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The topic of summability methods has been studied by many although the name of G. H. Hardy and his classical work "Divergent Series" is best known. Almost all of the early work was done using single sequences or series. In the past 30 years research has been done extending some of these results to double sequences or double series by Cheng for the circular Riesz means and by Ustina for the Hausdorff means. In this paper we extend some of these results for the Quasi-Hausdorff means. The results and methods of attack closely follow those of Ishiguro and Ramanujan, who worked with Quasi-Hausdorff means for single sequences and single series. The terminology is fairly standard although some new definitions are needed.

We shall first develop the Quasi-Hausdorff transformation of double sequences and double series, next find conditions to make it a regular transformation, thirdly apply it to the partial sums of a

double Fourier series to check the Gibbs phenomenon, and conclude by investigating the Lebesgue constants of the method. It is noted that the class of weight functions used in the definition of the Quasi-Hausdorff means contains the probability distribution functions of two variables. Therefore the results contained in this research could possibly be used in the area of probability.

Gibbs Phenomenon and Lebesgue Constants for the Quasi-Hausdorff Means of Double Series

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GIBBS PHENOMENON AND LEBESGUE CONSTANTS FOR THE QUASI-HAUSDORFF MEANS OF DOUBLE SERIES

I. INTRODUCTION AND TERMINOLOGY AND RESULTS

Introduction

The topic of summability methods has been studied by many although the name of G.H. Hardy and his classical work "Divergent Series" is best known. Almost all of the early work was done using single sequences or series. In the past 30 years research has been done extending some of these results to double sequences or double series by Cheng [3] for the circular Riesz means and by Ustina [26] for the Hausdorff means. In this paper we entend some of these results for the Quasi-Hausdorff means. The results and methods of attack closely follow those of Ishiguro [10-14] and Ramanujan [21-23], who worked with Quasi-Hausdorff means for single sequences and single series. The terminology is fairly standard although some new definitions are needed.

We shall first develope the Quasi-Hausdorff transformation of double sequences and double series, next find conditions to make it a regular transformation, thirdly apply it to the partial sums of a double Fourier series to check the Gibbs phenomenon, and conclude by investigating the Lebesgue constants of the method. It is noted that the class of weight functions used in the definition of the Quasi-Hausdorff means contains the probability distribution functions of two variables. Therefore the results contained in this research could possibly be used in the area of probability.

Terminology and Results

Definition 1.1. Let Ω consist of all sets $R_{mn} = \{(i,j) | i = 0,1,2,\ldots,m;\ j = 0,1,2,\ldots,n\}$. We say the double series $\sum_{m,n} a_{mn}$ is Pringsheim convergent to A if given any positive number $\epsilon > 0$ there exists a set $F_\epsilon \in \Omega$ such that if $F \in \Omega$ and $F_\epsilon \subset F$ then

$$\Big| \sum_{(m,n)\in F} a_{mn} - A \Big| < \epsilon .$$

Definition 1.2. Let $f(x,t)$ be defined on the cell $[a_1, a_2] \times [b_1, b_2]$, and let $\{x_i\}$, $\{t_j\}$ be two sequences such that

$$a_1 = x_0 < x_1 < x_2 < \ldots < x_m = a_2 ,$$

$$b_1 = t_0 < t_1 < t_2 < \ldots < t_n = b_2 .$$

If the double sum

$$\sum_{i,j=1}^{m,n} |f(x_i, t_j) - f(x_i, t_{j-1}) - f(x_{i-1}, t_j) + f(x_{i-1}, t_{j-1})|$$

is uniformly bounded for all such sequences and if M is the least upper bound, and if for some fixed t, respectively fixed x, $f(x,t)$ is of bounded variation in the variable x, respectively t, $(x, t) \in [a_1, a_2] \times [b_1, b_2]$, then $f(x,t)$ is said to be of <u>bounded variation</u> on the cell $[a_1, a_2] \times [b_1, b_2]$ in the <u>Hardy-Krause</u> sense, and M is the total variation [9].

Hobson also notes that if $f(x,t)$ is of bounded variation in the Hardy-Krause sense then

$$f(x, t) = P(x, t) - N(x, t) - f(a_1, b_1) ,$$

where $P(x, t)$, $N(x, t)$ are the positive and negative variation functions of $f(x,t)$ on $[a_1, a_2] \times [b_1, b_2]$. If we fix the value of one of the variables, say $t = B$, then

$$f(x, B) = P(x, B) - N(x, B) - f(a_1, b_1)$$

is by definition a function of bounded variation in a single variable.

It follows from concepts in one variable that $f(x, B)$, $P(x, B)$, $N(x, B)$ have identical sets of points of discontinuity on the line $t = B$. In particular if $f(x, B)$ satisfies

$$\lim_{x \to a_1^+} f(x, B) = f(a_1, B) \qquad \text{for all} \quad b_1 \leq B \leq b_2$$

then

$$\lim_{x \to a_1^+} P(x, B) = P(a_1, B)$$

and

$$\lim_{x \to a_1^+} N(x, B) = N(a_1, B) \quad \text{for all} \quad b_1 \leq B \leq b_2 .$$

Definition 1. 3. A function $f(x, y)$ is said to be normalized if we have

$$f(a, b) = (1/4)\{f(a^+, b^+)+f(a^+, b^-)+f(a^-, b^+)+f(a^-, b^-)\}$$

where (a, b) is in the domain of $f(x, y)$; and if one of the coordinates is fixed, say y, then

$$f(a, y) = (1/2)\{f(a^+, y)+f(a^-, y)\} .$$

Definition 1.4. Let $\{\epsilon_n\}$ be a sequence of positive numbers such that

$$\lim_{n \to \infty} \epsilon_n = 0,$$

and let $\{p_k\}$ be sequence of natural numbers such that

$$\lim_{k \to \infty} p_k = \infty.$$

Let $\{S_{m,n}(x,y)\}$ be a sequence of real valued functions defined for

$$0 < |x-x_0| + |y-y_0| < \epsilon_n .$$

Then

$$\limsup_{\substack{m,n\to\infty \\ (x,y)\to(x_0,y_0)}} S_{mn}(x,y) = \lim_{k\to\infty} [\sup\{S_{m,n}(x,y) | m,n > p_k , \; 0 < |x-x_0| + |y-y_0| < \epsilon_k\}] ,$$

with $\liminf S_{m,n}(x,y)$ defined in a similar manner.

