AN ABSTRACT OF THE THESIS OF

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(Name) (Degree) (Major)

Date thesis is presented May 12, 1965

Title AN EXTENSION OF THE METHOD OF HAAR FOR DETERMINING THE ASYMPTOTIC BEHAVIOR OF INTEGRALS OF THE INVERSE LAPLACE TRANSFORM TYPE

Abstract approved Redacted for Privacy
(Major professor)

An asymptotic expansion of an integral of the inverse Laplace transform type

$$F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)\, e^{st}\, ds$$

for large t is given under the assumption that $f(s)$ is analytic in a right half-plane. The thesis represents generalizations of the cases considered by A. Haar [16] and Hull and Froese [18]. As Haar has shown, the asymptotics for $F(t)$ are completely determined by the character and location of the singularity(ies) of $f(s)$ furthest to the right. Haar considered only singularities of algebraic logarithmic character; later, Hull and Froese extended this work to include essential singularities of $f(s)$ of rather simple character. The thesis presented here extends the latter work to cover a variety

of cases where $f(s)$ has essential singular points of a more complicated nature, primarily Whittaker functions of the first and second kinds. The cases considered by Hull and Froese follow readily as special cases of the functions considered here.

AN EXTENSION OF THE METHOD OF HAAR FOR DETERMINING THE ASYMPTOTIC BEHAVIOR OF INTEGRALS OF THE INVERSE LAPLACE TRANSFORM TYPE

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1965

APPROVED:

Redacted for Privacy

Professor of Mathematics

In Charge of Major

Redacted for Privacy

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Dean of Graduate School

Date thesis is presented May 12, 1965

Typed by Carol Baker

ACKNOWLEDGMENT

The author wishes to thank Professor F. Oberhettinger for his assistance in the completion of this thesis.

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INTRODUCTION

Integrals of the inverse Laplace transform type,

$$F(t) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} f(s)\, e^{ts} ds \,, \tag{1}$$

occur frequently in problems of applied mathematics or mathematical physics, especially boundary value problems. Numerous examples of the use of inverse Laplace transforms may be found in electrical circuit theory [5] , in problems in the theory of heat conduction [4] and wave propogation [19] .

Best known perhaps is the procedure to make a partial differential equation with given boundary and given initial conditions subject to a Laplace transform. The result of this operation is the reduction to an ordinary or partial differential equation with given boundary conditions. Suppose the latter problem can be solved. The Laplace inverse of this solution then yields the solution of the original problem. Other applications center around pulse diffraction and pulse propogation problems. While in the well known classical problems only time harmonic phenomena were considered, it has become more and more important in recent times to investigate such phenomena under the influence of arbitrary time dependency (signals).

For instance, denote by $G_{\frac{1}{2}}(P,Q,\gamma)$ the two Green's functions of the "modified" Helmholtz equation $\Delta u - \gamma^2 u = 0$, where P and Q are the locations of the point of observation and the unit source respectively. (This equation is obtained from Helmholtz's equation, $\Delta u + k^2 u = 0$, when the wave number k is replaced by $-i\gamma$ [20]). Then the solution for the case of a "Dirac" pulse stimulation of the source is given by

$$\Phi_D(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G_{\frac{1}{2}}(P, Q, \gamma)\, e^{\gamma t} d\gamma . \tag{2}$$

From this the case of an arbitrary time dependency of the source stimulation can be obtained by a further integration [21]. Finally, the corresponding Green's function, $\overline{G}_{\frac{1}{2}}$, for the heat conduction equation can be obtained from those $G_{\frac{1}{2}}$ of the modified Helmholtz equation as an inverse Laplace transform,

$$\overline{G}_{\frac{1}{2}} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G_{\frac{1}{2}}(P, Q, \sqrt{\gamma}\,)\, e^{\gamma(kt)} d\gamma , \tag{3}$$

(k is the heat conduction constant).

The parameter t in the inversion integrals usually represents the time. Since it is often not possible to give the solution of such inverse Laplace integrals in a closed form it is desirable to obtain approximate solutions. For instance, a method for asymptotic evaluation for large t would then lead to an approximate solution for the afore mentioned physical problems after the lapse of a sufficiently long time. Various methods of finding approximate solutions of $F(t)$ for large t have been employed. The method of stationary phase [9], the method of steepest descents [9], and certain other types of contour integration [3] can be used rather generally. For specific forms of $f(s)$ other methods are possible. Carson [6] determined an asymptotic expansion of $F(t)$ for large t when $f(s)$ is represented by a series of the form

$$f(s) \sim a_0 + a_1/s + a_2/s^2 + \cdots , \tag{4}$$

or

$$f(s) \sim a_0 + a_1/s^{1/2} + a_2/s + a_3/s^{3/2} + \cdots . \tag{5}$$

Similar representations were also given by Stachó [24] and Amerio [2]. If $f(s)$ is of the form

$$f(s) = \frac{a}{s^{\mu}} + \frac{\phi(s)}{s^n}, \quad \mu < 0, \quad n < 1, \quad |\phi(s)| < M \tag{6}$$

as $s \to \infty$, and satisfies certain other conditions, then Obreschkoff [22] gives an estimate of $F(t)$ valid for complex t. More general forms of asymptotic representations of Laplace transforms are given by Erdélyi [10, 13].

It is the method of Haar [16] which we shall consider. In this method we consider the singularity of $f(s)$ furthest to the right in the complex s-plane, and show that this singularity determines an asymptotic value of $F(t)$ as $t \to \infty$.

In Chapter I we generalize Haar's method and give a theorem for determining an asymptotic estimate of a function which is represented by an integral of the inverse Laplace transform type.

In Chapter II we apply the method to some specific comparison functions having essential singularities. We use functions involving Whittaker functions of the first and second kind for comparison functions.

In Chapter III we show how the previous result can be made valid for complex values of t.

In Chapter IV we tabulate the special cases which follow as a result of using Whittaker functions as comparison functions.

CHAPTER I

ASYMPTOTIC ESTIMATES OF FUNCTIONS REPRESENTED BY INTEGRALS OF THE INVERSE LAPLACE TRANSFORM TYPE

PART 1: DARBOUX'S METHOD

The method for the determination of the asymptotic behavior of an integral of the Laplace inverse type (1) for large t as developed by Haar [16] has its origin in the famous investigations by Darboux [25]. The latter's method is concerned with the determination of an approximate expression for a given sequence of numbers, a_n, for large n.

Suppose that

$$\underline{\lim}\, |a_n|^{-1/n} = r \tag{7}$$

is finite. Then the function $f(z)$ defined by the power series,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n , \tag{8}$$

represents an analytic function for $|z| < r$. For $z = \rho e^{i\phi} (\rho < r)$ we may write

$$f(\rho e^{i\phi}) = \sum_{n=0}^{\infty} a_n \rho^n e^{in\phi} , \tag{9}$$

and for the ℓ^{th} derivative we may write

$$f^{(\ell)}(\rho e^{i\phi}) = i^{\ell} \sum_{n=0}^{\infty} a_n n^{\ell} \rho^n e^{in\phi}. \tag{10}$$

(The differentiation in the expression is taken with respect to ϕ for fixed ρ). Then the properties

$$2\pi a_m \rho^m = \int_0^{2\pi} f(\rho e^{i\phi}) e^{-im\phi} d\phi \tag{11}$$

and

$$2\pi i^{\ell} a_m m^{\ell} \rho^m = \int_0^{2\pi} f^{(\ell)}(\rho e^{i\phi}) e^{-im\phi} d\phi, \tag{12}$$

which are the coefficients in the Fourier expansion of $f(\rho e^{i\phi})$ and $f^{(\ell)}(\rho e^{i\phi})$, tend to zero as $m \to \infty$ by the Riemann-Lebesgue lemma; and the asymptotic estimates

$$a_m \rho^m = o(1) \tag{13}$$

and

$$a_m m^{\ell} \rho^m = o(1) \qquad (\ell = 1, 2, 3, \cdots) \tag{14}$$

hold as $m \to \infty$. The second estimate is, of course, a refinement of the first estimate. Since the above statements are improved as ρ approaches r, the question arises whether it is possible to let ρ become equal to r. It is clear, likewise, by the Riemann-Lebesgue lemma, that the more precise estimates

$$a_m r^m = o(1) \qquad \text{as } m \to \infty \tag{15}$$

and

$$a_m m^{\ell} r^m = o(1) \qquad \text{as } m \to \infty \tag{16}$$

also hold if

$$\lim_{\rho \to r} f(\rho e^{i\phi}) , \tag{17}$$

(which is called the boundary function of $f(\rho e^{i\phi})$), is a continuous function of ϕ, or if its first ℓ derivatives with respect to ϕ have the same property as far as the second esimate is concerned. If the boundary function $f(re^{i\phi})$ is not continuous, one proceeds in the following manner: a new function,

$$g(z) = \sum_{m=0}^{\infty} b_m z^m , \tag{18}$$

with given coefficients, b_m, is constructed such that

$$\lim |b_m|^{-1/m} = \lim |a_m|^{-1/m}. \tag{19}$$

If, furthermore, the boundary function $f(z) - g(z)$ is continuous, then

$$(a_m - b_m) r^m = o(1) \qquad \text{as } m \to \infty , \tag{20}$$

and if the first ℓ derivatives of $f(z) - g(z)$ have the same property the above estimate is refined to

$$(a_m - b_m) r_m^m m^{\ell} = o(1) \qquad \text{as } m \to \infty . \tag{21}$$

PART 2: HAAR'S METHOD

The general procedure developed by Darboux can be carried over in an analogous manner to find an asymptotic estimate of an integral of the inverse Laplace transform type. Consider the integral

$$F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) e^{ts} ds , \tag{22}$$

where $f(s)$ is a known function. We have an analogous estimate of $F(t)$ for large t which follows from the Riemann-Lebesgue lemma for Fourier integrals [26], which may be stated as follows: let

$$g(y) = \int_{-\infty}^{\infty} e^{iyt} h(t)\, dt \tag{23}$$

be uniformly convergent for $|y| \geq Y$, then for $t \geq T$, $g(y) \to 0$ as $|y| \to \infty$.

Now let $s = c + iy$ in equation (22), so that

$$\begin{aligned} F(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)\, e^{st} ds \\ &= \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} f(c+iy)\, e^{iyt} dy, \end{aligned} \tag{24}$$

then by the Riemann-Lebesgue lemma we have

<u>Theorem 1.</u> If $\int_{-\infty}^{\infty} f(c+iy)e^{iyt}dy$ converges uniformly for $t \geq T$, then

$$F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)e^{st}ds = o(e^{ct}) \text{ as } t \to \infty . \qquad (25)$$

<u>Theorem 2.</u> If $\int_{-\infty}^{\infty} f(c+iy)e^{ity}dy$ converges uniformly for $t \geq -T$, then

$$F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)e^{st}ds = o(e^{-ct}) \text{ as } t \to -\infty . \qquad (26)$$

We see from Theorem 1 that the further we can shift the path of integration to the left, the better the estimate of $F(t)$ as t becomes a large positive number. For large negative t a better estimate of $F(t)$ is obtained by shifting the path of integration as far to the right as possible.

In investigating the behavior of $F(t)$ for large positive t, the question arises as to how far one can shift the path of integration to the left. By Cauchy's theorem we can shift the integral to the left until the new path of integration reaches the singular point of $f(s)$ furthest to the right. It should be noted, that in some cases, we may be able to shift the path of integration beyond the abscissa of convergence of the integrand. This differs from the analog to power series where it is necessary that there be a singularity on the radius of convergence.

Denote by x_0 the abscissa of the singular point or points of $f(s)$ which is furthest to the right. Can we shift the path of integration onto the line $\text{Re } s = x_0$? The answer is yes, if we can construct a function $g(s)$ which is analytic inside the strip $x_0 < \text{Re } s \leq c$ and two dimensionally continuous on the line $\text{Re } s = x_0$, i. e. $g(s)$ is continuous on the line $\text{Re } s = x_0$ as we approach the line in any direction from the inside. The less severe condition of continuity on the boundary is an improvement of Cauchy's theorem first given by Pollard [23], and then refined by Heibronn [17].

Haar stated conditions under which a comparison function, having a singularity of the same character as $f(s)$ and at the same point as the singularity of $f(s)$, could be used to estimate $F(t)$. In his paper Haar considered only functions, $f(s)$, which were results of Laplace transforms, i. e. $f(s)$ is given by

$$f(s) = \int_0^\infty F(t)e^{-st}dt. \tag{27}$$

Furthermore, the comparison functions which he assumed were of algebraic character, $(s-s_0)^a$, where $a \neq 1, 2, 3, \cdots$, and algebraic logarithmic character, $(s-s_0)^a \log(s-s_0)$. For functions, $f(s)$, possessing these types of singularities at the point $s = s_0$, an asymptotic estimate of $F(t)$ can be given for large t.

More recently another comparison function was given by Hull and Froese [18]. If $f(s)$ has a singularity of the above type multiplied by $\exp(1/s)$, then $F(t)$ has an asymptotic expansion for large t.

