AN ABSTRACT OF THE THESIS OF

ZENAS RUSSELL HARTVIGSON for the DOCTOR OF PHILOSOPHY
(Name) (Degree)

in MATHEMATICS presented on July 25, 1973
(Major) (Date)

Title: THE COMPLETENESS AXIOM OF THE LOBACHEVSKIAN GEOMETRY

Redacted for privacy

Abstract approved: ______________________
Harry Goheen

This paper gives a proof that the Completeness Axiom of Lobachevskian geometry--as formulated in the second English translation of David Hilbert's Foundations of Geometry (tenth German edition)--is a theorem in the three dimensional Poincaré model. An explicit canonical isomorphism between all models of Lobachevskian space is given.

This, together with the work of William Lee Zell (A Model of Non-Euclidean Geometry in Three Dimensions, Master's Thesis, Oregon State University, 1967) and Robert W. Eschrich (A Model of Non-Euclidean Geometry in Three Dimensions, II, Master's Thesis, Oregon State University, 1968), establishes that the three dimensional Poincaré model is a model of Lobachevskian geometry based upon Hilbert's axioms.

The Completeness Axiom of Lobachevskian Geometry

by

Zenas Russell Hartvigson

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1974

APPROVED:

Redacted for privacy

Professor of Mathematics

in charge of major

Redacted for privacy

Chairman of Department of Mathematics

Redacted for privacy

Dean of Graduate School

Date thesis is presented July 25, 1973

Typed by Clover Redfern for Zenas Russell Hartvigson

ACKNOWLEDGMENT

I wish to acknowledge both my debt and my gratitude to Nicholas Lobachevski and David Hilbert, in whose shadow any student of this subject must walk; to the National Science Foundation, whose Science Faculty Fellowship made these past three years financially possible; to my wife, Sharon, who so patiently helped proof read and so kindly and wisely often lifted me from discouragement; to my typist, Mrs. Clover Redfern, whose help and uncanny ability to read my awful penmanship so greatly speeded these final days.

TABLE OF CONTENTS

THE COMPLETENESS AXIOM OF LOBACHEVSKIAN GEOMETRY

INTRODUCTION

Any axiom system consists of a sequence of undefined notions, defined notions, and the axioms which describe the behavior of these notions. For example, the axiom, "There exist at least two points on a line" relates two undefined notions--point and line. It does this using a defined notion--"a point A is on a line ℓ iff there is a second point B so that the line incident on A and B is the line ℓ." Note that the definition is expressed using the undefined terms, point and line together with the undefined relation of incidence for lines.

Of the several considerations regarding axiom systems, this paper addresses only two in any detail. These are consistency and categoricity. Specifically, we will prove that the completeness axiom of Lobachevskian geometry is a theorem in the Poincaré model. All of the axioms of Lobachevskian geometry (as formulated in the tenth edition of Hilbert's Foundations of Geometry) [5] except for the axiom of completeness have already been shown to be theorems in the Poincaré model in work done by W. L. Zell [18] and R. W. Eschrich [3].

When the completeness axiom is proved to be a theorem in the

Poincaré model, the Poincaré model will have been shown to be a model of Lobachevskian geometry and the geometry will be known to be as consistent as the real number system. The proof that the completeness axiom is a theorem in the model involves a proof that all possible models of Lobachevskian geometry are isomorphic. This part of the proof of the completeness axiom gives us a proof of the categoricity of Lobachevskian geometry without further work.

Before we go further, we now specifically state what the Poincaré model is. This analytical formulation is found in Zell's and Eschrich's papers. The existence of a parametric formulation which is mentioned in "6." below, is proved as a theorem by Zell.

In the Poincaré model:

1. A <u>point</u> is an ordered triple of real numbers (x, y, z) so that $x^2 + y^2 + z^2 < 1$.

2. A <u>plane</u> is an equivalence class of equations having a representative of the form,

$$D(x^2+y^2+z^2+1) + Ax + By + Cz = 0,$$

where $A^2 + B^2 + C^2 > 4D^2$ with A, B, C, D real constants and x, y, z real variables.

3. A <u>line</u> is an equivalence class of pairs of equations having a representative pair of the form

$$* \begin{cases} D(x^2+y^2+z^2+1) + Ax + By + Cz = 0, \\ D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0 \end{cases}$$

where $A, B, C, D, A', B', C', D'$ are real constants, x, y, z are real variables and

(a) at least one point satisfies the system of equations *,

(b) $A^2 + B^2 + C^2 > 4D^2$ and $A'^2 + B'^2 + C'^2 > 4D'^2$, and

(c) $\operatorname{rank} \begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix} = 2$.

4. A <u>line is incident on a pair of distinct points</u> iff both points are in the solution set of the line.

5. A <u>plane is incident on three non-collinear points</u> iff each point is in the solution set of the plane.

6. If

   (a) $x = f(t)$, $y = g(t)$, $z = h(t)$ is a parametric representation of a line (where f, g, h are real, continuous, monotonic functions), and

   (b) k, k', k'' are the values of the parameter t associated with points P, P', P'',

   then P' is <u>between</u> P and P'' iff

   $k < k' < k''$ or $k'' < k' < k$.

7. Two segments PQ and RS are <u>congruent</u> iff there is a finite product of inversions taking P to R and Q to S, where an inversion is defined as follows in the model: Let α be a plane in the model with representative

$$D(x^2+y^2+z^2+1) + Ax + By + Cz = 0.$$

The <u>inverse</u> with respect to α of the point $P = (a, b, c)$ is

$$I(a,b,c) = \begin{cases} (a+Ak, b+Bk, c+Ck) \text{ if } D = 0 \text{ and } k = -2\dfrac{aA+bB+cC}{A^2+B^2+C^2} \\[2ex] \left(-\dfrac{A}{2D}+K\left[a+\dfrac{A}{2D}\right], -\dfrac{B}{2D}+K\left[b+\dfrac{B}{2D}\right], -\dfrac{C}{2D}+K\left[c+\dfrac{C}{2D}\right]\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } D \neq 0 \\[2ex] \text{and } K = \dfrac{\dfrac{A^2+B^2+C^2}{4D^2}-1}{\left(a+\dfrac{A}{2D}\right)^2+\left(b+\dfrac{B}{2D}\right)^2+\left(c+\dfrac{C}{2D}\right)^2} \end{cases}$$

8. $\angle PQR$ and $\angle P'Q'R'$ are called <u>congruent</u> angles iff there is a finite product of inversions, ψ, so that

(a) $\psi(Q) = Q'$

(b) P' is a point of the ray $\overrightarrow{\psi(Q)\psi(P)}$

(c) R' is a point of the ray $\overrightarrow{\psi(Q)\psi(R)}$.

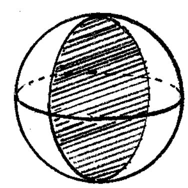


Figure 1

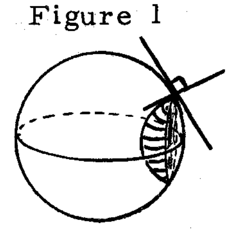


Figure 2

Graphically, a "plane" is the portion of a plane (through (0, 0, 0)) inside the open unit sphere (see Figure 1) or the portion of a sphere (with center outside the unit sphere) inside the unit sphere where the sphere meets the unit sphere orthogonally (see Figure 2). A line is the intersection of two planes. The congruence group elements are products of inversions where an inversion with respect to a plane can be visualized from what an inversion is with respect to a line in the planar case (Figure 3). The inverse of P, I(P) = P' is that point P' so that if A is the center of the circle (sphere) defining the line (plane), then $\|AP\| \|AP'\| = r^2$ where r is the radius of the circle (sphere) of inversion (see Figure 3). It is easily shown that orthogonal circles leave each other invariant with respect to inversions of this sort and thus the boundary of the unit ball is fixed under inversion with respect to any "plane." The "flat" planes can be thought of as spheres of infinite radius if one so desires. Their inversions are simply reflections across the plane.

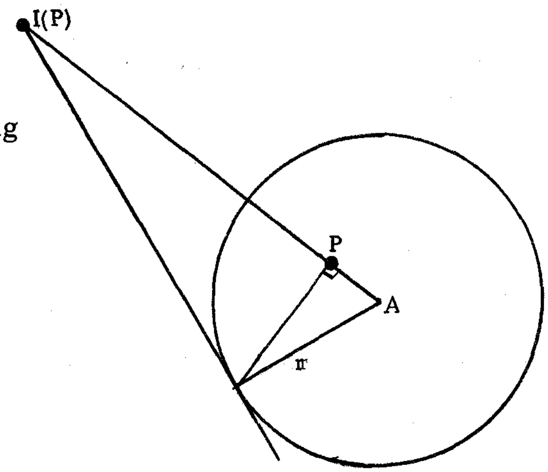


Figure 3.

To show that these eight definitions (or realizations) of the undefined terms of the Lobachevskian geometry do in fact define a model of this geometry, we must prove that every axiom of the Lobachevskian geometry is a theorem in the model. This is always the method that must be followed to prove that a given <u>thing</u> is a model of a given axiom system. As previously noted, in 1967 and 1968 W. L. Zell and R. W. Eschrich proved that all of the axioms of incidence (or connection), order, congruence, parallels, and Archimedes were theorems of this model. The one remaining axiom that needs to be proved to be a theorem in this model is the completeness axiom.

The completeness axiom as now formulated states:

> An extension of the set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follow from the axioms of Incidence, Order, Congruence, and Archimedes is impossible [5, p. 26].

This axiom was formulated by Paul Bernays in the 1950's and proved to give Hilbert's earlier completeness axiom as a theorem [5]. The notions of completeness were not formulated until after the death of Lobachevski so it is not any blot on his work to find in them no specific attention given to completeness considerations. His work is in fact a brilliant treatise in its insights and sense of what is necessary.

The specific argument used to prove that the completeness axiom is a theorem in the Poincaré model will be given shortly. First we briefly examine what has been done in this regard. The most complete work commonly available treating the completeness axiom and the questions of consistency and categoricity of Lobachevskian geometry is the <u>Foundations of Geometry</u> by Karol Borsuk and Wanda Szmielew [2] which uses a "completeness axiom" that leads immediately to the proof of a Dedekind property for the points on any ordered line. However, the Dedekind property is not shown to be a theorem based on Hilbert's completeness axiom. It appears that this may be a very hard theorem to prove without recourse to a parallel axiom. In fact it may be impossible to prove without essentially treating the two possible parallel axioms as cases to establish the proof. We already know it is a theorem in Euclidean geometry. We will show that it is also a theorem of Lobachevskian geometry.

Since Borsuk and Szmielew do not claim to be proving the completeness axiom as formulated in the Hilbert axioms (nor to use his other axioms as the basis of this proof) to be a theorem in their model, it is improper to criticize them for not doing so. Neither do they consider the Poincaré model in Euclidean space but rather the Beltrami-Klein planar model in the projective plane. No other published attempts appear in the literature which address this problem even so extensively as does the work of Borsuk and Szmielew.

In his study of Lobachevskian geometry, Hilbert developed some properties regarding the relationship of the axioms I-IV and the completeness axiom in the planar case [6]. He does not--in any published work--consider the question addressed in this paper.

Some persons, not fully understanding the problem addressed here, have suggested that this present problem was done by Curtis M. Fulton of the University of California at Davis in his paper <u>Linear Completeness and Hyperbolic Trigonometry</u> [4]. However, following my presentation of a paper on this work at the U. of C. at Davis in April 1973, Professor Fulton (who was present) informed me that he had assumed what I was proving in order to get his very short proof of the trigonometric results which will take so long to derive here.

Our proof that the completeness axiom is a theorem in the Poincaré model will be constructed as follows:

(1) We shall demonstrate that there is a one-to-one map of the set of points on any line <u>into</u> the real numbers. This will be done as a theorem in absolute geometry.

(2) Knowing of such an injection, we suppose that this injection is an order preserving one-to-one correspondence. We shall then show that on the basis of this assumption and upon the basis of the axioms, we can construct a model of the Lobachevskian geometry. This will show that the above assumption does not lead to a contradiction. We will show that the model constructed is the Poincaré model and that every model of Lobachevskian geometry is isomorphic to the Poincaré model. Hence the axiom of completeness is a theorem in the Poincaré model.

The establishment of the injection described in (1) above is achieved by constructing a one-to-one correspondence between every point on a line and a set of real numbers. This is done by pairing each point with a unique binary infinite sequence--the base two radix representation of the associated real number.

The process of proving the claim in (2) is much more difficult.

A list of the steps in the process should be helpful to one's following of the subsequent presentation.

Based upon the key assumption that a one-to-one correspondence exists between the points of a line in Lobachevskian geometry and the real numbers as described above we can:

a. construct a map $| \, |_S$: set of classes of segments $\rightarrow \mathbb{R}^+ \cup \{0\}$.

b. based on a., get a map

$| \, |_A$: set of classes of angles $\rightarrow (0, \pi)$

c. based on a. and b., get the "Lobachevskian function"

$\Pi : \mathbb{R} \rightarrow (0, \pi)$.

We then use the axioms to establish specific values for this last correspondence. We will define a coordinatization of Lobachevskian space and show that the set of triples that can be used to name the points of space (x, y, z) are exactly those triples so that

$$\cos^2\Pi(x) + \cos^2\Pi(y) + \cos^2\Pi(z) < 1 .$$

This will lead us to a one-to-one correspondence between the defined terms point, line, plane, incidence (for lines), incidence (for planes), betweenness, congruence (for segments) and congruence (for angles) in the Poincaré model and the corresponding undefined terms in the geometry. This last construction is canonical and gives the necessary isomorphism between all models of Lobachevskian geometry needed to finish the proof.

In the following pages the specific details will be given.

Before proceeding with the actual argument is seems worthwhile to include here the undefined notions, the defined notions, the axioms, and the theorems accepted without proof which shall be used in this paper.

0.1 The undefined terms are point, line, and plane.

0.2 The undefined relations are as follows:

a. Incidence for lines is a symmetric relation between the set of pairs of distinct points and the set of lines.

b. Incidence for planes is a symmetric relation between the set of triples of non-collinear points and the set of planes.

c. Betweenness is a non-symmetrical relation between the set of points and the set of pairs of points.

d. Congruence for segments is a relation between the set of segments and itself. We use "$\cong$" to denote this relation.

e. Congruence for angles is a relation between the set of angles and itself. We use "$\cong$" to denote this relation.

0.3 The defined notions are:

a. A point P is on a line ℓ iff there is a point Q so that the line incident on P and Q, denoted PQ, is ℓ. We also say P is a point of ℓ, P is in ℓ, ℓ contains P, etc.

b. A set of points each of which is on the same line ℓ, is said to be a collinear set.

c. A point P is on a plane α iff there are points Q and R so that $\{P, Q, R\}$ is a non-collinear set and the plane incident on P, Q, and R is α. We also say P is a point of α, P is in α, α contains P, etc.

d. If every point of a line ℓ is a point of plane α, we say ℓ is a line of α, ℓ is in α, α contains ℓ, etc.

e. If B is a point between points A and C we write A-B-C.

f. If every point [line] of a set of points [lines] is a point [line] of the same plane, they are called coplanar.

g. A segment, denoted AB, is the set of all points between A and B.

h. A ray with end point A, denoted $\overrightarrow{AB}$, is the set of all points P so that P = B or A-P-B or A-B-P.

i. If A, B, C are non-collinear, then triangle ABC, denoted $\triangle ABC$, is $AB \cup AC \cup BC \cup \{A, B, C\}$.

j. An angle is two non-collinear rays with a common end point together with their common end point (which is called the vertex of the angle).

k. $\triangle ABC$ is congruent to $\triangle A'B'C'$, denoted $\triangle ABC \cong \triangle A'B'C'$ if $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$,

$AB \cong A'B'$, $AC \cong A'C'$, and $BC \cong B'C'$.

1. If every line ℓ of plane α, containing a given point P, is perpendicular to line p at P, then we say p is perpendicular to α at P, p and α are perpendicular at P, or p is normal to P at ℓ.

0.4 The axioms are:

I, 1. For every two points A, B there exists a line a that is incident upon A and B.

I, 2. For every two points A, B of a line ℓ the line incident upon A and B is ℓ.

I, 3. There exist at least two points on a line. There exist at least three points that do not lie on a line.

I, 4. For any three points A, B, C that do not lie on the same line there exists a plane α that is incident upon them. For every plane there exists a point which it contains.

I, 5. For any three points A, B, C of plane α that do not lie on one and the same line, the plane incident upon them is α.

I, 6. If two points A, B of a line a lie in a plane α then every point of a lies in the plane α.

I, 7. If two planes α, β have a point A in common, then they have at least one more point B in common.

I, 8. There exist at least four points which do not lie in a plane.

II, 1. If a point B lies between a point A and a point C then the points A, B, C

are three distinct points of a line, and B then also lies between C and A.

II, 2. For two points A and C, there always exists at least one point B on the line <u>AC</u> such that C lies between A and B.

II, 3. Of any three points on a line there exists no more than one that lies between the other two.

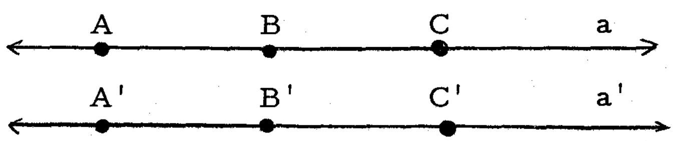
II, 4. Let A, B, C be three points that do not lie on a line and let a be a line in the plane ABC which does not meet any of the points, A, B, C. If the line a passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.

III, 1. If A, B are two points on a line a, and A' is a point on the same or on another line a' then it is always possible to find a point B' on a given side of the line a' through A' such that the segment AB is congruent or equal to the segment A'B'. In symbols

$$AB \cong A'B'$$

III, 2. If a segment $A'B'$ and a segment $A''B''$, are congruent to the same segment AB, then the segment $A'B'$ is also congruent to the segment $A''B''$, or briefly, if two segments are congruent to a third one they are congruent to each other.

III, 3. On the line a let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or on another line a' let



$A'B'$ and $B'C'$ be two segments which except for B' also have no point in common. In that case, if

$$AB \cong A'B' \quad \text{and} \quad BC \cong B'C'$$

then $$AC \cong A'C'.$$

III, 4. Let $\angle(\vec{h}, \vec{k})$ be an angle in a plane α and a' a line in a plane α' and let a definite side of a' in α' be given. Let $\vec{h}'$ be a ray on the line a' that emanates from the point $0'$. Then there exists in the plane α' one and only one ray $\vec{k}'$ such that the angle $\angle(\vec{h}, \vec{k})$ is congruent or equal to the angle $\angle(\vec{h}', \vec{k}')$ and at the same time all interior points of the angle $\angle(\vec{h}', \vec{k}')$ lie on the given side of a'.

Symbolically

$$\angle(\vec{h}, \vec{k}) \cong \angle(\vec{h'}, \vec{k'}).$$

Every angle is congruent to itself, i.e.,

$$\angle(\vec{h}, \vec{k}) \cong \angle(\vec{h}, \vec{k}),$$

is always true.

III, 5. If for two triangles ΔABC and $\Delta A'B'C'$ the congruences

$$AB \cong A'B', \quad AC \cong A'C', \quad \angle BAC \cong \angle B'A'C'$$

hold, then the congruence

$$\angle ABC \cong \angle A'B'C'$$

is also satisfied.

IV. (Lobachevski's Axiom). Let a be any line and A a point not on it. Then there are at least two lines in the plane, determined by a and A, that pass through A and do not intersect a.

V, 1. (Axiom of measure or Archimedes' Axiom). If AB and CD are any segments, then there exists a number n such that n segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B.

V, 2. (Axiom of line completeness). An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follow from Axioms I-III, and from V, 1 is impossible.

In our subsequent work we shall have need of several absolute-geometry results which are readily available in the literature. We shall state below those used. At times these results will not be referred to by number. Instead, the name of the theorem or a brief statement of the theorem will be given.

0.5 THEOREM. (The plane separation theorem) Given a line ℓ and a plane containing it, then the set of all points of the plane not on ℓ are partitioned into two classes called sides of the plane as determined by the given line ℓ. Two points A and B are on the same side of ℓ iff AB and ℓ have no point in common. Two points are on opposite sides if AB and ℓ have some point in common [5, p. 8].

0.6 THEOREM. (Crossbar theorem) If D is in the interior of $\angle$ BAC, then $\overrightarrow{AD}$ intersects BC [10, p. 69].

0.7 THEOREM. (Exterior angle theorem) Any exterior angle of a triangle is greater than either interior angle that is not adjacent to it [5, p. 21].

0.8 THEOREM. In every triangle the greater angle lies opposite the greater side [5, p. 22].

0.9 THEOREM. Every segment can be bisected [5, p. 23].

0.10 THEOREM. Given any finite number of three or more points on a line it is always possible to label them $A_1, A_2, \ldots, A_n$ in such a way that A_j is between A_i and A_k if $1 \leq i < j < k \leq n$. Besides this order of labeling there is only the reverse one that has the same property [5, p. 7-8].

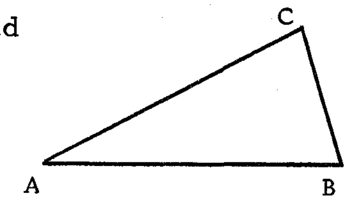
0.11 THEOREM. If two lines intersect, then there is a unique plane containing them [10, p. 39].

0.12 THEOREM. Every angle has exactly one bisector [10, p. 89].

0.13 THEOREM. There exists a unique perpendicular to a given line and containing a given point [10, p. 107].

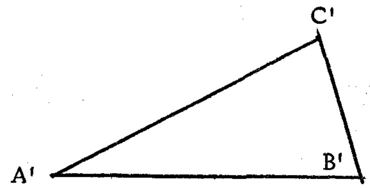
0.14 THEOREM. (Triangle Inequality) In any triangle $\triangle ABC$, $[AB] + [BC] > [AC]$ [10, p. 110].

0.15 THEOREM. (Polygonal Inequality) If $A_1, \ldots, A_n$ are any points then $[A_1A_2] + [A_2A_3] + \ldots + [A_{n-1}A_n] \geq [A_1A_n]$ [10, p. 125].



0.16 THEOREM. In triangles $\triangle ABC$ and $\triangle A'B'C'$, if

1. $AB \cong A'B'$, $BC \cong B'C'$, and $\angle B \cong \angle B'$ (called S.A.S.),



2. $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $AB \cong A'B'$, (called A.S.A.),
3. $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $BC \cong B'C'$ (called S.A.A.) or
4. $AB \cong A'B'$, $AC \cong A'C'$, and $BC \cong B'C'$ (called S.S.S.),

then $\triangle ABC \cong \triangle A'B'C'$ [5, p. 14-19; 10, p. 84-101].

0.17 THEOREM. If a line ℓ is perpendicular to each of two intersecting lines m and n at their point of intersection, then it is perpendicular to the plane containing m and n [10, p. 177].

0.18 THEOREM. Any two lines perpendicular to the same plane are coplanar [10, p. 179].

0.19 THEOREM. Given a point P and a plane α, there is a unique line ℓ perpendicular to α through P [10, p. 180].

0.20 THEOREM. If two planes have a point in common, then they have a line in common (a direct result of I, 6 and I, 7 above).

0.21 THEOREM. If ℓ is perpendicular to two distinct planes, then the planes do not meet [10, p. 183].

0.22 THEOREM. In plane α, if p, q, r are lines perpendicular to line ℓ at P, Q, and R, respectively, and p, q, r meet line ℓ' of α at points P', Q', R' respectively, then P-Q-R iff P'-Q'-R' [10, p. 136].

0.23 THEOREM. A line ℓ not containing any vertex of $\triangle ABC$, meets at most two of the sides of $\triangle ABC$ [10, p. 63].

0.24 THEOREM. If α is the perpendicular bisecting plane of a segment AB, then P is a point of α iff $AP \cong BP$ [10, p. 179].

Note: In 0.11 through 0.19 notions of segment and angle inequality are used in the proofs. These notions are explicitly given in Chapter 1 below.

I. SEGMENT CLASSES AND THE REAL NUMBERS--THE KEY ASSUMPTION

In this section we develop the map $| \ |_S$ which allows us to demonstrate an injection of the set of points on a line into the real numbers. This is done without recourse to the parallel axiom so the results are theorems of absolute geometry. Some other results of absolute geometry that will be assumed without proof in this and subsequent sections are:

1.1 THEOREM. The undefined relation "congruence" for segments, denoted by "$\cong$", is an equivalence relation [5, p. 18; 13; 14]. Also the undefined relation "congruence" for angles, denoted by "$\cong$", is an equivalence relation [5, p. 18; 14].

The following definitions from absolute geometry will be used (where the notation "A-B-C" is read "the point B is between the points A and C").

1.2 DEFINITION. The segment AB is <u>less than</u> the segment CD, denoted $AB < CD$, means there is a point P so that C-P-D and $AB \cong CP$.

1.3 DEFINITION. The angle $\angle ABC$ is <u>less than</u> the angle $\angle DEF$, denoted $\angle ABC < \angle DEF$, means there is a point P so that P is interior to $\angle DEF$ and $\angle ABC \cong \angle PEF$.

We use the symbols "[AB]" and "[$\angle$ ABC]" to denote the equivalence classes of congruent segments and angles containing the representatives AB and $\angle$ABC respectively. Further, we define addition for segments in the usual way [5, p. 51; 10, p. 247ff.].

1.4 DEFINITION. [AB] + [CD] is that class of segments [EF] so that there is a point P with E-P-F, AB $\cong$ EP, and CD $\cong$ PF. We write [AB] + [CD] = [EF].

The proof that this addition operation is well defined is also assumed as one of the "standard" results of absolute geometry. An ideal class [ ] is often introduced with the property that [ ] + [AB] = [AB]. This acts as an additive identity element. Furthermore [ ] < [AB] for every non-ideal class [AB].

It is possible to establish the existence of the midpoint of a segment without using the parallel axiom (0.9) so for any segment AB one can always find a segment CD so that [CD] + [CD] = [AB]. With this in mind, the next definition is intuitively very reasonable.

1.5 DEFINITION. For any segment AB, the class of segments denoted by "1/2[AB]" is the class of segments [CD] so that if M is the midpoint of AB, then CD $\cong$ AM. Inductively we define $1/2(1/2^n[AB]) = 1/2^{n+1}[AB]$.

We have an obvious ordering for the segment classes.

1.6 DEFINITION. $[AB] < [CD]$ means $AB < CD$.

The notation $n[AB]$ will, as usual, mean $[AB] + \ldots + [AB]$ with n summands.

These definitions are all independent of the representatives of the classes considered in any given instance. To conserve space, this verification is not carried out here since the method is well known and does not contribute any added insight into the arguments or statements.

1.7 LEMMA. If $[AB] < [CD]$, then for any class $[EF]$, $[AB] + [EF] < [CD] + [EF]$.

Proof: Let $[GH] = [CD] + [EF]$. Then by definition, there is a point P of GH so that $GP \cong EF$ and $PH \cong CD$. Now $[AB] < [CD]$ implies there is a point Q of PH so that $PQ \cong AB$. This gives us $G\text{-}P\text{-}Q\text{-}H$ with $[GQ] \cong [AB] + [EF]$ and we are done.

1.8 LEMMA. If $[AB] < [CD]$ and $[EF] < [GH]$, then $[AB] + [EF] < [CD] + [GH]$.

Proof: By 1.7, $[AB] + [EF] < [CD] + [EF]$ and $[CD] + [EF] < [CD] + [GH]$. By transitivity the proof is complete.

1.9 LEMMA. If $[AB] \leq [CD]$ then $n[AB] \leq n[CD]$.

Proof: For equality the proof is obvious. Otherwise use induction.

If $n = 1$ we are done. Suppose $k[AB] < k[CD]$. Then $(k+1)[AB] = [AB] + k[AB] < [CD] + k[CD] = (k+1)[CD]$ by 1.8 and by induction we are done.

1.10 LEMMA. If AB and CD are any segments, then there is an integer k so that $1/2^k[AB] < [CD]$.

Proof: Suppose that for every integer k $1/2^k[AB] \geq [CD]$. It then follows, by 1.9, that $[AB] \geq 2^k[CD]$. Now for every integer $m > 0$ there is an integer k so that $2^k > m$. We conclude that CD is a segment so that for every sequence $A = A_0, A_1, A_2, \ldots, A_{2^k}$ of points of $\overrightarrow{AB}$ with $A_i\text{-}A_j\text{-}A_k$ $(0 \leq i < j < k \leq m)$ and $A_{i-1}A_i \cong CD$, we have $AA_m < AA_{2^k} \leq AB$ contradicting Archimedes' axiom.

1.11 LEMMA. If $[AB] < [CD]$ then there is a class $[EF]$ so that $[AB] + [EF] = [CD]$.

Proof: $[AB] < [CD]$ implies that there is a point E of CD so that $[AB] = [CE]$. Now $[CE] + [ED] = [CD]$ by Axiom III, 3 in 0.4. $[AB] + [ED] = [CD]$ and we are done.

We now proceed with some definitions leading closer to the proof of the existence of an injection of the set of points on any line into the real numbers.

1.12 DEFINITION. Let $[AB]$ and $[CD]$ be any segment classes. $\mu([AB],[CD])$ = the greatest integer k so that $k[AB] \leq [CD]$.

Clearly the value of μ is 0 if $[AB] > [CD]$. If $[AB] \leq [CD]$ Archimedes' axiom assures us that there is a number m so that $m[AB] > [CD]$. Since the set of integers is well ordered, there is a least such number, M, so that k is always $M-1$. The fact that μ is well defined can be proved by the usual argument showing that the choice of representative can be arbitrary.

An intuitive formulation of the meaning of μ is that μ counts the maximum number of contiguous segments of a given class that do not reach beyond the end of a given segment.

1.13 DEFINITION. Let $[AB]$ and $[CD]$ be any segment classes so that $[CD] < [AB]$. Let n be any integer. We inductively define the three-variable function U to be

$$U([AB],[CD],n) = \begin{cases} 0 & \text{if } [CD] \text{ is the "ideal" class so that } [CD] + [AB] = [AB] \\ 0 & \text{if } n = 1 \text{ and } 1/2[AB] > [CD] \\ 1 & \text{if } n = 1 \text{ and } 1/2[AB] \leq [CD] \\ \left.\begin{matrix} 0 & \text{if } n > 1 \\ 1 & \text{if } n > 1 \end{matrix}\right\} & \text{and } 1/2^n[AB] + \displaystyle\sum_{i=1}^{n-1} U([AB][CD],i)\, 1/2^i[AB] \begin{cases} > [CD] \\ \leq [CD] \end{cases} \end{cases}$$

The function U gives us a way to assign a real number less than one and equal to or greater than zero to any class [CD] less than [AB] including the ideal segment class which acts as the additive identity in the arithmetic of the segment class addition. The proof that U is well defined is left out, again, because of the standard and messy nature of the argument.

1.14 THEOREM. If the number 1 is assigned to a given (non-ideal) segment class [AB], then for each class of segments [CD] < [AB] there is a unique real number $r \in [0, 1)$ denoted by $r = m([CD])$, so that if

$$S = \left\{ S_n : S_n = \sum_{i=1}^{n} 1/2^i U([AB], [CD], i) \right\},$$

then

$$r = \text{l.u.b.} S$$

Proof: a. S is a non-empty set of real numbers since

$$S_1 = 1/2\ U([AB], [CD], 1) = 0 \text{ or } 1.$$

b. S is bounded above by one since $S_n \leq \sum_{i=1}^{n} 1/2^i = 1 - 1/2^n < 1$ by a simple induction argument.

c. Since by "a" and "b" S is a non-empty set of real numbers bounded above by one, then r exists and is equal to or less than one.

d. $r < 1$ since $[CD] < [AB]$ means there is a non-ideal class $[DE]$ so that $[CD] + [DE] = [AB]$. Now for some positive integer k, $1/2^k[AB] < [DE]$ by 1.10. We observe that

$$r = \text{l.u.b. } S = \lim_{n \to \infty} S_n \leq \lim_{n \to \infty} \left(\sum_{i=1}^{n} 1/2^i \right) - 1/2^k < 1 - 1/2^{k+1} < 1$$

e. If $[CD]$ is not the ideal class, $m[CD] > 0$ since by 1.10 there is a k so that $1/2^k[AB] < [CD]$. Thus $U([AB],[CD],n) = 1$ for at least one value $n \geq k$ so $r > 0$.

f. If $[CD]$ is the ideal class, then $U([AB],[CD],n) = 0$ so $r = 0$.

g. For each class $[CD]$, r is unique. r has the infinite base-two numeral as its expression. The definition of U does not allow even two different expressions of r, let alone two different values.

h. r cannot be associated with two different classes either. This follows since if $r = m([CD]) = m([EF])$ and $[CD] \neq [EF]$ we show a contradiction occurs. Without loss of generality let $[CD] < [EF]$. This means there is a non-ideal class $[DG]$ so that $[CD] + [DG] = [EF]$ and there is an integer k so that $1/2^k[AB] < [DC]$. Let

$$S_n = \sum_{i=1}^{n} 1/2^i U([AB][CD], i) \quad \text{and} \quad T_n = \sum_{i=1}^{n} 1/2^i U([AB],[EF], i).$$

Then $r = \lim_{n\to\infty} S_n < \lim_{n\to\infty} S_n + 1/2^{k+1} < \lim_{n\to\infty} T_n = r$ which is impossible.

This theorem gives us a unique number r, $0 \leq r \leq 1$, associated with every class $[EF] \leq [AB]$ provided that 1 is associated with $[AB]$. We now extend this notion to give a unique number $r \in [0,\infty)$ associated with each segment class $[CD]$.

1.15 THEOREM. Let $[AB]$ be some (non-ideal) class assigned the number 1. Let $[CD]$ be any class. Let $[C'D']$ be the class so that $[C'D'] + \mu([AB][CD])[AB] = [CD]$. Then $[C'D'] < [AB]$ and there is a unique real number $r = |[CD]|_S = \mu([AB],[CD]) + m[C'D']$ associated with $[CD]$ and $r \in [0,\infty)$.

Proof: By the definition of μ in 1.12 $\mu([AB],[CD])[AB] \leq [CD]$ and $\mu([AB],[CD])$ is the largest integer so that this is true. If $\mu([AB],[CD])[AB] = [CD]$ then $[C'D']$ is the ideal class and by definition $[C'D'] < [AB]$. If $\mu([AB],[CD])[AB] < [CD]$, then $[C'D'] < [AB]$ since if not, we have a contradiction to the definition of μ.

If $[CD] < [AB]$, $\mu([AB][CD]) = 0$ and the real number $|[CD]|_S = m([CD])$ is unique by 1.14.

If $[CD] \geq [AB]$, $|[CD]|_S$ is unique by 1.14 and the definition of μ. It is clearly non-negative. Note: Implicitly, the definition of μ draws on Archimedes' axiom to insure that every segment determines a value for $\mu([AB],[CD])$ and that $|[CD]|_S$ can assume arbitrarily large values.

At this point we have proved that there is an injective map $|\ |_S$ from the set of segment classes into the non-negative real numbers. This is now extended to an injection of the set of points on a line into $\mathbb{R}$. From what has already been done we see that the following definition gives a unique assignment of points to real numbers.

1.16 DEFINITION. Let A and B be distinct points on a line ℓ. Let $[AB]$ be assigned the number one. If P is a point on the ray $\overrightarrow{AB}$ of ℓ, assign the number $|[AP]|_S$ to P. If Q is a point of ℓ so that Q-A-B then assign the number $-|[AQ]|_S$ to Q. Assign 0 to the point A.

We have at this point established an injection of the set of points on any line into the set of real numbers. To establish a proof that the completeness axiom is a theorem in the Poincaré model we will now assume that this injection can be extended to a one-to-one, order preserving correspondence without introducing a contradiction. To establish the freedom from contradiction we will show that this assumption allows us to construct a model upon the basis of the axioms of the geometry and show this model is necessarily isomorphic to the Poincaré model by giving a canonical method of establishing an isomorphism between these and any other models.

It is well to note that as one develops the arithmetic of addition of segment classes in the absolute geometry, a commutative group structure can be established (not carried out in this paper) which makes the injection given above into an injective group homomorphism. The one-to-one correspondence we assume is the order preserving map which is the extension of this homomorphism to an isomorphism.

The rationale for not including in this paper a verification that the above injection is also a group homomorphism is that, first, the method is essentially standard, and second, the fact that the extension is an isomorphism is not a central part of the argument of the remainder of the proof. Unlike in Euclidean geometry, we do not have an easily developed multiplication leading to a field structure [5, p. 131-149]. The field structure plays an important role in establishing the completeness axiom as a theorem in the analytic geometry model of Euclidean geometry but the Euclidean parallel axiom plays a central role in this proof [5]. Once we have a field structure for the ordered arithmetic of the segment classes, the completeness axiom in the geometry assures us that this field is the real number field (up to isomorphism) and the Dedekind property for segments, rays, lines, etc. follows. Such a derivation of the Dedekind property in Lobachevskian geometry seems impossible. The procedure outlined earlier (which is based on the assumption of the one-to-one correspondence described) is a method of establishing proof without having a field structure within the geometry. What we really need in our method is the Dedekind property. Our assumption gives us this.

It is often hard in a paper such as this to decide how detailed the background should be. One is disinclined to prove all background theorems unless their proofs involve an essentially different method of argument dictated by the hypotheses or axiom system adopted or unless the theorem or proofs are not readily accessible in the literature. The choice to leave out or include a given theorem is usually somewhat arbitrary. For example the proof that for $n \geq 3$, one can order n-points on a line (used in Lemma 1.7, Lemma 1.10, the definition of $n[AB]$, and the definition of addition of segment classes) is not included. This is not easy to prove, but is accessible in the literature (e.g. [5, p. 7-8]). Furthermore the proofs given in the literature are essentially the same as that which would be done here, so inclusion of such a proof would not be particularly instructive.

Step "(1)" of the argument proving that the completeness axiom is a theorem in the Poincaré model is now done. The construction of

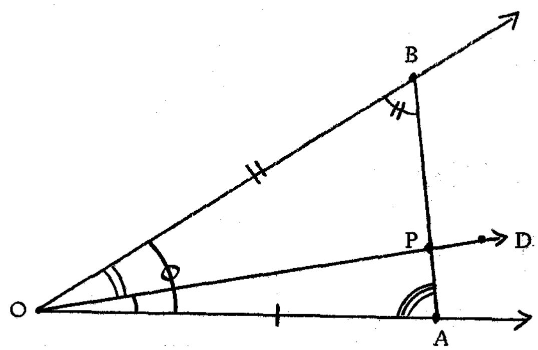
"$| \;|_S$" assures us that we can have no more points than real numbers. We will prove the hypothesis that, "There are exactly as many points as real numbers," does not lead to a contradiction. This hypothesis will be called our "key assumption."

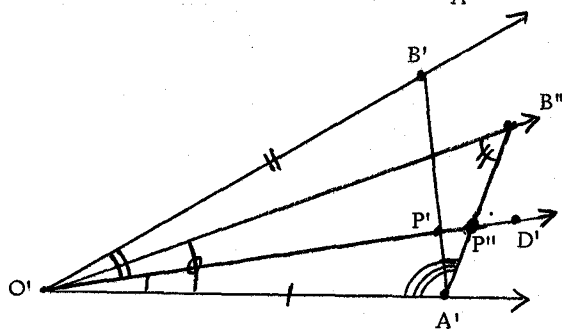
II. ANGLE CLASSES AND THE MEASURE OF ANGLES

The result from absolute geometry that congruence for angles is an equivalence relation [5, p. 18; 14] allows us to define equivalence classes of congruent angles. An addition for angle classes is harder to formulate than for classes of segments since one cannot reasonably expect the "sum" of two angles--whatever that means--to always be non-ambiguous (consider "adding" two obtuse angles). If two angles are acute then an addition can be described but in general it is not closed. Two acute angles, each larger than "half a right angle" could only "sum" to an angle greater than a right angle so the sum is not an acute angle. These observations, of course, draw upon the notion of the measure of the "angle" of rotation relative to some fixed reference position such as is used in trigonometry and analysis. This notion is quite different from the notion of angle in geometry, though both are certainly related (at least intuitively).

Considerable care must be exercised to avoid the inadvertent interchange of <u>angle</u>--the defined object of geometry--and <u>angle</u> in the sense of a directed rotation or the real value of a "wrapping" function as in trigonometry and analysis. To avoid this, it is sometimes necessary to include methods of argument that seem unnecessarily involved or even obscure unless the reader remembers that such an interchange of the separate notions of angle is at best logically dangerous. The so-called angle sum theorem of absolute geometry states:

2.1 THEOREM. If D and D' are points in the interior of angles $\angle AOB$ and $\angle A'O'B'$ respectively and $\angle AOD \cong \angle A'O'D'$ and $\angle DOB \cong \angle D'O'B'$, then $\angle AOB \cong \angle A'O'B'$.

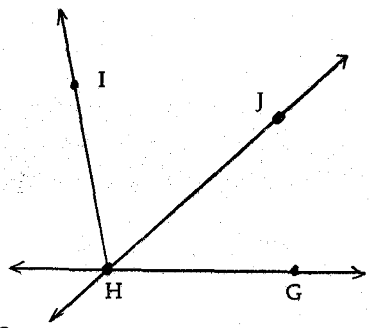




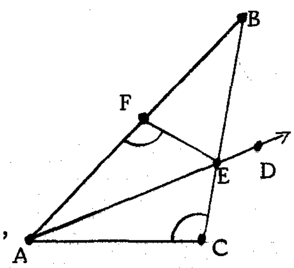
Proof: Without loss of generality we may suppose $OA \cong O'A'$ and $OB \cong O'B'$. By the cross-bar theorem (0.6) rays $\overrightarrow{OD}$ and $\overrightarrow{O'D'}$ meet AB and $A'B'$ in points P and P' respectively. Let P'' be the point of $\overrightarrow{O'D'}$ so that $OP \cong OP''$. Then $\triangle OAP \cong \triangle O'A'P''$ by SAS so $\angle OAP \cong \angle O'A'P''$. On $\overrightarrow{A'P''}$ let B'' be the point so that $AB \cong A'B''$. Then $\triangle OAB \cong \triangle O'A'B''$ by SAS and $OB \cong O'B''$. By "segment subtraction theorem $BP \cong B''P''$ and $\triangle OPB \cong \triangle O'P''B''$ by S.S.S. and therefore $\angle P''O'B'' \cong \angle POB$ and $\angle POB \cong \angle P'O'B'$ forces $\angle P''O'B'' \cong \angle P'O'B'$. Because B' and B'' are on the same side of line $\underline{O'D'}$, $B' = B''$ (result of Axioms III-1, III-2) and $\angle AOB \cong \angle A'O'B'$ as desired.

2.2 DEFINITION. Let $[\angle ABC]$ and $[\angle DEF]$ be classes of acute angles. The class $[\angle ABC] + [\angle DEF]$ is the class of angles $[\angle GHI]$ so that there is a ray $\overrightarrow{HJ}$ interior to $\angle GHI$ with $\angle GHJ \cong \angle ABC$ and $\angle JHI \cong \angle DEF$.

This definition is independent of representatives and clearly makes sense as we see below. Let $\overrightarrow{HG}$ be any ray. On a given side of line $\underline{HG}$ (for notation see 0.3) we have a unique ray $\overrightarrow{HJ}$ so that $\angle GHJ \cong \angle ABC$. Since $\angle GHJ$ is acute its supplement is obtuse and the interior of the supplement is defined by the ray of $\underline{HG}$ from H and on the opposite side of $\underline{HJ}$ from G, we have a ray $\overrightarrow{HI}$ so that $\angle JHI \cong \angle DEF$. The rays $\overrightarrow{HI}$ and $\overrightarrow{HJ}$ are on the same side of $\underline{GH}$ and thus $\angle GHI$ is an angle with the right characteristics to allow us to use the angle "sum" theorem to prove the definition is non-ambiguous and reasonable.



2.3 LEMMA. If $\triangle ABC$ is a triangle with a non-acute angle at C, and $\overrightarrow{AD}$ is the bisector of $\angle BAC$, then $\overrightarrow{AD}$ meets BC at E and $CE < EB$.

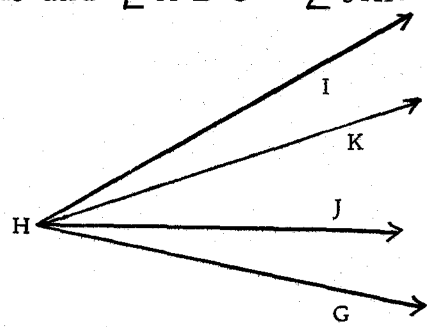


Proof: The existence of E is assured by the cross-bar theorem. Since $\angle ACE$ is non-acute, referring to $\triangle ABE$ we have $AC < AE$ and referring to $\triangle ABC$ we have $AC < AB$. (The two inequalities for segments follow from the absolute geometry theorems that state that the angle sum of any triange is no greater than two right angles and that the greatest side is opposite the greatest angle) $AC < AB$ implies there is a point F on AB so that $AF \cong AB$. Thus $\triangle AEC \cong \triangle AEF$ by SAS. Now $\angle ABC < \angle BFE$ since $\angle BFE$ is congruent to the exterior angle at C of $\triangle ABC$. Finally $EF < EB$ and $CE \cong EF$ implies $CE < EB$ as claimed.

2.4 LEMMA. If $[\angle ABC]$, $[\angle A'B'C']$, $[\angle DEF]$ are acute angle classes with $\angle ABC < \angle A'B'C'$, then

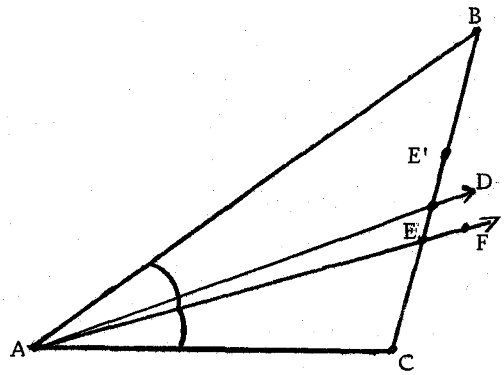
$$[\angle DEF] + [\angle ABC] < [\angle DEF] + [\angle A'B'C']$$

Proof: $[\angle A'B'C'] + [\angle DEF]$ is defined to be the angle class containing the representative $\angle GHI$ so that there is a ray $\overrightarrow{HJ}$ interior to $\angle GHI$ with $\angle DEF \cong \angle GHJ$ and $\angle A'B'C' \cong \angle JHI$. Since $[\angle ABC] < [\angle A'B'C']$ there is a ray $\overrightarrow{HK}$ interior to $\angle JHI$ so that $\angle JHK \cong \angle ABC$. By the angle "addition" theorem $[\angle GHK] = [\angle ABC] + [\angle DEF]$ and $[\angle GHK] < [\angle GHI] = [\angle A'B'C'] + [\angle DEF]$ and we are done.



2.5 LEMMA. If $\triangle ABC$ has a non-acute angle at C, and if D is the midpoint of BC, then $\angle BAD < \angle DAC$.

Proof: Let $\overrightarrow{AF}$ be the angle bisector of $\angle BAC$. $\overrightarrow{AF}$ meets BC at E by the cross-bar theorem. By Lemma 2.3 $CE < EB$. Let E' be the point of EB so that $BE' \cong EC$. Let M be the midpoint of EE'. By segment "addition" theorem we have $MB \cong MC$ so $M = D$. With the ordering given by these results we have C-E-D-E'-B [5, p. 7-8, th 6]. Thus $\angle DAC > \angle CAE$ and $\angle CAE \cong \angle BAE > \angle BAD$ and we are done by transitivity.



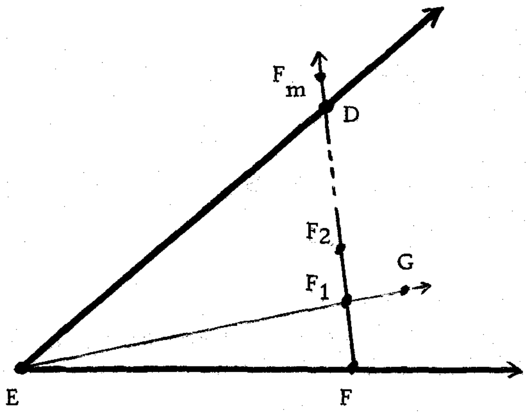
2.6 LEMMA. If $n[\angle ABC] < [\angle rt]$, then there is an angle class $(n+1)[\angle ABC] = n[\angle ABC] + [\angle ABC]$.

Proof: If $n = 1$ then by definition $(n+1)[\angle ABC]$ exists. If $n > 1$ then clearly $\angle ABC < \angle rt$ by definition and we are done by definition 2.2.

Clearly, $(n+1)[\angle ABC]$ may be a non-acute angle class and in that case $(n+2)[\angle ABC]$ may or may not make sense. The next lemma assures us that we can always make $n[\angle ABC]$ "exceed" any acute angle class provided $\angle ABC < \angle rt$.

2.7 THEOREM. If $[\angle ABC]$ and $[\angle DEF]$ are any acute angle classes, then there is a positive integer n so that $n[\angle ABC] > [\angle DEF]$.

Proof: We need to show that $n[\angle ABC]$ exists and that the given inequality holds for some n. If $\angle DEF \leq \angle ABC$ take $n = 2$. If $\angle DEF > \angle ABC$ let $\overrightarrow{EG}$ be the ray interior to $\angle DEF$ so $\angle FEG \cong \angle ABC$. Let F_1 be the point of $\overrightarrow{EG}$ on DF (whose existence is assured by the cross-bar theorem). By Archimedes' axiom



we have a sequence of points $F_0 = F, F_1, \ldots, F_m$ on $\overrightarrow{FD}$ so that $F_i\text{-}F_j\text{-}F_k$ if $0 \leq i < j < k \leq m$ and $F_{i-1}F_i \cong FF_1$ for each $i = 1, \ldots, m$, and $FF_m > FD$. By the angle addition theorem we get

$$\sum_{i=0}^{m} [\angle F_iEF_{i+1}] = [\angle FEF_m] > [\angle FED]$$

By Lemma 2.3 we know $[\angle F_iEF_{i+1}] \leq [\angle ABC]$, for $0 \leq i < m-1$ and by 2.4

$$[\angle F_{i-1}EF_i] + [\angle F_iEF_{i+1}] < 2[\angle ABC]$$

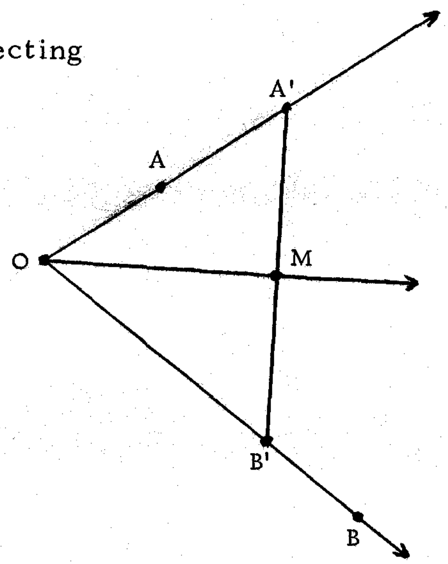
for $0 < i < m-1$. Now if $m[\angle ABC]$ is defined, we have

$$m[\angle ABC] > \sum_{i=1}^{m} [\angle F_{i-1}EF_i] = [\angle FEF_m] > [\angle FED]$$

and we are done.

From 2.6 we know that if $k[\angle ABC] < [\angle rt]$, then $(k+1)[\angle ABC]$ is defined so for $m[\angle ABC]$ to be undefined we must have some value $n < m$ so that $n[\angle ABC]$ is defined and $n[\angle ABC] \geq [\angle rt]$. Thus there is a number n satisfying the hypothesis because $[\angle DEF]$ is an acute angle class. This completes our proof.

Once the properties of segment, angle, and triangle congruence are developed even a little, one can show that every angle has a bisector. This is done by selecting points A' and B' on the sides of $\angle AOB$ so that $OA' \cong OB'$. Letting M be the midpoint of $A'B'$ gives one triangle $\triangle A'OM \cong \triangle B'OM$. This ensures that $\angle A'OM \cong \angle B'OM$ and $\overrightarrow{OM}$ is called the bisector of $\angle AOB$. (0.12)



2.8 DEFINITION. Let $[\angle AOB]$ be any angle class. Let $1/2[\angle AOB]$ be the angle class containing the representative $\angle AOM$ where $\overrightarrow{OM}$ is the bisector of angle $\angle AOM$. Inductively define the class $1/2^{k+1}[\angle AOB] = 1/2\,(1/2^{k}[\angle AOB])$ where k is a positive integer.

2.9 THEOREM. If $[\angle ABC]$ and $[\angle DEF]$ are any classes of angles, then there is an integer k so that $1/2^k[\angle ABC] < [\angle DEF]$.

Proof: If $\angle DEF \geq \angle rt$ then pick $k = 2$. Suppose $\angle DEF < \angle rt$. Without loss of generality we may suppose $\angle ABC$ is also acute (since in any case $1/2[\angle ABC]$ is a class of acute angles and the powers of k would at most be increased by one). If no such k existed we would always have $1/2^k[\angle ABC] \geq [\angle DEF]$, i.e., $[\angle ABC] \geq 2^k[\angle DEF]$ for all k. This contradicts Theorem 2.7 and we are done.

Putting Theorems 2.7 and 2.9 together, we can now establish the map "$| \ |_A$" described in the introduction.

2.10 DEFINITION. Let $[\angle rt]$ be the class of right angles. Let $[\angle ABC]$ be any non-obtuse angle class. Inductively define the two-variable function η to be

$$\eta([\angle ABC],n) = \begin{cases} 0 \text{ if } [\angle ABC] \text{ is the ideal class so that} \\ \qquad [\angle ABC] + [\angle rt] = [\angle rt] \\ 0 \text{ if } n = 1 \text{ and } 1/2[\angle rt] > [\angle ABC] \\ 1 \text{ if } n = 1 \text{ and } 1/2[\angle rt] \leq [\angle ABC] \\ \left.\begin{matrix} 0 \text{ if } n > 1 \\ 1 \text{ if } n > 1 \end{matrix}\right\} \text{ and} \\ \qquad 1/2^n[\angle rt] + \sum_{i=1}^{n-1} \eta([\angle ABC],i)1/2^i[\angle rt] \begin{cases} > [\angle ABC] \\ \leq [\angle ABC] \end{cases} \end{cases}$$

The function η gives us a way to assign a real number r in the half open interval $(0, \pi/2]$ uniquely to every angle class which is not obtuse. (Again the "well-definedness" argument is left out.)