Definition 1.5. If the sequence $\{S_{m,n}(x,y)$ converges pointwise to a limit function $f(x,y)$ in the region $0 < |x-x_0| + |y-y_0| < \epsilon$, then $\{S_{m,n}(x,y)\}$ is said to exhibit the Gibbs' phenomenon at (x_0,y_0) if one or both of the following inequalities hold.

$$\limsup_{\substack{m,n\to\infty \\ (x,y)\to(x_0,y_0)}} S_{m,n}(x,y) > \limsup_{(x,y)\to(x_0,y_0)} f(x,y)$$

$$\liminf_{\substack{m,n\to\infty \\ (x,y)\to(x_0,y_0)}} S_{m,n}(x,y) < \liminf_{(x,y)\to(x_0,y_0)} f(x,y) .$$

Ustina [26] was able to show the following:

Lemma 1.6. Let $f(x,y)$ be a normalized function, periodic in each variable, and of bounded variation in the Hardy-Krause sense in the period rectangle. The Gibbs' phenomenon for $f(x, y)$ at the point $(x, y) = (0, 0)$ is the same as the Gibbs' phenomenon for the function

$$\chi(f; x, y) = (c/\pi^2)\phi(x)\phi(y) + g_1(0)\phi(x) + g_2(0)\phi(y)$$

where

$$\phi(t) = \begin{cases} 0 & t = 0 \\ (\pi - t)/2 & 0 < t < 2\pi \\ \phi(t+2k\pi) & k = \pm 1, \pm 2, \ldots \end{cases}$$

$$c = f(0^+, 0^+) - f(0^+, 0^-) - f(0^-, 0^+) + f(0^-, 0^-) ,$$

$$g_1(y) = (1/\pi)\{f(0^+, y) - f(0^-, y)\} - (c/2\pi) \operatorname{sgn} y$$

$$g_2(x) = (1/\pi)\{f(x, 0^+) - f(x, 0^-)\} - (c/2\pi) \operatorname{sgn} x .$$

<u>Definition 1.7.</u> The forward difference operator Δ is given by

$$\Delta h_n = h_n - h_{n+1} ,$$

and

$$\Delta_{p+1} h_n = \Delta(\Delta_p h_n) .$$

We note that Δ distributes across addition and thus we can write

$$\Delta_p h_n = \Delta(\Delta_{p-1} h_n) = \Delta_k(\Delta_{p-k} h_n) = \Delta_{p-k}(\Delta_k h_n) ,$$

thus showing that the operator Δ_r commutes with Δ_s for integers $r, s \geq 0$.

The <u>double</u> forward difference operator $\bar{\Delta}$ is given by

$$\bar{\Delta} h_{rs} = h_{rs} - h_{r+1,s} - h_{r,s+1} + h_{r+1,s+1} .$$

If we let $\Delta_{1,0}$ represent the forward difference operator which obeys

$$\Delta_{1,0} h_{rs} = h_{rs} - h_{r+1,s}$$

then

$$\bar{\Delta} h_{rs} = \Delta_{1,0}(h_{rs} - h_{r,s+1}) .$$

If we also let $\Delta_{0,1}$ be given by

$$\Delta_{0,1} h_{rs} = h_{rs} - h_{r,s+1}$$

then we see

$$\bar{\Delta} h_{rs} = \Delta_{1,0}(\Delta_{0,1} h_{rs}) .$$

Hence we shall denote the double forward difference operator $\bar{\Delta}$ by

$$\bar{\Delta} = \Delta_{1,1} = \Delta_{1,0}\Delta_{0,1} = \Delta_{0,1}\Delta_{1,0} .$$

In light of this we make the following definition.

<u>Definition 1.8.</u> The double <u>forward difference operator</u> $\Delta_{i,j}$ is given by

$$\Delta_{i,j}h_{rs} = \Delta_{i,0}\Delta_{0,j}h_{rs} = \Delta_{0,j}\Delta_{i,0}h_{rs} ,$$

where $\Delta_{i,0}$, respectively $\Delta_{0,j}$, is the (single) forward difference operator acting on the first, respectively second, variable subscript.

We note that other authors have used the symbols $\overset{1}{\Delta}_i$, $\overset{2}{\Delta}_j$ to write

$$\Delta_{i,j} = \overset{1}{\Delta}_i\overset{2}{\Delta}_j .$$

Using the definition of the (single) forward difference operator we see the following identities are valid:

(i) $\Delta_{p+1,0} = \Delta_{1,0}(\Delta_{p,0}) = \Delta_{p,0}(\Delta_{1,0})$

(ii) $\Delta_{0,q+1} = \Delta_{0,1}(\Delta_{0,q}) = \Delta_{0,q}(\Delta_{0,1})$

(iii) $\Delta_{p,q}\Delta_{r,s} = \Delta_{p+r,q+s} = \Delta_{i,j}\Delta_{p+r-i,q+s-j} = \Delta_{r,s}\Delta_{p,q}$

(iv) $\Delta_p h_m = \sum_{s=0}^{p} (-1)^s \binom{p}{s} h_{s+m}$

(v) $\Delta_{p,q}h_{mn} = \Delta_{p,0}(\Delta_{0,q}h_{mn}) = \Delta_{0,q}(\Delta_{p,0}h_{mn})$

$$= \Delta_{0,q} \sum_{s=0}^{p} (-1)^s \binom{p}{s} h_{s+m,n} =$$

$$= \sum_{s=0}^{p} (-1)^s \binom{p}{s} \sum_{r=0}^{q} (-1)^r \binom{q}{r} h_{s+m,\, r+n}$$

$$= \sum_{s,\, r=0}^{p,\, q} (-1)^{s+r} \binom{p}{s}\binom{q}{r} h_{s+m,\, r+n} .$$

Here, of course, $\binom{p}{s}$ is the binomial coefficient, which obeys the identity

$$\binom{p+1}{s} = \binom{p}{s} + \binom{p}{s-1} .$$

From the identities

$$\Delta_{i,j} h_{mn} = \Delta_{i,j} h_{m+1,n} + \Delta_{i+1,j} h_{mn}$$

$$\Delta_{i,j} h_{mn} = \Delta_{i,j} h_{m,n+1} + \Delta_{i,j+1} h_{mn}$$

Hildebrandt and Schoenberg [8] have shown that

$$h_{mn} = \sum_{r=m}^{p} \sum_{s=n}^{p} \binom{p-m}{r-m}\binom{p-n}{s-n} \Delta_{p-r,\, p-s} h_{rs} .$$

Hence

$$h_{0,0} = \sum_{r,\, s=0}^{p} \binom{p}{r}\binom{p}{s} \Delta_{p-r,\, p-s} h_{rs} .$$

Adams [1] improved the result to show

$$h_{0,0} = \sum_{r,s=0}^{p,q} \binom{p}{r}\binom{q}{s} \Delta_{p-r,q-s} h_{rs} .$$

The concept of the mean value of an almost periodic function is contained in the following lemma [5].