We now generalize the method developed by Haar to obtain asymptotic expressions for functions, $F(t)$, represented as inverse Laplace transforms whose image functions, $f(s)$, have essential singularities. Furthermore, $F(t)$ will be assumed to be defined by an integral of the Laplace transform type,

$$F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)e^{st} ds, \tag{28}$$

but $f(s)$ itself isn't necessarily the result of a Laplace transform. We shall also show later that we do not need to restrict ourselves to real t. We first give a theorem which is the basis for results which shall follow.

Theorem 3. Let $F(t)$ be represented by an inverse Laplace transform type integral

$$F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)e^{st} ds . \tag{29}$$

Assume that $f(s)$ has the following properties:

(i) $f(s)$ is analytic for all values of $s = x+iy$ whose real part is larger than some number a. Furthermore, the function $f(x+iy)$ is for real y at $y = \pm\infty$ and large t of Fourier character, i. e.

$$\int_{Y}^{\infty} \quad \text{and} \quad \int_{-\infty}^{-Y} f(x+iy)e^{ity}dy \to 0 \tag{30}$$

uniformly for all $t \geq T$, $x > a$.

(ii) $F(s)$ is two dimensionally continuous on the line $\operatorname{Re} s = a$, and is of Fourier character for large t at $y = \pm\infty$ on this line.

(iii) The integrals

$$\int_{a+i\omega}^{c+i\omega} \quad \text{and} \quad \int_{a-i\omega}^{c-i\omega} f(s)e^{st}ds \tag{31}$$

tend to zero for large values of t as $\omega \to \infty$ for any finite $c > a$,

then, under these conditions

$$F(t) = o(e^{at}) \qquad \text{as } t \to \infty . \tag{32}$$

If in addition we have the further condition that

(iv) $f(a+iy)$ is ℓ times differentiable and each of the derivatives are continuous and such that

$$f(a+iy),\quad f'(a+iy),\quad \cdots,\quad f^{(\ell-1)}(a+iy)$$

tend to zero as $y \to \pm\infty$, and $f^{(\ell)}(a+iy)$ is of Fourier character for large t at $y = \pm\infty$,

then under conditions (i)-(iv) we have the asymptotic formula

$$F(t) = o(t^{-\ell} e^{at}) \qquad \text{as } t \to \infty \,. \tag{33}$$

The proof is based on an extention of Cauchy's theorem where we allow the function $f(s)$ to be two dimensionally continuous on the boundary [17]. Consider the integration around the closed path in Figure 1. When we let $\omega \to \infty$, condition (iii) requires the contributions on parts C_2 and C_4 to vanish. Hence, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{st}ds = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)e^{st}ds \,, \tag{34}$$

Now apply Theorem 1 and the result follows immediately.

When condition (iv) also holds, we write

$$\int_{-\infty}^{\infty} f^{(\ell)}(a+iy)e^{ity}dy = \lim_{\omega\to\infty} \int_{-\omega}^{\omega} f^{(\ell)}(a+iy)e^{ity}dy \tag{35}$$

and integrate by parts to obtain

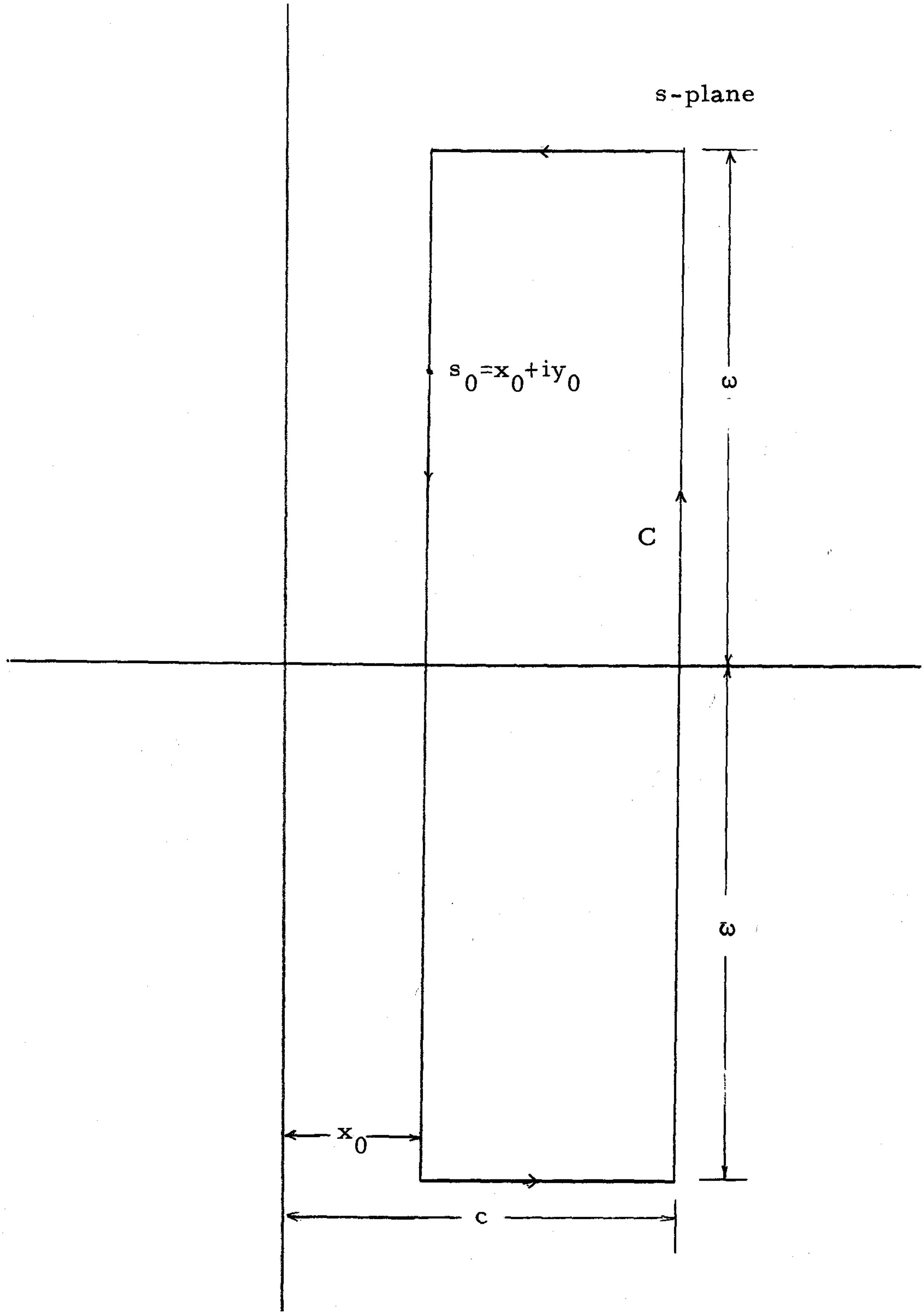

Figure 1. Contour of integration for determining $F(t)$ when t is real.

$$\lim_{\omega\to\infty} \int_{-\omega}^{\omega} f^{(\ell)}(a+iy)e^{ity}dy = \lim_{\omega\to\infty} \Big[\frac{\{e^{ity} f^{(\ell-1)}(a+iy)\}}{i} \Big|_{-\omega}^{\omega}$$

$$-t \int_{-\omega}^{\omega} e^{ity} f^{(\ell-1)}(a+iy)dy \Big] . \tag{36}$$

The first term on the right vanishes by condition (iv). Now, repeat the operation until

$$F(t) = \frac{(-t)^{-\ell} e^{at}}{2\pi} \int_{-\infty}^{\infty} f^{(\ell)}(a+iy)e^{ity}dy . \tag{37}$$

Apply Theorem 1 and the theorem is proved.

When $f(s)$ has a singularity on the line $\operatorname{Re} s = a$, we follow the procedure of Darboux to obtain an asymptotic estimate of $F(t)$ for large t. If $f(s)$ has more than one singularity, it will be the singularities on a line furthest to the right which will concern us. If we can form a difference function by introducing a new function $\phi(s)$ having the proper singularity (or singularities if $f(s)$ should have more than one singularity on this line) such that the difference function satisfies Theorem 3, then the behavior of $F(t)$ is essentially determined by the inverse Laplace transform of $\phi(s)$.

Let $F(t)$ be a function represented as an inverse Laplace transform type of integral whose asymptotic behavior is desired. Assume that $f(s)$ has a singularity at $s_0 = x_0 + iy_0$, and it can be decomposed into two parts, a singular part and a continuous part,

$$f(s) = f_s(s) + f_c(s) \,, \tag{38}$$

where $f_c(s)$ is analytic in $x_0 \leq \mathrm{Re}\, s \leq c$ and where c is some real number greater than x_0. Now define a new function, $\phi(s)$, whose inverse Laplace transform is known, namely,

$$\Phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} \phi(s) ds \,. \tag{39}$$

Furthermore, let $\phi(s)$ be analytic in $x_0 < \mathrm{Re}\, s \leq c$ and continuous in $x_0 \leq \mathrm{Re}\, s \leq c$ (save at the point $s = s_0$) and at s_0 possess a singularity of the same character as $f(s)$. Now let $g(s)$ denote the difference function

$$g(s) = f(s) - \phi(s) \,. \tag{40}$$

If $g(s)$ behaves suitably at infinity, i. e. fulfills the conditions (i)-(iii) of Theorem 3, then by the generalized Cauchy theorem we can shift the path of integration from the line $\mathrm{Re}\, s = c$ to the abscissa $\mathrm{Re}\, s = x_0$. Hence,

$$G(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} g(s) ds = \frac{1}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} e^{ts} g(s) ds \,, \tag{40}$$

so finally,

$$F(t) = \Phi(t) + o(e^{x_0 t}) \qquad \text{as } t \to \infty . \tag{42}$$

In addition, if $g(s)$ satisfies condition (iv), then

$$F(t) = \Phi(t) + o(e^{x_0 t} t^{-\ell}) \qquad \text{as } t \to \infty . \tag{43}$$

CHAPTER II

ASYMPTOTIC ESTIMATES OF INTEGRALS OF FUNCTIONS HAVING ESSENTIAL SINGULARITIES

Let us now consider the following Whittaker functions

$$\phi_1(s) = (s-s_0)^k e^{a/2(s-s_0)} M_{k,\mu}[a/(s-s_0)] \qquad \mathrm{Re}(k-\mu) < -1/2 \tag{44}$$

$$\phi_2(s) = (s-s_0)^k W_{k,\mu}[a/(s-s_0)] \qquad \mathrm{Re}\,(k \pm \mu) < 1/2 \tag{45}$$

$$\phi_3(s) = (s-s_0)^{-k} e^{-1/2(s-s_0)} W_{k,\mu}[1/(s-s_0)] \qquad \mathrm{Re}\,(k \pm \mu) > -1/2 \tag{46}$$

$$\phi_4(s) = (s-s_0)^{-\sigma} e^{a/2(s-s_0)} W_{k,\mu}[a/(s-s_0)] \,. \qquad \mathrm{Re}\,(1/2 \pm u + \sigma) > 0 \tag{47}$$

We shall show that these functions, $\phi_\ell(s)$ $\ell = 1, 2, 3, 4$, which have essential singularities at $s = s_0$, are suitable to use as comparison functions in the theory previously developed. We start by stating four lemmas.

<u>Lemma 1.</u> The functions $\phi_\ell(s)$ $\ell = 1, 2, 3, 4$, as defined by equations (44)-(47) are analytic for all values of $s = x+iy$ whose real part is larger than x_0. Furthermore, the functions $\phi_\ell(s)$ are such that the integrals

$$\int_{Y+y_0}^{\infty} \text{ and } \int_{-\infty}^{-Y+y_0} \phi_\ell[(x-x_0) + i(y-y_0)]\ e^{iyt} dy \tag{48}$$

tend to zero uniformly for $t \geq T$ and for some finite $Y > 0$ and any finite $x > x_0$.

Proof Since the functions $\phi_\ell(s)$ $\ell = 1, 2, 3, 4,$ are the Laplace transforms of functions $\Phi_\ell(t)$ $\ell = 1, 2, 3, 4$ [12], they are analytic in their half-plane of convergence, namely $\text{Re}\, s > x_0$ [8].

