinking of it just as  $n$ -tuples and that is such a nice idea. Maybe that's a more algebraic way of thinking about it too. But also I remember just thinking oh, ok here we go it's just these  $n$ -tuples. That was when I think I first clued in that that was a fair thing to do, or that was a legal math step to do then I became a lot more comfortable working with vectors, just thinking of them in terms of  $n$ -tuples. I remember last year when Bob told me, in graduate algebra, he was talking about Galois theory and he said something like, you can just think of this as a vector space, and I thought, what are you talking about? But this year Burt didn't even really say that you should be thinking about it, but it just seemed a lot more natural, that's how I started thinking about it, I thought of it in terms of the basis. And so I started looking at it in terms of that and then it seemed again more intuitive what it meant for it

to be a vector in that sense, not so much in the the arrow sense as it is more in just a linear combination idea and that type of thing.

Milestones. [No response]

## • Vector Space

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* The first thing I think of is the arrows, just because of the word vector. Then I just think of it in terms of the basis. And the reason I say that is because that is how it is easiest to work with for me, is to think of it in terms of linear combinations of these bases. You have these fixed things. Sometimes they look like vectors and sometimes they just look like bases written out, depending on what kind of problem I'm doing. And then I just think of little chunks. If it's a vector I think of adding one chunk, an interval vector. I mean, I think of it in terms of like  $1b, 2b, 3b$ , and how they combine to get to where you want to be. If I think of it in terms of these things, then I just think of them as polynomials and those coefficients, polynomials, I guess polynomials isn't the right word but you know what I mean, just like those as being attached and then you have some magnitude to how much you have of those.

Specific examples.

*Bill:* Just in using the Galois theory. That's where I really started thinking about it, just like an extension, algebraic extension. It's one way that I thought about it. And then the other way is just to think about it in terms of the vectors. Actually just the pictures of how you could. I really don't think of it as a vector added to a vector added to a vector. I just think of it like the components. So when I'm talking about thinking of a Galois extension as a vector space over the base field, I'm thinking of it more just in terms of a field, I think more of a field as just being a set of elements that I don't really know anything about, but if there's a vector space over it

then I know how that vector space affects it, or how the field below affects the vector space, how you can do it in terms of the bases and that.

Visual images. [No response]

Specific images. [No response]

First introduction.

*Bill:* Probably fourth quarter calculus, no I guess that would actually be probably in linear algebra, undergraduate linear algebra. I didn't really catch on to what it meant.

Changes in understanding.

*Bill:* My understanding has developed quite a bit just this year in taking the 600 level algebra class. Because when we did Galois theory then I really started to think of things in terms of vector spaces as opposed to when I thought of vector spaces before I just sort of chugged through it, then I thought of it more in terms of the bases and linear combinations. So that's how it's developed. It really changed quite a bit this year, from even last year. I mean it changed a lot last year for me. I really didn't have much of an idea of what it was. I just remember some axioms as an undergraduate. Then I got here and I had some idea of what it was. I just picture it as being round. I don't think they ever say that it's round like a sphere, like if it was a three dimensional vector space that would be a sphere, but it seems like it would be if it was finite, but it's not, it goes on for ever, it seems like everything can keep pushing out and away. But now I think if I would picture the whole vector space I would picture it being very round as a matter of fact. Not that it needs to be but.

Milestones. [No response]

**• Basis**

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I think of vector space. Those two are just interchangeable, vector space and basis are really interchangeable in my mind. And so I usually think of the vector space in terms of its basis, and in the same way I think of a basis in terms of a vector space. A basis is really just a strict set of points, but then how it extends out, I always think of the basis as extending out. Also I think of it in terms of a little linear combination idea. In that sense maybe it's more thinking of it in terms of vectors, that you can't add these two vectors to get the third one, there's no way you can combine them to fill out. When I think of the first kind [calculus idea of vector space] it's more in terms of vector calculus and it's also in terms of linear algebra. I associate more with the vector idea and the linear combinations. In terms of the Galois stuff I don't think of it in that way, I don't really think of it as being linearly independent because in a way for the Galois extensions it doesn't matter if it's not linearly independent you've added nothing new. Normally you don't find the order of your basis by thinking of it in terms of linearly independent you think of it in terms of what you know about the extension, you usually find it in a different way. How many elements you have and it doesn't seem that you'd go that direct route of thinking of them as linearly independent, but you use some other things to find out the number of things that are in your basis. You find that many things that are linearly independent but you don't really think of them in the same way as thinking of linearly independent as getting a combination to get the next ones. Or if you do, you just do it symbolically, you can't take the square root of 3 and the square root of 2 to get the square root of 5 or something like that.

Specific examples. [No response]

Visual images. [No response]

Specific images. [No response]

First introduction. [No response]

Changes in understanding.

*Bill:* It just matured a little bit especially during my linear algebra and here at OSU graduate school. But then also it all of a sudden split into a different direction and then it matured into other things, more algebra stuff.

Milestones. [No response]

### • Interpretive Analysis

Bill first describes an algebraic symbolic image of a transformation, in fact of a particular property of a transformation, that  $T(a\vec{v}) = \mathbf{a}T(\vec{v})$ . He also visualizes a transformation with a geometric image of a shift. In his mind, a transformation is linear. He describes a significant change in his understanding of transformation to a more intuitive understanding of them, particularly when he became comfortable with the idea of invariant subspaces.

His visual images of vector are of both algebraic symbols and a more diagrammatic image - ordered  $n$ -tuples, and the conventional arrow. He thinks of a vector as a linear combination of the basis vectors. He describes a significant event that changed his understanding when he went from a very concrete, applied understanding of a vector in three space, with magnitude and direction, often representing some force, often pulling something up an inclined plane, to a more abstract and algebraic way of thinking of a vector as an  $n$ -tuple. As he describes it, it seemed more intuitive and more natural to think of them in that way.

Although the very first image of vector space he evokes is of arrows, he immediately changes to thinking of a vector space in terms of its basis. He suggests



that, for him, vector space and basis are interchangeable. He visualizes a basis with algebraic symbols, as an ordered  $n$ -tuple, but he also describes a diagrammatic image of something round, a sphere if the space is three dimensional. He describes a significant paradigm shift occurring when he learned Galois theory. His understanding then changed from an 'apply the definition' approach, to a more intuitive understanding of a vector space as an extension of a base field.

Bill's conception of basis is closely linked with vector space in Bill's mind. He also associates it with linear algebra and Galois theory. He describes a significant change in his understanding occurring when the idea of a basis seemed to split into two different directions. One remained the understanding of basis as a collection of vectors, the other was a new understanding of a vector space over a base field in connection with Galois theory. Seeing this application of vector spaces in abstract algebra was a significant event in his mathematical experience, and caused a radical paradigm shift in his understanding of vector space and basis.

### • Function, Continuity, Derivative, Integral

These concepts were all first encountered in calculus class. However, Bill's understanding of them has broadened considerably since then, particularly as a result of tutoring.

#### • Function

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I think of, I was going to say broccoli! That's because that's how I explained it to my first class that I taught, so that was the first

time I ever used it. And I explained it in terms of broccoli because it's the whole composition of functions thing: given  $f(x)$  is  $x^2 + 1$ ,  $f(x + 1)$  is  $x^2 + 2$  you know, they just don't know. So I said, what would  $f$  of broccoli be, because that says, now don't think like you're thinking, it kind of removes that whole  $x, y$  mystery type thing. So then they always have to go, well, what do we do with  $x$ , I just plug it in there. Actually though, I decided broccoli is not the best thing. Well, broccoli's ok, but now I've started using this oval, so what's  $f$  of oval? And they say oval squared plus one, and then you can go into your oval and write in  $x + 1$  and that's fine. So it's more conducive than broccoli but it's the same idea. Outside of that I would think of it as a curve. That's just with one variable in mind. Having just taught vector calculus too I would say it has changed my opinion of it, of what it is.

Specific examples. [No response]

Visual images.

*Bill:* I would still say, I would either think of the image  $f(x)$  thing or I would think of just a curve but just in one dimension. But again it depends on what setting I'm looking at because when we talked about some of these functions in algebra, we would have infinitely many variables or countably many variables, and then of course for those we would just have to go away from the idea of a curve, and to the idea of just plugging in and thinking of it in a different sense, I'm not really sure what, but I think it is probably more just symbolic and just thinking of it as a relation, a relation between this  $n$ -tuple or this vector and what comes out, and what comes out you know is a scalar, so, instead of visualizing it that's just an idea that's in my head.

Specific images.

*Bill:* Of course I'm going to say the derivative one and the multiple zeros of a function. So that comes to mind, just because that seems to be what's going on today. As far as functions well, I'm reminded of the continuous function things in math, inverse function maps on open sets, but in general that's what comes to mind.

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

### • Continuity

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I guess I think of a curve. And it being very smooth. I think that is how you learn it too, you don't have to lift your pencil off the paper, there's no jumps, it just stays. But I actually don't think of a sharp corner either, I also think of a nice smooth curve, even though it could have a sharp corner.

Specific examples.

*Bill:* I usually, I think a curve is what I think of first when I think of continuity.

Visual images. [No response]

Specific images.

*Bill:* If  $f$  is a continuous function the inverse takes open sets to open sets. Before, actually I guess that's going to change, that I always thought of a function is continuous if, you know, the limit, la la la, just like you did in calculus, just like you teach your 241 kids. But now I think of it more in terms of sets and open sets. So I guess it's changed somewhat in that way. But that would also be considered the theorems that come into my head. Actually it

developed somewhat in complex analysis this last quarter because we were doing Riemann surfaces, and we kept talking about these local homeomorphisms and I don't know anything about topology, but I got a vision of just sort of blobs it over here it's still an open set and some properties stay invariant under it. So it's developed somewhat.

First introduction.

*Bill:* I suppose in calculus again was the first time. But then again, I think that idea developed a little bit through the calculus sequence. I just maybe got more familiar, or maybe I just learned not to worry about continuity, that could be it too. I think that I developed some sort of intuitive understanding of what it meant to be continuous, as opposed to having to check all those criteria. I think that you just thought to yourself oh well there's no place where it gets messed up, you just have a vision that there's no, especially if it's a polynomial, you know there's no place where it's going to blow up, you know it's all going to be nice and smooth, one place goes to the next, everything's fine. And then it developed more when I had analysis. I really didn't have analysis as an undergrad. I didn't have an advanced calc class. But my first ideas of continuity in a different sense, or a more mature sense was in analysis, when I thought what does it really mean for all epsilon there exists a delta. And as for a moment when I really think that hit me, I just know that working on problems, I think there came a point when I got tired of looking up the definitions of some things. I don't think I have a very good understanding of it though. I really don't. I think my understanding is still pretty immature as far as a mathematician goes. As far as Jo average on the street I think I'm ok about it. I have some understanding of it and I have an intuitive understanding but I really don't feel like I really, I mean I know that the inverse maps open sets to open sets but I think, pretty much, I need to remember what the theorems are as opposed to having to look them up.

Changes in understanding. [No response]

Milestones. [No response]

## • Derivative

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I think the first thing that comes to mind is symbolically,  $x^4 = 4x^3$ . And then what comes to mind after that is the idea of a tangent line and the slope, but I guess first what comes to mind though is that little  $x^4$  thing, but that's maybe just what you're reminded of.

Specific examples. [No response]

Visual images.

*Bill:* Yea, I think mostly I think of the max min ideas, so you picture a curve and how you get to the top, your tangent line and stuff like that. And even when I taught 254 last quarter I think of my tangent plane as coming out and even if it's an inflection point then that's when the tangent line comes up like this, and it's not the max but you're just changing direction. And now in higher dimensions it's that your tangent plane comes up to a point and it may be a max, it may be a min, maybe a saddle point or something.

Specific images.

*Bill:* In algebra if the derivative and its original function share a zero then it's a multiple zero. That is one. And that's an idea that I used in algebra and in complex, that was developed in several places. Also you take the derivative set it equal to zero. Find the critical point. I mean that definitely has come to mind. That is something that is pumped into us. It seems like there's a ton of examples, but those are the two that I would say come to mind. Right now though, it would be the multiple zero idea that comes into my mind first. I mean that seems to be what I used it for always.

First introduction. [No response]

Changes in understanding.

*Bill:* I think it's simplified actually. I think I tried to read too much into it when I was in calculus. You take the derivative, set it equal to zero, you do that about a zillion times and you do all those things where you use it as a tool but really you just forget what it's really telling you or what it's really saying, and all it's saying is that that tangent line is just good enough. I think that it happens to everyone that you are given so many tools that you can use the derivative for that you just forget what it's saying. And so I think I've kind of stripped away some of the extra things and now the derivative just tells me what the derivative tells me. Also, and maybe the reason that the polynomial idea came up first is because in algebra you don't really think of it in terms of curves, but you still use it if you're looking for multiple zeros or something then you take the greatest common divisor, and if it's true, da da lalala. So now they are a little more symbolic, you don't really think of it in terms of a curve, you think of it in terms of just how it's being factored and if there's more than one zero the derivative will have that same zero also. So that's the other thing that comes to mind. And I think that's two different points of view. I mean that's two different of my math points of view. It depends which area I look at. If I look at it from the algebraic point of view then I don't think of it in terms of tangent lines at all. I think of it more of as a function being factored and then it just has more than one zero and then, on the other side I think of it in terms of the tangent line I guess.

Milestones. [No response]

### • Integral

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* Area. I think of the area of the curve. I think of a curve and

the shaded region below. Yea definitely. But I tried to get away from that because in complex it doesn't mean that any more. I don't know exactly what it means in complex analysis, but just an integral does make me think of area.

Specific examples.

*Bill:* The examples that come to mind are the ones where you have to find the area bounded by these two curves and so you take the area of one and you subtract the area of the other. That's what comes to my mind when I think of integrals, that's very concrete though. It's almost as if you took the paper and you cut out what you want, you cut out the whole region and then you cut out the part you didn't want if that's something below the curve and so you are just really taking those areas and subtracting them. You're removing the areas.

Visual images.

*Bill:* I still just, the first thing that always pops into my head is just the area idea.

Specific images.

*Bill:* The name Fatou's Lemma is the one that seems to come up in my head, but I don't want to talk about that, because I really don't know much about it. It's one of those when can you switch the limit and the integral around but I don't know much about it. That just popped into my head. Not in terms of theorems, more in terms of conceptually, like this integral is like the normal distribution but that's not really a theorem.

First introduction. [No response]

Changes in understanding.

*Bill:* I think so. When I had probability, that's when I really thought of it more, not in terms of area, but more in terms of weight of a space. You use integrals to figure out the probability. That still told you the area or the portion you're cutting off but it seemed more like a weight now as opposed to an area. Also I remember, when, as an undergrad, when we took vector calculus what we did was a lot of this vector stuff, and we did double, triple integration. And I remember it was explained to me so well about how to do double and triple integrals. And I have always remembered that. It has never failed me. And when I explain it to other people, like in the learning center they go oh, and it's just like this little red light shining and I know all is well because they're just going to be able to do these problems. So that is one point when it really changed. So that I also think of it as just adding up all along the line, the y axis, and just cutting it up and add up all these things. You're just kind of cutting up the area. So, it changed when I thought of it in a probabilistic sort of sense. And I think that that's probably it. But I don't think it really got much deeper, it just kind of gave me a different point of view of thinking of it.

Milestones. [No response]

### • Interpretive Analysis

He first evokes the algebraic symbols  $f(x)$  commonly used to denote a function. With this he also associates the diagrammatic image of the graph of the function, if the function is a function of a single variable. He has this concept linked with broccoli because of some technique he was using in class to try to teach about function composition. It stuck with Bill, I wonder if it stuck with his students?!

Bill visualizes continuity graphically, by evoking a visual image of a smooth curve, one with no sharp points. Even though he is aware that this is not a necessary condition for continuity, he prefers this image. He also describes a topological understanding of a continuous function as one whose inverse maps open sets to opens sets. He describes his understanding of continuity as increasing and becoming



more intuitive. He describes a change from the standard introductory calculus definition of a continuous function of one variable, to a more general topological understanding. However, there does not seem to be any particular sudden event that changed his understanding. It seems to have been a more-gradual development, such as occurred when he took complex analysis. It is interesting to note that Bill's assessment of his own understanding is that it is still somewhat immature. One measure he gives of this is that he still has to look up theorems rather than having them at his fingertips.

He evokes algebraic symbols to represent a derivative, namely  $x^4 = 4x^3$ . Other visual images he evokes are diagrammatic, of the slope of the tangent line, and the tangent plane, and images associated with max-min theory. He also has an algebraic understanding of a formal derivative as a tool for determining multiplicities of roots of polynomials. Thus he has two distinct areas of understanding, one analytic, one algebraic. He describes his understanding of derivative as changing dramatically, by becoming condensed and simplified. It was quite an event when he first understood what a derivative was really about, with all the applications he had lost sight of what it really represented. The way he describes it, it seems that he was able to condense all the information that had been given him concerning derivative so that it made a condensed, compact, unified gestalt, and as a result became much more clear to him. It would seem that he is describing the condensation phase in his understanding of derivative.

His visual image of an integral is a diagrammatic one of the area under a curve. His prototypical model of an integral is of the area between two curves. He describes visualizing this with a very concrete analogy to paper cutting, by cutting out the area under the top function, and cutting away the area below the bottom function. He also links this with the application to volumes of solids of revolution, and has a visual image of this process. He connects this concept with probability theory, where an integral is interpreted as a weight rather than an area. This new application broadened his understanding but did not radically change it.

However, his understanding changed significantly when he began tutoring students in techniques of integration, particularly the use of double and triple integrals to compute area and volume.

### • Metric Space

Evoked images. (What first comes to mind when you think of concept?)

*Bill:* I usually just think of distance. I think my first initial reaction is to think of two points and see how far apart they are as in distance. Then I have to think 'don't think like that' because that's not always the way you should be thinking of the metric. So then I try and abstract it or just think of it in terms of a relationship between points, not so much as the distance that you can measure with a tape measure, but more of a relationship between where points are in relation to each other. I try and remove that idea of distance even though that is the first thing that pops into my head, and then I think 'no, that's not how,' I mean usually that's Euclidean distance, but then I try and put it out of the way.

Specific examples. [No response]

Visual images.

*Bill:* I'll think of the two points. Something that pops into my head when I think about it is I think of, and I've forgotten what theorem this is, but it's when you have a point outside of a curve, you take the minimum distance, that is what I think of for a metric. I mean where is the minimum of that function on that curve, you know, for a metric. So I guess that's what comes to mind visually, is that. Even though that is not the right example, that's what I keep in my head, that's how you use the metric but also it's the idea of the distance between the points but it may not just be a distance.

Specific images.

*Bill:* Well I can't remember the name of that theorem. I think what the theorem said was if you have a point outside of a bounded set that there is some point of that set that has the minimum distance in terms of the metric, they always say distance for the metric.

First introduction. [No response]

Changes in understanding.

*Bill:* Well, I guess it changed from not knowing anything for quite some time. I mean I really didn't think too much about it, to being introduced to the topic as a graduate student. I guess everyone has the idea of a metric, but you didn't know that there was anything else, you just found the distance between two points so, what's the distance between these two points in the plane and you drew a little triangle. But as an idea I think I was introduced to it here at graduate school, and as far as changing I think it is still developing, it's still a pretty immature idea I guess. It's hard to see how it's changed, because as an idea it hasn't been there long. Other than I broke that initial idea of it's just how far apart things are, I don't think it's changed. Because now when I read some things in cryptography that talk about distance between words, and there's no distance between them [in the Euclidean sense].

Milestones. [No response]

### • Interpretive Analysis

Bill's visual image of a metric space incorporates the idea of the metric, rather than the space. He visualizes two points and the distance between them, and he also thinks of the distance between a point and a set, or curve. He links this idea of metric with distance, and that to cryptography and the use of the word 'distance'

in that context. His understanding has broadened from the simple interpretation of a metric as the Euclidean distance between two points in the plane, to a much more general understanding of a metric. However, he does not describe any radical paradigm shift. He describes his understanding as still developing.

### CASE STUDY 6 - CRAIG (graduate)

#### • **Well Ordered Set, Partially Ordered Set, Zorn's Lemma, Group, Ring, Field, Set, Cardinality**

Craig did not contribute much about these concepts. The transcripts have been omitted.

#### • **Induction, Infinity**

Craig connects the concepts of induction and infinity by linking them both with high school introductions to them. He has a basic understanding of infinity as a large quantity, and induction as a proof technique.

#### • **Induction**

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* Something I just kind of flunked. Something that I neglect to use.

Specific examples.

*Craig:* Well I have weird examples of induction. My favorite induction thing is this little proof by induction that all horses are the same color. I guess when I think of induction I think of relatively straight forward algebraic or numeric facts establishing rules in general, like how to get from one over to the next.

Visual images.[No response.]

Specific images.[No response.]

First introduction.

*Craig:* Well again I'm not sure exactly but I know it was in high school if not before. Probably in high school.

Changes in understanding.

*Craig:* Not really that I'm aware of.

Milestones.[No response.]

### • Infinity

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* A whole lot. I mean infinity as being a whole lot. Not a lot of

things. The next thing that comes to mind, these may not be very mathematical, is a story about the nature of infinity that I was told in high school. It's a paradox of sorts about the infinity hotel which has infinite number of rooms. One night they're all filled up and another guest shows up, wants a room, the manager says no problem, just move everyone to the  $(n + 1)^{st}$  room. Now there's room 1 open.

Specific examples.

*Craig:* I'm not sure. It may actually have been in geometry, high school. In the context of straight lines moving to infinity. Something like that, I mean, using infinity as induction.

Visual images.

*Craig:* Maybe the plane and it's way out there.

Specific images.

*Craig:* I guess proof by induction isn't really a theorem.

First introduction.

*Craig:* I'm not sure. It may actually have been in geometry, high school. In the context of straight lines moving to infinity. Something like that, I mean, using infinity as induction.

Changes in understanding.

*Craig:* No, not really. I mean it seems intuitive now but I can't really remember. That might be a problem with me for a lot of these things. My mathematical background has a hole. Since being in highschool and as an undergraduate, there's a gap of 13 or

14 years between then and now, so in some sense not a lot of continuity there.

Milestones.[No response.]

• **Interpretive Analysis**

Craig's prototypical model for induction is actually a counter-example, of a false induction proof. This is helpful for him in recalling the limitations of induction.

He immediately links infinity with 'size', and then describes a metaphorical understanding of infinity. The analogy is with a hotel having infinitely many rooms, and thus always able to accommodate one more guest. His prototypical model of infinity is given by the natural numbers. He describes a visual image of infinity out beyond the plane, and links infinity with induction.

• **Metric Space, Vector, Vector Space, Basis**

• **Metric Space**

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* Distance. In the sense that there will be some kind of distance function on the space.

Specific examples.

*Craig:* Just the usual Euclidean three space, with the usual distance between points.

Visual images.[No response.]

Specific images.

*Craig:* I guess there should be but, not coming to mind at the moment.

First introduction.