2.11 THEOREM. Let $|[\angle rt]|_A = \pi/2$. Let $[\angle ABC]$ be any class of non-obtuse angles. Then there is a unique real number $r = |[\angle ABC]|_A$ so that $0 < r \leq \pi/2$ and

$$r = \text{l.u.b.} \left\{ S_n : S_n = \pi/2 \sum_{i=1}^{n} 1/2^i \eta([\angle ABC], i) \right\}$$

Proof: (Using 2.7 and 2.9 together with 2.10 the proof is essentially identical to the proof of 1.14 so will not be repeated here.)

2.12 THEOREM. If we assume the extension of the injection $|\ |_S$ to all real numbers, then the injection $|\ |_A$ is a surjection onto $(0, \pi/2]$.

Proof: Let $\angle POQ$ be a right angle. Let r be any number between 0 and $\pi/2$ and suppose there is no angle class $[\angle ABC]$ corresponding to r. Let $S = \{[\angle DEF]: |[\angle DEF]|_A < r\}$ and let $T = \{[\angle GHI]: |[\angle GHI]|_A > r\}$. By the use of the cross-bar theorem we immediately get two sets

$\underline{S} = \{[OF']: F'$ is the point of PQ so that $[\angle POF'] = [\angle DEF]\}$

$\underline{T} = \{[OI']: I'$ is the point of PQ so that $[\angle POI'] = [\angle GHI]\}$.

Clearly, every class in $\underline{S}$ is less than every class of $\underline{T}$ from the definition of $| \ |_A$ and ordering of angle classes. Using $| \ |_S$ we get two more sets

$$S' = \{s = |[OF']|_S : [OF'] \in \underline{S}\}$$
$$T' = \{t = |[OI']|_S : [OI'] \in \underline{T}\}$$

S' and T' are sets of reals so that the Dedekind hypothesis is satisfied so there is a number k so that k is the first element of T' or the last element of S'. By our "extension" hypothesis there is a point K on PQ so that $[PK]$ corresponds to k and hence an angle class $[\angle POK]$ corresponding to r (a contradiction) and the theorem is proved.

Based on our key assumption that "$| \ |_S$" is a bijection and the results of 2.12 we can extend "$| \ |_A$" to $(0, \pi)$ in the usual manner by the following definition of the extension of "$| \ |_A$" to all classes.

2.13 DEFINITION. Let $[\angle ABC]$ be any angle class. If $[\angle ABC]$ is a non-obtuse angle class then $E|[\angle ABC]|_A = |[\angle ABC]|_A$. If $[\angle ABC]$ is an obtuse angle class, then let $\underline{[\angle ABC]}$ be the class of supplements of representatives of $[\angle ABC]$. Define $E|[\angle ABC]|_A = \pi - |\underline{[\angle ABC]}|_A$. In all subsequent work we will use the symbol "$| \ |_A$" to mean "$E| \ |_A$".

This concludes part (b) of the steps of the argument outlined in the introduction since now $| \; |_A$: Set of angle classes $\rightarrow (0, \pi)$.

III. THE ANGLE OF PARALLELISM AND THE "LOBACHEVSKIAN FUNCTION" Π

In the two preceding chapters we have established an assignment of numbers to segment classes and to angle classes. Implicitly this gives us the notion of measure for both segments and angles and the coordinatization of any line. This is because (based on the key assumption) we can extend the injections "$| \, |_S$" and "$| \, |_A$" to surjections as described earlier. We can thus use results established on the basis of "metric" or "ruler" axioms such as in Moise [10] and even the results of measure concepts as used by Lobachevski [9]. We shall draw upon these sources to make possible a somewhat shorter development than would be possible if these same results were proved in this paper also. There seems no good reason to redo that work which can be obtained directly from the literature once the results of Chapters 1 and 2 are established. Our present goal is to establish the Lobachevskian function Π. Whenever the literature follows a path that would require significant other details than those given there, those details will be developed independently in this paper.

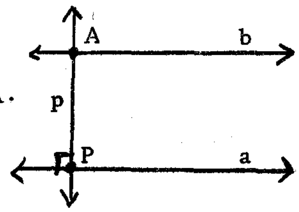
Euclid essentially proved the following theorems of absolute geometry. (These have later been proved on the more logically sound basis of Hilbert axioms) [10, p. 93-97].

If α is a plane, a is a line of α, and A is a point of α not on a, then

(1) There is a unique line p of α that is incident on A and perpendicular to line a at a point P of line a (P is called the foot of the perpendicular from A to a).

(2) There is a unique line b of plane α that is perpendicular to line p at A.

(3) The line b does not meet line a.

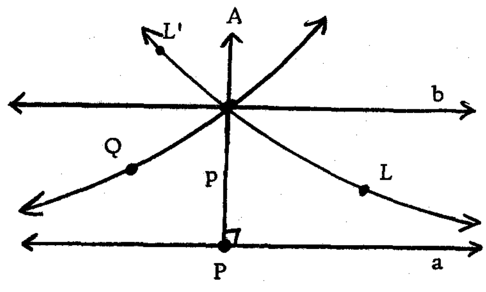


The following lemma is an easy result of the Lobachevskian parallel axiom and the results (1), (2), (3) above.

3.1 LEMMA. Let α, a, A, and p be as above. Then in each half plane of α determined by p, there are points L and Q on opposite sides of line p in plane α so that

(1) $\angle$ PAL and $\angle$ PAQ are both acute and

(2) $\overrightarrow{AL}$ and $\overrightarrow{AQ}$ neither meet line a.



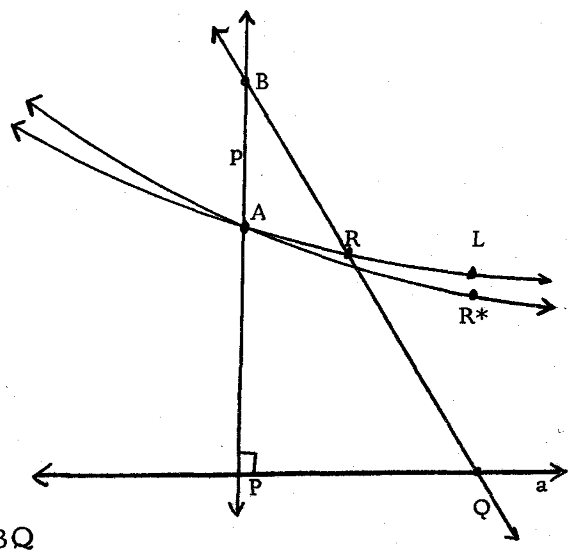
Proof: By (3) above, the perpendicular b to p at A in α does not meet line a. Lobachevski's parallel axiom assures us there is another line ℓ in α so that ℓ is different from b, incident on A, and does not meet a. Let $\overrightarrow{AL}$ and $\overrightarrow{AL'}$ be the opposite rays from A on ℓ. Since ℓ is not the perpendicular to p at A, either $\angle$ PAL or $\angle$ PAL' is acute. Renaming if necessary, suppose $\angle$ PAL is acute. $\overrightarrow{AL}$ does not meet a since

ℓ does not meet a. By Axiom III-4, there is a unique ray $\overrightarrow{AQ}$ in the half-plane of α determined by p and not containing L so that $\angle PAQ \cong \angle PAL$. $\overrightarrow{AQ}$ does not meet line a, because if it did meet line a at some point S, then $\overrightarrow{AL}$ must meet line a at a point T so that S-P-T and $SP \cong PT$. This proves the lemma.

3.2 THEOREM. Let a, α, A, and p be as above. On a given side of p in plane α, there is a unique ray $\overrightarrow{AR^*}$ so that

1. $\overrightarrow{AR^*}$ does not meet line a,
2. $\angle PAR^*$ is an acute angle,
3. if $\overrightarrow{AS}$ is any ray of α on the same side of p as R* and $\angle PAS < \angle PAR^*$, then $\overrightarrow{AS}$ meets line a at a point of the ray $\overrightarrow{PQ}$ of line a on R*'s side of p.

Proof: Let Q be a point of line a distinct from P on the given side of p. Let B be a point so that P-A-B. By Lemma 3.1, there is a point L of plane α on Q's side of p so that $\overrightarrow{AL}$ does not meet $\overrightarrow{PQ}$ and so that $\angle PAL$ is acute. Thus $\overrightarrow{AL}$ does not meet PQ, and by Pasch's axiom line $\underline{AL}$ meets BQ



at R. But R is on Q's side of p so R is a point of $\overrightarrow{AL}$. Let

$$D = \{S : S = R \text{ or } S \text{ is a point of } QR \text{ so that } \overrightarrow{AS} \text{ does not meet } \overrightarrow{PQ}\}$$

$$E = (RQ \cup \{R\}) - D.$$

Let us order these sets as follows, $S < T$ iff $QS < QT$. Now $D \neq \phi$ since $R \in D$, $E \neq \phi$ since for every Q' of $\overrightarrow{PQ}$ with P-Q-Q', $\overrightarrow{AQ'}$ meets RQ at a point Q''. Further, $D \cup E = RQ \cup \{R\}$, $D \cap E = \phi$ and $S < T$ for every element S in E and T in D.

Now E has no last element since if S' is such a last element $\overrightarrow{AS'}$ meets $\overrightarrow{AQ}$ at K. Let K' be any point so that A-K-K' and $\overrightarrow{AK'}$ meets RQ at a point S'' so that Q-S'-S'' and $S'' > S'$ in E.

We now make our first use of our assumption. We claim D necessarily has a first element since otherwise the correspondence with the real numbers will give us a Dedekind class of reals with no first element in a set that <u>must</u>, in the reals, have a first element. Hence there is a point $R*$ of D so that all these necessary conditions for the theorem are satisfied. Uniqueness follows immediately from the construction of the argument.

3.3 DEFINITION. In the notation of Theorem 3.2, the angle

$\angle$ PAR* is called the angle of parallelism associated with PA and line AR* is said to be parallel to line PQ in the direction of $\overrightarrow{PQ}$.

The results of Theorem 3.2 and the definition in 3.3 are included in Section 16 of Lobachevski's The Theory of Parallels [9]. His proof is valid if the Dedekind property is assumed instead of the completeness axiom of Hilbert. Lobachevski attaches metric notions to his treatment at this point by defining the angle of parallelism $\Pi(|[AP]|_S) = |[\angle APR^*]|_A$ (in terms of this paper's notation). Please realize that this is intended only to explain this paper's steps and no way is meant to play down the imaginative work done by Lobachevski. As noted earlier, we must justify the claim that there is in fact an angle which can rightfully be called THE angle of parallelism for a given segment class. The results given above justify such a claim.

Lobachevski's next result--in his Section 17--gives:

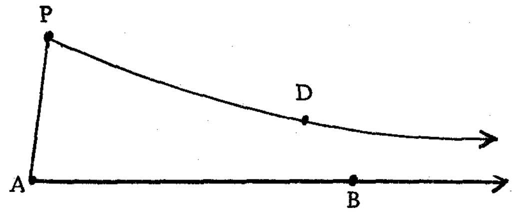
3.4 THEOREM. "A straight line maintains the characteristic of parallelism at all its points" [9, p. 15].

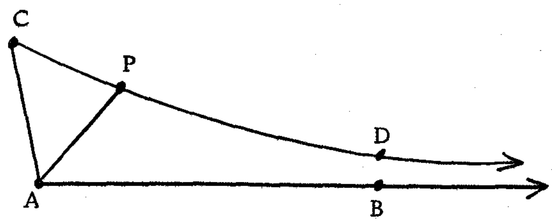
His proof is valid from the standpoint of our present work. He then gives the following theorem:

3.5 THEOREM. "Two lines are always mutually parallel" [9, p. 16].

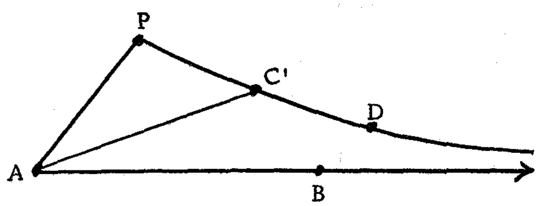
In the proof he gives for this theorem he gives one step in which one must "...slide the figure EFAB until it coincides with AG, ..." This "sliding" can be precisely stated and proved in several ways. One such way is given by Moise [10] in which he proves the following results. We use his more operational notation:

3.5.0 DEFINITION. Let $\underline{AB}$ and $\underline{PD}$ be two lines in a plane α which do not meet. Then $\overrightarrow{PD} \cup PA \cup \overrightarrow{AB} \cup \{A, P\}$ is called the <u>open triangle</u> $\Delta DPAB$ provided B and D are on the same side of $\underline{AP}$. If every interior ray of $\angle APD$ intersects $\overrightarrow{AB}$, we say $\overrightarrow{PD}$ is critically parallel to $\overrightarrow{AD}$ and write $\overrightarrow{PD} | \overrightarrow{AB}$ [10, p. 311-312].





3.5.1 THEOREM. If $\overrightarrow{PD} | \overrightarrow{AB}$ and C-P-D, then $\overrightarrow{CD} | \overrightarrow{AB}$ [10, p. 312].



3.5.2 THEOREM. If $\overrightarrow{PD} | \overrightarrow{AB}$ and P-C'-D, then $\overrightarrow{C'D} | \overrightarrow{AB}$ [10, p. 313].

3.5.3 DEFINITION. Two rays $\vec{r}$ and $\vec{r}\,'$ are <u>equivalent</u> rays if one contains the other [10, p. 313].

3.5.4 THEOREM. If $\vec{r}$ is equivalent to $\vec{r}\,'$ and $\vec{r} \mid \overrightarrow{AB}$, then $\vec{r}\,' \mid \overrightarrow{AB}$. [10, p. 313].

3.5.5 THEOREM. If $\vec{r}$ and $\vec{r}\,'$ are equivalent rays, $\vec{s}$ and $\vec{s}\,'$ are equivalent rays, and $\vec{r} \mid \vec{s}$, then $\vec{r}\,' \mid \vec{s}\,'$ [10, p. 313].

3.5.6 THEOREM. The critical parallel to a given ray through a given external point is unique [10, p. 314].

3.5.7 DEFINITION. If $\overrightarrow{PD} \mid \overrightarrow{AB}$, then the open triangle $\Delta DPAB$ is called a <u>closed triangle</u> with AP called the <u>finite side</u> and $\overrightarrow{PD}$ and $\overrightarrow{AB}$ called the <u>infinite sides</u> or simply <u>sides</u> [10, p. 317].

3.5.8 DEFINITION. Two closed triangles are called equivalent if the rays that form their infinite sides are equivalent in pairs for some pairing. Furthermore, $\Delta DPAB$ is called <u>isosceles</u> if $\angle P \cong \angle A$ [10, p. 314, 317]. Note: there is obviously no predictable relationship between the finite sides of equivalent triangles.

3.5.9 THEOREM. Closed triangle $\Delta DPAB$ is equivalent to an isosceles closed triangle $\Delta DPA'B'$ which has P as vertex [10, p. 314, 317].

3.5.10 THEOREM. Critical parallelism is a symmetric relation [10, p. 314].

It is clear that 3.5.10 is essentially another formulation of 3.5. However, the string of theorems giving 3.5.10 is needed in some form to justify the "sliding" used by Lobachevski. In fact a sort of transitivity of parallels also follows.

3.5.11 THEOREM. If $\overrightarrow{AB} | \overrightarrow{CD}$, $\overrightarrow{CD} | \overrightarrow{EF}$ and $\overrightarrow{AB}$ is not equivalent to $\overrightarrow{EF}$, then $\overrightarrow{AB} | \overrightarrow{EF}$ [10, p. 315].

As may be noted by the references given for 3.5.7-3.5.9, our introduction of the defintion of closed triangles is not sequenced exactly as in Moise. This is done with no introduction of ambiguity and saves some time later when we state the "External Angle Theorem" for closed triangles (3.12) which plays an important role in subsequent proofs.

Lobachevski's Section 19 begins with:

3.6 THEOREM. In any rectilinear triangle the sum of the three angles cannot be greater than two right angles [9, p. 16].

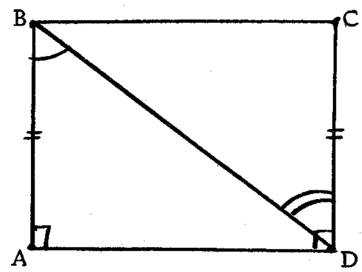
His proof involves supposing that this angle sum is greater than $\pi + a$ for $a > 0$. This uses the trigonometry-analysis notion of angles greater than a straight angle. This can easily be avoided by a method such as that used by Moise in his Chapters 7 and 10 [10]. It is interesting to note that Girolamo Saccheri [15] had established essentially this result one hundred years earlier but did not recognize the significance or even the validity of allowing the introduction of Lobachevski's axiom and the resulting strict "less-than" result for this angle sum and π. Moise's treatment, which is certainly not unique, proceeds by the following route.

3.6.1 DEFINITION. If A, B, C, D are coplanar points with $\underline{AB} \perp \underline{AD}$ at A, $\underline{DC} \perp \underline{AD}$ at D, $AB \cong CD$, then $\square ABCE = \{A, B, C, D\} \cup AB \cup BC \cup CD \cup AD$ is called a <u>Saccheri quadrilateral</u> with AD the <u>lower base</u>, BC the <u>upper base</u> and AB and CD the <u>sides</u>.

3.6.2 THEOREM. In any Saccheri quadrilateral, the upper base angles are congruent.

3.6.3 THEOREM. In any Saccheri quadrilateral, the upper base is congruent to the lower base or to a segment greater than the lower base.

3.6.4. THEOREM. In any Saccheri quadrilateral $\square ABCD$ with lower base AD, $\angle BDC \geqq \angle ABD$.



3.6.5 THEOREM. In any right triangle ΔABD with right angle at A,

$$|[\angle ABD]|_A + |[\angle ADB]|_A \leq \pi/2.$$

3.6.6 THEOREM. Every right triangle has two acute angles.

3.6.7 THEOREM. The hypotenuse of a right triangle is longer than either of its legs.

3.6.8 THEOREM. In $\triangle ABC$, let D be the foot of the perpendicular to $\underline{AC}$. If AC is the longest side of $\triangle ABC$, then A-D-C.

3.6.9 THEOREM. In any triangle $\triangle ABC$, we have $|[\angle A]|_A + |[\angle B]|_A + |[\angle C]|_A \leqq \pi$ [10, p. 125-130].

Lobachevski next establishes in Section 20,

3.7 THEOREM. "If in any rectilineal triangle the sum of the three angles is equal to two right angles, so is also the case for every other triangle" [9, p. 17].

He then observes, "From this it follows that only two hypotheses are allowable: Either is the sum of the three angles in all rectilineal triangles equal to π, or this sum is in all less than π" [9, p. 18].

His next result [9, p. 18] is, "From a given point we can always draw a straight line that shall make with a given straight line an angle as small as we choose." This is effectively our Theorem 2.9.

In his Section 22, Lobachevski establishes,

3.8 THEOREM. "If two perpendiculars to the same straight line are parallel [i.e., critically parallel] to each other, then the sums of the three angles in a rectilineal triangle is equal to two right angles" [9, p. 19].

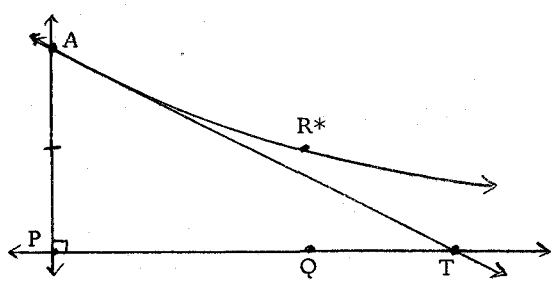
His argument, though somewhat "old-fashioned" in its choice of mathematical verbs is logically sound. The next two theorems, however, use a proof which seems to justify some substantially greater detail than that given by him. His proof involves limiting arguments which need a firmer base in light of our present mathematical notions of completeness and continuity. His theorem states:

> For every given angle (of measure) a there is a line (segment of length) p such that (the measure of the angle of parallelism) $\Pi(p) = a$ [9, p. 19].

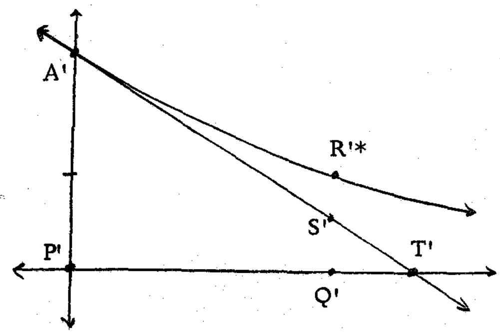
The parentheses are ours. To establish this result on our present foundation we define the Lobachevskian function Π. We then show Π is a decreasing function which can have any real value between 0 and $\pi/2$.

3.9 THEOREM. Let A, a and A', a' be points and lines of planes α and α' respectively. Let:

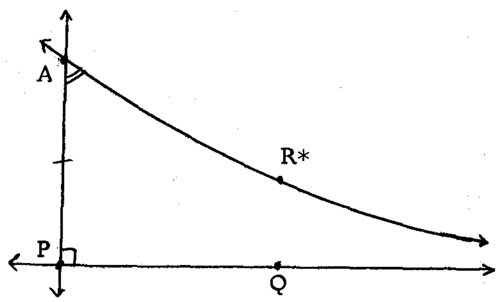
1. A and A' be points not on a and a' respectively, and
2. p and p' be the perpendiculars from A and A' to a and a' respectively with Q on p and Q' on p' so that $\overrightarrow{PQ} | \overrightarrow{AR^*}$ and $\overrightarrow{P'Q'} | \overrightarrow{A'R'^*}$.

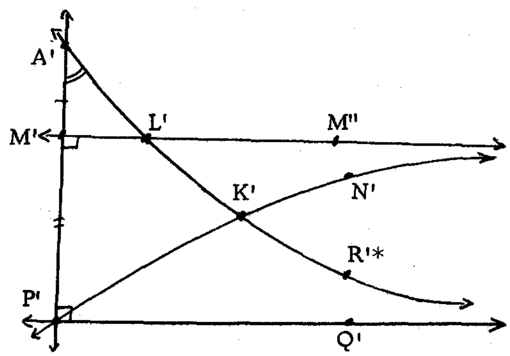


Then $PA \cong P'A'$ iff $\angle PAR^* \cong \angle P'A'R'^*$.



Proof: <u>"only if."</u> Suppose $\angle PAR^* < \angle P'A'R'^*$ given that $PA \cong P'A'$. Then interior to $\angle P'A'R'^*$ there is a ray $\overrightarrow{A'S'}$ so that $\angle PAR^* \cong \angle P'A'S'$. But by the definition of $\angle P'A'R'^*$, $\overrightarrow{A'S'}$ meets $\overrightarrow{P'Q'}$ at T'. Now there is a point T on $\overrightarrow{PQ}$ so that $PT \cong P'T'$ and by S.A.S. $\triangle APT \cong \triangle A'P'T'$ so that $\angle PAT \cong \angle PAR^*$. This forces $\overrightarrow{AR^*}$ to meet $\overrightarrow{PQ}$ at T which is impossible. By symmetry this part of the proof is done.





<u>"if"</u> Now suppose $AP < A'P'$ given that $\angle PAR^* \cong \angle P'A'R'^*$. Let M' be a point between A' and P' so that $A'M' \cong AP$. Let m' be the perpendicular to $\overline{A'P'}$ at M' in α'. Then there is a ray $\overrightarrow{P'N'}$ interior to $\angle M'P'Q'$ so that $\overrightarrow{P'N'} | \overrightarrow{M'M''}$ (where M'' is any point of m' on the same side of $\overline{A'P'}$ as Q'). But then (by 3.5.10) $\overrightarrow{P'N'}$ meets $\overrightarrow{AR'^*}$ at a point K'. By Pasch's axiom $\overline{M'M''}$ meets $A'R'^*$. By betweenness considerations $\overrightarrow{M'M''}$ meets $A'K'$ at L' (every point of

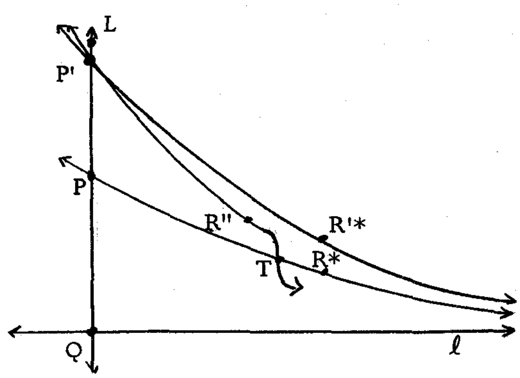
A'K' is on the same side of $\underline{A'P'}$ as $\overrightarrow{M'M''}$ and no point of m' except those on $\overrightarrow{M'M''}$ is so situated). Let L be the point of $\overrightarrow{AR^*}$ so $AL \cong A'L'$. Then $\Delta PAL \cong \Delta P'A'L'$ by S.A.S. Thus by uniqueness of perpendiculars to $\underline{AP}$ at P in α, $\overrightarrow{PQ} = \overrightarrow{PL}$ and again a contradiction results.

This theorem assures us that the following function is well defined:

3.10 DEFINITION. If [AB] is any segment class and $[\angle ABR^*]$ is the class of angles containing the angle of parallelism associated with AB, then the Lobachevskian function $\Pi: \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\Pi(|[AB]|_S) = |[\angle ABR^*]|_A$ where $|\ |_S$ is taken to be positive, i.e., with no reference point as in 1.16.

Lobachevski introduced this function and developed many results concerning it [9, p. 13].

3.11 LEMMA. If $|[AB]|_S = a$, $|[CD]|_S = b$, and $a > b$, then $\Pi(a) \leq \Pi(b)$.



Proof: Let α be the plane determined by line ℓ and a point L not on ℓ. Let p be the perpendicular from L to ℓ

with foot Q and let P and P' be points of p so that Q-P-P' with $PQ \cong CD$ and $P'Q \cong AB$. On a given side of p in α let R* and R'* be points so that $\angle QPR^*$ and $\angle QP'R'^*$ are the angles of parallelism for PQ and P'Q respectively. By 3.5.10 and 3.5.11, we know $\overrightarrow{PR^*} | \overrightarrow{P'R'^*}$ so line $\underline{PR^*}$ does not meet $\underline{P'R'^*}$. If $\Pi(a) > \Pi(b)$, then there is a ray $\overrightarrow{P'R''}$ interior to $\angle PP'R'^*$ such that $\angle PP'R'' \cong \angle QPR^*$. But $\overrightarrow{PR^*} | \overrightarrow{P'R'^*}$ forces $\overrightarrow{P'R''}$ to meet $\overrightarrow{PR^*}$ at some point T. In $\Delta PP'T$, $\angle QPT$ is an exterior angle so $\angle QPT > \angle PP'R''$ (by exterior angle theorem of absolute geometry). But $\angle QPT = \angle QPR^* \cong \angle PP'R''$ which is impossible so $\Pi(a) \leq \Pi(b)$ as claimed.

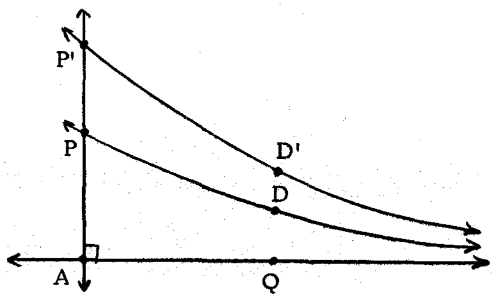
We now have the tools with which to draw upon Moise's formulation of the following theorem.

3.12 THEOREM. (The Exterior Angle Theorem for Closed Triangles) In every closed triangle, each exterior angle is greater than its remote interior angle [10, p. 317].

The proof he gives uses only materials so far included in this paper. 3.12 is then used together with 3.10 and 3.11 to prove that:

3.13 THEOREM. If $[AB] > [A'B']$, then $\Pi(|[AB]|_S < \Pi(|[A'B']|_S)$, i.e., the Lobachevskian function is strictly decreasing.

Proof: (This proof is in Moise [10, p. 319] but is sufficiently elegant as to justify its inclusion. To simplify its statement we use the figure at the right to schematically give the relationship of the rays, lines, points, etc.) Let AP and AP' be representatives of [AB] and [A'B'] respectively. Let $\angle APD$ and $\angle AP'D'$ be the angles of parallelism associated with [AB] and [A'B'] respectively. Now by previous work, $\overrightarrow{PD} | \overrightarrow{AQ}$ and $\overrightarrow{AQ} | \overrightarrow{P'D'}$ so by 3.5.11 $\overrightarrow{PD} | \overrightarrow{P'D'}$ and hence by 3.12 we are done.



The next two theorems follow easily and, with their proofs, are given in Moise [10, p. 319-320].

3.14 THEOREM. The upper base angles of a Saccheri quadrilateral are acute.

3.15 THEOREM. For every triangle, $\triangle ABC$, we have $|\angle A|_A + |\angle B|_A + |\angle C|_A < \pi$.

This leads to the definition of what has come to be known as the defect of a triangle.

3.16 DEFINITION. The defect of $\triangle ABC$, denoted $\delta(ABC)$, is $\delta(ABC) = \pi - |\angle A|_A - |\angle B|_A - |\angle C|_A$.

This leads to three theorems which have been stated and proved in Moise in a manner which is sound with respect to the mathematical foundation in this paper. These are:

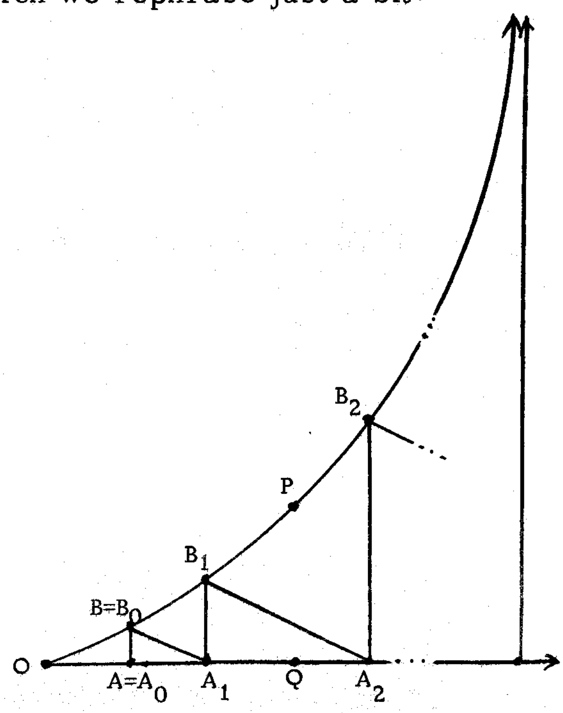
3.17 THEOREM. Let ΔABC be any triangle with B-D-C. Then $\delta(ABC) = \delta(ABD) + \delta(ACD)$.

3.18 THEOREM. If two triangles are similar, they are congruent.

3.19 THEOREM. $\lim_{a \to \infty} \Pi(a) = 0$.

(At this point Moise proceeds to develop area notions not relevant to this work.) We now have the machinery to prove Lobachevski's next theorem [9, p. 19, No. 23] which we rephrase just a bit.

3.20 THEOREM. For every acute angle $\angle POQ$, there is a line ℓ which is perpendicular to $\underline{OQ}$ and which is parallel to $\underline{OP}$ in the direction of $\overrightarrow{OP}$.



Proof: Let $\angle POQ$ be any acute angle. Let B be a point on $\overrightarrow{OP}$ and let A be the foot of the perpendicular from B to

$\underline{OQ}$. Since $\angle POQ$ is acute, A is on $\overrightarrow{OQ}$. Suppose all the perpendiculars from points on $\overrightarrow{OQ}$ in the plane α determined by $\underline{OQ}$ and P, meet $\overrightarrow{OP}$. Consider the following sequence of points on $\overrightarrow{OQ}$:

Let $A_0 = A$.

Let A_1 be the point of $\overrightarrow{OQ}$ so that $O\text{-}A_0\text{-}A_1$ and $OA_0 \cong A_0A_1$. Inductively define A_n to be that point of $\overrightarrow{OQ}$ so that $O\text{-}A_{n-1}\text{-}A_n$ and $OA_{n-1} \cong A_{n-1}A_n$. Let B_n be the point of $\overrightarrow{OP}$ where the perpendicular to $\underline{OQ}$ at A_n meets $\overrightarrow{OP}$.

This defines an infinite family of triangles which we consider as follows (cf. hypothesis and 3.17):

$$\delta(OA_1B_0) = 2\delta(OA_0B_0) < \delta(OA_1B_1),$$

$$\delta(OA_2B_1) = 2\delta(OA_1B_1) < \delta(OA_2B_2),$$

$$\vdots$$

$$\delta(OA_nB_{n-1}) = 2\delta(OA_{n-1}B_{n-1}) < \delta(OA_nB_n),$$

i.e., for any $n > 1$, $2^n\delta(OA_0B_0) < \delta(OA_nB_n)$ by induction. Now by definition $\delta(OA_nB_n) < \pi$ and $\delta(OA_0B_0) > 0$ and we have a contradiction to the Archimedian property for real numbers. Hence at least one of the perpendiculars from a point on $\overrightarrow{OQ}$ does not meet $\overrightarrow{OP}$.

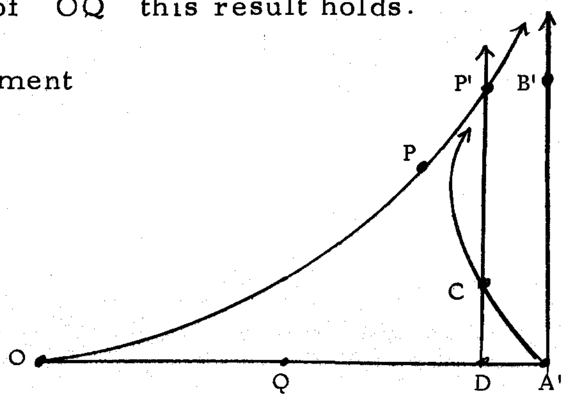
We have a natural ordering of the lines $\underline{Q_jR_j}$, $j \in \mathbb{R}^+$, which are perpendicular to $\underline{OQ}$ at points Q_j of $\overrightarrow{OQ}$ in plane α as follows: We say $\underline{Q_jR_j} < \underline{Q_kR_k}$ provided $O\text{-}Q_j\text{-}Q_k$. Now let

$\underline{L} = \{p: p$ is a line of α and $p \perp \underline{OQ}$ at a point of $\overrightarrow{OQ}\}$,

$\underline{S} = \{s: s \in \underline{L}$ and s meets $\overrightarrow{OP}\}$,

$\underline{T} = \{t: t \in \underline{L}$ and t does not meet $\overrightarrow{OP}\}$.

By our work above $s \in \underline{S}$ and $t \in \underline{T}$ implies $s < t$ with $\underline{S} \neq \phi$ and $\underline{T} \neq \phi$. By the Dedekind property for points on a line, which follows from our <u>key assumption</u>, $\underline{T}$ has a "first" element. (Obviously $\underline{S}$ cannot have a last element) and since $\underline{S}$ and $\underline{T}$ give rise to natural Dedekind classes of points of $\overrightarrow{OQ}$ this result holds.

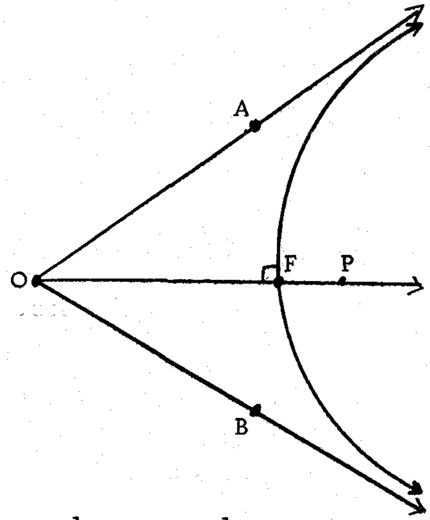


Let $\underline{A'B'}$ be the "first" element of $\underline{T}$ with A' on $\underline{OQ}$ and B' on P's side of $\underline{OQ}$. We claim $\overrightarrow{A'B'} | \overrightarrow{OP}$. Let $\overrightarrow{A'C}$ be any ray interior to angle $\angle OA'B'$, and suppose it does not meet $\overrightarrow{OP}$. Thus C is interior to $\angle POQ$ also. The perpendicular from C to $\underline{OQ}$ meets $\underline{OQ}$ at D of OA'. Then this perpendicular $\underline{CD}$ meets $\overrightarrow{OP}$ (since $\underline{A'B'}$ is the first element of $\underline{T}$) at P'. Hence $\underline{A'C}$ meets OP' by Pasch's axiom. By elementary betweenness arguments, $\overrightarrow{A'C}$ meets OP', a contradiction. Hence $\overrightarrow{A'B'} | \overrightarrow{OP}$ and hence $\underline{A'B'}$ is parallel to $\underline{OP}$ in the direction of $\overrightarrow{OP}$ and we are done.

3.21 COROLLARY. Every acute angle is the angle of parallelism for some segment.

Proof: In the notation above $|\angle POQ|_A = \Pi(|[OA']|_S)$.

3.21.1 COROLLARY. For every angle $\angle AOB$ there is a line ℓ that is parallel to $\underline{OA}$ in the direction of $\overrightarrow{OA}$ and also is parallel to $\underline{OB}$ in the direction of $\overrightarrow{OB}$.



Proof: Let $\overrightarrow{OP}$ be the bisector of $\angle AOB$. Let ℓ be the line perpendicular to $\underline{OP}$ and parallel to $\underline{OA}$ in the direction of $\overrightarrow{OA}$ and let F be the foot of this perpendicular on $\underline{OP}$. By 3.19 $|\angle AOF|_A = \Pi(|[OF]|_S) = |\angle BOF|_A$ and we are done.

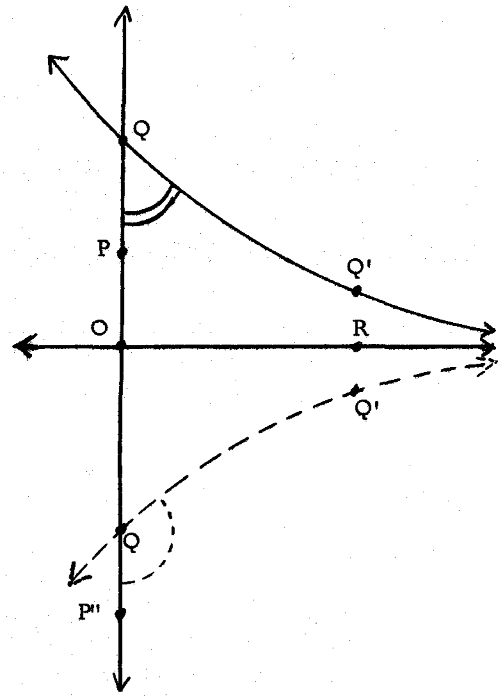
The line ℓ in 3.21.1 is said to be parallel to both sides of $\angle AOB$.

3.22 THEOREM. The function Π assumes all values between 0 and $\pi/2$.

Proof: Immediate from Corollary 3.21.

Let us now draw several results together. In 3.10 we defined $\Pi: \mathbb{R}^+ \to \mathbb{R}^+$ by defining $\Pi|[AB]|_S = |[\angle ABR^*]|_A$. Since $\angle ABR^*$ is an angle of parallelism, Π is clearly positive (cf. def. 3.3, 3.2, and 2.13). By 3.21, every acute angle is the angle of parallelism for some segment so, by definition, the supplement of any obtuse angle is the angle of parallelism for some segment. Let us assign angles to points on an ordered line as follows: In plane α let:

1. $\underline{PO}$ be any line,
2. $\underline{OR}$ be the perpendicular to $\underline{PO}$ at O in α,
3. Q be any point of $\underline{PO}$,
4. Q' be any point of α so that $\overrightarrow{QQ'} | \overrightarrow{OR}$.



Then

1. If $Q \epsilon \overrightarrow{OP}$, assign $\angle OQQ'$ to Q (i.e., assign the angles of parallelism associated with OQ),
2. If Q = P assign $\angle$ rt,
3. If P-O-Q, let P'' be any point of $\underline{PQ}$ so that $\overrightarrow{QP''}$ is equivalent to $\overrightarrow{PO}$ (cf. 3.5.3) then assign $\angle P''QQ'$ to Q (i.e., assign the supplement of the angle of parallelism associated with OQ).

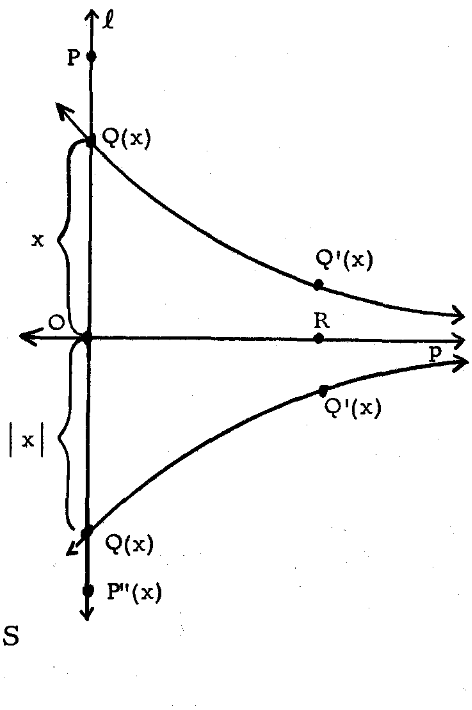
Theorems 3.13 and 3.21 assure us that such an assignment is continuous and one-to-one (based on our assumption that the injection from points on a line to the real numbers (1.16) is a bijection in the sense described in Chapter 1).

Using the extension of "$| |_A$" (described in 2.13) to all angle classes we are now in a position to state and accept the following extensions of Π which allows us to enlarge the range stated in 3.22:

3.22.1 DEFINITION. In plane α, let

1. $\underline{PO}$ be any line ℓ,
2. $\underline{OR}$ be perpendicular to $\underline{PO}$ at O in α,
3. x be any real number,
4. Q(x) be
$$\begin{cases} \text{the point } Q \text{ of } \overrightarrow{OP} \text{ so that } |[OQ]|_S = x & \text{if } x > 0 \\ \text{the point } O & \text{if } x = 0 \\ \text{the point } Q \text{ of } \ell \text{ so that } P\text{-}O\text{-}Q \text{ and } |[OQ]|_S = |x| & \text{if } x < 0, \end{cases}$$
5. Q'(x) be any point of α so that $\overrightarrow{Q(x)Q'(x)} \mid \overrightarrow{OR}$, and
6. P"(x) be any point of ℓ so that $\overrightarrow{Q''(x)P''(x)}$ is equivalent to $\overrightarrow{PO}$ (cf. 3.5.3).

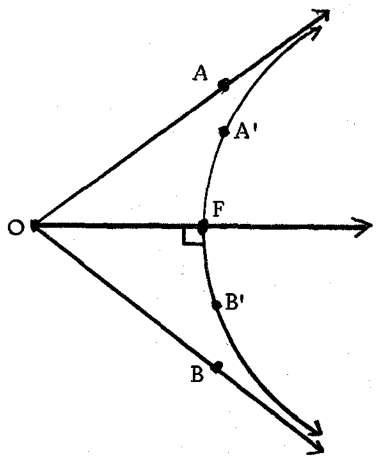
Then define $\boldsymbol{\Pi}(x) = |[\angle P''(x)Q(x)Q'(x)]|_A$.



It is clear that for $x > 0$ $\boldsymbol{\Pi}(x) = \Pi(x)$ and that $\boldsymbol{\Pi}: \mathbb{R} \to (0, \pi)$. For simplicity of notation, throughout the rest of this paper we will use the sumbol "Π" to mean "$\boldsymbol{\Pi}$". Since "$|\ |_S$" is always non-negative $\boldsymbol{\Pi}|[AB]|_S = \Pi|[AB]|_S$

so no ambiguity is possible here. The only difference is when we are using negative reals. The use of Π is not specifically needed until Chapter 6 and for all intermediate work the earlier meaning of the symbol Π is sufficient. However, this seems the best place to develop this extension to maintain continuity of exposition as much as is possible in so long an argument.

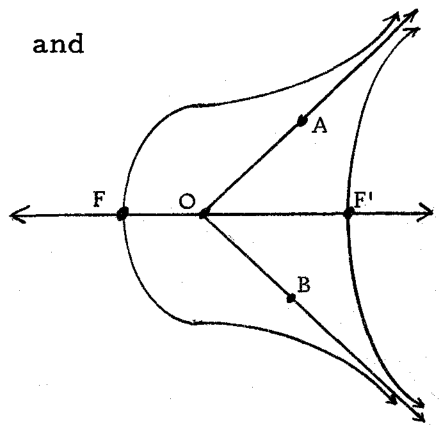
3.23 LEMMA. Let $\angle AOB$ be any angle. If ℓ is parallel to $\underline{OA}$ and $\underline{OB}$ in the direction of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, then ℓ is perpendicular to the bisector $\underline{OP}$ of $\angle AOB$ at a point F of $\overrightarrow{OP}$.



Proof: Let $\underline{OF}$ be the perpendicular from O to ℓ with foot F. Let A' and B' be points of ℓ so that $\overrightarrow{FA'} | \overrightarrow{OA}$ and $\overrightarrow{FB'} | \overrightarrow{OB}$. By definition $\angle FOA$ and $\angle FOB$ are angles of parallelism associated with OF and by 3.2 and 3.9, these are both acute and mutually congruent angles.

Thus the bisector of $\angle AOB$ is a ray of $\underline{OF}$. Let $\overrightarrow{OF'}$ be the bisector of $\angle AOB$. Suppose $OF \cong OF'$ and F-O-F'. Then by an argument such as that for 3.20 we get that the perpendicular to $\underline{OF}$ at F' is also parallel to $\underline{OA}$ and $\underline{OB}$ in the direction of $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Thus $\angle F'OA \cong \angle F'OB$ and both are acute by 3.2.

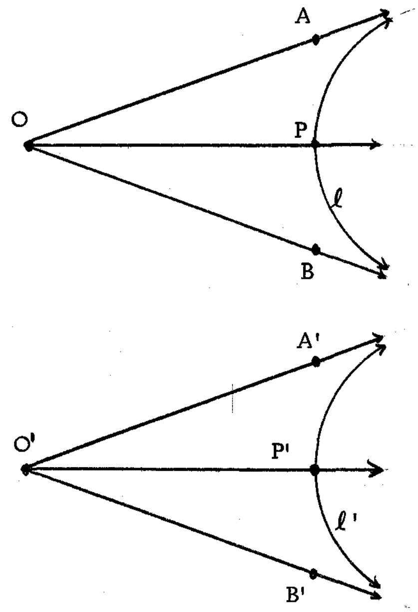
Thus at O we have acute angles $\angle F'OA$ and $\angle FOA$ as supplementary. That is impossible. Thus $\overrightarrow{OF}$ is the bisector of $\angle AOB$ as claimed.



3.24 THEOREM. If

1. $\angle AOB \cong \angle A'O'B'$,
2. ℓ and ℓ' are parallel to both sides of $\angle AOB$ and $\angle A'O'B'$ respectively,
3. P and P' are the feet of the perpendiculars from O and O' to ℓ and ℓ' respectively,

then $OP \cong O'P'$.



Proof: By 3.23 $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are the bisectors of $\angle AOB$ and $\angle A'O'B'$ respectively. If $OP < O'P'$, $\angle POA > \angle P'O'A'$ by 3.13, a contradiction. By symmetry $O'P' < OP$ is also impossible and we are done.

Theorem 3.24 is a key theorem in that it allows us to select one class of segments in a canonical way just as in absolute geometry it is possible to canonically select a certain class of angles. The ability to select a reference angle class, (specifically the class of right angles) allows us to choose, on the basis of the axioms, a reference standard for the function $|\ |_A$. Specifically $|[\angle rt]|_A = \pi/2$. Let [RT] be the class of segments which has as representation the segment OP so that if $\angle$ AOP is a right angle, then P is the foot of the perpendicular to the line ℓ which is parallel to both sides of $\angle$ AOB. This class contains all such segments for all right angles according to 3.24. It will be shown that the most reasonable value to select for $|[RT]|_S = \Pi^{-1}(|\frac{[\angle rt]}{2}|_A)$ is $\ln(\sqrt{2}+1)$ when we have elected to assign $\pi/2$ to the class of right angles.

3.25 DEFINITION. We will call the class [RT] (above) the <u>canonical class</u> or <u>standard class</u> of segments and its representaives <u>canonical segments</u> or <u>standard segments</u>.

We have now defined the Lobachevskian function Π and selected a reference class--the canonical class--of segments which relates angles and segments. We now proceed with developing the results to actually evaluate Π. Lobachevski developed, very cleverly, two figures which he called <u>oricycles</u> [9, p. 30] and <u>orispheres</u> [9, p. 33] and used spherical geometry and trigonometry to establish values for Π. This work was imaginatively done and justifies his name, rather than others', being attached to this geometry. Only he took the initative to develop the theorems to describe the basic properties so fully. However, in a foundations work such as this, care must be taken to build only on the stated foundation--in this case Hilbert's axioms. The development we use will somewhat parallel Lobachevski's but will supply proofs which are founded upon Hilbert's axioms.

IV. PENCILS AND ORICYCLES

In this section we lay the foundation for the development of the oricycles and orispheres together with their properties which are useful later in computing the formula for the "Lobachevskian function" Π.

As has been said already, Lobachevski follows a route not always adaptable to our foundation. In our subsequent work we will draw heavily on the definitions given by Shirokov [16] and will prove many of the theorems he proves. We will follow essentially the same path he does. Some of his proofs are not adaptable to this paper since he largely ignores the completeness axiom and uses arguments for which we have no justification. When this occurs we shall include appropriate additional proofs and results. These results will make it possible for us to compute the values for Π stated and used by Lobachevski. The development of the oricycles which we will make in this and the next chapter does--as noted above--require a substantial addition of proof above that given by Shirokov.

We shall--as near as is possible--make every attempt to specify what of the subsequent results are specifically Shirokov's. A majority of the proofs and results for oricycles are ours though the general direction is given by Shirokov. However, once the results for oricycles have been carefully developed to account properly for the completeness axiom, Shirokov's development for orispheres needs hardly any additional proof than that given by him. The orisphere results do not need any completeness results beyond that which we need and provide in the propositions and proofs given for oricycles. Thus, once we have established the results for oricycles needed by Shirokov to obtain the orisphere results, we will again draw heavily on the literature for the development of results for and from orispheres.

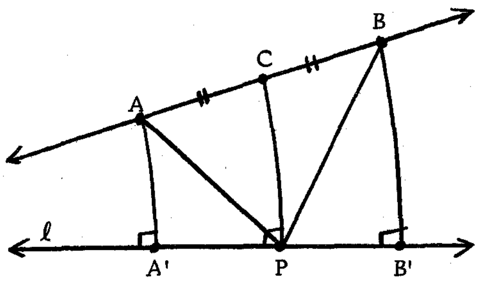
We now establish two helpful lemmas.

4.0.1 LEMMA. If

1. C is the midpoint of segment AB in plane α,
2. m is the perpendicular bisector of AB in α,
3. ℓ is the line of α which is perpendicular to m at P where $P \neq C$, and
4. A' and B' are the feet of the perpendicular to ℓ containing A and B, respectively,

then

1. $\square A'ABB'$ is a Saccheri quadrilateral with base $A'B'$, and
2. P is the midpoint of $A'B'$.

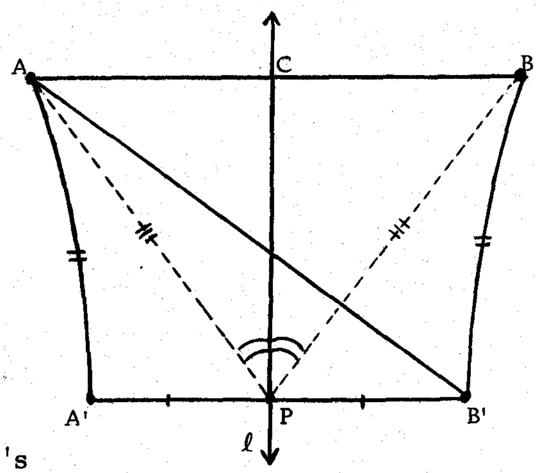


Proof: To avoid a contradiction to the exterior angle theorem we must conclude that ℓ and $\underline{AB}$ do not meet and also $\underline{AA'}$, $\underline{PC}$, and $\underline{BB'}$ do not meet each other since otherwise we have a triangle with an exterior angle congruent to a remote interior angle. Thus the triangles discussed below all exist:

$\triangle ACP \cong \triangle BCP$ by S.A.S. and thus $AP \cong BP$. Furthermore $\angle APA' \cong \angle BPB'$ since they are complements of congruent angles. Hence $\triangle APA' \cong \triangle BPB'$ by S.A.A. so that:

1. $AA' \cong BB'$ and by definition $\square A'ABB'$ is a Saccheri quadrilateral with base $A'B'$, and
2. $A'P \cong PB'$ so P is the midpoint of $A'B'$ as claimed.

4.0.2. LEMMA. In Saccheri quadrilateral $\square A'ABB'$ of plane α with base $A'B'$, a line ℓ of α is the perpendicular bisector of $A'B'$, iff ℓ is the perpendicular bisector of AB.



Proof: Let P be the foot of the perpendicular bisector ℓ of $A'B'$. As above we see that the exterior angle theorem ensures that ℓ not meet AA' and BB'. In triangles $\triangle A'AB'$ and $\triangle ABB'$ we use Pasch's axiom to conclude ℓ meets AB' and AB, respectively. Let C be the point of AB on ℓ.

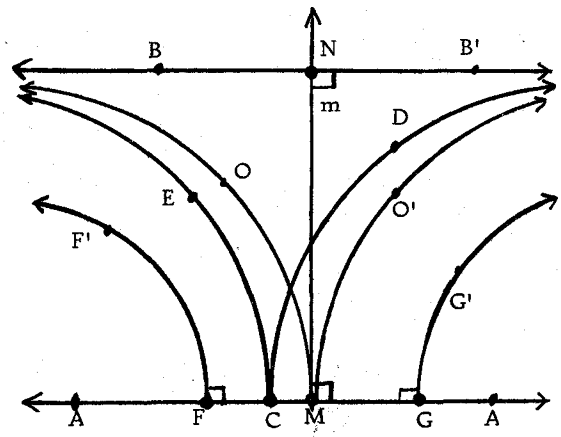
Now $\triangle AA'P \cong \triangle BB'P$ by S.A.S. and thus $AP \cong BP$. Also $\angle APC \cong \angle BPC$ since they are complements to congruent angles.