To establish the Fourier character of these comparison functions we point out the relation of the Whittaker functions to the Kummer or confluent hypergeometric functions. The Kummer functions are defined as

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

$$U(a, b, z) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - \frac{z^{1-b} M(1+a, 2-b, z)}{\Gamma(a)\ \Gamma(2-b)} \right\}, \tag{50}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0 = 1.$$

The Kummer functions are related to the Whittaker functions by

$$M_{k,\mu}(z) = e^{-1/2\,z} z^{1/2+\mu} M(1/2+\mu-k,\ 1+2\mu,\ z) \tag{51}$$

$$W_{k,\mu}(z) = e^{-1/2\,z} z^{1/2+\mu} U(1/2+\mu-k,\ 1+2\mu,\ z)\ . \tag{52}$$

Hence we may express our comparison functions as Kummer functions

$$\phi_1(s) = a^{\frac{1}{2}+\mu} (s-s_0)^{-(\frac{1}{2}+\mu-k)} \mathbf{M}(\tfrac{1}{2}+\mu-k,\ 1+2\mu,\ \tfrac{a}{s-s_0}) \tag{53}$$

$$\phi_2(s) = a^{\frac{1}{2}+\mu} (s-s_0)^{k-\mu-\frac{1}{2}} e^{-\frac{1}{2}\frac{a}{s-s_0}} U(\tfrac{1}{2}+\mu-k,\ 1+2\mu,\ \tfrac{a}{s-s_0}) \tag{54}$$

$$\phi_3(s) = (s-s_0)^{-k-\mu-\frac{1}{2}} e^{-\frac{1}{2}\frac{1}{s-s_0}} U(\tfrac{1}{2}+\mu-k,\ 1+2\mu,\ 1/s-s_0) \tag{55}$$

$$\phi_4(s) = a^{\frac{1}{2}+\mu} (s-s_0)^{-\sigma-\mu-\frac{1}{2}} U(\tfrac{1}{2}+\mu-k,\ 1+2\mu,\ 1/s-s_0)\ . \tag{56}$$

We may further express the $\phi_\ell(s)$ in terms of the first Kummer function by equation (50); hence,

$$\phi_\ell(s) = a^{1/2+\mu} e^{-\delta_\ell/(s-s_0)} [A_\ell (s-s_0)^{-\alpha_\ell}$$

$$M(1/2+\mu-k,\ 1+2\mu,\ a/(s-s_0)) - B_\ell (s-s_0)^{-\beta_\ell}$$

$$M(1/2+\mu-k,\ 1-2\mu,\ a/(s-s_0))]\ , \tag{57}$$

where

$$\delta_1 = 0 \qquad \alpha_1 = 1/2 + \mu - k \qquad \beta_1 = 0$$

$$\delta_2 = 1/2\,\alpha \qquad \alpha_2 = 1/2 + \mu - k \qquad \beta_2 = 1/2 - \mu - k$$

$$\delta_3 = 1/2 \qquad \alpha_3 = 1/2 + \mu + k \qquad \beta_3 = 1/2 - \mu + k$$

$$\delta_4 = 0 \qquad \alpha_4 = 1/2 + \mu + \sigma \qquad \beta_4 = 1/2 - \mu + \sigma$$

$$A_1 = 1$$

$$A_2 = A_3 = A_4 = \frac{\pi}{\sin[\pi(1+2\mu)]} \; \frac{1}{\Gamma(\frac{1}{2} - \mu - k)} \; \frac{1}{\Gamma(1+2\mu)}$$

$$B_1 = 0$$

$$B_2 = B_3 = B_4 = \frac{\pi}{\sin[\pi(1+2\mu)]} \; \frac{1}{\Gamma(\frac{1}{2} + \mu - k)} \; \frac{1}{\Gamma(1-2\mu)} .$$

We note from the restrictions on k, μ and σ in equations (44)-(47) that $\mathrm{Re}\,\alpha_\ell > 0$ and $\mathrm{Re}\,\beta_\ell > 0$. Since the confluent hypergeometric function, $M(a, b, z)$, is an entire function of z and tends to unity as $|z|$ tends to zero, the behavior of $\phi_\ell(s)$ at $s = s_0$ is simply

$$\lim_{s \to s_0} \phi_\ell(s) = A_\ell \, |s - s_0|^{-\lambda_\ell} , \tag{58}$$

where $\mathrm{Re}\,\lambda_\ell > 0$.

Consider now the integral

$$I = \int_{Y+y_0}^{\infty} \phi_\ell(s-s_0)e^{ity}dy \qquad \text{for } t \geq T. \tag{59}$$

when Y is large, I behaves like

$$A_\ell \int_{Y+y_0}^{\infty} e^{ity}[x-x_0) + i(y-y_0)]^{-\lambda_\ell} dy. \tag{60}$$

The above integral is absolutely convergent when $\mathrm{Re}\,\lambda_\ell > 1$.

For the case where $0 < \alpha_\ell \leq 1$ we may use partial integration,

$$A_\ell \int_{Y+y_0}^{\infty} e^{ity}[(x-x_0) + i(y-y_0)]^{-\lambda_\ell} dy =$$

$$\frac{A_\ell e^{ity}}{it} [(x-x_0) + i(y-y_0)]^{-\lambda_\ell} \Bigg|_{Y+y_0}^{\infty}$$

$$\frac{A_\ell \lambda_\ell}{t} \int_{Y+y_0}^{\infty} e^{ity}[(x-x_0)+i(y-y_0)]^{-\lambda_\ell -1} dy. \tag{61}$$

Now the first term on the right can be made arbitrarily small by taking Y to be large, and the second term on the right now converges absolutely to zero.

A similar argument holds for the integration from $-\infty$ to y_0-Y. This proves the lemma.

<u>Lemma 2.</u> The functions $\phi_\ell(s)$ $\ell = 1,2,3,4$ as defined by equations (44)-(47) are of Fourier character on the line $\mathrm{Re}\,s = x_0$.

The proof is exactly the same as for Lemma 1 except that we take $s = x_0 + iy$.

<u>Lemma 3.</u> The functions $\phi_\ell(s)$ $\ell = 1, 2, 3, 4$ as defined by equations (44)-(47) are such that the integrals

$$\lim_{\omega \to \infty} \int_{c+i\omega}^{x_0+i\omega} \text{ and } \lim_{\omega \to \infty} \int_{c-i\omega}^{x_0-i\omega} \phi_\ell(s) e^{st} ds \qquad (62)$$

tend to zero for $t \geq T$ and $\omega \to \infty$ for any finite $c > x_0$.

The proof of the lemma follows from the known asymptotic behavior of the confluent hypergeometric functions when the argument tends to zero. Let $s = x + i\omega$, then

$$\lim_{\omega \to \infty} \int_{c+i\omega}^{x_0+i\omega} \phi_\ell(s) e^{st} ds =$$

$$e^{iyt} \lim_{\omega \to \infty} \int_{c}^{x_0} \phi_\ell(x+i\omega) e^{xt} dx \,. \qquad (63)$$

For large ω we can replace $\phi_\ell(x+i\omega)$ by $A_\ell[(x-x_0)+(\omega-y_0)]^{-\lambda_\ell}$, then,

$$|A_\ell e^{ity} \lim_{\omega \to \infty} \int_{c}^{x_0} e^{xt}[(x-x_0) + i(\omega-y_0)]^{-\lambda_\ell} dx \,|$$

$$\leq A_\ell e^{x_1 t} (c-x_0) \lim_{\omega \to \infty} [(x_1-x_0) + i(\omega-y_0)]^{-\lambda_\ell} \quad x_0 \leq x_1 \leq c$$

$$\leq M \lim_{\omega \to \infty} [(x_1-x_0) + i(\omega-y_0)]^{-\lambda_\ell} = 0 \text{ if } \mathrm{Re}_\ell > 0 . \qquad (64)$$

The condition $\mathrm{Re}\,\lambda_\ell > 0$ was shown in lemma 1.

Lemma 4. $\phi_\ell(s)$ and its first $n-1$ derivatives tend to zero as $y \to \pm\infty$, (where n may arbitrarily be chosen) and $\phi_\ell^{(n)}(s)$ is of Fourier character.

Since the $\phi_\ell(s)$ are essentially of the form

$$\sum_{m=0}^{\infty} \frac{(b)_m}{(c)_m} \frac{1}{m!} \frac{a^m}{(s-s_0)^{m+\lambda_\ell}} \tag{65}$$

we see that their derivatives tend to zero more rapidly than the $\phi_\ell(s)$ themselves, hence the Fourier character is clearly established.

Theorem 4. Let $g_\ell(s)$ $\ell = 1,2,3,4$ be defined as

$$g_\ell(s) = f_\ell(s) - \phi_\ell(s) \tag{66}$$

where $f_\ell(s) = f_{\ell s}(s) + f_{\ell c}(s)$ such that $f_{\ell s}(s)$ is continuous in the strip $x_0 \leq \text{Re}\, s \leq c$ save at the point $s = s_0$, and analytic in $x_0 < \text{Re}\, s \leq c$, and $f_{\ell c}$ is analytic in $x_0 \leq \text{Re}\, s \leq c$. The $\phi_\ell(s)$ are the functions defined by equations (44)-(47). Furthermore, let

$$g_\ell(s_0) = f_{\ell c}(s_0) \tag{67}$$

be two dimensionally continuous at $s = s_0$. If $f(s)$ satisfies the conditions (i)-(iv) of Theorem 3, then

$$F_1(t) = \frac{a^{1/2}\Gamma(2\mu+1}{\Gamma(\mu-k-\frac{1}{2})} t^{-k-\frac{1}{2}} I_{2\mu}(2a^{\frac{1}{2}}t^{\frac{1}{2}}) + o(t^{-n}e^{x_0 t}) \tag{68}$$

$$F_2(t) = \frac{2a^{1/2}t^{-k-\frac{1}{2}} K_{2\mu}(2a^{\frac{1}{2}}t^{\frac{1}{2}})}{\Gamma(\frac{1}{2}-k+\mu)\,\Gamma(\frac{1}{2}-k-\mu)} + o(t^{-n}e^{x_0 t}) \tag{69}$$

$$F_3(t) = -t^{k-1/2}\{J_{2\mu}(2t^{1/2})\sin[(\mu-k)\pi] \tag{70}$$

$$+ Y_{2\mu}(2t^{1/2})\cos[(\mu-k)\pi]\} + o(t^{-n}e^{x_0 t})$$

$$F_4(t) = t^{-1}\frac{\Gamma(-2\mu)(at)^{\mu+1/2}}{+(1/2-k-\mu)\,\Gamma(1/2+\mu+\sigma)} \tag{71}$$

$${}_1F_2(1/2-k+\mu\,;\ 1+2\mu,\ 1/2+\mu+\sigma;\ at)$$

$$+\frac{\Gamma(2\mu)(at)^{-\mu+1/2}}{+(1/2-k+\mu)\,\Gamma(1/2-\mu+\sigma)}$$

$${}_1F_2(1/2-k-\mu\,;\ 1-2\mu,\ 1/2-\mu+\sigma\,;\ at)] + o(t^{-n}e^{x_0 t}).$$

The proof of the theorem follows as an immediate consequence of Lemmas 1-4 .

CHAPTER III

AN EXTENSION OF THE METHOD FOR COMPLEX ARGUMENT

The extension of Haar's method to include complex values of t is given by Doetsch [9], but only for a function, $f(s)$, whose behavior at $s = s_0$ is algebraic, or algebraic-logarithmic.

Doetsch gives two theorems related to these particular comparison functions. The first theorem requires the function $f(s)$ to be analytic in the sector $|\arg(s-s_0)| \leq \psi \, (\pi/2 < \psi \leq \pi)$ and near $s = s_0$ with the exception of $s = s_0$ itself; furthermore, $f(s)$ must be integrable in every finite interval on the line $|\arg(s-s_0)| = \psi$, and $f(s) \sim A(s-s_0)^{\lambda}$ uniformly as $s \to s_0$ in the sector (λ is arbitrarily complex). Then under these conditions $F(t) \sim A e^{s_0 t} t^{-\lambda-1}/\Gamma(-\lambda)$ for t tending to infinity in the sector $|\arg t| < \psi - \pi/2$.

To prove this he considers a transform $W(t)$ defined as

$$W(t) = \int_C f(s) e^{ts} ds, \tag{72}$$

where C is the contour shown in Figure 2. By considering the separate segments of the contour, each integration is shown to be less than $K\varepsilon t^{-\lambda-1}$, thus proving the theorem.