Lemma 1.9. If $f(x)$ is an almost periodic function, then there exists

$$\lim_{T \to \infty} \frac{1}{T} \int_{[a,a+T]} f(x)dx = \mathcal{M}\{f(x)\} ,$$

uniformly with respect to a. $\mathcal{M}\{f(x)\}$ is independent of a and is called the mean value of the almost periodic function $f(x)$.

Ishiguro used the mean value of an almost periodic function when he investigated the Lebesgue constants corresponding to the one dimensional Quasi-Hausdorff sequence to sequence transformation which is defined in the following manner:

Definition 1.10. The Quasi-Hausdorff summability matrix transforms the sequence $\{s_k\}$ into the sequence $\{h_n^*\}$ by means of the equation

$$h_n^* = \sum_{k=n}^{\infty} \binom{k}{n} s_k \int_{[0,1]} r^{n+1}(1-r)^{k-n} dg(r), \quad n = 0, 1, 2, \ldots$$

where $g(r)$ is of bounded variation on $[0, 1]$. This transformation is regular if and only if

$$\int_{[0,1]} dg(r) = g(1) - g(0) = 1 \quad \text{(Reference [21])}.$$

As the above is a Lebesgue-Stieltjes type integral we have need of two theorems stated in a Lebesgue-Stieltjes setting.

Dominated Convergence Theorem. If $\{f_n(x)\}$ is a sequence of functions defined and g-measurable on a set E, and there is a function $h(x)$ defined and g-summable on E such that $|f_n(x)| \leq h(x)$, and the functions $f_n(x)$ converge in measure or almost everywhere on E to a finite-valued limit function $f(x)$, then $f(x)$ is g-summable, and

$$\lim_{n \to \infty} \int_E f_n(x) dg(x) = \int_E f(x) dg(x),$$

and

$$\lim_{n \to \infty} \int_E |f_n(x) - f(x)| d\tau(x) = 0 ,$$

where $\tau(x)$ is the total variation function corresponding to $g(x)$ [19].

<u>Bounded Convergence Theorem</u>. If

(a) the functions $f_n(x)$ are defined, finite and g-summable on a set E^* of finite g-measure;

(b) the functions $f_n(x)$ converge in measure (or almost uniformly, or almost everywhere) on E^* to a limit function $f(x)$;

(c) for every positive ϵ there is a $\delta > 0$ such that $\int_E |f_n(x)| d\tau(x) < \epsilon$ for all n and for every subset E of E^* with $m_\tau E < \delta$;

(d) either the integrals $\int_{E^*} |f_n(x)| d\tau(x)$ are bounded, or $f(x)$ is finite almost everywhere;

then $f(x)$ is g-summable over E^*, and

$$\lim_{n \to \infty} \int_{E^*} f_n(x) dg(x) = \int_{E^*} f(x) dg(x),$$

and

$$\lim_{n \to \infty} \int_{E^*} |f_n(x) - f(x)| d\tau(x) = 0 \quad \text{(Reference [19]).}$$

Since the Quasi-Hausdorff means of a single sequence is defined by using two dimensional matrices it is to be expected that the means of a double sequence will be defined by using four dimensional matrices.

Definition 1.11. The matrix $\mathcal{P} = (\rho_{mnk\ell}) = ((-1)^{m+n}\binom{k}{m}\binom{\ell}{n})$ is termed a transposed difference matrix.

Definition 1.12. Let $\{\mu_{mn}\}$ be a given (double) sequence and let $\mathcal{M} = (\mu_{mnk\ell})$ be a "diagonal" matrix, the only nonzero elements being $\mu_{mnmn} \equiv \mu_{mn}$. The transformation matrix

$$H^* = \mathcal{P}\mathcal{M}\mathcal{P}$$

is called a Quasi-Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$.

We are able to show that the elements of any Quasi-Hausdorff matrix B corresponding to the sequence $\{\mu_{mn}\}$ must have the form

$$b_{mnk\ell} = \binom{k}{m}\binom{\ell}{n}\Delta_{k-m,\ell-n}\mu_{mn}.$$

By specializing the given sequence $\{\mu_{mn}\}$ to be a sequence of double moment constants corresponding to a weight function $g(u,v)$ we are able to find conditions for regularity of the resulting Quasi-Hausdorff transformation. Following Adams [2], Hildebrandt and Schoenberg [8] we have

Definition 1.13. The sequence $\{\mu_{mn}\}$, where

$$\mu_{mn} = \iint_{[0,1]\times[0,1]} u^m v^n dg(u,v), \quad m, n = 0, 1, 2, \ldots$$

is said to be a <u>sequence of double moment constants corresponding to the function $g(u,v)$</u>. Here $g(u,v)$ is of bounded variation in the sense of Hardy-Krause. If, in addition,

$$g(u,0) = g(u,0^+) = g(0,v) = g(0^+,v) = 0, \quad 0 \le u, v \le 1$$

$$g(1,1) - g(1,0) - g(0,1) + g(0,0) = 1,$$

then μ_{mn} is said to be a <u>regular moment constant</u>.

This places us in a position to prove

<u>Theorem 2.25.</u> The Quasi-Hausdorff matrix $H^*(\mu_{mn})$, which corresponds to the sequence $\{\mu_{mn}\}$, is an α_0 matrix (series to series regular) for convergent double series with bounded partial sums if $\{\mu_{mn}\}$ is a sequence of regular moment constants.

<u>Theorem 2.26.</u> A Quasi-Hausdorff matrix A is a T-matrix (sequence to sequence regular) if and only if

(a) μ_{mn} is a moment constant

(b) $$\iint_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv} = 1,$$

where $g(u,v)$ is a function which generates the sequence $\{\mu_{mn}\}$.