Thus $\triangle APC \cong \triangle BPC$ by S.A.S. and it follows that $AC \cong BC$ and $\ell \perp \underline{AB}$ at C.

The converse is immediate from 4.0.1.

4.0.3 DEFINITION. Two coplanar lines which do not meet and are not parallel in any direction are called <u>divergent</u> or <u>hyperparallel</u> lines.

4.0.4 LEMMA. Any two hyperparallel lines have a common perpendicular.



Proof: Let $\underline{AA'}$ and $\underline{BB'}$ be hyperparallel lines. From any point C of $\underline{AA'}$, let $\overrightarrow{CE} | \overrightarrow{B'B}$ and $\overrightarrow{CD} | \overrightarrow{BB'}$. Now, based on Theorem 3.20, we have lines $\underline{FF'}$ and $\underline{GG'}$ which are perpendicular to $\underline{AA'}$ at F and G respectively and so that $\overrightarrow{FF'} | \overrightarrow{CE}$ and $\overrightarrow{GG'} | \overrightarrow{CD}$. Let M be the midpoint of FG and let m be the perpendicular to $\underline{BB'}$ from M with foot N. Without loss of generality we may suppose B and B' are on F's and G's sides of $\underline{MN}$ respectively. Let $\overrightarrow{MO}$ and $\overrightarrow{MO'}$ be the rays so that $\overrightarrow{MO} | \overrightarrow{NB}$ and $\overrightarrow{MO'} | \overrightarrow{NB'}$. By 3.9 $\angle NMO \cong \angle NMO'$. By 3.5.11 ("transitivity" of parallels) $\overrightarrow{MO} | \overrightarrow{FF'}$ and $\overrightarrow{MO'} | \overrightarrow{GG'}$. Since $FM \cong MG$, $\underline{FF'} \perp \underline{AA'}$ and $\underline{GG'} \perp \underline{AA'}$ we have $\angle FMO \cong \angle GMO'$ by 3.9. Thus $\angle FMN \cong \angle GMN$ by "angle sum theorem" and thus $\underline{NM} \perp \underline{AA'}$ at M also and we are done.

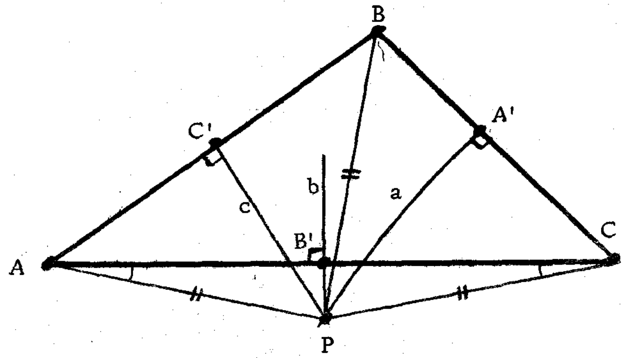
4.1 DEFINITION. The set of all lines of a plane containing a given point C is called a pencil of intersecting lines with center C [16, p. 35].

4.2 DEFINITION. The set of all lines of a plane which are perpendicular to a given line ℓ is called a pencil of divergent lines with axis ℓ [16, p. 35].

4.3 DEFINITION. The set of all lines of a plane which are parallel to a given line ℓ in a given direction $\overrightarrow{AB}$ of ℓ is called a pencil of parallel lines in the direction of $\overrightarrow{AB}$ (or simply a pencil of parallel lines if the direction is clear) [16, p. 35].

4.4 THEOREM. The perpendicular bisectors of the sides of a triangle belong to one pencil [9, p. 29; 16, p. 36].

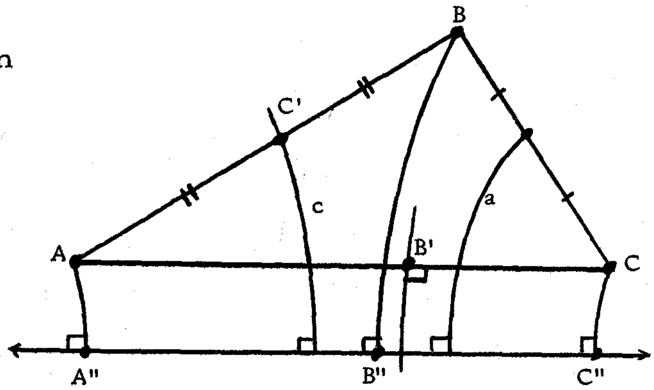
Proof: Let the perpendicular bisectors of the side opposite $\angle A$ be line a and let the midpoint of BC be A', etc. as shown.



Case 1. Suppose a and c meet in a point P. Consider segments AP, BP, and CB. In $\triangle AC'P$ and $\triangle BC'P$ we have $AC' \cong BC'$, $\angle AC'B \cong \angle BC'P$ and $C'P \cong C'P$ so by S.A.S. $\triangle AC'P \cong \triangle BC'P$. By symmetry we also have $\triangle BA'P \cong \triangle CA'P$. Hence $AP \cong BP \cong CP$ and $\triangle APC$ is isosceles making $\angle CAP \cong \angle ACP$.

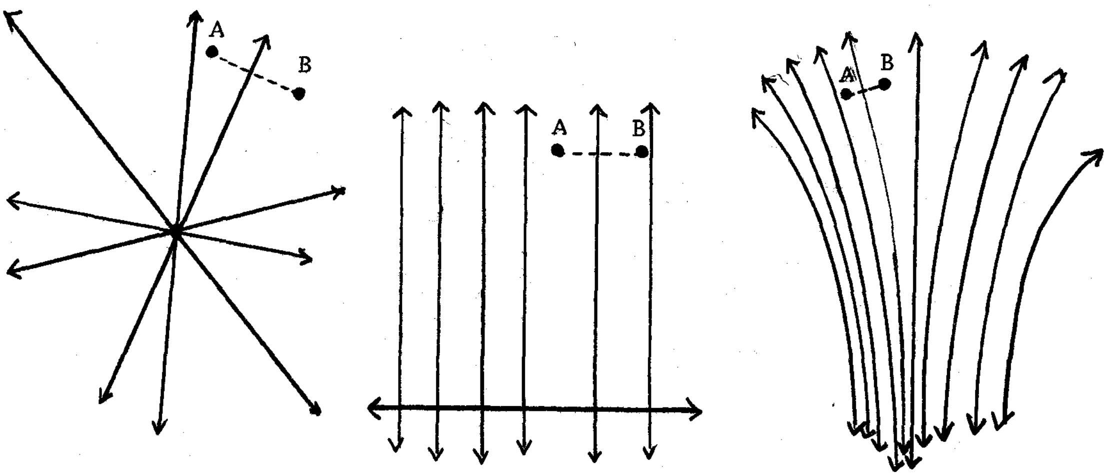
Therefore, if B' is the midpoint of AC, $\triangle AB'P \cong \triangle CB'P$ by **S**.A.S.. This makes $\underline{B'P}$ the perpendicular bisector of AC and a, b, and c are all in the pencil of intersecting lines with center P.

Case 2. Suppose a and c are hyperparallel and hence by Lemma 4.0.4 have a common perpendicular ℓ. Then $\underline{AB}$ and $\underline{BC}$ are both hyperparallel to ℓ so A, B, and C are not on ℓ. From A, B, and C construct the perpendiculars to ℓ with feet A", B", and C" respectively. By Lemma 4.0.1 □A"ABB" and □B"BCC" are Saccheri quadrilaterals with common side BB" so $AA'' \cong CC''$. Thus □A"ACC" is a Saccheri quadrilateral. Hence the perpendicular bisector b of AC is perpendicular to ℓ (by 4.0.2) and a, b, and c are all in the pencil of divergent lines with axis ℓ.



Case 3. If a and c are parallel in a given direction, then b is parallel to both a and c in this same direction, since otherwise we should have a contradiction to Cases 1 and 2 above (cf. 4.0.4). Hence a, b and c are all in the same pencil of parallels.

4.5 DEFINITION. Two points A and B <u>correspond to one another relative to a pencil of lines,</u> iff they are symmetrical with respect to some line of the pencil (denoted: A*B) [16, p. 39].



4.6 THEOREM. In a given plane α, if P*Q and R*Q then P*R.

Proof: If P = R, we are done trivially. Hence suppose P ≠ R.

Case 1. If P, Q and R are collinear, then the only pencil under which P*Q and R*Q is a divergent pencil with P, Q, R all on the axis of the pencil. The existence of the perpendicular bisector of PR gives P*R.

Case 2. If P, Q and R are noncollinear, then the perpendicular bisectors of the sides of ΔPQR are all in the same pencil (by Thm. 4.4) and the theorem is immediate.

4.7 THEOREM. For any given pencil ξ, the relation "*" is an equivalence relation on the set of points of the plane α containing ξ.

Proof: Let A be any point of α. Then A is on a unique line m of ξ. This follows because:

(a) if ξ is a pencil of intersecting lines with center C, $\underline{AC}$ is a line of ξ and is unique.

(b) If ξ is a divergent pencil with axis ℓ, there is a unique line m through A and perpendicular to ℓ. m is a line of ξ and perpendiculars are unique.

(c) if ξ is a parallel pencil in the direction of $\vec{r}$ on line r, there is a unique line m through A so that m is parallel to the line r in the direction of $\vec{r}$ provided A is not on r. Then m is a line of ξ and we are done. If A is on r we are done trivially. Uniqueness follows by 3.2.

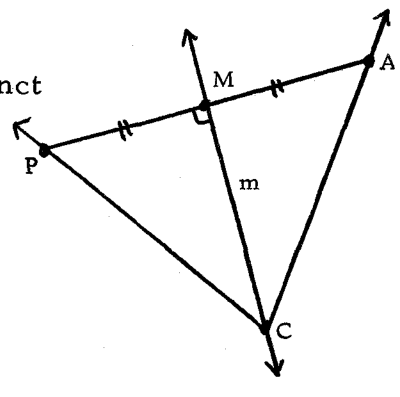
Reflexive. A is on a unique line m of ξ and A is trivially symmetric to itself with respect to m. Therefore A*A.

Symmetrical. If A is symmetric to B with respect to m, then the definition of symmetry to a line gives B symmetric to A. Thus A*B implies B*A.

Transitive. P*Q and Q*R imply P*Q and R*Q by symmetrical above. But P*Q and R*Q imply P*R by Theorem 4.6. Hence P*Q and Q*R imply P*R.

4.8 DEFINITION. Let ξ be a pencil in a plane α. Then denote by $\xi[A]$ the equivalence class of A with respect to ξ and "*".

4.9 THEOREM. If ξ is a pencil of intersecting lines in α with center C and A is a point of α different from C, then $\xi[A]$ is the circle Σ with radius AC and center C.



Proof: Let P*A. If A and P are distinct points, there is a line m of ξ which is the perpendicular bisector of AP at M. $\triangle CMP \cong \triangle CMA$ by S.A.S. and $PC \cong AC$ so $P \in \Sigma$. If P = A, $P \in \Sigma$. Thus $\xi[A] \subset \Sigma$.

Let P be a point of Σ. If P = A we have P*A. Otherwise let $\overrightarrow{CM}$ be the bisector of $\angle PCA$. $CP \cong CA$ so $\angle P \cong \angle A$

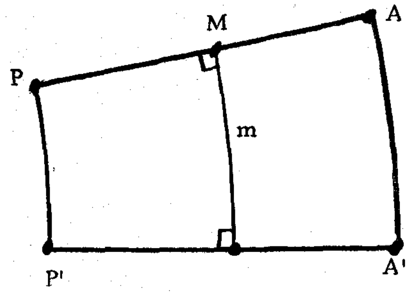
and thus $\Delta PMC \cong \Delta AMC$ by A.S.A. Thus $\underline{CM}$ is the perpendicular bisector of AP and $P*A$. This gives us $\Sigma \subset \xi[A]$ and we are done. If P-C-A, the proof is obvious.

4.10 THEOREM. If ξ is a divergent pencil with axis ℓ in plane α, and A is a point of α not on ℓ, then

$\xi[A] = E = \{P \mid P$ is a point of α and $PP' \cong AA'$ where P' and A' are the feet of the perpendicular to ℓ through P and A respectively$\}$.

Proof:

1. $(\xi[A] \subset E)$. If P is a point of $\xi[A]$ and $P = A$, then $P \in E$. If $P \neq A$ then there is a line m of ξ which is the perpendicular bisector of PA at M. By Lemma 4.0.1 $\square P'PAA'$ is a Saccheri quadrilateral with base $P'A'$ so $PP' \cong AA'$ and $\xi[A] \subset E$.



2. $(E \subset \xi[A])$. If $P = A$, then $P \in \xi[A]$. If $P \neq A$, then $\square P'PAA'$ is a Saccheri quadrilateral and by Lemma 4.0.2 the perpendicular bisector m of the base $A'P'$ is the perpendicular bisector of AP, so $P*A$. Hence $E \subset \xi[A]$ and we are done.

4.11 THEOREM. If ξ is a divergent pencil with axis ℓ in plane α, and A is a point of α not on ℓ, then $\xi[A]$ contains no point of ℓ.

Proof: Suppose $P \in \xi[A]$ and P is a point of ℓ. Then if P' is the foot of the perpendicular to ℓ through P, then $P = P'$. In that case PP' is not a segment so cannot be congruent to AA' as required by Theorem 4.10. This is impossible and the theorem is proved.

4.12 DEFINITION. If ξ is a divergent pencil with axis ℓ in a plane α and A is a point of α not on ℓ, then $\xi[A]$ is called an <u>equidistant curve associated with ℓ</u> [16, p. 39].

4.13 DEFINITION. If ξ is a parallel pencil in the direction of $\vec{r}$ in a plane π and A is a point of π, $\xi[A]$ is called an <u>oricycle</u>, or <u>limiting curve</u> <u>in the direction of $\vec{r}$</u> [16, p. 39].

4.14 THEOREM. If $\xi[A]$ is a circle, an equidistant curve or an oricycle and if P, Q, and R are distinct points of $\xi[A]$, then P, Q and R are non-collinear [16, p. 39].

Proof: If P, Q, and R are collinear, then the three lines of the pencil with which P, Q, and R are symmetric are perpendicular to the same line. Hence the pencil ξ is a divergent pencil with base

$\underline{PQ} = \underline{PR} = \underline{QR}$. But this is impossible since $\xi[A]$ is forced to be an equidistant curve and hence has no points in common with the base (4.11 and 4.12).

Note: Throughout the rest of this paper the symbol "$\xi[A]$" will be used to mean an oricycle.

V. PROPERTIES OF ORICYCLES AND ORISPHERES--THE VALUES OF Π

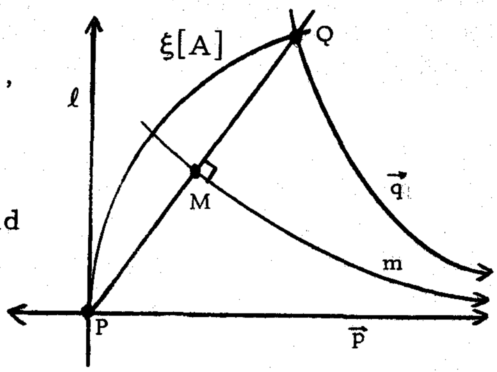
To simplify the following writing we will denote $|[\angle ABC]|_A$ by $m\angle ABC$ and $\Pi(|[AB]|_S)$ by $\Pi|AB|$ or $\Pi|[AB]|$.

5.1 THEOREM. Let

1. $\xi[A]$ be an oricycle in plane α and in the direction of $\vec{r}$,
2. P, Q be distinct points of this oricycle,
3. p, q be the lines of ξ incident on P and Q respectively, and
4. $\vec{p}$, $\vec{q}$ be the rays of p and q from P and Q in the direction of parallelism of p and q, respectively.

Then

1. $m\angle \overrightarrow{PQ}\,\vec{p} = m\angle \overrightarrow{QP}\,\vec{q} = \Pi|\frac{[PQ]}{2}|$,
2. $\ell \perp p$ at P in α iff ℓ has no other point of $\xi[A]$, and
3. $\angle \overrightarrow{PQ}\,\vec{p} \cong \angle \overrightarrow{QP}\,\vec{q} < \angle rt$.

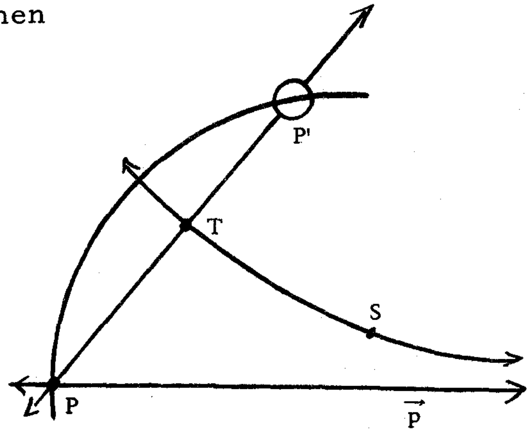


Proof: 1. $P, Q \in \xi[A]$ implies $P*Q$ so the perpendicular bisector m of PQ in α is in the pencil ξ (Definition 4.5). If M is the midpoint of PQ, then $m\angle \overrightarrow{QP}\,\vec{q} = \Pi|MQ| = \Pi|MP| = m\angle \overrightarrow{PQ}\,\vec{p}$. By definition $[MQ] = [AB]/2$.

2. If $\ell \perp p$ at P and ℓ is incident on a second point R of $\xi[A]$, then by 1. $m\angle \overrightarrow{RP}\,\vec{p} = \Pi|\frac{[AB]}{2}| < \pi/2$ so $\ell \not\perp p$. This

is impossible.

If $\ell \not\perp p$ then for one of the rays from P on ℓ, say $\vec{\ell}$, $\angle \vec{\ell}\, \vec{p}$, is acute. Thus by 3.21 there is a segment class [PT] so that $\Pi|PT| = m\angle \vec{\ell}\, \vec{p}$. Let T be the point of ℓ so PT is a representative of the given class. Let $\overrightarrow{TS} | \vec{p}$ and let P' be on ℓ so that P-T-P' and PT $\cong$ TP'. Then by definition, $\underline{TS} \perp \ell$ at T and P*P'. Hence $P' \in \xi[A]$ so ℓ has a second point of this oricycle. Thus by contrapositive, the statement is proved.



3. Immediate from Step 1.

5.2 THEOREM. If

1. $H = \xi[A]$, $H' = \xi'[C]$ are oricycles in planes α and α' respectively,
2. $\overrightarrow{AA'}$ and $\overrightarrow{CC'}$ are rays in the direction of ξ and ξ' respectively,
3. B is some point of H different from A, and
4. D is a point of α' so that $\angle A'AB \cong \angle C'CD$,

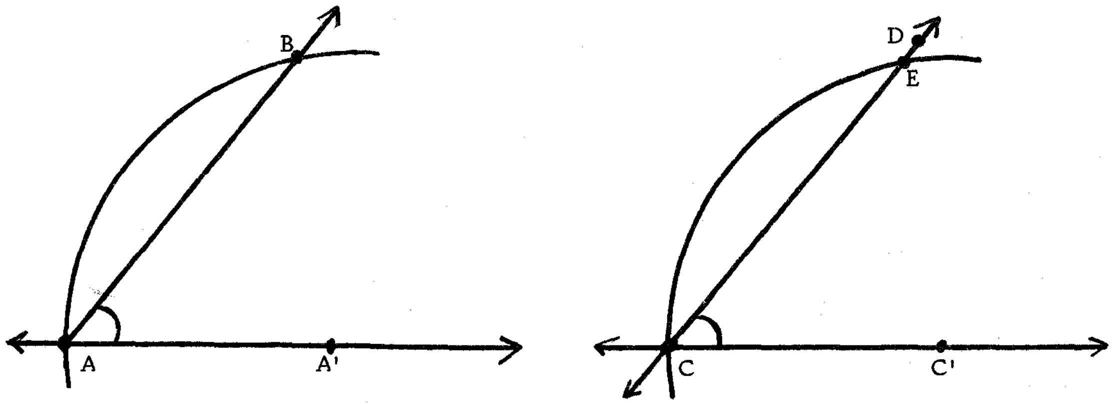
then

1. $\overrightarrow{CD}$ is incident on a point E of H' different from C, and

2. AB $\cong$ CE.

Proof: 1. By 5.1-3, $\angle$C'CD is acute and by 5.1-2 meets H' in a point E different from C.

2. By 5.1-1 we have $\Pi\left|\frac{[AB]}{2}\right| = \Pi\left|\frac{[CE]}{2}\right|$. Thus AB $\cong$ CE.



5.3 THEOREM. If

1. H = $\xi[A]$ is an oricycle in plane α and in the direction of $\overrightarrow{AA'}$,

2. P is a point of H different from A,

3. $\overrightarrow{PP'} | \overrightarrow{AA'}$, and

4. m is the perpendicular bisector of AP at M',

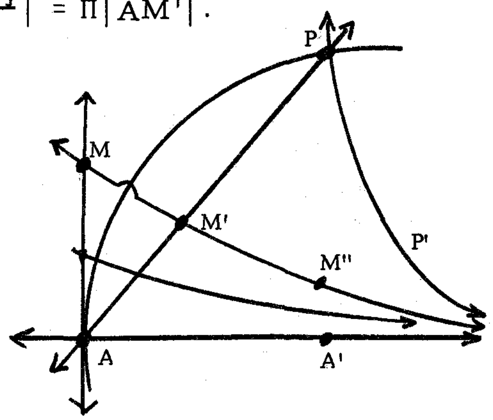
then

1. m is a line of ξ,

2. m is incident on a unique point M of H,

3. AM $\cong$ MP, and

4. M is exterior to $\angle$PAA'.



Proof: 1. By 5.1, $m\angle A'AP = \Pi\left|\frac{[AP]}{2}\right| = \Pi|AM'|$. Thus $\underline{M'M''}$ is parallel to $\underline{AA'}$ in the direction of $\overrightarrow{AA'}$ so $m \in \xi$.

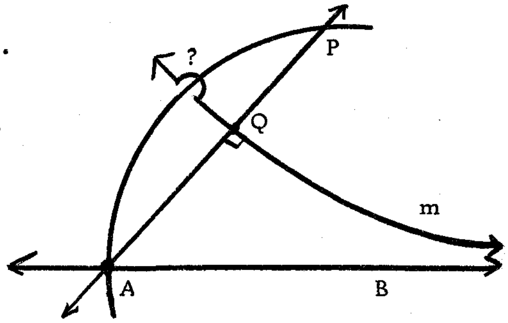
2. Let $\overrightarrow{M'M''} | \overrightarrow{AA'}$. Now by 3.5.9 closed triangle $\Delta A'AM'M''$ is equivalent to an isosceles closed triangle $\Delta A'AMM''$ (since $\angle AM'M''$ is a right angle, we know M-M'-M"). Furthermore by 5.1, we know $m\angle A'AM = \Pi\left|\frac{[AM]}{2}\right|$. Thus if D is the midpoint of AM, the perpendicular to $\underline{AM}$ at D is in ξ. Thus A*M and $M \in H$ as claimed. We now must show M is unique. Suppose m meets H in a second point N. Let n be the perpendicular bisector of MN. Then $n \in \xi$. But $m \in \xi$ (by 1.) and m meets n, a contradiction.

3. $\Delta AMM' \cong \Delta PMM'$ by S.A.S. so $AM \cong MP$.

4. As noted in Step 2., M-M'-M" so M is exterior to $\angle PAA'$.

5.3.1 COROLLARY. If $H = \xi[A]$ is an oricycle in plane α and in the direction of $\overrightarrow{AB}$ and m is any line of ξ, then m is incident on a unique point M of H.

Proof: If A is a point of m, we are done. Let Q be the foot of the perpendicular p from A to m. Let P be the point of p so that A-Q-P and $AQ \cong QP$. Then A*P by definition, $P \in H$, and by 5.3 m is incident on a point M of H.

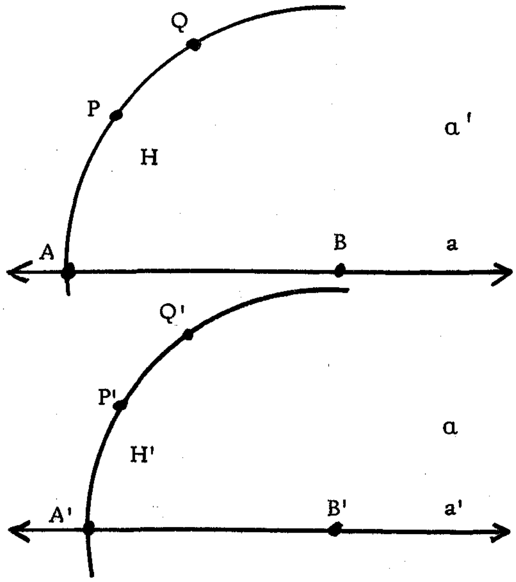


5.4 DEFINITION. Let $\xi[A] = H$ be an oricycle in plane α and let t be a line of α so that t meets H in exactly one point or equivalently (by 5.1) t is perpendicular to a line of ξ at some point P of H. Then we say t is tangent to H at P.

5.5 DEFINITION. Two oricycles H and H' are said to be congruent iff there is a one-to-one correspondence $f: H \rightarrow H'$ so that for any two distinct points P and Q of H, we have $PQ \cong f(P)f(Q)$. f is called a congruence [16, p. 41].

5.6 THEOREM. Any two oricycles are congruent [16, p. 41].

Proof: Let $H = \xi[A]$ and $H' = \xi'[A']$ be two oricycles in planes α and α' respectively with a and a' lines of ξ and ξ' through A and A' respectively. Let B and B' be points of a and a' so that $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are in the direction of



parallelism of ξ and ξ' respectively. Let $f: H \to H'$ be defined as follows:

(a) first choose a correspondence between the half planes defined by a and by a' in α and α' respectively.

(b) For each point P of H, let

$$f(P) = \begin{cases} A' & \text{if } P = A \\ P' & \text{so that } \angle BAP \cong \angle B'A'P' \text{ if } P \neq A \text{ and } P' \in H' \\ & \text{is in the half plane chosen to correspond to the} \\ & \text{half plane determined by a and containing } P. \end{cases}$$

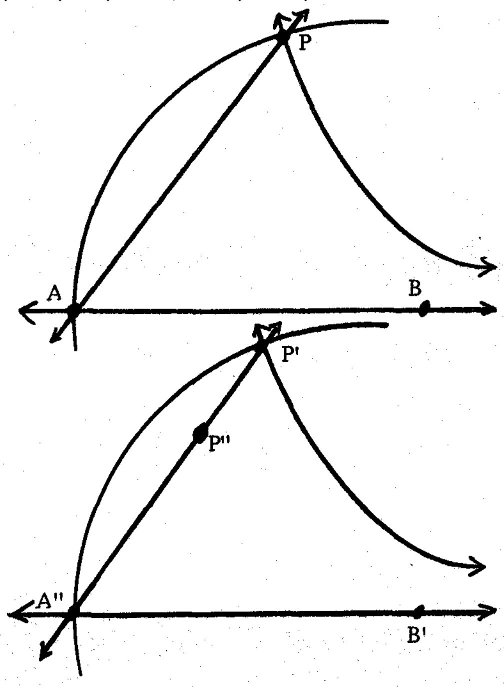
We claim f is a congruence between H and H'.

(1) <u>f is a bijection</u>: Suppose P and Q are distinct points of H. If one is A, we are done so suppose both are not A. If P and Q are in different half planes of α determined by a, then $f(P) \neq f(Q)$ so suppose they are in some half plane of α determined by a. Since A, P and Q are noncollinear (Thm. 4.14) $\angle BAP \not\cong \angle BAQ$ so $f(P) \neq f(Q)$ follows (4.14 and 5.1). 5.2 gives <u>onto</u>.

(2) <u>$f(P)f(Q) \cong PQ$</u>: A, P, Q and A', P', Q' are noncollinear (by 4.14) so consider $\triangle APQ$ and $\triangle A'P'Q'$. $AP \cong A'P'$ since $\Pi(\frac{[AP]}{2}) = m\angle BAP = m\angle B'A'P' = \Pi(\frac{[A'P']}{2})$ from 5.1, and 3.22.1 gives us $AP \cong A'P'$. By symmetry we also have $AQ \cong A'Q'$. By "angle difference" we also have

$\angle PAQ \cong \angle P'A'Q'$ so $\triangle APQ \cong \triangle A'P'Q'$ by S.A.S. Hence $PQ \cong P'Q' = f(P)f(Q)$ as claimed.

5.6.1 COROLLARY. Let H, A, B, a, H', A', B', and a' be as in 5.6.



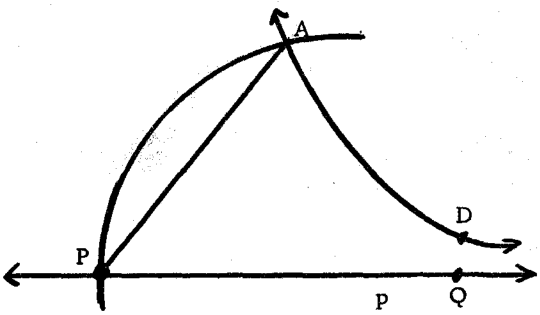
1. If $m\angle PAB = \Pi\left|\frac{[AP]}{2}\right|$ and P is in α, then P is on H.
2. If P is on H, P not A, and $\angle P''A'B' \cong \angle PAB$, then there is a unique point P' on $\overrightarrow{AP''}$ so P' is on H' and $AP \cong A'P'$.

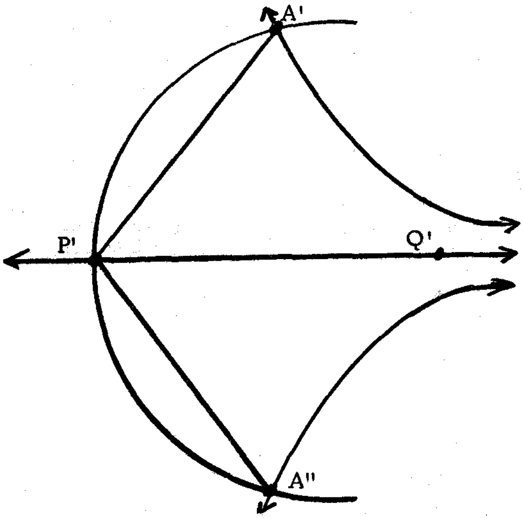
Proof: The argument for 5.6 handles both parts except the existence of P which follows from 5.1.

5.7 THEOREM. Let $\xi[A] = H$ and $\xi'[A'] = H'$ be two oricycles. There are exactly two possible choices for the values of $f: H \to H'$ so that:

1. f is a one-to-one correspondence,
2. For two specific points $P \in H$ and $P' \in H'$, $f(P) = P'$, and
3. $AB \cong f(A)f(B)$ for every two distinct points $A, B \in H$.

Proof: Let p and p' be the unique lines of ξ and ξ' which are incident upon P and P' respectively. Let A be any point of H different from P. A is thus not on p and if α is the plane containing the pencil ξ, A determines a unique half-plane of α as determined by p. Let a be the unique line of ξ incident on A. Let AD and PQ be rays of a and p so that $\overrightarrow{AD} | \overrightarrow{PQ}$. Then $\angle PAD \cong \angle APQ$ and each has "measure" $\Pi \left| \frac{[AP]}{2} \right|$ by 5.1. Let $\overrightarrow{P'Q'}$ be the ray of p' from P' in the direction of parallelism for ξ' in plane α'. Then on each side of p' in α' there is exactly one ray $\overrightarrow{P'A'}$ or $\overrightarrow{P'A''}$ so that $\angle Q'P'A' \cong \angle Q'P'A'' \cong \angle QPA$. On each ray there is exactly one point which we may name A' or A'' respectively so that $P'A' \cong P'A'' \cong PA$ with A' and A'' both on H' (by 5.6.1). Following the proof of 5.6 we see that f(A) assigned to either A' or A'' will lead to a congruence such as constructed in 5.6. A'' and A' are the only possible values of f(A).





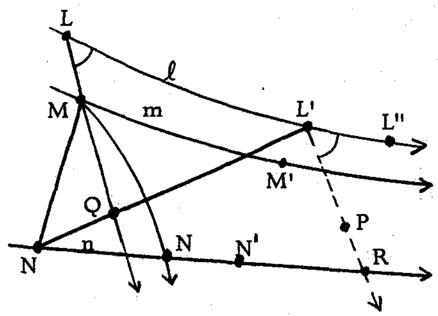
5.7.1 COROLLARY. There is a unique congruence $f: H \to H'$

so that for any two distinct points A and B of H and any two distinct points A' and B' of H' with $AB \cong A'B'$ we have $f(A) = A'$ and $f(B) = B'$.

Proof: By 5.7 we have exactly two possible congruences between H and H' with $f(A) = A'$. Exactly one of these also has $f(B) = B'$.

5.8 THEOREM. If ℓ, m, n are three distinct lines of a parallel pencil ξ in a plane α, then there is a line $p \notin \xi$ of α which meets all three of ℓ, m, n.

Proof: Let L, M and N be arbitrary points of ℓ, m, and n respectively. If L, M and N are collinear, we are done. Suppose L, M and N are not collinear. Let $\overrightarrow{LL'}$, $\overrightarrow{MM'}$ and $\overrightarrow{NN'}$ be rays of ℓ, m, and n in the direction of parallelism of ξ. If M and N are on opposite sides of ℓ then MN meets ℓ and $\underline{MN}$ is a line as desired. Suppose M, N are on the same side of ℓ. If N and L' are on the same side of $\underline{LM}$ then $\overrightarrow{LN}$ is a ray interior to $\angle L'LM$ and since $\overrightarrow{LL'} | \overrightarrow{MM'}$ we have $\overrightarrow{LN}$ meets $\overrightarrow{MM'}$ also and we are done. Suppose N and L' are on opposite sides of $\underline{LM}$. Then NL' meets $\underline{LM}$ at a point Q. Let L'' be a point of ℓ so that $L\text{-}L'\text{-}L''$. Then $\overrightarrow{L'L''} | \overrightarrow{NN'}$.



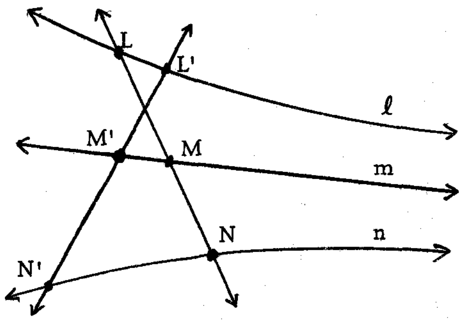
$\angle QL'L''$ is an exterior angle (of $\triangle LQL''$) opposite $\angle L$ of $\triangle QL'L$. Then there is a ray $\overrightarrow{L'P}$ interior to $\angle QL'L''$ so $\angle L'LQ \cong \angle L''L'P$ and then by definition of critical parallels, $\overrightarrow{L'P}$ meets $\overrightarrow{NN'}$ at some point R. $\underline{LM}$ meets $\triangle NL'R$ at Q' and otherwise satisfies Pasch's axiom so $\underline{LM}$ meets NR and we are done.

This theorem is surprisingly hard to prove, and unless considered carefully, seems so obvious as to not even need stating. However, it can be shown that, unlike Euclidean geometry in which every transversal meets every line of a parallel pencil, in Lobachevskian geometry no line meets every line of any parallel pencil. With this in mind one is lead to appreciate the significance of a theorem which allows us to show that for any three elements of any given pencil of parallels, there is some line which meets all three.

Theorem 5.8 gives rise to the following theorem leading to a betweenness relation for elements of a given parallel pencil.

5.9 THEOREM. If ℓ, m, n are distinct lines of a parallel pencil in a plane α and p and p' are two lines which meet ℓ, m, and n in points L, M, N and L', M', N' respectively, then L-M-N iff L'-M'-N'.

Proof: Suppose L-M-N. Then L and N are on opposite sides



of m. Now every point of ℓ is on the same side of m as is L and every point of n is on the same side of m as is N since the lines are parallel and thus do not intersect. Thus every point of ℓ is on the opposite side of m from N' and thus L'N' meets m. This is necessarily at M' so L'-M'-N'.

5.10 DEFINITION. If ℓ, m, and n are three distinct lines of a parallel pencil ξ of a plane α, we say m is between ℓ and n, written ℓ-m-n, iff some line of α meets ℓ, m, and n at L, M, and N respectively with L-M-N. (If $\ell_1, \ldots, \ell_n$ are n $(n \geq 3)$ distinct lines of a parallel pencil ξ of a plane α, we immediately get by induction, by 5.8, 5.9, and 0.10, that these lines may be labelled in such a way that ℓ_i-ℓ_j-ℓ_k iff $1 \leq i < j < k \leq n$. We write ℓ_1-ℓ_2-$\cdots$-ℓ_n).

The results of 5.8 and 5.9 assure us that 5.10 makes sense.

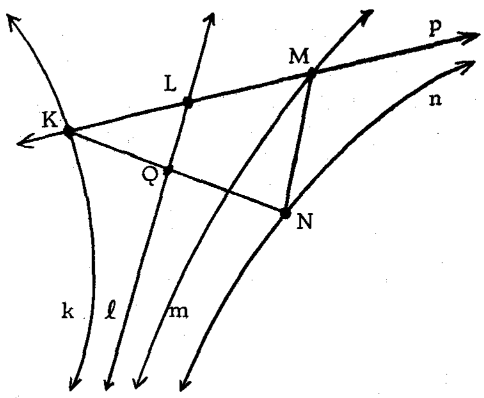
5.11 LEMMA. If k, ℓ, m, n are lines of the parallel pencil ξ of plane α, the following are true:

1. k-ℓ-m implies m-ℓ-k,
2. exactly one of k, ℓ, m is between the other two,
3. any four distinct lines can be named k, ℓ, m, n in such an

order that k-ℓ-m-n,

4. given k and ℓ, there are lines h and j of ξ so that k-h-ℓ and k-ℓ-j.

Proof: Because of the betweenness properties of points on a line, results 1., 2., and 4. follow immediately. Result 3. will be immediate if we can show there is a line which meets all four of the lines k, ℓ, m, n. By 5.8 there is a line p which meets k, ℓ and m in points K, L, M. If p meets n also, say at N, then the four points K, L, M, N can be renamed so that K-L-M-N and the obvious renaming of the lines gives the theorem. Suppose p does not meet n. Now rename as needed to get k-ℓ-m. We see that $K, L,$ and M on p (and thus k, ℓ, m) are all on the same side of n in α (otherwise KM meets n). Let N be any point of n. Then ℓ



meets either KN or MN in ΔKMN (by Pasch's axiom). If ℓ meets KN at Q, then M and N are on the same side of ℓ. This gives M on N's side of ℓ and on L's side of n. with the result that L and N are on opposite sides of m. Therefore QN meets m and $\underline{KN}$ meets all four of k, ℓ, m, n. As above these can be renamed and ordered k-ℓ-m-n. By symmetry we are done (i.e., if ℓ meets MN at Q').

5.11.1 COROLLARY. If $\ell_1, \ldots, \ell_n$ are n lines of a parallel pencil ξ in plane α, then these n lines can be named in such a way that $\ell_1 - \ell_2 - \ldots - \ell_n$. (For the proof, use induction and 5.11.)

These results give us a betweenness for elements of a parallel pencil which we will now use to develop an ordering for the points on any oricycle.

5.12 DEFINITION. If A, B, and C are distinct points of an oricycle $H = \xi[A]$, we say that B is between the points A and C on H, denoted $A*B*C$, iff the line b of ξ through B is between the lines a and c of ξ through A and C respectively in the sense of Definition 5.10. (Furthermore for any finite number of points $A_1, A_2, \ldots, A_n$, $n \geq 3$, we can rename these points so that for the corresponding lines $a_1, a_2, \ldots, a_n$ of ξ we have $a_1 - a_2 - \ldots - a_n$. We write $A_1 * A_2 * \ldots * A_n$ [cf. 5.10].)

5.13 THEOREM. If on oricycle H we have $A*B*C$, then

1. A, B, and C are distinct points of H, and
2. $C*B*A$.

Proof:

1. Follows directly from Definition 5.12,

2. Follows directly from 5.11.

5.14 THEOREM. For any two points A and C of oricycle H, there is at least one point B of H so that A*B*C, and there is at least one point D of H so that A*C*D.

The proof of 5.14 is immediate from 5.11 and so is the proof of the following theorem.

5.15 THEOREM. For any three distinct points A, B, C of oricycle H, exactly one of A*B*C, A*C*B, or B*A*C is true.

5.16 DEFINITION. If A and B are two distinct points on oricycle H, we define arc AB = {P: P $\in$ H and A*P*B}.

5.17 DEFINITION. Let H and H' be any two oricycles. We say arc AB of H is <u>congruent</u> to arc A'B' of H' iff there is a congruence f: H $\rightarrow$ H' so that f(A) = A' and f(B) = B'. We denote that arc AB is congruent to arc A'B' by

$$\text{arc } AB \cong \text{arc } A'B'.$$

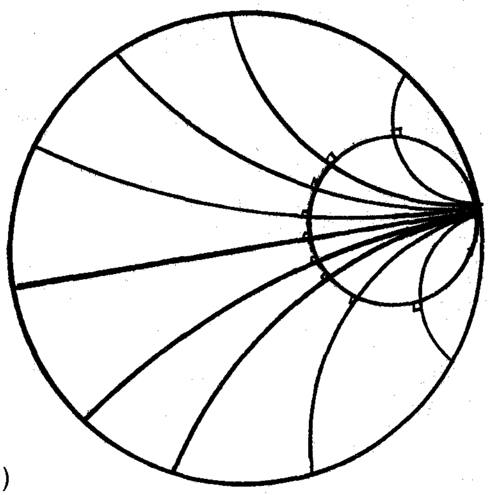
5.18 THEOREM. Congruence for arcs of oricycles is an equivalence relation.

Proof: Reflexive: arc AB $\cong$ arc AB since the identity correspondence, I, is a congruence between oricycles and I(A) = A, I(B) = B.

Symmetrical: arc AB $\cong$ arc CD implies there is a congruence of oricycles f so that f(A) = C and f(B) = D. f is a bijection so f^{-1} exists and is a congruence (cf. 5.5 and 5.7.1) so arc CD $\cong$ arc AB as desired.

Transitive: arc AB $\cong$ arc CD and arc CD $\cong$ arc EF means there are congruences of oricycles f and g so that f(A) = C, f(B) = D, g(C) = E and g(D) = F. Easily it is seen that $g \circ f$ is a congruence of oricycles so that $(g \circ f)(A) = E$, $(g \circ f)(B) = F$ and hence arc AB $\cong$ arc EF.

We can now develop an ordering for points on an oricycle, show that this ordering is a Dedekind ordering, and in general that an oricycle, while not a line (cf. 4.14), is remarkably line-like. It is probably wise (at this time) to point out that an oricycle in the Poincaré model (i.e., the realization of an oricycle in this model) is the set of points on a Euclidean circle tangent to the interior of the boundary of the model of the plane at the point (of the boundary of the model)



which is the "intersection" point of all the Poincaré lines of a given pencil. One example is shown.

5.19 DEFINITION. We say arc AB < arc CD iff there is a point E so that C*E*D and arc AB $\cong$ arc CE.

5.20 THEOREM. For every pair of oricycles $H = \xi[A]$ and $H' = \xi'[A']$, for any arc PQ of H, and for any point P' of H', there is a unique point Q' of H' on either side of the line p of ξ' incident on P' so that arc PQ $\cong$ arc P'Q'.

Proof: Let $\vec{p}'$ be the ray of p' from P' in the direction of parallelism of ξ'. On either side of p' in the plane of this pencil there is a unique ray $\vec{r}$ so that $m\angle \vec{p}'\vec{r} = \Pi\left|\frac{[PQ]}{2}\right|$. $\vec{r}$ meets H' in a unique point Q' (in either case) so that PQ $\cong$ P'Q' (by 5.1 and 5.2). By 5.7.1 there is a unique congruence of oricycles sending P to P' and Q to Q'. Thus arc PQ $\cong$ arc P'Q' for Q' on either side of p' and we are done.

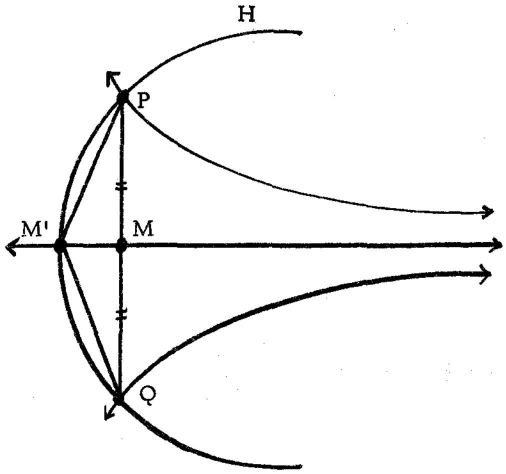
5.21 THEOREM. If

1. $\xi[A] = H$ is any oricycle in plane α,
2. P and Q are distinct points of H,
3. M is the midpoint of PQ,
4. m is the perpendicular bisector of PQ in α,

then

1. m meets arc PQ in some point M',

2. arc PM' $\cong$ arc M'Q.



Proof: 1. By 5.3, m meets H at a point M', PM' $\cong$ M'P and $m \in \xi$. Let p, q be lines of ξ incident on P, Q respectively. P-M-Q implies p-m-q and thus P*M'*Q so m meets arc PQ.

2. By 5.3 PM' $\cong$ M'Q. Let $f: H \to H$ be the congruence such that f(M') = M' and f(P) = Q (cf. 5.7.1). Thus arc PM' $\cong$ arc M'Q and we are done.

Theorem 5.21 leads immediately to the following:

5.22 DEFINITION. Let arc AC be any arc of oricycle H where H is in plane α. Define arc AB/2 to be the arc AM where M is the unique point of H on the perpendicular to AB in α. M is called the midpoint of arc AB.

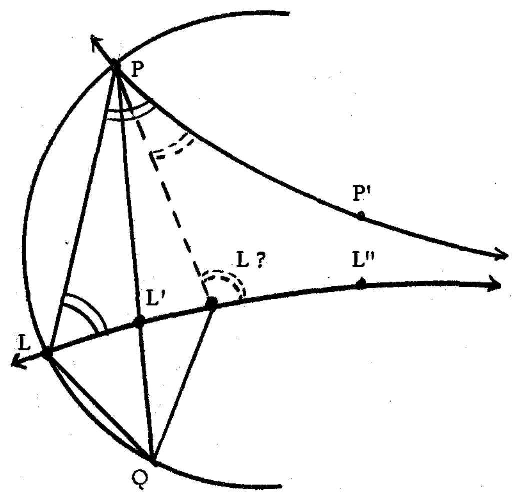
5.23 COROLLARY. If arc AB is any arc of oricycle H, then arc AB/2 $\cong$ arc BA/2.

Proof: By 5.22 we have arc AB/2 $\cong$ arc AM and arc BA/2 $\cong$ arc BM by definition. Now by 5.21, arc AM $\cong$ arc MB and by 5.18, arc MB $\cong$ arc BM. Thus, by 5.18, arc AM $\cong$ arc BM and arc AB/2 $\cong$ arc BA/2.

5.24 LEMMA. Let $\xi[P] = H$ be any oricycle in plane α in the direction of $\overrightarrow{PP'}$. Let Q be any other point of H and let p, q be the lines of ξ incident on P and Q respectively. If ℓ is any line of ξ between p and q, then ℓ meets arc PQ of H at a point L, meets PQ at a point L', and $\overrightarrow{LL'} | \overrightarrow{PP'}$.

Proof: By 5.3.1, ℓ is incident on a point L of H and by Defintion (5.12) P*L*Q making L a point of arc PQ. The betweenness for the parallel lines requires P and Q to be on opposite sides of ℓ and thus PQ meets ℓ at a point L'.

It remains to show that $\overrightarrow{LL'} | \overrightarrow{PP'}$. Let L" be a point of ℓ so that $\overrightarrow{LL''} | \overrightarrow{PP'}$. Suppose L'-L-L".



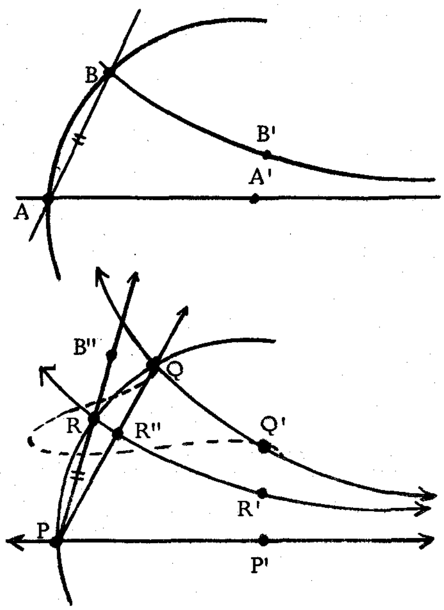
Now $\angle$PLL" and $\angle$QLL" are acute (by Theorem 5.1). But in ΔPQL we have $\angle$PLQ is the "sum" of $\angle$PLL' and $\angle$QLL', with both of the latter obtuse (since their complements are acute) which is impossible. Hence L' is on $\overrightarrow{LL''}$ and we are done.

5.25 THEOREM. If $\xi[A] = H$, $\xi'[P] = H'$ are oricycles in planes α and α' respectively, B and Q points of H and H' different from A and P respectively, $\overrightarrow{AA'} | \overrightarrow{BB'}$, $\overrightarrow{PP'} | \overrightarrow{QQ'}$ in the direction of parallelism of H and H' respectively, then the following are equivalent.

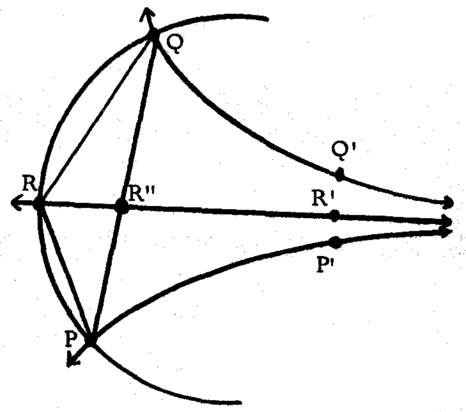
1. $AB < PQ$,
2. $\angle A'AB > \angle P'PQ$,
3. arc $AB <$ arc PQ [16, p. 44].

Proof: "1. iff 2." Immediate from 5.1 and 3.13 since $m\angle A'AB = \Pi\left|\frac{[AB]}{2}\right|$ and $m\angle P'PQ = \Pi\left|\frac{[PQ]}{2}\right|$ which, together with $AB < PQ$ implying $\frac{[AB]}{2} < \frac{[PQ]}{2}$, gives us $m\angle A'AB > m\angle P'PQ$ and the equivalence follows.

"2. implies 3." $\angle A'AB$ is acute by 5.1. Hence the unique ray $\overrightarrow{PB''}$ on Q's side of $\underline{PP'}$ such that $\angle A'AB \cong \angle P'PB''$ meets H' at a point say R, with $AB \cong PR$ (5.2). Let R' be a point of α' so that $\overrightarrow{RR'} | \overrightarrow{PP'}$. Thus #1. $\angle P'PQ < \angle P'PR$ so $\overrightarrow{PQ}$ meets $\overrightarrow{RR'}$ (by definition of critical parallels) at some point R". We need to show P*R*Q. Suppose P*Q*R. Then $\underline{PP'}$-$\underline{QQ'}$-$\underline{RR'}$ so $\overrightarrow{QQ'}$



meets $\overrightarrow{PR}$ as we already know, but also meets PR at point Q'' so that $\overrightarrow{QQ''} | \overrightarrow{PP'}$ (5.24). Hence Q'' is on $\overrightarrow{QQ'}$ requiring $\angle P'PR < \angle P'PQ$ contradicting #1 above. This gives $P{*}R{*}Q$ as desired so arc $AB <$ arc PQ.



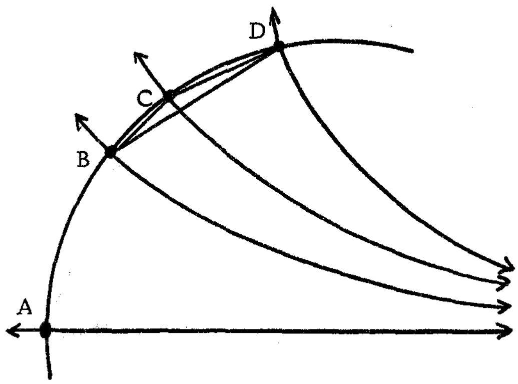
"3. implies 2." Let R be the point of arc PQ so that $P{*}R{*}Q$ and arc $AB \cong$ arc PR. Let R' be a point of α' so that $\overrightarrow{RR'} | \overrightarrow{PP'}$. Lemma 5.24 now requires $\overrightarrow{RR'}$ to meet PQ at R'' so that $\overrightarrow{RR''} | \overrightarrow{PP'}$. It is enough to show that $PR < PQ$. Since $P{*}R{*}Q$, we have $[\angle PRQ] = [\angle PRR''] + [\angle QRR'']$. But $\overrightarrow{RR''} | \overrightarrow{PP'}$ and $\overrightarrow{RR''} | \overrightarrow{QQ'}$ tells us $\angle RQR'' < \angle QRR''$ and $\angle RPR'' < \angle PRR''$ (since $\Delta P'PRR'$ and $\Delta Q'QRR'$ are isosceles closed triangles). Thus in ΔPQR, $\angle PRQ$ is the greatest angle and thus PQ the greatest side. Thus $PR < PQ$ giving $AB < PQ$, which in turn (by "1. iff 2.") gives $\angle A'AB > \angle P'PQ$ as desired.

5.25.1 COROLLARY. If $\xi[A] = H$, $\xi'[P] = H'$ are oricycles in planes α and α' respectively, and B and Q are points of H and H' respectively, then $AB \cong PQ$ iff arc $AB \cong$ arc PQ [16, p. 44].

Proof: By 5.25, arc $AB <$ arc PQ and arc $AB >$ arc PQ both require $AB \ncong PQ$, and we are done.

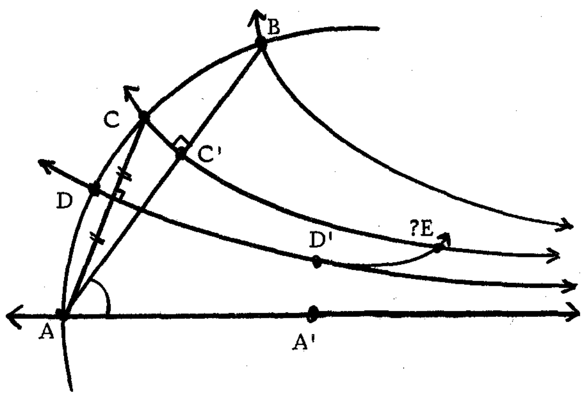
5.25.2 COROLLARY. If B and C are points of arc AD of oricycle H, and A*B*C*D, then arc BC < arc AD and BC < AD.