*Craig:* I don't really ever remember seeing it before this year, in the fall, in real analysis.

Changes in understanding.

*Craig:* Not a lot. I mean, well, probably not a lot.

Milestones.[No response.]

### • Vector

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* An arrow. Just an arrow.

Specific examples.

*Craig:* Oh I guess I tend to think about physical examples, vectors as



acceleration, positions, or inertia, things like that.

Visual images.[No response.]

Specific images.[No response.]

First introduction.

*Craig:* Certainly in freshman physics. I'm not sure if I'd seen it before that or not.

Changes in understanding.[No response.]

Milestones.[No response.]

### • Vector Space

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* Three-dimensional space.

Specific examples.[No response.]

Visual images.

*Craig:* Real space. I'm not sure if that's a visual image or not. I'm not seeing a set of axes.

Specific images.[No response.]

First introduction.

*Craig:* I'm not entirely sure although there was a one quarter course in vector calculus as an undergraduate where I might have seen it.

Changes in understanding.[No response.]

Milestones.[No response.]

• **Basis**

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* I guess two or three space and the usual basis vectors.

Specific examples.[No response.]

Visual images.

*Craig:* That's a visual image since, you know, it's a picture.

Specific images.[No response.]

First introduction.

*Craig:* I was actually introduced to the idea in physics or mechanics class

because I didn't take linear algebra.

Changes in understanding.

*Craig:* In the sense of realizing usually, in physics we would have first used it just sort of standard, the normal unit vectors, and later on they introduced the idea that you could use other things, special cases. And also I guess I generalized it in the sense of it being tied to three dimensional space, but it could be a little more abstract.

Milestones.[No response.]

### • Interpretive Analysis

Craig links the concept of metric space specifically with the metric itself, and begins by describing the idea of distance. His prototypical model for a metric space is Euclidean three space, with the usual metric. His diagrammatic model of a vector is of an arrow. He links this concept with many physics applications that he learned. His prototypical model of a vector space is also Euclidean three space. He describes a visual image of real space, but at the same time is not visualizing axes. His understanding has changed somewhat but no sudden event is discernible.

Not surprisingly, Craig's prototypical model for a basis is the standard basis in either two or three space. His understanding changed in that it became somewhat more general when he learned about  $n$ -dimensional vector spaces, rather than being restricted to 2 or 3 dimensions. However, his reflection indicates that he still relies heavily on his original prototype. This continues to provide the access point to his concept image of basis, from which he then accesses more recent information related to the concept of basis.

## • Limit, Convergence

### • Limit

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* Something that was made out to be very difficult when I took freshman calculus. But as far as I can tell isn't.

Specific examples.[No response.]

Visual images.

*Craig:* Probably something graphically, imagining it getting closer and closer to some point on a graph.

Specific images.[No response.]

First introduction.

*Craig:* High school.

Changes in understanding.

*Craig:* I'm not really aware of the idea having changed. I remember being told it was a concept that people have trouble with and so for quite a while I went around assuming I didn't really understand what it was. But then eventually I decided I really did. I decided that it really wasn't anything more. Maybe I really still don't understand it, but the more or less intuitive understanding that

I thought I had at the beginning hasn't changed much.

Milestones.[No response.]

• **Convergence**

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* I guess getting close and closer to some fixed points.

Specific examples.[No response.]

Visual images.

*Craig:* A number line, lines that are getting closer and closer to some point.

Specific images.[No response.]

First introduction.[No response.]

Changes in understanding, milestones.[No response.]

• **Interpretive Analysis**

Rather than evoking any symbolic or diagrammatic image, Craig first recalls a self-awareness of discomfort connected with limit. He recalls thinking that it ought

to be a difficult concept to understand, but he personally didn't find it to be so. He has a diagrammatic, graphical image of both limit and convergence.

## • Function, Continuity

### • Function

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* Well, examples of functions or a rule.  $x^2$  or log ones.

Specific examples. [No response.]

Visual images.

*Craig:* I'm seeing a parabola. The very first thing that came to mind was probably the formula. But when you asked about specific examples, the  $x^2$  and the parabola sort of go together.

Specific images. [No response.]

First introduction.

*Craig:* I don't really remember.

Changes in understanding.

*Craig:* Well, I've probably expanded the classic things that I've considered to be functions. So maybe it's generalized some.

Milestones.[No response.]

### • Continuity

Evoked images. (What first comes to mind when you think of concept?)

*Craig:* A graph. Something along the lines of a sine curve probably. A nice smooth curve.

Specific examples.

*Craig:* Lines, curves, simple curves, the kind that you know, run your pencil over.

Visual images.[No response.]

Specific images.

*Craig:* Yeah I guess actually. I mean I don't remember a lot of theorems. If I haven't seen it this year I probably won't remember, but things like intermediate value theorem, mean value theorem.

First introduction.

*Craig:* Again it's difficult for me to remember. I would guess that I first saw it in high school, precalculus, so the beginning of calculus, a course in highschool. If not then probably here, or in vector calculus in college.

Changes in understanding.

*Craig:* The first introduction to it would probably be just the geometric thing. Smooth curves and somehow there's additional, maybe I associate that with epsilon and delta and all that sort of stuff. Formal definition.

Milestones. [No response.]

### • Interpretive Analysis

Craig describes prototypical models of functions,  $x^2$  and  $\log x$ . He evokes the algebraic symbols for these functions first, rather than their graphs. Again, his understanding has broadened with the addition of examples, but does not seem to have undergone any sudden shift.

Craig first evokes a diagrammatic model of continuity, the graph of a function. The prototypical model for that is the sine function. His understanding changed from a very intuitive, geometric, diagrammatic understanding of continuity to an understanding of the formal  $\epsilon$ - $\delta$  definition of continuity.

### • Derivative, Integral

#### • Derivative

Evoked images. (What first comes to mind when you think of concept?)

*Craig:*  $\frac{d}{dx}$ . That was the first thing, but ok, now if I think about it for a



second longer. Slopes of tangent lines. Maximum.

Specific examples.

*Craig:* Some specific examples. (pause) Well, no specific examples that jump out when I think of it.

Visual images.

*Craig:* When I said slopes of tangent lines I was imagining a curve with a tangent.

Specific images.

*Craig:* The rules for finding derivatives in particular cases count as theorems.

First introduction.

*Craig:* I know I was first introduced to the concept in highschool, although I think that the first introduction is very formal. I'm not sure that that was the case in just learning the rule for differentiating polynomial type terms. And again, certainly, I mean the notion of derivative has extended somewhat since that first, I mean in terms of directional derivatives and surfaces, just broadened the concepts.

Changes in understanding, milestones.[No response.]

## • Integral

Evoked images. (What first comes to mind when you think of

concept?)

*Craig:* The symbol.

Specific examples.[No response.]

Visual images.[No response.]

Specific images.

*Craig:* Yes, the fundamental theorem of calculus. And then stuff related to Lebesgue integration. Lebesgue dominated converging theorem. Things like that, or maybe not quite, just about any of them but there are things that involve integrals and cover integrals.

First introduction.

*Craig:* That idea was I think first introduced to me in highschool, I'm not sure that we got that far actually, if not then in freshmen calculus.

Changes in understanding.

*Craig:* My ideas about integrals have definitely evolved some. This year. Starting out from just the idea of area under a curve and then in real analysis this year we looked into much more detail about the foundation. As in some of the other things I guess it's in some sense more general, in that they seem to define an intergral in ways that are extended beyond area under a curve, it's going to depend on how you define a measure on your sets. It's a generalization, again the converging idea.

Milestones.[No response.]

• **Interpretive Analysis**

Craig's understanding of these concepts derives predominantly from his calculus experiences, although he has broadened his understanding of integral recently. He evokes the algebraic symbol  $\frac{d}{dx}$  for derivative. His visual image is of the slope of the tangent line to a graph. His understanding has broadened to include derivatives of vector valued functions. He associates the symbol  $\int$  with this concept. His understanding now includes different types of integration, since he has been taking an analysis class. No sudden shift in his understanding is apparent.

**CASE STUDY 7 - DORIS (graduate)**

• **Well Ordered Set, Partially Ordered Set, Induction,  
Zorn's Lemma, Cardinality**

Doris does not have a clear recollection of these concepts, although she has encountered them all at one time or another. Her prototypical model for a well ordered set is the natural numbers. She is able to evoke some kind of container schema which models the ordering of the elements in the set, even though she is unable to evoke the formal concept definition for a well ordered set. Similarly, without having a clear idea of the formal concept definition of a partially ordered set, she is able to evoke both an accurate visual image, of a tree diagram, and an accurate prototypical model, of the power set of a set. She does not evoke any visual images associated with induction, other than the algebraic symbols used to denote the steps of a typical inductive process. She evokes the actual process of an inductive proof, and links this process with the natural numbers.

## • Set, Group, Ring, Field

Doris evokes the same diagrammatic image of a set for the algebraic structures in this group as for the concept of set itself.

### • Set

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* A set of numbers. A set of objects. I guess I get a picture of a finite set, circles around things. There was an immediate picture, a circle with things in it, numbers, set numbers.

Specific examples.

*Doris:* I mean to me the power sets, the natural numbers, the integers, and, to me it's just everything. Everything is a set.

Visual images. [No response]

Specific images. [No response]

First introduction.

*Doris:* A lot of these I'm going to say 7th grade and you know, actually, the main problem being there that at that period of time my life was pretty foggy. I remember tenth grade geometry, I had the same math teacher for seventh, eighth, and ninth grade. It was an advanced class and so I remember a few of the things that we were definitely introduced to under him, but where I'm relating all these

things to 7th grade there is a possibility that they were eighth or ninth grade too.

Changes in understanding.

*Doris:* It's evolved. There again, a field is a set, a ring is a set, as far as the size and the differences between different sets and the differences of the operations you can have on sets and all those intersections, unions, Boolean algebras, that stuff that you can have or not have, that's grown, I don't know that my basic notion of what a set is has though.

Milestones. [No response]

## • Group

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* I think of it initially just as a set with objects, numbers, that's the first thing that pops into my mind.

Specific examples.

*Doris:* The first examples that would come would be like  $D_5$  or  $D_n$ , or the integers mod something. There again finite groups are one of the first examples I think of.

Visual images.

*Doris:* The first thing that I visualize is a finite set.

Specific images. [No response]

First introduction.

*Doris:* Senior year in college.

Changes in understanding.

*Doris:* It's certainly grown.

Milestones. [No response]

### • Ring

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* The integers.

Specific examples. [No response]

Visual images. [No response]

Specific images. [No response]

First introduction.

*Doris:* Senior year in college and I didn't pay a lot of attention, we had 2 or 3 weeks and at the end was crammed with rings, fields. I mean,

it wasn't 3 weeks it was only 2 weeks.

Changes in understanding.

*Doris:* Yeah it hasn't necessarily changed, it's there again grown in some degrees, similiar to my knowledge of fields. It's not that it's really changed, it's made more sense along the way.

Milestones. [No response]

### • Field

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* Oh, a corn field. And then, second of all, I just think of a set of numbers, and probably the first one that comes to mind is rationals.

Specific examples. [No response]

Visual images.

*Doris:* Yeah, I've got a visual image of that. I mean there is something visual that goes on, a scattering of numbers, like when you say a field, I see one of those Disney cartoon things where they would just have a bunch of numbers up there. Yeah, I have a picture like that I guess, of a field of numbers. Like a corn field.

Specific images. [No response]

First introduction.

*Doris:* My senior year.

Changes in understanding.

*Doris:* I wouldn't say that it ever changed. I came to understand fields better. I took another course where we studied fields, then my first year in graduate school. The first introduction as far as to fields at the senior level, I was real distracted at the time and didn't pay a lot of attention and didn't particularly get what was going on, as far as I didn't really care what was going on, as far as what was the difference between a ring and a field and all that. So I'd just say that my understanding of fields has grown. It grew mainly just through studying, and right now we've got lectures on quadratic fields, and I see now where the term associates in a field and what they mean and I remember going through that definition before and yesterday it was real clear why there was a need for that, and I don't think that I realized that even when I was first going through those definitions, so you see it grows but I don't think it's ever really changed. It's just being flicked into place more solidly.

Milestones. [No response]

### • Interpretive Analysis

Doris visualizes a set with a diagrammatic image of a container, a circle around the set elements, which she describes as numbers or objects, and this turns out to be a useful image for algebraic structures such as groups. She evokes several prototypical images of sets; the power set of a set, the natural numbers, the integers, but goes on to say that *everything* may be construed as a set. Her concept image has grown to include different algebraic structures as sets, such as groups, rings and fields, and to include properties of and operations on sets.

She links the concept of a group with that of a set of objects or numbers. Her prototypical image of a group is of a finite group such as the dihedral groups, or the



group of integers modulo  $n$ , for some  $n$ . Her visual image is the same diagrammatic image - a container schema of a circle around the elements in the group - that she evokes for a set. Her prototypical image of a ring is the integers.

With the concept of field, Doris draws an interesting analogy with a non-mathematical understanding of a field, and uses this to help her to visualize a mathematical field. She describes an analogy with a corn field, and explains that she sees the elements in the (mathematical) field scattered around inside a container of some sort. She describes this image as being like a cartoon balloon with words inside denoting the words that the character is speaking. These words are analogous to the elements in the field. Her prototypical image of a field is the rational numbers.

Her understanding of these algebraic structures seems to have grown gradually, rather than having undergone any sudden change, particularly for her concept image of field. She suggests that these pieces slowly fell into place, the reasons for definitions and the subtle differences between rings and fields became apparent, the different algebraic structures became linked. She describes her understanding as being more solid, perhaps suggesting a condensation of the pieces of information associated with a field, to form a compact concept image, ready for the next (reification) stage.

### • Metric Space, Infinity, Limit, Convergence

Doris has these concepts linked with a general topological thread.

#### • Metric Space

Evoked images. (What first comes to mind when you think of concept?)

*Doris:*  $R^2$ . I mean that's what I see.

Specific examples. [No response]

Visual images.

*Doris:* I think of a disc, a ball. That's when you have open sets and you have a measure on your open sets.

Specific images.

*Doris:* No, no specific theorems. It's a lot easier to talk about convergence, it's a lot easier to talk about whether or not a set has limit points, it's a lot easier topologically to talk about in a metric space. It's fairly difficult to talk about the limit points in spaces where you don't have a metric.

First introduction.

*Doris:* First year in graduate school.

Changes in understanding.

*Doris:* No, nope same stuff.

Milestones. [No response]

### • Infinity

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* The first thing that comes to mind is the symbol infinity, then immediately followed by north pole, south pole, a globe. Probably a quick flash of the number line and it's way out there, those types of things.

Specific examples. [No response]

Visual images.

*Doris:* A globe, and the north pole and the south pole are points at infinity.

Specific images.

*Doris:* No particular theorems that come to mind associated with infinity. I can't even think of a theorem right now.

First introduction.

*Doris:* I think that I was probably first introduced to the concept of infinity in the seventh grade.

Changes in understanding.

*Doris:* The only time that infinity went through any sort of serious change for me was in the REU program where they first really started throwing around the terms, countably infinite versus uncountably infinite and I didn't even know there was a difference. So this was in my junior year of college that I was still unaware that there was a difference between the two. And that term was thrown around in classes more and more frequently. I certainly went through my senior year hearing that but never knowing what the difference really was. I guess it was in my first year of graduate school I took topology and real analysis and I came to understand the difference between countable and uncountable infinity.

Milestones. [No response]

• **Limit**

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* A limit as  $n$  goes to infinity. I guess, what first comes to mind is taking the limit as something goes to infinity, kind of looking down, I see convergence I guess, looking down a funnel or something, where it goes way out there. Approaches something, a cone. So I guess like the picture I get, or what I think of, but there's no, I don't think of a specific example. That's it.

Specific examples. [No response]

Visual images. [No response]

Specific images.

*Doris:* Oh, I know right off the top that there are things like the ratio test and the comparison test and those sorts of things. They sort of pop into mind fairly rapidly, that there are ways to test to see if the limit exists. I guess to me that's the main thing, is being this, surrounding whether or not limits exist.

First introduction.

*Doris:* Probably in calculus my senior year.

Changes in understanding.

*Doris:* Yes, my understanding of limits, there again it has not changed as much as it's been added on to. There again, one of the main problems I started running into was people started talking about limsups and liminfs, and that was probably my first year in college, when I supposedly took advanced calculus. And that was when I was really not there, I got straight C's in that class. It was at 8:30 in the morning, I fell asleep a lot. So I'm sure, how I talk about uncountably and countably infinite, I should have probably learned, or was told about all that stuff in there and didn't get it. I just wasn't there, I was very possibly asleep, so then it was in graduate school where I finally started understanding limsup and liminf and that kind of stuff. But as far as the limit existing, whether or not the limit exists, no I wouldn't say that my concept of that has changed, or what it is, you know what I mean.

Milestones. [No response]

### • Convergence

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* I think of convergence of a sequence as real similar to the limit. The first thing that came to mind was an infinite sum and so that was wrong. What I see is just that it telescopes right back. And the first thing is like thinking a set of numbers for  $1/n$  where  $n$  goes to zero.

Specific examples. [No response]

Visual images. [No response]

Specific images.

*Doris:* There's topologies that come to mind. I was thinking, box topology

versus product topology. Along the lines of specific examples, I was looking at, I think it was the  $1/n$  sequence, or looking at some different infinite series in the box topology and product topology, and then comparing how the sequence converged under different topologies.

First introduction. [No response]

Changes in understanding.

*Doris:* It changed in topology as far as that when we talk about convergence of a sequence it can be pretty well dependent on what we're talking about as far as for measure, what's distance. So my idea of convergence of a sequence changed as I realized that there could be a change in how you would measure distance. That convergence of a sequence is not a yes or no answer.

Milestones. [No response]

### • Interpretive Analysis

Doris' prototypical image of a metric space is of  $R^2$ . The visual image she has associated with this concept is a metaphorical image of a ball or disc, that is an open set in the space and not the space itself. She links this concept with convergence and topology.

She first evokes a visual image of the algebraic symbol,  $\infty$ , for infinity, along with a diagrammatic image of the north pole and the south pole of the sphere. She also visualizes infinity as sitting at the end of the real number line. Her understanding seems to have undergone a significant change when she learned about countable and uncountable infinity, in topology and in real analysis classes, where she came to understand their meaning and the subtleties involved.

She first evokes the algebraic symbol  $\lim$  used for limit. She describes visualizing a limit using an analogic image of looking down a funnel to see what's at the end of the cone. She links this concept with convergence, and the various tests for convergence. Again, she feels that her conception of limit has broadened to include related properties and concepts, but has not undergone a significant and sudden change. In fact, her basic understanding of a limit itself has remained unchanged.

She links the concept of convergence with limit, and with topology, her prototypical model of convergence being of the sequence  $\frac{1}{n}$  as  $n$  goes to  $\infty$ . Her understanding changed significantly as a result of studying convergence under different topologies, and using different metrics, and this seems to have been a significant factor in broadening her associated concept images.

### • Function, Continuity, Derivative, Integral

At one time, Doris considered that every function was continuous, most likely because she had not encountered any discontinuous functions. As a result, the concept images of function and continuity remain linked, and share common links with other concepts in calculus. She evokes visual images of algebraic symbols for these concepts.

#### • Function

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* The first thing is  $f(x)$ , pictures of graphs, trig functions, polynomials.

Specific examples. [No response]

Visual images.

*Doris:* I don't see the graphs [of those polynomial functions]. I see the generic graph. When I start thinking about it, I guess the first graph is always just a wavy line and it's like ok, that's cosine and sine, so that's where you get trig functions coming into it and then it's just polynomials,  $x^2$ .

Specific images. [No response]

First introduction.

*Doris:* Junior high. Seventh and eighth grade. I know by the ninth grade that I had had all the algebra, because we went into tenth grade geometry. Maybe we were doing some sort or other of lightweight analysis in ninth grade. So, I understood algebra by ninth grade and so whatever I was doing was in preparation for doing geometry, and geometry then was where there were proofs involved and all that stuff.

Changes in understanding.

*Doris:* Maybe a little bit. I think that maybe for a long time I thought of all functions as being continuous functions. Things like the greatest integer function I know for a long time I never even thought of things like that as functions. There again I don't know that there was any great lightbulb or that it was just sort of an evolving you know, that there are a lot of different types of functions.

Milestones. [No response]

• Continuity



Evoked images. (What first comes to mind when you think of concept?)

*Doris:* Smooth functions.

Specific examples. [No response]

Visual images.

*Doris:* A wavy line.

Specific images.

*Doris:* [No response]

First introduction.

*Doris:* I'm sure it had to be introduced in calculus, just because when I teach calculus I see it, I know that I was given a definition of continuity in calculus, or had to have been given one then, had to have been given one considerably before that. I took algebra and all that in high school so I'm sure that I was given a definition for that early on. As far as the definition of continuous where a function is continuous if the inverse image of an open set goes to an open set that concept you know, in graduate school.

Changes in understanding.

*Doris:* I guess I'd have to think about it right now if that was even changing or if it was just going from a notion of continuity in 2 space versus the notion of continuity in other types of spaces.

Milestones. [No response]

**• Derivative**

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* Not much.

Specific examples.

*Doris:* Well, velocity, tangent, the tangent line to the graph, but that's not what first came to mind, it was just like the derivative.

Visual images.

*Doris:* I don't get a visual image, I thought about  $x^2$  being a  $2x$ , real briefly but there's not a lot there. I can think of a lot of things right now that I could've thought of immediately but that's not the idea.

Specific images. [No response]

First introduction.

*Doris:* Freshman year, I guess actually I take that back, senior year in high school. So that's when I was first introduced to it. I don't know if my knowledge of derivatives has changed at all. I've seen additional uses involving roots of polynomials and things along those lines. But aside from that, or just using it as a tool in maybe more advanced settings, but my idea of a derivative and that type of thing hasn't changed much. I think that, for me, I don't look at the derivative as being a concept, I look at the derivative as being a tool. To me a field is a concept, a derivative is a tool you use.

Changes in understanding. [No response]

Milestones. [No response]

### • Integral

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* The little symbol.

Specific examples.

*Doris:* You say specific examples I think of  $\frac{1}{x}$  and  $\frac{1}{x^2}$ . I can think of the answer to those pretty quickly but I never...

Visual images.

*Doris:* Area under the curve. There's a curve and then just a shaded region.

Specific images.

*Doris:* The only thing, and that's just because I'm dreading it is Lebesgue dominated convergence theorems. My masters orals are coming up, so it's all those theorems that I don't know, that have to do with mainly Lebesgue integration. Aside from that, I can think of different things, but the integral is a tool. It does this, series converge and if you look at the term as an integral and you can use it as a tool to see if certain infinite series converge or not. I think of it as a tool.

First introduction.

*Doris:* Senior year in high school.

Changes in understanding.

*Doris:* It's grown from the standpoint of that typically how I think of integration, I still think of integration as Riemann integration not Lebesgue integration. The integral is just like basically first year calculus. But when I think about it on any other type of level I think of it as representing an infinite series, and so I guess that's more what it means.