This theorem leads us to investigate the connection between Quasi-Hausdorff T-matrices and Quasi-Hausdorff α matrices. We find the following definition to logically follow from previous work:

Definition 1.14. The sequence to sequence Quasi-Hausdorff matrix, corresponding to the real double moment sequence $\{\mu_{mn}\}$, is given by means of the equation

$$h^*_{mn} = \sum_{k,\ell=m,n}^{\infty} \{\binom{k}{m}\binom{\ell}{n} \iint_{[0,1]\times[0,1]} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n} dg(u,v)\} s_{k\ell}$$

$$= \sum_{k,\ell=m,n}^{\infty} f_{mnk\ell}\, s_{k\ell} \,,$$

where $\{s_{k\ell}\}$ is the sequence being transformed into $\{h^*_{mn}\}$. The matrix

$$F = (f_{mnk\ell}) \,,$$

which will be denoted by

$$H^*(\mu_{m+1,n+1}) \,,$$

is the sequence to sequence Quasi-Hausdorff matrix.

Relations between $H^*(\mu_{mn})$, the series to series matrix, and $H^*(\mu_{m+1,n+1})$, the sequence to sequence matrix, lead to conditions for regularity of $H^*(\mu_{m+1,n+1})$.

We apply this transform to the sequence of partial sums of a double Fourier series of a function which is of bounded variation in the sense of Hardy-Krause, and study the convergence properties of the transformed sequence.

If $x = A$ is a jump discontinuity of a function then the sequence of graphs corresponding to the partial sums of the Fourier series of the function seem to "condense" on a line which is orthogonal to the x axis and passing through this point of discontinuity. If the length of this interval of "condensation" is larger than the jump of the function then the Gibbs' phenomenon is said to be present. Fejer [30] was able to show that if one considers the Cesàro $(C, 1)$ sums of the Fourier series then the Gibbs' phenomenon was not present at the point of discontinuity and in fact the Fourier series converges uniformly on every compact interval on which the function is continuous. We treat the corresponding problem for the Quasi-Hausdorff method.

We find

Proposition 1.15. The regular Quasi-Hausdorff means of the partial sums of the Fourier series of $f(x,y)$, where $f(x,y)$ is of bounded variation in the sense of Hardy-Krause, do not exhibit the Gibbs' phenomenon at any point of continuity of $f(x,y)$.

Proposition 3.9. For the two dimensional regular Quasi-Hausdorff means of the function $\phi(x,y)$ we have

$$\lim_{m,n\to\infty} h^*_{mn}(\phi; x_m, y_n)$$

$$= \iint_{[0,1]\times[0,1]} \left\{ \int_{[0,\tau]} \frac{\sin(y/u)}{y}\, dy \right\} \left\{ \int_{[0,\hat{\tau}]} \frac{\sin(y/v)}{y}\, dy \right\} dg(u,v)$$

provided $g(u, v)$ is continuous at the axes,

$$mx_m \to \tau \leq \infty, \quad ny_n \to \hat{\tau} \leq \infty$$

$$mx_m^2 \to 0, \quad ny_n^2 \to 0 \quad \text{as} \quad m, n \to \infty.$$

Here $h^*_{mn}(\phi; x_m, y_n)$ is the transformed sequence of partial sums of $\phi(x, y)$.

We conclude this research into the Quasi-Hausdorff means by finding a representation for the Lebesgue constants for this method.

It is well known that the growth rate of the Lebesgue constants for Fourier series plays a key role in the convergence properties of such series. For instance, the fact that these constants diverge implies, by the Uniform Boundedness Theorem, that there exists a continuous function whose Fourier series diverges at some point. Likewise the asymptotic behavior of the Lebesgue constants for the Quasi-Hausdorff method plays a basic role in the convergence properties of the Quasi-Hausdorff means.

The next theorem describes the growth rate of the Lebesgue constants for the Quasi-Hausdorff method.

Theorem 4.23. If the weight function $g(u, v)$ which generated the regular Quasi-Hausdorff matrix associated with the Lebesgue constant $L^*(M, N; g)$ is a function which is continuous and zero on some cross neighborhood $\{(x, y) \mid 0 \leq x \leq \delta \text{ or } 0 \leq y \leq \delta\}$ for some δ, then

$$L^*(M, N; g) = C^*(g) \ln M \ln N + o(\ln M \ln N), \quad M, N \to \infty,$$

where

$$C^*(g) = (4/\pi^2) \left| \iint_{\{1\}\times\{1\}} dg(u, v) \right| + (2/\pi^3)\{\mathcal{M}\{f_1\} + \mathcal{M}\{f_2\} + (\pi/2)\mathcal{M}\{f_3\}\},$$

where

$$f_1(w) = \left| \iint_{\{1\}\times[\delta, 1)} \sin(w/v)\, dg(u, v) \right|$$

$$f_2(z) = \left| \iint_{[\delta, 1)\times\{1\}} \sin(z/u)\, dg(u, v) \right|$$

$$f_2(z, w) = \left| \iint_{[\delta, 1)\times[\delta, 1)} \sin(z/u) \sin(w/v)\, dg(u, v) \right|$$

and it is assumed that these mean values exist.

Upon specializing the function $g(u, v)$ to be a countable linear combination of two dimensional interval functions we derive the two dimensional analogue of the one dimensional theorem proved by Ishiguro.

II. QUASI-HAUSDORFF MEANS AND REGULARITY OF THE TRANSFORMATION

In this chapter we give a development of the Quasi-Hausdorff means for the double sequences. Proofs are provided only for those theorems for which we have no references. In general, the theory is a logical extension of the corresponding theory for Quasi-Hausdorff means of a simple sequence.

Definition 2.1. Let $A = (a_{mnk\ell})$ be a four dimensional matrix, and let $S = (s_{mn})$ be a two dimensional matrix whose elements are the elements of the double sequence $\{s_{mn}\}$. The two dimensional matrix

$$T = AS \tag{A-2}$$

whose elements are the elements of the double sequence $\{t_{mn}\}$, where

$$t_{mn} = \sum_{k,\ell=0}^{\infty} a_{mnk\ell} s_{k\ell} \; ; \quad m, n = 0, 1, 2, \ldots \tag{A-3}$$

is meaningful for every m, n is a transformation of the matrix S. The matrix A is said to provide a sequence to sequence transformation or a series to series transformation or a series to sequence transformation according as it converts

$$\{s_{mn}\} \text{ to } \{t_{k\ell}\} \text{ or } \sum_{m,n} s_{mn} \text{ to } \sum_{k,\ell} t_{k\ell} \text{ or } \sum_{m,n} s_{mn} \text{ to } \{t_{k\ell}\}.$$

Definition 2.2. The sequence $\{s_{mn}\}$ is said to be summable by the matrix A to the sum t if t_{mn} exists for every m, n and if

$$\lim_{m,n \to \infty} t_{mn} = t < \infty$$

where convergence is in the sense of Pringsheim.