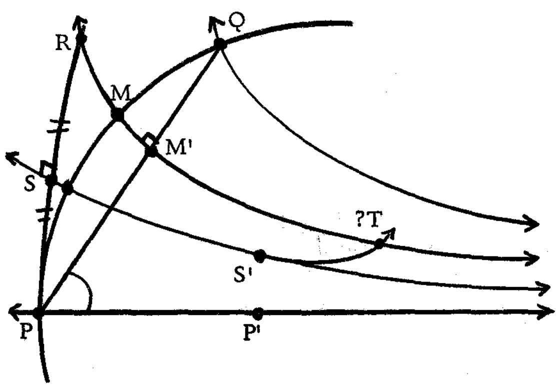
Proof: By definition arc BC < arc BD so BC < BD (5.25). Also arc BD < arc AD so BD < AD (5.25). Thus BC < AD and arc BC < arc AD by 5.25.



5.25.3 COROLLARY. If arc AB $\cong$ arc PQ, then arc AB/2 $\cong$ arc PQ/2.

Proof: Let $\xi[A] = H$ and $\xi'[P] = H'$ be oricycles in planes α and α' and in the direction of $\overrightarrow{AA'}$ and $\overrightarrow{PP'}$ respectively. By 5.25.1 AB $\cong$ PQ hence by 5.1 $m\angle BAA' = m\angle QPP'$ so from the definition of measure, $\angle BAA' \cong \angle QPP'$. Let c and m be the perpendicular bisectors of AB and PQ at C' and





M' in α and α' respectively. Then (by 5.24) c and m meet H and H' at C and M respectively so that $\overrightarrow{CC'} | \overrightarrow{AA'}$ and $\overrightarrow{MM'} | \overrightarrow{PP'}$.

Let R be a point of m on the side of $\underline{PQ}$ containing M so that $RM' \cong CC'$. Thus $\triangle ACC' \cong \triangle PRM'$ by S.A.S.

Now let d and s be the perpendicular bisectors of AC and PR at D and S in α and α' respectively. By 5.3, $d \in \xi$. Also $s \in \xi'$ since if s meets $\overrightarrow{MM'}$ at T then d meets $\overrightarrow{CC'}$ at E so that $CE \cong RT$ with $\triangle CDE \cong \triangle RST$ by S.A.S. which is impossible. By a symmetrical argument (using the result that $\angle RPM' \cong \angle CAC'$ by corresponding parts of congruent triangles and "angle sum" theorem) s is parallel to $\underline{PP'}$ and m in the direction of $\overrightarrow{MM'}$. (Note: to verify direction of parallels just consider any ray $\overrightarrow{SU}$ interior to $\angle RSS'$ where S' is on s and on the P' side of $\underline{PR}$. Then use critical parallelism of d in the $\overrightarrow{CC'}$ direction.) Hence R is on H' and on m so R = M. Thus $PM \cong AC$ and by 5.25.1 and Definition 5.22 we are done.

We are now about ready to give a Dedekind ordering to the points on any given oricycle. This can be done by constructing an "order-preserving" map as done below.

5.26 LEMMA. Let $\xi[A] = H$ be any oricycle in plane α and in the direction of $\overrightarrow{AA'}$. Let P and Q be distinct points of

H. We define a mapping $g: \text{arc } PQ \to PQ$ as follows: for each point R of arc PQ, let R' be the point of PQ so that $\overrightarrow{RR'} | \overrightarrow{AA'}$, and define $g(R) = R'$. Then

a) g is a one-to-one correspondence,

b) if R, S, T are any points of arc PQ, then $g(R)-g(S)-g(T)$ if $R*S*T$, and

c) if R is any point of arc PQ, then $P-g(R)-Q$.

Proof:

a) Clearly g makes sense since 5.24 assures us of the existence of both R and R' given either one and the uniqueness is trivial. This also assures us g is one-to-one and onto.

b) Follows immediately from Definitions 5.10 and 5.12.

c) R in arc PQ iff $P*R*Q$ so g(R) is a point of PQ and we are done.

5.27 THEOREM. Let $\xi[A] = H$ be any oricycle in plane α with arc AB any arc of H. If

1. $\underline{S}, \underline{T}$ are non-empty subsets of arc AB,
2. $\underline{S} \cup \underline{T} = \text{arc } AB$ and $\underline{S} \cap \underline{T} = \phi$, and
3. $P \in \underline{S}$ and $Q \in \underline{T}$ implies arc AP < arc AQ,

then

1. $P \in \underline{S}$ and $A*P_1*P$ (with $P_1 \in H$) implies $P_1 \in \underline{S}$,

2. $P \in \underline{T}$ and $P{*}P_2{*}B$ (with $P_2 \in H$) implies $P_2 \in \underline{T}$, and
3. There is a point K of arc AB so that if $P \in$ arc AB then arc AP < arc AK implies $P \in \underline{S}$ and arc AP > arc AK implies $P \in \underline{T}$, i.e., either $\underline{S}$ has a "last" element or $\underline{T}$ has a "first" element.

Proof: Let g: arc AB $\rightarrow$ AB be as defined in Lemma 5.26 and immediately we get

$$g(\underline{S}) \cup g(\underline{T}) = AB,$$

$$g(\underline{S}) \cap g(\underline{T}) = \phi,$$

$$g(\underline{S}) \neq \phi \quad \text{and} \quad g(\underline{T}) \neq \phi\,.$$

Furthermore arc AP < arc AQ gives us $A{*}P{*}Q$ and consequently a-p-q in which a, p, and q are the lines of ξ incident on A, P, and Q respectively so that $g(A)$-$g(P)$-$g(Q)$ and $AP' < AQ'$ (cf. Lemma 5.26). But segments have the Dedekind property, from our key assumption, so we know

1. $P' \in g(\underline{S})$ and A-P_1'-P' implies $P_1' \in g(\underline{S})$,
2. $P' \in g(\underline{T})$ and P'-P_2'-B implies $P_2' \in g(\underline{T})$, and
3. There is a point K' of segment AB so that if $P' \in AB$ then $AP' < AK'$ implies $P' \in g(\underline{S})$ and $AP' > AK'$ implies $P' \in g(\underline{T})$.

From these results and 5.26, we get the proof immediately.

This result leads directly to the Archimedian property which is our next theorem about arcs of oricycles. First we introduce another, very natural, definition. Since any line ℓ of the defining pencil of an oricycle H meets H exactly one time and ℓ separates the plane into two disjoint half planes, we state:

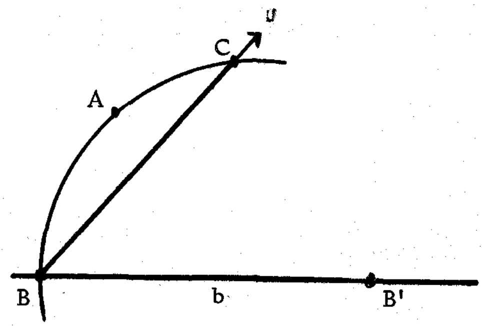
5.28 DEFINITION. Let $\xi[A] = H$ be any oricycle in plane α and in the direction of $\overrightarrow{AA'}$. Furthermore let

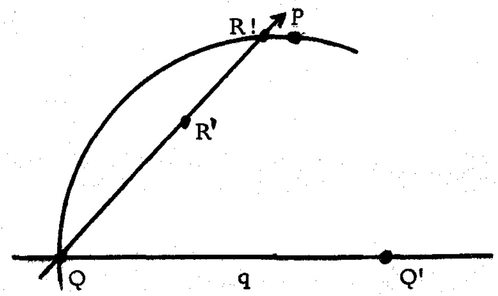
1. P, Q, R be distinct points of H, and
2. p, q, r be the lines of ξ incident on P, Q, and R respectively.

Then we say R is on the same (opposite) side of P as (from) Q in H iff r is on the same (opposite) side of p as (from) q in the pencil ξ in plane α.

5.29 LEMMA. Let $\xi[A] = H$ and $\xi'[P] = H'$ be oricycles in α and α' in the directions of $\overrightarrow{AA'}$ and $\overrightarrow{PP'}$ respectively. If arc BC is any arc of H and Q is any point of H', then on a given side of Q in H' there is a unique point R of H' so that arc BC $\cong$ arc QR.

Proof: Let b and q be the unique lines of ξ and ξ' incident on B and Q, respectively. Let B' and Q' be points of b and q, respectively, so $\overrightarrow{BB'} | \overrightarrow{AA'}$ and $\overrightarrow{QQ'} | \overrightarrow{PP'}$. Now on a given side of q there is a unique ray $\overrightarrow{QR'}$ so that $\angle Q'QR' \cong \angle B'BC$. Then by 5.6.1 there is a unique point R on $\overrightarrow{QR'}$ so that R is on H' and $BC \cong QR$, which is the same as requiring arc $BC \cong$ arc QR (by 5.25.1) and we are done.





5.30 THEOREM. (Archimedian property for arcs of oricycles) For any arc AB of oricycle H and any arc CD of oricycle H', there is a positive integer n so that, given the sequence of points of H, $A_0 = A, A_1, \ldots, A_n$ so that

1. A_i is on B's side of A, for $i = 1, \ldots, n$,
2. arc $A_{i-1}A_i \cong$ arc CD, for $i = 1, \ldots, n$,
3. $A_i{*}A_j{*}A_k$, for $0 \leq i < j < k \leq n$,

then arc $AA_n >$ arc AB.

Proof: If arc $CD >$ arc AB then on B's side of A in H there is a point A_1 of H so that arc $CD \cong$ arc AA_1 by 5.29.

If arc $CD \cong$ arc AB, let $A_1 = B$. On the side of B opposite from A on H, there is a point A_2 of H so that arc $BA_2 \cong$ arc CD by 5.29 giving $A_0 = A$, $A_1 = B$, A_2 and with $A_0{*}A_1{*}A_2$ and arc $AA_2 >$ arc AB as desired.

If arc CD < arc AB we prove by contradiction using 5.27. Suppose that for every integer n we have arc AA_n < arc AB. Let us define two sets $\underline{S}$ and $\underline{T}$ as follows:

$\underline{S} = \{P : P \in \text{arc } AB \text{ and arc } AP < \text{arc } AA_k \text{ for some } k = 1, 2, \ldots\}$

$\underline{T} = (\text{arc } AB) - \underline{S}$.

$\underline{S} \neq \phi$ since $A_1 \in \underline{S}$. In fact every A_i is in S since $AA_i < AA_{i+1}$. Also $\underline{T} \neq \phi$ since arc CD < arc AB means there is a point B' so that B*B'*A and arc CD $\cong$ arc BB'. If B' were in $\underline{S}$, then for some k arc AB' < arc AA_k < arc AB, i.e., $B'{*}A_k{*}B$ so arc A_kB < arc CD. Thus by 5.29 there is a point A_{k+1} satisfying all the above conditions and with $A{*}A_k{*}B{*}A_{k+1}$ so B is in $\underline{S}$ by 5.25.2, which is impossible. All the hypotheses of 5.27 are satisfied and hence there is a point K of arc AB so that whenever $P \in$ arc AB then arc AP < arc AK implies $P \in \underline{S}$ and arc AP > arc AK implies $P \in \underline{T}$. K is not in $\underline{S}$ because $K \in \underline{S}$ requires there be a point $A_m \in \underline{S}$ with arc AK < arc AA_m. But this means $A_m \in \underline{T}$ also which is impossible. Since $\underline{T} = (\text{arc } AB) - \underline{S}$, $K \in \underline{T}$.

arc CD < arc AK because otherwise either A_1 or A_2 would be in $\underline{T}$ which is false. Thus there is a point B' of arc AK so that (#1) arc CD $\cong$ arc B'K. Now $B' \in \underline{S}$ so there is an integer k so arc AB' < arc AA_k. This requires $A{*}B'{*}A_k{*}K$. We see that from this we get $B'*A_k*A_{k+1}*K$ by the definition of S. Thus arc CD $\cong$ arc A_kA_{k+1} < arc B'K by 5.25.2 contradicting (#1) above. Thus we have a point B' of $\underline{S}$ so that arc AB' > arc AA_k for every k so $\underline{S}$ is not a possible set and the theorem follows.

5.31 LEMMA. If

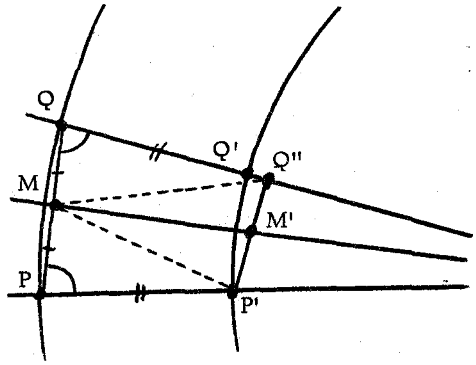
1. ξ is a parallel pencil in plane α in a given direction $\vec{a}$,
2. p and q are distinct lines of ξ, and
3. P and P' are distinct points of p,

then

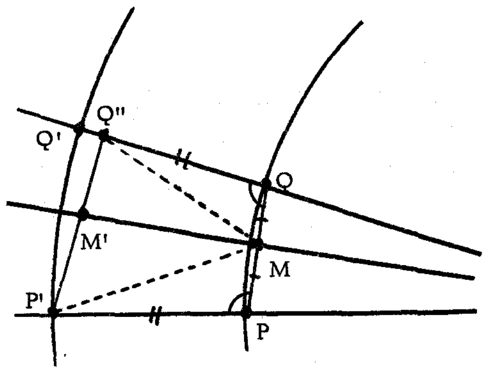
1. q is incident on two distinct points Q and Q' of oricycles $H = \xi[P]$ and $H' = \xi[P']$ respectively,
2. PP' $\cong$ QQ', and
3. if in addition $\overrightarrow{PP'}|\vec{a}$, then
   (a) $\overrightarrow{QQ'}|\vec{a}$,
   (b) PQ > P'Q',
   (c) arc PQ > arc P'Q'.

Proof: 1. Since equivalence class H is distinct from equivalence class H' and by 5.3 q is incident on a unique point Q of H and a unique point Q' of H', then Q and Q' are distinct.

2. On line q there is a point Q" so that Q" is on the same side of $\underline{PQ}$ in plane α as is P' and PP' $\cong$ QQ". (See figures below.) Now using 5.1 and supplementary angles if



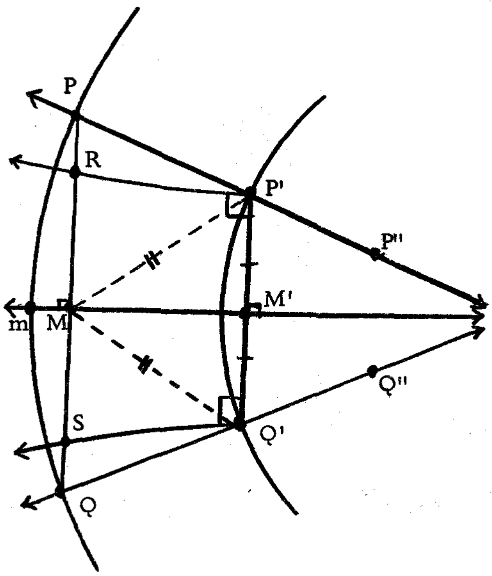
necessary, we have $\angle PQQ'' \cong \angle QPP'$. Let m be the perpendicular bisector of PQ at M and we have, by 5.3, that m is a line of ξ between p and q (by definition) so m meets $P'Q''$ at some point M' (5.3.1 and Definition 5.12). We now have $\triangle MQQ'' \cong \triangle MPP'$ by S.A.S. and by the "angle difference" theorem and corresponding congruent parts of triangles, we get $\angle Q''MM' \cong \angle P'MM'$



and $MQ'' \cong MP'$, so $\triangle Q''MM' \cong \triangle P'MM'$ by S.A.S. Thus m is the perpendicular bisector of $P'Q''$ and $P'*Q''$ by definition. But then $Q'' \in H'$ and Q'' is a point of q so by uniqueness of the symmetrical point $Q'' = Q'$ and hence $PP' \cong QQ'$ as desired.

3. (a) By the argument in 2. above we immediately see that $\overrightarrow{PP'}|\vec{a}$ forces $\overrightarrow{QQ'}|\vec{a}$ since P' and Q' are on the same side of line $\underline{PQ}$ in plane α and (a) is proved.

(b) For reference let P'' and Q'' be points of p and q so that $P\text{-}P'\text{-}P''$ and $Q\text{-}Q'\text{-}Q''$. As in (2) let m be the perpendicular bisector of PQ at M. The proof of (2) gives us that m is the perpendicular bisector of $P'Q'$ at M' as well. Now $\angle M'P'P'' \cong \angle M'Q'Q''$ are acute (by 5.1) so the lines r and s perpendicular to $\underline{P'Q'}$ at P' and Q' meet line $\underline{PQ}$ in points R and S of MP and MQ respectively. The proof of this is an easy argument using rays of r and s interior to $\angle PP'M'$ and $\angle QQ'M'$, hyperparallelism of r, s, and m and the cross-bar theorem. Then



$\triangle MM'P' \cong \triangle MM'Q'$ by S.A.S.. This gives $\angle RMP' \cong \angle SMQ'$ and $\angle MP'R \cong \angle MQ'S$ by complements of congruent angle so that $\triangle RMP' \cong \triangle SMQ'$ by A.S.A. Hence $\square P'Q'SR$ is a Saccheri quadrilateral so $P'Q' < RS < PQ$ by 3.6.3 so arc $P'Q'$ < arc PQ by 5.25 and we are done.

This theorem allows us to formulate the following definition.

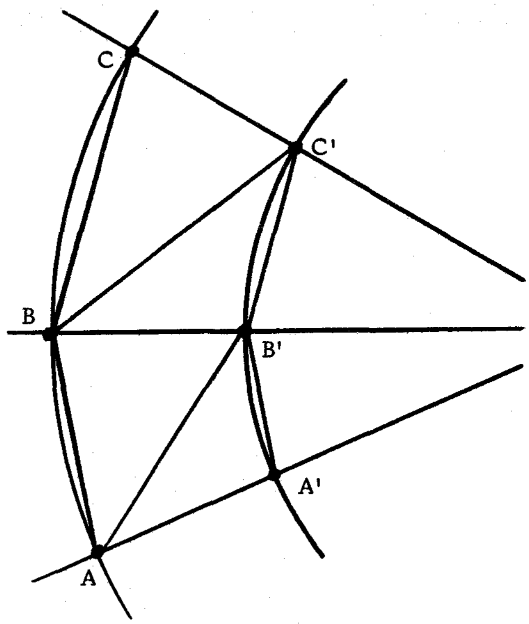
5.32 DEFINITION. Given

1. ξ is a pencil of parallels in α in the direction of $\vec{a}$,
2. p and q are distinct lines of ξ,
3. P and P' distinct points of p, and
4. Q and Q' the distinct points of $\xi[P]$ and $\xi[P']$ where q meets each of these oricycles.

The <u>distance</u> between arc PQ and arc $P'Q'$ is $|[PP']|_S$.

Furthermore we say arc PQ and arc $P'Q'$ are <u>concentric</u> arcs of $\xi[P]$ and $\xi[P']$ [16, p. 44].

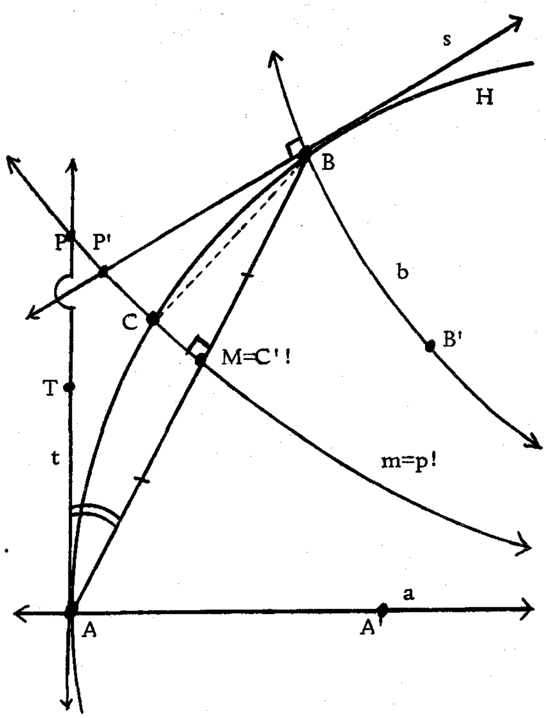
5.33 LEMMA. Let arc AB and arc A'B' be concentric arcs of $\xi[A] = H$ and $\xi[A'] = H'$, respectively, with $\overrightarrow{AA'}$ in the direction of ξ. Let C be any point of H so that A*B*C and arc AB $\cong$ arc BC. Finally let c be the line of ξ incident on C. Then c meets H' in a point C' so that A'*B'*C' and arc A'B' $\cong$ arc B'C'.



Proof: By 5.3.1, c meets H' at C' and, by Definition 5.12, $\underline{AA'}$-$\underline{BB'}$-$\underline{CC'}$ giving A'*B'*C'. Now 5.31 gives us AA' $\cong$ BB' $\cong$ CC' while 5.1, 5.2, and 5.25.1 give us that AB $\cong$ BC and $\angle$ABB' $\cong$ $\angle$BCC' so $\triangle$ABB' $\cong$ $\triangle$BCC' by S.A.S. Thus, using corresponding parts and the "angle subtraction" theorem we have also $\triangle$AA'B' $\cong$ $\triangle$BB'C'. Hence A'B' $\cong$ B'C' and by 5.25.1 we have arc A'B' $\cong$ arc B'C' as needed.

We leave this particular discussion for a time to proceed with the development of the measure of arcs of oricycles. First we establish two very useful theorems.

5.34 THEOREM. If A, B are distinct points of oricycle $H = \xi[A]$ in plane α and in the direction of $\overrightarrow{AA'}$ and if $|[AB]|_S < 2\Pi^{-1}(\pi/4)$, then t and s, the tangent lines to H at A and B, respectively, meet at a point P. Furthermore, if p is the line of ξ incident on P, then p is incident on a unique point C of arc AB, on a point C' of AB and $\overrightarrow{CC'}|\overrightarrow{AA'}$.



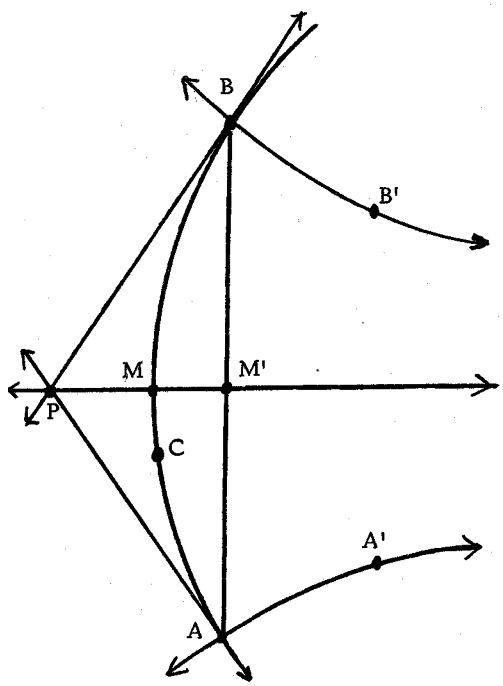
Proof: Let M be the midpoint of AB and let a, m, b be the lines of ξ incident on A, M, B respectively and let B' be a point of b so that $\overrightarrow{BB'}|\overrightarrow{AA'}$. Now $m\angle A'AB = \Pi|\frac{[AB]}{2}| = \Pi|AM|$ by 5.1 and $|[AM]|_S < \Pi^{-1}(\pi/4)$. Thus $[\angle A'AB] > 1/2[\angle rt]$ (by 3.13). Let T be a point of t on M's side of a. Then $[\angle TAM] < 1/2[\angle rt]$ so $\overrightarrow{AT}$ meets m at a point P (by definition of critical angles). By symmetry S meets m at a point P'. But $\triangle AMP \cong \triangle BMP'$, by A.S.A., so $P = P'$ and m is the perpendicular bisector of AB. By definition, m is between a and b so m meets arc AB at C and AB at C'. Then $\overrightarrow{CC'}|\overrightarrow{AA'}$ as desired, by 5.24. Finally, let

$p = m$ and the theorem is proved.

5.34.1 COROLLARY. For the points P, C, and C' of Theorem 5.34, $P\text{-}C\text{-}C'$.

Proof: C is on H so $\angle A'AC$ and $\angle B'BC$ are both acute, by 5.1. Further C is between a and b, so C is interior to both $\angle B'BP$ and $\angle A'AP$ giving us C interior to $\angle APB$. By 5.25 we are done.

5.35 THEOREM. Let $\xi[A] = H$ be any oricycle in plane α and in the direction of $\overrightarrow{AA'}$. Let A, B be distinct points of H with $|AB|_S < 2\Pi^{-1}(\pi/4)$. If P is the point of intersection of the lines t and s which are tangent to H at A and B respectively, then every point C of arc AB is in the interior of $\angle APB$.



Proof: Let B' be a point of α so $\overrightarrow{BB'} | \overrightarrow{AA'}$. Then, by 5.1 for every point C of arc AB, $\angle B'BC$ and $\angle A'AC$ are acute so C is on the same side of $\overline{AP}$ as A', the same side of $\overline{BP}$ as A' and, by the definition of arc BC, C is between $\overline{AA'}$ and $\overline{BB'}$ so C is interior to $\angle APB$ as claimed.

The argument has now progressed to the place where we can develop a measure for arcs of oricycles that will give the equations tying the values of the Lobachevskian function Π to the values of $|\ |_A$ already selected, will give trigonometric identities allowing precise study of the coordinatized Lobachevskian space, and ultimately give the canonical isomorphism needed to finish the proof that the completeness axiom is a theorem in the Poincaré model. The length of this argument is long but seems necessary to establish the validity of the "obvious" interrelated properties of oricycles, lines of defining pencils, "chords" of oricycles, arcs of oricycles, etc.

Lobachevski reaches the conclusions developed so far concerning oricycles in an argument covering about three pages [9, p. 30-33]. He does not establish the validity of these results in either an exhaustive manner nor in a way that easily adapts to the Hilbert formulation of the axioms. In Shirikov [16] one sees an outline of some of these results but again large parts of the necessary argument are not even mentioned as needing to be done. Borsuk [2] follows a development which requires a considerable further development of the topology than is necessary in this paper's approach. His argument is not noticeably shorter even after the topology is developed so it seems reasonable to follow a more classical approach.

Before continuing further, we draw upon Theorem 5.18, which states that congruence of arcs of oricycles is an equivalence relation to formulate the following definitions. These definitions will make the subsequent work more concise.

5.36 DEFINITION. [arc AB] is the equivalence class of arcs of oricycles congruent to arc AB. We define [arc AB] < [arc CD] iff arc AB < arc CD.

5.37 DEFINITION. [arc AB] + [arc CD] is the equivalence class of arcs of oricycles [arc EF] so that if arc EF is a representative, then there is a point G of arc EF so that arc AB $\cong$ arc EG and arc CD $\cong$ arc GF. As usual we will use the symbol n[arc AB] to mean [arc AB] + ... + [arc AB] with n-summands.

5.38 DEFINITION. 1/2[arc AB] is the class of arcs [arc AM] so that M is the midpoint of arc AB (in the notation of 5.22 1/2[arc AB] = [arc AB/2].). Inductively we define $1/2^k$[arc AB] = 1/2($1/2^{k-1}$[arc AB]).

5.39 DEFINITION. We define 0[arc AB] to be the empty class of arcs of oricycles with the property that 0[arc AB] + [arc CD] = [arc CD].

5.40 LEMMA. If arc AB is an arc of oricycle H in plane α, $A_0 = A, A_1, \ldots, A_n$ is a sequence of points of H so that

1. A_i is on B's side of A, for $i = 1, \ldots, n$,

2. arc $A_{i-1}A_i \cong$ arc A_0A_1, for $i = 1, \ldots, n$, and

3. $A_i*A_j*A_k$, for $0 \leq i < j < k \leq n$,

then [arc AA_n] = n[arc AA_1].

Proof: [arc AA_2] = 2[arc AA_1] by Defintion 5.38 and by induction [arc AA_n] = n[arc AA_1] using Definition 5.38.

5.41 DEFINITION. A finite set of distinct points $\underline{P} = \{A_0 = A, A_1, \ldots, A_n = B\}$ so that $A_1, \ldots, A_{n-1}$ are points of arc AB of oricycle H in plane α, and $A_0*A_1*\ldots*A_n$ is called a partition of arc AB.

5.42 DEFINITION. Let $\underline{P} = \{A_0 = A, A_1, \ldots, A_n = B\}$ be a partition of arc AB and let $L(\underline{P}) = \sum_{i=1}^{n} |A_{i-1}A_i|_S$. Let $S = \{L(\underline{P}) : \underline{P}$ is a partition of arc AB$\}$. Then we say the <u>length of arc AB</u>, denoted $\widehat{AB}$, is the l.u.b.S.

(Note: If l.u.b.S exists we will say <u>$\widehat{AB}$ exists.</u>) This definition is a standard sort of formulation. It makes sense provided S is in fact bounded above. The following lemmas will verify that S is in fact always bounded above.

5.43 LEMMA. If $\underline{P}$ and $\underline{Q}$ are partitions of arc AB, and

$\underline{P} \subsetneq \underline{Q}$, then $L(\underline{P}) < L(\underline{Q})$.

Proof: For at least two points P_{i-1}, P_i of $\underline{P}$ there is at least one point Q_j of $\underline{Q}$ so that $P_{i-1}*Q_j*P_i$. P_{i-1}, Q_j and P_i are noncollinear (4.14) so by the triangle inequality of absolute geometry, $[P_{i-1}Q_j] + [Q_jP_i] > [P_{i-1}P_i]$, so $L(\underline{P}) < L(\underline{P} \cup \{Q_j\})$. Repeating this argument a finite number of times using the properties of betweenness for points on an arc of an oricycle we get a finite sequence of inequalities $L(\underline{P}) < L(\underline{P} \cup \{Q_j\}) < \ldots < L(\underline{Q})$ with transitivity giving the desired result.

5.44 LEMMA. Let arc AB be an arc of oricycle H in plane α. Let arc CD be any arc of an oricycle with arc AB $\cong$ arc CD. Suppose $\widehat{AB}$ exists. Then $\widehat{CD}$ exists and $\widehat{AB} = \widehat{CD}$.

Proof: Let $\underline{P} = \{C_0, \ldots, C_n\}$ be any partition of arc CD. Let $\underline{Q} = \{A_0, \ldots, A_n\}$ be defined by

(a) $A_0 = A$ and $A_n = B$,

(b) A_i is on B's side of A on oricycle H, $0 < i < n$,

(c) $[\text{arc } AA_i] = [\text{arc } CC_i]$, $0 < i \leq n$.

From the definition of a partition, we know arc $CC_i <$ arc CC_{i+1} for $0 < i < n$ and thus by (c) we know arc $AA_i <$ arc AA_{i+1}, for $0 < i < n$, and $\underline{Q}$ is thus a partition of arc AB by definition.

We will now show $A_{i-1}A_i \cong C_{i-1}C_i$ for $0 < i \leq n$. Using (b), (c), 4.14, 5.1, 5.25, 5.25.1, 5.25.2, and the "angle difference" theorem from absolute geometry, we have

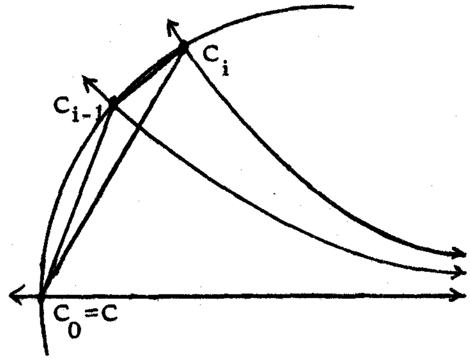
(in $\triangle AA_{i-1}A_i$ and $\triangle CC_{i-1}C_i$) that

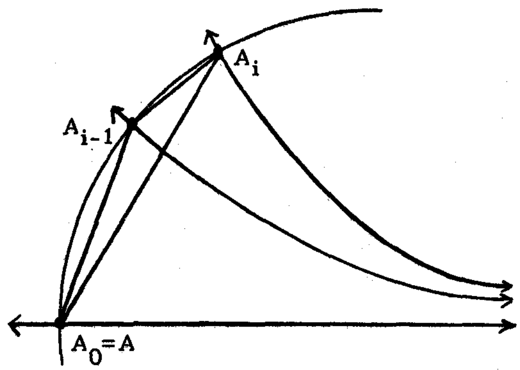
$\angle A_{i-1}AA_i \cong \angle C_{i-1}CC_i$, $1 < i \leq n$

$AA_{i-1} \cong CC_{i-1}$, $2 \leq i \leq n$

$AA_i \cong CC_i$, $2 \leq i \leq n$

thus $\triangle AA_{i-1}A_i \cong \triangle CC_{i-1}C_i$ by S.A.S. giving $A_{i-1}A_i \cong C_{i-1}C_i$ as desired.





Thus by symmetry we can conclude that every partition $\underline{P}$ of arc CD corresponds to a partition $\underline{Q}$ of arc AB so that $L(\underline{P}) = L(\underline{Q})$ and conversely. Hence, by definition, $\widehat{CD}$ exists and $\widehat{AB} = \widehat{CD}$.

5.45 LEMMA. If B is a point of arc AC and $\widehat{AB}$, $\widehat{AC}$, and $\widehat{BC}$ exist, then $\widehat{AB} + \widehat{BC} = \widehat{AC}$.

Proof: B a point of arc AC implies that for every point D of arc BC, A*B*D*C (by definition). Thus if

$\underline{P} = \{A_0 = A, A_1, \ldots, A_m = B\}$ and $\underline{Q} = \{B_0 = B, B_1, \ldots, B_n = C\}$ are partitions of arc AB and arc BC respectively, then

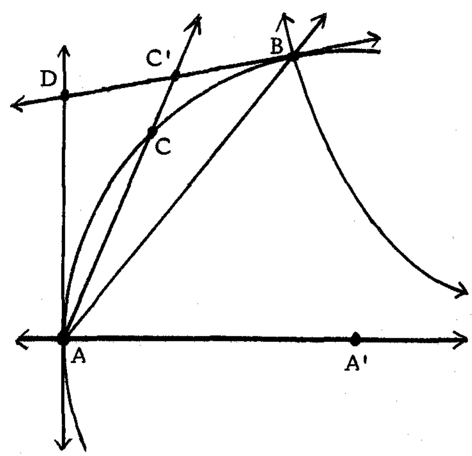
$\underline{R} = \{D_0, \ldots, D_{m+n}\}$ with

$$D_i = \begin{cases} A_i & \text{for } 0 \leq i \leq m, \ A_i \in \underline{P}, \\ B_{i-m} & \text{for } m \leq i \leq m+n, \ B_{i-m} \in \underline{Q}, \end{cases}$$

is a partition of arc AB and $L(\underline{P}) + L(\underline{Q}) = L(\underline{R})$.

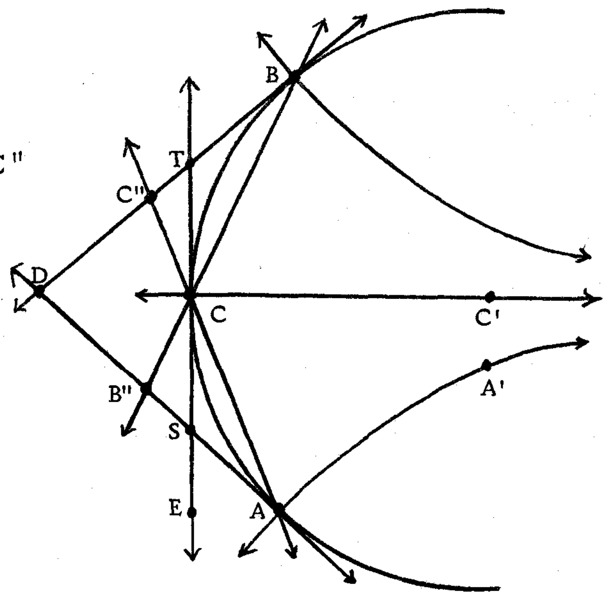
Using Lemma 5.43, we see that any partition $\underline{R}$ of arc AC can be enlarged (if necessary) to $\underline{R}' = \underline{R} \cup \{B\}$ giving partition $\underline{P}'$ and $\underline{Q}'$ of arc AB and arc BC respectively in which $L(\underline{P}') + L(\underline{Q}') = L(\underline{R}') \geq L(\underline{R})$. By Definition 5.42 the theorem follows.

5.46 LEMMA. Let H be an oricycle in plane α and in the direction of $\overrightarrow{AA'}$ and let arc AB be any arc of H so $|[AB]|_S < 2\Pi^{-1}(\pi/4)$. If D is the point of intersection of the tangents to H at A and B in α, and C is any point of arc AB, then $\overrightarrow{AC}$ meets BD at some point C'.



Proof: By 5.34 and 5.35, the point D exists and C is interior to $\angle ADB$. By 5.1, 5.16, 5.25 we have $\angle A'AB < \angle A'AC < \angle A'AD$ so $\overrightarrow{AC}$ is interior to $\angle DAB$ and thus meets BD at a point C' by the cross-bar theorem applied to $\triangle ABD$.

5.47 LEMMA. Let H be an oricycle in plane α and in the direction of $\overrightarrow{AA'}$. Let arc AB be any arc of H so that $|[AB]|_S < 2\Pi^{-1}(\pi/4)$. Let D be the point of intersection of the tangents to H at A and B in α. Let C be any point of arc AB. If B" and C" are the points of $\overrightarrow{BC}$ on AD and $\overrightarrow{AC}$ on BD respectively, then the tangent t to H at C in α meets AB" at S and BC" at T.



Proof: Let $\overrightarrow{CC'} | \overrightarrow{AA'}$. (The existence of B", C" and D is assured by 5.34 and 5.46.) Let E be any point of t on A's side of $\underline{CC'}$. By 5.1 we know that t is perpendicular to $\underline{CC'}$ while $\angle C'CA$ and $\angle C'CB$ are acute. Furthermore, by the definition of supplementary angles $\angle C'CB''$ is obtuse since $\angle C'CB$ is acute. Thus we have

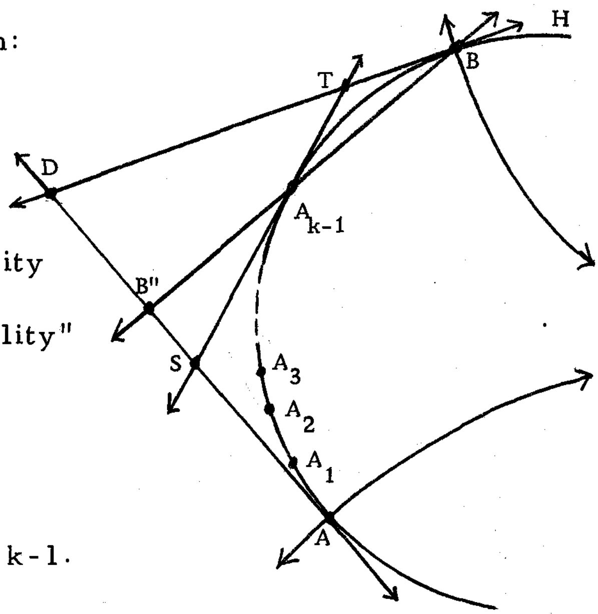
$$\angle C'CA < \angle C'CE < \angle C'CB'' .$$

Then by the cross-bar theorem $\overrightarrow{CE}$ meets AB" at S, and by symmetry we are done.

5.48 LEMMA. Let arc AB be an arc of oricycle H in plane α and in the direction of $\overrightarrow{AA'}$. Let $|[AB]|_S < 2\Pi^{-1}(\pi/4)$. Then $\widehat{AB}$ exists.

Proof: By 5.34 and 5.35 the tangents to H at A and B in α meet in a point D and every point of arc AB is interior to $\angle ADB$. Let $\underline{P}_n = \{A_0, \ldots, A_n\}$ be a partition with n+1 distinct elements $A_0 = A, \ldots, A_n = B$ of arc AB. We use induction on n to prove the following proposition:

$L(\underline{P}_n) < |[AD]|_S + |[BD]|_S$.



Let n = 1: Then $L(\underline{P}_1) = |[AB]|_S$ and the inequality follows from the "triangle inequality" theorem of absolute geometry.

Suppose the proposition is true for every partition with n = k-1. Let $\underline{P}_k$ be any partition as above. Then, by 5.46 $\overrightarrow{BA_{k-1}}$ meets AD at B" and the tangent to H at A_{k-1} in α meets AB" at S, by 5.47. By the induction hypotheses and 5.35 every point of $\underline{P}_k - \{B\}$ is interior to $\angle ASA_{k-1}$ and $L(\underline{P}_k - \{B\}) < |[AS]|_S + |[SA_{k-1}]|_S$. Using the "triangle inequality" we get

$$[AD] + [DB] = [AS] + [SB''] + [B''D] + [DB]$$
$$> [AS] + [SB''] + [B''B]$$
$$= [AS] + ([SB'']+[B''A_{k-1}]) + [A_{k-1}B]$$
$$> [AS] + [SA_{k-1}] + [A_{k-1}B]$$

Hence

$$|[AD]|_S + |[DB]|_S > |[AS]|_S + |[SA_{k-1}]|_S + |[A_{k-1}B]|_S$$
$$> L(P_k - \{B\}) + |[A_{k-1}B]|_S$$
$$= L(P_k) .$$

So by induction $L(P_n)$ is bounded above for all n and by Definition 5.42, $\widehat{AB}$ exists.

5.49 THEOREM. For any arc PQ of oricycle H in plane α, $\widehat{PQ}$ exists.

Proof: If $|[PQ]|_S < 2\Pi^{-1}(\pi/4)$, we are done, using 5.47. Let arc AB be any arc of H so that $|[AB]|_S < 2\Pi^{-1}(\pi/4)$. Then by 5.30, 5.40, and well-ordering, there is a number $n = 1, 2, \ldots$ so that $n[\text{arc } AB] > [\text{arc } PQ]$ and so that $(n-1)[\text{arc } AB] \leq [\text{arc } PQ]$. If $(n-1)[\text{arc } AB] = [\text{arc } PQ]$ then using induction, 5.45, and 5.48, $n\widehat{AB} = \widehat{PQ}$ and we are done. If $(n-1)[\text{arc } AB] < [\text{arc } PQ] < n[\text{arc } AB]$ then there is an arc class $[\text{arc } BC] < [\text{arc } AB]$ so that $(n-1)[\text{arc } AB] + [\text{arc } BC] = [\text{arc } PQ]$. Thus, again by induction, 5.45, and 5.48, $(n-1)\widehat{AB} + \widehat{BC} = \widehat{PQ}$. This completes the argument.

It is interesting to look back over the necessary machinery needed to translate the three pages of Lobachevski's work (mentioned earlier) into an appropriate sequence of arguments based on the axioms of Hilbert. It was not easy, but the results will prove to be well worth the effort.

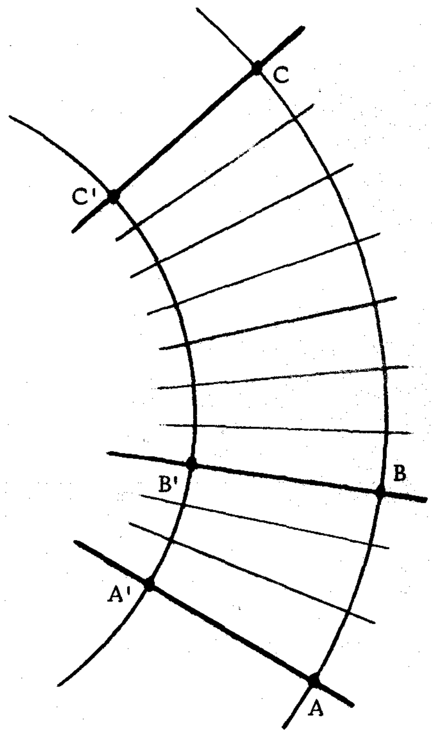
5.50 THEOREM. arc AB $\cong$ arc CD iff $\widehat{AB} = \widehat{CD}$.

Proof: By 5.49, $\widehat{AB}$ and $\widehat{CD}$ both exist. The "only if" part of this statement is just Lemma 5.44. To establish the "if" part we establish the contrapositive. If arc AB < arc CD then there is a point E of arc CD so that arc AB $\cong$ arc CE making $\widehat{AB} = \widehat{CE}$ by 5.44. However, by 5.45, $\widehat{CE} + \widehat{ED} = \widehat{CD}$ so that $\widehat{AB} < \widehat{CD}$. By symmetry we have that arc AB $\not\cong$ arc CD implies $\widehat{AB} \neq \widehat{CD}$ and by this contrapositive we conclude the theorem is true.

5.51 THEOREM. Let arc AB and arc A'B' be concentric arcs (cf. 5.35). Then the ratio $\widehat{AB}/\widehat{A'B'}$ depends only on the distance between the arcs [16, p. 45].

Proof: There is no loss of generality to suppose A*B*C on H and do the proof as follows.

Suppose arc AB and arc BC are commensurable, i.e., there is an arc PQ so that m[arc PQ] = [arc AB] and n[arc PQ] = [arc BC] for some positive integers m, n. Implicitly,



we thus get sequences of points $\{A_i\}_0^m$ and $\{B_j\}_0^n$, of arc AB and arc BC respectively so that $A = A_0 * A_1 * \ldots * A_m = B = B_0 * B_1 * \ldots * B_n$ with arc $A_{i-1}A_i \cong$ arc PQ $\cong$ arc $B_{j-1}B_j$, $1 \leq i \leq m$, $1 \leq j \leq n$ or (by 5.50) $\widehat{A_{i-1}A_i} = \widehat{PQ} = \widehat{B_{j-1}B_j}$. By 5.12 and 5.11.1 this induces sequences $\{A_i'\}_0^m$ and $\{B_j'\}_0^n$, of arc A'B' and arc B'C' respectively, with $A' = A_0' * A_1' * \ldots * A_m' = B' = B_0' * B_1' * \ldots * B_n'$. Furthermore by 5.33 and 5.50, $\widehat{A_{i-1}'A_i'} = \widehat{P'Q'} = \widehat{B_{j-1}'B_j'}$, $1 \leq i \leq m$, $1 \leq j \leq n$, where [arc P'Q'] = [arc $A_0'A_1'$]. By 5.31, arc PQ $\not\cong$ arc P'Q' and if $\overrightarrow{AA'}$ is in the direction of the pencil defining arc AB, then arc P'Q' < arc PQ. However, by 5.37, 5.45, and 5.50 we have

$$\frac{\widehat{AB}}{\widehat{BC}} = \frac{m\widehat{PQ}}{n\widehat{PQ}} = \frac{m}{n} = \frac{m\widehat{P'Q'}}{n\widehat{P'Q'}} = \frac{\widehat{A'B'}}{\widehat{B'C'}}$$

so that $\dfrac{\widehat{AB}}{\widehat{A'B'}} = \dfrac{\widehat{BC}}{\widehat{B'C'}}$ and by 5.31 this ratio depends on the distance between the arcs.

If arc AB and arc BC are non-commensurable we can apply the classical and well known limiting arguments used in such arguments as this [10, Chap. 20].

The results of 5.51 were observed and stated by Lobachevski [9, p. 32-33]. He also stated the results of the following corollary:

5.51.1 COROLLARY. If arc AB and arc A'B' are concentric arcs and $\overrightarrow{AA'}$ is in the direction of the associated pencil of parallels, then $\widehat{AB} / \widehat{A'B'} > 1$.

Proof: Immediate from 5.31, 5.50, and 5.51.

Lobachevski further observes (in our notation) that:

> If we therefore for $|[AA']|_S = 1$ put $\widehat{AB} = e\widehat{A'B'}$, then we must have for every x, $\widehat{A'B'} = \widehat{AB}e^{-x}$. Since e is an unknown number only subjected to the condition $e > 1$ and further the linear unit for x may be taken at will, therefore we may, for the simplification of reckoning, so choose it that by e is to be understood the base of Napierian logarithms [9, p. 33].

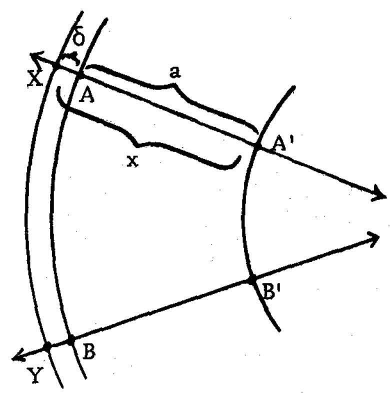
This choice of e can be more directly justified on the basis of our present foundation. Shirokov [16, p. 46] gives basically the following argument.

5.52 THEOREM. Let arc AB and arc A'B' be concentric arcs so that $\overrightarrow{AA'}$ is in the direction of parallelism of the associated pencil of parallels. If $x = |[AA']|_S$ then $\widehat{AB}/\widehat{A'B'} = e^{x/k}$ where k is a positive real number [16, p. 46].

Proof: Let arc A"B" be a third concentric arc so that A-A'-A" and let $y = |[A'A'']|_S$. Then by 5.51:

$$\frac{\widehat{AB}}{\widehat{A'B'}} = f(x)\ , \quad \frac{\widehat{A'B'}}{\widehat{A''B''}} = f(y)\ , \quad \frac{\widehat{AB}}{\widehat{A''B''}} = f(x+y)$$

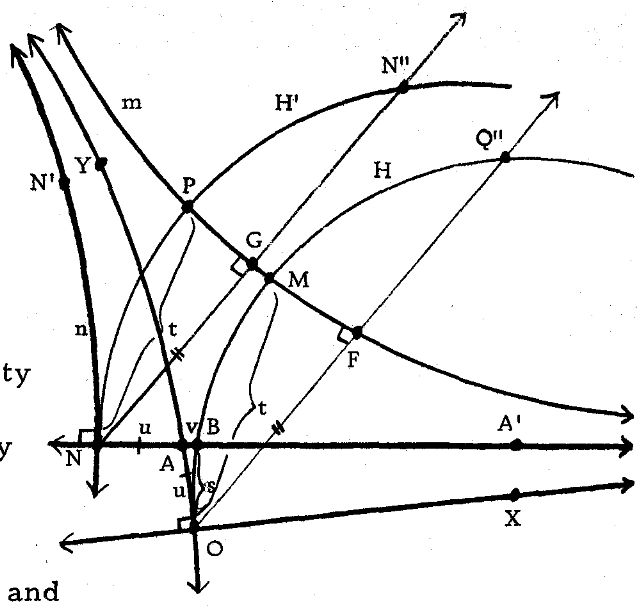
and in each case the value of f is greater than 1 (by 5.51.1). Thus (#1) $f(x) \cdot f(y) = f(x+y)$ and, by 5.31, f is increasing. To show f is an exponential function, we need only show f is also continuous and positive. From our "key assumption" we know every real number corresponds to a point on line $\underline{AA'}$ so that for every $x > 0$, there is an arc XY with X on $\underline{AA'}$ and concentric with arc AB. Thus for every x, $f(x)$ is defined. It remains to show $\lim_{x \to a} f(x) = f(a)$ for every $a > 0$. Intuitively this is obvious but we require a formal argument. Let arc XY, arc AB and arc A'B' be concentric arcs with $|[AA']|_S = a$ and $|[XA]|_S = \delta$. If $\overrightarrow{XA}$ is in the direction of parallelism, then by 5.31 $XY > AB$. By the polygonal inequality from absolute geometry [10, p. 124] and from 5.31 $[XY] \leq [AB] + 2[XA]$, i.e., $[XY] - [AB] \leq 2[XA]$ so that $|[XY] - [AB]|_S \leq 2\delta$. Thus in the limit $|[XY]|_S = |[AB]|_S$ and thus by 5.25.1 arc XY $\cong$ arc AB in the limit, so that in the limit $\widehat{XY} = \widehat{AB}$ and $f(x) = f(a)$. By a symmetrical argument with $\overrightarrow{AX}$ in the direction of parallelism, the argument is complete.



5.53 THEOREM. In any plane α, let

1. ξ be the pencil in the direction of $\overrightarrow{OX}$,
2. $\underline{OY} \perp \underline{OX}$ at O,
3. A be any point of $\underline{OY}$ different from O,
4. $\underline{AA'}$ be the line of ξ incident on A with $\overrightarrow{AA'} | \overrightarrow{OX}$,
5. B be the unique point of $\overrightarrow{AA'}$ on $H = \xi(O)$ (cf. 5.3.1 and 5.1),
6. the line m be the unique line of ξ which is parallel to both sides of right angle $\angle YOX$ (cf. 3.21.1),
7. M be the unique point of m on H (cf. 5.31), and
8. $s = \widehat{OB}$, $t = \widehat{OM}$, $u = |[OA]|_S$, $v = |[AB]|_S$,

then $s = t \tanh(u/k)$ and $e^{(v/k)} = \cosh(u/k)$ where k is the constant in 5.52 [16, p. 48-49].



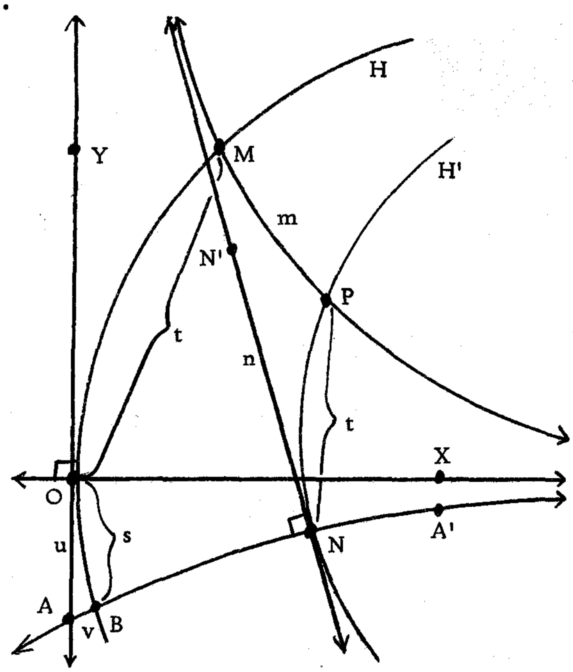
Proof: Case I. A is on $\overrightarrow{OY}$. Without loss of generality we may suppose O-A-Y. By 5.1 $\angle OAA'$ is acute and by the "vertical-angle" theorem and 3.20, there is a line n on the side of α opposite A' with respect to $\underline{OY}$ so that n is perpendicular to $\underline{AA'}$ at some point

N and parallel to $\underline{OY}$ in the direction of $\overrightarrow{AY}$. Let $\overrightarrow{NN'} | \overrightarrow{AY}$ with N' on n. Let $H' = \xi[N]$. By 5.3.1, m meets H' at a point P. Since m is parallel to $\underline{OY}$ in the direction of $\overrightarrow{OY}$ (6. of hypothesis) and $\overrightarrow{NN'} | \overrightarrow{OY}$ and $\overrightarrow{NA} | \overrightarrow{OX}$, from 3.5.11 ("transitivity" of critical parallels) m is parallel to both sides of right angle $\angle A'NN'$. If F and G are the feet of the perpendiculars from O and N, respectively, to m, then $OF \cong NG$, by 3.24.