The second theorem allows $f(s)$ to have one of the three

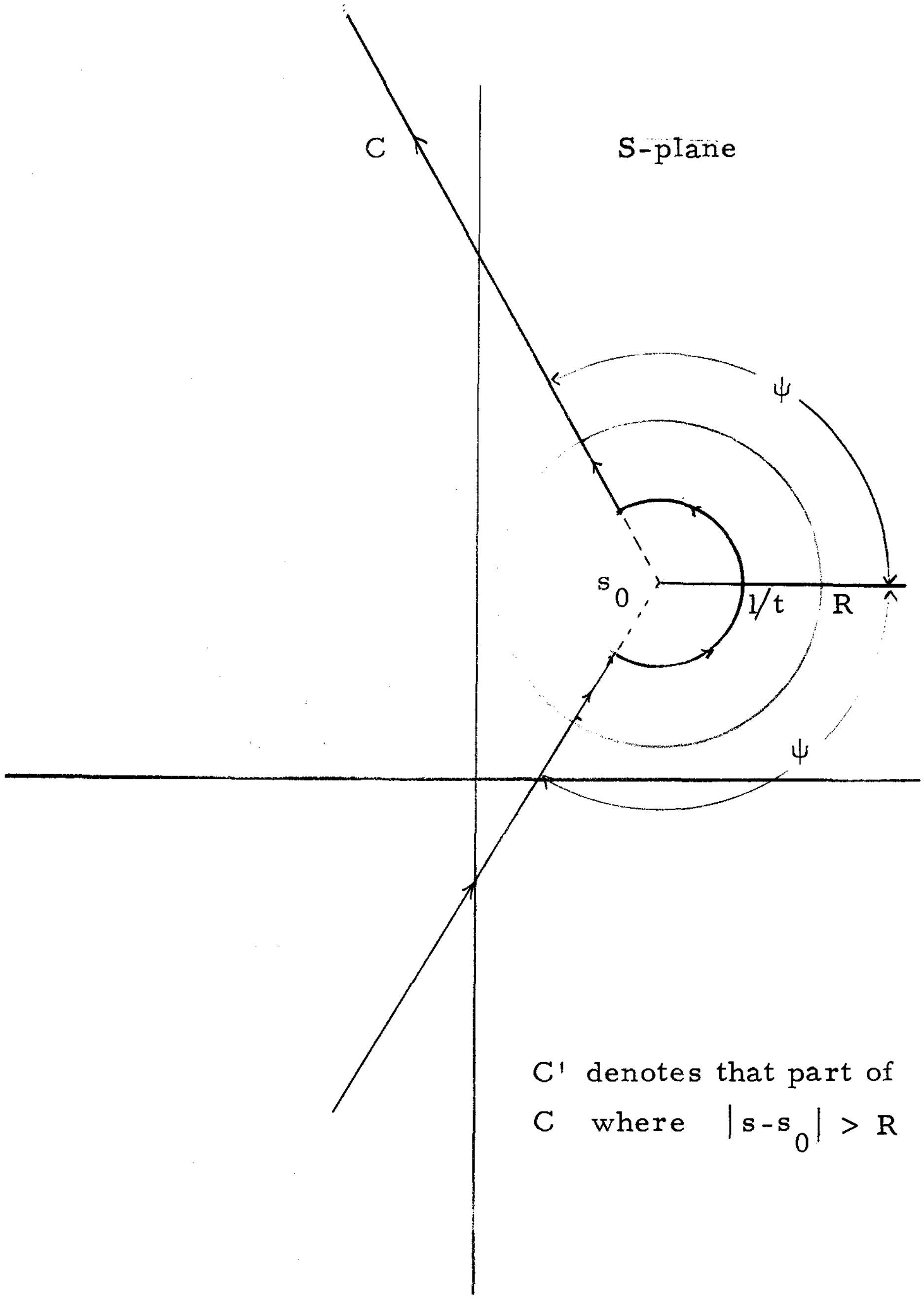

Figure 2. Contour path for W(t).

following expansions near $s = s_0$:

$$\text{A.} \qquad f(s) \approx \sum_{\nu=0}^{\infty} c_\nu (s-s_0)^{\lambda\nu} (\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_1 < \cdots)$$

$$\text{B.} \qquad f(s) \approx \log(s-s_0) \sum_{\nu=0}^{\infty} c_\nu (s-s_0)^{\nu}$$

$$\text{C.} \qquad f(s) \approx \log(s-s_0) \sum_{\nu=0}^{\infty} c_\nu (s-s_0)^{\lambda\nu} (\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_1 < \cdots)$$

$$\operatorname{Re}\lambda_\nu \neq 0, 1, \cdots .$$

If $f(s)$ satisfies the other conditions of the previous theorem, then as $t \to \infty$

$$\text{A}'. \qquad F(t) \approx e^{s_0 t} \sum_{\nu=0}^{\infty} \frac{c_\nu t^{-\lambda_\nu - 1}}{\Gamma(-\lambda\nu)}$$

$$\text{B}'. \qquad F(t) \approx -e^{s_0 t} \sum_{\nu=0}^{\infty} c_\nu (-1)^\nu \nu! \, t^{-\nu-1}$$

$$\text{C}'. \qquad F(t) \approx -e^{s_0 t} \sum_{\nu=0}^{\infty} \frac{c_\nu t^{-\lambda_\nu - 1}}{\Gamma(-\lambda\nu)} \left[\log t - \frac{\Gamma'(-\lambda\nu)}{\Gamma(-\lambda\nu)}\right],$$

where the expansions are now valid for $|\arg t| < \psi - \pi/2$.

Hull and Froese [18] give a theorem similar to Doetsch's, showing that under suitable conditions on $f(s)$, the contributions on the part of C for which $|s-s_0| \geq R > 0$, tend to zero, i.e. the integral

$$\frac{1}{2\pi i}\int_{C'} e^{st}f(s)ds = o(e^{-\varepsilon t}) \tag{73}$$

for some $\varepsilon > 0$. We shall now show that the comparison functions used in Chapter II are suitable to allow the asymptotic representations for $F(t)$ to be extended to complex t.

Theorem 4a Let $g(s)$ satisfy the following conditions:

(i) $g_\ell(s)$ is an analytic function of s for $|\arg(s-s_0)| < \psi$ and continuous in $|\arg(s-s_0)| \leq \psi$ where $1/2\,\pi < \psi < \pi$, except possibly for a singularity at $s = s_0 = x_0 + iy_0$.

(ii) $e^{ps}g(s)$, $e^{ps}g'(s), \cdots, e^{ps}g^{(n)}(s)$ are continuous and tend to zero as $|s| \to \infty$ in the sector $\frac{1}{2}\pi \leq \arg(s-s_0)| \leq \psi$, where p is real and finite.

(iii) $$G(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}g(s)ds \tag{74}$$

exists for some real $a > x_0$.

(iv) $g(s)$, $g'(s), \cdots, g^{(n)}(s)$ tend to zero uniformly in x as $y \to \pm\infty$ in the strip $0 \leq \mathrm{Re}\, s \leq a$.

Then, under these conditions,

$$G(t) = \frac{1}{2\pi i} \int_C e^{st} g(s) ds, \tag{75}$$

where C is the contour shown in Figure 3.

The proof of this theorem follows as an immediate consequence of Cauchy's integral theorem. By condition (iv) the contributions to the integral over paths C_1 and C_4 vanish. The integration also vanishes on paths C_2 and C_3 by condition (ii). By Cauchy's integral theorem,

$$\int_{a-i\infty}^{a+i\infty} + \int_{C_1} + \int_{C_2} - \int_{C} + \int_{C_3} + \int_{C_4} = 0, \tag{76}$$

and since

$$\int_{C_1} = \int_{C_2} = \int_{C_3} = \int_{C_4} = 0, \tag{77}$$

we have

$$G(t) = \int_{a-i\infty}^{a+i\infty} g(s) e^{st} ds = \int_C g(s) e^{st} ds. \tag{78}$$

Theorem 4b Define the functions $g_\ell(s)$ $\ell = 1, 2, 3, 4$ to be

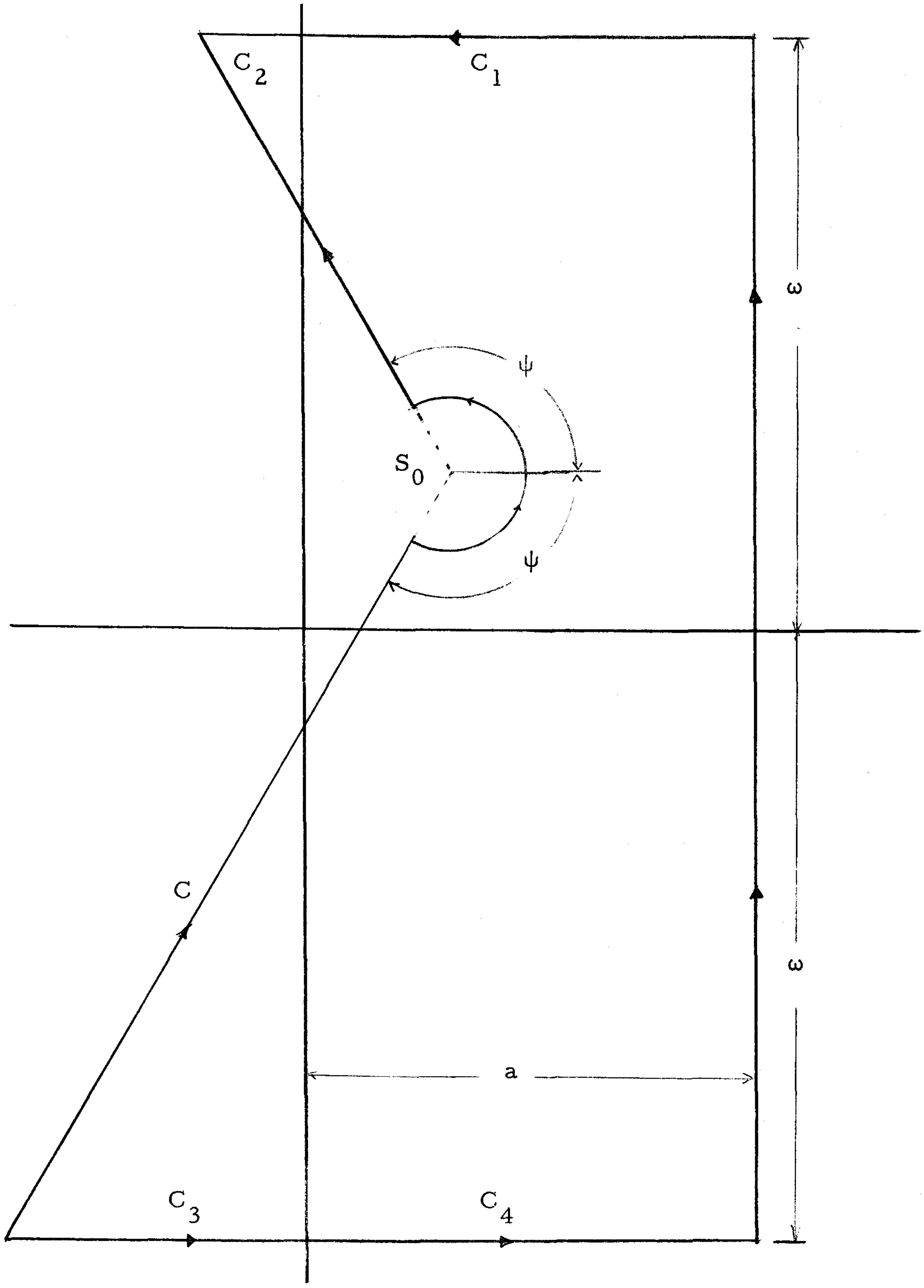

Figure 3. Contour of integration for determining $F(t)$ when t is complex.

$$g_1(s) \;=\; f_1(s) - (s-s_0)^k \, e^{\frac{1}{2}\frac{a}{(s-s_0)}} \, M_{k,\mu}\left(\frac{a}{s-s_0}\right) \tag{79}$$

$$\mathrm{Re}\,(k-\mu) < \frac{1}{2}$$

$$g_2(s) \;=\; f_2(s) - (s-s_0)^k W_{k,\mu}\left(\frac{a}{s-s_0}\right) \tag{80}$$

$$\mathrm{Re}\,(k\pm\mu) < \frac{1}{2}$$

$$g_3(s) \;=\; f_3(s) - (s-s_0)^{-k} \, e^{-\frac{1}{2}\frac{1}{s-s_0}} \, W_{k,\mu}\left(\frac{1}{s-s_0}\right) \tag{81}$$

$$\mathrm{Re}\,(k\pm\mu) > -\frac{1}{2}$$

$$g_4(s) \;=\; f_4(s) - (s-s_0)^{-\sigma} \, e^{\frac{1}{2}\frac{a}{s-s_0}} \, W_{k,\mu}\left(\frac{a}{s-s_0}\right) \tag{82}$$

$$\mathrm{Re}\left(\frac{1}{2} \pm \mu + \sigma\right) > 0 .$$

Furthermore, let the $g_\ell(s)$ satisfy the conditions of Theorem 4a, then,

$$F_1(t) = \frac{a^{1/2}\Gamma(2\mu+1)}{\Gamma(\mu-k-\frac{1}{2})} t^{-k-\frac{1}{2}} I_{2\mu}(2a^{\frac{1}{2}}t^{\frac{1}{2}}) + o(t^{-n}e^{x_0 t}) \tag{83}$$

$$F_2(t) = \frac{2a^{1/2}t^{-k-\frac{1}{2}}K_{2\mu}(2a^{\frac{1}{2}}t^{\frac{1}{2}})}{\Gamma(\frac{1}{2}-k+\mu)\,\Gamma(\frac{1}{2}-k-\mu)} + o(t^{-n}e^{x_0 t}) \tag{84}$$

$$F_3(t) = -t^{k-1/2}\{J_{2\mu}(2t^{1/2})\sin[(\mu-k)\pi] \tag{85}$$

$$+ Y_{2\mu}(2t^{1/2})\cos[(\mu-k)\pi]\} + o(t^{-n}e^{x_0 t})$$

$$F_4(t) = t^{\sigma-1} \frac{\Gamma(-2\mu(at)^{\mu+1/2}}{\Gamma(1/2-k-\mu)\Gamma(1/2+\mu+\sigma)} \tag{86}$$

$${}_1F_2(1/2-k+\mu\,;\ 1+2\mu,\ 1/2+\mu+\sigma;\ at)$$

$$+ \frac{\Gamma(2\mu)\,(at)^{-\mu+1/2}}{\Gamma(1/2-k+\mu)\ \Gamma(1/2-\mu+\sigma)}$$

$${}_1F_2(1/2-k-\mu\,;\ 1-2\mu,\ 1/2-\mu+\sigma;\ at)] + o(t^{-n}e^{x_0 t})$$

where the asymptotic representations, $F_\ell(t)$, are valid for $|\arg t| < \psi - \pi/2$, $\pi < \psi < \pi/2$.