Definition 2.3. The transformation (A-2) is said to be

(i) convergence preserving if every convergent sequence $\{s_{mn}\}$ is transformed into a convergent sequence $\{t_{mn}\}$,

(ii) regular if in addition

$$\lim_{k,\ell \to \infty} t_{k\ell} = \lim_{m,n \to \infty} s_{mn} \tag{A-6}$$

(iii) totally regular if (A-6) holds even when s_{mn} diverges to positive or negative infinity.

Similar terminology is used for series to sequence and series to series transformations.

Definition 2.4. The matrix $A = (a_{mnk\ell})$ is said to be a

(i) K-matrix if it is sequence to sequence convergence preserving,

(ii) T-matrix if it is sequence to sequence regular,

(iii) β-matrix if it is series to sequence convergence preserving,

(iv) γ-matrix if it is series to sequence regular,

(v) δ-matrix if it is series to series convergence preserving,

(vi) α-matrix if it is series to series regular.

Lemma 2.5. The matrix $A = (a_{mnk\ell})$ is a K-matrix, for bounded convergent sequences $\{x_{mn}\}$ into bounded convergent sequences $\{y_{pq}\}$, if and only if

(i) $\lim_{m,n\to\infty} a_{mnk\ell} = \beta_{k\ell}$ (each k and ℓ)

(ii) $\lim_{m,n\to\infty} \sum_{k,\ell=0}^{\infty} a_{mnk\ell} = B$

(iii) $\sup_{m,n} \sum_{k,\ell=0}^{\infty} |a_{mnk\ell}|$ is finite

(iv) $\lim_{m,n\to\infty} \sum_{k=0}^{\infty} |a_{mnk\ell} - \beta_{k\ell}| = 0$, (for each ℓ)

(v) $\lim_{m,n\to\infty} \sum_{\ell=0}^{\infty} |a_{mnk\ell} - \beta_{k\ell}| = 0$, (for each k).

When the above is satisfied,

$$\lim_{m,n\to\infty} y_{mn} = Bx + \sum_{k,\ell=0}^{\infty} \beta_{k\ell}(x_{k\ell} - x),$$

where

$$x = \lim_{m,n\to\infty} x_{mn},$$

the series $\sum_{k,\ell} \beta_{k\ell}(x_{k\ell}-x)$ being always absolutely convergent. The matrix is a T-matrix, if, in addition, $\beta_{k\ell} = 0$ and $B = 1$ [24].

Lemma 2.6. The matrix $A = (a_{mnk\ell})$ transforms bounded sequences $\{x_{mn}\}$ into bounded sequences $\{y_{k\ell}\}$ if and only if

$$\sup_{m,n} \sum_{k,\ell=0}^{\infty} |a_{mnk\ell}| \quad \text{if finite [24].}$$

Lemma 2.7. The matrix $A = (a_{mnk\ell})$ is a β-matrix for convergent double series with bounded partial sums if and only if

(i) $\sum_{i,j=0}^{\infty} |\Delta_{11} a_{mnij}| < \mathcal{K}_{mn}$ $(m, n=0, 1, 2, \ldots)$

(ii) $\lim_{j\to\infty} \Delta_{10} a_{mnij} = 0,$ (for all i; $m, n = 0, 1, 2, \ldots$)

(iii) $\lim_{i\to\infty} \Delta_{01} a_{mnij} = 0,$ (for all j; $m, n = 0, 1, 2, \ldots$).

The matrix A is a γ-matrix if and only if, in addition,

(iv) $\lim_{m,n\to\infty} \sum_{j=0}^{\infty} |\Delta_{11} a_{mnij}| = 0,$ (for all i)

(v) $\lim_{m,n\to\infty} \sum_{i=0}^{\infty} |\Delta_{11} a_{mnij}| = 0,$ (for all j)

(vi) $\lim_{m,n\to\infty} a_{mnij} = 1,$ $(i, j = 0, 1, 2, \ldots)$

(vii) $\sum_{i,j=0}^{\infty} |\Delta_{11} a_{mnij}| < \mathcal{K},$ (independent of m, n, but only for $m \geq M$, $n \geq N$) [20].

<u>Proposition 2.8</u>. The matrix $A = (a_{mnk\ell})$ is an α-matrix if and only if the matrix $G = (g_{mnk\ell})$ is a γ-matrix, where the elements of G are defined by

$$g_{mnk\ell} = \sum_{r,s=0}^{m,n} a_{rsk\ell} .$$

<u>Proof</u>. Let

$$v_{mn} = \sum_{k,\ell=0}^{\infty} a_{mnk\ell} u_{k\ell} \tag{4.1}$$

exist for $m, n = 0, 1, 2, \ldots$

and

$$\sigma_{pq} = \sum_{m,n=0}^{p,q} v_{mn} . \tag{4.2}$$

But

$$\sigma_{pq} = \sum_{m,n=0}^{p,q} \sum_{k,\ell=0}^{\infty} a_{mnk\ell} u_{k\ell}$$

$$= \sum_{k,\ell=0}^{\infty} \sum_{m,n=0}^{p,q} a_{mnk\ell} \quad u_{k\ell} \; .$$

Let

$$g_{pqk\ell} = \sum_{m,n=0}^{p,q} a_{mnk\ell} \; . \tag{4.3}$$

Then

$$\sigma_{pq} = \sum_{k,\ell=0}^{\infty} g_{pqk\ell} u_{k\ell} \; . \tag{4.4}$$

Thus (4. 1) implies the existence of (4. 4).

Conversely if (4. 4) exists for each p, q then

$$\sigma_{mn} - \sigma_{m,n-1} - \sigma_{m-1,n} + \sigma_{m-1,n-1}$$

$$= \sum_{k,\ell=0}^{\infty} \{g_{mnk\ell} - g_{m,n-1,k\ell} - g_{m-1,nk\ell} + g_{m-1,n-1,k\ell}\} u_{k\ell}$$

$$= \sum_{k,\ell=0}^{\infty} a_{mnk\ell} u_{k\ell} \, , \qquad \text{by using (4. 3)}$$

$$= v_{mn} \; .$$

Hence the existence of (4. 4) implies the existence of (4. 1) and the validity of (4. 2), (4. 3).