We now observe that on the opposite side of m from O and N there are points O" and N" of $\underline{OF}$ and $\underline{NG}$ respectively with $OF \cong FO"$ and $NG \cong GN"$. Thus by definition (4.5 and 4.8) O" and N" are on H and H' respectively. But m is the perpendicular bisector of OO" and of NN" so (using 5.25.1) arc NN" $\cong$ arc OO" and (by 5.25.3) arc OM $\cong$ arc NP. This gives us $\overset{\frown}{OM} = \overset{\frown}{NP} = t$ by hypothesis 8. and by 5.50, $\angle NAY \cong \angle OAB$ by the "vertical-angle" theorem. From the facts that $\overrightarrow{NN'} | \overrightarrow{AY}$ and $\overrightarrow{OX} | \overrightarrow{AA'}$, we see by definition that $\angle NAY$ and $\angle OAB$ are the angles of parallelism for segments AN and AO which requires $AN \cong AO$ (3.9) so $|[AN]|_S = u$.

We now have arc NP and arc BM as concentric arcs with $\overrightarrow{NB}$ in the direction of the pencil associated with H and H'. Also we have N-A-B. Thus the distance between these arcs is u+v (from our work above) and by 5.52

$$\frac{t}{t-s} = e^{(u+v)/k} \quad \text{or} \quad t-s = te^{(-u-v)/k} \tag{i}$$



Case II. Suppose A-O-Y. Now on X's side of α with respect to line OY, there is a line n perpendicular to AA' at N and there is a point N' of n so that $\overrightarrow{NN'} | \overrightarrow{OY}$ (by 3.20). Let $H' = \xi[N]$. Then since $m \in \xi$ there is a unique point P of H' on m (5.3.1). Just as in Case 1, arc OM $\cong$ arc NP and AN $\cong$ OA. Thus $\widehat{NP} = t$ and we get

$$\frac{t+s}{t} = e^{(u-v)/k} \quad \text{or} \quad t+s = te^{(u-v)/k} \ . \tag{ii}$$

Now adding (i) and (ii) we get $2t = t(e^{(u-v)/k} + e^{(-u-v)/k})$ so that

$$e^{v/k} = \frac{e^{u/k} + e^{-u/k}}{2} = \cosh(u/k) \tag{iii}$$

Subtracting (i) from (ii) gives us

$$s = t\,\frac{e^{u/k} - e^{-u/k}}{2e^{v/k}} \ .$$

Thus by (iii) we have

$$s = t \tanh(u/k).$$

From our construction we see t is constant so s is a function of u. This completes the argument.

Our next step is to extend the notion of oricycles to orispheres. When that is done we shall return to the main argument of the proof of the completeness axiom as a theorem in the Poincaré model.

It is interesting to note that Shirokov has gotten the above results from a different base in just 15 pages having provided proof of each step. He uses Hilbert's axioms but does not consider the Archimedian and completeness axioms however. Doing so requires substantial additional effort. As has already been mentioned, Shirokov's proofs given for the development of orispheres are sound once the above "fill-in" has been done. We shall not copy the proofs of his work on orispheres into this paper but rather give the results and provide proof only when necessary to develop a result not previously done or not in the spirit of Hilbert's axiomatic treatment.

Lobachevski defined orispheres in terms of the revolving of an oricycle about any line of the defining pencil [9, p. 33]. Shirokov defines them in a manner analogous to that which he gives for oricycles. Before stating his definition he develops some results relating to the notion of parallelism for lines in space and the notion of parallel as extended to planes. His first lemma is, when translated into our notation:

5.54 LEMMA. Let a and b be parallel lines in plane γ, parallel in the direction of $\vec{b}$. If α and β are distinct intersecting planes containing a and b respectively, then their line of intersection c is parallel to both a and b in the direction of $\vec{b}$ [16, p. 50; cf. also 9, p. 22-23].

The next result is Lobachevski's proposition 25 which he proves in a manner which fits our criterion and which includes 5.54 as a part of the argument.

5.55 THEOREM. Two lines which are parallel to a third line in the same direction are parallel to each other in this same direction [16, p. 51; 9, p. 22-23].

A standard result of absolute geometry is [10, p. 180, Theorem 17]: given any point P not on a given plane α there is a unique line p on P that meets α, say at A, and is perpendicular to every line of α incident on A. (In this case we say <u>p is perpendicular to α at A.</u>) Also if we have two lines p and q perpendicular to α, then p and q are coplanar [10, p. 179, Theorem 12]. Thus the plane determined by any line ℓ not in α and the line p from any point P of ℓ so that p is perpendicular to α at A

is the plane containing all lines q from a point Q of ℓ and perpendicular to α.

These results allow us to talk accurately about the perpendicular projection of ℓ onto α as is done by Shirokov. The only time this projection is not a line is in case ℓ is already perpendicular to α [16, p. 51].

Shirokov states that in general, "...under this projection we obtain the line $\underline{A'B'}$ in the plane α - i.e. the projection of the line $\underline{AB}$. Since they lie in the same plane, the lines $\underline{AB}$ and $\underline{A'B'}$ can either 1) intersect, ... 2) diverge, ... or 3) be parallel." This gives:

5.56 DEFINITION. If lines $\underline{AB}$ and $\underline{A'B'}$ are as above, and 1) these lines intersect, we say that $\underline{AB}$ and α intersect, 2) these lines are divergent (i.e., hyperparallel) we say that $\underline{AB}$ and α are divergent, and 3) these lines are parallel in the direction of $\vec{r}$, we say that $\underline{AB}$ and α are parallel in the direction of $\vec{r}$ [16, p. 52].

The results of 5.55 immediately give the

5.57 THEOREM. If a line is parallel to some line lying in plane α, then it is parallel to plane α [16, p. 52].

Shirokov's next definition is:

5.58 DEFINITION. Two planes are said to be parallel if it is possible to construct a third plane which is perpendicular to both of them and which intersects them in parallel lines [16, p. 53].

He justifies the validity of this definition by proving--though not stating the theorem as a specific theorem in his work--the absolute geometry result that for any two distinct planes it is always possible to construct a plane perpendicular to both given planes [16, p. 52-53]. This leads to a theorem, "...of great significance for the construction of Lobachevskian geometry..." [16, p. 54].

5.59 THEOREM. Through a line $\underline{AA'}$ parallel to the plane α, there is exactly one plane parallel to α; all other planes containing $\underline{AA'}$ intersect α [16, p. 54].

Shirokov then defines pencils in space, corresponding points relative to these pencils, and orispheres.

5.60 DEFINITION. The set of all lines and planes in space which

1. are incident on a given point C is called a pencil of intersecting lines and planes with center C;
2. are perpendicular to a given plane α is called a pencil of divergent lines and planes with carrier plane α;
3. are parallel to a given line ℓ in a given direction $\vec{\ell}$ is

called a pencil of parallel lines and planes in the direction of $\vec{l}$ [16, p. 54-55].

He then observes:

All these three types of pencils possess certain common properties. Thus through every point of space [excluding the center in the case of an intersecting pencil of lines and planes] there passes one and only one line of the pencil; two points of the space which do not lie on the same line of the pencil determine a unique plane of the pencil; lines of the pencil belonging to planes of the pencil form a pencil [of lines] of the corresponding type; two lines of the pencil determine a plane of the pencil; if two planes which pass through two lines of the pencil intersect, then their line of intersection belongs to the pencil; two lines of the pencil determine a pencil, as well as do three independent planes of the pencil, and so on [16, p. 55].

Next comes a definition and theorem analogous to 4.5 and 4.7:

5.61 DEFINITION. Two points are said to correspond relative to the given pencil of lines and planes if they are situated symmetrically with respect to some line belonging to this pencil [16, p. 55].

5.62 THEOREM. The relation correspond--denoted as A*B--means that A and B correspond with respect to a given pencil of lines and planes. It is an equivalence relation.

Proof: The reflexive and symmetric properties are proved exactly as in 4.7. Shirokov gives a proof of transitivity [16, p. 55-56].

The resulting equivalence classes are given special names.

5.63 DEFINITION. Let Σ be a pencil of lines and planes in space and let $\Sigma[A]$ be the equivalence class of points corresponding to point A with respect to Σ. Then

1. if Σ is an intersecting pencil and A is not the center C of Σ, $\Sigma[A]$ is called a <u>sphere</u> with center C and radius AC,
2. if Σ is a divergent pencil, $\Sigma[A]$ is called an <u>equidistant surface,</u> and
3. if Σ is a parallel pencil in the direction of $\vec{r}$, $\Sigma[A]$ is called a <u>limiting surface</u> or <u>orisphere</u> in the direction of $\vec{r}$ [16, p. 57-58].

Note: From now on we will always use Σ to mean a parallel pencil of lines and planes.

5.64 DEFINITION. If Σ is a parallel pencil in the direction of $\vec{r}$ then the lines of Σ are called its <u>axes</u> and the planes of Σ are called the <u>diametral planes</u> of the orisphere $\Sigma[A]$ [16, p. 58].

Shirokov then proves (as does Lobachevski)

5.65 THEOREM. If a non-diametral plane has a point in common with an oricycle, then it either intersects this surface in a circle or is tangent to it at one point [16, p. 58-59; 9, p. 35]. (Note: Compare with the corresponding results for lines and oricycles in 5.1

See also [12].)

Then comes the basic theorem necessary to get us back into the results proved by Lobachevski. In fact he gives essentially the same theorem from another development.

5.66 THEOREM. Let $\Sigma[A]$ be any orisphere. With the following realizations of the undefined terms, the axioms of Euclidean plane geometry are theorems, i.e., Euclidean plane geometry holds on $\Sigma[A]$:

1. P is a <u>point</u> if $P \in \Sigma[A]$,
2. $\ell = \xi[P]$ is a <u>line</u> if $\xi[P]$ is an oricycle determined by a diametral plane of Σ and $\Sigma[A]$,
3. $\ell = \xi[P]$ is <u>incident</u> on point Q iff $Q \in \ell$,
4. point Q is <u>between</u> points P and R iff P, Q, and R are distinct points of ℓ and $P{*}Q{*}R$,
5. segment $\overline{AB}$ = arc AB is congruent to segment $\overline{CD}$ = arc CD iff arc $AB \cong$ arc CD.
6. By ray $\overset{\circ\!\!\longrightarrow}{AB}$ we will mean all the points P of the oricycle $\xi[A]$ containing B so that P is on B's side of A in the sense of 5.28. We then define angle as $o\angle ABC = \{B\} \cup \overset{\circ\!\!\longrightarrow}{BA} \cup \overset{\circ\!\!\longrightarrow}{BC}$. $o\angle ABC$ is congruent to $o\angle A'B'C'$ iff the dihedral angles determined by the diametral planes defined by A, B and B, C or by A', B'

and B', C' respectively are congruent in the sense of absolute geometry [16, p. 58-61].

Note: Let oΔABC denote a triangle on an orisphere.

Shirokov proves all the plane Euclidean axioms of Hilbert except the Archimedian and completeness axioms are theorems in this model. However, we have done the latter in this paper as 5.27 and 5.30. It is in this argument that Theorem 5.59 is used--recall 5.59 is the theorem referred to by Shirokov as "...of great significance... ."

Shirokov next develops the basic identities for Lobachevskian geometry.

5.67 THEOREM. Let ΔABC be a right triangle with acute angles $\angle A$ and $\angle B$. Let $\alpha = m\angle A$ and $\beta = m\angle B$. Further let $a = |[BC]|_S$, $b = |[AC]|_S$, and $c = |[AB]|_S$ where AB is the hypotenuse. Let k be the constant described in 5.53. Then

1. $\cosh(c/k) = \cosh(b/k)\cosh(a/k)$,

2-a. $\tanh(b/k) = \tanh(c/k)\cos\alpha$,

2-b. $\tanh(a/k) = \tanh(c/k)\cos\beta$,

3-a. $\sinh(a/k) = \sinh(c/k)\sin\alpha$,

3-b. $\sinh(b/k) = \sinh(c/k)\sin\beta$,

4-a. $\tanh(a/k) = \sinh(b/k)\tan\alpha$, 4-b. $\tanh(b/k) = \sinh(a/k)\tan\beta$,

5. $\cosh(c/k) = \cot\alpha\cot\beta$,

6-a. $\cos\alpha = \cosh(a/k)\sin\beta$, and

6-b. $\cos\beta = \cosh(b/k)\sin\alpha$ [16, p. 62-66].

Although the proof of this is given in Shirokov, it seems useful to reproduce parts of this proof so the flavor of the argument can be available here. The method revolves about two constructions as follows. These are illustrated in Figures (i) and (ii) below. Let $\underline{AA'}$ be the perpendicular to the plane of ΔABC and let $\underline{BB'}$ and $\underline{CC'}$ be the lines incident on B and C so that $\overrightarrow{BB'} | \overrightarrow{AA'}$ and $\overrightarrow{CC'} | \overrightarrow{AA'}$. Let $\Sigma[A] = S$ be the orisphere determined by the pencil Σ defined by these parallel lines. The three diametral planes determined by $\underline{AA'}$, $\underline{BB'}$ and $\underline{CC'}$ intersect the orisphere S in three oricycles which meet $\underline{AA'}$ at A, $\underline{BB'}$ at B_1 and $\underline{CC'}$ at C_1, thus defining a triangle $o\Delta AB_1C_1$ on the orisphere S.

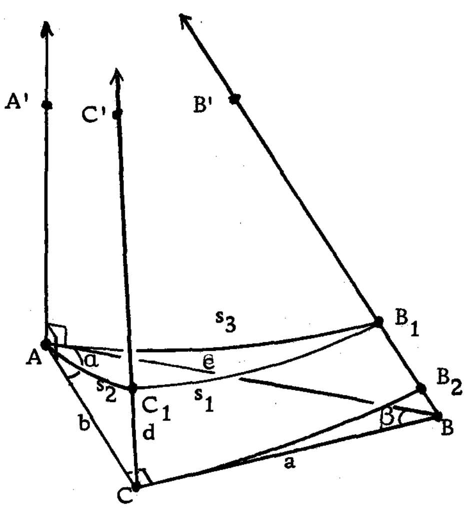


Figure (i).

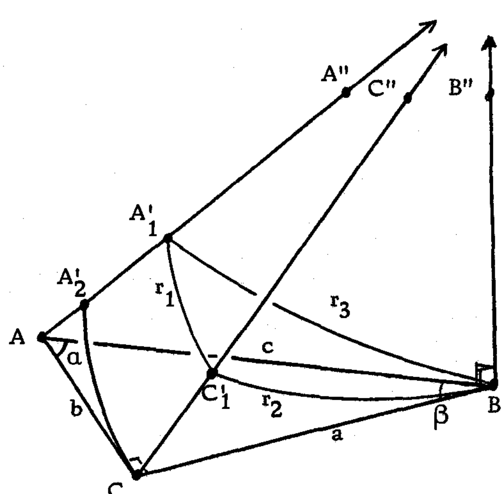


Figure (ii).

Since S is tangent to the plane of A, B and C at A, then the measure of $o\angle B_1AC_1$ is also α and $o\triangle AB_1C_1$ is a right triangle with $o\angle C$ as right angle. Let $\widehat{AB_1} = s_3$, $\widehat{B_1C_1} = s_1$ and $\widehat{AC_1} = s_2$. Let arc CB_2 be concentric to arc C_1B_1 in the plane of $\underline{BB'}$ and C as shown.

In a completely symmetrical manner with $\underline{BB''}$ perpendicular to the plane of $\triangle ABC$ at B, $\overrightarrow{AA''} | \overrightarrow{BB''}$ and $\overrightarrow{CC''} | \overrightarrow{BB''}$ we get $o\triangle BA_1'C_1'$ with $o\angle B$ of measure β and $o\angle C_1'$ a right angle. Point A_2' and arc CA_2' are defined symmetrically to B_2 and arc CB_2.

By 5.52, $\widehat{CB_2}/\widehat{C_1B_1} = e^{d/k}$ where $d = |[CC_1]|_S$, giving $s_1 = \widehat{CB_2}e^{-d/k}$. Using t as in 5.53, and the results of 5.53,

$$s_1 = t \tanh(a/k)e^{-d/k} .$$

Further, applying 5.53 directly,

$$s_2 = t \tanh(b/k),$$

$$s_3 = t \tanh(c/k).$$

Again drawing on 5.53 we get $e^{d/k} = \cosh(b/k)$ so that

$$s_1 = t \frac{\tanh(a/k)}{\cosh(b/k)} .$$

From 5.66 we have that the Pythagorean theorem holds for $o\triangle AB_1C_1$

giving $s_3^2 = s_1^2 + s_2^2$ or

$$\tanh^2(c/k) = \frac{\tanh^2(a/k)}{\cosh^2(b/k)} + \tanh^2(b/k)$$

$$= \operatorname{sech}^2(b/k)\tanh^2(a/k) + \tanh^2(b/k)$$

$$= (1-\tanh^2 b/k)\tanh^2(a/k) + \tanh^2(b/k)$$

$$= \frac{\sinh^2(a/k)\cosh^2(b/k) - \sinh^2(a/k)\sinh^2(b/k) + \sinh^2(b/k)\cosh^2(a/k)}{\cosh^2(a/k)\cosh^2(b/k)}$$

$$= \frac{\sinh^2(a/k)[\cosh^2(b/k) - \sinh^2(b/k)] + \sinh^2(b/k)\cosh^2(a/k)}{\cosh^2(a/k)\cosh^2(b/k)}$$

$$= \frac{\cosh^2(a/k) - 1 + \sinh^2(b/k)\cosh^2(a/k)}{\cosh^2(a/k)\cosh^2(b/k)}$$

$$= \frac{\cosh^2(a/k)[1+\sinh^2(b/k)] - 1}{\cosh^2(a/k)\cosh^2(b/k)}$$

$$= \frac{\cosh^2(a/k)\cosh^2(b/k) - 1}{\cosh^2(a/k)\cosh^2(b/k)}$$

$$= 1 - \frac{1}{\cosh^2(a/k)\cosh^2(b/k)} .$$

Thus

$$\frac{1}{1-\tanh(c/k)} = \cosh^2(a/k)\cosh^2(b/k) ,$$

so

$$\cosh^2(c/k) = \cosh^2(a/k)\cosh^2(b/k),$$

giving $\cosh(c/k) = \cosh(a/k)\cosh(b/k)$ as desired for "1.".

From the formula $s_2 = s_3 \cos \alpha$ [see Figure (i)] we get

2.a. $$\tanh(b/k) = \tanh(c/k) \cos \alpha.$$

Using symmetrical arguments for the configuration pictured in Figure (ii) we get $r_2 = r_3 \cos \beta$ and

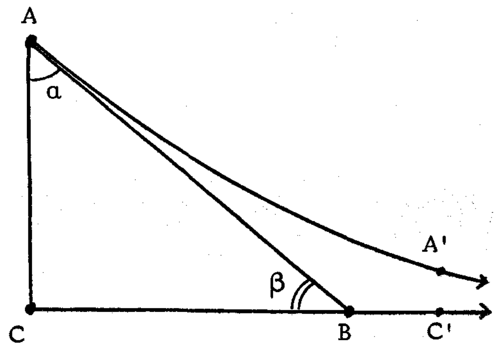
2.b. $$\tanh(a/k) = \tanh(c/k) \cos \beta.$$

In a like manner, using the appropriate configuration and the Euclidean results for $o\Delta AB_1C_1$ or $o\Delta BA_1'C_1'$ as appropriate, the remaining results are verified.

These same results are also proved by Norden [12. p. 169-174] in a somewhat different manner.

The next result is called "Lobachevski's fundamental formula [the function $\Pi(x)$]" by Shirokov [16, p. 69]. It is developed by Borsuk [2, p. 331-334], Norden [12, p. 176-177], Shirokov [16, p. 69-71], and Lobachevski [9, p. 39-41], each using a quite different method of justification but clearly all inspired by Lobachevski's insights.

Shirokov's approach is to consider the closed right triangle $\Delta A'ACC'$ with right angle, $\angle ACC'$. He lets B be any point of $\overrightarrow{CC'}$ and considers ΔABC with acute angles $\angle A$ and $\angle B$ so that $m\angle A = \alpha$, $m\angle B = \beta$, $|[AC]|_S = b$, $|[BC]|_S = a$, and



$|[AB]|_S = c$. From absolute geometry we know $AB > BC$ as AB is the side opposite the greatest angle $\angle C$. Thus, knowing $\cos\alpha = \cosh(a/k)\sin\beta$ from 5.67, $\lim_{a\to\infty} \cos\alpha = \lim_{a\to\infty} \cosh(a/k)\sin\beta \le 1$. This requires $\lim_{a\to\infty} \sin\beta = 0$ or $\lim_{a\to\infty} \beta = 0$ so that $\lim_{a\to\infty} (\alpha) = \Pi(b)$. From 2.a of 5.67 we have $\cos\alpha = \tanh(b/k)/\tanh(c/k)$, so

$$\lim_{a\to\infty} \cos\alpha = \cos\Pi(b) = \lim_{a\to\infty} \tanh(b/k)/\tanh(c/k)$$

$$= \lim_{c\to\infty} \tanh(b/k)/\tanh(c/k)$$

$$= \tanh b/k.$$

In other terms, $\tanh b/k = \cos\Pi(b)$. But

$$\tan\frac{\Pi(b)}{2} = \frac{\sqrt{1-\cos\Pi(b)}}{\sqrt{1+\cos\Pi(b)}}$$

$$= \frac{\sqrt{1-\tanh(b/k)}}{\sqrt{1+\tanh(b/k)}}$$

$$= \frac{\sqrt{e^{b/k}+e^{-b/k}-e^{b/k}+e^{-b/k}}}{\sqrt{e^{b/k}+e^{-b/k}+e^{b/k}-e^{-b/k}}}$$

$$= \frac{\sqrt{e^{-b/k}}}{\sqrt{e^{b/k}}}$$

$$= \sqrt{(e^{-b/k})^2} = e^{-b/k}$$

This is "Lobachevski's fundamental formula":

$$\tan\frac{\Pi(b)}{2} = e^{-b/k}$$

Shirokov does nothing more toward the examination of the constant k. However, we now consider it further and justify our earlier comment that the canonical segment would best be assigned the number, $\ln(\sqrt{2}+1)$.

If $b = \Pi^{-1}(\pi/4)$, "Lobachevski's fundamental formula" gives us:

$$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{1+\cos\pi/4}}$$

$$= \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$$

$$= \sqrt{(\sqrt{2}-1)^2}$$

$$= \sqrt{2}-1 = e^{-b/k}$$

Simple algebra gives $e^{b/k} = \sqrt{2}+1$ so that $b/k = \ln(\sqrt{2}+1)$. The choice of $b = \ln(\sqrt{2}+1)$ requires $k = 1$. This is the best choice to simplify computation and to simplify all of the results for the relationships in 5.67. Recall that the segment class associated with $\Pi^{-1}(|\frac{[\angle rt]}{2}|_A)$ is the canonical class of segments (3.25). We have assigned $\pi/2$ to the class of right angles, so the segment class associated with $\Pi^{-1}(\pi/4)$ is the canonical class. Thus we have justified the following:

5.68 DEFINITION. The number associated with the canonical segment class is $\ln(\sqrt{2}+1) = \Pi^{-1}(\pi/4)$.

Immediately this gives

5.69 THEOREM. $\tan\frac{\Pi(x)}{2} = e^{-x}$ for all x.

VI. THE CONCLUDING ARGUMENT

From 2.13 we can conclude that for any angle $\angle ABC$, $|[\angle ABC]|_A = \pi - |\underline{[\angle ABC]}|_A$ where $\underline{[\angle ABC]}$ is the class determined by the supplement of $\angle ABC$. This follows directly from 2.13, the definition of supplements, and the assignment of $\pi/2$ to the class of right angles. Thus we may conclude from the definition of Π (3.22.1) that, for all x, $\Pi(-x) = \pi - \Pi(x)$. This immediately gives:

6.1 LEMMA.

1. $\sin \Pi(-x) = \sin \Pi(x)$, and
2. $\cos \Pi(-x) = -\cos \Pi(x)$ [9, p. 19-21; 2, p. 334].

By using the results of 5.69, i.e., $\tan \frac{\Pi(x)}{2} = e^{-x}$ or equivalently $\cot \frac{\Pi(x)}{2} = e^{x}$ we can compute the following results:

6.2 THEOREM. For any two real numbers x and y,

1. $\sin \Pi(x \pm y) = \dfrac{\sin \Pi(x) \sin \Pi(y)}{1 \pm \cos \Pi(x) \cos \Pi(y)}$, and
2. $\cos \Pi(x \pm y) = \dfrac{\cos \Pi(x) \pm \cos \Pi(y)}{1 \pm \cos \Pi(x) \cos \Pi(y)}$ [9, p. 42; 2, p. 334-335].

Proof: "1." $\sin \Pi(x) = \sin(2\text{Arccot } e^x)$ by 5.69

$$= 2 \sin(\text{Arccot } e^x) \cos(\text{Arccot } e^x)$$

$$= 2[e^x/(1+e^{2x})^{1/2}][1/(1+e^{2x})^{1/2}]$$

$$= 2e^x/(1+e^{2x}) .$$

Thus

$$\sin \Pi(x+y) = 2e^{x+y}/(1+e^{2x+2y})$$

$$= \frac{4e^{x+y}}{2+2e^{2x}e^{2y}}$$

$$= \frac{4e^{x+y}}{e^{2x}e^{2y}+e^{2x}+e^{2y}+1+e^{2x}e^{2y}-e^{2x}-e^{2y}+1}$$

$$= \frac{4e^{x+y}}{(e^{2x}+1)(e^{2y}+1)+(e^{2x}-1)(e^{2y}-1)}$$

(i) $$= \frac{\dfrac{2e^{x}2e^{y}}{(e^{2x}+1)(e^{2y}+1)}}{1+\dfrac{e^{2x}-1}{e^{2x}+1}\,\dfrac{e^{2y}-1}{e^{2y}+1}} .$$

Now

$$\cos \Pi(x) = \cos(2 \operatorname{Arccot} e^{x})$$

$$= \cos^{2}(\operatorname{Arccot} e^{x}) - \sin^{2}(\operatorname{Arccot} e^{x})$$

$$= \frac{1}{1+e^{2x}} - \frac{e^{2x}}{1+e^{2x}}$$

$$= \frac{1-e^{2x}}{1+e^{2x}} ,$$

and hence

$$\sin \Pi(x+y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1+\cos \Pi(x) \operatorname{cox} \Pi(y)} .$$

By 6.1 we get $$\sin(x\pm y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 \pm \cos \Pi(x) \cos \Pi(y)} .$$

Similarly

$$\cos \Pi(x+y) = \frac{2}{2}\,\frac{1-e^{2x}e^{2y}}{1+e^{2x}e^{2y}}$$

$$= \frac{1-e^{2x}+e^{2y}-e^{2x}e^{2y}+1+e^{2x}-e^{2y}-e^{2x}e^{2y}}{1+e^{2x}+e^{2y}+e^{2x}e^{2y}+1-e^{2x}-e^{2y}+e^{2x}e^{2y}}$$

$$= \frac{(1-e^{2x})(1+e^{2y}) + (1+e^{2x})(1-e^{2y})}{(1+e^{2x})(1+e^{2y}) + (1-e^{2x})(1-e^{2y})}$$

$$= \frac{\dfrac{(1-e^{2x})(1+e^{2y})}{(1+e^{2x})(1+e^{2y})} + \dfrac{(1+e^{2x})(1-e^{2y})}{(1+e^{2x})(1+e^{2y})}}{1+ \dfrac{(1-e^{2x})(1-e^{2y})}{(1+e^{2x})(1+e^{2y})}} .$$

This, together with 6.1, gives

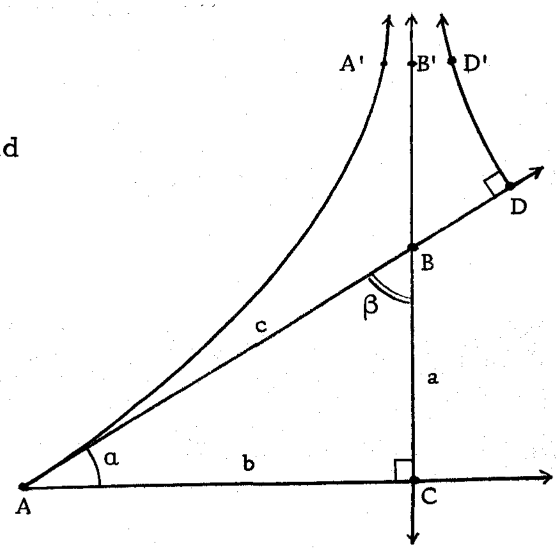
$$\cos \Pi(x\pm y) = \frac{\cos \Pi(x) \pm \operatorname{cox} \Pi(y)}{1 \pm \cos \Pi(x)\ \cos \Pi(y)} .$$

6.3 LEMMA. Let $\triangle ABC$ be a right triangle with $\angle C$ the right angle, $m\angle A = \alpha$, $m\angle B = \beta$, $|[AB]|_S = c$, $|[AC]|_S = b$, and $|[BC]|_S = a$. Then

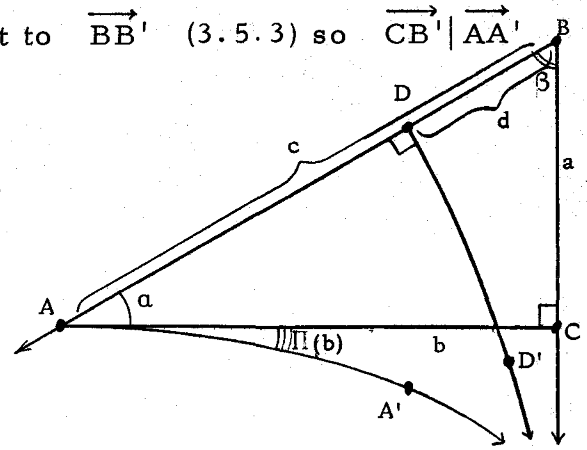
1. $\Pi(c+\Pi^{-1}(\beta)) + \alpha = \Pi(b)$, and
2. $\Pi(b) + \alpha = \Pi(c-\Pi^{-1}(\beta))$ [9, p. 39-41; 2, p. 335-336].

Proof: "1." In the plane of $\triangle ABC$, let B' be a point of $\underline{CB}$ so that C-B-B'; let D be a point of $\underline{AB}$ so that A-B-D and

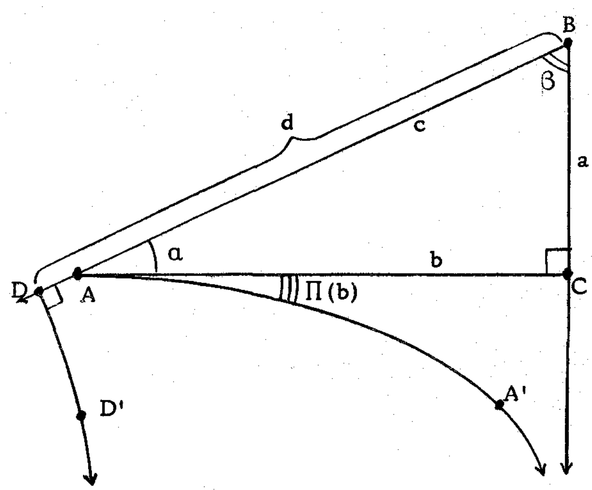
$|[BD]|_S = \Pi^{-1}(\beta)$; on the B' side of $\underline{AB}$ let $\overrightarrow{AA'}$ be the ray so that $m\angle BAA' = \Pi(c+\Pi^{-1}(\beta))$; and finally let $\underline{DD'}$ be the line perpendicular to $\underline{AB}$ at D with $\overrightarrow{DD'}$ on the B' side of $\underline{AB}$.



By the definition of $\angle BAA'$, $\overrightarrow{AA'}|\overrightarrow{DD'}$ and by the definition of segment BD and by vertical angles $\overrightarrow{DD'}|\overrightarrow{BB'}$. By "transitivity" of critical parallels (3.5.11), $\overrightarrow{AA'}|\overrightarrow{BB'}$. But $\overrightarrow{CB'}$ is equivalent to $\overrightarrow{BB'}$ (3.5.3) so $\overrightarrow{CB'}|\overrightarrow{AA'}$ and "1." is proved.



"2." In the plane of $\triangle ABC$, let D be a point of $\overrightarrow{BA}$ and $|[BD]| = d = \Pi^{-1}(\beta)$; opposite B's side of $\underline{AC}$ let $\overrightarrow{AA'}$ be the ray such that $m\angle CAA' = \Pi(b)$, and let $\underline{DD'}$ be perpendicular to $\underline{AB}$ at D with $\overrightarrow{DD'}$ on C's side of $\underline{AB}$.



Then $\overrightarrow{AA'}|\overrightarrow{BC}$, by definition of $\angle CAA'$ and by 3.5.3, and $\overrightarrow{BC}|\overrightarrow{DD'}$, by choice of

D; so, by 3.5.11, "transitivity" of parallels, $\overrightarrow{AA'} | \overrightarrow{DD'}$ (provided $\overrightarrow{AA'} \neq \overrightarrow{DD'}$ in which case we are trivially done since $\Pi(0) = \pi/2$ by definition and $\underline{AA'} \perp \underline{AB}$).

If A-D-B we are done immediately.

If D-A-B, then

$$\begin{aligned}\pi - (\Pi(b)+\alpha) &= \Pi(\Pi^{-1}(\beta)-c)\\ &= \Pi(d-c)\\ &= \pi - \Pi(c-d), \quad \text{by } 3.22.1,\\ &= \pi - \Pi(c-\Pi^{-1}(\beta)).\end{aligned}$$

Thus

$$\Pi(b) + \alpha = \Pi(c-\Pi^{-1}(\beta))$$

as claimed.

6.4 THEOREM. Let ΔABC be a right triangle with $\angle C$ a right angle, $m\angle A = \alpha$, $m\angle B = \beta$, $|[AB]|_S = c$, $|[AC]|_S = b$, and $|[BC]|_S = a$. Then

1. $\sin \Pi(c) = \sin \Pi(a) \sin \Pi(b)$, and
2. $\cos \Pi(b) = \cos \Pi(c) \cos \alpha$ [9, p. 42 ff.; 2, p. 339].

Proof: From 6.3, letting $d = \Pi^{-1}(\beta)$, we have

$$\Pi(b) = \Pi(c+d) + \alpha,$$

and

$$\Pi(b) = \Pi(c-d) - \alpha.$$

This gives us

(i) $$\cos \Pi(b) = \cos \Pi(c+d) \cos \alpha - \sin \Pi(c+d) \sin \alpha,$$

and

(ii) $$\cos \Pi(b) = \cos \Pi(c-d) \cos \alpha + \sin \Pi(c-d) \sin \alpha .$$

Using 6.2 and the fact that $d = \Pi^{-1}(\beta)$, we get

(iii) $$\cos \Pi(c+d) - \cos \Pi(c-d) = \frac{\cos \Pi(c) + \cos \beta}{1+ \cos \Pi(c) \cos \beta} - \frac{\cos \Pi(c) - \cos \beta}{1- \cos \Pi(c) \cos \beta}$$

$$= \frac{2 \cos \beta(1-\cos^2 \Pi(c))}{1- \cos^2 \Pi(c) \cos^2 \beta}$$

$$= \frac{2 \sin^2 \Pi(c) \cos \beta}{1- \cos^2 \Pi(c) \cos^2 \beta},$$

and

(iv) $$\sin \Pi(c+d) + \sin \Pi(c-d) = \frac{\sin \Pi(c) \sin \beta}{1+ \cos \Pi(c) \cos \beta} + \frac{\sin \Pi(c) \sin \beta}{1- \cos \Pi(c) \cos \beta}$$

$$= \frac{2 \sin \Pi(c) \sin \beta}{1- \cos^2 \Pi(c) \cos^2 \beta}.$$

Now subtracting (ii) from (i) and substituting the results of (iii) and (iv) into the resulting equation, we get

(v) $$\frac{2 \sin \Pi(c) \sin \beta}{1- \cos^2 \Pi(c) \cos^2 \beta} \sin \alpha = \frac{2 \sin^2 \Pi(c) \cos \beta}{1- \cos^2 \Pi(c) \cos^2 \beta} \cos \alpha .$$

This immediately gives

(v') $$\sin \beta \sin \alpha = \sin \Pi(c) \cos \beta \cos \alpha ,$$

or, equivalently

$$\text{(v'')} \qquad \sin \Pi(c) = \tan \alpha \tan \beta .$$

Again, from 6.3 we have

$$\Pi(b) - \alpha = \Pi(c+d),$$

and

$$\Pi(b) + \alpha = \Pi(c-d).$$

As above, we use 6.2 and the definition of $d = \Pi^{-1}(\beta)$ to get

$$\text{(vi)} \qquad \sin \Pi(b) \cos \alpha - \cos \Pi(b) \sin \alpha = \frac{\sin \Pi(c) \sin \beta}{1+\cos \Pi(c) \cos \beta},$$

and

$$\text{(vii)} \qquad \sin \Pi(b) \cos \alpha + \cos \Pi(b) \sin \alpha = \frac{\sin \Pi(c) \sin \beta}{1-\cos \Pi(c) \cos \beta}.$$

Adding (vi) and (vii) gives us

$$2 \sin \Pi(b) \cos \alpha = \frac{\sin \Pi(c) \sin \beta}{1+ \cos \Pi(c) \cos \beta} + \frac{\sin \Pi(c) \sin \beta}{1- \cos \Pi(c) \cos \beta}$$

$$= \frac{2 \sin \Pi(c) \sin \beta}{1- \cos^2 \Pi(c) \cos^2 \beta}$$

$$= \frac{2 \tan \alpha \tan \beta \sin \beta}{1-(1-\tan^2 \alpha \tan^2 \beta)\cos^2 \beta}, \quad \text{from (v'') above,}$$

$$= \frac{2 \tan \alpha \tan \beta \sin \beta}{\cos^2 \alpha(1-\cos^2 \beta) + \sin^2 \alpha \sin^2 \beta} \cos^2 \alpha$$

$$= 2 \frac{\sin \alpha}{\cos \beta} \cos \alpha .$$

Thus we have

(viii) $$\sin \Pi(b) = \sin \alpha / \cos \beta .$$

By a symmetrical argument we get

(ix) $$\sin \Pi(a) = \sin \beta / \cos \alpha .$$

We can now use (v''), (viii), and (ix) to conclude

(x) $$\sin \Pi(c) = \sin \Pi(a) \sin \Pi(b) .$$

Using (x), together with (v''), (viii), and (ix) we see that

(xi)
$$\begin{aligned}
\cos^2 \Pi(b) &= 1 - \sin^2 \alpha / \cos^2 \beta \\
&= [\cos^2 \beta - \sin^2 \alpha] / \cos^2 \beta \\
&= [\cos^2 \beta - \sin^2 \alpha(\cos^2 \beta + \sin^2 \beta)] / \cos^2 \beta \\
&= [\cos^2 \beta (1 - \sin^2 \alpha) + \sin^2 \alpha \sin^2 \beta] / \cos^2 \beta \\
&= [\cos^2 \alpha \cos^2 \beta + \sin^2 \alpha \sin^2 \beta] / \cos^2 \beta \\
&= \cos^2 \alpha + [\sin^2 \alpha \sin^2 \beta] / \cos^2 \beta \\
&= [1 + \frac{\sin^2 \alpha \sin^2 \beta}{\cos^2 \alpha \cos^2 \beta}] \cos^2 \alpha \\
&= [1 + \sin^2 \Pi(a) \sin^2 \Pi(b)] \cos^2 \alpha \\
&= [1 + \sin^2 \Pi(c)] \cos^2 \alpha \\
&= \cos^2 \Pi(c) \cos^2 \alpha .
\end{aligned}$$

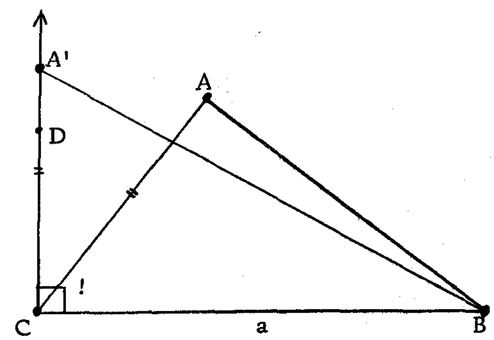
Now b and c are both positive and $0 < \alpha < \pi/2$ so we conclude from (xi) that

$$\cos \Pi(b) = \cos \Pi(c) \cos a,$$

and we are done.

6.4.1 COROLLARY. Let $\triangle ABC$ be a triangle with $|[AB]|_S = c$, $|[AC]|_S = b$, and $|[BC]|_S = a$. Then $\angle C$ is a right angle iff $\sin \Pi(c) = \sin \Pi(b) \sin \Pi(a)$.

Proof: The "only if" is just 6.4. Let $\overrightarrow{CD}$ be the $\perp$ ray on A's side of $\underline{BC}$ in the plane determined be A, B and C. Let A' be the point of $\overrightarrow{CD}$ so that $|[CA']|_S = b$. Then in $\triangle A'BC$ we have $\sin \Pi(a) \sin \Pi(b) = \sin \Pi |[A'B]|_S$.

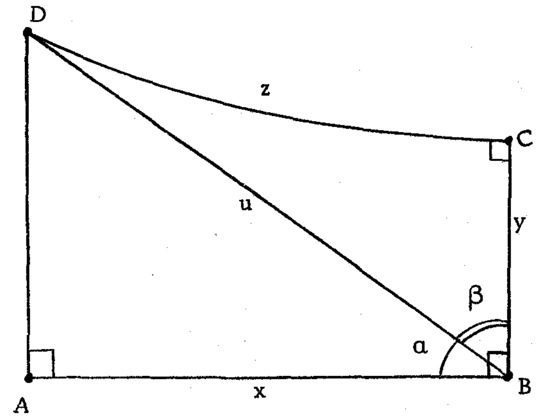


Hence $A'B \cong AB$ and $\triangle ABC \cong \triangle A'BC$ by S.S.S. and $\angle C$ is a right angle as claimed.

The next lemma is very useful later.

6.5 LEMMA. Let A, B, C, and D be points of plane α. If $\square ABCD$ is the quadrilateral so that $\angle A$, $\angle B$, $\angle C$ are all right angles with $|[AB]|_S = x$, $|[BC]|_S = y$, and $|[CD]|_S = z$, then $\cos \Pi(x) = \sin \Pi(y) \cos \Pi(z)$ [2, p. 138].

Proof: Let $|[BD]|_S = u$, $m\angle ABD = \alpha$, and $m\angle CBD = \beta$. Clearly



$\alpha + \beta = \pi/2$. In ΔABD we apply 6.4, part 2, to get

(i) $\cos \alpha = \cos \Pi(x)/\cos \Pi(u)$.

In ΔCBD we similarly get

(ii) $\cos \beta = \cos \Pi(y)/\cos \Pi(u)$.

But we know $\beta = \pi/2 - \alpha$ so, from (ii), we get

(iii) $$\sin \alpha = \cos \Pi(y)/\cos \Pi(u).$$

Now from (i) and (iii) we have

(iv) $$1 = \cos^2 \alpha + \sin^2 \alpha = \frac{\cos^2 \Pi(x) + \cos^2 \Pi(y)}{\cos^2 \Pi(u)},$$

i.e., we have

(v) $$\cos^2 \Pi(u) = \cos^2 \Pi(x) + \cos^2 \Pi(y) .$$

If we apply 6.4, part 1 to ΔBCD, we get

(vi) $$\sin^2 \Pi(u) = \sin^2 \Pi(y) \sin^2 \Pi(z) .$$

From (v) we conclude that

(vii) $$1 - \sin^2 \Pi(u) = \cos^2 \Pi(x) + 1 - \sin^2 \Pi(y) .$$

Combining (vi) and (vii) we have

(viii) $$\sin^2 \Pi(y) \sin^2 \Pi(z) = \sin^2 \Pi(y) - \cos^2 \Pi(x) .$$

Equation (viii) is equivalent to

$$\text{(ix)} \qquad \sin^2 \Pi(y) \cos^2 \Pi(z) = \cos^2 \Pi(x)$$

Since $x > 0$, $y > 0$, and $z > 0$, we conclude from (ix) that

$$\cos \Pi(x) = \sin \Pi(y) \cos \Pi(z) \; .$$

In Chapter I we described how we could use "$| \;|_S$" to assign numbers to the points on any line (1.16). This assignment allows us to define a coordinate system for Lobachevskian space in the usual way.

6.6 DEFINITION. Let

1. α be any plane,
2. $\underline{OX}$ be any line of α,
3. $\underline{OY}$ be the line of α perpendicular to $\underline{OX}$ at O,
4. $\underline{OZ}$ be the line perpendicular to α at O, and
5. zero be the real number associated with O on each line.

Then

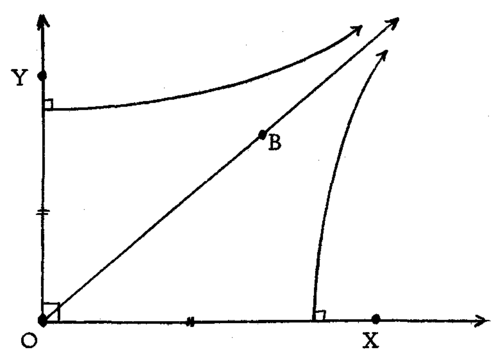
1. $\underline{OX}$, $\underline{OY}$, and $\underline{OZ}$ are called the X-, Y- and Z-axes respectively,
2. (x, y, z) is called the <u>coordinate triple</u> or the <u>coordinates</u> of point P iff x, y, and z are the numbers associated with the feet of the perpendicular from P to the X-, Y-,

and Z-axes respectively, and

3. the plane determined by the W_1- and W_2-axes (where $W_1 \in \{X, Y, Z\}$ and $W_2 \in \{X, Y, Z\} - \{W_1\}$) is called the <u>$W_1W_2$-plane</u> or the <u>$W_2W_1$-plane</u>. These are each called <u>coordinate planes</u>.

Since there is a unique line incident on P and perpendicular to each axis, every point has a unique coordinate triple associated with it. However, unlike Euclidean geometry, every ordered triple of reals is not the coordinate triple of some point. To see that this is so, consider the following example in the XY-plane.

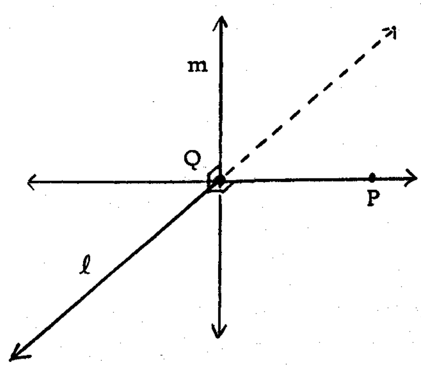
Let $\overrightarrow{OB}$ be the bisector of $\angle XOY$ where, without loss of generality, $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are chosen as positive rays. It is impossible to have $(x_0, y_0, 0)$ name any point in the XY-plane if $x_0 \geq \ln(\sqrt{2}+1)$ and $y_0 \geq \ln(\sqrt{2}+1)$. This is true since the perpendiculars in the XY-plane from the points associated with $\ln(\sqrt{2}+1)$ are parallel to <u>OB</u> in the direction of $\overrightarrow{OB}$ and are on opposite sides of <u>OB</u>. This follows from our arbitrary assignment $\Pi^{-1}(\pi/4) = \ln(\sqrt{2}+1)$.



We shall now show that (a, b, c) can be the coordinates of a point P iff $\cos^2\Pi(a) + \cos^2\Pi(b) + \cos^2\Pi(c) < 1$. To do this we first establish some useful lemmas. In the subsequent work we shall assume some planes, lines, and directions have been taken in space to give us a coordinate system in the sense of definition 6.6 above.

6.7 LEMMA. If ℓ is any line and P is any point then there is a unique plane λ incident on P and perpendicular to ℓ.

Proof: By 0.12, there exists a unique line p incident on P and perpendicular to ℓ at Q. The lines p and ℓ intersect at Q and thus, by 0.11, determine a plane α. By 0.19, there is a unique line m incident on Q and perpendicular to α. By 0.17, the plane λ determined by m and p is perpendicular to ℓ, since ℓ is perpendicular to both m and p at their point of intersection. Any other plane μ perpendicular to ℓ and containing P, say at point Q' of ℓ, must have $\underline{PQ'} \perp \ell$ through P. Thus Q' = Q (since perpendiculars from P to ℓ are unique). Hence $\mu = \lambda$ and we are done.



6.8 LEMMA. If

1. ℓ and m are lines of plane α,

2. ℓ and m are perpendicular at O,

3. n is the perpendicular to α at O,

4. P is any point, and

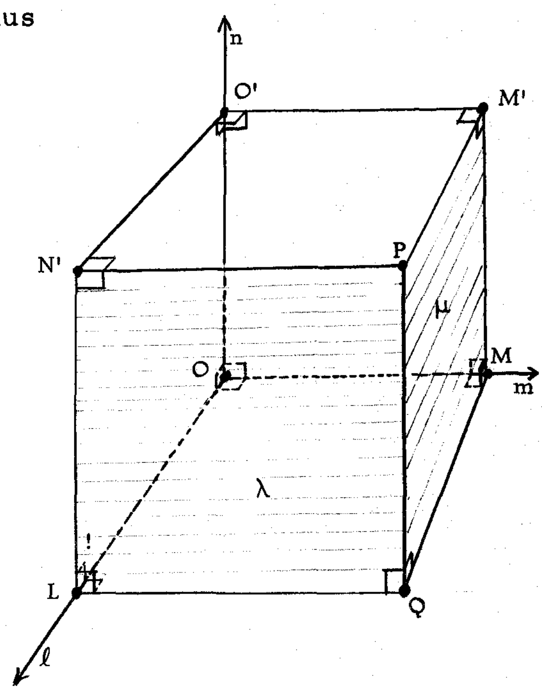
5. λ and μ are the unique planes containing P and perpendicular to ℓ and m, say at L and M, respectively,

then

1. the line p containing P and perpendicular to α, say at Q, is the line of intersection of λ and μ, and

2. p and n are coplanar.

Proof: "1." λ meets the plane of ℓ and m in a line $\underline{LQ'}$. Let $\underline{LN'}$ be perpendicular to $\underline{LQ'}$ at L in λ. Then, by 0.17, $\underline{LN'}$ is perpendicular at L to the plane determined by ℓ and m ($\underline{LN'}$ is perpendicular to ℓ since λ is perpendicular to ℓ as well as perpendicular to $\underline{LQ'}$.) Thus $\underline{LN'}$ and $\underline{PQ}$ are perpendicular to the same plane and are coplanar. This means $\underline{PQ}$ is in λ. By a symmetrical argument, $\underline{PQ}$ is in μ, and hence $\underline{PQ}$ is the line of intersection.



"2." From our hypothesis n and $\underline{PQ}$ are both perpendicular to the plane of ℓ and m so, by 0.17 they are coplanar.

It is interesting to observe that while $\underline{LQ} \perp \underline{PQ}$ and $\underline{MQ} \perp \underline{PQ}$, in general, $\underline{LQ}$ is not perpendicular to $\underline{MQ}$. In fact, they will be perpendicular only when Q is on ℓ or m. In the figure above we have sketched the points O' and M' with the perpendiculars at all vertices marked. In general all three of the angles at P are acute.

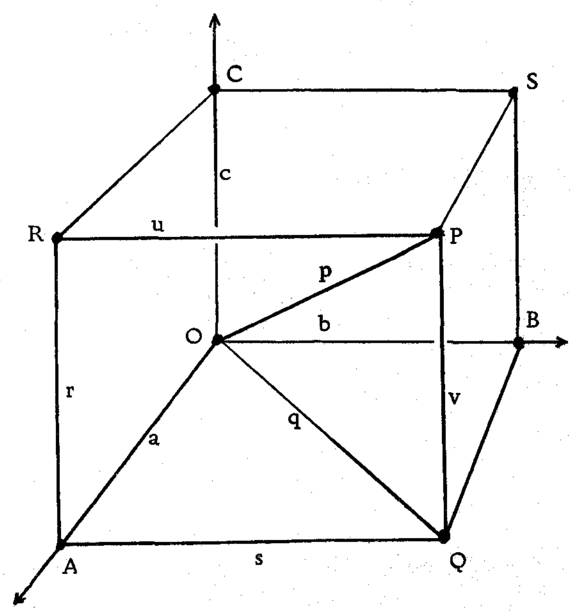
6.8.1 COROLLARY. If P is any point in space, and P' is the foot of the perpendicular to the XY-plane [or XZ- and YZ-plane, respectively] and P has coordinates (x, y, z), then P' has coordinates (x, y, 0) [or (x, 0, z) and (0, y, z) respectively].

Proof: By 6.8, P' is a point of the planes perpendicular to the X- and Y-axes at the points of these axes associated with x and y respectively. But this requires the lines from P' perpendicular to the X-, Y-, and Z-axes to have feet associated with x, y, and 0 respectively. Thus by Definition 6.6, P' has coordinates (x, y, 0). The remaining two cases follow immediately by symmetry.

6.9 THEOREM. P is a point with coordinates (a, b, c) iff $\cos^2\Pi(a) + \cos^2\Pi(b) + \cos^2\Pi(c) < 1$.

Proof: "only if": If P has coordinates (0, 0, 0) we are done, trivially. Let us suppose P has coordinates $(a, b, c) \neq (0, 0, 0)$. Then let $p = |[OP]|_S$. Let A, B, and C be the points of the X-, Y-, and Z-axes associated with a, b, c respectively and by Definition 1.16 conclude $|[OA]|_S = |a|$, $|[OB]|_S = |b|$ and $|[OC]|_S = |c|$. Let Q, R, and S be the feet of the perpendiculars through P to the XY-, XZ-, YZ-planes respectively.