Milestones. [No response]

#### • Interpretive Analysis

Doris first evokes the algebraic symbol  $f(x)$  for a function. She quickly evokes visual images of graphs of various functions, her prototypical visual image being the graph of a trig function, a wavy line. She describes her understanding as having broadened, rather than having undergone a sudden shift, as a result of expanding her examples to contain such discontinuous functions as the greatest integer function, which at one time she did not think of as a function, because of its discontinuity. Her prototypical visual image for continuity is a smooth graph. Her understanding changed after taking topology, to include an understanding of continuity in  $n$  space rather than just in two space, indicating a broadening of her conception of continuity.

Doris immediately evokes the visual image of algebraic symbols, such as  $x^2 \rightarrow 2x$  associated with derivative. Later on she describes the tangent line to a graph. It is clear that she considers a derivative to be a very useful tool rather than a concept from her use of operational language to describe her understanding, which broadened to include the use of the formal derivative in an algebraic setting.

Doris evokes the visual image of the algebraic symbol  $\int$  for integral. Her prototypical examples are of the integration of simple functions such as  $\int \frac{1}{x}$  and  $\int \frac{1}{x^2}$ . She also visualizes an integral diagrammatically as the area under a curve. Her understanding has broadened to include different types of integrals, and she links the concept of integrals with the concept of series. She primarily thinks of an integral as a computational tool, particularly useful as a test for determining convergence of infinite series. This would indicate a predominantly operational understanding of integral, even though it is clear that Doris also understands an integral structurally. She seems to be more at ease with integrals in an operational setting.

### • Transformation, Vector, Vector Space, Basis

Doris links these concepts because of their common links with linear algebra.

#### • Transformation

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* Probably a matrix.

Specific examples. [No response]

Visual images.

*Doris:* No. There's no picture that goes with that. I mean a transformation, to a certain extent, the first visualization and, I mean it's almost associated with electricity. Ok, so you say transformation it's like almost like think transform electricity and then it's like ok we're going to transform something and that's it. That's as far as a visualization goes, then it's like, then it goes into ok so that's a matrix.

Specific images. [No response]

First introduction.

*Doris:* I don't know. I took linear algebra in my sophomore year in college. They must have called it a transformation at that point but I don't ever recall that. And then it was junior year as far as that it was a *linear* transformation. I guess I feel like, there again, that was pretty well words and it wasn't until I got to graduate school where that started to make more sense as far as changing something from one basis to another, where it was taking something from one space to another space perhaps, where it was transformed, a vector and getting more of a feel for what that was. So, there's another one where I guess I felt like lots of times there were words going around and words never got really explained, like infinity, like transformation, and you just sort of played as if you knew what they meant, and it wasn't until much later that anything ever really got explained, fully. I'm sure that a matrix got called a linear transformation and I'm sure that those words were used there, but I never thought of a matrix as a function, as something that transformed a vector into something else, at all, probably until I was studying it in graduate school. And I aced those classes.

Changes in understanding.

*Doris:* [No response]

Milestones. [No response]

**• Vector**

Evoked images. (What first comes to mind when you think of concept?)

*Doris:* A little pointy arrow in three space.

Specific examples. [No response]

Visual images. [No response]

Specific images. [No response]

First introduction.

*Doris:* It would be sophomore year in college. Although actually, it probably would have been sophomore year in high school because I was probably introduced to vectors in geometry. I was thinking of vectors as being introduced in linear algebra but actually it might have been introduced in geometry.

Changes in understanding. [No response]

Milestones. [No response]

**• Vector Space**

Evoked images. (What first comes to mind when you think of concept?)

*Doris:*  $R^n$ , just  $R^n$ . I might see something as far as just the little graph and how they always draw the arrow up from the origin, and the representation in matrix form.

Specific examples. [No response]

Visual images. [No response]

Specific images. [No response]

First introduction.

*Doris:* Probably sophomore year in college.

Changes in understanding.

*Doris:* It's just one of those things where it's grown. Everything that was happening in the linear algebra courses that I was taking was in  $R^n$  and then as far as with algebra and stuff and starting where you're taking vector spaces over the integers mod  $p$ . Those sorts of vector spaces, and then starting to take a look at modules and various things like that, my notion of vector spaces or almost vector spaces started to change, or just grow. I guess one of the other things as far as just with transformations and vector spaces, part of my algebra class my senior year we took non-euclidean geometry, we took a term of non-euclidean geometry and that's where transformations and matrices as transformations and vector spaces when you didn't, in non-euclidean space, it didn't really change but it certainly got a lot more involved than the vector spaces we learned about in math 311.

Milestones. [No response]

• **Basis**



Evoked images. (What first comes to mind when you think of concept?)

*Doris:* Linear algebra. That anything in the space that you're talking about can be written as a linear combination of those elements. That bases aren't unique, that it's a set of objects, and I guess there again I think of bases generally as finite, however I know that that's not necessarily true.

Specific examples. [No response]

Visual images.

*Doris:* This isn't numbers, this is a set, where it's got, you know, like little letters and things like  $e_1$ ,  $e_2$ .

Specific images. [No response]

First introduction.

*Doris:* It would have to have been sophomore year in college.

Changes in understanding.

*Doris:* Yes, it hasn't changed it's grown. My knowledge of linear algebra just did. I was taking it that sophomore year, I took some more my junior year and then I took it in graduate school and that's where a lot of it came together pretty well. And that was after going through some  $R$ -modules stuff. The thing with similar matrices and just characteristic polynomial, change of basis and all that stuff, tying things together there.

Milestones. [No response]

### • Interpretive Analysis

Doris links the concept of transformation with that of a matrix. She does not have an accompanying visual image of a matrix, but describes an analogy with an electrical transformation that gives her a model for a mathematical transformation. For her, a transformation is linear. Again, her understanding seems to have condensed into a compact coherent whole, in which different pieces of information linked together gradually over time.

Her prototypical image for a vector space is  $R^n$ . She visualizes a vector space by visualizing its basis, in this case the standard basis for  $n$  space, as arrows emanating from the origin. She also visualizes the algebraic symbols of the matrix representation of the basis. Her understanding has broadened, but again there is no indication of a sudden shift having occurred.

Doris first evokes certain key properties of a vector space basis, linking this concept with linear algebra, and visualizing the algebraic symbols for the standard basis for Euclidean space, in the form  $e_1, e_2$ . Again her understanding has grown through taking a graduate linear algebra course, where many things came together, allowing her to pass through a condensation phase, forging links between different topics in linear algebra, and making the subject a more unified whole.

### CASE STUDY 8 - ALAN (faculty)

#### • **Set, Well Ordered Set, Partially Ordered Set, Zorn's Lemma, Induction**

The concept of set provides a gradual broadening of Alan's conceptions, and links among them.

• Set

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* A bunch of objects, off the top of my head. That's what first comes to my mind. That's a very innocent question because I'm teaching math 231 this summer, I've taught it a couple of times before and we stress being precise and accurate in our definitions, except when we define set. In the book, a set is a collection of objects. Oh yeah, but what's a collection? A set. And we never really define 'set' in a precise way. A set is a bunch of objects that you're studying currently and that's what comes to mind. I guess it comes to mind increasingly because I've been teaching that.

Specific examples.

*Alan:* I guess about 25 examples in Math 231, sets of integers, sets of real numbers. Nowadays we have a lot of 'Oh gee' articles about fractals, those are very interesting sets. I don't know how useful they are at the moment, but they will be.

Visual images.

*Alan:* The beautiful fractals, unfolding on a color monitor on the computer. We're taught, and I try to get the students away from that, that sets are sort of three circles, you have three sets, you have three circles in a Venn diagram and sets don't always look like that. So, I guess the visual image I have of these sets is three lumpy circles intersecting and then the classical Venn diagram. But once you get beyond that visual image there are many sets that are very interesting, very useful and impossible to visualize. That's a pet peeve of mine, that you can't always do that and that sometimes you have to understand what a set is without a picture. You can't always draw a picture.

Specific images.

*Alan:* I think one theorem that might come to my mind in that regard is the distinction between countable sets and infinite sets and the fact that the real numbers are so big compared to finite sets or countable sets. I don't know, I'm a little vague about that.

First introduction.

*Alan:* I think talking about sets and set theory came in graduate school. I didn't see much of sets with undergraduate math, I took a lot of applied courses and we talked about sets there but, I'll take that back, I guess it was probably an upper division math course that I took that in Wisconsin where they talked not too much about set theory but about very practical sets of objects, like vectors and matrices and things like that, but the set aspect wasn't really emphasized in college at that time. It was emphasized when I got into graduate work.

Changes in understanding.

*Alan:* It's changed quite a bit from just mention of it in college courses, sets, to lots of examples of different kinds of sets, sets that broaden from just being sets of numbers, into sets of functions, sets of transformations over function spaces, sets of linear transformations over Banach spaces. I think the main thing is many more examples of different situations where there are sets of objects that are interesting.

Milestones.

*Alan:* No, I don't think of any. I don't have lots of milestones. It's just sort of evolved. It didn't come about all at once.

## • Well Ordered Set

Evoked images. (What first comes to mind when you think of con-

cept?)

*Alan:* I think I associate the positive integers with that, because they are, and it's obvious. You take any subset of them, there is a smallest element and so I guess that's sort of the prototype I have in my mind. Beyond that, I don't use that so I don't know that I have too much more about that.

Specific examples.

*Alan:* Oh, all sorts of examples, hierarchies of functions, function spaces. I guess the examples that come to mind are mainly from the reals and the complexes. No, I'm sorry, the integers and subsets that disobey it, and I've forgotten how to construct some of those, though I know there are a whole bunch of examples where you have a set that didn't have the property.

Visual images. [No response]

Specific images.

*Alan:* The specific theorems that come to mind would be induction, basically, that's the whole point of ordinary induction. If you look at the exceptions there's a first one and therefore that contradicts the previous one, the proof about exception is not really an exception. With induction it's very easy to prove, once you have a well ordered set.

First introduction.

*Alan:* That was in graduate school, I don't remember which course.

Changes in understanding.

*Alan:* I don't use that concept very often. It's a very theoretical kind of thing. We did talk about it a bit in a real analysis course, but just mentioned it in passing.

Milestones.

*Alan:* When I was first introduced to it, just one course in it.

### • Partially Ordered Set

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* A set which is a collection of sets.

Specific examples.

*Alan:* [Set containment] is a very nice example. There are obviously subsets that don't contain, one doesn't contain the other and the other doesn't contain the other and so it's partially ordered. Those are intersecting sets, sets that are not one completely within the other. Another example is functions, one function bigger than another and that's having to do with the graphs. Well, that's obviously a partially ordering when they cross, when neither is higher than the other.

Visual images.

*Alan:* You can visualize the order of containment, so if one set contains another, it's considered bigger. I think I visualize these things probably with the subset analogy, that's one example. The fact that I clearly have two sets not containing one another.

Specific images.

*Alan:* I don't know, I can't think of any specific theorems about partial ordering.

First introduction.

*Alan:* That certainly wasn't in graduate school. We did that in undergraduate algebra, then in physics.

Changes in understanding.

*Alan:* That's something that evolved quite a bit in graduate school because I saw more examples of where this was important, especially in function spaces and measure theory, stuff like that. We talked about collections of objects, order is very important originally.

Milestones.

*Alan:* A particular course in graduate school helped a lot, an analysis course.

### • Zorn's Lemma

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* An analysis course I took as a graduate student that I was in we spent a month on Zorn's lemma and various things you can prove with that. That's what first comes to mind. We spent an inordinate amount of time on Zorn's lemma, and theorems that are based on assumptions, not always accepted.

Specific examples.

*Alan:* Actually, we use this every day. One application of Zorn's lemma is the proof that every linear space has a basis. That's important to understand what's happening in a particular example. You can come up with a basis for the space, Zorn's lemma says you can. That's a problem of a rather theoretical nature, but they're always there.

Visual images.

*Alan:* Sort of, probably this hierarchy of things. I think the main application of Zorn's lemma is the extended induction that you can do with it and you have the integers proving that everything in this small universe here, showing that you can get outside it and that's where you can prove everything. That's the best sort of image I have of that.

Specific images. [No response]

First introduction.

*Alan:* In a first year graduate course in real analysis or topology. No, it must have been real analysis.

Changes in understanding.

*Alan:* My understanding certainly went up after I took that course. My understanding of it was zero before that time and I don't think it's changed much after that course.

Milestones. [No response]



## • Induction

**Evoked images.** (What first comes to mind when you think of concept?)

*Alan:* The summation of the squares of numbers from one to  $n$  and I'm not sure, although I do know other ways of proving that, the easiest way to prove that is using induction. That's one thing that comes to mind is that example, I have no idea why.

Specific examples.

*Alan:* All kinds of examples come to mind. One example that came to me recently, about a year and a half ago I went to a contest of high school students and somebody asked them a question. The person had to give a ten minute talk on this question. The question was if you divide the plane,  $R^2$ , by straight lines, how many regions do you get when you have ten lines? It's pretty easy to come up with a conjecture, and then they were asked to prove it. The easiest way that you can do it, the proof is quite easy, surprisingly enough, surprising to me anyway. And it's just by induction. How many if you have just one line? There are two, and if you add another line, you split two regions into their parts, and so adding one line, the effect is very obvious and so you can do the inductive step and the proofs were straightforward, and that's an interesting example. Instead of the usual algebra one where sums are a powerful tool. But a couple of those high school students did it quite surprisingly.

Visual images. [No response]

Specific images. [No response]

First introduction.

*Alan:* Oh boy, probably in a math course in, maybe sophomore or junior year in college and reintroduced in graduate school in some course

that I don't remember specifically. It was an elementary thing so I don't remember.

### Changes in understanding.

*Alan:* I think I've broadened, over time, gradually, in some way. The kinds of problems, it isn't just numbers and algebra, algebraic problems, but geometry problems. I've used it also in computer science. You can prove that this program will produce a certain output, if you put  $n$  in. And you can do that by induction. I guess the examples have increased primarily. One real increase in concept was the effect of applying induction to \*\*\*\*, you can use Zorn's lemma in that instance, transfinite induction actually. I guess it's an increase in my understanding.

### Milestones.

*Alan:* I guess one minor milestone was that exam, I was one of the judges. That's a minor one.

### • Interpretive Analysis

Alan's prototypical images of set are of sets of numbers that he usually sees in finite mathematics when he teaches it, a visual image of a fractal that he has been investigating these lately, and a diagrammatic visual image of Venn diagrams. He realizes the limitations of using such images, and tries to discourage their use by students, or at least to make them aware that there are interesting sets that may not be visualized in any meaningful way. He describes his conception of sets as having undergone a gradual change and broadened to include many different examples, rather than undergoing any sudden, earthquake-like shift.

Alan's prototypical image (or prototype, to use his words) for a well ordered set is the positive integers, with the salient property being the least element principle.

He describes a link with the idea of induction, saying that the well ordering property is what makes induction work.

For partially ordered set, he describes a visual image of set containment, but does not make clear how he visualizes the ordering, perhaps using a linear order schema. He describes two prototypical images, both of which have visual aspects. One is the set containment model, the other is a partial ordering on a set of functions. This graphical diagrammatic model of a partially ordered set clearly shows the nature of the partial ordering - any two functions whose graphs cross are not comparable. This is actually an analogic model, where both the concept and its analog are mathematical ideas. Alan is using the graphical model for functions to provide an understanding of another mathematical concept, namely a partially ordered set. He describes a graduate course as a milestone that broadened his conception to encompass many more examples of partially ordered sets. No radical shift is evident in what he says, however.

Alan visualizes Zorn's lemma as a hierarchy of things, again not explicitly describing the nature of his visual image. I conjecture that these images are more of an intuitive 'feeling' about the salient features of the concept, rather than any clear, concrete, static picture. He also has this concept linked to the concept of induction. The example he gives is actually an example of an application of Zorn's lemma to his area of mathematics, in proving the existence of a basis.

Alan links induction with various finite sums which are best proved by induction. He provides an example of a problem he encountered recently where induction is used in the solution. He describes his understanding as having changed over time, mainly by seeing applications of induction in many different areas of mathematics and computer science. This broadening of his understanding seems to have been gradual, rather than a sudden earthquake-like event. He does say, however, that learning transfinite induction caused a noticeable increase in his understanding.

- Algebraic structures

Alan encountered these concepts in a graduate algebra class.

- Group

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* Probably my algebra course, and, you can see my prejudices here, I'm not an algebraist. I do work with some groups, groups of linear transformations and with computer science I've done semi-groups. So, I guess the first thing that comes to mind with groups is matrices. A non-singular matrix is an invertible element. Also, I've worked more with linear spaces, linear spaces with a group operation is very trivial but very important.

Specific examples.

*Alan:* Specific examples, I got that already.

Visual images.

*Alan:* Really, I don't have much of a visual image of a group. The only groups I remember from my algebra course were always in a table. They were finite groups and we had a,b,c up here and a,b,c down here and get all our group operation results on the table. I think of a grade school multiplication table, visualize a group that way. That works only with finite groups. I use infinite groups.

Specific images. [No response]

First introduction. [No response]

Changes in understanding.

*Alan:* Not too much, I don't know much about groups. Group theory is way out in the abstract department. I know some rather simple properties of groups.

Milestones.

*Alan:* Yeah, algebra course.

### • Ring

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* My wedding! No, my wife's wedding ring! No, ok, you can tell, not too much. I don't work with rings very often. The rings I work with are linear spaces and I suppose that's the only example I ever work with, and so I don't think much about them. I guess the first thing that comes to mind is a real or complex linear space.

Specific examples.

*Alan:* I gave you one.

Visual images.

*Alan:*  $R^n$

Specific images. [No response]

First introduction.

*Alan:* It was in this algebra course, and then it was kind of downhill. That was my peak.

Changes in understanding.

*Alan:* Not much, it kind of peaked when I took the course. I got an A in the course.

Milestones.

*Alan:* That course, I guess, was the peak of the mountain and it was downhill from there.

### • Field

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* Wheat! I think of how little algebra I took when I was a graduate student. I did take an algebra course and the only field that I saw was a number field like the real numbers and complex numbers. It's stuff that I don't use every day, not like vectors. But, I guess I think of a very specific few that I work with.

Specific examples. [No response]

Visual images.

*Alan:* I really think just of the number line when I try to visualize a field. But I don't work alot with fields.

Specific images.

*Alan:* There's one, I remember. Fields with certain properties, but anyway, there are only three examples, real numbers, complex numbers, and quaternions, and then after that you lose something, one property.

First introduction.

*Alan:* I think that algebra course I took junior or senior year in college and I was more interested in physics at the time. I did o.k. They're important in, oh, things like measure theory and so forth, at least at a very elementary level. Fields and properties of fields, structure and things like that.

Changes in understanding.

*Alan:* Not too much because I haven't worked with it.

Milestones.

*Alan:* I guess when I was taking a serious, high-level, senior-level course in algebra. That was a milestone because I hadn't had any before then.

### • Interpretive Analysis

Alan's prototypical image of a group is of a group of linear transformations, or matrices, since these are the groups that he works with. His visual image is one

that he recalls from his undergraduate days, a diagrammatic image of a group table (or possibly only an algebraic symbol, if there is no ability to interpret the implicit information, such as commutativity, inverses of elements, contained in such a table). But it is not an image that he finds particularly useful, since he works with infinite groups.

Alan first evokes a non-mathematical model of a ring, but quickly evokes a mathematical prototypical image of a ring, either a real or complex linear space. His visual image of a ring is  $R^n$ , but he does not explain how he visualizes this. Again I would conjecture that it is more of an intuitive feeling, somewhat out-of-focus, dynamic, rather than any static, concrete picture of a ring.

As with a ring, Alan first evokes a non-mathematical model for a field. His prototypical, mathematical model is of the real or the complex numbers. He visualizes a field by visualizing the real number line, which might suggest a linear order schema to represent the elements.

### • Linear algebra concepts

Alan describes a basis as analogous to a set of building blocks. This analogy provides him with useful visual images for basis, and a rich but non-technical language for describing a basis.

### • Vector

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* Just a finite array of objects, objects in order, finite. I've worked quite a bit recently, I've been playing around with languages that



help mathematicians do things, like Derive, and Matlab. A very important concept for them is the fact that they should deal with vectors as objects and in some cases the objects are restricted just to numbers, but you can also have other kinds of objects. In a computer, infinite things are hard to deal with. Computers are very finite, whatever that means. And a vector is a very convenient unit. It's an object that, an ordered  $n$ -tuple if you want to get fancy language, just  $n$  things in a row and their order is important.

Specific examples. [No response]

Visual images.

*Alan:* Of course, the visual images come from physics, maybe a fluid flowing, things moving in a certain direction, velocity vector, much less a position vector, how to go from here to there, you go in that direction, or acceleration, or force, associated with moving in a direction. Really, two and three dimensional vectors are what physics and beginning calculus talk about and they're quite different from the general idea of just a set of objects. There's at least three kinds of vectors that I know about, because yesterday I nearly got ran down with a truck which said 'vector control' and I said go get those vectors. And it was rodent control, a private rodent control company, and a vector is something that carries germs. This is real, but it's not what you meant. But the first two concepts are as an ordered  $n$ -tuple and as a length and direction. The concepts of mathematics.

Specific images.

*Alan:* Lots of specific theorems, in general with vectors. I'm not sure, I mean here there's a whole microcosm of theorems we use and which ones do we want to pick on. I've been interested in solving equations so I look at theorems that will guarantee you the solution of a problem so one that stands out might be the linear matrix theorem which says that when a matrix is non-singular and when it's defined that you can actually solve for vectors in an equation, in a matrix vector equation. So I mean that's just one. That doesn't stand out much, there are lots of theorems about vectors. I'm not

quite sure if any specific theorems come to mind. Not too many that stand out from the others.

First introduction.

*Alan:* I was introduced to this in physics, as an undergraduate where there they had sort of an aura over a vector. A vector is three dimensional, except when they solve real problems they usually reduce it to two dimensions, because they can do that. It's a three dimensional thing and it's a point and a direction and that's quite a different concept than first comes to mind because there it's an object that you visualize with a direction and a length and to me that's secondary but in physics it's primary.

Changes in understanding.

*Alan:* I think I sort of first was introduced to vectors as lengths and angles, positions, directions, and have gradually gone beyond that, I think, that's debatable, to the idea of an  $n$ -tuple, a finite sequence of objects that are useful for describing some kind of system. Engineers do this, they call them system vectors, that's a statement that describes a satellite, you need to know six things about its position and velocity, including two angles for orientation and so, that's how I started, but I started with a vector as something in three space and I evolved to studying them as a much more general concept.

Milestones.

*Alan:* I don't know. Perhaps changing from physics to math helped. Other than that, the end of my undergraduate college and beginning graduate school I thought more about vectors as a sequence of things rather than direction and magnitude.

## • Vector Space

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* I guess  $R^3$ .

Specific examples.