Now let A be a regular series to series transformation of $\sum u_{k\ell}$ into $\sum v_{mn}$, and let $\sum v_{mn} = s$. Using (4.2) we then see $\sigma_{pq} \to s$ as $p, q \to \infty$, so that the matrix G whose elements are given by (4.3) defines a regular series to sequence transformation given by (4.4). On the other hand if G defines a regular series to sequence transformation then (4.4) yields

$$\sigma_{pq} \to s \quad \text{as} \quad p, q \to \infty,$$

and so by (4.2)

$$\sum_{m,n=0}^{\infty} v_{mn} = s ;$$

hence A defines a regular series to series transformation.

Definition 2.9.

(i) An α-matrix with $\lim_{k,\ell \to \infty} a_{mnk\ell} = 0$ for all m, n is said to be an α_0-matrix.

(ii) A γ-matrix with $\lim_{k,\ell \to \infty} g_{mnk\ell} = 0$ for all m, n is said to be a γ_0-matrix.

Corollary 2.8. The matrix A is an α_0-matrix if and only if the matrix G is a γ_0-matrix.

Proof. If $\lim_{k,\ell \to \infty} a_{mnk\ell} = 0$ then (4.3) implies

$\lim_{k,\ell \to \infty} g_{mnk\ell} = 0$. On the other hand if $\lim_{k,\ell \to \infty} g_{mnk\ell} = 0$ then since

$$g_{mnk\ell} - g_{m,n-1,k\ell} - g_{m-1,nk\ell} + g_{m-1,n-1,k\ell} = a_{mnk\ell} \qquad (4.5)$$

we see that $\lim_{k,\ell \to \infty} a_{mnk\ell} = 0$.

Definition 2.10. The matrix $\rho = (\rho_{mnk\ell})$, whose elements are defined by

$$\rho_{mnk\ell} = \begin{cases} (-1)^{m+n}\binom{k}{m}\binom{\ell}{n}, & k \geq m,\ \ell \geq n \\ 0, & \text{otherwise} \end{cases}$$

we shall term a transposed difference matrix.

Proposition 2.11. The transposed difference matrix is its own inverse.

Proof. Let

$$h_{mnrs} = \sum_{k,\ell=0}^{\infty} \rho_{mnk\ell}\rho_{k\ell rs}, \quad \text{thus} \quad H = \rho\rho.$$

We shall apply the matrix H to the double sequence $\{u_{rs}\}$ and show that the transformed sequence is the same. We find

$$\sum_{r,s=0}^{\infty} h_{mnrs} u_{rs} = \sum_{r,s=0}^{\infty} \sum_{k,\ell=0}^{\infty} \rho_{mnk\ell} \rho_{k\ell rs} u_{rs}$$

$$= \sum_{r,s=0}^{\infty} u_{rs} (-1)^{m+n} \sum_{k,\ell=m,n}^{\infty} (-1)^{k+\ell} \binom{k}{m}\binom{\ell}{n}\binom{r}{k}\binom{s}{\ell}$$

$$= \sum_{r,s=0}^{\infty} u_{rs} (-1)^{m+n} \binom{s}{n}\binom{r}{m} \sum_{k,\ell=m,n}^{r,s} (-1)^{k+\ell} \binom{r-m}{k-m}\binom{s-n}{\ell-n}$$

$$= u_{rs}$$

since

$$\sum_{k,\ell=m,n}^{r,s} (-1)^{k+\ell} \binom{r-m}{k-m}\binom{s-n}{\ell-n} = \begin{cases} (-1)^{m+n}, & r = m,\ s = n \\ 0, & \text{otherwise} \end{cases} .$$

Definition 2.12. Let $\{\mu_{mn}\}$ be a given sequence and let $\mathcal{M} = (\mu_{mnk\ell})$ be a "diagonal matrix" whose only nonzero elements are $\mu_{mnmn} = \mu_{mn}$. The transformation matrix

$$H^* = \rho \mathcal{M} \rho$$

is called a Quasi-Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$.

The sequence $\{s_{mn}\}$ is said to be summable to s in the Quasi-Hausdorff sense corresponding to the sequence $\{\mu_{mn}\}$, if the sequence $\{t_{mn}\}$, where

$$T = H^*S,$$

approaches $\underline{s}$ as m, n become infinite.

The matrix H^* is well defined since if we let:

(i) $H^* = \rho[\mu\rho]$ then the elements of $[\mu\rho]$ are given by

$$\sum_{r,s=0}^{\infty} \mu_{k\ell rs}\rho_{rsmn} = \mu_{k\ell k\ell}\rho_{k\ell mn}$$

because μ is a "diagonal" matrix. Thus the elements of $\rho[\mu\rho]$ are given by

$$\sum_{k,\ell=0}^{\infty} \rho_{rsk\ell}\mu_{k\ell k\ell}\rho_{k\ell mn} = \sum_{k,\ell=m,n}^{r,s} \rho_{rsk\ell}\mu_{k\ell k\ell}\rho_{k\ell mn}.$$

(ii) $H^* = [\rho\mu]\rho$ then similarly

$$\sum_{m,n=0}^{\infty} \rho_{rsmn}\mu_{mnk\ell} = \rho_{rsk\ell}\mu_{k\ell k\ell},$$

and so $[\rho\mu]\rho$ has elements given by

$$\sum_{k,\ell=0}^{\infty} \rho_{rsk\ell}\mu_{k\ell k\ell}\rho_{k\ell mn} = \sum_{k,\ell=m,n}^{r,s} \rho_{rsk\ell}\mu_{k\ell k\ell}\rho_{k\ell mn}.$$

We note since the matrix $\mathcal{M}$ defined above has its "diagonal elements" described by μ_{mnmn} that if we define the transpose of the four dimensional matrix $A = (a_{mnk\ell})$ by $A^t = (a_{mnk\ell})^t = (a_{k\ell mn})$, then $(H^*)^t$, where H^* is as previously defined, is a Hausdorff two dimensional transformation matrix. We also note that since the transposed difference matrix is its own inverse it is trivial to show that Quasi-Hausdorff matrices commute.