Again we assume $f_\ell(s)$ is composed of a singular and a regular part such that $f_{\ell s}(s)$ is continuous in the region $-\psi \leq \arg(s-s_0) \leq \psi$ and analytic in the region $-\psi < \arg(s-s_0) < \psi$ except possibly in the point $s = s_0$; $f_{cs}(s)$ is analytic in the region $-\psi \leq \arg(s-s_0) \leq \psi$.

That the integration of $\phi_\ell(s)$ along the line $a-i\infty$ to $a+i\infty$ may be replaced by the integration on the contour C is easily seen. By lemma 3 the integrals over $\phi_\ell(s)$ vanish on paths C_1 and C_4. On paths C_2 and C_3 the integrals

$$\lim_{\omega \to \infty} \int_{i\omega}^{-X+i\omega} e^{st}\phi_\ell(s)ds \quad \text{and} \quad \lim_{\omega \to \infty} \int_{-X-i\omega}^{-\omega} e^{st}\phi_\ell(s)ds \tag{87}$$

vanish for $X > 0$ since $\phi_\ell(s) \to A_k(s-s_0)^{-\lambda_\ell}$ for large s, where $\lambda_\ell > 0$, and since the exponential term is never positive.

Since $f_\ell(s) = \phi_\ell(s) + g_\ell(s)$ we see that $f(s)$ satisfies the assumptions of Theorem 4a, hence,

$$\begin{aligned} F_\ell(t) &= \int_C [\phi_\ell(s) + g_\ell(s)]\, e^{ts}ds \\ &= \Phi_\ell(t) + o(t^{-n}e^{x_0 t}) \, . \end{aligned} \tag{88}$$

We now establish the domain of t for which the above equation holds. To do this consider the integration along the line $s = re^{i\psi}$,

$$\int_R^\infty g(re^{i\psi})\, e^{tre^{i\psi}}\, e^{i\psi}dr\,, \tag{89}$$

where R is a large positive real number. Let us allow t to be complex and denote it by $t = \tau e^{i\beta}$, then the above integral becomes

$$\int_R^\infty g(re^{i\psi})\, e^{Tr\cos(\psi+\beta)}\, e^{i[\psi+ rT\sin(\psi+\beta)]}$$

$$\leq \int_R^\infty g(re^{i\psi})\, e^{Tr\cos(\psi+\beta)} dr\,. \qquad (90)$$

By condition (iv) of the theorem this integral is finite as long as the exponential term in the integrand is negative. This leads to the condition

$$\pi/2 - \psi < \beta < \frac{3\pi}{2} - \psi\,, \qquad (91)$$

Considering the integration along the line $s = re^{-i\psi}$, we are led by a similar argument to the simultaneous condition on β that

$$-\frac{3\pi}{2} + \psi < \beta < \psi - \pi/2\,. \qquad (92)$$

The common domain for the argument of t is

$$-\psi + \pi/2 < \beta = \arg t < \psi - \pi/2\,. \qquad (93)$$

For the functions $\phi_\ell(s)$ we may let ψ extend to π, hence the asymptotic representations for $F_\ell(t)$ are valid for $-\pi/2 < \arg t < \pi/2$. In a similar manner we can find the asymptotic estimate of $F_\ell(t)$ as t tends to minus infinity in any direction in the left half-plane.

The comparison functions introduced in Chapter II are not the only functions, having essential singularities, that can be used for comparison functions. In fact, any function whose inverse Laplace transform is known and which also satisfies the conditions of Theorem 3 or Theorem 4a can be used as a comparison function.

A general procedure for estimating a function $F(t)$ which is represented by an integral of the inverse Laplace transform type would be to first determine the singularities of $f(s)$ furthest to the right and determine the character of these singularities. Secondly, we would check a table of inverse Laplace transforms and see if there exists a function (or functions) $\phi(s)$ having singularities of the same character as $f(s)$ which are located at the same points or can be translated to the same points as the singularities of $f(s)$. Thirdly, the function $\phi(s)$ must satisfy the conditions of either Theorem 3 or Theorem 4a; if it does satisfy these conditions, then an estimate of $F(t)$ can be given.

When it occurs that a comparison function is of the form

$$\phi(s) = s^{\beta} \psi(s) , \tag{94}$$

we can often obtain logarithmic comparison functions by differentiation with respect to β,

$$\frac{\partial^n}{\partial \beta^n} \phi(s) = (\log s)^n s^{\beta} \psi(s) . \tag{95}$$

One must check to see if this new comparison satisfies the conditions of the theorem and verify that $\frac{\partial^n \Phi(t)}{\partial \beta^n}$ exists.

An interesting problem still to be investigated is that of finding an estimate of $F(t)$ when $f(s)$ has an infinite number of singularities spaced on a line $\mathrm{Re}\, s = x_0$.

CHAPTER IV

SPECIAL CASES OF THE GENERAL COMPARISON FUNCTIONS

The advantage of choosing Whittaker functions for comparison functions is evidenced by the large number of functions which can be obtained from the Whittaker functions by specializing their parameters. By suitable choice of k and μ we obtain as comparison functions Bessel functions, parabolic cylinder functions, error functions, Hermite functions and several other functions.

The number of examples of comparison functions previously given by Haar [16], is extended to several types of functions having essential singularities. The results of Hull and Froese [18] also follow as a special case.

We first list the functions which are obtained from $M_{k,\mu}(s)$ and $W_{k,\mu}(s)$ by specializing the parameters k and μ. These are listed in Tables 1 and 2. We then apply Theorem 4b to each of special cases which lie in the range of validity of the theorem. These special cases and their inversion formulas are listed in Tables 3 through 6. The numbers in the left hand column identify the special case which is numbered in Tables 1 and 2.

The notation is the same as that used in the Handbook of Mathematical Functions [1], and the properties of the functions listed may be found there.

Table 1. Special cases of $M_{k,\mu}(z)$.

	$M_{k,\mu}(z)$			Relation	Function
	k	μ	z		
1.	0	ν	2iz	$2^{2\nu+1/2}\, i^{\frac{1}{2}+\nu}\, \Gamma(1+\nu) z^{\frac{1}{2}}\, J_{\nu}(z)$	Bessel
2.	0	$-\nu$	2iz	$2^{\frac{1}{2}-2\nu}\, i^{\frac{1}{2}-\nu}\, \Gamma(1-\nu) z^{1/2}$ $[\,J_{\nu}(z)\cos(\pi\nu)-Y_{\nu}(z)\sin(\pi\nu)]$	Bessel
3.	0	ν	2z	$2^{1/2+2\nu}\, z^{\frac{1}{2}}\, \Gamma(1+\nu) I_{\nu}(z)$	Modified Bessel
4.	0	$n+\frac{1}{2}$	2iz	$2^{2n+3/2}\, i^{n+1}\, \Gamma(\frac{3}{2}+n) z^{1/2} J_{n+\frac{1}{2}}(z)$	Spherical Bessel
5.	0	$-n-\frac{1}{2}$	2iz	$2^{-2n-1/2}\, i^{-n}\, \Gamma(\frac{1}{2}-n) z^{1/2} J_{-n-1/2}(z)$	Spherical Bessel
6.	0	$n+\frac{1}{2}$	2z	$2^{2n+\frac{1}{2}}\, \Gamma(\frac{3}{2}+n) z^{1/2} I_{n+\frac{1}{2}}(z)$	Spherical Bessel

Table 1. (Continued)

	$M_{k,u}(z)$			Relation	Function
	k	μ	z		
7.	0	n	$-2(iz)^{1/2}$	$2^{2n+\frac{1}{2}} i^{\frac{1}{2}n+1/4} \pi^{-n}\Gamma(1+n)$ $e^{-2\pi z+(iz)^{1/2}} x^{\frac{1}{2}n+\frac{1}{4}}(\mathrm{ber}_n z + i\,\mathrm{bei}_n z)$	Kelvin
8.	$i\eta$	$L+\frac{1}{2}$	2iz	$(2i)^{L+1}\, F_L(\eta, z)/\, C_L(\eta)$	Coulomb Wave
9.	$\frac{1}{2}\alpha+\frac{1}{2}+n$	$\frac{1}{2}\alpha$	z	$\frac{n!}{(\alpha+1)_n}\, e^{-\frac{1}{2}z}\, z^{\frac{1}{2}\alpha+\frac{1}{2}}\, L_n^{(\alpha)}(z)$	Laguerre
10.	$-\frac{1}{2}\alpha+\frac{1}{2}$	$\frac{1}{2}\alpha$	-z	$\alpha e^{\frac{1}{2}i\pi(\alpha+1)}\, e^{\frac{1}{2}z}\, z^{-\frac{1}{2}\alpha+1/2}\, \gamma(\alpha, z)$	Incomplete Gamma
11.	$\frac{1}{2}(1+\nu+n)$	$\frac{1}{2}(\nu-n)$	z	$\frac{(n!)^{1/2}}{(1+\nu-n)_n}\, e^{-\frac{1}{2}z}\, z^{\frac{1}{2}\nu+1/2}\, \rho_n(\nu, z)$	Poisson-Charlier
12.	$-\frac{1}{2}\alpha$	$\frac{1}{2}(\alpha-1)$	z	$z^{\frac{1}{2}\alpha}\, e^{\frac{1}{2}z}$	Exponential

Table 1. (Continued)

	$M_{k,\mu}(z)$			Relation	Function
	k	μ	z		
13.	0	$\frac{1}{2}$	$-2iz$	$-2i \sin z$	Trignometric
14.	0	$\frac{1}{2}$	$2z$	$2 \sinh z$	Hyperbolic
15.	$\frac{1}{4}+\frac{1}{2}\nu$	$-\frac{1}{4}$	$\frac{1}{2}z^2$	$2^{-3/4} z^{1/2} E_\nu^{(0)}(z)$	Weber or Parabolic Cylinder
16.	$\frac{1}{4}+\frac{1}{2}\nu$	$\frac{1}{4}$	$\frac{1}{2}z^2$	$2^{-7/4} z^{1/2} E_\nu^{(1)}(z)$	Weber or Parabolic Cylinder
17.	$\frac{1}{4}+n$	$-\frac{1}{4}$	$\frac{1}{2}z^2$	$\frac{n!}{(2n)!}(-1)^n 2^{n-1/4} e^{-\frac{1}{4}z^2} z^{\frac{1}{2}} He_{2n}(z)$	Hermite
18.	$\frac{3}{4}+n$	$\frac{1}{4}$	$\frac{1}{2}z^2$	$\frac{n!}{(2n+1)!}(-1)^n 2^{n-3/4} e^{-\frac{1}{4}z^2} z^{3/2} He_{2n+1}(z)$	Hermite
19.	$\frac{1}{4}$	$\frac{1}{4}$	$-z^2$	$2^{-1}\pi^{1/2} e^{i\frac{3}{4}\pi} e^{\frac{1}{2}z^2} z^{1/2} \operatorname{erf}(x)$	Error Integral
20.	$\frac{1}{2}n-\frac{1}{2}m$	$\frac{1}{2}n$	z^2	$\frac{n!}{\Gamma(\frac{1}{2}m+1/2)} e^{\frac{1}{2}z^2} z^{-n-m} T(m,n,z)$	Toronto

Table 2. Special cases of $W_{k,\mu}(z)$.