*Alan:* A specific example of a vector space is  $R^3$ . Specific examples of applications of  $R^3$  I just told you. Since then I've enlarged my concepts of vector fields to include more abstract ideas of sets of functions, higher dimensions, the set of all continuous functions is a vector field, is a vector space. That came later, of course.

Visual images.

*Alan:* I always visualize a set of axes, I don't know why, I shouldn't. I can only go through about three dimensions that way. But a set of objects I visualize as three dimensional things, even when they're functions. I have sort of vague problems with pictures.

Specific images.

*Alan:* The famous one of Banach has to do with maps and transformations between vector spaces, saying that under certain circumstances linear transformations are automatically continuous. As a matrix, linear transformations are automatically continuous. but it doesn't come for free. You see when you get introduced to this through physics, they don't stress specific theorems, or any theorems, physics doesn't work with theorems, they tell you about things and you understand how to operate with math, not how to prove things. You do lots of crazy things in physics like subtract infinity and you get a finite answer, I've never understood how they did that.

First introduction.

*Alan:* In both beginning physics and more advanced physics. A three-dimensional vector space, essentially  $R^3$ , was important for positions and velocities and accelerations in mechanics. Definition of fields, electric type field descriptions would be impossible without vectors or some substitute for vectors.

Changes in understanding.

*Alan:* I think, mainly, in this case, just many, many more examples of vector spaces rather than just  $R^3$ . Of course, there's some theorems that go with that because what's obvious in  $R^3$  isn't so obvious in others.

Milestones.

*Alan:* No, I don't think so. Not many milestones, I'm sorry.

### • Basis

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* Basis is a very fundamental thing. It's very useful in applied math. The idea is, I guess the word that comes to mind is simple. A basis is a bunch of simple objects which by linearly combining them you get more complicated. And so I think of a basis as a sort of building blocks, with which you can build, or approximately build much more complicated things. So, for example, when I teach some of the applied courses on methods for engineers and so forth, I talk a lot about bases. Very practical bases. The polynomial basis for approximating functions. We use sines and cosines to approximate functions. And so I think of those two examples, I guess, of a basis. Or, if you have a basis for fifth degree polynomials, you can approximate a lot of functions that are used in real problems. If there isn't enough accuracy, you have a bigger list, twenty of them instead of five.

Specific examples.

*Alan:* Polynomials and \*\*\*\* polynomials, that's a very specialized one for using to satisfy certain boundary conditions in differential equations.

Visual images.

*Alan:* Sort of the building block idea. Here we have this set that's available on the shelf. I'm thinking, we can take these things down and build. I don't know if that's what you had in mind.

Specific images.

*Alan:* Lots of those. Diffy q's. Theorems that tell you that a certain basis, a countable basis usually, all of the powers of  $t$  are complete in some sense and that says that, in particular applied problems, or theoretical problems, that you can take the function and approximate it to any degree of accuracy, arbitrary degree. And the fact that let's say a continuous function from zero to one, that this set of powers of  $t$  is complete, is important. The completeness theorem, about bases.

First introduction.

*Alan:* I had a senior linear course at Wisconsin and we talked about this idea of a basis as being important, as building blocks.

Changes in understanding.

*Alan:* It's a continuous process. Bases were mentioned in college. I don't think their importance was stressed on that level. And then, as I've worked with it more and more and taught students with it, it has become very important, I feel, and valuable. I don't put that in every course or teach it when I shouldn't, but certainly when

you're doing applied math it's extremely important.

Milestones. [No response]

### • Interpretive Analysis

Alan describes many different visual images of vectors from physics applications, which was where he was first introduced to vectors. This introduction forms the foundation of his conceptual image of vectors, to which he has added further information, particularly of a mathematical nature. His prototypical image for a vector is an ordered  $n$ -tuple, which is more general than the two or three dimensional physical model of something having length and direction. In fact, he describes this as a milestone in his understanding of a vector, when he began to think of a vector as a sequence of objects in order.

His prototypical image of a vector space is  $R^3$ , although he also says that his conception has broadened to include spaces of functions. His visual image is of a set of axes in three dimensions, but again he recognizes the limitations of such an image. He describes a theorem involving a transformation between vector spaces, using language that indicates his ability to conceive of a vector space as an object in this higher level process.

He evokes an analogic model for basis, that of a set of building blocks on a shelf. This provides a clear, concrete, non-mathematical model for understanding a basis as the basic building blocks of a vector space, and thus carries with it many of the abstract features of a basis, while having a simple, sensible metaphorical interpretation. His prototypical image of a basis is of the algebraic symbols used to represent a polynomial basis  $\{1, x, x^2, \dots\}$ , but his visual image remains that of the building blocks on a shelf. Again he describes his understanding as changing gradually rather than it undergoing a radical quantum jump.

• Concepts with prototypical images related to  $R^n$

For each of these five concepts Alan relies heavily on Euclidean space as a source for his prototypical images.

• Metric Space

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* I guess, examples. I'm not sure, I think of just many examples of metric spaces.  $L_p$  spaces, sequences in  $L_p$  spaces are functions and, on a simpler level,  $R^n$ .

Specific examples. [No response]

Visual images.

*Alan:* I think my visual image of a metric space is the blackboard,  $R^2$ , so I measure two objects, visualize the objects being close together and far apart and realizing I can use different kinds of rulers, they don't always have to be uniform rulers. And get different concepts for it. Metrics, even on the set of  $R^2$ , as I told you, I have trouble visualizing beyond  $R^3$  and so I use  $R^2$  and  $R^3$  as guides.

Specific images.

*Alan:* Sure. There's one that comes to mind. Certain compactness theorems come to my mind on metric spaces. What does it mean for a set to be compact? And in a particular metric space. For example, an easy example in  $R^n$ , what is a closed and bounded set? What does it mean when the metric space, which is all the continuous

functions and the metric used is the maximum absolute value of the difference of the two functions, and from there there is a theorem by Ascoli that tells you, test for uniform continuity and so forth. I come from a sort of tradition of applied math, and I like to know if there's a chance, at least, if a subsequence of [a sequence] will converge, and compactness gives me that. So, compactness theorems is one of a bunch of theorems. Obviously, there are many, many answers. But that's one.

First introduction.

*Alan:* I would think probably in the first year of graduate school.

Changes in understanding. [No response]

Milestones.

*Alan:* Milestone was certainly graduate school, and metric space as we worked with abstract spaces in thesis stuff.

### • Infinity

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* Very large.

Specific examples.

*Alan:* We use infinity all the time in elementary courses for delimiting the real numbers and so I suppose quite often you might see it as some null or bad result in some computation, calculating one over zero is infinity, and in some computer languages.



Visual images.

*Alan:* I don't know. Pass on that one.

Specific images.

*Alan:* I think of infinity as the completion or the closure of sets. For instance you have all of the real numbers and at the end you add on the infinity so that you can talk about what happens when things go to the ends. In complex numbers infinity is the sort of circular edge of all the complex numbers and so that's the concept I have of that. I guess that's partly a visual image.

First introduction.

*Alan:* Boy that's long ago. Probably in a high school math course, where the teacher talked about infinity in some sort of a gee whiz sort of way rather than any mathematically profound way.

Changes in understanding.

*Alan:* I think it's changed from the idea of infinity that's just bigger than a lot of things to an idea of a limit process. Ordinary objects like complex numbers and real numbers, that's what's at the edge. That could be made very topologically sound and I've done that. I'm not sure it's helped me a lot. Maybe the original idea was pretty good of infinity as way out there.

Milestones. [No response]

## • Cardinality

Evoked images. (What first comes to mind when you think of con-

cept?)

*Alan:* Positive integers. That's the famous example of a set with countable cardinality. The second example is the real numbers.

Specific examples. [No response]

Visual images.

*Alan:* You think of countable cardinality as sort of discrete, spaces in between, and then the real numbers are just on a continuum and so most of the examples that are important in applied math are subsets of  $R^n$  and they're indiscrete, they're continuous. Beyond that, it helps me to visualize it, limits of cardinality, and subsets of  $R$ , that's much bigger, I mean I can't visualize it, but I know it's bigger.

Specific images.

*Alan:* The first theorem was the real numbers are not countable and interestingly that was something that as a beginning grad student I had no idea how you prove. It was a simple proof, but it certainly wasn't to me then.

First introduction.

*Alan:* Before graduate school, cardinality was a finite arrangement and I didn't call it cardinality then. It was twenty things here, lots of things that I am counting, or there was an infinite number, that kind of cardinality. But then in college I had a real good course in analysis, we talked about cardinality of various kinds. I took a course from Walter Rudin, who was an interesting teacher, by the way. He gave us about mid-term time a handout, take home exam, and there were twenty questions, fifteen were open questions in mathematics that he had been unable to solve!

Changes in understanding. [No response]

Milestones.

*Alan:* I think this was from Rudin's exercises. Of course, in graduate school, a milestone.

### • Transformation

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* I guess in my mind that's associated always with the word 'linear' and I've gone beyond that, I think, but linear transformation is what is stressed in the theoretical course on matrices and linear spaces. I guess a linear, a matrix transformation is what I would first think of.

Specific examples.

*Alan:* I guess a linear transformation would be the obvious one, a matrix transformation. I've worked quite a bit with linear integral transformations. And then, of course, generalized some of that to non-linear transformations and integrals, not just functions, where transformation is just a function and a linear function is one special case. I don't use the word 'transformation' very much except in the situation of linear transformation.

Visual images.

*Alan:* There's sort of a picture of a space here and a space here and a transformation of this point to that point which is in the text book. And that helps some people. I'm not sure it helped me very much but it's an idea of transforming this into that. I can see those circles

on the text book page somewhere, little arrows and so forth. So that's the visual image.

Specific images.

*Alan:* I think one that's true for both the linear integral transformation and the matrix transformation is the fact which says that if a linear transformation is onto it's one to one or if it's one to one it is onto, it tells you that if you have one of those properties then you have both and that's been useful especially with integral operators of transformations.

First introduction.

*Alan:* This is hard for older faculty. Probably it was in college math or somewhere in my four years there. I don't know exactly.

Changes in understanding.

*Alan:* I've peaked out and understood more and more about different kinds of transformations over time and I can't be very specific on that one.

Milestones.

*Alan:* I think I learned quite a bit about this when I wrote a thesis as a PhD student. We talked about transformations quite a bit. The milestone there would be coming to grips with writing a paper having to do with non-linear transformations. My work was to approximate a non-linear problem with a linear transformation and that in turn in this particular problem leads to linear equations. The important thing is you can go to a computer and push a button and do linear equations. Non-linear, you don't push buttons on a computer very well and I was trying to make the connection so you could push a few buttons for non-linear equations, so I was using this approximation of non-linear things with linear transformations.

## • Function

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* The first thing that comes to mind is the sort of cookbook calculus definition of a function as a rule that assigns to some value of the domain some value of the range. That, I know, is how I first came across a function. After that, my development would be lots of examples - sines and cosines and exponentials and polynomials and a bunch of functions. It was probably in graduate school that I came across more of the idea of a function as a subset of the crossproduct of the domain and range space, that kind of idea. With that a much more general idea of a function where the domain is the real line or something more abstract. There's still that old definition with the rule and domain that's still quite valid in the more general situations. With a product space definition, you can take a lot of topological theorems and apply it where that's not so easy to do with a rule.

Specific examples.

*Alan:* I have more examples nowadays than I had then. Back then I talked about transformations between sets of functions. That's important in integral equations which transform functions into functions. So my number of examples was extended greatly by that. Differential equations, you take functions and transform them into functions and so I think it's an advantage. I've gone to where functions are very general, the domain is much more general.

Visual images.

*Alan:* Function is very similar to a transformation, which we've talked about and I have that same visual image. In fact, in my mental dictionary, a function and a transformation are the same thing. My linear functions are linear transformations. I use the word 'function' much more often than 'transformation'.

Specific images. [No response]

First introduction. [No response]

Changes in understanding. [No response]

Milestones.

*Alan:* Maybe going to graduate school changed that some.

### • Interpretive Analysis

Alan describes several different examples of metric spaces, but his prototypical visual image for a metric space is either  $R^2$  (which he describes metaphorically as the blackboard), or  $R^3$ . Alan is clearly aware that the limitation of this model is that it implies a uniform ruler. He describes a milestone in his understanding of metric spaces that occurred during graduate school, particularly while working on his thesis. This seems to have had a significant impact on his understanding.

His prototypical visual image for infinity is the circular edge of the complex numbers. Here he is visualizing the complex numbers as points in the plane with infinity at the edge, in contrast to others who describe the complex numbers as represented by a sphere, and infinity as the north pole of the sphere. His understanding has changed from an *object* (a number, something very large) that is just bigger than lots of other numbers, to a limit *process*, a way of delimiting or completing the real and complex number systems. This seems to be a significant shift in his understanding despite the fact that he does not recognize it as such.

Alan's prototypical image of cardinality is of infinite sets of numbers with different cardinalities, such as the positive integers, and the real numbers. He visualizes

countable cardinality with a discrete model where the numbers have spaces in between, while the real line provides the prototypical visual image of an uncountable set, with the numbers on a continuum, and so, in Alan's words, they are indiscrete. He links cardinality and denseness.

His experience in a graduate analysis class provided an earthquake-like event in his understanding of cardinality. Before then, he understood cardinality as the number of elements in a set, and he describes it as being a finite entity. It wasn't until he took this class that he learned about different types of infinite cardinality.

Alan clearly has the term transformation linked with linear transformation, most likely because his work deals almost exclusively with linear approximations of non-linear transformations. His prototypical image of a transformation is provided by a matrix, although his visual image is one he recalls from textbooks, of two circles and a transformation in between. It is clear that he does not find such an image useful himself, but it does form part of his concept image. He describes transformation as being synonymous with function in his mind. He describes a significant change in his understanding which occurred while writing his PhD thesis, when he had to 'come to grips with' the concept of transformation in order to write it. This seems to represent a significant paradigm shift for Alan, but the nature of the change is not clear.

When talking about function he first evokes the formal concept definition of a function as a rule, but says that later he learned to think of a function as a subset of a cross product, a far more structural way of describing a function. In fact, by describing functions (integral and differential equations) which act on functions he is using language that makes it clear that he has a dual operational structural understanding of a function as both object and process. Although he does not look on this as a significant change, I would suggest that it resulted in a reification of his conception of function.

## • Concepts linked with calculus

Alan describes different parts of his concept images associated with these concepts, one part which he uses for teaching calculus, and another part, with a more sophisticated structural understanding, which he uses in his own work.

### • Convergence

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* The ratio test. And various tests for determining those things. I remember that from doing advanced calculus when I was first in college. We had all sorts of tests - the integral test, the ratio test, the root test and things like that. They were again from an applied point of view.

Specific examples.

*Alan:* I think the examples are of series that don't converge. A lot of people may start out, at the very beginning, they think that any series with terms that go to zero is converging, and it's interesting to have like harmonic series. You can tell very easily by grouping the terms that the sum is larger than anything else, than any number we've mentioned. That's a specific example.

Visual images.

*Alan:* I see the [harmonic] series being longer and longer and that part of it being very consistent at about 7.92, and therefore the rest must be negligible and I see that as a process. That's partially a fraud, because you really don't know how much that tail end is because it's infinite, even though it starts off small and so that's the kind



of visual thing.

Specific images.

*Alan:* There are theorems associated with the tests I just said. The square root test, the comparison theorem, there are many theorems here, integral theorem.

First introduction. [No response]

Changes in understanding.

*Alan:* I think I probably peaked on that one in college and understood it pretty well then but I'm not sure there's any more to understand now. A few things have changed. I've talked about series in a broader setting that goes beyond what I learned in college. A broader setting being series of functions instead of a number, but those same tests work, well, the ratio test doesn't work, but the root test works and the integral test can be of value, maybe it's just frosting.

Milestones.

*Alan:* I don't know, I think that was advanced calculus when we did a whole bunch of that, and it was a joke, and then gradually, a little bit more after that.

## • Continuity

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* The first thing that comes to mind when I think of continuity is

when I took a first year graduate course that talked about I think seventeen definitions of continuity. After that I remember two of those but I don't remember the seventeen.

Specific examples.

*Alan:* Thousands of examples, say, continuous functions from Banach spaces onto Banach spaces. It was important that continuity was there but there were many things beyond that that were more important. So, most of the maps and functions that one talks about are continuous and that's important, but in the background.

Visual images.

*Alan:* I don't have a lot of visual images but, if I have to teach this in a course, I try to give the idea, and I do this with a graph or something, that if you have small changes in  $x$  then there should be a small change in  $y$  value and that can be a beast in a lower division course. It can be made into a picture, kind of a graph and so forth. [One can] associate continuity with the lack of jumps and that's one picture. It isn't the whole thing but there are some places where you can't picture it that way in a more abstract setting.

Specific images. [No response]

First introduction.

*Alan:* I was first introduced to that probably in college math in calculus but in a very loose way and all the things you work with are continuous anyway and they don't worry about it too much. You hear the term all the time but you only have a vague idea of what they're talking about. I was a physics major as an undergraduate so that may influence my conclusion because we got into, as I say, into a first year college course in sort of analysis at the first year graduate level and there we really got into what continuity was. I took a course in point set topology and got into a more abstract definition of continuity.

Changes in understanding.

*Alan:* Back in college it was sort of a vague thing, always in the background and no one stressed it very much, and neither did I. And then when I was a graduate student in mathematics I learned what continuity is and so that's how it changed. Ever since graduate school I don't think it's changed particularly much.

Milestones.

*Alan:* No, I don't think there was anything like that. I don't think there was a particular milestone where my understanding changed. The original idea is small changes in the domain make small changes in the range. How my understanding has changed was the fact that 'small' was more clearly defined in terms of a topology and that's the important part of that. I mean 'small' is vague and you can go to very abstract situations and measure 'small' even in topology and you get a new domain and your definition of 'small' is incorrect and that's how that changed. It seems pretty early. I mean, I haven't changed my concept of continuity in the last twenty years.

### • Derivative

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* Slope. Well, I've been teaching calculus too long. You know, rate of change, slopes. Slope was the first one I said, and maybe rate of change. Then, a third concept might be approximation, linear approximation.

Specific examples.

*Alan:* There are many, many, many examples and you know them as well as I do. But maybe it's interesting if you think about derivatives in

a more advanced setting, for instance if you have a transformation which is an integral, a non-linear transformation, it's a non-linear operator. When you think of the derivative as an operator that is linear and approximates these locally, these functions and that kind of example, I try to go bring back to calculus and talk about \*\*\*\* Well, that's the kind of visual image I have for that concept is that we're approximating non-linear functions and that's what a derivative is.

Visual images. [No response]

Specific images.

*Alan:* Oh, quite a few with derivatives. There are theorems that are used in approximation theory, used in Taylor series. There is a pair of theorems, one's an inverse function theorem where the derivative is essential to understand it. I think one remarkable theorem in analysis is that every differentiable complex function of a complex variable is analytic ... Taylor series ... that converges in some small enough ball, and that always seemed remarkable to me, the fact that if you can differentiate it once you can differentiate it many times. Do you want more of this? There's lots of theorems.

First introduction.

*Alan:* Oh, in a very operational way, in my last year of high school, in a calculus course. Then I took a more advanced calculus course my first year of college and it was still pretty operational, a derivative was when you have  $x^n$  you get  $nx^{n-1}$ . That's a derivative, and it's used in many circumstances. I was a physics major, so we used a lot of ... we had ... I was involved with using the derivatives as descriptions of physical models, partial differential equations and things. The derivative as an operation was essential to theoretical physics.

Changes in understanding.

*Alan:* Well, it started out as an operation you learn how to do in calculus by rote and the ones who got an A in the course knew how to do a lot more of them than the ones who got C's. I'm not sure how important that was. Afterwards the concept has changed into more of an approximation type of idea, where the derivative is approximating, either a linear operator, a linear transformation, the non-linear operation.

Milestones.

*Alan:* No, I think this was a gradual thing as I learned more about it.

### • Integral

Evoked images. (What first comes to mind when you think of concept?)

*Alan:* Integration. There's several things that compete in my mind, I'm trying to make an arbitrary decision which one was first. I think the first thing was summation. And maybe this is because I've always taught it as an extension of summation, of a Riemann sum idea. This is a physics application of it too. Riemann sum is very important for physics because you're approximating things, and so you're summing up the total charge here, the total charge here, the total charge here, and the total charge here, and you get finer and finer. But that's very awkward to do by hand, so you have integration as a way of doing it and that's easier. Integration is an extension of summation. It shares a lot of things with linear transformations. It's the same as summation and it does extend that a little. Other ideas I talk about or would think of is anti-derivative. And that I learned quite a bit later, how important the anti-derivative is. I learned that first operationally. Later, actually to just define the integral I was thinking of an anti-derivative. That's a whole interesting concept which is pursued in a couple of books.

Specific examples.

*Alan:* There are just lots of examples of integration. Maybe ones you can't integrate, how to approximate them. The integral  $e^{x^3}$ , you don't see that in the back of a book or in one of those tables. What to do about those? There are just lots of examples of integration.

Visual images.

*Alan:* I visualize that sort of Riemann sums, area under a curve. Of course you can go way beyond if you can do integrals over Banach spaces, simple idea of summation, little rectangles.

Specific images.

*Alan:* One that comes to mind is the limitation of Riemann's theory of integration and that is any bounded function with not too many discontinuities, maybe zero discontinuities, is integrable according to the definition in the calculus book, math 252. You don't do that in math 252, but that's an interesting theorem and if you have a finite number, a countable number, or even a little bit more, a measure zero number of discontinuities, you can still do that Riemann sum process and that applies to many but not to all. That leads to the idea of 'can you do better with some other theorem?'

First introduction. [No response]

Changes in understanding.

*Alan:* Essentially I've learned quite a bit about different kinds of integrals, beyond the usual one. I'm not sure how that's helped too much, but I've learned that integration is arbitrary and all that sort of thing. I think the understanding of integration as an inverse to differentiation was important, and that came probably later at the end of my undergraduate career.

Milestones.

*Alan:* I don't know, no particular milestones.

### • Interpretive Analysis

Alan's prototypical image of convergence is actually an example of a series that does not converge, namely the harmonic series, which he visualizes as an intuitive, dynamic process. In fact he says in his own words that he sees it as a process.

Alan's visual image of continuity is a graphical diagrammatic one of small changes in the independent variable giving rise to a small change in the dependent variable, (intuitively, lack of jumps) that he uses for teaching purposes only. Although he insists that there were no particular milestones, he describes a rather fundamental change in his understanding that occurred in an introductory topology course, where the intuitive idea of 'small' was made rigorous, providing him with a formal concept definition of continuity, rather than just an intuitive, graphical model.

Alan first evokes prototypical images of slope and rate of change, again arising from his teaching experiences. He goes on to describe his prototypical image for a derivative as a local linear approximation (or linear operator) for non-linear functions (or non-linear operators). Although he claims to have no particular milestones in his understanding it is clear from his discussion that his understanding underwent a fundamental change from a rote, operational (his word) and algorithmic understanding of a derivative, especially in physics, to a much deeper understanding of a derivative as a linear operator, that is, a change from operational to structural.