<u>Case 1.</u> P is not in any coordinate plane: Let $|[OQ]|_S = q$, $|[AR]|_S = r$, $|[AQ]|_S = s$, $|[RP]|_S = u$, $|[QP]|_S = v$.



By 6.8 and 6.5 we have

1. $\cos \Pi(s) = \cos \Pi(|b|)/\sin \Pi(|a|)$
2. $\cos \Pi(r) = \cos \Pi(|c|)/\sin \Pi(|a|)$
3. $\cos \Pi(v) = \cos \Pi(r)/\sin \Pi(s)$

$$= \frac{\cos \Pi(|c|)/\sin \Pi(|a|)}{\sin[\mathrm{Arccos}(\cos \Pi(|b|)/\sin \Pi(|a|))]}, \text{ from 1. and 2.,}$$

$$= \frac{\cos \Pi(|c|)/\sin \Pi(|a|)}{\sqrt{1-\cos^2 \Pi(|a|)-\cos^2 \Pi(|b|)}/\sin \Pi(|a|)}$$

$$= \frac{\cos \Pi(|c|)}{\sqrt{1-\cos^2 \Pi(a)-\cos^2 \Pi(b)}},$$

since

$$\cos^2 \Pi(x) = \cos^2 \Pi(|x|),$$

4. $\sin \Pi(q) = \sin \Pi(|a|) \sin \Pi(s)$ by 6.4 and 6.8

$$= \sin \Pi(|a|)\sqrt{1-\cos^2\Pi(a)-\cos^2\Pi(b)}/\sin \Pi(|a|)$$

(cf. computation of 3.)

$$= \sqrt{1-\cos^2\Pi(a)-\cos^2\Pi(b)}$$

5. $\sin \Pi(p) = \sin \Pi(q) \sin \Pi(v)$ by 6.4 and 6.8

$$= \sin \Pi(q) \sin \operatorname{Arcsin}\left(\frac{\cos \Pi(|c|)}{\sqrt{1-\cos^2\Pi(a)-\cos^2\Pi(b)}}\right)$$

$$= \frac{\sqrt{1-\cos^2\Pi(a)-\cos^2\Pi(b)}\sqrt{1-\cos^2\Pi(a)-\cos^2\Pi(b)-\cos^2\Pi(c)}}{\sqrt{1-\cos^2\Pi(a)-\cos^2\Pi(b)}}$$

$$= \sqrt{1-\cos^2\Pi(a)-\cos^2\Pi(b)-\cos^2\Pi(c)}$$

Now since $\Pi(p) \in (0, \pi)$, $\sin \Pi(p) > 0$ we have

$$\cos^2\Pi(a) + \cos^2\Pi(b) + \cos^2\Pi(c) < 1.$$

Case 2. P is in a coordinate plane: By symmetry we may suppose $P = Q$. Then by Step 4 above we are done if $Q \neq A$.

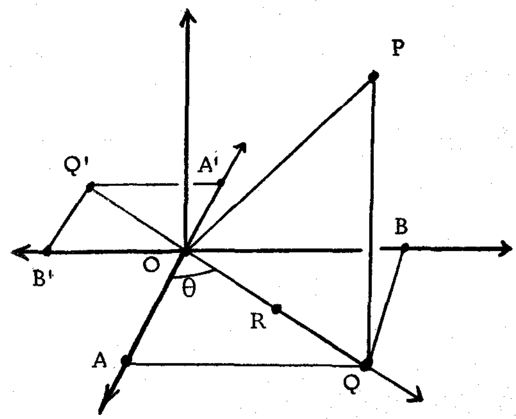
Case 3. If $P = A$ we are trivially done.

By symmetry we have completed this half of the proof.

"if": Suppose (a, b, c) is an ordered triple so that

$$\cos^2\Pi(a) + \cos^2\Pi(b) + \cos^2\Pi(c) < 1.$$

Case 1. Suppose a, b, c are all non-zero. Let A be the point on the positive X-axis associated with $|a|$. Let A' be associated with $-|a|$ on the X-axis.



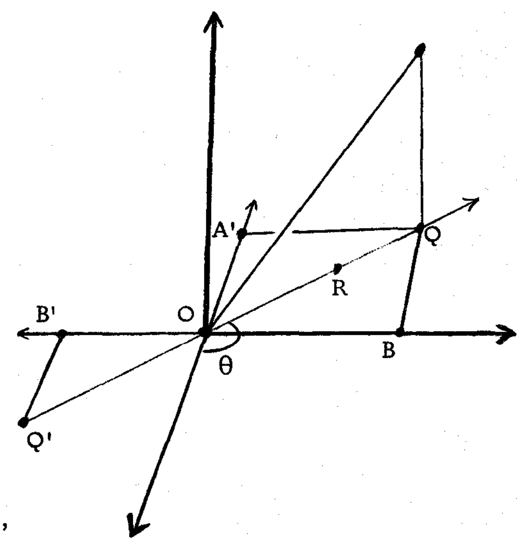
In the XY-plane there is a ray $\overrightarrow{OR}$, R being on the side of the XY-plane determined by the X-axis and containing the positive part of the Y-axis, so that

1. $\theta = m\angle AOR = \text{Arccos}\left(\dfrac{\cos \Pi(a)}{\sqrt{\cos^2\Pi(a)+\cos^2\Pi(b)}}\right)$

On $\overrightarrow{OR}$ there is a point Q so that

2. $\cos \Pi(q) = \sqrt{\cos^2\Pi(a)+\cos^2\Pi(b)}$

   with $q = |[OQ]|_S$.



Let F be the foot of the perpendicular from Q to the X-axis. Let $f = |[OF]|_S$. If $a > 0$, $\cos \Pi(a) > 0$ and thus $\angle AOR$ is acute so $\cos\theta \cos \Pi(q) = \cos \Pi(f)$ by 6.4, i.e.,

$$\frac{\cos \Pi(a)}{\sqrt{\cos^2\Pi(a)+\cos^2\Pi(b)}} \sqrt{\cos^2\Pi(a)+\cos^2(b)} = \cos \Pi(a) = \cos \Pi(f)$$

so F is A and thus Q has first coordinate a. If $a < 0$, $\cos \Pi(a) < 0$ and $\angle A'OR$ is acute, so $\cos(\pi-\theta) = -\cos\theta$ and as above $-\cos \Pi(a) = \cos \Pi(f)$ so $F = A'$. Thus Q has first coordinate a.

Now let B be the point on the positive Y axis which is the foot of the perpendicular from Q to the Y-axis. Then if $\theta < \pi/2$, $m\angle BOQ = \pi/2 - \theta$ and

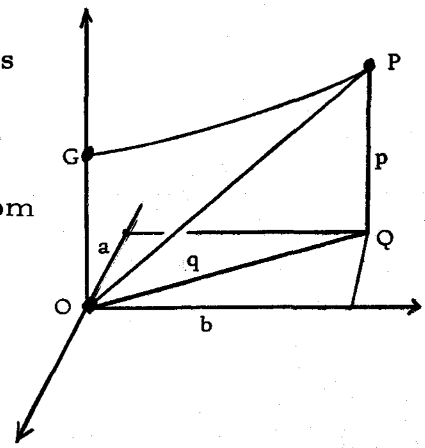
3. $$\cos(\pi/2-\theta)\cos\Pi(q) = \sin\theta\cos\Pi(q)$$
$$= \frac{\cos\Pi(|b|)}{\sqrt{\cos^2\Pi(a)+\cos^2\Pi(b)}}\sqrt{\cos^2\Pi(a)+\cos^2\Pi(b)}$$
$$= \cos\Pi(|b|) = \cos\Pi(|[OB]|_S)$$

since $|\cos\Pi(b)| = \cos\Pi(|b|)$. Let Q' be the point of $\underline{OQ}$ so that $Q\text{-}O\text{-}Q'$ and $OQ \cong OQ'$. Let B' be the point of the Y-axis so that $B\text{-}O\text{-}B'$ and $OB \cong OB'$. Then by vertical angles, $\angle BOQ \cong \angle B'OQ'$ and by S.A.S., $\Delta BOQ \cong \Delta B'OQ'$. Thus by 3., if $b > 0$, Q has coordinate b. If $b < 0$ Q' has coordinate b. For $\theta > \pi/2$, we use a symmetrical argument with appropriate changes of signs.

** We thus know that there is a point Q in the XY-plane with coordinates $(a, b, 0)$ when a, b and c are as given.

Now let $\underline{OS}$ be the perpendicular to the XY-plane, and let P be the point of $\underline{OS}$ so P is in the same half plane (of the plane containing the Z axis and $\underline{OQ}$) determined by $\underline{OQ}$ as is the positive part of the Z-axis, with P selected so that

$\cos \Pi(p) = \dfrac{\cos \Pi(|c|)}{\sqrt{1-\cos^2(q)}}$ (which we know exists by our hypothesis and our key assumption).



Let G be the foot of the perpendicular from P to the Z-axis. Then by 6.5

$$\cos \Pi(|[OG]|_S) = \sin \Pi(q) \cos \Pi(p)$$

$$= \sqrt{1-\cos^2 \Pi(q)} \frac{\cos \Pi(|c|)}{\sqrt{1-\cos^2 \Pi(q)}}$$

$$= \cos \Pi(|c|).$$

If $c > 0$, P has coordinates (a, b, c) using 6.8. If $c < 0$ let P' be symmetrical to P with respect to <u>OQ</u>. Then by 6.8, P' has coordinates (a, b, c).

<u>Case 2.</u> By symmetry we need only consider when $c = 0$. In that case the point Q described in ** above satisfies the conditions required.

<u>Case 3.</u> If two coordinates are zero the proof is immediate by our key assumption. Thus we have completed the proof.

6.9.1 COROLLARY. Let P be any point and let (a, b, c) be the coordinates of P. Let $p = |[OP]|_S$. Then

1. $\sin \Pi(p) = \sqrt{1-\cos^2 \Pi(a)-\cos^2 \Pi(b)-\cos^2 \Pi(c)}$, and
2. $\cos \Pi(p) = \sqrt{\cos^2 \Pi(a)+\cos^2 \Pi(b)+\cos^2 \Pi(c)}$.

Proof: Suppose $P \neq O$ since $p = 0$ gives trivial proof.

1. In Case 1 of the "only if" part of the argument in 6.9, Equation 5. gives $\sin \Pi(p) = \sqrt{1-\cos^2\Pi(a)-\cos^2\Pi(b)-\cos^2\Pi(c)}$.
2. Since $p > 0$ $\cos \Pi(p) = \sqrt{1-\sin^2\Pi(p)}$

$$= \sqrt{\cos^2\Pi(a)+\cos^2\Pi(b)+\cos^2\Pi(c)}$$

It is becoming quite tiresome to use the notation $\cos \Pi(a)$, $\cos \Pi(x)$, etc. In the subsequent development a much more frequent use of $\cos \circ \Pi$ is necessary. For this reason we introduce a shorthand notation for the values of this very important function.

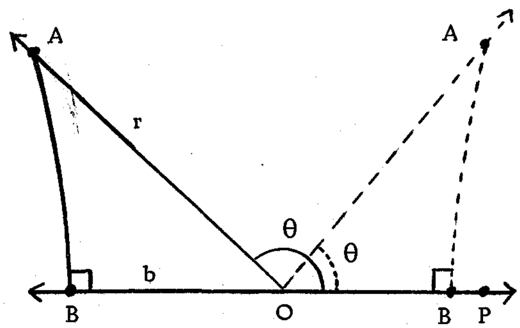
NOTATION: If x is any real number "$\underline{x}$" will be used to denote $\cos \Pi(x)$.

At times we will not use this shorthand if it seems necessary to place greater stress on the use of $\cos \circ \Pi$ than the shorthand seems to provide.

It is of extreme importance to avoid any carelessness in the reading of this shorthand. The results of 6.9.1-2, for example, appear remarkably Euclidean when written in this shorthand, i.e., if $|[OP]|_S = p$, $\underline{p} = \sqrt{\underline{a}^2+\underline{b}^2+\underline{c}^2}$ where (a, b, c) is the coordinate triple for P.

It is only when we keep in mind that, for example, $\underline{p}$ is the cosine of the number assigned to the angle of parallelism associated with the segment class $[OP]$, that the above formula comes to us with appropriate meaning.

6.10 LEMMA. Let O and P be distinct points of line ℓ. Choose the assigment of reals to ℓ so that O is associated with 0 and P is associated with a positive number. Let A be any point not on ℓ. Finally let B be the foot of the perpendicular to ℓ through A and let b be associated with B. Then if $\theta = m\angle AOP$ and $\|[AO]\|_S = r$, $\cos\theta = \underline{b}/\underline{r}$.



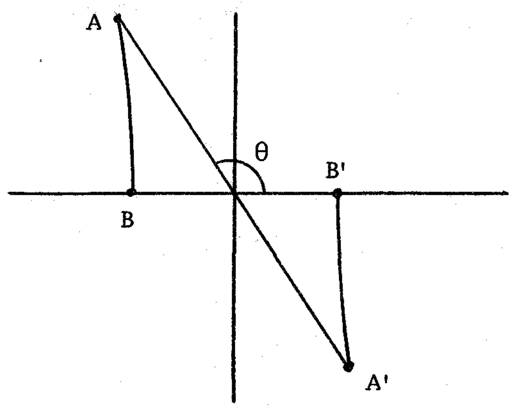
Proof: Case 1. Suppose B is a point of $\overrightarrow{OP}$. Then in triangle $\triangle AOB$ we have, by 6.4, $\cos\theta = \underline{b}/\underline{r}$.

Case 2. Suppose $B = O$. Then $b = 0$ and $\theta = \pi/2$ and $0 = \cos\pi/2 = \cos\pi/2/\underline{r}$.

Case 3. Suppose B-O-P. Then $\angle AOB$ is supplementary to $\angle AOP$ and by Definition 2.13, $m\angle AOB = \pi - \theta$ and, by 6.4, $\cos(\pi-\theta) = \cos\Pi(|b|)/\underline{r}$. But $\cos(\pi-\theta) = -\cos\theta$ and $b < 0$ so $\Pi(b) = \pi - \Pi|b|$ (cf. 3.22.1). Thus $\cos\Pi(|b|) = -\cos\Pi(b) = -\underline{b}$ giving us $\cos\theta = \underline{b}/\underline{r}$, as desired.

6.10.1 COROLLARY. Let $A, B, O, P, Q, \ell, r, b$, and θ be as above. If A' is any point so that A-O-A', B' is the foot of the perpendicular to ℓ through A', $\|[OB']\|_S = b'$, and $\|[OA']\|_S = r'$, then $\cos\theta = -\underline{b}'/\underline{r}'$.

Proof: Let $\varphi = m\angle A'OP$. By 6.10, $\cos\varphi = \underline{b}'/\underline{r}'$. However, $\cos\varphi = \cos(\pi-\theta) = -\cos\theta$ so we are done.



6.10.2 COROLLARY. Let

1. A be a point of the XY-plane different from the origin,
2. A have coordinates (a, b, 0),
3. $\overrightarrow{OX}$ and $\overrightarrow{OY}$ be the positive X-axis and Y-axis, respectively,
4. $m\angle AOX = \theta$, $m\angle AOY = \varphi$, and
5. $|[AO]|_S = r$.

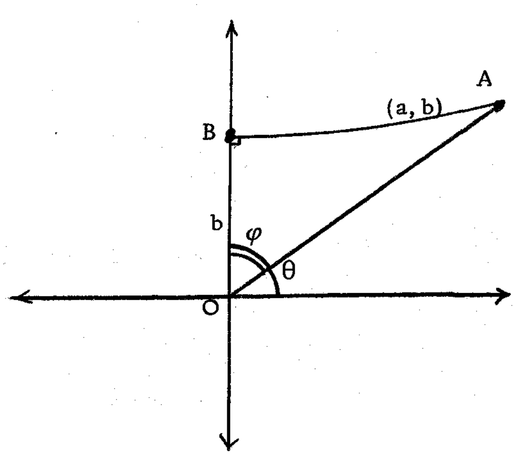
Then

1. if A is on Y's side of $\underline{OX}$, $\underline{b} = \underline{r}\sin\theta$, and
2. if A is on the opposite side of $\underline{OX}$ from Y, $\underline{b} = -\underline{r}\sin\theta$.

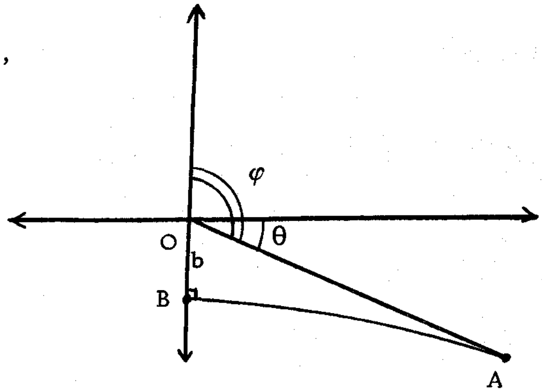
Proof: (1) if $\theta \neq \pi/2$ $\varphi = \pm(\pi/2-\theta)$.

By 6.10
$$\underline{b} = \underline{r}\cos(\pm[\pi/2-\theta])$$
$$= \underline{r}\cos(\pi/2-\theta)$$
$$= \underline{r}\sin\theta .$$

If $\theta = \pi/2$
$$\underline{b} = \underline{r}$$
$$= \underline{r}\sin\pi/2 .$$



(2) Let B be the foot of the perpendicular to the Y-axis from A. Then by 6.4,



$$\cos \Pi|[OB]|_S = \underline{r}\cos(\pi-\varphi)$$
$$= -\underline{r}\cos(\varphi).$$

Now $\varphi = \pi/2 + \theta$ so

$$\cos \Pi|[OB]|_S = +\underline{r}\sin\theta .$$

Since B is on the negative Y-axis, $\underline{b} = -\underline{r}\sin\theta$, since $b = -|[OB]|_S$ by definition. If $\theta = \pi/2$, $\underline{b} = -\underline{r}\sin\pi/2$.

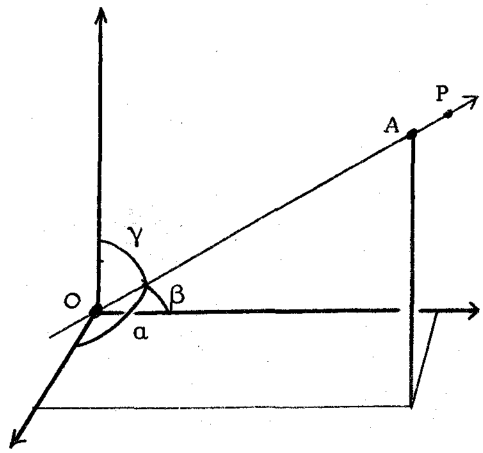
6.11 DEFINITION. Let A, O, and P be collinear with A and O distinct and with (a, b, c), (0, 0, 0) and (x, y, z) their respective coordinates. Then

$$\sigma[(a,b,c),(x,y,z)] = \begin{cases} 1 & \text{if } P \text{ is on } \overrightarrow{OA} \text{ or } P = O \\ -1 & \text{if } P\text{-}O\text{-}A \end{cases}$$

6.12 LEMMA. If ℓ is the line incident on the points O and A where O has coordinates (0, 0, 0) and A has coordinates (a, b, c) so $a^2+b^2+c^2 > 0$, and if P is any point of ℓ, say with coordinates (x, y, z), then as vectors $(\underline{x}, \underline{y}, \underline{z}) = t(\underline{a}, \underline{b}, \underline{c})$ where

$$t = \sqrt{\frac{\underline{x}^2+\underline{y}^2+\underline{z}^2}{\underline{a}^2+\underline{b}^2+\underline{c}^2}}\ \sigma[(a,b,c),(x,y,z)]$$

Proof: Suppose A is not on a negative axis. Let α, β, and γ be m∠AOX, m∠AOY, and m∠AOZ for any points X, Y, Z on the positive rays of the X-, Y-, and Z-axes respectively. Let P be any other point of $\underline{OA}$. Finally let $|[OA]|_S = r$ and let $|[OP]|_S = s$ where the



coordinates of A and P are (a, b, c) and (x, y, z) respectively.

Case 1. If P is on $\overrightarrow{OA}$, by 6.10 we have

$$\cos \alpha = \underline{a}/\underline{r} = \underline{x}/\underline{s},$$

$$\cos \beta = \underline{b}/\underline{r} = \underline{y}/\underline{s},$$

$$\cos \gamma = \underline{c}/\underline{r} = \underline{z}/\underline{s},$$

i.e., $(\underline{x}, \underline{y}, \underline{z}) = t(\underline{a}, \underline{b}, \underline{c})$ where $t = \underline{s}/\underline{r}$.

Case 2. If P = O, $\underline{x} = \underline{y} = \underline{z} = 0$.

Case 3. If P-O-A, by 6.11.1 we have (as above)

$(\underline{x}, \underline{y}, \underline{z}) = -t(\underline{a}, \underline{b}, \underline{c})$ where $t = \underline{s}/\underline{r}$. Now, by 6.9.1,

$t = \sqrt{\underline{x}^2+\underline{y}^2+\underline{z}^2} \;/\; \sqrt{\underline{a}^2+\underline{b}^2+\underline{c}^2}$.

This together with 1., 2., and 3. and the definition of σ completes the proof for A not on a negative axis. Suppose A is on the negative X-axis. Then (a, b, c) = (a, 0, 0), ℓ is the X-axis and thus (x, y, z) = (x, 0, 0). Thus we see

$(\underline{x}, 0, 0) = (\underline{x}^2/\underline{a}^2)(\underline{a}, 0, 0)\sigma[(a, 0, 0), (x, 0, 0)]$ and by symmetry of argument we are done.

We now consider necessary and sufficient conditions for given coordinates to name a point on a given line through the origin O.

6.13 LEMMA. Let O, P, and A be points with coordinates (0, 0, 0), (x, y, z), and (a, b, c) respectively. P is a point of line $\underline{OA}$ iff (in vector notation) there is a real number t so that $(\underline{x}, \underline{y}, \underline{z}) = t(\underline{a}, \underline{b}, \underline{c})$ with $|t| < (\underline{a}^2+\underline{b}^2+\underline{c}^2)^{-1/2}$.

Proof: "only if": If P is on $\underline{OA}$, this is just Lemma 6.12 and we are done.

"if": On $\overrightarrow{OA}$ there is a point Q so that $|[OQ]|_S = q$ with $\cos \Pi(q) = \sqrt{\underline{x}^2+\underline{y}^2+\underline{z}^2}$. Let Q have coordinates (x', y', z'). Then by 6.12 we have $(\underline{x}', \underline{y}', \underline{z}') = t'(\underline{a}, \underline{b}, \underline{c})$ where (by using 6.9.1)

$$t' = (\underline{q}/\sqrt{\underline{a}^2+\underline{b}^2+\underline{c}^2})\sigma[(a, b, c),(x', y', z')].$$

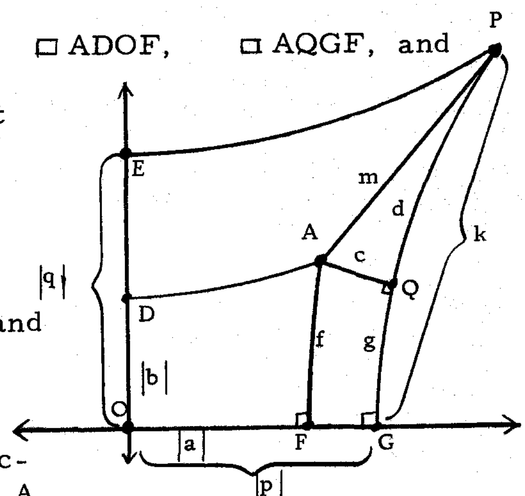
Thus $\cos \Pi(w) = \pm \cos \Pi(w')$ for $(w, w') = (x, x'), (y, y')$ or (z, z') respectively. Examining the construction of the argument for 6.12 we see that either P has the same coordinates as Q or P has the same coordinates as the point Q' of $\underline{OA}$ so that Q-O-Q' with $OQ \cong OQ'$. Since the planes perpendicular to the X-, Y-, and Z-axes which determine the points Q and Q' are unique, P = Q or P = Q' and we are done.

6.14 LEMMA. If A and P are distinct points of the XY-plane with coordinates (a, b, 0) and (p, q, 0), respectively, and if $|[AP]|_S = m$, then $\sin \Pi(m) = (\sqrt{1-\underline{a}^2-\underline{b}^2}\sqrt{1-\underline{p}^2-\underline{q}^2})/(1-\underline{ap}-\underline{bq})$ [2, p. 349].

Proof: Let D and E be the feet of the perpendiculars to the Y-axis containing A and P respectively. Let F and G be the feet of the perpendiculars, s and t, to the X-axis containing A and P respectively. Finally, let Q be the foot of the perpendicular to line t containing A (cf. figures below).

By definition we know that $|[OF]|_S = |a|$, $|[OD]|_S = |b|$, $|[OG]|_S = |p|$, and $|[OE]|_S = |q|$. Let $|[AQ]|_S = c$, $|[PQ]|_S = d$, $|[AF]|_S = f$, $|[GQ]|_S = g$, and $|[GP]|_S = k$.

Suppose A and P are each not on the X-axis or Y-axis and $F \neq G$. Then consider quadrilaterals $\square ADOF$, $\square AQGF$, and $\square PEOG$. We apply Lemma 6.5 to get



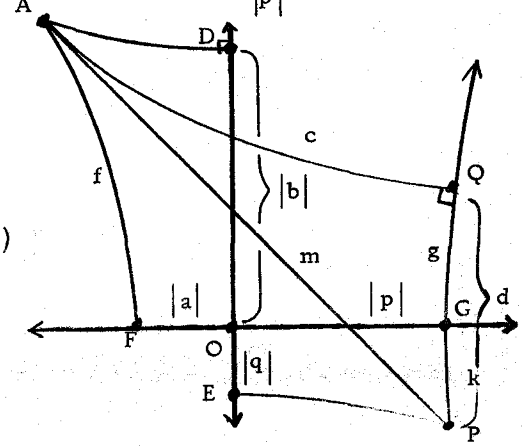
(i) $\cos \Pi(f) = \cos \Pi(|b|)/\sin \Pi(|a|)$,

(ii) $\cos \Pi(k) = \cos \Pi(|q|)/\sin \Pi(|p|)$,

(iii) $\cos \Pi(g) = \sin \Pi(|a-p|) \cos \Pi(f)$, and

(iv) $\cos \Pi(|a-p|) = \sin \Pi(g) \cos \Pi(c)$.

Now letting sgn be the signum function and using 6.1, Equations (i) and (ii) can be rewritten as



(i') $\cos \Pi(f) = \mathrm{sgn}(b) \cos \Pi(b)/\sin \Pi(a)$

(ii') $\cos \Pi(k) = \mathrm{sgn}(q) \cos \Pi(q)/\sin \Pi(p)$

Examination of (i), (ii), (iii), (iv), (i'), and (ii') will assure us that these equations are all true when either of A or P is on the Y-axis.

Now, if $\operatorname{sgn}(b)\operatorname{sgn}(q) = 1$ or 0, $d = |k-g|$. However, if $\operatorname{sgn}(b)\operatorname{sgn}(q) = -1$, $d = k+g$. Thus by 6.2, we have

$$\text{(v)} \qquad \sin \Pi(d) = \frac{\sin \Pi(k) \sin \Pi(g)}{1-\operatorname{sgn}(b)\operatorname{sgn}(q) \cos \Pi(k) \cos \Pi(g)} .$$

If $P \neq Q$, in $\triangle APQ$ we apply 6.4 to get

$$\text{(vi)} \qquad \sin \Pi(m) = \sin \Pi(d) \sin \Pi(c).$$

If $P = Q$, then $d = 0$, $\sin \Pi(d) = 1$, $m = c$ and Equation (vi) still holds.

Now using (v) and (vi), we get

$$\text{(vii)} \qquad \sin \Pi(m) = \frac{\sin \Pi(k) \sin \Pi(g) \sin \Pi(c)}{1-\operatorname{sgn}(b)\operatorname{sgn}(q) \cos \Pi(k) \cos \Pi(g)} .$$

From (ii'), we get

$$\begin{aligned} \sin^2 \Pi(k) &= 1 - \cos^2 \Pi(k) \\ &= 1 - \cos^2 \Pi(q)/[1-\cos^2 \Pi(p)] \\ &= \frac{1-\cos^2 \Pi(p)-\cos^2 \Pi(q)}{1-\cos^2 \Pi(p)} . \end{aligned}$$

Thus we have

(viii) $\sin^2\Pi(k) = (1-\underline{p}^2-\underline{q}^2)/(1-\underline{p}^2)$.

Using (i'), (iii), and (iv), we get

$$\begin{aligned}
\sin^2\Pi(g)\sin^2\Pi(c) &= \sin^2\Pi(g) - \sin^2\Pi(g)\cos^2\Pi(c) \\
&= \sin^2\Pi(g) - \cos^2\Pi(a-p), \quad \text{by (iv)}, \\
&= \sin^2\Pi(a-p) - \cos^2\Pi(g) \\
&= \sin^2\Pi(a-p)[1-\cos^2\Pi(f)], \quad \text{by (iii)}, \\
&= \frac{\sin^2\Pi(a)\sin^2\Pi(p)}{[1-\cos\Pi(a)\cos\Pi(p)]^2}\left[1-\frac{\cos^2\Pi(b)}{\sin^2\Pi(a)}\right], \\
&\qquad \text{by 6.2 and (i')}, \\
&= \frac{\sin^2\Pi(a)\sin^2\Pi(p)[\sin^2\Pi(a)-\cos^2\Pi(b)]}{[1-\cos\Pi(a)\cos\Pi(b)]^2\sin^2\Pi(a)} \\
&= \frac{[1-\cos^2\Pi(p)][1-\cos^2\Pi(a)-\cos^2\Pi(b)]}{[1-\cos\Pi(a)\cos\Pi(b)]^2}
\end{aligned}$$

i.e.,

(ix) $\sin^2\Pi(g)\sin^2\Pi(c) = (1-\underline{p}^2)(1-\underline{a}^2-\underline{b}^2)/(1-\underline{a}\,\underline{b})^2$.

Thus, in the expression of $\sin\Pi(m)$, the square of the numerator is computed, by (viii) and (ix), to be

(x) $[(1-\underline{p}^2-\underline{q}^2)/(1-\underline{p}^2)][(1-\underline{p}^2)(1-\underline{a}^2-\underline{b}^2)/(1-\underline{a}\,\underline{b})^2$

$= (1-\underline{p}^2-\underline{q}^2)(1-\underline{a}^2-\underline{b}^2)/(1-\underline{a}\,\underline{b})^2$.

For the denominator we see that

$$1 - \operatorname{sgn}(b)\operatorname{sgn}(q)\cos\Pi(k)\cos\Pi(g)$$

$$= 1 - \frac{\cos\Pi(q)}{\sin\Pi(p)}\sin\Pi(a-p)\frac{\cos\Pi(b)}{\sin\Pi(a)}, \quad \text{by (i'), (ii'), and (iii),}$$

$$= 1 - \frac{\cos\Pi(b)\cos\Pi(q)}{\sin\Pi(a)\sin\Pi(p)}\,\frac{\sin\Pi(a)\sin\Pi(p)}{1-\cos\Pi(a)\cos\Pi(p)}, \quad \text{by 6.2,}$$

$$= \frac{1-\cos\Pi(a)\cos\Pi(p) - \cos\Pi(b)\cos\Pi(q)}{1-\cos\Pi(a)\cos\Pi(p)}.$$

i.e.

(xi) $\quad 1 - \operatorname{sgn}(b)\operatorname{sgn}(q)\cos\Pi(k)\cos\Pi(g) = (1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})/(1-\underline{a}\,\underline{p})$.

Since $m > 0$, from (vii), (x), and (xi), we have

$$\sin\Pi(m) = [\sqrt{1-\underline{a}^2-\underline{b}^2}\,\sqrt{1-\underline{p}^2-\underline{q}^2}/(1-\underline{a}\,\underline{b})]/[(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})/(1-\underline{a}\,\underline{b})]$$

$$= \sqrt{1-\underline{a}^2-\underline{b}^2}\,\sqrt{1-\underline{p}^2-\underline{q}^2}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}),$$

as desired.

Now suppose $F = G \neq O$. Then $A = Q$ and we have

$$m = \begin{cases} k+f & \text{if } \operatorname{sgn}(b)\operatorname{sgn}(q) = -1 \\ |k-f| & \text{if } \operatorname{sgn}(b)\operatorname{sgn}(q) = 1 \text{ or } 0. \end{cases}$$

Thus, by 6.1 and 6.2 we have

(xii) $\quad \sin\Pi(m) = \dfrac{\sin\Pi(k)\sin\Pi(f)}{1-\operatorname{sgn}(b)\operatorname{sgn}(q)\cos\Pi(k)\cos\Pi(f)}$

By Lemma 6.5 we have

(xiii) $\quad \cos\Pi(f) = \operatorname{sgn}(b)\cos\Pi(b)/\sin\Pi(a)$, and

(xiv) $\cos \Pi(k) = \operatorname{sgn}(q) \cos \Pi(q)/\sin \Pi(p),$

so that

$$\text{(xv)} \qquad \sin^2 \Pi(m) = \frac{[1-\cos^2 \Pi(k)][1-\cos^2 \Pi(f)]}{[1-\frac{\cos \Pi(b)}{\sin \Pi(a)} \frac{\cos \Pi(q)}{\sin \Pi(p)}]^2}$$

$$= \frac{[\sin^2 \Pi(a)-\cos^2 \Pi(b)][\sin^2 \Pi(p)-\cos^2 \Pi(q)]}{[\sin \Pi(a) \sin \Pi(p) - \cos \Pi(b) \cos \Pi(q)]^2} .$$

But $a = p$, so from (xv) we have

$$\sin^2 \Pi(m) = \frac{[1-\underline{a}^2-\underline{b}^2][1-\underline{p}^2-\underline{q}^2]}{[1-\underline{a}^2-\underline{b}\,\underline{q}]^2}$$

$$= \frac{[1-\underline{a}^2-\underline{b}^2][1-\underline{p}^2-\underline{q}^2]}{[1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}]^2}$$

i.e., since $m > 0$,

$$\sin \Pi(m) = \sqrt{1-\underline{a}^2-\underline{b}^2}\sqrt{1-\underline{p}^2-\underline{q}^2}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}) .$$

Suppose A and P are on the X-axis. We consider three cases.

Case 1. A or P is the origin: $m = \max\{|a|, |p|\}$ since one of a or $p = 0$. By symmetry, let us suppose $a \neq 0$. If $a \neq 0$,

$$\sin \Pi(|a|) = \sin \Pi(a), \quad \text{by } 6.1,$$
$$= \sqrt{1-\underline{a}^2}$$
$$= \sqrt{1-\underline{a}^2-\underline{b}^2}\sqrt{1-\underline{p}^2-\underline{q}^2}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}),$$
since $b = p = q = 0$.

Case 2. O-A-P or O-P-A: $m = |a-p|$ so

$$\sin \Pi(m) = \sin \Pi(a-p), \quad \text{by } 6.1,$$
$$= \sqrt{1-\underline{a}^2}\sqrt{1-\underline{p}^2}/(1-\underline{a}\,\underline{p}), \quad \text{by } 6.2$$
$$= \sqrt{1-\underline{a}^2-\underline{b}^2}\sqrt{1-\underline{p}^2-\underline{q}^2}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}), \quad \text{since } b = q = 0.$$

Case 3. A-O-P: $m = |a| + |p|$ so

$$\sin \Pi(m) = \sqrt{1-\underline{a}^2}\sqrt{1-\underline{p}^2}/(1-\underline{a}\,\underline{p})$$
by 6.2 and because a and p differ in signs.

Thus, just as in Case 2 above we are done since $b = q = 0$.

Finally, if A and P are both on the Y-axis, we use an argument symmetrical to that for the case of both A and P on the X-axis to complete the last step of the proof.

6.15 THEOREM. If A and P are distinct points with coordinates (a, b, c) and (p, q, r) and if $|[AP]|_S = t$, then $\sin \Pi(t) = \sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}-\underline{c}\,\underline{r})$.

Proof: Let D and E be the feet of perpendiculars to the Z-axis containing A and P respectively. Let F and G be the feet of the perpendiculars, σ and τ, to the XY-plane containing A and P respectively. Let Q be the foot of the perpendicular to the line τ containing the point A.

We know, by 6.8.1, that F and G have coordinates $(a, b, 0)$ and $(p, q, 0)$ respectively. Let $|[AQ]|_S = h$, $|[PQ]|_S = d$, $|[AF]|_S = f$, $|[GQ]|_S = g$, and $|[GP]|_S = k$. We know that $|[OD]|_S = |c|$ and $|[OE]|_S = |r|$. We let $|[OF]|_S = u$ and $|[OG]| = v$.

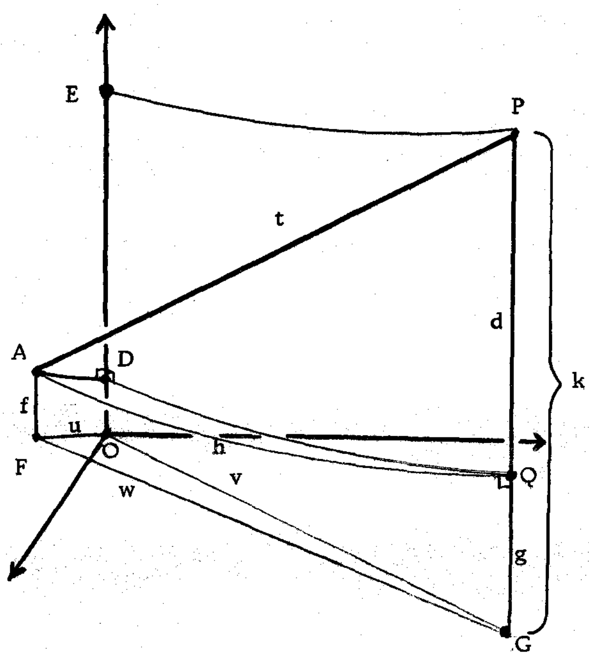
Suppose A and P are not both in XY-plane and $F \neq G$. Then by 6.9.1

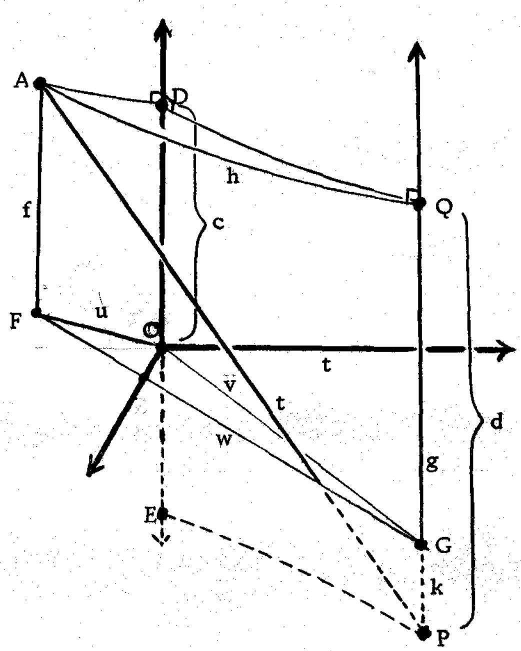
(i) $\sin \Pi(u) = \sqrt{1-\underline{a}^2-\underline{b}^2}$, and

(ii) $\sin \Pi(v) = \sqrt{1-\underline{p}^2-\underline{q}^2}$.

If we let $w = |[FG]|_S$, then, by 6.14, we have

(iii) $\sin \Pi(w) = \sqrt{1-\underline{a}^2-\underline{b}^2}\,\sqrt{1-\underline{p}^2-\underline{q}^2}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})$.





Examination of quadrilaterals □ADOF, □PEOG, and □AQGF, gives,

(iv) $\cos \Pi(f) = \cos \Pi(|c|)/\sin \Pi(u)$

$= \operatorname{sgn}(c)\ \underline{c}/\sqrt{1-\underline{a}^2-\underline{b}^2}$, by 6.5 and (i),

(v) $\cos \Pi(k) = \cos \Pi(|r|)/\sin \Pi(v)$

$= \operatorname{sgn}(r)\ \underline{r}/\sqrt{1-\underline{p}^2-\underline{q}^2}$, by 6.5 and (ii), and

(vi) $\cos \Pi(g) = \sin \Pi(w) \cos \Pi(f)$ by 6.5

$= \operatorname{sgn}(c)\ \underline{c}\sqrt{1-\underline{p}^2-\underline{q}^2}/(1-\underline{a}\ \underline{p}-\underline{b}\ \underline{q})$ by 6.14 and (iv).

Finally we get

(vii) $\cos \Pi(w) = \sin \Pi(g) \cos \Pi(h)$ by 6.5.

Examination of (iv), (v), (vi), and (vii) will show that these all hold when either of A or P is on the Z-axis.

If $P \neq Q$, in right triangle ΔAPQ, we have

(viii) $\sin \Pi(t) = \sin \Pi(h) \sin \Pi(d)$.

We see that (viii) is also true if $P = Q$.

Now just as in the proof of 6.14

if $\operatorname{sgn}(c)\operatorname{sgn}(r) = 0$ or 1, $d = |k-g|$, while

if $\operatorname{sgn}(c)\operatorname{sgn}(r) = -1$, $d = k + g$.

Thus

(ix) $$\sin \Pi(d) = \frac{\sin \Pi(k) \sin \Pi(g)}{1 - \operatorname{sgn}(c) \operatorname{sgn}(r) \cos \Pi(k) \cos \Pi(g)}$$

and Equation (viii) becomes

(x) $$\sin \Pi(t) = \frac{\sin \Pi(h) \sin \Pi(k) \sin \Pi(g)}{1 - \operatorname{sgn}(c) \operatorname{sgn}(r) \cos \Pi(k) \cos \Pi(g)}$$

Consideration of the numerator of (x) squared gives us

(xi) $$\sin^2 \Pi(h) \sin^2 \Pi(k) \sin^2 \Pi(g)$$

$$= (1 - \cos^2 \Pi(h)) \sin^2 \Pi(k) \sin^2 \Pi(g)$$

$$= [1 - \cos^2 \Pi(w) / \sin^2 \Pi(g)] \sin^2 \Pi(k) \sin^2 \Pi(g) \quad \text{from (iv)}$$

$$= [\sin^2 \Pi(g) - \cos^2 \Pi(w)] \sin^2 \Pi(k)$$

$$= [\sin^2 \Pi(w) - \cos^2 \Pi(g)][1 - \cos^2 \Pi(k)]$$

$$= \left[\frac{(1-\underline{a}^2-\underline{b}^2)(1-\underline{p}^2-\underline{q}^2)}{(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})^2} - \frac{\underline{c}^2(1-\underline{p}^2-\underline{q}^2)}{(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})^2}\right]\left[1 - \frac{\underline{r}^2}{1-\underline{p}^2\underline{q}^2}\right]$$

by 6.14, (v) and (vi),

$$= (1-\underline{a}^2-\underline{b}^2-\underline{c}^2)(1-\underline{p}^2-\underline{q}^2-\underline{r}^2)/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})^2 .$$

Considering the denominator of (x), we get

(xii) $$1 - \operatorname{sgn}(c) \operatorname{sgn}(r) \cos \Pi(k) \cos \Pi(g)$$

$$= 1 - [\underline{r}/\sqrt{1-\underline{p}^2-\underline{q}^2}][\underline{c}(\sqrt{1-\underline{p}^2-\underline{q}^2})/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})] \quad \text{from (ii), (vi),}$$

$$= 1 - \underline{r}\,\underline{c}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})$$

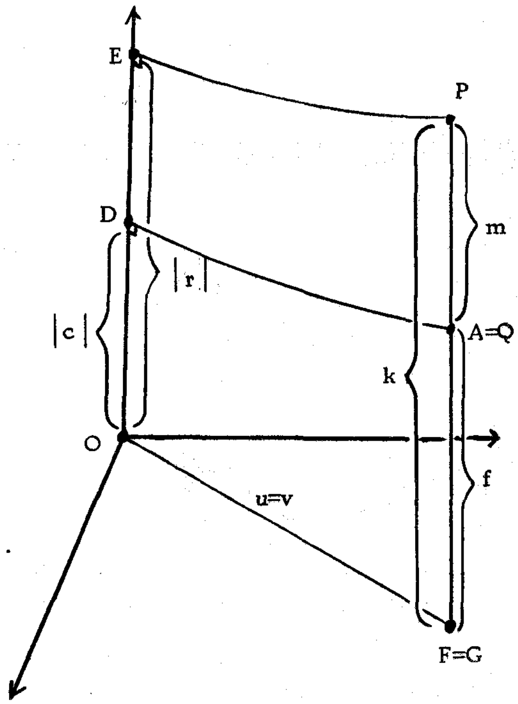
$$= (1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}-\underline{r}\,\underline{c})/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}).$$

combining (x), (xi), and (xii), we have

$$\sin \Pi(t) = \frac{\sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}\,\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})}{(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}-\underline{r}\,\underline{c})/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q})}$$

$$= \sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}\,\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2}/(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}-\underline{r}\,\underline{c})$$

as claimed.



Now suppose $F = G \neq O$.

Then m is determined to be

$$m = \begin{cases} k+f & \text{if } \operatorname{sgn}(c)\operatorname{sgn}(r) = -1 \\ |k-f| & \text{if } \operatorname{sgn}(c)\operatorname{sgn}(r) = 1 \text{ or } 0. \end{cases}$$

Thus by 6.1 and 6.2 we have

(xiii) $$\sin \Pi(m) = \frac{\sin \Pi(k)\,\sin \Pi(f)}{1-\operatorname{sgn}(c)\operatorname{sgn}(r)\underline{k}\,\underline{f}}\ .$$

By Lemma 6.5 and 6.9.1 we have

(xiv) $\underline{f} = \operatorname{sgn}(c)\,\underline{c}/\sqrt{1-\underline{a}^2-\underline{b}^2}$, and

(xv) $\underline{k} = \operatorname{sgn}(r)\,\underline{r}/\sqrt{1-\underline{p}^2-\underline{q}^2}$.

Thus (xiii) and the fact that $F = G$ allows us to write

(xvi) $$\sin^2 \Pi(m) = \frac{(1-\underline{f}^2)(1-\underline{k}^2)}{[1-\underline{c}\,\underline{r}/(1-\underline{a}^2-\underline{b}^2)]^2}$$

$$= \frac{(1-\underline{a}^2-\underline{b}^2-\underline{c}^2)(1-\underline{p}^2-\underline{q}^2-\underline{r}^2)\,/(1-\underline{a}^2-\underline{b}^2)^2}{(1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}-\underline{c}\,\underline{r})^2/(1-\underline{a}^2-\underline{b}^2)^2}\ .$$

since $m > 0$ we have

$$\sin \Pi(m) = \frac{\sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2}}{1-\underline{a}\,\underline{p}-\underline{b}\,\underline{q}-\underline{c}\,\underline{r}}$$

as claimed.

If A and P are on the Z-axis, then the result follows just as the case for A and P on the X-axis in 6.14 so we shall leave out the details here. If A and P are on the XY-plane we apply 6.14 directly. Thus we are done.

6.15.1 COROLLARY. Let P be any point different from the origin. Let P and I have coordinates (p, q, r) and (i, j, k). Then I is the midpoint of segment OP iff $(\underline{i}, \underline{j}, \underline{k}) = t(\underline{p}, \underline{q}, \underline{r})$ where $t = (1+\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2})^{-1}$.

Proof: By 6.13, I is on $\underline{OP}$ iff $(\underline{i}, \underline{j}, \underline{k}) = t(\underline{p}, \underline{q}, \underline{r})$ for some appropriate choice of t. I is the midpoint of OP iff

(i) OI = IP,

Let $\underline{I} = (\underline{i}, \underline{j}, \underline{k})$, $\underline{P} = (\underline{p}, \underline{q}, \underline{r})$. We now see that (i) is true iff

(ii) $\sqrt{1-\underline{I}\cdot\underline{I}} = \sqrt{1-\underline{I}\cdot\underline{I}}\sqrt{1-\underline{P}\cdot\underline{P}}/(1-\underline{I}\cdot\underline{P})$, by 6.9.1 and 6.15.

This equation is valid iff

(iii) $\sqrt{1-\underline{P}\cdot\underline{P}} = 1 - \underline{I}\cdot\underline{P}$. Now $\underline{I}\cdot\underline{P} = t\underline{P}\cdot\underline{P}$ for some t and

(iv) holds iff $t = (1-\sqrt{1-\underline{P}\cdot\underline{P}})/\underline{P}\cdot\underline{P}$

or equivalently $t = (1+\sqrt{1-P\cdot P})^{-1}$ as claimed.

6.16 LEMMA. If

1. ℓ is any line of the XY-plane,

2. ℓ does not contain the origin O,

3. A is the foot of the perpendicular from O to ℓ,

4. A has coordinates (a, b, 0), and

5. P is a point with coordinates (x, y, 0),

then P is a point of ℓ iff $\underline{a}\,\underline{x} + \underline{b}\,\underline{y} = \underline{a}^2 + \underline{b}^2$.

Proof: "<u>only if</u>". Let

1. $|[OA]|_S = r$, $|[OP]|_S = p$,

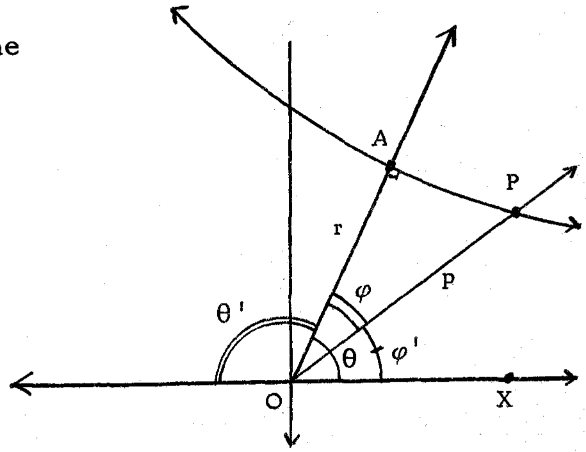
2. X and Y be on the positive X- and Y-axes respectively,

3. X' be on the X-axis so that X'-O-X,

4. $\theta = m\angle AOX$, $\theta' - m\angle AOX'$, $\varphi = m\angle AOP$, $\varphi' = m\angle POX$, and

   $\varphi'' = m\angle POX'$.

Case 1: $\overrightarrow{OP}$ is interior to $\angle AOX$: A and P are on the same side of the X-axis so $\varphi' = \theta - \varphi$. By 6.10 and 6.10.2:



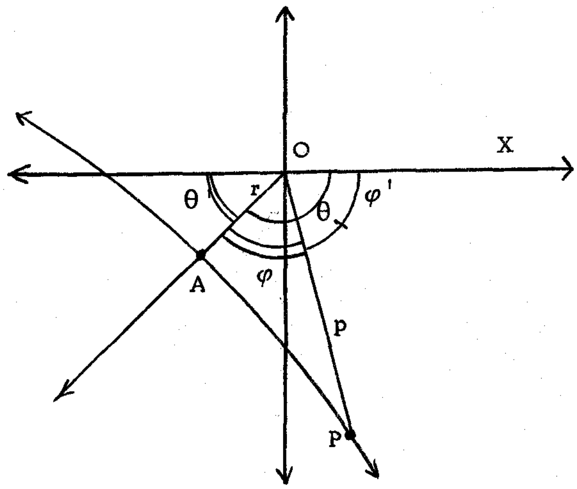
(i) $\underline{a} = \underline{r} \cos \theta$,

(ii) $\underline{b} = \pm \underline{r} \sin \theta$,

(iii) $\underline{x} = \underline{p} \cos(\theta - \varphi)$,

(iv) $\underline{y} = \pm \underline{p} \sin(\theta - \varphi)$,

where the signs used for $\underline{r}$ and $\underline{p}$ are always the same in (ii) and (iv). By 6.4 we conclude,



(v) $\underline{p} = \underline{r}/\cos \varphi$.

Using (iii), (iv), and (v), we get

(vi) $\underline{x} = \underline{r}(\cos \theta + \sin \theta \tan \varphi) = \underline{a} \pm \underline{b} \tan \varphi$,

(vii) $\underline{y} = \pm \underline{r}(\sin \theta - \cos \theta \tan \varphi) = \underline{b} \mp \underline{a} \tan \varphi$.

Thus

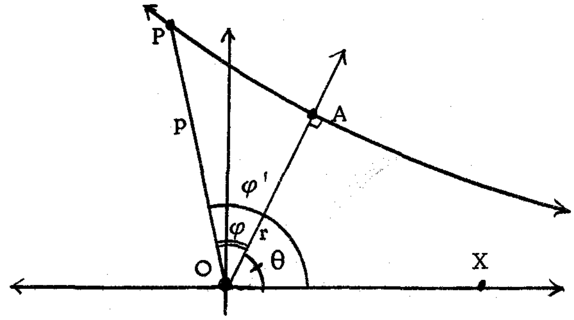
(viii) $\pm \tan \varphi = (\underline{x} - \underline{a})/\underline{b}$ provided $b \neq 0$ or

(viii') $\pm \tan \varphi = (\underline{b} - \underline{y})/\underline{a}$ provided $a \neq 0$.

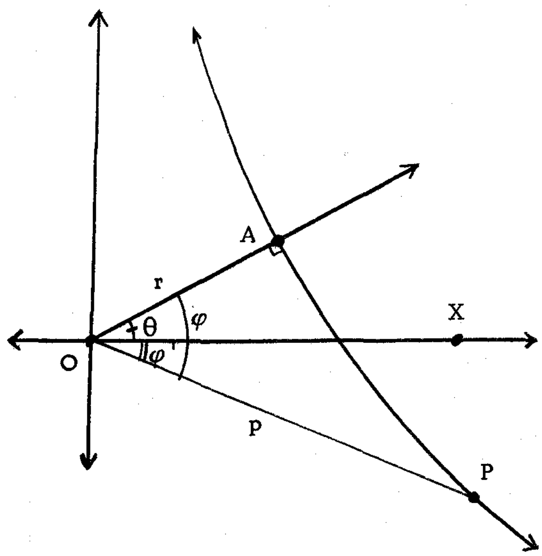
Since one of a or b is different from zero, first suppose $b \neq 0$. Then using (vii) and (viii)

$$\underline{b}\,\underline{y} = \underline{b}^2 + \underline{a}^2 - \underline{a}\,\underline{x}.$$

By a symmetrical use of (vi) and (viii') with $a \neq 0$, we also get the same result.