	$W_{k,\mu}(z)$				
	k	μ	z	Relation	Function
21.	0	ν	$2z$	$2^{1/2}\pi^{-1/2}z^{1/2}K_{\nu}(z)$	Modified Bessel
22.	0	ν	$-2iz$	$2^{-1/2}(-i)^{\nu+1/2}\pi^{1/2}e^{i\pi(\nu+1/2-z)+iz}z^{1/2}H_{\nu}^{(1)}(z)$	Hankel
23.	0	ν	$2iz$	$2^{-1/2}(i)^{\nu+1/2}\pi^{1/2}e^{-i\pi(\nu+1/2-z)-iz}z^{1/2}H_{\nu}^{(2)}(z)$	Hankel
24.	0	$n+\frac{1}{2}$	$2z$	$2^{1/2}\pi^{-1/2}z^{1/2}K_{n+\frac{1}{2}}(z)$	Spherical Bessel
25.	0	$\frac{1}{3}$	$\frac{4}{3}z^{3/2}$	$2\pi^{1/2}z^{1/4}\mathrm{Ai}(z)$	Airy
26.	0	n	$(iz)^{1/2}$	$2^{-n}i^{n+\frac{1}{4}}\pi^{-1/2}e^{\frac{1}{2}(iz)^{1/2}}z^{1/4}(\mathrm{ker}_n z+i\,\mathrm{kei}_n z)$	Kelvin
27.	$\frac{1}{2}(a+1)+n$	$\frac{1}{2}a$	z	$(-1)^n n!\, z^{\frac{1}{2}a+\frac{1}{2}}e^{-\frac{1}{2}z}L_n^{(a)}(z)$	Laguerre
28.	$\frac{1}{2}a-\frac{1}{2}$	$-\frac{1}{2}a$	z	$z^{-\frac{1}{2}-\frac{1}{2}a}e^{\frac{1}{2}z}\Gamma(a,z)$	Incomplete Gamma

Table 2. (Continued)

	$W_{k,\mu}(z)$			Relation	Function
	k	μ	z		
29.	$-\frac{1}{2}$	0	$-z$	$-e^{-\frac{1}{2}z}(-z)^{1/2}\,Ei(z)$	Exponential Integral
30.	$-\frac{1}{2}$	0	z	$e^{\frac{1}{2}z}\,z^{1/2}\,E_1(z)$	Exponential Integral
31.	$-\frac{1}{2}$	0	$-\ln z$	$-z^{-1/2}(-\ln z)^{1/2}\,\ell i\,(z)$	Logarithmic Integral
32.	$\frac{1}{2}+n$	$\frac{1}{2}m$	z	$\Gamma(1+n-\frac{1}{2}m)\;e^{-i\pi(\frac{1}{2}m-n)}$ $z^{\frac{1}{2}(m+1)}\;e^{\frac{1}{2}z}\;\omega_{n,m}(z)$	Cunningham
33.	$\frac{1}{2}\nu$	$-\frac{1}{2}$	$2z$	$\Gamma(1+\frac{1}{2}\nu)k_{\nu}(z)$ for $z>0$	Bateman
34.	$-\frac{1}{2}$	0	iz	$(iz)^{1/2}\,e^{\frac{1}{2}iz}\,[-\frac{1}{2}\pi i+iSi(x)-Ci(z)]$	Sine and Cosine Integral
35.	$-\frac{1}{2}$	0	$-iz$	$(-iz)^{\frac{1}{2}}\,e^{-\frac{1}{2}iz}\,[\frac{1}{2}\pi i-iSi(z)-Ci(z)]$	Sine and Cosine Integral

Table 2. (Continued)

	$W_{k,\mu}(z)$			Relation	Function
	k	μ	z		
36.	$\frac{1}{2}\nu+\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}z^2$	$2^{-\frac{1}{2}\nu-\frac{1}{4}} z^{\frac{1}{2}} D_\nu(z)$	Weber or Parabolic Cylinder
37.	$\frac{1}{2}\nu+\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}z^2$	$2^{-\frac{1}{4}-\frac{1}{2}\nu} z^{3/2} D_\nu(z)$	Weber or Parabolic Cylinder
38.	$\frac{1}{2}n+\frac{1}{4}$	$\frac{1}{4}$	z^2	$2^{-n} e^{-\frac{1}{2}z^2} z^{3/2} H_n(z)$	Hermite
39.	$-\frac{1}{4}$	$-\frac{1}{4}$	z^2	$\pi^{1/2} e^{\frac{1}{2}z^2} z^{1/2} \operatorname{erfc}(z)$	Error Integral

Table 3. Inversion formulas for special cases of first comparison functions.

	General comparison formula	
	$\phi_1(s)$	$\Phi_1(t)$
	$s^k e^{\frac{1}{2}\frac{a}{s}} M_{k,\mu}(a/s)$ $\mathrm{Re}(k-\mu) < 1/2$	$\dfrac{a^{\frac{1}{2}}\,\Gamma(2\mu+1)}{\Gamma(\mu-k+\frac{1}{2})}\, t^{-k-1/2} I_{2\mu}(2a^{1/2}t^{1/2})$
	Special cases	
1	$s^{-\frac{1}{2}} e^{\frac{1}{2}\frac{a}{s}} J_\nu(\frac{a}{2is})$ $\mathrm{Re}\,\nu < -\frac{1}{2}$	$2\pi^{-\frac{1}{2}} e^{-\frac{1}{2}i\pi\nu} t^{-\frac{1}{2}} I_{2\nu}(2a^{1/2}t^{1/2})$
2	$s^{-\frac{1}{2}} e^{\frac{1}{2}\frac{a}{s}} [J_\nu(\frac{a}{2is})\cos(\pi\nu)$ $-Y_\nu(\frac{a}{2is})\sin(\pi\nu)]$ $\mathrm{Re}\,\nu < 1/2$	$-2\nu^{-1}\pi^{-1/2} e^{-\frac{1}{2}i\pi\nu} t^{-\frac{1}{2}} I_{2\nu}(2a^{1/2}t^{1/2})$
3	$s^{-\frac{1}{2}} e^{\frac{1}{2}\frac{a}{s}} I_\nu(\frac{a}{2s})$ $\mathrm{Re}\,\nu > -\frac{1}{2}$	$2\pi^{-1/2} t^{-1/2} I_{2\nu}(2a^{1/2}t^{1/2})$
4	$s^{-\frac{1}{2}} e^{\frac{1}{2}\frac{a}{s}} J_{n+\frac{1}{2}}(\frac{a}{2is})$ $n > -1$	$2\pi^{-1/2} e^{-\frac{1}{2}i\pi(n+1/2)} t^{-1/2} I_{2n+1}(2a^{1/2}t^{1/2})$
6	$s^{-\frac{1}{2}} e^{\frac{1}{2}\frac{a}{s}} I_{n+\frac{1}{2}}(\frac{a}{2s})$ $n > 0$	$2\pi^{-1/2} t^{-1/2} I_{2n+1}(2a^{1/2}t^{1/2})$

Table 3. (Continued)

	$\phi_1(s)$	$\Phi_1(t)$
7	$s^{n-\frac{1}{2}} e^{\frac{1}{2}i\frac{a^2\pi}{s^2} + \frac{a}{s}}$ $(\mathrm{ber}_n \frac{a^2}{4is^2} + i\,\mathrm{bei}_n \frac{a^2}{4is^2})$ $n > 0$	$2^{-n+1} a^n \pi^{n-\frac{1}{2}} i^{-n} t^{-\frac{1}{2}} I_{2n}(2a^{1/2}t^{1/2})$
8	$s^{i\eta} e^{\frac{1}{2}\frac{a}{s}} F_L(\eta, \frac{a}{2is})$ $\mathrm{Re}\, L + \mathrm{Im}\,\eta > 1$	$a^{1/2}(2i)^{-L+1} C_L(\nu)$ $\frac{\Gamma(2L+2)}{\Gamma(L+1+i\eta)} t^{-i\eta-\frac{1}{2}} I_{2i\eta}(2a^{1/2}t^{1/2})$
10	$\gamma(\alpha, -a/s)$ $\mathrm{Re}\ \alpha > 0$	$a^{\frac{1}{2}\alpha} e^{-i\pi} t^{\frac{1}{2}\alpha - 1} I_\alpha(2a^{1/2}t^{1/2})$
12	$s^{-\alpha} e^{\frac{a}{s}}$ $\mathrm{Re}\,\alpha > 0$	$a^{\frac{1}{2}(1-\alpha)} t^{\frac{1}{2}(\alpha-1)} I_{\alpha-1}(2a^{1/2}t^{1/2})$
13	$e^{\frac{1}{2}\frac{a}{s}} \sin(\frac{-a}{2is})$	$-\frac{1}{2}a^{1/2} e^{-\frac{1}{2}i\pi} t^{-\frac{1}{2}} I_1(2a^{1/2}t^{1/2})$
14	$e^{\frac{1}{2}\frac{a}{s}} \sinh(\frac{a}{2is})$	$\frac{1}{2}a^{1/2} t^{-\frac{1}{2}} I_1(2a^{1/2}t^{1/2})$
15	$s^{\frac{1}{2}\nu} e^{\frac{1}{2}\frac{a}{s}} E_\nu^{(0)}(\frac{2^{1/2}a^{1/2}}{s^{1/2}})$ $\mathrm{Re}\,\nu < 0$	$\frac{2^{1/2}}{\Gamma(-\frac{1}{2}\nu)} t^{-\frac{1}{2}\nu} \sinh(2a^{1/2}t^{1/2})$

Table 3. (Continued)

	$\phi_1(s)$	$\Phi_1(t)$
16	$s^{\frac{1}{2}\nu} e^{\frac{1}{2}\frac{a}{s}} E_{\nu}^{(1)}\left(\frac{2^{1/2}a^{1/2}}{s^{1/2}}\right)$ Re $\nu < 1$	$\frac{2^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)} t^{-\frac{1}{2}\nu-1} \sinh(2a^{1/2}t^{1/2})$
19	$\mathrm{erf}[(-\frac{a}{s})^{1/2}]$	$e^{\frac{1}{2}i\pi} t^{-1} \sinh(2a^{1/2}t^{1/2})$
20	$s^{\frac{3}{4}n+\frac{1}{4}m+\frac{3}{4}} e^{\frac{a}{s}} T(m,n,a^{1/2}s^{-1/2})$ $m > -1$	$\frac{a^{\frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m} \Gamma(\frac{1}{2}m+1/2)}{\Gamma(\frac{1}{4}n+\frac{1}{4}m+\frac{1}{4})} t^{\frac{1}{4}m-\frac{1}{4}n-3/4}$ $I_n(2a^{1/2}t^{1/2})$

Table 4. Inversion formulas for special cases of second comparison function.

General comparison formula

$\phi_2(s)$	$\Phi_2(t)$
$s^k W_{k,\mu}(a/s)$ $\mathrm{Re}(k\pm\mu) > 1/2$	$\dfrac{2a^{1/2}t^{-k-1/2}K_{2\mu}(2a^{1/2}t^{1/2})}{\Gamma(\frac{1}{2}-k+\mu)\,\Gamma(\frac{1}{2}-k-\mu)}$

Special cases

	$\phi_2(s)$	$\Phi_2(t)$
21	$s^{-1/2}K_\nu(\frac{a}{2s})$ $-\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}$	$2\pi^{-1/2}\cos(\pi\nu)$ $t^{-1/2}K_{2\nu}(2a^{1/2}t^{1/2})$
22	$e^{\frac{1}{2}\frac{a}{s}(\pi-1)}\ s^{-1/2}H_\nu^{(1)}(\frac{-a}{2is})$ $-\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}$	$4e^{-i\pi(\nu+1/2)}\pi^{-3/2}e^{\frac{1}{2}\pi i}\cos(\pi\nu)$ $t^{-1/2}K_{2\nu}(2a^{1/2}t^{1/2})$
23	$e^{\frac{a}{2s}(\pi-1)}\ s^{-1/2}H_\nu^{(2)}(\frac{a}{2is})$ $-\frac{1}{2} < \mathrm{Re}\,\nu < 1/2$	$4e^{i\pi(\nu+1/2)}\ e^{-\frac{1}{2}\pi i}\cos(\pi\nu)$ $t^{-1/2}K_{2\nu}(2a^{1/2}t^{1/2})$
25	$s^{-1/6}\mathrm{Ai}[(\frac{3a}{4s})^{2/3}]$	$2^{-2/3}3^{-1/6}a^{-1/3}\pi^{-3/2}t^{-1/2}K_{2/3}(2a^{1/2}t^{1/2})$
28	$s^{\alpha-1}e^{\frac{1}{2}\frac{a}{s}}\Gamma(\alpha, a/s)$ $\mathrm{Re}\,\alpha < 2/3$	$\dfrac{2a^{\frac{1}{2}\alpha}t^{-\frac{1}{2}\alpha}}{\Gamma(1-\alpha)}\ K_\alpha(2a^{1/2}t^{1/2})$
29	$s^{-1}e^{-\frac{1}{2}\frac{a}{s}}\mathrm{Ei}(\frac{a}{s})$	$2e^{\frac{1}{2}\pi i}K_0(2a^{1/2}t^{1/2})$
30	$s^{-1}e^{\frac{1}{2}\frac{a}{s}}E_1(a/s)$	$2K_0(2a^{1/2}t^{1/2})$

Table 4.(Continued)