Alan describes a link between integration and summation, explaining that integration is an extension of the summation idea, and is also an inverse to differentiation. He evokes examples of non-elementary functions which may not be integrated in closed form, again emphasizing the importance of non-examples. His visual image is a diagrammatic one of the area under a curve, approximated by rectangles.

CASE STUDY 9 - BRAD (faculty)

• Partially Ordered Set, Zorn's Lemma, Induction

Brad has these concept images connected with each other and with the Well Ordering Principle, the Hausdorff Maximal Principle, and Transfinite Induction.

• Partially Ordered Set

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* The weird word 'poset' because I remember, when I was working in Germany, going to a lecture with some of my German friends and the guy started off saying 'let  $X$  be a poset,' and Germans all speak very good English and understand mathematical terms very well but none of them knew what a poset was. And I had to explain to them that it was a partially ordered set, a wonderful English invention. I suppose I think of Zorn's lemma to a certain extent. And I think of direct limits. Partially ordered sets are probably the one fundamental topic beyond a set itself.

Specific examples. [No response]

Visual images.

*Brad:* I picture trees for posets.

Specific images. [No response]



First introduction.

*Brad:* I think I heard the word partially ordered set in graduate school. In fact, when I talked about this student on Zorn's lemma, what she wanted to do was the theorem that every partial ordering can be extended to a total ordering. It's not that clear. The area of mathematics that I've done most of my stuff in, real algebraic geometry, is often related to putting some categories in some relation, and ordered algebraic structures are important. I have a feel for partially ordered sets. I also think of soft mathematics. Partially ordered sets are, this is a value judgement, I think of it, a lot of these kinds of very, very general terms are theorems that are so general that they apply to quite an enormous number of things, that they also really don't have a lot of intrinsic content. Zorn's lemma is an example of that. It's a general thing, it's a wonderful tool. And there are a few things like that.

Changes in understanding. [No response]

Milestones. [No response]

### • Zorn's Lemma

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Zorn's lemma. I don't know. Actually I have a whole bunch of weird images. Paul Zorn the person, there used to be a quarterback named Zorn, for the Seattle Seahawks. Well, Zorn's lemma. I think that's a big step in mathematical literature, understanding Zorn's lemma really, it frees you from, first of all you learn induction, because induction is nice, but Zorn's lemma really frees you from induction in a way that, it's extremely important. Zorn's lemma is a fundamental tool. I know that in discussing qualifying exam problems we feel that Zorn's lemma is something that should be tested for sure. It's just one of the basic requirements for understanding algebra but it could be on any exam. It's a funny

thing because Zorn's lemma was one of several different principles that were proposed at the time and were not really understood; the Hausdorff maximum principle, Zorn's lemma, transfinite induction, the well ordering principle. I don't think when these were all proposed it was immediately realized that they were all equivalent, or in what sense they were, or not. Zorn's lemma is just so incredibly, incredibly basic that it's hard to think of your life without it, at least at that level. I loved it when I first learned Zorn's lemma.

Specific examples. [No response]

Visual images. [No response]

Specific images.

*Brad:* The theorem, it's actually given as a theorem that needs the axiom of choice to formulate and that's take a solid ball of radius 1 in  $R^3$ . Then you can partition the ball into a finite number, in fact, I think the finite number is given as four, seven, eleven, some small finite number of disjoint sets and rigidly translate these sets. So you're allowed just to move them and rotate them, so you don't change any metric properties of the sets. And rearrange them and put them back together into two solid balls of radius 1. That's the Banach Tarski paradox. That's another level of sophistication. When I first heard that as an undergraduate, that really seemed like a paradox. It was presented cleverly, the person who presented it was an amateur theatrical player, and I left that lecture thinking this really does call into question the axiom of choice, the circle of ideas, because it's so counter-intuitive. But then later on, when we studied measure theory, and if I said that there's no finitely additive measure that's a translation invariant measure, on the  $\sigma$ -algebra of all subsets of  $R^3$ , suddenly this doesn't sound so weird. Because you experience non-measurable sets in lower dimensions and run into problems of countable additivity and, as you go on, this paradox doesn't seem so paradoxical. It seems more like a stage presentation to me now than a fundamental problem. I still use it and still talk to people about it, and tell people about it, and, depending on their level of sophistication, they have the same reaction, of course at some levels it doesn't mean anything. Most professional mathematicians have encountered this and thought about it, or have

chosen not to, but it's one of those statements that seems weird at a certain time in your life, at a certain level of sophistication, and then later on, says something completely different. So, Zorn's lemma itself is something like that, I suppose. The kind of conclusions that you draw. I use it freely. I use the axiom of choice freely, I don't worry about it at all.

First introduction.

*Brad:* I think I must have learned it in college, certainly. I think it was before my graduate algebra class, certainly either my first or second year in college. It was just neat. We really used it in algebra class. That was a lot of fun. Then in functional analysis. So, I always liked Zorn's lemma, it's a wonderful lemma. Some of the students, last year at one of the summer programs, someone tried to give a talk on the existence of partially ordered sets, and it was very interesting to watch, because she really didn't understand it. And, no comment on her ability whatever, but it was something that was just beyond her experience, not her grasp but her experience. She eventually did master the details of this topic but really didn't get a global picture of what was going on. And so I gave a talk on Zorn's lemma, just for the summer program students, and I mean these are students who, some of them had ostensibly seen it, and I always wondered if at the end of the summer they understood it. It was great to see that, because, here's Zorn's lemma and now here are five applications of it. And they just thought it was neat. I guess it's quite a mechanical tool that, like integration and differentiation and some of these other things that there's a crank you can turn, and it's fun. You know, you put the pieces in place and turn the crank and then you get the result. It all happens and the amazing thing about Zorn's lemma is that not only don't you see what happens but you can't see what happens, there's no way to see what happens. Now, it's independent of the axioms of mathematics.

Changes in understanding. [No response]

Milestones. [No response]

## • Induction

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Right now I'm just thinking of a collection of problems where you would use induction to solve them. We've been playing around with the Tower of Hanoi, for ways to solve it. It's still basically an open question. I think of induction as another version of a lot of other theorems. It's certainly connected to the axiom of choice. I also am somewhat fond of saying that induction is the only non-trivial property of the finite natural numbers. The irrationality of the square root of two. The principle of the least element, which is how I think of induction in a mathematical sense.

Specific examples. [No response]

Visual images.

*Brad:* Visual images... oh, I don't know... I guess some proofs I've seen on blackboards. Maybe... we used to explain it as dominos. A visual proof by induction says that if every domino will hit the next one and if you can knock over the first one you can knock them all over. But that's not really what I see when I think about it. I might explain it that way.

Specific images.

*Brad:* I just mentioned a few. As I said, I find that I use the principle of the least element much more than anything else. I mean, in algebra you prove things about Euclidean domains it's always somehow you look... you don't do it by induction, well it is induction, it's equivalent, you use least elements, well ordering.

First introduction.

*Brad:* I guess I first learned mathematical induction, again, in high school. I don't remember where, I suspect we were taught it when I took analytic geometry. When I was in college and worked as an instructor in the summer program for gifted high school students, that was one of the first things that we did. A lot of them didn't really know it, before then, they did a lot of stuff with induction and number theory, but I find induction is indeed something that's hard. But at this point, induction is one of those funny things that, by the time you get to graduate school, students here, for example, if in a class you say 'prove this by induction' that's it, that's the end of the proof. It's a proof technique that's so ingrained after you do mathematics for a while that you don't really think about it very much. Although in advanced and senior level classes here you still need to worry about it. You don't do much of that in calculus and so forth. It's a basic proof technique and transfinite induction is another basic proof technique. I can't think of too many proofs I've ever done that use transfinite induction.

Changes in understanding.

*Brad:* Well, it's changed. I thought of it as something you apply to a group of problems, to solve problems and not much more. Just a fundamental technique.

Milestones.

*Brad:* No, I can't think of any

### • Interpretive Analysis

Brad's visual image of a partially ordered set is a diagrammatic one of a tree, along with the word 'poset', and a link with Zorn's Lemma. In response to Zorn's Lemma, he evokes a non-mathematical link, and claims he does not have any visual images, because Zorn's lemma is a black box: just apply the lemma, and await the outcome. He links this concept with induction, claiming that since it is more

general, it frees you from ordinary induction, and also with several other equivalent mathematical ideas - the Well Ordering Principle, the Hausdorff Maximal Principle and Transfinite Induction, evoking many areas of application of these fundamental mathematical ideas. He describes his understanding of Zorn's Lemma as having changed, but either he is too far removed from the significant event responsible for this that he does not recollect it, or there have been several stages in his development, and from this perspective it would seem that any change has been gradual. In any case, it is not possible to identify any earthquake-like event that caused a sudden change.

He evokes an analogic model of a row of dominoes for induction, but emphasizes that he himself does not think in those terms, that it is something he would only use if he had to describe induction to someone unfamiliar with the concept. He describes lots of uses of induction in several areas of mathematics and links it with the least element principle, which he uses more often than induction.

### • Metric Space - an interesting visual image

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* I think of a metric space as a metric space, of course. A place where you measure distances. I guess I use metric spaces, I use p-adic measure. So actually, I have a bizarre notion of a metric, bizarre. I remember a great picture that someone had in Math Magazine. Usually, when you define metric space you define metric as something that's symmetric, and he wanted to discuss metrics that weren't necessarily symmetric, so he drew this picture, simply a bicyclist at the bottom of a hill. And I thought that was a great picture. I remember, or almost remember, most of the theorems from topology. I've never been particularly fond of those sorts of ideas. If you ever meet a theorem and you don't like it, you know that you're going to need to use it, that's just the way it goes. That's the way mathematics goes. I guess I think metric spaces

really are the first examples of topological spaces that students encounter. That's probably the first ones that I saw. Except that I took these Moore notes, so that we had to come up with examples of this and that, so that we ended up learning a lot of examples. Our favorite topological space was the natural numbers with the co-finite topology.

Specific examples. [No response]

Visual images.

*Brad:* I suppose I do. When I see a metric space I guess I really do tend to picture  $R^n$  in alot more detail. Metrics come up in other contexts as well.

Specific images.

*Brad:* I guess one of the theorems that I've proven is the non-existence of a metric on certain spaces.

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

Brad's first sentence 'I think of a metric space as a metric space' indicates that this concept has become a compact, unified object with its own meaning, which he understands wholistically. In order to describe aspects of it he has to take a

moment to unpack all the pieces, find all the links. His paradigmatic model for a metric space is  $R^n$ . He recalls a visual image of a non-symmetric metric, a diagram of a bicyclist at the bottom of a hill. This simple image gets to the heart of the non-symmetry, in that the hill looks very different to the cyclist from the top - an easy, effortless run down, than from the bottom - a steep, strenuous climb. The effort required is different, providing the non-symmetry.

### • Group, Ring Field

These abstract algebraic structures are linked. Brad distinguishes two aspects of groups, the algebraic and the geometric.

#### • Group

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Well, I, again, two things, one is just the algebraic concept, finite, abelian, matrices, and the other is geometric. So, on one hand, groups are really nice, fun things to play with. They really are what got me into mathematics in the sense that I was, as I said before, kind of burned out or turned off by my studies in multivariate calculus and then came abstract algebra where we studied groups and just the sheer joy of finding some finite groups. I still like group theory although group theory is very hard and there are lots of parts that I don't know. Group theorists know a lot about groups and Lie groups, and Galois theory, and infinite groups. I think if I speak of groups to more general audience, I would think of them purely as symmetry groups and discuss groups of symmetries of common objects like a book, a square. I'm sometimes asked the question 'what good is group theory?' and one of the answers I give is (something about symmetry groups in the every day world). They come up in this Ramsey theory stuff I'm thinking about and if you analyze game theory, you want to analyze a game like Qubic, the first thing you should notice is that



any corner ball has 3 moves. And then it would appear that there were edge moves and then the middle of a face moves and then there's the cube inside. But it turns out that there's a symmetry of the whole board which distinguishes the inside cube from the outside cube and preserves straight lines. So that, then, in fact, that symmetry will cut down the number of real first moves there are. And it's things like that that really. So I think of groups as a way of, actually, I think of modding out by groups, in other words, groups of symmetries are a way of, if you understand the symmetries of some space, it certainly cuts down on the number of cases that you have to consider.

Specific examples. [No response]

Visual images.

*Brad:* Visual images... I suppose the symmetry example is visual. I don't have many visual images, I wish I did. I'd probably be better at using them.

Specific images. [No response]

First introduction.

*Brad:* Well, again, my first introduction, I don't think I learned group theory very well. There's another thing that comes up with this. My office mate in college, in graduate school, tells a story, it's a great story, and it really had a great effect on me. The story was that he had taken a group theory class in college as an undergrad and was really excited by it. Groups attract a lot of people who eventually become mathematicians, something so clean and so abstract. And a lot of people who eventually do very hard analysis in what I consider mathematics, started out being attracted to group theory. And they had just done this class in group theory and this guy was on the train, he was going home to Long Island, and sitting next to him was one of his class mates and he was really excited and trying to explain to him about all this stuff. And he started talking about groups, and isomorphisms, and after he went through trying to explain them, a person who happened to be seated behind them just tapped him on the shoulder and said 'excuse me, but I overheard your conversation and I think what you mean to say is that an isomorphism is a renaming of the

elements.' And that really had a great effect on him, and that story had a great effect on me too. That group theory is a good place to try to understand that the axiomatic approach is extremely powerful, and what mathematicians work with, but none the less it's important not to lose sight of the simple concepts.

Changes in understanding. [No response]

Milestones. [No response]

## • Ring

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Well, I guess I think of algebra, rings of polynomials. I mean a ring is, again, I could go through and say the same thing that I said about fields for rings. There's a difference, but not a whole lot of difference.

Specific examples. [No response]

Visual images.

*Brad:* I don't have a a real visual image. First of all the word ring gets you this other visual image. I think that in all of these things, you start off studying rings, fields, groups, whatever, and the first impression is that it's just an axiom system. The second impression is there aren't that many examples. And then you really deal with it at a sophisticated level and you suddenly discover how many incredible examples of these things there are. It's just like continuous functions. At first the only ones you can think of are polynomials and maybe transcendental functions, and it's not until you study real analysis that you build these Cantor functions and suddenly you realize that continuity isn't that great a restriction and that there are examples that you never dreamed of. But it is very difficult to build an understanding of what rings are, especially non-commutative rings, for me they're very difficult.

Specific images.

*Brad:* Here's a nice theorem that I think I did in the algebra class, about euclidean rings, and one nice thing is the internal characterization of those rings follows particularly the theory of a group of units. You build up this ring in levels by looking at... so there are some units, and you hope there are finitely many, and now you try to find elements  $a$  such that  $R \text{ mod } aR$  is a complete set of representatives among the units, then you get another finite set like this. This builds up the levels of the ring. If the ring is euclidean then the level of an element actually defines a norm, and that's the minimum Euclidean norm. That was kind of nice. So, for example, for the integers, the units are plus or minus one. There are two of them. So  $Z \text{ mod } n$ , so then you ask when is  $Z \text{ mod } n$ , oh, plus zero, so when does  $Z \text{ mod } n$  have representatives among plus or minus one, zero, that is if  $n$  is plus or minus two or plus or minus three. And so the first set is things having absolute value less than one and the next, absolute value less than three, and next is the absolute value less than seven, so you're constructing log base 2. So, it's kind of neat. There's a lot of nice little things on that level. On a more complicated level, rings get very very complicated. I'm very frightened by ring theory. My knowledge of commutative algebra is probably not what it should be, and there's a lot of interesting stuff in ring theory.

First introduction.

*Brad:* I probably learned the concept in high school.

Changes in understanding. [No response]

Milestones. [No response]

## • Field

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Probably a finite field.

Specific examples.

*Brad:* Sure,  $\mathbb{Z} \text{ mod } 2\mathbb{Z}$  or something, I don't know. Actually, I work with real closed fields most of the time and more with number fields these days or finite fields, so I guess I think of those examples. The examples I think of less are function fields, they are what I probably should think of first and then after that some sort of abstract, more abstract fields. I tend to think of the following things - either finite fields or rationals, real closed fields, archimedean, and maybe I guess I should mention complex numbers, or the reals, and those are basically the fields that I actually think of.

Visual images.

*Brad:* Actually, I think the most visual images I have are diagrams of field extensions. I don't know that I have lots of visual images. I do have a few visual images of finite fields, of curves over finite fields, I actually sat down and looked at them, not learning a whole lot other than verifying theorems that were there, and hoping to maybe see something that I never did. I guess I do have very visual images of real closed fields as well and I sort of see the different Archimedean parts of the field. I could draw a picture of it... so the real line, a real field I can think of as the real line, complex field as the real plane, and then when I think of the non-archimedean fields I tend to think of this sort of fuzzy, fuzzy area which is a whole field and a big gap and then another fuzzy area of some sort and I do sort of view these things that way. I guess those are ordered fields, and even the view of the complex numbers is ordered and it's a very bad line.

Specific images.

*Brad:* One of the specific theorems that comes to mind is that there are infinitely many, in fact more than, uncountable, sub fields of the complex numbers embedded as fields, and only one of them is the real numbers. They all sit in different ways inside the complex plane, that's what comes to mind. I suppose if you have field theory, you think of finite Galois correspondence. Another theorem, the fact that any two finite fields of the same order are isomorphic. Actually, I sometimes picture finite field extensions in terms

of the basis and I think of a lattice in particular. I tend to, in the case of number fields I picture the ring of integers not the field itself and then that's a more reasonable set to picture in some sort of discrete setting.

First introduction.

*Brad:* I think it was eleventh grade, I took an abstract algebra class, and we used Fraleigh.

Changes in understanding.

*Brad:* My understanding is still changing. I still think about questions in field theory and people ask questions and so forth, so it has certainly changed an awful lot. I guess when I first saw it it was just a definition. And now, maybe this is just with hindsight, but I guess the definition of a field is linked closely with some sort of notion of the difference between the real numbers and the rational numbers. I don't know that my understanding of what a field is, in terms of that definition has changed much. My understanding of the properties of fields, of more complicated fields, what they mean and where they arise and so forth has changed drastically.

Milestones.

*Brad:* I suppose the algebra class that I took in college, I just did many many problems and spent many hours sitting around trying to solve these problems and I worked very hard on them, they were very challenging. Four of us spent a lot of time... Teaching abstract algebra, being there, working with all this non-archimedean stuff. And now I'm working on a project with some faculty and we're working with number fields and we're starting to learn a lot different point of view on fields than just computation and that's really when, just the last few years that I was thinking about fields as not just abstract objects, but as objects which can be represented and worked with on a computer, both in terms of the real fields, and in terms of algebraic number theory. Computation, understanding how they work, both in theory and in practice. So my concept has really changed alot actually.

### • Interpretive Analysis

Brad divides his conception of groups into two distinct areas - one algebraic, the other geometric. The algebraic part includes prototypical examples of groups, such as finite, abelian, matrices. The geometric part includes symmetry groups, rotations, reflections and translations. His visual image is a diagrammatic one of symmetries of common objects such as a book or a square. Groups are a tool he uses in algebra to simplify problems by exploring their symmetry. This is also the model he would use to describe the concept of a group to a novice.

Brad's prototypical example of a ring is a ring of polynomials. He links this concept very closely with that of a field, saying that what is true of a field may almost always be said of a ring. He doesn't have a mathematical visual image of a ring, preferring to think of a ring as an axiom system, with a concept image built up from a rich set of examples.

For the concept of field, Brad first evokes a prototypical example of a finite field,  $Z/2Z$ . He describes various visual images - field extensions, and lattices over a base field, curves over finite fields, and a fuzzy image associated with a non-archimedean field. He prefers to evoke the more discrete ring of integers associated with a given field, rather than the field itself.

A significant event occurred for Brad when he realized that, although it is important to understand the axiomatic approach to abstract algebra, it is equally important not to lose sight of the intuitive and simple underlying concepts. In other words, while formal concept definitions are essential ingredients in one's understanding, they are not sufficient without a less formal concept image, rich with intuitive examples, models and analogies with everyday experiences. For example, the formal concept definition of isomorphism, as a one-to-one, onto, operation preserving map between two groups is all well and good, but it is only really *understood* if it is recognized as a formalization of the simple idea of 'a renaming or relabelling' of the elements.

He describes his understanding of fields as having changed significantly several times as the result of several different events. His graduate algebra class provided the first significant change, teaching algebra provided the second, and looking at non-archimedean fields provided the third. His understanding changed from just being aware of the formal concept definition to actually understanding the properties, and knowing many more examples of fields. In his words, what has occurred is a deepening of his understanding of what fields mean and where they arise and how they work in theory and in practice.

### • Infinity, Limit, Continuity, Convergence

Brad draws a distinction between the images he evokes for teaching these concepts, and those that he himself uses to visualize them.

#### • Infinity

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Boy that's a tough one. I think of infinity in many different ways. It's hard to say. I guess the first thing I think of, actually I changed my mind when you asked the question, from some kind of the picture of the infinity symbol, and then I thought of this tape, this videotape that showed yesterday called 'Not Knot,' with a description of moving links, like infinity. So that's maybe it, I mean infinity's just another knot.

Specific examples.

*Brad:* I mean I really do have a geometric picture of it. There are two kinds of infinity, there's a mathematical cardinality kind of infinity, like the natural numbers set, and real numbers are larger and I know exactly when I studied that, because that was my senior year in high school. I took a course on logic and set theory and started all that kind of thing. Then there's also the sort of calculus infinity, which I don't think I understood. I think I understood the cardinality infinity before I understood the calculus infinity which is somehow a notion of something getting arbitrarily large or the real line going out to infinity or the plane stretching out to infinity, the boundary of some unbounded infinite set without a boundary. I guess the geometric infinity is again, that same sort of picture of something that's large, something with volume that goes on forever. When I try to explain infinity or use the concept of infinity with high school students, for example, or elementary school teachers, I have told them 'think big', I didn't find it something they had difficulty grasping. Everybody seemed to understand what infinity meant. I'd draw a plane on the blackboard and say 'out at infinity', and everybody copied, they don't necessarily understand what the concept is. And another example of infinity is for example take the plane and put a point at infinity and you get a sphere, and then in that sense you see infinity. I guess those are some visual images. I mean I'm not seeing visual images, but it's, I don't know, it's hard to describe visually what goes on in my head. I see a point at infinity, and I see very different types of infinity. A plane and curves going out to infinity in various different ways. One of the things that really helps to visualize infinity, if you want to work with it, is to visualize it as zero and invert things, and look at what happens when things come to the origin or to a point that's not there and then just think of that point as infinity. That's a really easy way to visualize what's happening. A more mundane picture is sort of railroad tracks going to infinity. I don't know what I picture it as, because I mean, do I picture the number 17? I mean it's just a concept that I live with.

Visual images. [No response]

Specific images. [No response]

First introduction.