Case 2: $\overrightarrow{OA}$ is interior to $\angle POX$: In this case $\varphi' = \theta + \varphi$. The proof is just as for Case 1 with the appropriate changes in sign for (iii), (iv), (vi), and (vii), with the condition that the signs for $\underline{r}$ and $\underline{p}$ used in (ii) and (iv) always be chosen the same.



Case 3: $\overrightarrow{OX}$ is interior to $\angle AOP$: In this case $\varphi' = \varphi - \theta$. As above we get, by 6.4, 6.10, 6.10.2 with A and P on opposite sides of the X-axis,

(i) $\underline{a} = \underline{r} \cos \theta$,

(ii) $\underline{b} = \pm \underline{r} \sin \theta$,

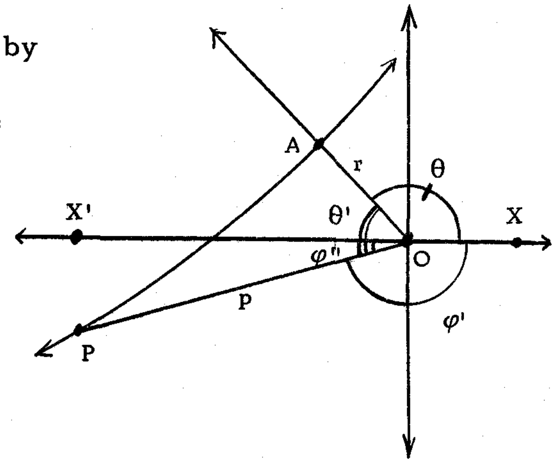
(iii) $\underline{x} = \underline{p} \cos (\varphi - \theta)$,

(iv) $\underline{y} = \pm \underline{p} \sin (\varphi - \theta)$,

where the signs for $\underline{r}$ and $\underline{p}$ in (ii) and (iv) are always chosen opposite.

The desired result then follows just as in Case 1.

Case 4: $\overrightarrow{OX'}$ is interior to $\angle AOP$: $\varphi'' = \varphi - \theta'$. Again A and P are on opposite sides of the X-axis and the computation is



just as in Case 3 with θ replaced by θ' and φ' replaced by φ''. The selection of signs is the same as in Case 3.

Case 5: A is on the X-axis: Proof is immediate from Cases 3 and 4 with θ and θ' set equal to zero respectively.

Case 6: A = P: Proof is trivial.

"if". Suppose P has coordinates $(x, y, 0)$ and $\underline{a}\,\underline{x} + \underline{b}\,\underline{y} = \underline{a}^2 + \underline{b}^2$. If P = A, P is on ℓ, as desired. If $P \neq A$, let $|[AP]|_S = q$. Then by Theorem 6.15

$$\sin \Pi(q) = (\sqrt{1-\underline{a}^2-\underline{b}^2}\,\sqrt{1-\underline{x}^2-\underline{y}^2})/(1-\underline{a}\,\underline{x}-\underline{b}\,\underline{y}).$$

By hypothesis

$$\begin{aligned}\sin \Pi(q) &= (\sqrt{1-\underline{a}^2-\underline{b}^2}\,\sqrt{1-\underline{x}^2-\underline{y}^2})/(1-\underline{a}^2-\underline{b}^2)\\ &= \sqrt{1-\underline{x}^2-\underline{y}^2}\,/\,\sqrt{1-\underline{a}^2-\underline{b}^2} = \sin \Pi(p)/\sin \Pi(r),\end{aligned}$$

so $\sin \Pi(p) = \sin \Pi(q)\sin \Pi(r)$. Thus by 6.4.1 ΔOPA is a right triangle with right angle at A so P is on ℓ and we are done.

It is well to note that the choice of signs described in Cases 1-4 is not a free choice. The pairing of signs is forced from the juxtaposition of the rays $\overrightarrow{OA}$, $\overrightarrow{OP}$, $\overrightarrow{OX}$ and $\overrightarrow{OX'}$ and the results of 6.10.2. Lemmas 6.13 and 6.16 together with the following corollary essentially give equations for all lines through the origin and lines in any coordinate plane.

6.16.1 COROLLARY. If ℓ is any line of the W_1W_2-plane not containing the origin O, in which $W_1 \in \{X, Y, Z\}$, $W_2 \in \{X, Y, Z\} - \{W_1\}$, A is the foot of the perpendiculars from O to ℓ, and A has coordinates (a, b, c), and P coordinates (x, y, z), the P is on ℓ iff $\underline{a}\,\underline{x} + \underline{b}\,\underline{y} + \underline{c}\,\underline{z} = \underline{a}^2 + \underline{b}^2 + \underline{c}^2$.

Proof:

1. If W_1W_2-plane is the XY-plane, c and z are zero by definition and 6.16 gives the result immediately.
2. If W_1W_2-plane is the XZ-plane, b and y are zero and by an argument symmetrical to that for 6.16 we are done.
3. If W_1W_2-plane is YZ-plane, we get the result just as in 2.

The following results provide us with equations relating the components of the coordinates of certain collinear points.

6.17 LEMMA. If A and P are distinct points of the XY-plane with coordinates (a, b, 0) and (p, q, 0) respectively, then U is a point of the line $\overline{AP}$ with coordinates (x, y, 0), iff $(\underline{q}-\underline{b})\underline{x} + (\underline{a}-\underline{p})\underline{y} = \underline{a}\,\underline{q} - \underline{p}\,\underline{b}$.

Proof: Let (c, d, 0) be the coordinate triple associated with the foot of the perpendicular to $\underline{AP}$ and containing the origin. Suppose $\underline{AP}$ is not a line through the origin. Then one of c or d is different from zero. By 6.16 we have

(i) $\quad ca + db = cp + dq$

or equivalently

(ii) $\quad c(a-p) = d(q-b)$.

Let us suppose $d \neq 0$. Then $p = a$ forces $q = b$ in Equation (ii) and thus $P = P'$. This contradicts the hypothesis. Thus when $d \neq 0$, $p \neq a$ and we get,

(ii) $\quad c/d = -(b-q)/(a-p)$.

Using 6.16 and dividing by d, we get U is on $\underline{AP}$ iff

(iv) $\quad (c/d)x + y = (c^2/d) + d$.

Substituting (iii) into (iv) and multiplying the result by $(a-p)$ gives us,

(v) $\quad (q-b)x + (a-p)y = (q-a)c + (a-p)d$.

Since (a, b, 0) also satisfies the hypothesis of 6.16, in particular, (v) becomes

(vi) $\quad (q-b)a + (a-p)b = (q-a)c + (a-p)d$,

or equivalently

(vii) $\quad aq - bp = (q-a)c + (a-p)d$.

Substituting (vii) into (v) we get the desired equation.

(viii) $(\underline{q}-\underline{b})\underline{x} + (\underline{a}-\underline{p})\underline{y} = \underline{a}\,\underline{q} - \underline{b}\,\underline{p}$

If $c \neq 0$, we get the same result by symmetry.

Now suppose $\underline{AP}$ contains the origin. Since A and P are distinct, either $a^2+b^2 \neq 0$ or $p^2+q^2 \neq 0$. Suppose the former. Then by 6.13 we immediately get

(ix) $\underline{x} = \underline{a}\sqrt{(\underline{x}^2+\underline{y}^2)/(\underline{a}^2+\underline{b}^2)}$,

and

(x) $\underline{y} = \underline{b}\sqrt{(\underline{x}^2+\underline{y}^2)/(\underline{a}^2+\underline{b}^2)}$, iff U is on $\underline{AP}$.

Suppose that $a \neq 0$. Then from (ix) and (x) we have,

(xi) $\underline{y} = \underline{b}\,\underline{x}/\underline{a}$ or $\underline{a}\,\underline{y} - \underline{b}\,\underline{x} = 0$

If P is also not the origin, by a symmetrical argument, we get

(xii) $\underline{p}\,\underline{y} - \underline{q}\,\underline{x} = 0$.

Either (xi) or (xii) applied to the specific coordinates (a, b, 0) and (p, q, 0) will give,

(xiii) $\underline{a}\,\underline{p} - \underline{b}\,\underline{q} = 0$.

Thus from (xi), (xii), and (xiii),

(xv) $(\underline{q}-\underline{b})\underline{x} + (\underline{a}-\underline{p})\underline{y} = \underline{a}\,\underline{p} - \underline{b}\,\underline{q}$ as desired.

This is also valid when not both A and P are different from the origin, and we are done.

6.17.1 COROLLARY. If P and P' are distinct points of the XZ-plane [YZ-plane] with coordinates (p, 0, q) and (p', 0, q') [(0, p, q) and (0, p', q')] respectively, then U is any point of line $\underline{PP'}$ with coordinates (x, 0, y) [(0, x, y)], if and only if

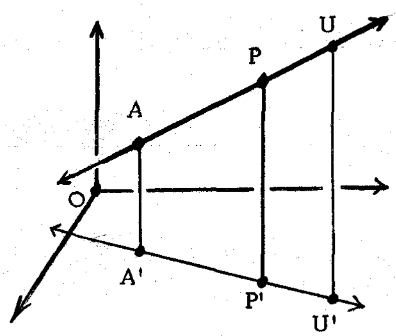
$$(\underline{q}'-\underline{q})\underline{x} + (\underline{p}-\underline{p}')\underline{y} = \underline{p}\,\underline{q}' - \underline{p}'\underline{q}\,.$$

Proof: Immediate by symmetry from 6.17.

6.18 THEOREM. If A and P are distinct points with coordinates (a, b, c) and (p, q, r), respectively, then U is any point of line $\underline{AP}$ with coordinates (x, y, z), iff

$(\underline{x}, \underline{y}, \underline{z}) = (\underline{p}, \underline{q}, \underline{r}) + t(\underline{a}-\underline{p}, \underline{b}-\underline{q}, \underline{c}-\underline{r})$ [in vector notation] with $t = (\underline{w}-\underline{s})/(\underline{d}-\underline{s})$ where $(w, d, s) \in \{(x, a, p), (y, b, q), (z, c, r)\}$ and $d-s \neq 0$.

Proof: Let A', P', U'; A'', P'', U'', and A''', P''', U''' be the "feet" of the perpendiculars to the XY-, XZ- and YZ axes, respectively, and containing A, P, and U respectively. From 0.18, 0.20, and 6.8.1 we see that these points have coordinates as follows:



A' --(a, b, 0), P' --(p, q, 0), U' --(x, y, 0),

A'' --(a, 0, c), P'' --(p, 0, r), U'' --(x, 0, z),

A''' --(0, b, c), P''' --(0, q, r), U''' --(0, y, z).

Applying 6.17 and 6.17.1, we get the results that U', U" and U'" are as described iff

$$(\underline{q}-\underline{b})\underline{x} + (\underline{a}-\underline{p})\underline{y} = \underline{a}\,\underline{q} - \underline{p}\,\underline{b},$$

$$(\underline{r}-\underline{c})\underline{x} + (\underline{a}-\underline{p})\underline{z} = \underline{a}\,\underline{r} - \underline{p}\,\underline{c},$$

and

$$(\underline{r}-\underline{c})\underline{y} + (\underline{b}-\underline{q})\underline{z} = \underline{b}\,\underline{r} - \underline{q}\,\underline{c},$$

which in turn give, respectively,

(i) $(\underline{q}-\underline{b})\underline{x} + (\underline{a}-\underline{p})\underline{y} = (\underline{q}-\underline{b})\underline{p} + (\underline{a}-\underline{p})\underline{q}$,

(ii) $(\underline{r}-\underline{c})\underline{x} + (\underline{a}-\underline{p})\underline{z} = (\underline{r}-\underline{c})\underline{p} + (\underline{a}-\underline{p})\underline{r}$,

(iii) $(\underline{r}-\underline{c})\underline{y} + (\underline{b}-\underline{q})\underline{z} = (\underline{r}-\underline{c})\underline{q} + (\underline{b}-\underline{q})\underline{r}$.

Now if each of the differences $a-p, b-q, c-r \neq 0$ we get

(iv) $(\underline{x}-\underline{p})/(\underline{a}-\underline{p}) = (\underline{y}-\underline{q})/(\underline{b}-\underline{q}) = (\underline{z}-\underline{r})/(\underline{c}-\underline{r}) = t$

or

(v) $(\underline{x}, \underline{y}, \underline{z}) = (\underline{p}, \underline{q}, \underline{r}) + t(\underline{a}-\underline{p}, \underline{b}-\underline{q}, \underline{c}-\underline{r})$.

If exactly one of the differences is zero, say $a-p = 0$, we have for example (p, b, c), (p, q, r), (p, y, z) are the coordinates of A, P, and U respectively and examination of Equation (i), (ii), and (iii) will give $(\underline{y}-\underline{q})/(\underline{b}-\underline{q}) = (\underline{z}-\underline{r})/(\underline{c}-\underline{r}) = t$ and Equation (v) holds iff U is a point of $\underline{AP}$.

Since A and P are distinct, not all the differences can be zero. Hence t is always defined. Our above argument (by symmetry) always assures us that t exists and has the same values for given points A and P, thus removing any possible ambiguity in defining t as the given quotient and we are done.

6.18.1 COROLLARY. Let A, P, U be points with coordinates (a, b, c), (p, q, r) and (x, y, z) respectively, then A-U-P iff $(\underline{x}, \underline{y}, \underline{z}) = (\underline{p}, \underline{q}, \underline{r}) + t(\underline{a}-\underline{p}, \underline{b}-\underline{q}, \underline{c}-\underline{r})$ with $0 < t < 1$.

Proof: If A-U-P, then A, U, and P are collinear and by 6.18, $(\underline{x}, \underline{y}, \underline{z}) = (\underline{p}, \underline{q}, \underline{r}) + t(\underline{a}-\underline{p}, \underline{b}-\underline{q}, \underline{c}-\underline{r})$ where $t = (\underline{w}-\underline{s})/(\underline{d}-\underline{s})$ where $(w, d, s) \in \{(x, a, p), (y, b, q), (z, c, r)\}$ and $d-s \neq 0$.

Since A and P are distinct, one of a-p, b-q, c-r is non-zero. Suppose $a-p \neq 0$. Then the planes α, β, and γ, perpendicular to the X-axis at A', U', and P' and containing the points A, U, and P, respectively, contain the points A, U, and P of line $\underline{AP}$ with A-U-P. Hence by Appendix A-2, A'-U'-P'. The points A', U', and P' have coordinates (a, 0, 0), (x, 0, 0) and (p, 0, 0). Now x is between a and p from the definition of the assignment of numbers to the coordinate axes. Thus $a < x < p$ or $a > x > p$. Thus $a-p < x-p < 0$ or $a-p > x-p > 0$ i.e., either $0 < -1(x-p) < (-1)(a-p)$ or $0 < (x-p) < a-p$. Hence $0 < \frac{x-p}{a-p} = t < 1$. By symmetry we are done.

Let $(\underline{x}, \underline{y}, \underline{z}) = (\underline{p}, \underline{q}, \underline{r}) + t(\underline{a}-\underline{p}, \underline{b}-\underline{q}, \underline{c}-\underline{r})$ with $0 < t < 1$. Since A and P are distinct, one of a-p, b-q, c-r is not zero. Suppose $a \neq p$. Then by 6.18 and direct computation

$t = (\underline{x}-\underline{p})/(\underline{a}-\underline{p})$ and U is a point of AP. By an argument analogous to that above we get A-U-P. By symmetry of argument we are done.

6.18.2 COROLLARY. Let A, A', A" be points with coordinates (a, b, c), (a', b', c'), (a", b", c"), respectively, on line ℓ which has an equation $(\underline{x}, \underline{y}, \underline{z}) = (\underline{p}, \underline{q}, \underline{r}) + t(\underline{u}-\underline{p}, \underline{v}-\underline{q}, \underline{w}-\underline{r})$. Then A-A'-A" iff the values k, k', k" of t associated with A, A', A" have the property that $k < k' < k''$ or $k'' < k' < k$.

Proof: From 6.18.1 we know A-A'-A" iff

(i) $(\underline{a}', \underline{b}', \underline{c}') = (\underline{a}, \underline{b}, \underline{c})(1-j) + (\underline{a}'', \underline{b}'', \underline{c}'')j$, where $0 < j < 1$,

and

(ii) $(\underline{a}', \underline{b}', \underline{c}') = (\underline{a}'', \underline{b}'', \underline{c}'')(1-i) + (\underline{a}, \underline{b}, \underline{c})i$, where $0 < i < 1$.

From our hypothesis and (i) and (ii), we have

(iii) $$(\underline{a}', \underline{b}', \underline{c}') = [(\underline{p}, \underline{q}, \underline{r})(1-k)+(\underline{u}, \underline{v}, \underline{w})k](1-j) + [(\underline{p}, \underline{q}, \underline{r})(1-k'')+(\underline{u}, \underline{v}, \underline{w})k'']j,$$

and

(iv) $$(\underline{a}', \underline{b}', \underline{c}') = [(\underline{p}, \underline{q}, \underline{r})(1-k'')+(\underline{u}, \underline{v}, \underline{w})k''](1-i) + [(\underline{p}, \underline{q}, \underline{r})(1-k)+(\underline{u}, \underline{v}, \underline{w})k]i .$$

From (iii) and (iv),

(v) $$(\underline{a}', \underline{b}', \underline{c}') = (\underline{p}, \underline{q}, \underline{r}) + [k+j(k''-k)](\underline{u}-\underline{p}, \underline{v}-\underline{q}, \underline{w}-\underline{r}) ,$$

(vi) $\quad (\underline{a}',\underline{b}',\underline{c}') = (\underline{p},\underline{q},\underline{r}) + [k''+i(k-k'')](\underline{u}-\underline{p},\ \underline{v}-\underline{q},\ \underline{w}-\underline{r})$.

Thus from the hypothesis

(vii) $\quad k' = k + j(k''-k), \quad \text{for} \quad 0 < j < 1,$

(viii) $\quad k' = k'' + i(k-k'') \quad \text{for} \quad 0 < i < 1.$

If $k'' > k$, (vii) gives $k < k'$ while (viii) gives $k' < k''$.

If $k > k''$, (vii) gives $k > k'$ while (viii) gives $k' > k''$.

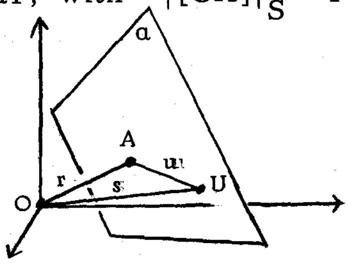
This completes the proof.

6.19 LEMMA. If α is a plane, so that

1. A is the foot of the perpendicular to α containing the origin,
2. A has coordinates $(a, b, c) \neq (0, 0, 0)$, and
3. U is any point with coordinates (x, y, z),

then U is a point of α iff $\underline{a}\,\underline{x} + \underline{b}\,\underline{y} + \underline{c}\,\underline{z} = \underline{a}^2 + \underline{b}^2 + \underline{c}^2$.

Proof: If O, A, and U are noncollinear, with $|[OA]|_S = r$ $|[OU]|_S = s$, and $|[AU]|_S = u$, then



$\sin \Pi(r) = \sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}$, by 6.9.1,

$\sin \Pi(s) = \sqrt{1-\underline{x}^2-\underline{y}^2-\underline{z}^2}$, by 6.9.1,

$\sin \Pi(u) = \sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}\,\sqrt{1-\underline{x}^2-\underline{y}^2-\underline{z}^2}\,/(1-\underline{a}\,\underline{x}-\underline{b}\,\underline{y}-\underline{c}\,\underline{z})$, by 6.15.

Then we know ΔOUA is a right triangle with right angle at A iff $\sin \Pi(s) = \sin \Pi(r) \sin \Pi(t)$, (cf. 6.4.1). But

$$\sqrt{1-\underline{x}^2-\underline{y}^2-\underline{z}^2} = [(1-\underline{a}^2-\underline{b}^2-\underline{c}^2)\sqrt{1-\underline{x}^2-\underline{y}^2-\underline{z}^2}]/(1-\underline{a}\,\underline{x}-\underline{b}\,\underline{y}-\underline{c}\,\underline{z})$$

iff

$$1-\underline{a}^2-\underline{b}^2-\underline{c}^2 = 1-\underline{a}\,\underline{x}-\underline{b}\,\underline{y}-\underline{c}\,\underline{z}, \quad \text{i.e.,}$$

iff

$$\underline{a}\,\underline{x}+\underline{b}\,\underline{y}+\underline{c}\,\underline{z} = \underline{a}^2+\underline{b}^2+\underline{c}^2, \quad \text{as claimed.}$$

By the definition of perpendicularity for a line and a plane and the uniqueness of perpendiculars to a line at a point A, we conclude U, as given, is on α iff $\underline{a}\,\underline{x}+\underline{b}\,\underline{y}+\underline{c}\,\underline{z} = \underline{a}^2+\underline{b}^2+\underline{c}^2$.

By 6.12, O, A, and U are collinear iff $(\underline{x}, \underline{y}, \underline{z}) = t(\underline{a}, \underline{b}, \underline{c})$ where $t = (\sqrt{\underline{x}^2+\underline{y}^2+\underline{z}^2}\,/\sqrt{\underline{a}^2+\underline{b}^2+\underline{c}^2})\sigma[(a, b, c),(x, y, z)]$.

$$\underline{a}\,\underline{x}+\underline{b}\,\underline{y}+\underline{c}\,\underline{z} = (\underline{a}^2+\underline{b}^2+\underline{c}^2)t = \underline{a}^2+\underline{b}^2+\underline{c}^2 \quad \text{iff} \quad t = 1.$$

Now $t = [\cos \Pi(s)/\cos \Pi(r)]\sigma[(a, b, c), (x, y, z)] = 1$ iff $s = r$. To have $\sigma[(a, b, c), (x, y, z)] = 1$ under these circumstances we necessarily have $(a, b, c) = (x, y, z)$. Thus U = A (cf. 6.9.1, 6.11, 6.12). Thus the equation $\underline{a}\,\underline{x}+\underline{b}\,\underline{y}+\underline{c}\,\underline{z} = \underline{a}^2+\underline{b}^2+\underline{c}^2$ is trivially true when the above conditions are given for U and A.

6.19.1 COROLLARY. The perpendicular bisecting plane of the segment OP (where O is the origin and P has coordinates $(p, q, r) \neq (0, 0, 0)$) is the plane α which has an equation of the form $\underline{a}\,\underline{x}+\underline{b}\,\underline{y}+\underline{c}\,\underline{z} = \underline{a}^2+\underline{b}^2+\underline{c}^2$ where $(\underline{a},\underline{b},\underline{c}) = (\underline{p},\underline{q},\underline{r})(1+\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2})^{-1}$.

Proof: Direct from 6.19 and Corollary 6.15.1.

Before stating the next theorem we make some relevant observations. If $1 > \underline{a}^2 + \underline{b}^2 + \underline{c}^2 > d^2$ and $k = d/(\underline{a}^2+\underline{b}^2+\underline{c}^2)$, then both $(\underline{a}, \underline{b}, \underline{c})$ and $(\underline{a}\,k, \underline{b}\,k, \underline{c}\,k)$ determine points of Lobachevskian geometry. The former triple determines the point A whose coordinates are (a, b, c). This follows from 6.9 since A has coordinates (a, b, c) iff $\underline{a}^2 + \underline{b}^2 + \underline{c}^2 < 1$. The latter triple also determines a point. Clearly

$$(\underline{a}k)^2 + (\underline{b}k)^2 + (\underline{c}k)^2 = (\underline{a}^2+\underline{b}^2+\underline{c}^2)d^2/(\underline{a}^2+\underline{b}^2+\underline{c}^2)^2$$

$$= d^2/(\underline{a}^2+\underline{b}^2+\underline{c}^2) < 1 .$$

Thus we know $|\underline{a}k| < 1$, $|\underline{b}k| < 1$, and $|\underline{c}k| < 1$.

Let $\alpha = \cos^{-1}(\underline{a}k)$, $\beta = \cos^{-1}(\underline{b}k)$, $\gamma = \cos^{-1}(\underline{c}k)$. By the definition of the Lobachevskian function Π, there are unique numbers a', b', and c' so that

$$a' = \Pi^{-1}(\alpha), \quad b' = \Pi^{-1}(\beta), \quad c' = \Pi^{-1}(\gamma) .$$

Then by definition, $\underline{a}' = \underline{a}k$, $\underline{b}' = \underline{b}k$, and $\underline{c}' = \underline{c}k$ and, by 6.9, $(\underline{a}k, \underline{b}k, \underline{c}k)$ determines the point A' of Lobachevskian geometry whose coordinates are (a', b', c').

Furthermore the point A' is on the line $\underline{OA}$. This follows because

$$|k| = \frac{|d|}{\underline{a}^2+\underline{b}^2+\underline{c}^2} < \frac{\sqrt{\underline{a}^2+\underline{b}^2+\underline{c}^2}}{\underline{a}^2+\underline{b}^2+\underline{c}^2} = (\underline{a}^2+\underline{b}^2+\underline{c}^2)^{-1/2}$$

and hence by 6.13, A' is on $\overline{OA}$ as claimed.

These results allow us to state the following theorem.

6.20 THEOREM. Let A be a point, different from the origin, with coordinates (a, b, c). Let d be any number such that

$$a^2 + b^2 + c^2 > d^2.$$

Let

$$\$ = \{X : X \text{ is a point with coordinates } (x, y, z) \text{ so that } ax + by + cz = d\}.$$

Then P is a point of \$ iff P is a point of the plane α which is perpendicular to the line $\overline{OA}$ at the point A' whose coordinates are (a', b', c') where $a' = ak$, $b' = bk$, $c' = ck$ with $k = d/(a^2+b^2+c^2)$.

Proof: As was observed above, the point A' is certainly determined under the stated hypotheses of this theorem.

"if": Suppose A' is not the origin. Then by 6.19, P with coordinates (x, y, z) is on α iff

$$a'x + b'y + c'z = a'^2 + b'^2 + c'^2,$$

i.e.,

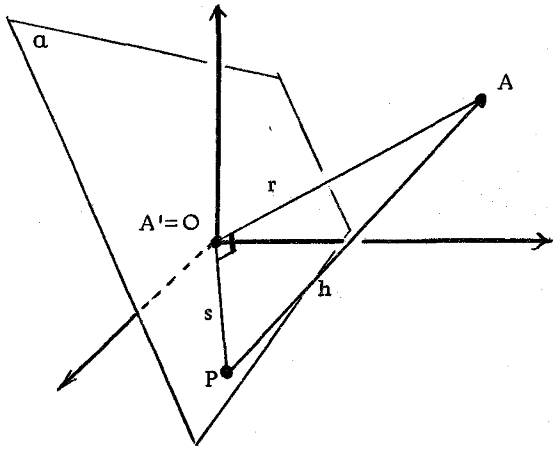
$$(ax+by+cz)k = (a^2+b^2+c^2)k^2,$$

or

$$ax + by + cz = (a^2+b^2+c^2)d/(a^2+b^2+c^2) = d,$$

Thus P is in \$.

Now suppose A' is the origin. Then $k = 0$ and $d = 0$ by necessity. Suppose P, O = A', and A are non-collinear. Then let $r = |[OA]|_S$, $s = |[OP]|_S$ and $h = |[AP]|_S$. By the definition of perpendicularity for lines and planes ΔAOP is a right triangle with right angle at O. Thus by 6.4.1 we must have



(i) $\sin \Pi(h) = \sin \Pi(r) \sin \Pi(s)$.

Using 6.9.1 and 6.15 Equation (i) becomes

(ii) $$\frac{\sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}\sqrt{1-\underline{x}^2-\underline{y}^2-\underline{z}^2}}{1-\underline{a}\,\underline{x}-\underline{b}\,\underline{y}-\underline{c}\,\underline{z}} = \sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}\sqrt{1-\underline{x}^2-\underline{y}^2-\underline{z}^2}.$$

Equation (ii) is true iff $1 - \underline{a}\,\underline{x} - \underline{b}\,\underline{y} - \underline{c}\,\underline{z} = 1$ or, equivalently, $\underline{a}\,\underline{x} + \underline{b}\,\underline{y} + \underline{c}\,\underline{z} = d = 0$.

If P is on $\underline{OA}$ then P = O and trivially $\underline{a}\,\underline{x} + \underline{b}\,\underline{y} + \underline{c}\,\underline{z} = d = 0$.

Thus P is in \$, so this half of the theorem is proved.

"only if": Under the same conditions for (a, b, c) and d as given in the hypothesis, consider the Euclidean analytic geometry equation

(iii) $\underline{a}\,\chi + \underline{b}\,\psi + \underline{c}\,\zeta = d$.

In Euclidean analytic geometry, (iii) describes the set of all points of the Euclidean plane normal to the line with equation

(iv) $\quad (\chi, \psi, \zeta) = t(\underline{a}, \underline{b}, \underline{c})$

where the parameter t has domain the set of all real numbers.

The line described by (iv) meets the plane described by (iii) in the Euclidean point $(\underline{a}k, \underline{b}k, \underline{c}k)$ where $k = d/(\underline{a}^2+\underline{b}^2+\underline{c}^2)$. . . . This all in Euclidean geometry of course.

However, our observations preceding the statement of this theorem assure us that, under the hypothesis of this theorem, $(\underline{a}k, \underline{b}k, \underline{c}k)$ determines a Lobachevskian point A' with coordinates (a', b', c') so that $(\underline{a}', \underline{b}', \underline{c}') = (\underline{a}k, \underline{b}k, \underline{c}k)$, A' is a point of $\underline{OA}$, and $A' \neq A$ since $k \neq 1$.

Now let P be any point of $\underline{OA}$ in $\$$. Then $(\underline{x}, \underline{y}, \underline{z}) = (\underline{a}, \underline{b}, \underline{c})j$ and $(\underline{a}^2+\underline{b}^2+\underline{c}^2)j = d$ i.e., $j = d/(\underline{a}^2+\underline{b}^2+\underline{c}^2) = k$ and thus $P = A'$. Thus A' is the only point of $\underline{OA}$ in $\$$ and conversely.

Now let P be any point of $\$$ different from A'. Furthermore, let: $h = |[AP]|_S$, $r = |[AA']|_S$, $s = |[A'P]|_S$. By 6.15 and the definition of $\$$, d, and k, we have

$$\sin \Pi(h) = [(1-\underline{a}^2-\underline{b}^2-\underline{c}^2)(1-\underline{x}^2-\underline{y}^2-\underline{z}^2)]^{1/2}/(1-d),$$

$$\sin \Pi(r) = [(1-\underline{a}^2-\underline{b}^2-\underline{c}^2)(1-\underline{a}^2k^2-\underline{b}^2k^2-\underline{c}^2k^2)]^{1/2}/(1-d),$$

$$\sin \Pi(s) = [(1-\underline{x}^2-\underline{y}^2-\underline{z}^2)(1-\underline{a}^2k^2-\underline{b}^2k^2-\underline{c}^2k^2)]^{1/2}/(1-dk),$$

since

$$1\text{-}d = 1 - \underline{a}\,\underline{x} - \underline{b}\,\underline{y} - \underline{c}\,\underline{z},$$

$$1\text{-}d = 1 - (\underline{a}^2 + \underline{b}^2 + \underline{c}^2)k\,,$$

$$1\text{-}dk = 1 - (\underline{a}\,\underline{x} + \underline{b}\,\underline{y} + \underline{c}\,\underline{z})k.$$

But

$$1 - \underline{a}^2k^2 - \underline{b}^2k^2 - \underline{c}^2k^2 = 1 - (\underline{a}^2 + \underline{b}^2 + \underline{c}^2)k^2$$

$$= 1 - dk\,.$$

Thus by substitution and direct computation we have

$$\sin \Pi(h) = \sin \Pi(r) \sin \Pi(s)\,.$$

Thus by 6.4.1 $\underline{A'P}$ is perpendicular to $\underline{A'A}$ at A' and thus P is in α and we are done.

The theorem above tells us a great deal about the "equation" of a given plane α. It is not constructive in the sense that a method is given for explicitly writing an equation of the plane determined by three specific non-collinear points. Such an explicit constructive formulation can be readily given if we make use of some Euclidean results on the triples associated with the coordinates of the three points given. One must carefully read the next few remarks to keep fully in mind when the results are Euclidean on the triples $(\underline{x}, \underline{y}, \underline{z})$, etc. and when they are Lobachevskian results.

Let $\vec{V} = (\underline{p}\text{-}\underline{i}, \underline{q}\text{-}\underline{j}, \underline{r}\text{-}\underline{k})$, $\vec{W} = (\underline{u}\text{-}\underline{i}, \underline{v}\text{-}\underline{j}, \underline{w}\text{-}\underline{k})$ be two vectors determined by the coordinates of I, P, U. The vector $\vec{A} = \vec{V} \times \vec{W}$

is in the direction of the Euclidean line normal to the Euclidean plane determined by the Euclidean points $(\underline{i}, \underline{j}, \underline{k})$, $(\underline{p}, \underline{q}, \underline{r})$, $(\underline{u}, \underline{v}, \underline{w})$. In particular,

$$\vec{A} = [(\underline{q}-\underline{j})(\underline{w}-\underline{k})-(\underline{r}-\underline{k})(\underline{v}-\underline{j}), (\underline{r}-\underline{k})(\underline{u}-\underline{i})-(\underline{p}-\underline{i})(\underline{w}-\underline{k}), (\underline{p}-\underline{i})(\underline{v}-\underline{j})-(\underline{q}-\underline{j})(\underline{u}-\underline{i})]$$

$$\doteq (\underline{a}', \underline{b}', \underline{c}') .$$

From analytical Euclidean geometry [cf. 11, p. 87 ff.] we know that the plane containing the given points has equation,

(i) $\underline{a}'x + \underline{b}'y + \underline{c}'z = d'$,

where the perpendicular distance from the plane to the origin is $|d| / \sqrt{\underline{a}^2+\underline{b}^2+\underline{c}^2}$. Now each of $(\underline{i}, \underline{j}, \underline{k})$, $(\underline{p}, \underline{q}, \underline{r})$, $(\underline{u}, \underline{v}, \underline{w})$ is interior to the unit ball so necessarily $\underline{a}'^2 + \underline{b}'^2 + \underline{c}'^2 > d'^2$.

Furthermore $|\vec{A}|^2 = \underline{a}'^2 + \underline{b}'^2 + \underline{c}'^2$ = square of "the area of the parallelogram, two of whose adjacent sides are $\vec{V}$ and $\vec{W}$" [17, p. 68]. Elementary computation assures us $|\vec{A}|^2 < 4$ since each of the points defining $\vec{V}$ and $\vec{W}$ are inside the unit ball. Thus, letting $(\underline{a}, \underline{b}, \underline{c}) = (\underline{a}', \underline{b}', \underline{c}')1/2$ and $d = d'/2$ gives us $1 > \underline{a}^2 + \underline{b}^2 + \underline{c}^2 > d^2$ and (i) becomes

(ii) $\underline{a}x + \underline{b}y + \underline{c}z = d$.

Restriction of (x, y, z) to values of $(\underline{x}, \underline{y}, \underline{z})$ so that $\underline{x}^2 + \underline{y}^2 + \underline{z}^2 < 1$ gives us the following result from 6.20:

6.20.1 COROLLARY. Let I, P, and U be any three non-collinear points of Lobachevskian space with coordinates (i, j, k), (p, q, r), (u, v, w), respectively. X, with coordinates (x, y, z), is a point of the plane determined by I, P, U iff $\underline{a}\,\underline{x} + \underline{b}\,\underline{y} + \underline{c}\,\underline{z} = d$ where $(\underline{a}, \underline{b}, \underline{c}) = 1/2(\vec{V} \times \vec{W})$ with $\vec{V} = (\underline{p}-\underline{i}, \underline{q}-\underline{j}, \underline{r}-\underline{k})$, $\vec{W} = (\underline{u}-\underline{i}, \underline{v}-\underline{j}, \underline{w}-\underline{k})$ and where $d = \underline{a}\,\underline{i} + \underline{b}\,\underline{j} + \underline{c}\,\underline{k}$. Note that in particular $d = (\underline{i}\,\underline{q}\,\underline{w} - \underline{i}\,\underline{r}\,\underline{v} + \underline{j}\,\underline{r}\,\underline{u} - \underline{j}\,\underline{p}\,\underline{w} + \underline{k}\,\underline{p}\,\underline{v} - \underline{k}\,\underline{q}\,\underline{u})1/2$.

We now recall a common relation which is generally used without being explicitly written down. Since it is so well known we do not assign it a specific number in our sequencing.

An equation $f(x, y, z) = c$ is said to be equivalent to equation $g(x, y, z) = d$ iff their solution sets are the same. This is an equivalence relation on the set of equations with three independent real variables. This is a common relation and the verification that it is an equivalence relation is both simple and obvious so it will not be formally given here.

This relation allows us to extend the results of 6.20 to

6.20.2 COROLLARY. $(\underline{x}, \underline{y}, \underline{z})$ is in the solution set of an equation from the equivalence class of equations of three independent real variables having the equation $\underline{a}\,\underline{x} + \underline{b}\,\underline{y} + \underline{c}\,\underline{z} = d$ with $1 > \underline{a}^2 + \underline{b}^2 + \underline{c}^2 > d^2$ as a representative iff the point P with coordinates (x, y, z) is a point of the plane α which is perpendicular to $\underline{OA}$ at the point A' whose coordinates are $(\underline{a}k, \underline{b}k, \underline{c}k)$ with $k = d/(\underline{a}^2+\underline{b}^2+\underline{c}^2)$.

Proof: (a, b, c) names some point since $\underline{a}^2 + \underline{b}^2 + \underline{c}^2 < 1$ and the remainder of the theorem is direct from 6.20.

At this time we pause for a moment in our development to consider what we have developed so far. We have shown upon the basis of our axioms and/or basic assumption that:

1. P is a point with coordinates (x, y, z) iff $\underline{x}^2 + \underline{y}^2 + \underline{z}^2 < 1$ (6.9).

2. If A and P are distinct points with coordinates (a, b, c) and (p, q, r), and $t = |[AP]|_S$, then $\sin \Pi(t) = \sqrt{(1-\underline{a}^2-\underline{b}^2-\underline{c}^2)(1-\underline{p}^2-\underline{q}^2-\underline{r}^2)}/(1-\underline{a}\,\underline{p} - \underline{b}\,\underline{q} - \underline{c}\,\underline{r})$ thus essentially giving us a distance formula (6.15). (In fact we can do exactly that using 5.69 and the natural log function.)

3. If A and P are distinct points, the points of line $\underline{AP}$ have coordinates (x, y, z) satisfying the formulas $(\underline{x}, \underline{y}, \underline{z}) = (\underline{p}, \underline{q}, \underline{r}) + t(\underline{a}-\underline{p}, \underline{b}-\underline{q}, \underline{c}-\underline{r})$ for appropriate values of t (6.8) which, by 6.18.2, gives an analytic expression for betweenness.

4. Finally we have been able to characterize the relation of incidence between planes and three non-collinear points in

terms of solution sets of certain classes of equation (6.20.2).

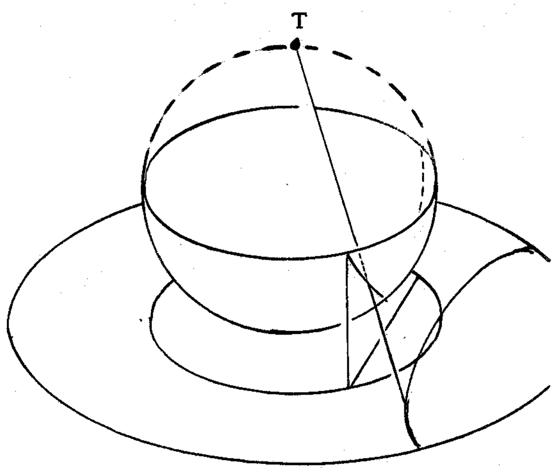
These theorems characterize various notions about points and their interrelationship with lines, planes, and numbers. Their planar counterparts have been considered by Beltrami [1, p. 284-342], Klein [7, p. 573-625] and Borsuk [2, p. 334-345] in various degrees of detail and upon various axiomatic bases. Beltrami and Klein did the necessary work to allow us to describe what Borsuk calls the "Beltrami coordinate system on [a] plane. . . " [2, p. 341-344]. Borsuk, using his own earlier work relating to projective planes, considers analytic (Cartesian) geometry of the plane, C_2, as a sub-space of projective (analytic) two space, P_2, and defines what he calls Klein space, K_2, to be the interior of the unit disk in C_2. He uses the points of K_2 as the points for what he calls the "Klein-Beltrami" model [2, p. 245 ff.]. In this model he develops a measure using preservation of crossratio by projective transformations together with the properties of the subset of projective transformations which leave K_2 fixed. He develops an isometry between K_2 and a Lobachevskian plane with coordinatization so that point P with coordinates (x, y) (by our coordinatization) has Beltrami coordinates $(\underline{x}, \underline{y})$ [2, p. 341-344]. From this he shows Lobachevskian geometry of a plane as described by his axioms is categorical [2, p. 344-345]. However, our problem has been harder in that Hilbert's axioms are much more primitive (as we have noted earlier) and also we are concerned with space Lobachevskian geometry and the Poincaré model which is in Euclidean space.

If we extend the notion of the Beltrami coordinate system to assign P (with our coordinates (x, y, z)) the coordinates ($\underline{x}, \underline{y}, \underline{z}$), 6.9 assures us we have an obvious map from Lobachevskian space into the interior of the unit ball B^3 in $E^3 = \mathbb{R}^3$. 6.18 and 6.20.2 assure us that the lines and planes are mapped in an obvious way to the intersection of lines and planes (in E^3) with B^3. It is neither our desire nor of any real value to our development to further consider this model except to explain the evolution of the map from Lobachevskian space to the open unit ball B^3 which we will describe shortly.

As one examines the ways one might possibly construct an isomorphism between Lobachevskian space and the Poincaré model, the results summarized above and a study of the so called Beltrami-Klein model direct one to an examination of the plane Poincaré model and the Beltrami-Klein model for ideas. Kutuzov discusses various interpretations of Lobachevskian <u>plane</u> geometry [8, p. 560-570]. Kutuzov states:

> If we construct a hemisphere the equator of which coincides with the circumference of the Beltrami map and orthogonally project this map upon the hemisphere, and if we then stereographically project the hemisphere from a pole S which lies on the equator onto a plane α perpendicular to the diameter passing through the pole of projection S, we obtain on the plane α Poincaré's ...model of the geometry of Lobachevskii (sic) [8, p. 570].

(His accompanying figure makes clear what this translation leaves somewhat unclear. His figure is shown at the right.) This leads to the consideration of the three-space analog of this construction, i.e., a projection of the unit ball B^3 into the lower "hemisphere" of the "four sphere" $\Sigma_4 = \{(x,y,z,w): x^2+y^2+z^2+(w-1)^2=1\}$, and stereographic projection of this hemisphere from $(0,0,0,2)$ into the "hyperplane" with equation $w = 0$. This maps B^3 into the open ball with radius 2. Shrinking by a factor of $1/2$ gives a map from $B^3 \to B^3$.



We now describe analytically the construction of this pairing of points so roughly outlined above.

Let $(a,b,c,0)$ be a point of the unit three-ball B^3 viewed as a manifold of E^4. Let

$$\Sigma_4 = \{(x,y,z,w): x^2+y^2+z^2+(w-1)^2 = 1 \text{ and } w < 1\}.$$

Let $T = (0,0,0,2)$. Then the projection of $(a,b,c,0)$ onto Σ_4 is the point $(a,b,c,1-\sqrt{1-a^2-b^2-c^2})$. Now the "line" through $(0,0,0,2)$ and (a,b,c,d) of Σ_4 has equation:

$$(x,y,z,w) = (0,0,0,2) + t(a,b,c,d-2) \quad \text{where} \quad t \in \mathbb{R},$$

i.e. $(x, y, z, w) = (ta, tb, tc, 2+t(d-2))$

For any point on this line with fourth coordinate 0,

$$2 + t(d-2) = 0, \quad \text{i.e.,} \quad t = 2/(2-d).$$

Thus

$$(x, y, z, 0) = (2a, 2b, 2c, 0)1/(2-d).$$

Thus the stereographic image of $(a, b, c, 1-\sqrt{1-a^2-b^2-c^2})$ is $(2a, 2b, 2c, 0)(1/(1+\sqrt{1-a^2-b^2-c^2}))$. Shrinking by $1/2$ gives

$$(a, b, c, 0) \rightarrow (a, b, c, 0)[1/(1+\sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2})].$$

This leads to the following definition:

6.21 DEFINITION. For each point P with coordinates (p, q, r), we define

$$(\overline{p}, \overline{q}, \overline{r}) = (\underline{p}, \underline{q}, \underline{r})[1/(1+\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2})].$$

6.22 LEMMA. Let A and P be any points with coordinates (a, b, c) and (p, q, r), respectively. Then $(a, b, c) = (p, q, r)$ iff $(\overline{a}, \overline{b}, \overline{c}) = (\overline{p}, \overline{q}, \overline{r})$.

Proof: The "only if" part of the argument is obvious from the Definition 6.21.

Now suppose $(\overline{a}, \overline{b}, \overline{c}) = (\overline{p}, \overline{q}, \overline{r})$. Then by 6.21 $(\underline{a}, \underline{b}, \underline{c})k = (\underline{p}, \underline{q}, \underline{r})j$ where

$$k = \frac{1}{1+\sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2}} \quad \text{and} \quad j = \frac{1}{1+\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2}}.$$

Suppose $j \leq k$. Then $j/k \leq 1$ and $(\underline{a}, \underline{b}, \underline{c}) = (\underline{p}, \underline{q}, \underline{r})j/k$ so $\underline{a} \leq \underline{p}$, $\underline{b} \leq \underline{q}$, $\underline{c} \leq \underline{r}$. Thus if $\alpha = |[OA]|_S$ and $\beta = |[OP]|_S$ then $\Pi(\alpha)$, $\Pi(\beta) \in (0, \pi/2]$ and by 6.9.1,

$$\sin \Pi(\alpha) = \sqrt{1-\underline{a}^2-\underline{b}^2-\underline{c}^2} \geq \sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2} = \sin \Pi(\beta).$$

Hence

$$1 + \sin \Pi(\beta) \leq 1 + \sin \Pi(\alpha)$$

and

$$1 \leq \frac{1+\sin \Pi(\alpha)}{1+\sin \Pi(\beta)} = \frac{\frac{1}{1+\sin \Pi(\alpha)}}{\frac{1}{1+\sin \Pi(\beta)}} = \frac{j}{k}.$$

so $1 \leq j/k \leq 1$ giving $j = k$ and $(\underline{a}, \underline{b}, \underline{c}) = (\underline{p}, \underline{q}, \underline{r})$. By symmetry, if $j \geq k$, we have $(\underline{a}, \underline{b}, \underline{c}) = (\underline{p}, \underline{q}, \underline{r})$. Since $\cos \circ \Pi$ is one-to-one, we are done.

6.23 THEOREM. P is a point of Lobachevskian geometry with coordinates (x, y, z) iff $\bar{x}^2 + \bar{y}^2 + \bar{z}^2 < 1$.

Proof: If P is a point with coordinates (x, y, z) and $p = |[OP]|_S$ then

$$\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = (\underline{x}^2+\underline{y}^2+\underline{z}^2)[1+2\sqrt{1-\underline{x}^2-\underline{y}^2-\underline{z}^2}+1-\underline{x}^2-\underline{y}^2-\underline{z}^2]^{-1}$$

$$= \cos^2\Pi(p)/[1+\sin\Pi(p)]^2 \quad \text{by 6.9.1}$$

$$= \frac{1-\sin\Pi(p)}{1+\sin\Pi(p)} < 1$$

since $0 < \sin\Pi(p) \leq 1$.

Now let $(\bar{x}, \bar{y}, \bar{z})$ be as defined in 6.21, with $\bar{x}^2 + \bar{y}^2 + \bar{z}^2 < 1$. Then clearly $(u, v, w) = (\bar{x}, \bar{y}, \bar{z})2/(1+\bar{x}^2+\bar{y}^2+\bar{z}^2)$ has the property that

$$u^2 + v^2 + w^2 = \frac{4(\bar{x}^2+\bar{y}^2+\bar{z}^2)}{[1+\bar{x}^2+\bar{y}^2+\bar{z}^2]^2} < 1 .$$

Furthermore each of u, v, w is less than one in absolute value. Now let $\underline{x}_0 = u$, $\underline{y}_0 = v$, $\underline{z}_0 = w$. Using 6.9 let (x_0, y_0, z_0) be the unique point corresponding to $(\underline{x}_0, \underline{y}_0, \underline{z}_0)$. Then $\bar{x}_0 = \bar{x}$, $\bar{y}_0 = \bar{y}$ and $\bar{z}_0 = \bar{z}$ since

$$(\bar{x}_0, \bar{y}_0, \bar{z}_0)$$

$$= (u, v, w)[1/(1+\sqrt{1-u^2-v^2-w^2})]$$

$$= (\bar{x}, \bar{y}, \bar{z}) \frac{2}{1+\bar{x}^2+\bar{y}^2+\bar{z}^2} \frac{1}{1+\sqrt{\dfrac{1+2(\bar{x}^2+\bar{y}^2+\bar{z}^2)+(\bar{x}^2+\bar{y}^2+\bar{z}^2)^2-4(\bar{x}^2+\bar{y}^2+\bar{z}^2)}{(1+\bar{x}^2+\bar{y}^2+\bar{z}^2)^2}}}$$

$$= (\bar{x}, \bar{y}, \bar{z}) \frac{2}{(1+\bar{x}^2+\bar{y}^2+\bar{z}^2)+\sqrt{[1-(\bar{x}^2+\bar{y}^2+\bar{z}^2)]^2}}$$

$$= (\bar{x}, \bar{y}, \bar{z})$$

as claimed and by 6.22 we are done.

6.23.1 COROLLARY. Let P and U have coordinates (p, q, r) and (u, v, w) respectively. Let $|[PU]|_S = \delta$ and $\overline{P} = (\overline{p}, \overline{q}, \overline{r})$ $\overline{U} = (\overline{u}, \overline{v}, \overline{w})$. Then

$$\sin \Pi(\delta) = \frac{(1-\overline{P}\cdot\overline{P})(1-\overline{U}\cdot\overline{U})}{(1+\overline{P}\cdot\overline{P})(1+\overline{U}\cdot\overline{U})-4\overline{P}\cdot\overline{U}}.$$

Proof: Referring to the computation used in 6.23 we know $(\underline{p}, \underline{q}, \underline{r}) = 2(\overline{p}, \overline{q}, \overline{r})/(1+\overline{P}\cdot\overline{P})$. Hence by symmetry and 6.15 we have

$$\sin \Pi(\delta) = \frac{\dfrac{[1+2\overline{P}\cdot\overline{P}+(\overline{P}\cdot\overline{P})^2-4\overline{P}\cdot\overline{P}]^{1/2}[1+2\overline{U}\cdot\overline{U}+(\overline{U}\cdot\overline{U})^2-4(\overline{U}\cdot\overline{U})]^{1/2}}{(1+\overline{P}\cdot\overline{P})(1+\overline{U}\cdot\overline{U})}}{1-\dfrac{4\overline{P}\cdot\overline{U}}{(1+\overline{P}\cdot\overline{P})(1+\overline{U}\cdot\overline{U})}}$$

$$= \frac{(1-\overline{P}\cdot\overline{P})(1-\overline{U}\cdot\overline{U})}{(1+\overline{P}\cdot\overline{P})(1+\overline{U}\cdot\overline{U})-4\overline{P}\cdot\overline{U}}$$

as claimed.

Examination of the argument of 6.23 will assure us that for any point P with coordinates (x, y, z), we have $(\underline{x}, \underline{y}, \underline{z}) = (\overline{x}, \overline{y}, \overline{z})2/(1+\overline{x}^2+\overline{y}^2+\overline{z}^2)$. From Corollary 6.20.2 we thus have

6.24 THEOREM. $(\overline{x}, \overline{y}, \overline{z})$ is in the solution set of an equation from the equivalence class of equations of three real variables having the representative equation

$$D(\bar{x}^2+\bar{y}^2+\bar{z}^2+1) + A\bar{x} + B\bar{y} + C\bar{z} = 0 \quad \text{with} \quad A^2+B^2+C^2 > 4D^2$$

iff the point P with coordinates (x, y, z) is a point of the plane α which is perpendicular to the line $\underline{OQ}$ at the point Q' whose coordinates are

$$\left(-\frac{A}{2}k, -\frac{B}{2}k, -\frac{C}{2}k\right) \quad \text{with} \quad k = 4D/A^2+B^2+C^2.$$

Proof: Just let $A = -2\underline{a}$, $B = -2\underline{b}$, $C = -2\underline{c}$, $D = d$ in 6.20.2 and use the observation preceding the statement of this theorem.

6.24.1 COROLLARY. If P is any point, different from the origin O, with coordinates (p, q, r), then the perpendicular bisecting plane α of the segment OP has a representative equation of the form

$$(1/2)(\bar{p}^2+\bar{q}^2+\bar{r}^2)(\bar{x}^2+\bar{y}^2+\bar{z}^2+1) - \bar{p}\bar{x} - \bar{q}\bar{y} - \bar{r}\bar{z} = 0.$$

Proof: From 6.19.1 we know α has an equation of the form

$\underline{a}\,\underline{x} + \underline{b}\,\underline{y} + \underline{c}\,\underline{z} = \underline{a}^2 + \underline{b}^2 + \underline{c}^2$, where

$(\underline{a}, \underline{b}, \underline{c}) = (\underline{p}, \underline{q}, \underline{r})(1+\sqrt{1-\underline{p}^2-\underline{q}^2-\underline{r}^2})$. But, by Definition 6.21

$(\underline{a}, \underline{b}, \underline{c}) = 1/2(\bar{p}, \bar{q}, \bar{r})$ and the corollary follows directly from 6.24.

From 0.20 we know that two distinct planes with a point in common have a line in common. If ℓ is any line we may use axiom I, 3, 8 to get, first, a point not on ℓ, thus a plane α containing ℓ, and, second, a point not on α, and thus a second plane β, distinct from α, which also contains ℓ. We conclude that for any two distinct intersecting planes we have a unique line and for any line ℓ there exist distinct planes whose common line is ℓ. This, together with 6.24, leads us to state:

6.25 THEOREM. U is a point of line ℓ, with coordinates (x, y, z), iff $(\bar{x}, \bar{y}, \bar{z})$ is in the solution set of a representative pair of equations from the equivalence class of pairs of equations having a representative pair of the form

$$D(\bar{x}^2+\bar{y}^2+\bar{z}^2+1) + A\bar{x} + B\bar{y} + C\bar{z} = 0$$

$$D'(\bar{x}^2+\bar{y}^2+\bar{z}^2+1) + A'\bar{x} + B'\bar{y} + C'\bar{z} = 0$$

where

1. $A^2 + B^2 + C^2 > 4D^2$ and $A'^2 + B'^2 + C'^2 > 4D'^2$.
2. The equations have at least one solution .
3. rank $\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix} = 2$.