	$\phi_2(s)$	$\Phi_2(t)$
31	$s^{-1} e^{\frac{1}{2}\frac{a}{s}} \ell i(e^{-a/s})$	$2e^{\frac{1}{2}\pi i} K_0(2a^{1/2}t^{1/2})$
32	$s^{n-\frac{1}{2}m} e^{\frac{1}{2}\frac{a}{s}} \omega_{n,m}(\frac{a}{s})$ $n \pm \frac{1}{2}m < 0$	$\frac{2a^{-\frac{1}{2}m} \pi^{-1} e^{i\pi(\frac{1}{2}m-n)}}{\Gamma(-n-\frac{m}{2}) \csc[\pi(m/2-n)]}$ $t^{-n-1} K_m(2a^{1/2}t^{1/2})$
33	$s^{\frac{1}{2}\nu} k_\nu(\frac{a}{2s})$ $\mathrm{Re}\,\nu < 0$	$\frac{2a^{1/2} t^{-\frac{1}{2}(\nu+1)} K_1(2a^{1/2}t^{1/2})}{\Gamma(-\nu/2)\,\Gamma(1-\nu/2)\,\Gamma(1+\nu/2)}$
34	$s^{-1} e^{\frac{1}{2}\frac{a}{s}} [-\frac{1}{2}\pi i + iSi(\frac{a}{is})$ $-Ci(\frac{a}{is})]$	$2K_0(2a^{1/2}t^{1/2})$
35	$s^{-1} e^{\frac{1}{2}\frac{a}{2}} [\frac{1}{2}\pi i - iSi(\frac{-a}{is})$ $-Ci(\frac{-a}{is})]$	$2K_0(2a^{1/2}t^{1/2})$
36	$s^{\frac{1}{2}\nu} D_\nu[(\frac{2a}{s})^{1/2}]$ $\mathrm{Re}\,\nu < 0$	$\frac{2^{\frac{1}{2}\nu} \pi^{1/2} t^{-\frac{1}{2}\nu+1/2} e^{-2a^{1/2}t^{1/2}}}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)\;\Gamma(-\frac{1}{2}\nu)}$
37	$s^{\frac{1}{2}\nu-1/2} D_\nu[(\frac{2a}{s})^{1/2}]$ $\mathrm{Re}\,\nu < 0$	$\frac{2^{\frac{1}{2}+\frac{1}{2}\nu} \pi^{1/2} t^{-\frac{1}{2}\nu-1} e^{-2a^{1/2}t^{1/2}}}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)\;\Gamma(-\frac{1}{2}\nu)}$
39	$s^{-\frac{1}{2}} e^{\frac{1}{2}\frac{a}{s}} \mathrm{erfc}[(\frac{a}{s})^{1/2}]$	$2\pi^{-1/2} t^{-1/4} e^{-2a^{1/2}t^{1/2}}$

Table 5. Inversion formulas for special cases of third comparison function

• General comparison formula

	$\phi_3(s)$	$\Phi_3(t)$
	$s^{-k} e^{-\frac{1}{2}s^{-1}} W_{k,\mu}(s^{-1})$ $\mathrm{Re}(k \pm \mu) > -1/2$	$-t^{k-1/2}\{J_{2\mu}(2t^{1/2})\sin[(\mu-k)\pi]$ $+ Y_{2\mu}(2t^{1/2})\cos[(\mu-k)\pi]\}$
Special cases		
21	$s^{-\frac{1}{2}} e^{-\frac{1}{2}\frac{1}{s}} K_\nu(\frac{1}{2s})$ $-1/2 < \mathrm{Re}\,\nu < 1/2$	$-\pi^{1/2} t^{-1/2}[J_{2\nu}(2t^{1/2})\sin(\pi\nu)$ $+ Y_{2\nu}(2t^{1/2})\cos(\pi\nu)]$
22	$e^{\frac{1}{s}(2\pi-1)} s^{-1/2} H_\nu^{(1)}(\frac{-1}{2is})$ $-1/2 < \mathrm{Re}\,\nu < 1/2$	$-2e^{-i\pi(\nu+1)}(-i)^{-\nu}\pi^{-1/2}t^{-1/2}$ $[J_{2\nu}(2t^{1/2})\sin(\pi\nu) + Y_{2\nu}(2t^{1/2})\cos(\pi\nu)]$
23	$e^{\frac{1}{s}(2\pi-1)} s^{-1/2} H_\nu^{(2)}(\frac{1}{2is})$ $-1/2 < \mathrm{Re}\,\nu < 1/2$	$-2e^{i\pi(\nu+1)}(i)^{\nu}\pi^{-1/2}t^{-1/2}$ $[J_{2\nu}(2t^{1/2})\sin(\pi\nu)+Y_{2\nu}(2t^{1/2})\cos(\pi\nu)]$
25	$s^{-1/6} e^{-\frac{1}{2s}} \mathrm{Ai}[(\frac{3a}{4s})^{2/3}]$	$-\pi^{-1/2}3^{-1/6}2^{-2/3}[3^{\frac{1}{2}}J_{2\nu}(2t^{1/2})+Y_{2\nu}(2t^{1/2})]$
27	$s^{-n} e^{-\frac{1}{s}} L_n^{(a)}(s^{-1})$ $-2(n+1) < \mathrm{Re}\,a < 2n$	$\frac{t^{n+1/2}}{n!} J_a(2t^{1/2})$
28	$\Gamma(a, \frac{1}{s})$ $\mathrm{Re}\,a > 0$	$-t^{a-1}\{J_{-a}(2t^{1/2})\sin[(\frac{1}{2}-\frac{3}{2}a)]$ $+ Y_{-a}(2t^{1/2})\cos[(\frac{1}{2}-\frac{3}{2}a)]\}$

Table 5. (Continued)

	$\phi_3(s)$	$\Phi_3(t)$
32	$s^{-\frac{1}{2}n-\frac{1}{2}m-1}\,\omega_{n,m}(1/s)$ $n \pm \frac{1}{2}m > -1$	$\frac{e^{i\pi(\frac{1}{2}m-n)}}{\Gamma(1+n-1/2m)}\,t^{n}\{J_m(2t^{1/2})\sin[(\frac{1}{2}m-n-1/2)\pi]$ $+Y_m(2t^{1/2})\cos[(\frac{1}{2}m-n-1/2)\pi]\}$
33	$s^{-\frac{1}{2}\nu}\,e^{-\frac{1}{2s}}\,k(\frac{1}{2s})$ $\mathrm{Re}\,\nu > 0 \quad s > 0$	$\frac{\pi^{1/2}t^{-1/4}}{\Gamma(\nu)}\,\sin[2t^{1/2}-\frac{1}{2}\pi\nu-\frac{1}{2}\pi]$
36	$s^{-\frac{1}{2}\nu-1/2}\,e^{-\frac{1}{2s}}\,D_\nu(2^{1/2}s^{-1/2})$ $\mathrm{Re}\,\nu > -1$	$-\pi^{1/2}2^{-1/2\nu-1}t^{\frac{1}{2}\nu}\sin[2t^{1/2}-(\frac{1}{2}\nu+\frac{1}{2})\pi]$
37	$s^{-\frac{1}{2}\nu-1}\,e^{-\frac{1}{2s}}\,D_\nu(2^{1/2}s^{-1/2})$ $\mathrm{Re}\,\nu > -1$	$-\pi^{1/2}2^{-1/2\nu-3/2}t^{\frac{1}{2}\nu}\cos(2t^{1/2}+\frac{1}{2}\pi\nu)$
38	$e^{-\frac{1}{2}}\,s^{-\frac{1}{2}n-1}\,H_n(s^{-1/2})$ $n \geq 0$	$-2^{n}\,\pi^{1/2}\,t^{\frac{1}{2}n}\cos(2t^{1/2}+\frac{1}{2}n\pi)$
39	$\mathrm{erfc}(s^{-1/2})$	$\pi^{-1}\,t^{-1}\,\sin(2t^{1/2})$

Table 6. Inversion formulas for special cases of fourth comparison function

General inversion formula

$\phi_4(s)$	$\Phi_4(t)$
$s^{-\sigma} e^{\frac{1}{2}\frac{a}{s}} W_{k,\mu}(a/s)$ $\mathrm{Re}\,(\frac{1}{2}+\sigma\pm\mu)>0$	$t^{\sigma-1}\Bigg[\dfrac{\Gamma(-2\mu)(at)^{\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-k-\mu)\Gamma(\frac{1}{2}+\mu+\sigma)}$ $F(\frac{1}{2}-k+\mu;\,1+2\mu,\,1/2\ \mu+\sigma;\ at)$ $+\dfrac{\Gamma(2\mu)(at)^{-\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+\mu)\,\Gamma(1/2-\mu+\sigma)}$ $F(\frac{1}{2}-k-\mu;\ 1-2\mu,\ \frac{1}{2}-\mu+\sigma;\ at)\Bigg]$

Special cases

21	$s^{-\sigma-\frac{1}{2}} e^{\frac{1}{2}\frac{a}{s}} K_\nu(\frac{a}{2s})$ $\mathrm{Re}\,(\sigma\pm\nu)>-1/2$	$a^{-\frac{1}{2}}\pi^{1/2}t^{\sigma-1}\Bigg[\dfrac{\Gamma(-2\nu)(at)^{\nu+1/2}}{\Gamma(1/2-\nu)\Gamma(1/2+\nu+\sigma)}$ $F(\frac{1}{2}+\nu;\ 1+2\nu,\ \frac{1}{2}+\nu+\sigma;\ at)$ $+\dfrac{\Gamma(2\nu)(at)^{-\nu+1/2}}{\Gamma(1/2+\nu)\,\Gamma(\frac{1}{2}-\nu+\sigma)}$ $F(1/2-\nu;\ 1-2\nu,\ 1/2-\nu+\sigma;\ at)\Bigg]$

Table 6.(Continued)

	$\phi_4(s)$	$\Phi_4(t)$
22	$s^{-\sigma-1/2}\, e^{\frac{\pi a}{2s}}\, H_\nu^{(1)}\left(\frac{-a}{2is}\right)$ $\operatorname{Re}(\sigma \pm \nu) > -\frac{1}{2}$	$2e^{-i\pi(\nu+1/2)}(-i)^{-\nu}\pi^{-1/2}a^{-1/2}t^{\sigma-1}$ $\left[\frac{\Gamma(-2\nu)(at)^{\nu+1/2}}{\Gamma(1/2-\nu)\Gamma(1/2+\nu+\sigma)}\right.$ $F(\frac{1}{2}+\nu;\ 1+2\nu, \frac{1}{2}+\nu+\sigma;\ at)$ $+\frac{\Gamma(2\nu)\,(at)^{-\nu+1/2}}{\Gamma(1/2+\nu)\,\Gamma(\frac{1}{2}-\nu+\sigma)}$ $\left.F(1/2-\nu;\ 1-2\nu,\ 1/2-\nu+\sigma;\ at)\right]$
23	$s^{-\sigma-1/2}\, e^{\frac{\pi a}{2s}}\, H_\nu^{(2)}\left(\frac{a}{2is}\right)$ $\operatorname{Re}(\sigma \pm \nu) > -1/2$	$2e^{i\pi(\nu+1/2)}i^{\nu}\pi^{-1/2}a^{-1/2}t^{\sigma-1}$ $\left[\frac{\Gamma(-2\nu)(at)^{\nu+1/2}}{\Gamma(1/2-\nu)\,\Gamma(\frac{1}{2}+\nu+\sigma)}\right.$ $F(\frac{1}{2}+\nu;\ 1+2\nu, \frac{1}{2}+\nu+\sigma;\ at)$ $+\frac{\Gamma(2\nu)(at)^{-\nu+1/2}}{\Gamma(1/2+\nu)\Gamma(\frac{1}{2}-\nu+\sigma)}$ $\left.F(1/2-\nu;\ 1-2\nu,\ 1/2-\nu+\sigma;\ at)\right]$
24	$s^{-\sigma-\frac{1}{2}}\, e^{\frac{1}{2}\frac{a}{s}}\, K_{n+1/2}\left(\frac{a}{2s}\right)$ $n < \operatorname{Re}\sigma < n+1$	$a^{-1/2}\pi^{1/2}t^{\sigma-1}\left[\frac{\Gamma(-2n-1)(at)^{n+1}}{\Gamma(-n)\Gamma(n+1+\sigma)}\right.$ $F(n+1;\ 2n+2,\ n+1+\sigma;\ at)$ $+\frac{\Gamma(2n+1)(at)^{-n}}{\Gamma(n+1)\,\Gamma(-n+\sigma)}$ $\left.F(-n;\ -2n,\ -n+\sigma;\ at)\right]$

Table 6. (Continued)