*Brad:* I think I was first introduced to infinity when I was in grade school. I remember my parents talking to me about it. My Dad was a really active amateur astronomer, I remember somehow discovering cosmology. I'm trying to think when I first really thought about infinity. I think it was just as simple as something that got bigger. I can't remember when I really started dealing with infinity. I can't remember not knowing about it, let's put it that way.

Changes in understanding.

*Brad:* Certainly my understanding of the concept has changed and evolved over time. It's a lot different now than it was. I guess as a child I think that infinity was just this bizarre word that meant 'bigger than'. Now it means a lot of different things. So as I said, there's a difference.

Milestones.

*Brad:* I think the milestone in my understanding of these algebraic definitions was taking logic and set theory. Really understanding cardinality in terms of one to one maps. That made a lot of difference, and as I said I understood that a lot better than the limit as  $x$  goes to zero or as  $x$  goes to infinity. I think that the time that I really started to understand these other concepts of infinity was in college. I think with teaching calculus and then of course after that my notions have become more sophisticated, but they ought to.

## • Limit

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* When I think of a limit, I guess I think of what happens eventually. A limit describes what happens when something gets large or asymptotically big. And that could be infinitesimal as well, but at the moment, the direction I'm thinking is the large scale so the

limit is describing what happens when you ignore all the minor fluctuations that are going on at any finite time. I studied some phenomena and I'm looking at this phenomenon from an unreal, external point of view. Let me give you an example of that. Because I don't think of, of course at times I think of it as 'let epsilon go to zero and see what happens there', but really, in most of my thinking, my mathematical thinking I think a different way.

Specific examples.

*Brad:* An example that came up last year is that of an unbiased, random law where you flip a coin and move right if it's heads let's say and left if it's tails. This is a really interesting example in that, if you analyze this random walk there are two things that you find. One of them is that you'll cross, with probability one, almost surely, you'll cross the origin infinitely many times. So that if you let this thing go on forever, you'll cross, you'll be positive half the time, so to speak, and negative half the time. On the other hand, if you take any interval about the origin, then the proportion of time you spend in that interval goes to zero. So in other words as you flip this coin, most of the time, it's very rare, if you're winning a dollar and losing a dollar, it's very rare that you have, that your winnings or losings are within \$10.00. Most of the time, for any, if you give me any finite amount of money, like a million dollars, then the proportion of time that you're plus or minus any number less than a million is zero, it goes to zero. So what this does, already, we're seeing some sort of concept of a limit, but the limit of this picture is looking at the whole line instead of looking at the line and thinking of it as an interval with two end points.

Visual images.

*Brad:* What this picture is is that if you think of this entire process as having occurred, and you open your eyes and just start looking at some random point in time then with probability  $1/2$  this randomly walking person will be either at plus 1 or at minus 1 and you'll never see this person ever cross, ever in the middle of the interval, but you'll always see him or her at plus 1 or minus 1 and yet you know that they must cross the origin pretty often. This is kind of, I like to think of this, and it's probably not right, but it at least gives

me a way that maybe makes sense to me, I assume when people say things like 'electrons are in shells and they're either always, they're in this shell or this shell and there's nothing in between.' Well, here's an example of something where, this is some sort of stochastic process. So that's sort of what I think of as a limit as looking at something from that perspective. In other words, you're unreal outside the perspective, and the actual process, you can't see this because you're flipping the coin, you're living in the coin world and you can only see finitely many times and for you you're taking steps, but for this outside observer the picture looks completely different and that's sort of what I view as a limit. That's not what I would tell a calculus student, but that's what a limit is. So when you take the limit as  $x$  goes to zero of  $\sin x$  over  $x$  and ask what's going on here you're standing back and seeing what happens if you could see what happens at an infinite, or arbitrarily small zero time, and that's what I, that's how I think of limits. I have some visual concepts of calculus pictures, keep getting smaller, functions getting smaller, somehow things get small, or things go out to infinity, picturing what happens if you, like at this point on a sphere, and another picture of a limit as going out along the lines.

Specific images.

*Brad:* I suppose that limits are linked with continuity.

First introduction. [No response]

Changes in understanding.

*Brad:* My understanding has changed considerably. I'm not sure I ever learned the concept of a limit very well, other than with epsilons and deltas, which was a fun definition to work with. I think it was much later that I started to realize that it wasn't an intuitive definition, sometimes. I can understand things intuitively and one of the things I try to do if possible when I see sequences and series and beyond, is to get this, to develop a feel for the rates of growths of things. I find that students, even very good students, many of them will not have the same sort of feel for which of these functions grows faster or which of these series converge, that a working mathematician

has. Almost all working mathematicians will know whether the summation of  $1/n \log n$ , for example, converges or diverges or these sorts of things or can figure them out.

Milestones.

*Brad:* I mean just doing mathematics at various points, category theory makes changes. Recently, just the whole business of dynamical systems has come up. The limit is another basic concept, mathematicians need to be able to work with things from an outside perspective and see what happens in the long run. Computing, getting into analysis of algorithms, looking at the asymptotic speed of algorithms, big(O), little (O), different complexities of algorithms. If you ask me these questions five years from now I suspect that I'd still be talking about limits but different types of limits. Or if not then, in between.

### • Continuity

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* I guess years of teaching calculus. That's probably what comes to mind. Continuity is a great concept.

Specific examples.

*Brad:* A line being drawn without lifting the pencil, but more generally, functions on some sort of topological space. There are some other concepts I've played around with, the concept of continuity in a much more abstract, non-Archimedean setting, where continuity is a little bit different from algebraic notions of continuity which may make more sense more generally than in some more restricted settings.

Visual images.

*Brad:* I think of them in terms of functions having no holes, because there are certain contexts in which there are holes like that, like rational functions. I try not to envision this concept like this, it's much more convenient to look at more abstract definitions, and they tend to, the visual images sometimes confuse you. I'm thinking about when I deal with calculus and analysis, then I do have some fairly standard visual images that I use to teach students, the process of converging, to plugging in values in a calculator and making a small mistake leads to a large error. The results shouldn't be off by a lot.

Specific images.

*Brad:* Well if you ask me about theorems I'll tell you it's continuous and basic theorems in math, distance function, property of functions, those are the sort of things that come to mind. I don't really think of much in terms of theorems.

First introduction.

*Brad:* I think I was first introduced to that in calculus, so that was in high school. I actually liked the epsilon delta business. The notion of continuity makes certain sense.

Changes in understanding.

*Brad:* Continuity is certainly one of the fundamental concepts in mathematics, so you can't work in mathematics for long without thinking about it at all. I do remember some very specific things, in particular, going from this epsilon delta business and drawing a line without lifting the pencil off the paper, like in grade school, and then in calculus learning epsilon delta, to college and struggling with abstract topology because there were a group of us that learned topology from Moore notes, and all we had were notes that said definition, definition, definition, and then a bunch of questions that we had to answer. So it was a struggle to think of all these exam-

ples and counter-examples. Then of course, after a while that stuff was ingrained, very easy and it was real analysis then where things got, I learned about things like continuous everywhere nowhere differentiable functions, sets of measure zero, almost immeasurable functions. There are rules of thumb that help you deal with continuous functions in the real domain and I think in recent years I've thought much more in terms of continuity in a number theoretic way. 44So, I mean again, continuity is something that's just an ever pervasive concept. An example of that sort of thing is, is there are fixed point theorems which say that a continuous map from a compact set to a compact set has a fixed point. Take something like that and divorce it of its original geometric content and transplant it to a different setting, and you get some result that's very strange, so that it's changed an awful lot.

Milestones.

*Brad:* I pointed to one particular milestone. The other milestone is somewhat embarrassing. There was a qualifying exam that had a very easy question on it that had to do with continuity and compactness and stuff like that and I remember not getting this question! I'm not going to give the details but that was another milestone at least in the real analysis part. I don't know when I really developed that fully.

## • Convergence

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* It's a sort of a limit, I suppose. I think of tests for convergence, I think, if a sequence converges. I prefer to deal with positive sequences and absolute convergence. I deal with conditional convergence a whole lot. It makes me think of very interesting problems on the boundary. I guess that was a theorem that I really loved because you have a series and it has this radius of convergence. First of all it *has* a radius of convergence, secondly that the radius of convergence goes up to the first pole, and thirdly that you don't know

what happens on the boundary. That was a real nice theorem, I must have learned that in college. I probably saw it in my first real analysis class. I've taught sequences and series here sometimes and I never was very satisfied with the results, and no one was really excited about it. One of the nice things about infinite series and convergence is that there's an intuitive notion which is this notion of a limit but in fact it's a little bit removed from what happens because, for example, the harmonic series diverges but it doesn't diverge on a calculator, and the divergence of a harmonic series is of fundamental and vital importance to many parts of mathematics, and physics probably, in structural engineering, but it's not something that diverges on a calculator. I deal more with divergent sequences, and they're very sophisticated notions.

Specific examples. [No response]

Visual images.

*Brad:* I suppose I can call up some visual images, but I don't get, the visual images are at least kind of vague. I didn't really deal with sequences and series very much. As I said I didn't take a whole year of calculus, and in advanced calculus did some of the same stuff. But already it was at a higher level. So I really skipped most of calculus and most of real analysis, and when I skipped that and went to grad school, felt very bad for doing that when I did it, so I didn't learn a lot of this stuff when I should have. But I really do like it now. I marvel at it. I get the right answers and I just marvel. So I think of sort of Putnam type and recreation type math. qfour  
[No response]

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

Brad's visual images of infinity are of the algebraic symbol  $\infty$ , and of diagrammatic images of a knot, and a metaphorical image of railroad tracks. At the same time, Brad indicates an unwillingness to visualize infinity, claiming that it is just a concept that he lives with, rather than something he visualizes. He describes two distinct prototypical examples of infinity, one associated with mathematical cardinality, and the other with calculus. The mathematical type is modeled by the cardinality of the natural numbers and the reals, in other words the cardinality of countable and uncountable sets. In contrast, the calculus type of infinity is represented by something getting arbitrarily large, such as the real line, or the plane, or curves going to infinity. It is also represented by the prototypical model of the plane as a sphere, with infinity at the north pole. Despite his apparent reluctance to visualize, he suggests that one way to visualize infinity is to invert (replace  $x$  with  $\frac{1}{x}$ ), and thus if  $x$  previously approached infinity, then it now approaches zero. This makes it possible to investigate behavior at the origin, rather than at infinity.

He describes a significant event in his understanding when infinity changed from being a word that meant 'bigger than', to a much broader understanding which included many examples. He indicates that this came about initially while teaching calculus. As he discusses this concept, he uses links between infinity and cardinality and limits.

Brad evokes the usual calculus pictures associated with limit which he uses when teaching, but which he does not use himself when thinking about limits. He describes prototypical examples illustrating the intuitive idea of limits, such as a random walk, or a coin toss, which at any finite stage appears different from the limiting result. He draws the analogy between these models and the model of electrons in shells. He links limits with continuity via the formal  $\epsilon - \delta$  calculus definition of a continuity. His understanding changed from this formal definition to a more intuitive idea linked to many different areas of mathematics.



Brad's prototypical example for convergence is the conditional convergence of a series, and its radius of convergence. In describing convergence of the harmonic series, he's saying that intuition is suspect, so that you have to know when to go with it and when not. In other words, it is a sophisticated notion. You must *understand* the formal definition first, rather than having just a rote memorization of it, before you can rely on an intuitive understanding. He links this concept with the concept of a limit. He says that he *can* evoke visual images for this concept but is again reluctant to do so. The images he has are associated with his teaching of the concept. He prefers to evoke the feeling of fun and the link with recreational mathematics that he feels associated with convergence.

Brad first evokes a diagrammatic visual image of drawing a continuous function without lifting the pencil from the paper, a function with no holes in it, a standard image used in teaching. He also evokes a more general example of functions on a topological space rather than the visual teaching models, claiming it is better for him to consider abstract definitions, unless teaching. He describes his understanding as having changed from the teaching model, to an understanding of the  $\epsilon - \delta$  definition of continuity, and then to a more general understanding of continuity through topology via the Moore method. His understanding also changed as a result of transferring analysis theorems involving continuity to other settings such as number theory and studying the results.

### • Set, Cardinality

Any discussion of set necessarily includes mention of cardinality, the interesting sets are infinite.

### • Set

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* I think I picture a finite set first. The next thing that comes to mind, I mean it's just a fundamental concept. It's a word I use all the time. In fact, it's the word with *the* most definitions in Webster's dictionary, mathematical definitions, it has an enormous number. It's really a not defined concept, and when I think of talking to other people at all different levels, mathematicians at all levels from elementary school on up use the word 'set' and don't worry about what it is. It seems to be a convenient term for grouping things together. It's hard to imagine a mathematical discussion where the word 'set' doesn't get used in a variety of different ways. 'A set of this, and a set of that', so it's kind of a way of meaning a collection of objects with common properties. Then there's set theory itself which is much more complicated.

Specific examples.

*Brad:* Specific examples that come to mind of sets. Now I think when I picture sets I picture diagrams of dots or sets of vertices of a graph. It's just such a basic concept, it's so meaningful.

Visual images.

*Brad:* I guess I could picture a finite set as sort of dots with circles around them, pictures of that sort. I don't know that I really use them. I usually visualize what's in the set, not the set itself. The set of real numbers, I don't think of them with curly brackets around the real numbers. I think of the real numbers that I want to focus on.

Specific images.

*Brad:* Theorems, well, yeah, there are lots of biggies. Theorems I think if you ask about that, let's see, some of the theorems in logic, theorems about ordinal numbers and so forth. I think of recursive function theory to a certain extent. Right now we've been fooling around

with Ramsey theory, so there's a use of lots of little theorems. A nice definition of that, given any set of three ordinary people, two of them with the same name determine what relation they are. So we can, for example, one of the open questions that we've been looking at is this if you color the plane with four colors, arbitrary coloring with four colors, are there two points, one unit apart of the same color? So there are these kinds of compactness theorems I suppose that I think of when I think of sets. So I guess the fact that the natural numbers and the reals are not the same cardinality. That was perhaps a turning point in a way. Probably learned that too early and accepted it right away without really playing with it, I mean I accepted it as just something that we're told. What else was there? The direct limit of a sequence of finite sets is a finite set that's nonempty. That theorem is very important. It's also a theorem you can state using trees. If you have a tree which is finite at every node, such that the tree itself is infinite then ...

First introduction.

*Brad:* I was first really introduced to the concept of a set before I can remember... I can't remember not knowing what a set was. So I was nine years old or something. I think it, I don't know when children are told the word set. I had a little bit of the new math so I think they might have told me then, but we were told about sets and intersections and unions and so forth. When I first was introduced to the formal concept was when I took logic and I started to think a little bit about understanding what a set was, and then more recently I've been fooling around with ordinals and that's a bit more complicated.

Changes in understanding.

*Brad:* If you ask me this question about any concept in mathematics it's changed, it's complicated, is your understanding different now than it once was, so I can't imagine that my understanding would be the same now as it once was. So I can't imagine that my understanding would be the same now as it was five years ago. I can't imagine that it will be the same in five years. In anything that, any basic concept at all. That's an observation. How has it changed? I suppose from a naive point of view of a collection of objects to going, passing,

through a stage where I was very concerned with the precise notion of what was a set and what wasn't a set, back to a probably more naive notion which I use on a daily basis, but being aware of some of the problems that arise. The other thing that really made a big difference was that I at one point was, I still am, very interested in logic. And I liked model theory alot, and I liked to work with it. In graduate school I had a wonderful teacher to work with. The third quarter of the sequence was set theory. The person who taught set theory was somebody I do appreciate enormously, he is an excellent philosopher and a very fine individual, but was also an extremely non-dynamic speaker and he just put me to sleep and I just dropped out of there. So I never really learned set theory in graduate school and it wasn't until later that I began learning a little bit. I don't know if that's a good milestone.

Milestones. [No response]

## • Cardinality

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* We sort of talked a little bit about cardinality. I mean cardinality is just the number of things in a set and/or another notion of cardinality is simply you dare take the collection of all sets, and call two of them equivalent if there's a one-to-one correspondence between them. I mean I guess I talked a little bit about diagonalizing, the reals and rational numbers and cardinality and so forth. Those arguments certainly were one of the things that turned me on to logic. I remember in graduate school one of the professors said that somebody came to his office and asked questions about cardinality, but couldn't accept the conclusion. Really, literally, could not accept completeness and thought obviously there's something wrong here but it doesn't prove anything. All it proves is that that number wasn't on your list in the first place so you didn't make the list right. Now, I mean, if you think about that, you might think that that's stupid, but I don't think it is at all. I think it's a very valid argument.

Specific examples.

*Brad:* In more recent times, I have constructed real numbers and things like that where, again, from the external mathematician's point of view, there are only countably many computable real numbers, let's say numbers which can be specified via a computer program in a finite time interval. And yet, they are uncountable. They're intrinsically uncountable, in the sense that there's no computer program that can list them. So that these notions of cardinality become very sophisticated and very interesting to deal with even in a world where, from an external point of view, everything is countable, everything is an interval. You can still, within that, see it in terms of cardinality and that's very fascinating. The other thing that I think of with cardinality is good notation in mathematics. The set of functions from the set  $X$  to the set  $Y$  and you call that  $Y^X$ , that sort of thing is really good notation to introduce at an early stage because it's right. I mean, it's not just a matter of taste, I don't think. I think it's a matter of this is the notation that generalizes exponentiation and that really is a more abstract and fundamental concept of what we mean by exponentiation. So, you say 2 to the  $n$ , I mean that's the number of subsets of a set with  $n$  elements, in other words the number of functions from  $n$  to the set  $\{0, 1\}$ , and it's all there lurking in the background and there's no need to go in and say this is why we do this in a very detailed fashion but I think it's a good idea to use notation to try to do things. Cardinality is accepted in places which at least don't conflict later on with other concepts.

Visual images. [No response]

Specific images. [No response]

First introduction. [No response]

Changes in understanding. [No response]

Milestones. [No response]

### • Interpretive Analysis

Brad interprets the word set to mean a term for grouping things, a generic container word, used intuitively. He first evokes the prototypical example of a finite set, and also evokes visual images of a diagrammatic type, a circle with dots inside, or a set of vertices of a graph. However, if he has to visualize a set such as a set of real numbers he would normally visualize the elements inside the set, and not the set itself.

He describes his understanding as having changed from a naive understanding of a set as just a collection of objects, to being concerned about a precise and formal definition of a set, back to a naive understanding, while at the same time being aware of some of the problems with defining a set rigorously. I would argue that this, instead of being a cyclic event where Brad finds himself back to his original understanding, is an example of a learning spiral. He has come back to the original intuitive understanding of a set and yet at the same time his understanding is on a different level because he has had time to consider the nuances and subtleties involved in giving a formal definition. This adds a new dimension to his original understanding.

Brad links this concept with computability issues that he has been investigating lately. He also evokes the notation  $Y^X$  for the set of functions from the set  $X$  to the set  $Y$ , explaining that this is a welcome generalization of exponential notation. He does not have any visual images of cardinality. He links this concept with logic, and computability.

### • Transformation, Function

These two concepts are more or less equivalent in Brad's mind, although the images Brad evokes are different. He links a function with its graph, while a trans-

formation is not linked with a graphical image.

### • Transformation

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Probably these days I think of a linear transformation. A map between spaces. I guess I think of it in a mathematical sense, I don't think of werewolves, like people into animals. I guess the word transformation is one of those sophisticated words that mathematicians throw around and isn't a particularly well defined word very often. It means just a map from one thing to another that preserves some property that you're interested in.

Specific examples.

*Brad:* Specific examples are linear transformations, affine transformations, I guess there are a lot of other things I might think of as transformations, but I wouldn't call them transformations. Personally, I don't use the word transformation.

Visual images.

*Brad:* Do I have any visual images? Yeah, I have visual images. Somehow, a space being transformed into another, maybe even topologically, doughnuts into coffee cups. But more from linear algebra, so I have pictures from linear algebra, when you say linear transformation I don't see matrices.

Specific images.

*Brad:* I guess my favorite theorem is the principal axis theorem. Rotat-

ing a sphere about two different axes. There are a lot of classical theorems that are useful in the sense of the composition of transformations. Oh I forgot Moebius transformations. The word transformation is such a bizarre word. As I said before, it's a technical word, I think that certainly I've heard the word transformation in school somewhere, but I don't know when I first heard it in the mathematical sense.

First introduction.

*Brad:* I guess the first time that I heard the word transformation in a mathematical sense was probably linear algebra, I never really had a formal course in linear algebra.

Changes in understanding. [No response]

Milestones. [No response]

## • Function

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* I think of a mapping. I mean, to me a function is, a function, map, whatever, is the same sort of thing. I might even just think of a morphism. So a function is just a rule that assigns elements of one set to element of some other set. Certainly in teaching I find that most students don't know what a function is. But at least they've worked with them. They really have no idea. I mean, it's one of those simple concepts that is so simple it takes a long time at the beginning to work with. I find that even at the advanced level people are a little bit confused about functions. If you ask them to compose two functions they're never quite sure how to do it. And the reason, and that's really a pretty serious indictment of the way we teach calculus and so forth, that I guess I feel that students come through thinking of functions as a name, they think of the name



of the function -  $\sin x$  - and manipulate it by differentiating or integrating or adding or subtracting and they never think of the  $\sin$  function, they think of  $\sin x$ . And as you go more and more towards technology-based things, I don't think that's going to change much, in fact we may go in the wrong direction because that's what you type into the calculator is  $\sin$  and I am very concerned that they don't understand the basic thing, that a function is a rule and what it means and that the function itself is an object.

Specific examples.

*Brad:* In some sense, everything in mathematics is a set and everything is a function. A function is just another set. A colleague tells me this story, and I might get some of the details wrong but the basic idea is that he went to high school with a friend and they went to different colleges. And very soon after they started taking the math sequence at these two colleges they got back together and one asked 'how do they define function over there?' The guy said 'well, you know, it's a set of ordered pairs'. 'Oh, well that's how we define it over here too.' It's kind of hard to think of examples of functions, lots of things are functions of other things. In the real world they come up all the time. How well you perform in a race is a function of training and so forth and so on.

Visual images.

*Brad:* I have an image for all of the stuff in calculus, all of us teach a lot of calculus during the course of our lives. I think I taught more calculus in graduate school than here but I still have these pictures of drawing graphs, graphs of functions. And in fact replacing a function by its graph is not a bad idea in general. I often try to think of a function as a graph. In algebraic geometry in some sense you have to think of functions as graphs. One of the things that really changed things since I've gone on is I've learned to associate a function with its graph. We do that in calculus, but we don't really associate the two as the same object. The other kinds of things that happen with functions is when you start realizing that functions with different co-domains, with different ranges, there's a difference between the range and the co-domain, and all these sort of technical little things that make a difference in terms of the way

a function is defined and what its range is and the function as a set of ordered pairs. I think often of associations instead of functions. I mean you associate to, to a manifold you associate its homology and I guess those are functions but I don't think of them as functions, the same way that I would think of  $y = f(x)$ . So there's something that's changed a little bit.