The following is an example of a Quasi-Hausdorff matrix:

Let $A = (a_{mnk\ell})$ where

$$a_{mnk\ell} = \begin{cases} 1/(k+1)(\ell+1), & k \geq m, \quad \ell \geq n \\ 0\ , & \text{otherwise.} \end{cases}$$

To show that this is a Quasi-Hausdorff matrix consider the sequence

$$\mu_{mn} = \mu_{mnmn} = 1/(m+1)(n+1).$$

Let C be the Quasi-Hausdorff matrix corresponding to this sequence. Then its elements are of the form

$$c_{mnk\ell} = \sum_{r,s=0}^{\infty} \rho_{mnrs}\mu_{rs}\rho_{rsk\ell}\ .$$

We shall show that $C = A$.

We apply the matrix C to the double sequence $\{u_{rs}\}$. We find

$$\sum_{k,\ell=0}^{\infty} c_{mnk\ell}u_{k\ell} = \sum_{k,\ell=0}^{\infty}\sum_{r,s=0}^{\infty} \rho_{mnrs}\mu_{rs}\rho_{rsk\ell}u_{k\ell}$$

$$= \sum_{k,\ell=0}^{\infty}\sum_{r,s=0}^{\infty} (-1)^{m+n}(-1)^{r+s}\binom{r}{m}\binom{s}{n}\mu_{rs}\binom{k}{r}\binom{\ell}{s}u_{k\ell}$$

$$= \sum_{k,\ell=0}^{\infty} (-1)^{m+n}u_{k\ell}\binom{k}{m}\binom{\ell}{n}\sum_{r,s=m,n}^{k,\ell} (-1)^{r+s}\binom{k-m}{r-m}\binom{\ell-n}{s-n}\mu_{rs} \tag{A-13}$$

$$= \sum_{k,\ell=m,n}^{\infty} (-1)^{m+n}u_{k\ell}\binom{k}{m}\binom{\ell}{n}\sum_{r,s=m,n}^{k,\ell} (-1)^{r+s}\binom{k-m}{r-m}\binom{\ell-n}{s-n}\frac{1}{r+1}\,\frac{1}{s+1}$$

$$= \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n}u_{k\ell}\sum_{r,s=m,n}^{k,\ell} (-1)^{r+s-m-n}\binom{k-m}{r-m}\binom{\ell-n}{s-n}$$

$$\times \int_{[0,1]} u^r du \int_{[0,1]} v^s dv$$

$$= \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n}u_{k\ell} \int_{[0,1]} u^m(1-u)^{k-m}du \int_{[0,1]} v^n(1-v)^{\ell-n}dv$$

$$= \sum_{k,\ell=m,n}^{\infty} u_{k\ell}\binom{k}{m}\binom{\ell}{n}B(m+1,k-m+1)B(n+1,\ell-n+1) =$$

where $B(,)$ is the Beta function

$$= \sum_{k,\ell=m,n}^{\infty} u_{k\ell} \frac{1}{k+1} \frac{1}{\ell+1}$$

upon using $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Hence $c_{mnk\ell} = a_{mnk\ell}$, and we now see that A is a Quasi-Hausdorff matrix.

The above example also helps to generate the form of an arbitrary element of a Quasi-Hausdorff matrix.

<u>Theorem 2.13.</u> A matrix $B = (b_{mnk\ell})$ is a Quasi-Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$ if and only if its elements have the form

$$b_{mnk\ell} = \binom{k}{m}\binom{\ell}{n} \sum_{r,s=0}^{k-m,\ell-n} (-1)^{r+s} \binom{k-m}{r}\binom{\ell-n}{s} \mu_{r+m,s+n}$$

$$= \binom{k}{m}\binom{\ell}{n} \Delta_{k-m,\ell-n}\mu_{mn} .$$

<u>Proof.</u> Let $B = \rho\mu\rho$ be a Quasi-Hausdorff matrix. Applying this to a double sequence matrix S we have $T = BS$ where

$$t_{mn} = \sum_{k,\ell=0}^{\infty} b_{mnk\ell} s_{k\ell} .$$

Using (A-13) of the example with $s_{k\ell}$ replacing $u_{k\ell}$ we find

$$t_{mn} = \sum_{k,\ell=0} \binom{k}{m}\binom{\ell}{n} \sum_{p,q=m,n}^{k,\ell} (-1)^{p-m+q-n} \binom{k-m}{p-m}\binom{\ell-n}{q-n} \mu_{pq} \quad s_{k\ell}$$

$$= \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} \sum_{r,s=0}^{k-m,\ell-n} (-1)^{r+s} \binom{k-m}{r}\binom{\ell-n}{s} \mu_{r+m,\, s+n} \quad s_{k\ell}$$

Therefore

$$b_{mnk\ell} = \binom{k}{m}\binom{\ell}{n} \sum_{r,s=0}^{k-m,\,\ell-n} (-1)^{r+s} \binom{k-m}{r}\binom{\ell-n}{s} \mu_{r+m,\, s+n}$$

$$= \binom{k}{m}\binom{\ell}{n} \Delta_{k-m,\,\ell-n} \mu_{mn}$$

by definition of the difference operator.

Then upon reversing the steps we have the remaining part of the statement.

Definition 2. 14. We call

$$\mu_{mn} = \iint_{[0,\,1]\times[0,\,1]} u^m v^n dg(u,v), \quad m,n = 0,1,2,\ldots$$

a sequence of double moment constants corresponding to the function g(u, v), if $g(u,v)$ is of bounded variation in the sense of Hardy-Krause in $[0,1] \times [0,1]$. If in addition,

$g(u,0) = g(u,0^+) = g(0^+,v) = g(0,v) = 0$, for (u,v) in $[0,1]\times[0,1]$ and $\{g(1,1)-g(1,0)-g(0,1)+g(0,0)\} = 1$, then μ_{mn} is said to be <u>regular</u> [2].

For notational purposes we will write $H^*(\mu_{mn})$ to represent the Quasi-Hausdorff matrix which is generated by a sequence $\{\mu_{mn}\}$.

We now find conditions for $H^*(\mu_{mn})$ to be a regular matrix (Theorem 2.25).

We shall show that when μ_{mn} is a regular moment constant then $H^*(\mu_{mn})$ is a regular series to series matrix. We use Proposition 2.8 and Lemma 2.7.