	$\phi_4(s)$	$\Phi_4(t)$
25	$s^{-\sigma-1/6} e^{\frac{1}{2}\frac{a}{s}} \mathrm{Ai}[(\frac{3a}{4s})^{2/3}]$	$2^{-2/3} 3^{-1/6} a^{-1/6} \pi^{-1/2} t^{\sigma-1}$ $[\frac{\Gamma(-2/3)(at)^{5/6}}{\Gamma(1/6)\Gamma(5/6+\sigma)} F(5/6; 5/3, 5/6+\sigma; at)$ $+ \frac{\Gamma(2/3)(at)^{1/6}}{\Gamma(5/6)\Gamma(1/6+\sigma)} F(1/6; -1/3, 1/6+\sigma; at)]$
26	$e^{\frac{a}{s}} s^{-\sigma-1/2}\left[\mathrm{ker}_n(\frac{a^2}{is^2}) + \mathrm{kei}_n(\frac{a^2}{is^2})\right]$ $\mathrm{Re}\ \sigma > n - 1/2$	$2^n a^{-1/2} i^{-n} \pi^{1/2} t^{\sigma-1}\left[\frac{\Gamma(-2n)(at)^{n+1/2}}{\Gamma(1/2-n)\Gamma(\frac{1}{2}+n+\sigma)}\right.$ $F(1/2+n;\ 1+2n,\ 1/2+n+\sigma;\ at)$ $+\frac{\Gamma(2n)(at)^{-n+1/2}}{\Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n+\sigma)}$ $\left.F(\frac{1}{2}-n;\ 1-2n,\ \frac{1}{2}-n+\sigma;\ at)\right]$
27	$s^{-\sigma-\frac{1}{2}\alpha-1/2} L_n^{(\alpha)}(a/s)$ $\mathrm{Re}(\ \pm\frac{1}{2}\alpha) > -\frac{1}{2}$	$\frac{a^{-\frac{1}{2}\alpha-\frac{1}{2}}(-1)^n}{n!} t^{\sigma-1}\left[\frac{\Gamma(-\alpha)(at)^{\frac{1}{2}\alpha+1/2}}{\Gamma(-\alpha-n)\Gamma(\frac{1}{2}+\alpha+\sigma)}\right.$ $F(n;\ 1+\alpha,\ 1/2+\frac{1}{2}\alpha+\sigma;\ at)$ $\frac{\Gamma(\alpha)(at)^{\frac{1}{2}-\frac{1}{2}\alpha}}{\Gamma(-n)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha+\sigma)}$ $\left.F(n-\alpha;\ 1-\alpha, \frac{1}{2}-\frac{1}{2}\alpha+\sigma;\ at)\right]$
28	$e^{\frac{a}{s}} s^{-\sigma-\frac{1}{2}\alpha+\frac{1}{2}} \Gamma(\alpha, a/s)$ $\mathrm{Re}(\sigma \mp \frac{1}{2}\alpha) > -\frac{1}{2}$	$a^{-\frac{1}{2}+\frac{1}{2}\alpha} t^{\sigma-1} \frac{\Gamma(\alpha)(at)^{-\frac{1}{2}\alpha+1/2}}{\Gamma(1/2-\frac{1}{2}\alpha+\sigma)}$ ${}_0F_1(\frac{1}{2}+\alpha+\sigma;\ at)$

Table 6. (Continued)

	$\phi_4(s)$	$\Phi_4(t)$
32	$e^{\frac{a}{s}} s^{-\sigma+\frac{1}{2}m-\frac{1}{2}} \omega_{n,m}(a/s)$ $\mathrm{Re}\,\sigma > \frac{1}{2}m-1/2$	$e^{i\pi(\frac{1}{2}m-n)} a^{-\frac{1}{2}m-1/2} \Gamma(1+n-1/2m)$ $t^{\sigma-1}\left[\frac{\Gamma(-m)(at)^{\frac{1}{2}m+1/2}}{\Gamma(-n-\frac{1}{2}m)\Gamma(\frac{1}{2}+\frac{1}{2}m+\sigma)}\right.$ $F(-n+1/2m;\ 1+m,\ \frac{1}{2}+\frac{1}{2}m+\sigma;\ at)$ $+\frac{\Gamma(m)(at)^{-\frac{1}{2}m+1/2}}{\Gamma(\frac{1}{2}m-n)\Gamma(1/2-1/2m+\sigma)}$ $\left.F(-n-1/2m;\ 1-m,\ 1/2-1/2m+\sigma;\ at)\right]$
36	$s^{-\sigma-1/4} e^{\frac{1}{2}\frac{a}{s}} D_\nu[(\frac{2a}{s})^{1/2}]$ $\mathrm{Re}\,\sigma > -1/4$	$2^{\frac{1}{2}\nu} a^{-1/4} t^{\sigma-1}\left[\frac{\pi^{1/2}(at)^{1/4}}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)\Gamma(\frac{1}{4}+\sigma)}\right.$ $F(-\frac{1}{2}\nu;\ 1/2,\ 1/4+\sigma;\ at)$ $-\frac{2\pi^{1/2}(at)^{3/4}}{\Gamma(-\frac{1}{2}\nu)\Gamma(3/4+\sigma)}$ $\left.F(\frac{1}{2}-\frac{1}{2}\nu;\ 1;\ 3/4+\sigma;\ at)\right]$

Table 6. (Continued)

	$\phi_4(s)$	$\Phi_4(t)$
37	$s^{-\sigma-3/4}\, e^{\frac{1}{2}\frac{a}{s}}\, D_\nu[(\frac{2a}{s})^{1/2}]$ $\operatorname{Re}\sigma > -1/4$	$2^{-\frac{1}{2}+\frac{1}{2}\nu}\, a^{-3/4}\, t^{\sigma-1} \left[\frac{-2\pi^{1/2}(at)^{3/4}}{\Gamma(-\frac{1}{2}\nu)\Gamma(\frac{3}{4}+\sigma)} \right.$ $F(\frac{1}{2}-\frac{1}{2}\nu;\ 1,\ 3/4+\sigma;\ at)$ $+\ \frac{\pi^{1/2}(at)^{1/4}}{\Gamma(1/2-\frac{1}{2}\nu)\,\Gamma(1/4+\sigma)}$ $\left. F(-\frac{1}{2}\nu;\ 1/2,\ 1/4+\sigma;\ at) \right]$
38	$s^{-\sigma-3/4} H_n[(\frac{a}{s})^{1/2}]$ $\operatorname{Re}\sigma > -1/4$	$2^n a^{-3/4} t^{\sigma-1} \left[\frac{-2\pi^{1/2}(at)^{3/4}}{\Gamma(-1/2n)\,\Gamma(3/4+\sigma)} \right.$ $F(\frac{1}{2}-\frac{1}{2}n;\ 1,\ 3/4+\sigma;\ at)$ $+\ \frac{\pi^{1/2}(at)^{1/4}}{\Gamma(1/2-1/2n)\,\Gamma(1/4+\sigma)}$ $\left. F(-\frac{1}{2}n;\ 1/2,\ 1/4+\sigma;\ at) \right]$
39	$s^{-\sigma-1/4}\, e^{\frac{a}{s}}\, \operatorname{erfc}[(\frac{a}{s})^{1/2}]$ $\operatorname{Re}\sigma > -\frac{1}{4}$	$a^{-1/4} t^{\sigma-1} \left[\frac{(at)^{1/4}}{\Gamma(1/4+\sigma)}\ {}_0F_1(\frac{1}{4}+\sigma;\ at) \right.$ $\left. -\ \frac{2(at)^{3/4}}{\Gamma(3/4+\sigma)}\ {}_0F_1(3/4+\sigma;\ at) \right]$

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APPENDIX

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Since the behavior of $F(t)$ for large t is determined primarily by the behavior of the inverse Laplace transform of the comparison functions, it is helpful to know the asymptotic behavior of the $\Phi_\ell(t)$ $\ell = 1, 2, 3, 4$ for large t. For $\phi_1(s)$ we found that $\Phi_1(t)$ contained the factor $t^{-k-1/2} I_{2\mu}(2a^{1/2}t^{1/2})$. An asymptotic expansion of $I_\nu(z)$ for large z is known [11, p. 86].

$$I_\nu(z) = (2\pi z)^{-1/2} \{ e^z [\sum_{m=0}^{M-1} (-1)^m (\nu, m)(2z)^{-m} + O(|z|^{-M})$$

$$+ ie^{-z+i\pi\nu} [\sum_{m=0}^{M-1} (\nu, m)(2z)^{-m} + O(|z|^{-M})]\}$$

$$-\pi/2 < \arg z < 3\pi/2 .$$

Hence for large t we have

$$\Phi_1(t) \sim A_1 t^{-k-3/4} [e^{2a^{1/2}t^{1/2}} + ie^{-2a^{1/2}t^{1/2} + 2i\pi\mu}] .$$

Similarly, $\Phi_2(t)$ contains the factor $K_{2\mu}(2a^{1/2}t^{1/2})$ and again an asymptotic expansion for the modified Bessel function is known [11, p. 86],

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left[\sum_{m=0}^{M-1} (\nu, m)(2z)^{-m} + O(|z|^{-M}) \right)$$

$$-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2} \, .$$

Hence we have for the leading term in the asymptotic expansion of $\Phi_2(t)$

$$\Phi_2(t) \sim A_2 t^{-k-3/4} e^{-2a^{1/2} t^{1/2}} .$$

An asymptotic representation of $\Phi_3(t)$ requires asymptotic expansions for $J_\nu(z)$ and $Y_\nu(z)$ for large z [11, p. 85],

$$J_\nu(z) = \left(\frac{1}{2}\pi z\right)^{-1/2} \{\cos(z - 1/2\nu\pi - 1/4\pi)$$

$$\left[\sum_{m=0}^{M-1} (-1)^m (\nu, 2m)(2z)^{-2m} + O(|z|^{-2M}) \right]$$

$$- \sin(z - 1/2\nu\pi - 1/4\pi) \left[\sum_{m=0}^{M-1} (-1)^m (\nu, 2m+1)(2z)^{-2m-1} \right.$$

$$\left. + O(|z|^{-2M-1}) \right] \}$$

$$-\pi < \arg z < \pi$$

and

$$Y_\nu(z) = \left(\frac{1}{2}\pi z\right)^{-1/2} \{\sin(z - \frac{1}{2}\pi\nu - 1/4\,\pi)$$

$$\left[\sum_{m=0}^{M-1} (-1)^m (\nu, 2m)(2z)^{-2m} + 0(|z|^{-2M})\right]$$

$$+ \cos(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi)\left[\sum_{m=0}^{M-1} (-1)^m(\nu, 2m+1)(2z)^{-2m-1}\right.$$

$$\left. + O(|z|^{-2M-1})\right]\} .$$

Combining the above terms we obtain an asymptotic expression for $\Phi_3(t)$,

$$\Phi_3(t) \sim A_3 t^{k-3/4} \sin(2t^{1/2} - k\pi - \frac{1}{4}\pi) .$$

The asymptotic representations for generalized hypergeometric functions have been obtained by Fox [14] and Wright [27], [28]. Defining the generalized hypergeometric function by the series

$${}_pF_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n\beta_1)\Gamma(\alpha_2+n\beta_2)\cdots\Gamma(\alpha_p+n\beta_p)}{\Gamma(\rho_1+n\sigma_1)\Gamma(\rho_2+n\sigma_2)\cdots\Gamma(\rho_q+n\sigma_q)} \frac{z^n}{n!} ,$$

we see that in Table 6, the asymptotic values of $F(t)$ for large t are expressed as hypergeometric functions of the type ${}_1F_2(t)$

with $\beta_1 = 1$ and $\sigma_1 = \sigma_2 = 1$. We can specialize the parameters in an asymptotic expansion for the generalized hypergeometric function to obtain an asymptotic expansion for $\Phi_4(t)$. Using a theorem by Fox[14] we have

$$
\begin{aligned}
{}_1F_2(t) = \frac{t^{\frac{1}{2}\theta}}{2\pi^{1/2}} \Big\{ \Big[& \frac{A_1(0)}{(4t)^{1/2}} + \frac{A_2(0)}{4t} + \cdots \\
& + \frac{A_N(0)}{(4t)^{1/2N}} + o(t^{-N/2}) \Big] \exp(2t^{1/2}) \\
& + \Big[1 + \frac{A_1(1) e^{-\pi i}}{(4t)^{1/2}} + \frac{A_2(1) e^{-2\pi i}}{(4t)} + \cdots \\
& \frac{A_N(1) e^{-N\pi i}}{(4t)^{N/2}} + o(t^{-N/2}) \Big] \exp(2t^{1/2} e^{\pi i} + \theta \pi i) \Big\} \\
& + \sum_{\lambda=0}^{1} K_\lambda \; t^{-\alpha-\lambda} \, e^{i\pi(\alpha+\lambda)} + o(t^{\frac{\theta - N}{2}}),
\end{aligned}
$$

where $A_\lambda(\ell)$ is determined from the expansion

$$
\sqrt{2\pi}\,\frac{\Gamma(1+t)}{\Gamma(\rho_1+\ell+t)\Gamma(\rho_2+\ell+t)} = \sum_{\lambda=0}^{1} A_\lambda(\ell)\,\frac{\Gamma(t+1)}{\Gamma(t+2\ell+\lambda+1-\theta)}
$$

$$
\cdot\, O(t^{\theta-1}) \qquad -\pi < \arg t < \pi \,.
$$

Denoting the left hand side of the above equation by $H(t)$, then $K_\lambda z^{-\alpha+t+\lambda}$ is the residue of $z^{i}\Gamma(-t)\,H(t)/\sqrt{2\pi}$ at the pole $t = -(\alpha+\ell+\lambda)$. We have further denoted $\theta = \alpha - \rho_1 - \rho_2 + 1/2$

and λ as the largest positive integer such that $\mathrm{Re}\,[\,\theta + 2\alpha + 2\lambda\,] < N$, where N is any positive integer. The leading terms in the expression are then

$$ {}_1F_2(t) \sim \frac{\pi^{-1/2}}{2} \, [\exp(2it^{1/2} + i\pi\theta) + Az^{-1/2}\exp(2t^{1/2})] \, . $$

Hence we can determine the asymptotic behavior of $\Phi_4(t)$ and any of the special cases given in Table 6.