Specific images. [No response]

First introduction.

*Brad:* I suppose sometime in elementary school. I don't, again, that's something that I remember thinking of when I was in elementary school and I don't remember not knowing what a function was.

Changes in understanding.

*Brad:* I'm sure that at one time I had a very naive point of view of a function, just in terms of however I first learned it. I don't really know how I first learned it. In calculus, I don't know when I really started looking at a function as a rule. It probably wasn't until pretty late, judging from what I see other people do. I don't suppose I was much different. I must have had some kind of vague confusion of functions and it changed at some point, and I'm not quite sure when that changed. Certainly by the time I got out of high school I knew what a function was. But then, I don't know, I really don't know when I started thinking of a function as a rule. It's probably still evolving. Category theory had a lot to do with that. It's the same, you replace functions with morphisms and suddenly functions become special cases of a more abstract object which is in some sense easier to manipulate or at least to manipulate more formally, by more formal rules as opposed to trying to derive some geometric content.

Milestones. [No response]

### • Interpretive Analysis

Brad's prototypical model for a transformation is a linear or affine transformation. He describes a visual image of transformation as something that turns doughnuts into coffee cups, or more generally an image of one space being transformed into another. He explicitly states that he does *not* visualize matrices, although he links transformations with linear algebra. In fact, he does not use the word transformation much at all, suspecting that it is not particularly well defined in mathematics. He would prefer to talk about function.

Brad evokes images of functions that he has associated with this concept from teaching calculus, particularly graphs of functions. He links the concept of function with that of a mapping, a morphism, and a rule, and links it visually with the graph of a function, in fact his understanding changed when he began to think of a function and its graph as one and the same object. When he expresses his concern that students 'do not understand the basic thing, that a function is a rule, and what it means, and that the function itself is an object' he is in actuality observing that reification is very difficult for students.

He describes a significant change in his understanding as it progressed from a naive understanding of a function (he does not give details) to an understanding of a function as a rule, and also when he learned to formally manipulate functions, rather than relying on the geometric (probably graphical) content of his understanding. It seems that this last change occurred when he was able to work with functions, without some visual, geometric aid, requiring a structural understanding of function as an abstract, manipulable object rather than a process.

### • **Vector, Vector Space, Basis - linear algebra concepts**

These concepts share a common link with linear algebra, a basis providing the

means for representing vectors, which are points in a vector space.

### • Vector

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* I suppose I think of something accelerating. If a student were to ask, I would give a standard definition of an object with magnitude and direction. I think of a vector space, a vector is a point in a vector space. What's a vector space, well a vector space is a set of axioms.

Specific examples.

*Brad:* The specific examples that come to mind are points in  $R^n$ , points in  $K^n$  where  $K$  is a field. Functions can be vectors, a function is a vector. Right now we're doing linear algebra on a very large scale dealing with factoring numbers so I think of exponent vectors of products of primes.

Visual images.

*Brad:* To draw a vector on my Macintosh I have to draw a little arrow. That's a visual image. I visualize often in 2-space, even though the vector might be infinite.

Specific images.

*Brad:* It depends on what moment you catch me. If I'm thinking about a problem in linear algebra or in quadratic forms or whatever, I might be thinking of a completely different type of theorem. I can think of one that, well simply the existence of a basis and that all

bases have the same cardinality. That's a very general theorem.

**First introduction.**

*Brad:* I had a teacher in high school, a guy who looked like, as kids we would play old maid and on the old maid set of cards was Dapper Dan with a handle bar mustache, this guy looked kind of like Dapper Dan and he wanted to teach us all linear algebra. I don't know why. It was just something that wasn't done but he always used to come around and talk to us and say 'you've got to learn vector space'. By the time I got to graduate school I really didn't know much about them. Linear algebra was something that I just didn't like, didn't appreciate at all. The first time I ever really dealt with linear algebra in a significant way was in Lie groups and then I started to appreciate it much later and now I love linear algebra. It's great and I see it all over the place and get frustrated with undergraduates who can't understand it, it's such an enormously rich field and for that reason I think of a vector in a mathematical way, as a point in a vector space as opposed to the way that I first learned it which was as magnitude and direction in physics, the velocity vector and that sort of thing.

Changes in understanding.

*Brad:* My understanding, as I said, I have indicated before, it's completely different now than it was before. The one I first used was just some sort of object in a mathematical formula, and now vectors rule my world, they're all over the place.

Milestones. [No response]

## • Basis

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Well, I suppose I think of a vector space basis. A vector space basis is simply a way of seeing vectors. You just really define what a vector is in terms of the basis.

Specific examples.

*Brad:* Bases for finite dimensional vector spaces are sometimes interesting. An interesting example arises in determining the characteristic polynomial for a group. The basis is the geometric series which is a very useful basis which allows you to get nice closed forms for calculations. Other interesting ones are, in number theory, basis for rings of integers. I was just reviewing something in the last few days about computing when a ring of polynomials, counting the number of positive roots, or the number of roots of one polynomial and using them for another polynomial. There you can trace it in different quadratic forms, the polynomial of interest, and in order to see what's the right number, you compute the basis. I don't think too much of infinite dimensional bases, although certainly one of my favorite theorems is that every vector space has a basis. One of my favorite applications these days is, pedagogically, a fairly easy construction of a non-measurable set, which simply goes that if you take a basis for the reals and for the rationals and make sure that one of the basis elements is one so that the projection of the rational number is itself onto this basis vector, then you just break up the real numbers into the sets which are, which have different projections onto these coordinates, so there are countably many possibilities, and these are translation invariant, because each one is just gotten by adding one rational number to another so the reals are now a countable union of translated sets and all of them are of finite measure.

Visual images. [No response]

Specific images.

*Brad:* I guess they're all theorems of the form that things are independent of the choice of a basis. I guess those are the nice ones. I don't use too many theorems of that form, except the integral basis theorem, which is an abstract form of the existence of a basis of a certain type.

Although, certainly the existence of orthogonal bases is very nice. So I guess I also think of Gramm Schmidt basis. In fact, that came up in some research I did a couple of years ago finding, in various abstract settings, just trying to understand what a particular space that we constructed was and it turned out that it was important that Gramm Schmidt existed. They just come up all the time. So that's what I think of bases. But on the level of what is a basis, it's just a way to name vectors or vector spaces. When I teach linear algebra at first, a basis is a linearly independent spanning set. I'll mention another theorem that, that's not a theorem but one of the students here had asked, when I taught algebra, and I still don't understand this. When you prove that every vector space has a basis you apply Zorn's lemma to linearly independent sets, and take a maximal linearly independent set and show that that spans the space and that gives you the basis. So why can't you do the same thing for minimal spanning sets? In other words, why can't you start at the other end, with the finite dimensional case there's some sort of duality, you could in fact start with a spanning set. In the finite dimensional case you prove a little theorem that says that if you, if a space is spanned by some number of vectors then the linearly independent set can't have more than that number. And that's, that sets up a duality. And you can also prove that if there's a linearly independent set of some numbers then a spanning set has to have that many, but somehow in the infinite dimensional case there's no convenient way to start with the spanning set and pare it down to get a basis.

First introduction.

*Brad:* I was first introduced to the concept of a basis whenever I learned, when I took linear algebra, during the process of *not* learning linear algebra. And then later on I actually studied functional analysis in college as well. I guess bases came up there.

Changes in understanding. [No response]

Milestones.

*Brad:* Just going through algebra, number theory, and so forth, really

changed my understanding of a basis.

### • Interpretive Analysis

Brad has two prototypical models for vectors, one for explanation to students, one for himself. For students he would describe a vector as an object having both magnitude and direction, whereas for himself, he thinks of a vector as a point in a vector space. His prototypical model for a vector is actually points in the vector spaces,  $R^n$  or, more generally,  $K^n$ , where  $K$  is any field, in fact, functions can be vectors. A visual image that he evokes is the arrow that he sees on his computer screen when he sketches vectors. His understanding changed significantly from understanding vectors as objects with magnitude and direction, to understanding them just as points in a vector space.

He describes a basis as a means of seeing or portraying vectors, and his prototypical example for a basis is a vector space basis. His understanding changed as a result of courses in algebra and number theory, but does not seem to have undergone a sudden shift. Again he describes a wide variety of examples in different area, algebra and number theory, analysis, and theorems in linear algebra.

### • Derivative, Integral

Although he learned of these two concepts as inverses, he does not like to think of them in that way.

### • Derivative



Evoked images. (What first comes to mind when you think of concept?)

*Brad:* Oh boy. The best linear approximation to a function.

Specific examples.

*Brad:* The derivative of a polynomial, the standard functions and their derivatives, I guess I think of a derivative now more as a linear approximation.

Visual images.

*Brad:* The visual image, I guess, other than the usual tangent line idea, I don't have many, I have some other images like inflations in mind in terms of derivative, derivative is a rate, inflation is a rate. And things like velocity and acceleration and stuff like that. I guess you see derivatives all over. The other thing I think about derivatives is from the purely formal point of view. Let's see, it was very interesting to have him lecture the other day on knot polynomials and I might have asked this question. There are various coefficients of these polynomials and the student giving the lecture just talked about the linear term and the coefficients of the  $z$ -term, which looked like it was related to the linking number. He asked the question 'is the derivative at zero the linking number?' And in fact, that's the way I would have phrased that as well. In other words, rather than say the coefficient of the linear term, I would think of it as the derivative at zero. Because the derivative of a polynomial from an algebraist's point of view is such an important formal tool, it counts whether there are formally multiple zeros, so that then in a sense I think of the derivative, the linear approximation, most of the time, but from the algebraist's point of view I actually think of derivative, formal derivatives of polynomials or rational functions as some distinct set of rules. Sort of as a generating function, a probability distribution of generating function, the derivative gives the coefficients, this allows you to find out various moments and so forth by looking, by putting all the data together into one function. So these are some of the images associated with derivatives.

Specific images.

*Brad:* L'Hopital's rule is one that comes to mind. There's the inverse function theorem, I suppose, various mapping theorems. Those kinds of theorems, as well as the theorem about a polynomial's derivative.

First introduction.

*Brad:* That would be back when I was in calculus. And in physics, I was taking some physics in high school.

Changes in understanding.

*Brad:* Yes my understanding of this concept has changed a lot throughout this time.

Milestones.

*Brad:* I don't know when I first understood the formal power of derivatives, but that must have been pretty late. Just the definition I knew in high school and actually, there was another problem that I did that was kind of a milestone when I was in my middle year in college. I was going to pass some exam and I was given a list of problems to solve in a twenty four hour period. One of the problems was some fairly offensive looking integral and some other function and the idea was to show that these two functions were equal, although it was clear you couldn't integrate this thing. I struggled with that for a long time and finally got it but I remembered a trick which I had seen and that was that you differentiate the integral. And when you differentiated the integral and you differentiated the function then you have the derivatives and I recognized that and evaluated it at a point and thereby proved the two functions were equal, and that was quite a revelation, even though, I guess, that's a simple theorem that if you have two equal differentiable functions then the derivatives are equal, and that was quite a revelation. After that, taking algebra was the next milestone, and then using the

derivative all over the place.

## • Integral

Evoked images. (What first comes to mind when you think of concept?)

*Brad:* South Los Angeles! Actually that's true, but let's see, when I think of integration I think of sums. When I think of integration these days I really try to see integrals as, the interesting cases of integrals as being finite sums or sums over, like discrete sums, on one hand and integration over some more complicated space is another, I think of it as an averaging process. I think of it as a uniform, as a way of uniformizing a lot of theorems that give you an infinite series as well as real analysis and probability. The stuff I've done most recently with integrals is analytic number theory. So, in particular I *don't* think of it as the inverse operation of differentiation, which is certainly how I thought about it when I first learned it. That's kind of a little bit historically inaccurate too, because from my understanding, I mean integration is this kind of summing and averaging process, and the wonderful thing about the fundamental theorem of calculus is that it relates these two apparently different things, and as a result we learn, when we learn calculus, when I learned calculus I learned them as related objects, related operations, and perhaps never developed, for that reason, at that time, a good appreciation of what each was separately until much much later.

Specific examples.

*Brad:* Well, the standard example of series that I think of is the infinite series, Riemann sum stuff like that. Although I'm not really an analyst, I've been working with trying to define integration on surreal fields, surreal numbers, integrate functions, I've thought about that for a while. Not only what sort of answer that you get when you do the right kind of thing, but also the most precise way of doing it.

## Visual images.

*Brad:* One visual image I have is the picture of areas under curves, but another one that ties back to Riemann sums is trying to figure out the limit of the area of a regular  $n$ -gon and showing that that approaches the area of a circle. I remember that problem particularly, and sort of getting this Riemann sum and it turned out to be the limit. That's somewhat visual. I don't have too many, well I guess I sort of picture functions I don't know. What I'd like to do is be able to have a good visualization of convolution. One of the things that I haven't been able to is to develop a really satisfactory picture of a convolution in my mind.

## Specific images.

*Brad:* I mentioned the Fundamental Theorem of Calculus so that's one. I think there are a lot of tricks I've been telling the students here that have to do with summation of  $\frac{1}{p \log p}$ , (the sum of all the primes less than  $p$ ). So right now I have a few little theorems that are tricks for summing series or integrating functions that you see, that are useful in probability and number theory, and Putnam type problems.

## First introduction.

*Brad:* I remember the definition of Riemann integral, and an experience I had was I took calculus in the tenth grade, and in eleventh grade I saw there was a course in advanced calculus offered at the high school I was at, and I started this class. One of the first questions that the instructor asked was what is an integral. And I think I was the only one in the class who remembered the definition as a Riemann sum. Real analysis, by Miss Cunningham from Michigan. Later, in graduate school, one of my friends, it turned out had her as his calculus teacher as well, in Michigan. But I quickly dropped out of the class, I couldn't, in my mid-sixties terminology my head was somewhere else, and I didn't get turned back on to mathematics until later that year when I took probability and abstract algebra which I liked a lot. But the calculus, the computations really turned me off at that particular time. I remember that was certainly an important time because I remembered the definition, and I still like

the definition, the Riemann sum definition. You can play with it. But now I think of integrals as much more general things. For one thing there's the various line integrals where you integrate over a curve, integration really is just an averaging, an averaging, a summing, you might integrate over paths in some very abstract space, with some sort of measure. Probability integrals are fun to play with. There are so many different kinds of integrals.

Changes in understanding.

*Brad:* Yeah, my understanding has changed of course. I spent in college a lot of time reading about mathematical integration, these various theories about mathematics of integration, and I was at the time learning from an advisor who was an analyst and I was studying analysis. I guess that's, and that's really about as sophisticated as I got, understanding a few theories of integration. A graduate course in real analysis, we did some of that. I believe that that's probably the main thing that makes a hole in my background. I don't think I really understand integration at the level that's really necessary. I'm not sure, I suppose if I needed to use it I could do that. One of the funny things about mathematics is that when you're a student, you still have time to learn all these things, but when you start working you learn things by working with them and by needing them and it's very rare that any of us have the time to go back and say well I ought to really learn this theory just for itself and understand some of these justifications unless we particularly need it, at least at my level. I think the big revelation that every theorem I could write down for integration possible was possible for series - was a revelation.

Milestones. [No response]

### • Interpretive Analysis

Two visual images that Brad evokes of a derivative are the symbolic, algebraic notation for the derivative of a polynomial, and the tangent line to a function. He describes two different interpretations of derivative, one calculus based - the idea of

the best linear approximation to a function, the other algebraic - formal derivatives in algebra. These two interpretations led to significant events in his understanding. One was the notion of a formal derivative, and its link to multiplicities of polynomials, the other was the revelation that if  $f = g$ , and  $f, g$  differentiable, then  $f' = g'$ , which was a sudden but significant development.

Brad's prototypical model for integral is Riemann sums, and he evokes a visual image of the area under a curve, or the areas of regular polygons converging to the area of a circle, but he prefers to think of it as a summing, or averaging process. In particular, he does not think of integral as an inverse to derivatives, although he learned it that way initially. A significant event in his understanding was when he realized that any theorems for integrals were also applicable to series.

## CHAPTER V SUMMARY AND CONCLUSIONS

In this final chapter we summarize our findings and discuss their implications for the nature of visual imagery and of reification in advanced mathematical thinking, as well as their implications for both the teaching and the learning of advanced mathematics. We also discuss the limitations of this particular study by clearly addressing what this study was *not* intended to do, and conclude by pointing to directions for future research.

Let's consider once again our research questions, and discuss our findings in light of each question:

1. What is the nature of visual images used by advanced mathematical thinkers? How may they be classified as to the type of visual image? Are they prototypical? Metaphoric (analogic)? Diagrammatic, including algebraic symbols?
2. Can we identify any structure - hooks, which provide access to information, links, which forge connections among different pieces of information, or schema, which provide the scaffolding - within an individual's concept image?
3. Can we identify different stages in the learning processes of an advanced mathematical thinker by looking at the associated visual images and structures? In particular, can we distinguish whether a given mathematical concept is in the interiorization, condensation, or reification stage of understanding? Is there evidence of an earthquake-like event indicative of the reification process?

### **The Nature of Visual Imagery in Advanced Mathematical Thought**

In order to discuss the important types of visual images found in this study it is helpful to distinguish four categories: *prototypes, diagrams, (including algebraic*

*symbols*), *metaphors (analogies)*, and *personal symbols*. This classification is more one of *use* or *purpose* of the image, rather than of the nature of the image itself. For example, a mathematical prototypical image could be a diagram, a string of algebraic symbols, a more fuzzy, hazy featureless image, or a metaphorical, possibly extra-mathematical image. In any case, its purpose is to provide a ready example of the given concept, an example that encapsulates all the important characteristics of the given concept. For example, the natural numbers was a common prototypical example cited for the concept of a well ordered set, but it was represented differently by different individuals, some preferring to visualize it as algebraic symbols  $\{1, 2, 3, \dots\}$ , while others visualized it diagrammatically with a linear order schema, such as a number line. On the other hand, a *group* was sometimes visualized as a rather fuzzy blob with one distinguished element - the identity. We examine the purpose of each of these types of images in advanced mathematical thought.

The purpose of a diagrammatic image is to provide a clear image without any superfluous details, much as a stick figure gives a clear diagrammatic image of the essential characteristics of a human, without all the details. However, skill is required both in deciding which features of a diagram are inessential, and also in the interpretation of a diagrammatic image when given. Many such mathematical images require prior knowledge of the conventions used in the construction of the diagram in order that the implicit information is correctly extracted. For example, the graph of a function in the Cartesian coordinate system is not readily understood by anyone unfamiliar with the particular conventions used in that type of representation.

The purpose of an algebraic symbol is to provide a convenient mathematical shorthand representation of the concept. Examples include the use of  $\infty$  for infinity,  $f'(x)$  for derivative and  $\int$  for integral. There is no intrinsic pictorial meaning in the symbols themselves, and the interpretation of the symbols relies on a prior knowledge of their meaning. Thus algebraic symbols may be classified as a type of diagrammatic image inasmuch as they both require an understanding of the



convention rather than its being evident in the image. The dividing line between these two is murky at best, as a matrix could be construed as either a diagrammatic image, or an algebraic symbol, or both. Also, a diagram may be combined with algebraic symbols as labels.

The purpose of a metaphorical image is to provide an informal representation of a concept, and as such it provides a connection with prior experiences, both mathematical and non-mathematical. It is often, though not always, extra-mathematical in nature, and aids in the assimilation and incorporation of new concepts into the existing cognitive structure. This type of image is shared and understood by the mathematics community, for example the metaphorical image of falling dominoes for the mathematical concept of induction, or the Riemann sphere for points in the plane, including infinity. Of course, it is important to realize that an image may have properties of more than one type, and that a metaphorical image may be diagrammatic in nature, as is the Riemann sphere.

The purpose of a personal symbol is similar to that of a metaphor except that this is a highly idiosyncratic and personal image, and not generally shared by the mathematics community. Such a symbol most likely has meaning *only* to the specific individual who constructed it, for whom it acts as a mnemonic device for the concept in question. For example, one subject made use of a corn field to represent a number field, where the ears of corn are the numbers in the field.

All of these types of images may be used as *hooks* or initial access points to additional information of all kinds in the individual's concept image, and in fact, examples abound of the use of all types in this way. Often these images are the common property of the entire mathematics community, even though, as Thurston(1994) points out, they are infrequently found in formal mathematics forums such as mathematics books and papers. At times, however, these images - particularly metaphors and personal symbols - may be extremely individualized and even idiosyncratic, as we saw above. They may be of an entirely non-mathematical nature and may seem totally unrelated to the mathematical concept which they are

designed to represent. In spite of this they nevertheless are indispensable in providing access to the individual's knowledge of and understanding of the given concept. The idiosyncratic nature of the development of these personal images means that no two individuals necessarily have the same mental representations associated with any given concept, nor do they necessarily access or hook into them in the same way, nor do they have the same connections or links among different parts or even to other conceptions.

### The Case of Infinity

To illustrate, we look at some examples from the case studies, the first ones drawn from responses to the concept of infinity. Common visual images associated with infinity are very intuitive, vague and hazy ones connoting *big*, or *a whole lot*, or *huge*, rather than any well-defined, concrete pictorial images. The infinity symbol,  $\infty$  itself was evoked frequently, clearly an example of an algebraic symbolic image being used as an access hook into the concept image associated with infinity.

In Adam's case, he exhibited a general pattern of evoking only a symbolic image for those concepts about which he did not have a clear understanding, or whenever he experienced some kind of conflict associated with the concept, such as with infinity. He evoked the infinity symbol as his visual image, at the same time describing the difficulty he encountered in trying to understand infinity, and the conflict he developed when he learned of different 'sizes' of infinity (countable and uncountable cardinality). He is still battling with the conflicting prototypes of countable infinity and a continuum.

However, for others, even those with a seemingly clear understanding of infinity, the infinity symbol was a powerful visual symbol that was evoked, usually along with some other visual image, such as the north pole of the Riemann sphere. This particular image of infinity, an example of a metaphorical image connecting the

mathematical idea of infinity with the everyday idea of the spherical world in which we live, occurred frequently, and seemed particularly potent in providing reification of infinity. Being able to 'get one's hands on' infinity, to concretize it, to be able to draw it, was a very significant reification event. The diagrammatic image of infinity as the 'end' of, or a delimiter for, the real number line or the circular 'edge' of the complex plane, or a graphical image of a function going off to infinity, provided yet more different images of infinity.

For both Bill and Brad, their visual images of infinity consisted of two distinctly different types. Bill described an image of spinning around in place and yet never getting out of reach, as well as one of going on forever and eventually getting out of reach, both metaphorical images, related to personal experience, representing infinity as some never-ending process. Brad described a cardinality type of infinity such as the countable cardinality of the natural numbers, as distinct from a calculus type of infinity of something getting arbitrarily large, such as the real line going off to infinity, both mathematical metaphors connecting infinity with other mathematical ideas. As a mathematical concept becomes reified (like the real numbers as the numbers line) then it becomes as real to the person using it as a glass of water. As such, it now has the potential to serve as the basis for another metaphor. The use of such powerful metaphorical images for infinity seems to indicate a firm and deep structural understanding of the concept.