Proof: We only need to observe that condition 3. is necessary and sufficient to assure the normal lines to the two planes are not perpendicular to a common plane. This will assure us that the planes meet in at most a line if they meet in a point. A direct application of 6.24 completes the proof.

We now turn our attention to an analytic formulation for betweenness in terms of the "over-bar" triples. Let A, P, U be any points with coordinates (a, b, c), (p, q, r) and (x, y, z) respectively. From 6.18, we know U is on line $\underline{AP}$ iff

(i) $(\underline{x}, \underline{y}, \underline{z}) = (\underline{p}, \underline{q}, \underline{r})(1-t) + (\underline{a}, \underline{b}, \underline{c})t$

with $t = (\underline{w}-\underline{s})/(\underline{d}-\underline{s})$, where $(w, d, s) \in \{(x, a, p), (y, b, q), (z, c, r)\}$ and $d-s \neq 0$. From (i) we see that for t in an appropriate open interval we have

(ii) $\underline{x} = \underline{p} + t(\underline{a}-\underline{p}) = f(t)$,

(iii) $\underline{y} = \underline{q} + t(\underline{b}-\underline{q}) = g(t)$,

(iv) $\underline{z} = \underline{r} + t(\underline{c}-\underline{r}) = h(t)$,

where f, g, and h are monotomic by 6.18.2 and the proof of 6.18.1.

Thus, by Definition 6.21,

(ii') $\bar{x} = F(t) = f(t)k(t)$,

(iii') $\bar{y} = G(t) = g(t)k(t)$,

(iv') $\bar{z} = H(t) = h(t)k(t)$,

where $k(t) = 1/(1+\sqrt{1-f^2(t)-g^2(t)-h^2(t)}$. Clearly F, G, and H are also monotomic and A-U-P iff $t_a < t_u < t_p$ or $t_a > t_u > t_p$ where t_x is the parameter associated with point X of $\underline{AP}$. We have thus proved most of the following theorem.

6.26 THEOREM. Let A, P, U have coordinates (a, b, c), (p, q, r), and (x, y, z), and U be on line $\underline{AP}$. Then there exist monotomic functions $\bar{x} = F(t)$, $\bar{y} = G(t)$, $\bar{z} = H(t)$ with t in an open interval. If such a parametric formulation is given, A-U-P iff $t_a < t_u < t_p$ or $t_p < t_u < t_a$.

Proof: It is sufficient to apply 6.18.2 directly to complete the argument.

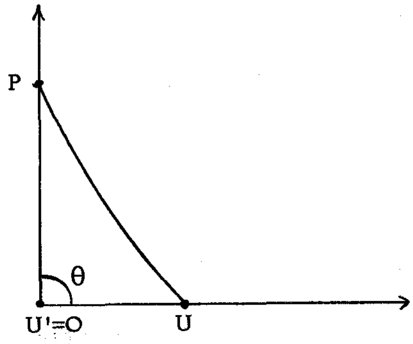
6.27 LEMMA. Let P and U be distinct points different from the origin O and not collinear with O. Let them have coordinates (p, q, r) and (u, v, w). Further, let $\underline{P} = (\underline{p}, \underline{q}, \underline{r})$ and $\underline{U} = (\underline{u}, \underline{v}, \underline{w})$. Then if $\theta = m\angle POU$, $\cos\theta = \underline{P}\cdot\underline{U}/(\sqrt{\underline{P}\cdot\underline{P}}\sqrt{\underline{U}\cdot\underline{U}})$.

Proof: Let U' be the foot of the perpendicular from P to $\underline{OU}$. Let $\underline{U}' = t\underline{U}$. (The existence of an appropriate t is assured by 6.13.)

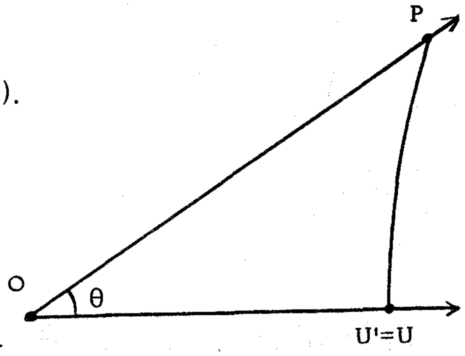
<u>Case 1.</u> U' = O: This means $\underline{OP} \perp \underline{OU}$ and $\cos\theta = 0$. In right triangle $\triangle OPU$ we use the sine formula of 6.4 and 6.15 to get

$$\frac{\sqrt{1-\underline{P}\cdot\underline{P}}\sqrt{1-\underline{U}\cdot\underline{U}}}{1-\underline{P}\cdot\underline{U}} = \sqrt{1-\underline{P}\cdot\underline{P}}\sqrt{1-\underline{U}\cdot\underline{U}},$$

which, since $\underline{P}\cdot\underline{P}$ and $\underline{U}\cdot\underline{U}$ are both between 0 and 1, is true iff $\underline{P}\cdot\underline{U} = 0$.



Thus the theorem is true for $\angle POU$ a right angle.



Case 2. $U' = U$: By 6.4 we have

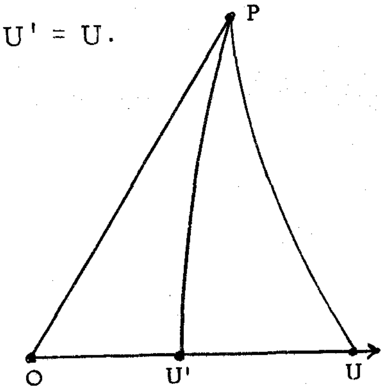
$\cos\theta = \sqrt{\underline{U}\cdot\underline{U}}/\sqrt{\underline{P}\cdot\underline{P}} = \underline{U}\cdot\underline{U}/(\sqrt{\underline{P}\cdot\underline{P}}\sqrt{\underline{U}\cdot\underline{U}})$.

Using the sine formula in 6.4 and 6.15

we get $\sqrt{1-\underline{P}\cdot\underline{P}} = \dfrac{(1-\underline{U}\cdot\underline{U})\sqrt{1-\underline{P}\cdot\underline{P}}}{1-\underline{P}\cdot\underline{U}}$

so that $1-\underline{U}\cdot\underline{U} = 1-\underline{P}\cdot\underline{U}$ or equivalently

$\underline{U}\cdot\underline{U} = \underline{P}\cdot\underline{U}$. Thus the theorem is proved for $U' = U$.



Case 3. $O\text{-}U'\text{-}U$: $0 < t < 1$. By 6.18, using 6.4 and 6.15 again, we have for triangle $\Delta PUU'$,

$$\frac{\sqrt{1-\underline{P}\cdot\underline{P}}\sqrt{1-\underline{U}\cdot\underline{U}}}{1-\underline{P}\cdot\underline{U}} = \frac{\sqrt{1-\underline{P}\cdot\underline{P}}\sqrt{1-\underline{U}\cdot\underline{U}}(1-\underline{U}'\cdot\underline{U}')}{(1-\underline{P}\cdot\underline{U}')(1-\underline{U}\cdot\underline{U}')},$$

giving us $(1-\underline{P}\cdot\underline{U}t)(1-\underline{U}\cdot\underline{U}t) = (1-\underline{P}\cdot\underline{U})(1-\underline{U}\cdot\underline{U}t^2)$. This is equivalent to $\underline{U}\cdot\underline{U}t^2 - (\underline{P}\cdot\underline{U}+\underline{U}\cdot\underline{U})t + \underline{P}\cdot\underline{U} = 0$. Thus

$$t = \frac{(\underline{P}\cdot\underline{U}+\underline{U}\cdot\underline{U}) \pm \sqrt{(\underline{P}\cdot\underline{U}+\underline{U}\cdot\underline{U})^2 - 4(\underline{P}\cdot\underline{U})(\underline{U}\cdot\underline{U})}}{2\underline{U}\cdot\underline{U}}$$

$$= \frac{(\underline{P}\cdot\underline{U}+\underline{U}\cdot\underline{U}) \pm (\underline{P}\cdot\underline{U}-\underline{U}\cdot\underline{U})}{2\underline{U}\cdot\underline{U}}$$

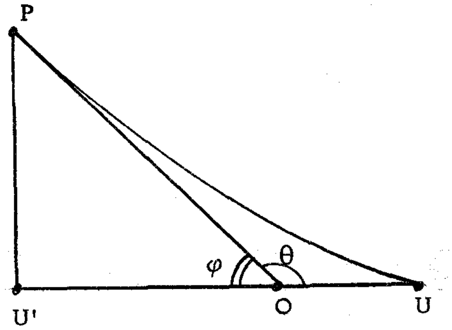
$$= 1 \quad \text{or} \quad \frac{\underline{P}\cdot\underline{U}}{\underline{U}\cdot\underline{U}} .$$

But $0 < t < 1$ so $t = \dfrac{\underline{P}\cdot\underline{U}}{\underline{U}\cdot\underline{U}} > 0$ and $|\underline{P}\cdot\underline{U}| = \underline{P}\cdot\underline{U}$. Hence

$$\cos\theta = \frac{\sqrt{\underline{U}'\cdot\underline{U}'}}{\sqrt{\underline{P}\cdot\underline{P}}} = \frac{\sqrt{\underline{U}\cdot\underline{U}t^2}}{\sqrt{\underline{P}\cdot\underline{P}}} = \frac{\sqrt{(\underline{P}\cdot\underline{U})^2/\underline{U}\cdot\underline{U}}}{\sqrt{\underline{P}\cdot\underline{P}}}$$

and we are done for $O\text{-}U'\text{-}U$.

Case 4. O-U-U': $\angle POU$ is acute and from absolute geometry considerations, the perpendicular from U to OP will meet OP at a point P'. Apply Case 3 to this symmetrical case.



Case 5. U'-O-U: By 6.18.2 we know $t < 0$. In right triangle $\triangle PUU'$, we apply 6.4 and 6.15 to get $t = \frac{\underline{P}\cdot\underline{U}}{\underline{U}\cdot\underline{U}}$ just as we did in Case 3. Now

$$\cos\theta = -\cos\varphi = -\frac{\sqrt{\underline{U}'\cdot\underline{U}'}}{\sqrt{\underline{P}\cdot\underline{P}}}$$

$$= -\frac{\sqrt{\underline{U}\cdot\underline{U}\ t^2}}{\sqrt{\underline{P}\cdot\underline{P}}} = -\frac{\sqrt{(\underline{P}\cdot\underline{U})^2}}{\sqrt{\underline{P}\cdot\underline{P}}\sqrt{\underline{U}\cdot\underline{U}}} = -\frac{|\underline{P}\cdot\underline{U}|}{\sqrt{\underline{P}\cdot\underline{P}}\sqrt{\underline{U}\cdot\underline{U}}}$$

since $\underline{U}\cdot\underline{U} > 0$. But $t < 0$ forces $\sqrt{(\underline{P}\cdot\underline{U})^2} = -\underline{P}\cdot\underline{U}$ and we are done.

6.27.1 COROLLARY. Let P and U be distinct points different from the origin and not collinear with the origin O. Let P and U have coordinates (p, q, r) and (u, v, w) respectively and let $\overline{P} = (\overline{p}, \overline{q}, \overline{r})$ and $\overline{U} = (\overline{u}, \overline{v}, \overline{w})$. Then if $\theta = m\angle POU$,

$$\cos\theta = \overline{P}\cdot\overline{U}/(\sqrt{\overline{P}\cdot\overline{P}}\sqrt{\overline{U}\cdot\overline{U}}) .$$

Proof: From 6.21 we know

$$\overline{P} = \underline{P}k \quad \text{where} \quad k = (1+\sqrt{1-\underline{P}\cdot\underline{P}})^{-1} .$$

Furthermore

$$\overline{U} = \underline{U}\ell \quad \text{where} \quad \ell = (1+\sqrt{1-\underline{U}\cdot\underline{U}})^{-1} .$$

Now by 6.27,

$$\cos\theta = \underline{P}\cdot\underline{U}/(\sqrt{\underline{P}\cdot\underline{P}}\sqrt{\underline{U}\cdot\underline{U}}$$

$$= \underline{P}\cdot\underline{U}k\ell/(k\ell\sqrt{\underline{P}\cdot\underline{P}}\sqrt{\underline{U}\cdot\underline{U}}$$

$$= \overline{P}\cdot\overline{U}/(\sqrt{\overline{P}\cdot\overline{P}}\sqrt{\overline{U}\cdot\overline{U}}$$

since $k, \ell > 0$, and the claim is proved.

In his thesis devoted to a proof that the congruence and Archimedes axioms were theorems in the Poincaré model, Eschrich defined a transformation which he called "inversion." This transformation is an extension of the plane inversion maps common in the study of the planar Poincaré model [cf. 8, p. 347ff; 10, p. 348 ff.]. Eschrich's definition is:

6.28 "DEFINITION: Given a 'plane' α: $D(\overline{x}^2+\overline{y}^2+\overline{z}^2+1) + A\overline{x} + B\overline{y} + C\overline{z} = 0$, the <u>inverse</u> $\overline{P}'$ with respect to α of the 'point' $\overline{P} = (\overline{p}, \overline{q}, \overline{r})$ is defined as

$$\overline{P}' = \begin{cases} (\overline{p}+sA, \overline{q}+sB, \overline{r}+sC) \text{ where } s = \dfrac{-2(A\overline{p}+B\overline{q}+C\overline{r})}{A^2+B^2+C^2}, & \text{if } D = 0, \\ (-A/2D + t[\overline{p}+A/2D], -B/2D+t[\overline{q}+B/2D], -c/2D+t[\overline{r}+C/2D]) & \\ \quad \text{where } t = \dfrac{[(A^2+B^2+C^2)/4D^2]-1}{(\overline{p}+A/2D)^2+(\overline{q}+B/2D)^2+(\overline{r}+C/2D)^2}, & \text{if } D \neq 0 \text{"} \end{cases}$$

[3, p. 1-2].

Our next lemma will prove that the points associated with $\overline{P}$ and $\overline{U}$ are the same "distance apart" as the points associated with $\overline{P}'$ and $\overline{U}'$. Please note that the $\overline{P}$, $\overline{U}$, $\underline{P}$, $\underline{U}$, etc. which we see used here are not points. They are triples which are associated with the points of the geometry. The points are still undefined objects. We have only associated a triple of one sort or another with the points of the geometry.

6.29 LEMMA. Let P and U be distinct points. Let P' and U' be the points associated with the $\overline{P}'$ and $\overline{U}'$ which are defined in 6.28. If $\sigma = |[PU]|_S$ and $\sigma' = |[P'U']|_{S'}$ then $\sin \Pi(\sigma) = \sin \Pi(\sigma')$.

Proof: Let $\overline{P} = (\overline{p}, \overline{q}, \overline{r})$, $\overline{U} = (\overline{u}, \overline{v}, \overline{w})$, $\overline{P}' = (\overline{p}', \overline{q}', \overline{r}')$, $\overline{U}' = (\overline{u}', \overline{v}', \overline{w}')$.

Case 1. $D = 0$: We first observe that

$$\overline{P}' \cdot \overline{P}' = \overline{P} \cdot \overline{P} + 2s(A\overline{p}+B\overline{q}+C\overline{r}) + s^2(A^2+B^2+C^2)$$

$$= \overline{P} \cdot \overline{P} - 4\,\frac{(A\overline{p}+B\overline{q}+C\overline{r})^2}{A^2+B^2+C^2} + 4\,\frac{(A\overline{p}+B\overline{q}+C\overline{r})^2}{A^2+B^2+C^2}$$

from definition of s, 6.28

$$= \overline{P} \cdot \overline{P}.$$

By symmetry we have $\overline{U}' \cdot \overline{U}' = \overline{U} \cdot \overline{U}$. Furthermore we have

$$\overline{P}'\cdot\overline{U}' = \overline{P}\cdot\overline{U} - \frac{2(A\overline{u}+B\overline{v}+C\overline{w})}{A^2+B^2+C^2}(A\overline{p}+B\overline{q}+C\overline{r}) - \frac{2(A\overline{p}+B\overline{q}+C\overline{r})}{A^2+B^2+C^2}(A\overline{u}+B\overline{v}+C\overline{w})$$

$$+ 4\frac{(A\overline{u}+B\overline{v}+C\overline{w})(A\overline{p}+B\overline{q}+C\overline{r})}{A^2+B^2+C^2}$$

$$= \overline{P}\cdot\overline{U} \ .$$

Thus, by 6.23.1, $\sin \Pi(\sigma) = \sin \Pi(\sigma')$.

Case 2. $D \neq 0$. From 6.28 we have

$$\overline{P}' = [-\frac{A}{2D} + t(\overline{p}+\frac{A}{2D}), -\frac{B}{2D} + t(\overline{q}+\frac{B}{2D}), -\frac{C}{2D} + t(\overline{r}+\frac{C}{2D})] \ .$$

Let $k = \dfrac{A^2+B^2+C^2}{4D^2} - 1$,

$$j = (\overline{p}+\frac{A}{2D})^2 + (\overline{q}+\frac{B}{2D})^2 + (\overline{r}+\frac{C}{2D})^2 ,$$

and hence, $t = k/j$.

(i) $$\overline{P}'\cdot\overline{P}' = \frac{A^2+B^2+C^2}{4D^2} - 2t[\frac{A}{2D}(\overline{p}+\frac{A}{2D})+\frac{B}{2D}(\overline{q}+\frac{B}{2D})+\frac{C}{2D}(\overline{r}+\frac{C}{2D})] + t^2(j)$$

$$= 1 + \frac{A^2+B^2+C^2}{4D^2} - 1 - 2t[\frac{A}{2D}(\overline{p}+\frac{A}{2D})+\frac{B}{2D}(\overline{q}+\frac{B}{2D})+\frac{C}{2D}(\overline{r}+\frac{C}{2D})] + tk$$

$$= 1 + k - 2\frac{k}{j}[\frac{A}{2D}(\overline{p}+\frac{A}{2D})+\frac{B}{2D}(\overline{q}+\frac{B}{2D})+\frac{C}{2D}(\overline{r}+\frac{C}{2D})] + \frac{k^2}{j}$$

$$= 1 + \frac{k}{j}[(\overline{p}+\frac{A}{2D})^2+(\overline{q}+\frac{B}{2D})^2+(\overline{r}+\frac{C}{2D})^2 - 2\frac{A}{2D}(\overline{p}+\frac{A}{2D}) - 2\frac{B}{2D}(\overline{q}+\frac{B}{2D})$$

$$-2\frac{C}{2D}(\overline{r}+\frac{C}{2D}) + \frac{A^2}{4D^2} + \frac{B^2}{4D^2} + \frac{C^2}{4D^2} - 1]$$

$$= 1 - k/j(1-\overline{P}\cdot\overline{P}) \ .$$

In a symmetrical way we get that

(ii) $\overline{U}'\cdot\overline{U}' = 1 - k/\ell(1-\overline{U}\cdot\overline{U})$

where $\ell = (\overline{u}+A/2D)^2+(\overline{v}+B/2D)^2+(\overline{w}+C/2D)^2$.

Thus $(1-\overline{P}'\cdot\overline{P}')(1-\overline{U}'\cdot\overline{U}') = k^2/j\ell(1-\overline{P}\cdot\overline{P})(1-\overline{U}\cdot\overline{U})$.

This expression is the numerator in the formula for $\sin\Pi(\sigma')$ given in 6.23.1.

Now let

$$H' = -(A/2D)(\overline{p}+A/2D) - (B/2D)(\overline{q}+B/2D) - (C/2D)(\overline{r}+C/2D) ,$$

$$K' = -(A/2D)(\overline{u}+A/2D) - (B/2D)(\overline{v}+B/2D) - (C/2D)(\overline{w}+C/2D) ,$$

$$L' = (\overline{p}+A/2D)(\overline{u}+A/2D) + (\overline{q}+B/2D)(\overline{v}+B/2D) + (\overline{r}+C/2D)(\overline{w}+C/2D),$$

$$H = \overline{P}\cdot\overline{P} - 1,$$

$$K = \overline{U}\cdot\overline{U} - 1.$$

We compute the denominator for the expression of $\sin\Pi(\sigma')$ as described in 6.23.1 as follows:

$$(1+\overline{P}'\cdot\overline{P}')(1+\overline{U}'\cdot\overline{U}') - 4\overline{P}'\cdot\overline{U}'$$

$$= (2+kH/j)(2+kK/\ell) - 4\overline{P}'\cdot\overline{U}' \quad \text{from (i) and (ii) above}$$

$$= 4 + 2\frac{k}{j}H + 2\frac{k}{\ell}K + \frac{k^2}{j\ell}HK - 4 - 4k - 4\frac{k}{j}H' - 4\frac{k}{\ell}K' - 4\frac{k^2}{j\ell}L'$$

$$= 2(k/j)(H-2H') + 2(k/\ell)(K-2K') + (k^2/j\ell)(HK-4L') - 4k$$

$$= 2(k/j)(j+k) + 2(k/\ell)(\ell+k) + (k^2/j\ell)(HK-4L') - 4k \quad =$$

$= (k^2/j\ell)(2j+2\ell+HK-4L')$

$= (k^2/j\ell)(HK+2[(\bar{p}+A/2D)^2+(\bar{u}+A/2d)^2+(\bar{q}+B/2D)^2+(\bar{v}+B/2D)^2+(\bar{r}+C/2D)^2$

$+(\bar{w}+C/2D)^2-2(\bar{p}+A/2D)(\bar{u}+A/2D)$

$-2(\bar{q}+B/2D)(\bar{v}+B/2D)-2(\bar{r}+C/2D)(\bar{w}+C/2D)])$

$= (k^2/j\ell)(HK+2[(\bar{p}-\bar{u})^2+(\bar{q}-\bar{v})^2+(\bar{r}-\bar{w})^2])$

$= (k^2/j\ell)[(\bar{P}\cdot\bar{P})(\bar{U}\cdot\bar{U})+1-\bar{P}\cdot\bar{P}-\bar{U}\cdot\bar{U}+2\bar{P}\cdot\bar{P}+2\bar{U}\cdot\bar{U}-4\bar{P}\cdot\bar{U}]$

$= (k^2/j\ell)[(1+\bar{P}\cdot\bar{P})(1+\bar{U}\cdot\bar{U})-4\bar{P}\cdot\bar{U}]$.

Thus, by 6.23.1 and these computations, we have

$$\sin \Pi(\sigma') = \frac{(k^2/j\ell)(1-\bar{P}\cdot\bar{P})(1-\bar{U}\cdot\bar{U})}{(k^2/j\ell)[(1+\bar{P}\cdot\bar{P})(1+\bar{U}\cdot\bar{U})-4\bar{P}\cdot\bar{U}]}$$

$$= \sin \Pi(\sigma)$$

as claimed, and we are done.

6.30 DEFINITION. Let points P and P' have coordinates (p, q, r) and (p', q', r') respectively. Let $\bar{P} = (\bar{p}, \bar{q}, \bar{r})$ and $\bar{P}' = (\bar{p}', \bar{q}', \bar{r}')$. We define $F(P, \alpha) = P'$ iff P' is the point associated with the triple $\bar{P}'$ and $\bar{P}'$ is the inverse of $\bar{P}$ with respect to α as defined in 6.28. Denote the identity map by $F(P, 0)$. We will call $F(\cdot, \alpha)$ a <u>reflection map</u>.

6.31 THEOREM. For any given plane α, the correspondence F defined above is a bijection of Lobachevskian space onto itself which maps lines into lines, rays into rays, and congruent segments into congruent segments.

Proof: F is a bijection by 6.22 and 6.23. That it maps lines to lines follows from 6.25 and Eschrich's Lemma 10 [3, p. 6] which states that if ℓ is the set of all $(\bar{x}, \bar{y}, \bar{z})$ satisfying a pair of equations such as we have in 6.25, $F(\ell, \alpha)$ is also such a set. That F maps rays into rays follows from the definition of rays, 6.26, and Eschrich's Lemma 11 [3, p. 10]. That F maps congruent segments onto congruent segments follows from Eschrich's Lemma 11 and 6.29.

6.32 LEMMA. If P is any point, different from the origin O, with coordinates (p, q, r), and α is the perpendicular bisecting plane of the segment OP, then $F(P, \alpha) = O$.

Proof: By 6.24.1 α has a representative equation of the form $(1/2)(\bar{p}^2+\bar{q}^2+\bar{r}^2)(\bar{x}^2+\bar{y}^2+\bar{z}^2+1) - \bar{p}\,\bar{x} - \bar{q}\,\bar{y} - \bar{r}\,\bar{z} = 0$. Thus by Eschrich's Lemma 8 [3, p. 4-5], we immediately have the result that $\bar{P}' = (0,0,0)$ and thus by 6.22, 6.23, and 6.31, we have $F(P, \alpha) = O$ as claimed.

6.33 LEMMA. Let P and U be distinct points with coordinates (p, q, r) and (u, v, w). If O is the origin and P, U, and O are non-collinear, $\bar{P} = (\bar{p}, \bar{q}, \bar{r})$, and $\bar{U} = (\bar{u}, \bar{v}, \bar{w})$, then the point Q corresponding to

$$\overline{Q} = 1/2[(\overline{P}/\sqrt{\overline{P}\cdot\overline{P}}) + (\overline{U}/\sqrt{\overline{U}\cdot\overline{U}})]$$

is on the bisector of the angle $\angle POU$.

Proof: From Euclidean geometric considerations we know $\overline{Q}$, as a Euclidean point, is on the Euclidean segment joining the points on the unit sphere corresponding to the unit vectors $\overline{P}/\sqrt{\overline{P}\cdot\overline{P}}$ and $\overline{U}/\sqrt{\overline{U}\cdot\overline{U}}$. As such, $\overline{Q}$ is a Euclidean point interior to the unit sphere so $\sqrt{\overline{Q}\cdot\overline{Q}} < 1$ and thus, by 6.23, there is a point Q of Lobachevskian space corresponding to the triple $\overline{Q}$.

Let $D(\overline{x}^2+\overline{y}^2+\overline{z}^2+1) + A\overline{x} + B\overline{y} + C\overline{z} = 0$ be a representative of the class of equations describing the plane α determined by P, U, and O (6.24). Then $D = 0$ (6.20.1 and proof of 6.24) and since $\overline{P}$ and $\overline{U}$ both satisfy this equation, then so does $\overline{Q}$ and Q is a point of α (6.24).

Let

$$\theta = m\angle POU,$$

$$\varphi = m\angle POQ,$$

$$\psi = m\angle UOQ.$$

From 6.27.1 we have

$$\cos\varphi = \overline{P}\cdot\overline{Q}/\sqrt{(\overline{P}\cdot\overline{P})(\overline{Q}\cdot\overline{Q})}$$

$$= \frac{\overline{P}\cdot(\overline{P}K+\overline{U}L)}{\sqrt{\overline{P}\cdot\overline{P}[(\overline{P}K+\overline{U}L)\cdot(\overline{P}K+\overline{U}L)}} \quad \text{where} \quad K = \frac{1}{2\sqrt{\overline{P}\cdot\overline{P}}}$$

$$\text{and} \quad L = \frac{1}{2\sqrt{\overline{U}\cdot\overline{U}}}$$

$$= \frac{\overline{P}\cdot\overline{P}K+\overline{P}\cdot\overline{U}L}{\sqrt{\overline{P}\cdot\overline{P}(\overline{P}\cdot\overline{P}K^2+\overline{U}\cdot\overline{U}L^2+2\overline{P}\cdot\overline{U}LK)}}$$

$$= \left[\frac{(\sqrt{\overline{P}\cdot\overline{P}} + \overline{P}\cdot\overline{U}/\sqrt{\overline{U}\cdot\overline{U}})^2}{4[\overline{P}\cdot\overline{P}(1/2+1/2\overline{P}\cdot\overline{U}/\sqrt{(\overline{P}\cdot\overline{P})(\overline{U}\cdot\overline{U})})]}\right]^{1/2}$$

$$= \left[\frac{(\sqrt{\overline{P}\cdot\overline{P}}\sqrt{\overline{U}\cdot\overline{U}}+\overline{P}\cdot\overline{U})^2/\overline{U}\cdot\overline{U}}{2\overline{P}\cdot\overline{P}(\sqrt{\overline{P}\cdot\overline{P}}\sqrt{\overline{U}\cdot\overline{U}}+\overline{P}\cdot\overline{U})/\sqrt{(\overline{P}\cdot\overline{P})(\overline{U}\cdot\overline{U})}}\right]^{1/2}$$

$$= \left[\frac{1+\overline{P}\cdot\overline{U}/(\sqrt{\overline{P}\cdot\overline{P}}\sqrt{\overline{U}\cdot\overline{U}})}{2}\right]^{1/2}$$

$$= \cos(\theta/2), \quad \text{by 6.27.1.}$$

In a symmetrical way we get $\cos\psi = \cos\theta/2$. Since $0 < \theta < \pi$ and Q is in the plane α, absolute geometry considerations force $\overrightarrow{OQ}$ to be the bisector of $\angle POU$ and we are done.

6.34 LEMMA. Let P and U be distinct points so that $OP \cong OU$. If O, P, and U are not collinear, then the plane α having a representative equation of the form:

$$\begin{vmatrix} x & y & z \\ \dfrac{\bar{p}+\bar{u}}{\sqrt{\bar{P}\cdot\bar{P}}} & \dfrac{\bar{q}+\bar{v}}{\sqrt{\bar{P}\cdot\bar{P}}} & \dfrac{\bar{r}+\bar{w}}{\sqrt{\bar{P}\cdot\bar{P}}} \\ \begin{vmatrix} \bar{q} & \bar{r} \\ \bar{v} & \bar{w} \end{vmatrix} & -\begin{vmatrix} \bar{p} & \bar{r} \\ \bar{u} & \bar{w} \end{vmatrix} & \begin{vmatrix} \bar{p} & \bar{q} \\ \bar{u} & \bar{v} \end{vmatrix} \end{vmatrix} = 0$$

is the plane determined by any point on the bisecting ray of $\angle POU$ and the line perpendicular to the plane determined by O, P, and U at O. Furthermore $F(P, \alpha) = U$.

Proof: Let P and U have coordinates (p, q, r) and (u, v, w). Let $\bar{P} = (\bar{p}, \bar{q}, \bar{r})$ and $\bar{U} = (\bar{u}, \bar{v}, \bar{w})$. By 6.33, $\bar{Q} = 1/2(\bar{P}/\sqrt{\bar{P}\cdot\bar{P}} + \bar{U}/\sqrt{\bar{U}\cdot\bar{U}})$ is on the angle bisector of $\angle POU$. Since $OP \cong OU$, we know, by 6.9.1, $\sqrt{\underline{P}\cdot\underline{P}} = \sqrt{\underline{U}\cdot\underline{U}}$. Then, by elementary algebra, using Definition 6.21, $\sqrt{\bar{P}\cdot\bar{P}} = \sqrt{\bar{U}\cdot\bar{U}}$. Thus, we have

$$\bar{Q} = 1/2(\bar{P}+\bar{U})/\sqrt{\bar{P}\cdot\bar{P}}) .$$

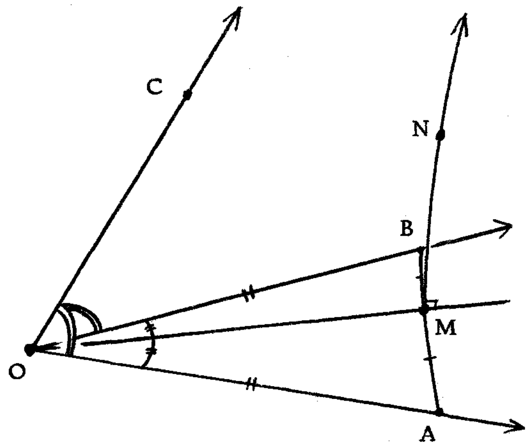
From 6.20.1, 6.20, 6.24 we see that the plane α determined by $\bar{Q}$ and the line ℓ perpendicular, at O, to the plane of O, P and U is the plane having the representative equation above. $\underline{OQ}$ is the perpendicular bisector of PU by S.A.S. and the cross-bar theorem, and the line perpendicular to the plane determined by O, P and U at the midpoint of PU is, by 0.18, coplanar with ℓ. Hence α is the perpendicular bisecting plane of PU as claimed.

To complete the proof, we observe that Eschrich's Theorem 11 [3, p. 10-11] proves that the set of all triples of the form

$$(\bar{x}, \bar{y}, \bar{z}) + t(\bar{p}, \bar{q}, \bar{r})$$

is mapped by inversion into the set of all triples of the form $(\bar{x}, \bar{y}, \bar{z}) = s(\bar{u}, \bar{v}, \bar{w})$ with the explicit pairing given by $s = t(\bar{P} \cdot \bar{P} / \bar{U} \cdot \bar{U})$ (which in our case makes $s = t$) when an inversion is "across" a "plane" whose equation is (in our case) in the above form. Thus $F(P, \alpha) = U$ as claimed and the lemma is proved.

6.35 LEMMA. If $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are distinct non-equivalent rays, $OA \cong OB$, and $\angle AOC \cong \angle BOC$, then line $\underline{OC}$ is in the plane β determined by the bisector of $\angle AOB$ and the line $\underline{OP}$ perpendicular to the plane α determined by A, O, and B.



Proof: We first show that $\underline{OC}$ is a line of the perpendicular bisector plane β of AB. Let M be the midpoint of AB and let $\underline{MN}$ be the line perpendicular to α at M. $\triangle AOM \cong \triangle BOM$ by S.A.S. so $\underline{OM} \perp \underline{AB}$ at M. Thus the plane β determined by O and $\underline{MN}$ is the perpendicular bisecting plane of AB. By 0.24,

Q is a point of the perpendicular bisecting plane of segment AB iff $AQ \cong BQ$. Hence O and C are both points of β since in $\triangle AOC$ and $\triangle BOC$ we have congruence by S.A.S. By Axiom I, 6, the line $\underline{OC}$ is in β.

Let $\underline{OP}$ be perpendicular to α at O. Then by 0.18, $\underline{OP}$ is in β and, by I, 5, β is the plane determined by $\underline{OP}$ and $\underline{OM}$. Thus the theorem is proved.

6.36 LEMMA. If P is a point of plane α, then $F(P, \alpha) = P$.

Proof: Let P have coordinates (p, q, r). If $D = 0$, by 6.28, we get $A\bar{p} + B\bar{q} + C\bar{r} = 0$ so $s = 0$ and $\bar{P}' = \bar{P}$. By 6.30 $F(P, \alpha) = P$ as desired.

If $D \neq 0$, in 6.28 we get the denominator of the parameter t is

$$(\bar{p}+A/2D)^2 + (\bar{q}+B/2D)^2 + (\bar{r}+C/2D)^2$$

$$= \bar{p}^2 + \bar{q}^2 + \bar{r}^2 + 1 + (1/D)(A\bar{p}+B\bar{q}+C\bar{r}) + (1/D^2)(A^2+B^2+C^2) - 1$$

$$= (1/D^2)(A^2+B^2+C^2) - 1,$$

since P is a point of α. Hence $t = 1$ and hence $\bar{P}' = \bar{P}$. Thus $F(P, \alpha) = P$ by 6.30.

6.37 LEMMA. Let O be the origin. If $PO \cong UO$, P-O-U, and α is the plane perpendicular to $\underline{PU}$ at O, then $F(P, \alpha) = U$.

Proof: By 6.20 and 6.24 we see that α has an equation of the form $\overline{p}\,x + \overline{q}\,y + \overline{r}\,z = 0$ where P has coordinates (p, q, r). P and U are collinear with the origin so by 6.18, $\underline{U} = t\underline{P}$ for some number t. By 6.9.1 $\sqrt{\underline{P}\cdot\underline{P}} = \sqrt{\underline{U}\cdot\underline{U}} = |t|\,\sqrt{\underline{P}\cdot\underline{P}}$ so $|t| = 1$. Since U-O-P, 6.18.2 tells us $t = -1$ since $t = 1$ corresponds to P, $t = 0$ corresponds to O.

Thus $\overline{U} = -\overline{P}$, and, in 6.28, $\overline{P}' = (\overline{p}+s\overline{p}, \overline{q}+s\overline{q}, \overline{r}+s\overline{r})$, where $s = -2(\overline{P}\cdot\overline{P}/\overline{P}\cdot\overline{P}) = -2$. Thus, $\overline{P}' = -\overline{P} = \overline{U}$. Hence by 6.30 $F(P, \alpha) = U$ as claimed.

We now introduce some notation to make the statement of the next theorem less cumbersome. If α is any plane denote $F(P, \alpha)$ by $F_\alpha(P)$. This allows us to speak of the composition of two reflections.

6.38 LEMMA. If P is any point and α is any plane, then $(F_\alpha \circ F_\alpha)(P) = P$.

Proof: If P is a point of α, we are done by 6.36. If P is not in α, let $\overline{P} = (\overline{p}, \overline{q}, \overline{r})$ (where (p, q, r) are the coordinates of P) and by Eschrich's Lemma 6 [3, p. 3] and 6.29 if $\overline{Q} = \overline{P}'$, then $\overline{Q}' = \overline{P}$ and by 6.30 we are done.

We now recapitulate those results which we have just proved and which allow us to prove the next two very important results.

6.29 tells us every reflection map is congruence preserving for segments.

6.30 provides us with an identity reflection map which we now denote by I.

6.31 tells us reflection maps are line, ray, betweenness, preserving maps.

6.32 tells us we can always send one end P of a segment to the original O by using α as the perpendicular bisecting plane (p.b.p.) of OP.

6.34 tells us we can always send P to U when $OP \cong OU$, if $P \neq U$, by using α as the p.b.p. of PU when P, O, U are non-collinear.

6.35 tells us that given distinct non-equivalent rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ with $\angle AOC \cong \angle BOC$, then $\overrightarrow{OC}$ is in the p.b.p. of AB.

6.36 tells us that points of the reflecting plane are fixed.

6.37 tells us that whenever P-O-U and $PO \cong OU$, and α is the p.b.p. of PU, then $F(P, \alpha) = U$.

6.38 tells us $F_\alpha^{-1} = F_\alpha$.

6.39 THEOREM. $PU \cong P'U'$, iff there are a finite number of reflection maps $F_1, \ldots, F_n$ so that if $F = F_n \circ \cdots \circ F_1$, $F(P) = P'$ and $F(U) = U'$.

Proof: "only if":

Case 1. $P = P'$ and $U = U'$: Let $F_1 = I$ and we are done.

Case 2. $P = O$:

Subcase 2.i. $P' = O$: If $U = U'$ use Case 1.

a.1. If $U \neq U'$ and U, O, U' are collinear then $U\text{-}O\text{-}U'$. Let γ be the p.b.p. of PP'. Then by 6.36 and 6.37 we are done if $F_1 = F_\gamma$.

a.2. If $U \neq U'$ and U, O and U' are non-collinear, let γ be the p.b.p. of UU' and let $F_1 = F_\gamma$ and we are done by 6.34 and 6.37 (since γ contains O).

Subcase 2.ii. $P' \neq O$: Let β be the p.b.p. of OP' and let $F_2 = F_\beta$. By 6.32 this puts $F_\beta(P') = O$. Now we are in the configuration of Subcase 2.i and we are done.

Case 3. $P \neq O$: Let α be the p.b.p. of PO. By 6.32 we have $F_\alpha(P) = O$, so let $F_1 = F_\alpha$ and apply Case 2.

"if": Proof is immediate by 6.29 and transitivity of congruence for segments.

6.40 THEOREM. $\angle PQR \cong \angle P'Q'R'$ iff there are a finite number of reflection maps $F_1, \ldots, F_n$ so that if $F = F_n \circ \cdots \circ F_1$, $F(Q) = Q'$, $F(\overrightarrow{QR}) = \overrightarrow{Q'R'}$, and $F(\overrightarrow{QP}) = \overrightarrow{Q'P'}$.

Proof: <u>"only if":</u> With no loss of generality let $QP \cong Q'P'$, $QR \cong Q'R'$.

<u>Case 1.</u> $\overrightarrow{QP} = \overrightarrow{Q'P'}$ and $\overrightarrow{QR} = \overrightarrow{Q'R'}$: Let $F_1 = I$ and we are done.

<u>Case 2.</u> $Q = O$:

<u>Subcase 2.i.</u> $Q' = O$ and $P = P'$:

(a) If $R = R'$, let $F_1 = I$ and we are done.

(b) If $R \neq R'$ and R' is in the plane determined by P, Q, and R, then let δ be the p.b.p. of RR'.

b.1. If $\overrightarrow{QR}$ and $\overrightarrow{QR'}$ are opposite rays on $\underline{RR'}$, $\underline{QP}$ is in δ since $\angle PQR \cong \angle P'Q'R'$. Let $F_1 = F_\delta$ and by 6.37, 6.36 and 6.31 we are done.

b.2. If $\overrightarrow{QR}$ and $\overrightarrow{QR'}$ are not opposite rays, $R \neq R'$ implies R and R' are on opposite sides of QP and by S.A.S. the point M of RR' on $\underline{QP}$ is the midpoint of RR' and $\underline{QP} \perp \underline{RR'}$ at M. Thus $\underline{QP}$ is in δ and by 6.36, 6.34 and 6.31 $F_1 = F_\delta$ is an appropriate choice which finishes the proof.

(c) If $R \neq R'$ and R' is not in the plane determined by P, Q and R, then let δ be the p.b.p. of RR'. By 6.35 $QP = OP$ is in the plane δ and by 6.34, 6.37, and 6.31 $F_1 = F_\delta$ is an appropriate choice which finishes the proof.

Subcase 2.ii. $Q' = O$ and $P \neq P'$: Let γ be the p.b.p. of PP'. Then let $F_1 = F_\gamma$ and by 6.36, 6.37, and 6.31 or by 6.36, 6.34, and 6.31 (depending on whether $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are opposite rays or not, respectively) $F_1(P) = P'$ and we are placed in the configuration of Subcase 2.i and are done.

Subcase 2.iii. $Q' \neq O$: Let β be the p.b.p. of $Q'O$. Then let $F_n = F_\beta$ (for appropriate n) and by 6.34 $F_n(Q') = O$ and we are placed in the configuration of either 2.i or 2.ii and are done using 6.38.

Case 3. $Q \neq O$: Let α be the p.b.p. of OQ. Set $F_1 = F_\alpha$ and by 6.32 $F_1(Q) = O$ and we are in the configuration of Case 2, and thus we are done.

"if": With no loss of generality we may let $QP \cong Q'P'$ and $QR \cong Q'R'$. By 6.29 we know $F(P) = P'$ and $F(R) = R'$ so again by 6.29, $PR \cong P'R'$ using transitivity of congruence of segments. Thus by S.S.S., $\Delta PQR \cong \Delta P'Q'R'$ and thus $\angle PQR \cong \angle P'Q'R'$ as claimed.

6.41 THEOREM. The completeness axiom is a theorem in the Poincaré model.

Proof: Our basic assumption has allowed us to prove the preceding results. We have shown that there is a model of the geometry based on the assumption in the following way:

By 6.23 we know P is a point iff there exist coordinates associated with the point, of the form $(\bar{p}, \bar{q}, \bar{r})$ with $\bar{p}^2 + \bar{q}^2 + \bar{r}^2 < 1$.

By 6.24 we know that α is a plane iff there exists an equivalence class of equations having a representative of the form $D(\bar{x}^2+\bar{y}^2+\bar{z}^2+1) + A\bar{x} + B\bar{y} + C\bar{z} = 0$ with $A^2 + B^2 + C^2 > 4D^2$ and X is a point of α iff $(\bar{x}, \bar{y}, \bar{z})$ is in the solution set of a member of this class of equations.

By 6.25 we know that ℓ is a line iff there exists an equivalence class of pairs of equations having a representative pair of the form

$$D(\bar{x}^2+\bar{y}^2+\bar{z}^2+1) + A\bar{x} + B\bar{y} + C\bar{z} = 0,$$

$$D'(\bar{x}^2+\bar{y}^2+\bar{z}^2+1) + A'\bar{x} + B'\bar{y} + C'\bar{z} = 0,$$

with

1. $A^2 + B^2 + C^2 > 4D^2$, $A'^2 + B'^2 + C'^2 > 4D'^2$,
2. the pair of equations has at least one common solution,
3. $\text{rank}\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix} = 2$,

and X is on ℓ iff $(\bar{x}, \bar{y}, \bar{z})$ is in the solution set of a representative pair from the class.

By 6.26 we know that for A, P, X any collinear distinct points, there exists monotonic functions $\bar{x} = F(t)$, $\bar{y} = G(t)$, $\bar{z} = H(t)$, with t in an open interval. For such a prametric formulation, A-U-P iff $t_A < t_U < t_P$ or $t_A > t_U > t_P$.

By 6.39 and 6.30 $PU \cong P'U'$ iff there are a finite number of inversions mapping $\bar{P}$ to $\bar{P}'$ and $\bar{U}$ to $\bar{U}'$.

Finally, by 6.40 and 6.30 $\angle PQR \cong \angle P'Q'R'$ iff there are a finite number of inversions mapping $\bar{Q}$ to $\bar{Q}'$, $\overrightarrow{\bar{Q}\bar{P}}$ onto $\overrightarrow{\bar{Q}'\bar{P}'}$, and $\overrightarrow{\bar{Q}\bar{R}}$ onto $\overrightarrow{\bar{Q}'\bar{R}'}$.

Thus we have an isomorphism between the model constructed from the geometry and the Poincaré model. Clearly the construction used is available in any model of Lobachevskian geometry so we have a canonical isomorphism between all models, and the Poincare model is isomorphic to any model of Lobachevskian geometry. Our basic or key assumption does not lead to a contradiction so the completeness axiom is a theorem in the Poincaré model and we are done.

In summary, we recall our method. We have proved that our key assumption (first discussed on pages 8 and 9 and further discussed and described explicitly on pages 28 through 31) does not lead to a contradiction. We have done this by showing that if the set of points on a line is the set of real numbers with their ordinary field properties, the axioms of Lobachevskian geometry force the meanings of the undefined terms to be those of the Poincaré model. Hence the completeness axiom is a theorm of the Poincaré model and the Poincaré model is a model of Lobachevskian geometry based on Hilbert's axioms.

BIBLIOGRAPHY

1. Beltrami, E. Saggio Di Interpetrazione Della Geometria Non-Euclidea. Giornale Di Matematiche VI (1868), 284-312.

2. Borsuk, Karol and Wanda Szmielew. Foundations of Geometry. Amsterdam, North-Holland Pub. Co., 1960. 444 p.

3. Eschrich, Robert W. A Model of Non-Euclidean Geometry in Three Dimensions, II. Master's Thesis. Corvallis, Oregon State University, 1968. 33 numb. leaves.

4. Fulton, C.M. Linear Completeness and Hyperbolic Trigonometry. Proceedings of A.M.S. (5) 9 (1958), 726-728.

5. Hilbert, David. Foundations of Geometry. Tr. by Leo Unger. La Salle, The Open Court Publishing Co., 1971. 226 p.

6. _______________. Neue Begrundung der Bolyai-Lobatschefskyschen Geometry. Math. Annalen 57 (1903), 137-150.

7. Klein, Felix. Uber die sogenannte Nicht-Euklidische Geometrie. Math. Annalen. 4 (1871), 573-625.

8. Kutuzov, B.V. Geometry. Tr. by Louis I. Gordon and Edward S. Shater. S.M.S.G., Studies in Mathematics. V. 4, 1960. 577 p.

9. Lobachevski, Nicholas. The Theory of Parallels. Tr. by George B. Halsted. La Salle, The Open Court Publishing Co., 1914. 50 p.

10. Moise, Edwin E. Elementary Geometry from an Advanced Standpoint. Reading, Addison-Wesley Publishing Co. 1964. 419 p.

11. Murdoch, D.C. Analytic Geometry. New York, John Wiley and Sons, Inc. 1965. 294 p.

12. Norden, A.P. Elementare Einfuhrung in die Lobatschewskische Geometrie. Berlin, VEB Deutscher Verlag der Wissenschaften. 1958. 259 p.

13. Rosenthal, Artur. Uber das dritte Hilbertsche Axiom der Verknupfung. Math. Annalen 69 (1910) 223-226.

14. ________________. Vereinfachungen des Hilbertschen Systems der Kongruenzaxiome. Math. Annalen 71 (1912) 257-274.

15. Saccheri, Girolamo. Euclides Vindicatus. Tr. by George B. Halsted. Chicago, The Open Court Publishing Co., 1920. 246 p.

16. Shirokov, P. A. A Sketch of the Fundamentals of Lobachevskian Geometry. Tr. by Leo F. Boron and Ward D. Bouwsma. Groningen, P. Noordhoff, 1964. 88 p.

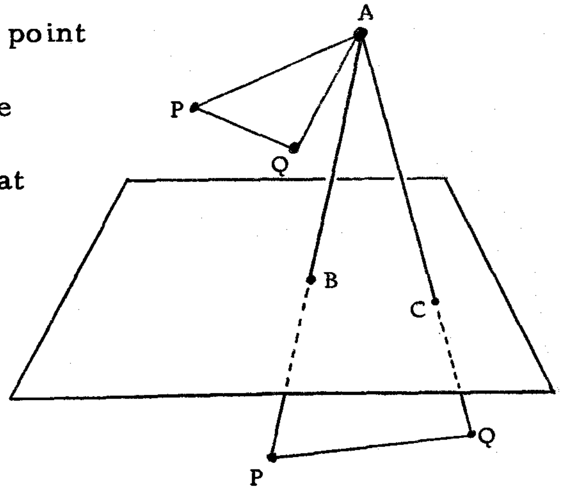
17. Widder, David V. Advanced Calculus (2nd ed.). Englewood Cliffs, Prentice-Hall, Inc., 1965. 520 p.

18. Zell, Wm. Lee. A Model of Non-Euclidean Geometry in Three Dimensions. Master's Thesis. Corvallis, Oregon State University, 1967. 34 numb. leaves.

APPENDIX

APPENDIX

A-1. THEOREM. The points of space not on a plane α may be separated into two classes, called the half spaces determined by α, so that I) any two points P, Q are in the same class iff PQ contains no point of α, whereas II) two points P, Q are in difference classes iff PQ contains a point of α (compare with planar case in Hilbert, p. 9, Theorem 10, [5]).



Proof: I. By Axiom I, 8 there is a point A not on α. Let P and Q be any points of space not on α so that PQ has no point of α.

Case 1: A, P, and Q are collinear. Then by Axioms II, 1, 3 exactly one of A-P-Q, A-Q-P, or P-A-Q is true. If AP or AQ has a point B of α, then in the first two possibilities, we necessarily have A-B-P-Q or A-B-Q-P respectively, since P-B-Q is impossible and A-B-P and A-B-Q are necessary. Thus each of AP and AQ has a point of α is either does. If P-A-Q, then clearly neither AP nor AQ can meet α since PQ does not.

Case 2: Suppose A, P, Q are not collinear. Then consider $\triangle$APQ. If AP contains a point B of α, then by, Axioms I, 4, 5, the plane β determined by A, P, and Q, meets α in a line ℓ containing B, by 0.20. The line ℓ meets AQ, by Pasch's axiom since PQ has no point of α. Clearly, by symmetry, if AP has no point of α, AQ has no point of α.

II. Suppose PQ contains exactly one point B of α.

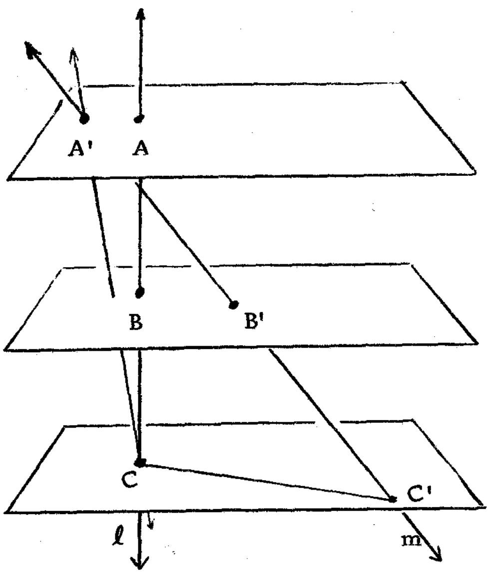
Case 1. A, P, Q are collinear: As above we have exactly one of A-P-Q, A-Q-P, P-A-Q. We are given P-B-Q so if either of the first two occurs, AQ does not contain a point of α and AP does. If P-A-Q, then either P-A-B-Q or P-B-A-Q since $A \neq B$. In any case exactly one of AP or AQ contains a point of α.

Case 2. A, P, Q are non-collinear: Then as in Case I-2 above, we consider the plane β determined by A, P, Q and in $\triangle APQ$ observe that the line ℓ, determined by α and β and containing B of PQ, meets exactly one of AP or AQ by Pasch's axiom and 0.23.

We are now ready to define the relation r between the set of points of space not on α and itself by A r B iff AB does not meet α or B = A. "r" is an equivalence relation since i) A r A, ii) A r B implies B r A, and iii) A r B and B r C implies A r C. This latter is an immediate result of the argument above. Let $S = \{P: A \text{ r } P$ for some fixed point A not on α and P any point not on $\alpha\}$

Let $\overline{S}$ be the complement of S in the set of all points not on α. Then $\overline{S}$ is not empty (by II, 2) and by the argument above S and $\overline{S}$ partition the set of all points not on α.

A-2. THEOREM. If α, β, and γ are three distinct planes all perpendicular to line ℓ at A, B, and C respectively and if line m meets each of α, β, and γ at A', B', and C' respectively, then A-B-C iff A'-B'-C'.



Proof: Since, by 0.21, any two distinct planes perpendicular to the same plane do not meet α, β, and γ have no points in common. If A-B-C, then, from the above, A, B, A', B' are all on the same side of plane α as A and B, by Theorem A-1 above, since AB, BB', A'A, all contain no points of γ. Thus we have A'-B'-C' or B'-A'-C'. If A'-B'-C', we are done. B'-A'-C' is impossible since A and C are in opposite half spaces determined by β, A' and A are in the same half space as are C' and C, and hence by A-1, A'C' contains a point of β. B' is the only possible such point and hence, A'-B'-C'. Thus A-B-C implies A'-B'-C' and by symmetry we are done.