These descriptions of the various visual images associated with infinity serve to illustrate how individual and personal these images are. At the same time, the promotion of collaborative learning and group activities may encourage students to share their experiences, providing an opportunity for all students to develop their own personal, individual metaphorical images. The common thread of the infinity symbol, and the power of the image of the north pole to represent infinity in such a way as to cause its reification, indicate that there are some instructive pedagogical ideas and issues here, which we discuss below in the section dealing with the implications for teaching.

## Prototypes and Metaphors as Community Property

Both prototypical images and metaphorical images are abundant in mathematical circles, at least at an informal level. There is a commonality of prototypes, and to a lesser extent, of metaphors, which are handed down from one 'generation' of mathematicians to the next. In the former case, they are fairly uniform from one group to the next, and standard images of certain mathematical concepts are easily found. In the case of metaphors, although they too may become community property, they may often be highly personalized in nature and less apt to be widely shared amongst the community at large. They tend to be more individualized and even idiosyncratic, with their meaning not widely understood.

Prototypical images and examples abound - such as the use of  $N$  for a well ordered set, the power set of a set for a partially ordered set,  $Z$  or  $Z_n$  for a group,  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  and  $\{\vec{i}, \vec{j}, \vec{k}\}$  for a basis,  $\frac{1}{x}$  for a limit - even to the extent that one may rely on a single, particularly rich and versatile prototype to provide examples of several linked concepts. For example, Alan employed  $R^n$  as a single source of many different images and examples for several concepts. It served as his prototypical image for both a ring and a field, a metric space, a vector space, infinity (as the delimiter for the real line), cardinality (of the continuum), and transformations (linear, on  $R^n$ ).

Metaphorical images were widespread also. Some interesting examples were Alan's use of the metaphor of building blocks to describe the purpose of a basis, Andy's description of a set as a group of people, and a partially ordered set as a family tree complete with leaves and branches. Also, his use of a function machine to describe a function, Bill's use of a ladder metaphor for induction, and of the relationship of a ring to a field as that of a younger brother are all useful metaphors. These types of metaphorical examples are extremely helpful to the novice in understanding new and difficult mathematical concepts because they help in connecting these strange new ideas to already familiar ones.

## A Historical Trace of Concept Image Development

Another striking observation is of the temporal development and subsequent recall of parts of the concept image. We observed many instances where the first access hook into the concept image was provided by the earliest image or prototypical example encountered by that individual, even though a far more sophisticated and well developed structure of the concept image may have been assimilated during the intervening time. This earliest experience is usually not abandoned or torn down, but is used as the entry point to the structure. From this vantage point, the individual is able to then move around to the newer parts of the concept image.

Some of these instances may be found in Adam's discussion of the concepts of continuity, derivative, integral, limit, convergence and function. For each of these functions he describes an early image associated with the concept which he uses to hook into his concept image, followed later by a deeper, more mathematically sophisticated understanding of the concept. For example he describes a derivative initially as a rote process, such as that of finding  $nx^{n-1}$  when given  $x^n$ , but later describes revisiting this concept in real analysis class and re-learning the limit definition. He had been exposed to it at an earlier time, but had not retained it. It was not until he revisited it later that the links were forged between the earlier and later ideas. Bill describes a similar experience, also with the concept of derivative, while Craig describes an early intuitive understanding of continuity as a smooth curve with no breaks, followed by a later understanding of the formal  $\epsilon - \delta$  definition.

This construction and retention of mathematical scaffolding to support the collection of ideas, images, definitions and examples associated with a given concept is an extremely important one. The scaffolding provides a way of connecting information and of moving around in the concept image in order to retrieve relevant information. Often the access point to the structure is an early image or example, but this is often connected via the scaffolding to more recent, richer information. Rather than destroying the scaffolding once the concept image is constructed, there

seems to be a preference for developing a rich and complex scaffolding to facilitate the connection of components of the concept image, and of different concept images.

There are few examples of personal idiosyncratic images in this study, but Doris provides us with two. She uses the metaphor of a corn field to understand a mathematical field, where the numbers in the field are represented by the ears of corn in the field, and the metaphor of an electrical transformation provides the basis for her understanding of a mathematical transformation, where the notion of changing one quantity into another is the key. Thurston(1994) comments on the importance of personal and idiosyncratic images in the development of his own mathematical understanding:

My mathematical education was rather independent and idiosyncratic, where for a number of years I learned things on my own, developing personal mental models for how to think about mathematics. This has often been a big advantage for me in thinking about mathematics, because it's easy to pick up later the standard mental models shared by groups of mathematicians. This means that some concepts that I use freely and naturally in my personal thinking are foreign to most mathematicians I talk to. My personal models and structures are similar in character to the kinds of models group of mathematicians share - but they are often different models.(pp 174-175).

Commonly shared metaphors may aid us in communicating mathematical ideas to one another, but they also impose constraints. New metaphors, perhaps quite personal ones, can provide the individual with creative insights that ultimately mark progress for the entire community. While we cannot 'teach' creativity in general, perhaps we can better *encourage* students to search for their own metaphors. This emphasizes the importance of developing and using personal idiosyncratic models, but it also makes it clear that it is an individual responsibility, rather than a group one. Because of its very personal nature it may not be taught. These implications for the teaching and learning of mathematics are discussed below.

## The Nature of Reification in Advanced Mathematical Thought

Sfard emphasizes that a reification event is a sudden, earthquake-like event, and as such is so monumental that it would certainly be unforgettable. One might therefore expect that advanced mathematical thinkers would, if only by virtue of their continued exposure to the study of advanced mathematics, have many such 'earth-shaking' experiences to recount. This does not seem to be the case in this study. More often than not, individuals describe a gradual deepening and broadening of their understanding of a particular concept, but seldom an earthquake-like event that suddenly and immediately changed the nature of their understanding.

This is not to suggest that such events are completely absent, but that perhaps they are far more infrequent than one might expect. I will suggest several explanations. It may be that any such events occurred so long ago in these individuals' mathematical experiences that they have simply forgotten the impact of the leap to a new understanding. On the other hand, it may be that these events occur so frequently in the experience of the advanced mathematician as to no longer draw continued surprise. They are no longer of such huge proportions simply because they are too routine. Perhaps there are degrees of these earthquake-like events. Just as one who lives in Southern California is no longer roused out of bed by tremors of 4 and 5 points on the Richter scale, but only when 'the Big One' threatens, so it may be with mathematical reification events. The minor events no longer arouse the advanced mathematician in the same way that a student struggling to understand may be aroused by an 'aha!' event. The advanced mathematician may merely say 'ah yes', and continue on. So the question becomes one of whether there are degrees of reification events, and whether advanced mathematicians are merely overlooking the more mundane events, waiting for the 'big one'.

An alternative explanation may be that, at least for the advanced mathematician, a reification event is no longer always a sudden quantitative leap, but instead sometimes a more gradual, drawn out process of continual deepening and broad-

ening of understanding until one gradually begins to understand the concept as an object. On the other hand, perhaps the advanced mathematician has the ability to understand a new concept immediately as an object, and the long and drawn out deepening and broadening of understanding so often reported is indicative of a post-reification stage, in which the concept, once objectified, begins to take on a fuller and more comprehensive meaning as its place in the body of mathematical knowledge becomes more firmly established. It is clear that there are more questions here than answers.

### **Implications for the Teaching and Learning of Advanced Mathematics**

There are several implications for teaching that arise from this study. First of all, it is clear that visual images are pervasive in advanced mathematical thought. Seldom was any individual found to be without a visual component to the concept image, even though at times it was no more than the associated mathematical symbol for the concept. It is also apparent that if students are not provided with a clear and meaningful visual image, they will go to great lengths to develop a meaningful one, even though it may be extra-mathematical and highly idiosyncratic, such as Doris' image of a field as a corn field, where the numbers were represented by the ears of corn.

Thus, even though Gauss may have wished for all scaffolding (presumably including intuitive visual images) to be removed so that only the finished logical, linear symbolic argument remains, this is obviously not the way in which the individuals in this study developed their mathematical understanding. The scaffolding plays an essential role in the development and maintenance of the concept image as well as providing a structure by means of which one may move around within various components of the concept image and among different concept images. Since



the first images are often retained as the access point to this structure, these play an especially important role. The instructor does not always have control of this (for example, students encounter infinity before and outside instruction). The instructors of entry level and introductory courses are not always familiar with the long term implications and applications of mathematical concepts. Ways should be sought to encourage students' development of scaffolding to support well developed concept images, and links among them. There is not much point in having a tree house without an access ladder, or a house with many rooms but no doors or passageways to connect them.

A visual image can indeed be an important trigger for reification. For example, it is most clear that, from a pedagogical point of view, the representation of infinity as the north pole of the Riemann sphere played this role for many of the subjects in this study. This particular metaphorical image should be emphasized and revisited many times, so that all students have ample opportunity to experience the reification of this concept with the aid of this image at the appropriate time.

In general, it is important to provide students with a variety of visual examples and images of all types for all mathematical concepts, if possible, to facilitate the development of their own conceptions, because it is from these that they begin to construct their own understanding, build their own scaffolding, and embark on the progression from procedural to structural understanding. In light of Thurston's comments about the desirability of developing one's own personal and idiosyncratic images, this type of image building should be encouraged. This may, however, be a rather difficult task, since providing such examples defeats the purpose, and yet students need to know that this is something to attempt. Perhaps if we share what personal images we have or are aware of, this will provide sufficient encouragement for students to develop their own. One wonders, also, where the more widely accepted images come from. It would seem that they originally must have been the product of some individual's mind, which was then subsequently shared among the mathematical community at large. Thus we need to encourage a more fertile

breeding ground for personal images. Should personal images be shared property or not? Perhaps the development of such images should be generally encouraged but it should be left up to each individual to form them. Otherwise they are no longer what they claim to be!

It seems that timing of the presentation of these images is also extremely important, yet it may not be possible to ascertain exactly when students are ready for the important transitions, from interiorization to condensation, and particularly for the quantum leap from condensation to reification of a concept. Certain examples and images may need to be revisited over and over again to give the repeated opportunities for all students to reach this stage. It is clear from Beth's experience with metric spaces that it cannot be assumed that all students assimilate everything said, whenever it is said. A revisiting of images (other than blobs) of metric spaces, along with examples and a formal definition would no doubt have helped Beth and her classmates to begin to construct a meaningful understanding of a metric space.

Adam's and Calvin's different experiences with the  $\epsilon - \delta$  definition of continuity also illustrate the critical nature of timing and the importance of revisiting ideas. Adam learned the formal definition early, at the same time as he learned the intuitive 'draw it without lifting your pencil' image of continuity, but did not see any connection between these two until much later when he revisited the formal definition in a real analysis class. It was not until this much later time that he forged the link in his concept image between the visual image and the formal definition, and consequently was able to make sense of both the visual image and the formal definition, and to see the important links between the two. In contrast, Calvin learned the intuitive pictorial image first and only much later learned the formal definition, at which time, he claimed, this made the concept of continuity clear in his mind. Thus an argument may be made for repeatedly emphasizing both the intuitive visual images and the formal images of continuity in the hopes that more and more students will be assimilating them into their concept images. Again, collaborative learning would encourage and facilitate this by the sharing of ideas.

As we have seen, the types of visual imagery evoked by the individuals in this study cover the spectrum from algebraic symbols, to diagrams and graphs, to rich analogies with non-mathematical ideas, as well as metaphors linking two different mathematical systems or concepts, and idiosyncratic personal symbols. However there is another side to the development and nurturing of visual imagery - the affective side. This aspect of visual imagery in mathematics cannot be ignored. There seem to be differing levels of individual preference for and indeed differing levels of personal comfort with visualization. Some mathematicians rely heavily on visual imagery, while others regard its use with scepticism and wariness. Some individuals, like Calvin, feel lost when they cannot visualize a concept, and the absence of an appropriate visual image becomes a major obstacle in the further development of a concept image. Hence, any further construction is put on hold until this defect is repaired by the acquisition of a suitable visual image. Others, like Alan, feel wary regarding visual imagery and consider it to be restrictive and potentially misleading. It is possible that this is a pedagogical point of view that Alan is expressing. While not abandoning visual images completely, Alan has divided his concept image into two parts - one for himself, where he does not rely on pictorial visual images but instead uses algebraic symbolism extensively, and one for his teaching - where he has a ready catalog of visual images which he provides for students to use in building their concept images. Although he has made a personal decision to minimize his reliance on visual imagery, he recognizes its importance in any introduction to mathematical concepts. Thus we must accept that some individuals will be more inclined towards visual imagery than others.

This also brings up the issue of pedagogical content versus personal content. It was apparent that both faculty members in this study distinguished between two 'parts' of their concept image - one part for their own personal use, and the other part for use in teaching. For example, faculty members Alan and Brad described visual images for such concepts as sets, limits, continuity, derivatives and integrals that they themselves do not rely upon but that they employ when teaching.

Prototypes and diagrams are 'internal' to mathematics, in the sense that they themselves are mathematical in nature and content, while metaphors and personal hooks may be extra-mathematical in the way that they draw on experiences and images outside the mathematical realm. Nevertheless, they are very prevalent in advanced mathematical thinking. The use of analogies and metaphors seems to accompany increased comfort with and familiarity with the concept, as well as an intuitive understanding of the concept, to such an extent that I am led to conjecture that the use of such imagery is an essential component of advanced mathematical thought. In fact, I would suggest that this may be one measure of that elusive 'mathematical maturity' that we require of students before they may take certain mathematics courses. Is it that what we are really attempting to measure is whether students are able to rephrase important mathematical ideas in their own everyday words, that is by using metaphors, and by drawing analogies with other mathematical and non-mathematical ideas? Is this one measure that may indicate 'mathematical maturity'? It is by associating new ideas with more familiar mathematical concepts, or even with familiar ideas beyond the mathematical realm, by connecting mathematics with personal experience, by linking new with existing, that the advanced mathematical thinker gains an understanding of new mathematical ideas. Such understanding would be unattainable without the linking mechanism of metaphorical imagery and personal idiosyncratic symbols.

### **Limitations of the Study**

This study and these particular research questions were designed to document *only* what each individual voluntarily evoked in connection with each concept. No probing or follow-up questions were asked if nothing was voluntarily evoked, since I wanted to document those parts of the concept image readily and voluntarily accessible in connection with a given concept, and not that information accessible only

by provision of further external hooks. I also wanted to document the individuals' own self-provided hooks, and how these hooks were used to access the concept image. Certainly, the subjects might have evoked additional or different parts of their concept images *given additional cues or prompts*.

In particular, it is important not to infer that the responses given reflect the totality of the cognitive structure that each individual may possess associated with the concept, only that material which was easily and voluntarily evoked, without any prompting beyond the above seven questions. It is also important to understand that the interview was designed to uncover each individual's *own* hooks to information, not to see what was evoked, *if additional cues were provided*. In many instances it is quite possible that far more information may have been available, once some prompt was provided, but that was not the purpose of this study.

Particular care has been taken to avoid superimposing my own visual images associated with these concepts. For this reason I decided against including illustrations based on verbal descriptions of visual images in the transcript pages and in the subsequent analysis. (For example, the Riemann sphere was mentioned by several subjects in connection with the concept of infinity), and we can easily imagine the general nature of the visual images held by the subjects. However, there is the risk of assuming extra or insufficient details of such an image when presenting an illustration. While participants were not discouraged from using pencil and paper to describe their images, it was not mandatory, and few took advantage of this medium even when it was available to them. I have therefore chosen to rely on their verbal descriptions of their mental visual images.

It would, however, be an interesting and productive exercise to ask these individuals to render physical representations of their mental imagery, so that we may begin the huge task of cataloging such images for teachers and learners of mathematics. What a rich and invaluable volume that would be!

## Directions for Future Research

This documentation of the nature of the concept images of these nine advanced mathematical thinkers through reflective interviews, is essentially a self-assessment by these nine individuals of the nature and content of their conceptions about twenty-one mathematical concepts, and the links and connections among them. However, as was noted above, no probing was done in these interviews since we were primarily interested in the individuals' own hooks into their concept image. A fuller picture of the cognitive structure could be gained by further probing individuals' concept images *upon provision of external cues* as prompts, such as key words, ideas, images or examples.

Another approach might be to observe how these same individuals *use* their knowledge, their images, their prototypes and analogies in *doing* mathematics. How do these visual image structures help their mathematical problem solving? In what ways do they use their linked concept images when attacking mathematical problems? How does a metaphorical understanding of a concept, or the existence of a personal symbol, impact one's ability to solve problems? How does the presence or absence of reification affect their problem solving strategies? What parts of the concept image evoked in the problem solving setting? Are the same links forged and utilized in the problem solving process? These questions, and a host of others, await an answer.

One of the most striking and significant observations from these nine case studies is the degree of individuality exhibited in the visual conceptions of these nine people. No two individuals have the same mental representations associated with any given concept, nor do they access or hook into them in exactly the same way, nor do they have the same connections or links among different parts or even to other conceptions. This supports the view of mathematics learning as a constructive process, wherein each individual builds a personal meaning and understanding. No two individuals share the same 'experience', for even if they share access to

'information' (e.g. take the same class from the same instructor) the manner in which they assimilate and accommodate this 'data' into their existing concept image may be quite different, based on pre-existing images.

In particular, the visual images associated with any mathematical concept show wide variety from person to person. This is not to say that there are not some commonalities of purpose and types of visual images in use among mathematicians, but simply that there is more difference than sameness, and from these we may learn important implications for teaching. The Platonic view of mathematics as some body of predetermined universal truths is certainly not supported by this study. Rather it strongly favors a constructivist view of mathematical knowledge wherein each individual builds up a very personal structure of knowledge, incorporating visual images in a unique way.

## BIBLIOGRAPHY

- Anderson, J. (1978). Arguments Concerning Representations for Mental Imagery. *Psychological Review*, 85, 4, 249-277.
- Arnheim, A. (1969). *Visual Thinking*. University of California Press, Berkeley.
- Bruner, J., Olver, R., Greenfield, P. (1966). *Studies in Cognitive Growth*. Wiley: New York.
- Dreyfus, T. (1990). Advanced Mathematical Thinking. In P. Nesher and J. Kilpatrick (Eds.), *Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge University Press, pp113-134.
- Dreyfus, T. (1991). *Advanced Mathematical Thinking*. David Tall, Ed. Dordrecht: Kluwer Academic Publishers.
- Fischbein, E. (1987). *Intuition in Science and Mathematics*. Dordrecht: D. Reidel.
- Fischbein, E., Tirosh, D., Melamed, U. (1981). Is it Possible to Measure the Intuitive Acceptance of a Mathematical Statement? *Educational Studies in Mathematics* 12, 491-512.
- Goldin, G. A. (1987). Cognitive Representational Systems for Mathematical Problem Solving. In C. Janvier, (Ed.). *Problems of Representation in the Teaching and Learning of Mathematics*. New Jersey: Lawrence Erlbaum.
- Hadamard, J. (1945). *The Psychology of Invention in the Mathematical Field*. Princeton: Princeton University Press.



- Johnson, Mark. (1987). *The Body in the Mind*. University of Chicago Press.
- Kosslyn, S. M. (1983). *Ghosts in the Mind's Machine*. New York: W. W. Norton.
- Lakoff, George. (1987). *Women, Fire and Dangerous Things*. University of Chicago Press.
- Neisser, U. (1978). Perceiving, Anticipating, and Imagining. In *Perception and Cognition: Issues in the Foundations of Psychology*. C. W. Savage, Ed. *Minnesota Studies in the Philosophy of Science*, vIX.
- Paivio, A. (1971). *Imagery and Verbal Processes*. Holt, Rinehart and Winston, Inc. NY.
- Piaget, A. and Inhelder, B. (1971). *Mental Imagery and the Child*. Routledge and Kegan Paul. London.
- Presmeg, N. (1986). Visualization and Mathematical Giftedness. *Educational Studies in Mathematics* 17, 297-311.
- Presmeg, N. (1992). Prototypes, Metaphors, Metonymies and Imaginative Rationality in High School Mathematics, *Educational Studies in Mathematics*, 23: 595-610.
- Pylyshyn, Z. (1973). What the Mind's Eye Tells the Mind's Brain: A Critique of Mental Imagery, *Psychological Bulletin* 80, 1, 1-24.
- Schmalz, R. (1988). The Role of Intuition in Doing Mathematics. *Journal of Mathematical Behavior* 7, 33-44.
- Schwarz, B. and Bruckheimer, M. (1988). Representations and Analogies. In A. Borbàs (Ed.), *Proceedings of the 12th International Conference for the Psychology of Mathematics Education (Vol. 2, pp. 552-559)*. Veszprém, Hungary: Ferenc

Genzwein, OOK.

Senechal, M.(1990). Shape. In L. A. Steen (Ed.), *On the Shoulders of Giants*. Washington: National Academy Press.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics* 22, 1-36.

Sfard, A. (1992). Operational Origins of Mathematical Objects and the Quandary of Reification - The Case of Function. In Harel, G., and Dubinsky, Ed., eds. *The Concept of Function: Aspects of Epistemology and Pedagogy. MAA Notes, Volume 25*. Mathematical Association of America.

Sfard, A., Linchevsky, L. (1994). The Gains and the Pitfalls of Reification - The Case of Algebra. *Educational Studies in Mathematics* 26, 191-228.

Shepard, R. (1978). The Mental Image, *American Psychologist*, February, 125-137.

Steen, L. A. (1990). Pattern. In L. A. Steen (Ed.), *On the Shoulders of Giants*. Washington: National Academy Press.

Tall, D. (1991). *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers.

Tall, D. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof. In *Handbook of Research on Mathematics Teaching and Learning, NCTM*. McMillan.

Tall, D., Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity, *Educational Studies in Mathematics* 12, 151-169.

Thomas, H. L. (1975). The Concept of Function. In M. F. Roszkopf (Ed.), *Children's Mathematical Concepts: Six Piagetian Studies in Mathematics Education* (pp. 145-172). New York: Teachers College Press.

Thurston, W. P. (1994). On Proof and Progress in Mathematics, *Bulletin of the American Mathematical Society*, 30, 2, 161-177.

Vinner, S. (1983). Concept Definition, Concept Image and the Notion of Function, *International Journal of Mathematical Education in Science and Technology*, 14, 293-305.

Vinner, S., Dreyfus, T. (1989). Images and Definitions for the Concept of Function, *Journal for Research in Mathematics Education* 20, 4, 356-366.

Werner, H., Kaplan, B. (1963). *Symbol Formation: An Organismic Developmental Approach to the Psychology of Language and the Expression of Thought*. New York: Wiley.

Zimmerman, W. and Cunningham, S. (1991). Editor's Introduction: What is Mathematical Visualization? In W. Zimmerman and S. Cunningham (Eds.), *Visualization in Teaching and Learning Mathematics*. Mathematics Association of America.