<u>Proposition 2.15</u>. If

$$g_{mnij} = \sum_{r,s=0}^{m,n} h_{rsij} ,$$

where h_{rsij} is an element of H^*, then

$$\Delta_{11} g_{mnij} = \binom{i}{m}\binom{j}{n}\Delta_{i-m,j-n}\mu_{m+1,n+1} .$$

<u>Proof</u>. By definition

$$h_{rsij} = \binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{rs} .$$

Thus

$$\Delta_{11}h_{rsij} = \binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{rs} - \binom{i}{r}\binom{j+1}{s}\Delta_{i-r,j+1-s}\mu_{rs}$$

$$- \binom{i+1}{r}\binom{j}{s}\Delta_{i+1-r,j-s}\mu_{rs} + \binom{i+1}{r}\binom{j+1}{s}\Delta_{i+1-r,j+1-s}\mu_{rs}$$

$$= (1) + (2) + (3) + (4) .$$

Making use of the relation $\binom{q+1}{z} = \binom{q}{z} + \binom{q}{z-1}$ we rewrite

$$(2) = \binom{i}{r}\binom{j}{s}\Delta_{i-r,j+1-s}\mu_{rs} + \binom{i}{r}\binom{j}{s-1}\Delta_{i-r,j+1-s}\mu_{rs} \quad \text{(A)}$$

and

$$(4) = \binom{i+1}{r}\binom{j}{s}\Delta_{i+1-r,j+1-s}\mu_{rs} + \binom{i+1}{r}\binom{j}{s-1}\Delta_{i+1-r,j+1-s}\mu_{rs} . \quad \text{(B)}$$

We now use the relation $\Delta_{1,t+1} = \Delta_{1,t}(\Delta_{0,1})$ on the first terms of (A) and (B) to arrive at

$$(A_1) = \binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{rs} - \binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{r,s+1}$$

$$+ \binom{i}{r}\binom{j}{s-1}\Delta_{i-r,j+1-s}\mu_{rs}$$

and

$$(B_1) = \binom{i+1}{r}\binom{j}{s}\Delta_{i+1-r,j-s}\mu_{rs} - \binom{i+1}{r}\binom{j}{s}\Delta_{i+1-r,j-s}\mu_{r,s+1}$$

$$+ \binom{i+1}{r}\binom{j}{s-1}\Delta_{i+1-r,j+1-s}\mu_{rs} .$$

We note that the first terms of (A_1) and (B_1) are precisely terms (1) and (3).

Combining these results, making use of the relations between binomial numbers we arrive at

$$\Delta_{11} h_{rsij} = \binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{r+1,s+1} - \binom{i}{r}\binom{j}{s-1}\Delta_{i-r,j+1-s}\mu_{r+1,s}$$
$$- \binom{i}{r-1}\binom{j}{s}\Delta_{i+1-r,j-s}\mu_{r,s+1} + \binom{i}{r-1}\binom{j}{s-1}\Delta_{i+1-r,j+1-s}\mu_{rs} .$$

We sum on r, s:

$$\sum_{r,s=0}^{m,n} \Delta_{11} h_{rsij} = \sum_{s=0}^{n} \binom{j}{s} \sum_{r=0}^{m} \left\{ \binom{i}{r}\Delta_{i-r,j-s}\mu_{r+1,s+1} - \binom{i}{r-1}\Delta_{i+1-r,j-s}\mu_{r,s+1}\right\}$$
$$- \sum_{s=0}^{n} \binom{j}{s-1} \sum_{r=0}^{m} \left\{ \binom{i}{r}\Delta_{i-r,j+1-s}\mu_{r+1,s} - \binom{i}{r-1}\Delta_{i+1-r,j+1-s}\mu_{rs}\right\}$$
$$= \binom{i}{m} \sum_{s=0}^{n} \left\{ \binom{j}{s}\Delta_{i-m,j-s}\mu_{m+1,s+1} - \binom{j}{s-1}\Delta_{i-m,j+1-s}\mu_{m+1,s}\right\}$$
$$= \binom{i}{m}\binom{j}{n}\Delta_{i-m,j-n}\mu_{m+1,n+1} .$$

<u>Proposition 2.16</u>. If $g_{mnk\ell}$ is as defined in Proposition 2.15 then

$$\lim_{m,n\to\infty} g_{mnij} = \mu_{00} .$$

Proof. By definition

$$\lim_{m,n\to\infty} g_{mnij} = \lim_{m,n\to\infty} \sum_{r,s=0}^{m,n} \binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{rs} ,$$

$$= \sum_{r,s=0}^{i,j} \binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{rs}$$

$$= \mu_{00}$$

by definition of the difference operator and binomial numbers.

Proposition 2.17. If μ_{mn} is a regular moment constant then

$$\lim_{j\to\infty} \binom{i}{m}\binom{j}{q}\Delta_{i-m,j-q}\mu_{m+1,q} = 0, \quad \text{fixed } q$$

$$\lim_{i\to\infty} \binom{i}{p}\binom{j}{n}\Delta_{i-p,j-n}\mu_{p,n+1} = 0, \quad \text{fixed } p.$$

Proof. Considering the first relationship we have

$$\binom{i}{m}\binom{j}{q}\Delta_{i-m,j-q}\mu_{m+1,q} = \iint_{[0,1]\times[0,1]} \binom{i}{m}\binom{j}{q} u^{m+1}(1-u)^{i-m} v^{q}(1-v)^{j-q} dg(u,v).$$

Now

$$\binom{j}{q} v^q (1-v)^{j-q} \le \sum_{q=0}^{j} \binom{j}{q} v^q (1-v)^{j-q} = 1,$$

so that for a δ chosen such that $0 < \delta < 1$ we can write

$$0 \le \binom{i}{m}\binom{j}{q} \left| \Delta_{i-m,\, j-q} \mu_{m+1,\, q} \right|$$

$$\le \iint_{[0,1]\times[0,\delta]} u \left| dg(u,v) \right| + \iint_{[0,1]\times[\delta,1]} u \binom{j}{q} (1-\delta)^{j-q} \left| dg(u,v) \right|$$

$$\le \iint_{[0,1]\times[0,\delta]} \{dP(u,v) + dN(u,v)\}$$

$$+ \iint_{[0,1]\times[\delta,1]} \binom{j}{q} (1-\delta)^{j-q} \{dP(u,v) + dN(u,v)\}$$

where $P(u,v)$ and $N(u,v)$ are the positive and negative variations of the function $g(u,v)$.

$$\le \{P(1,\delta) - P(1,0) + P(0,0) - P(0,\delta) + N(1,\delta) - N(1,0)$$

$$+ N(0,0) - N(0,\delta)\} + \binom{j}{q} (1-\delta)^{j-q} \mathrm{Var}\, g .$$

Because $\binom{j}{q}(1-\delta)^{j-q} \to 0$ as $j \to \infty$ and by the continuity of $P(1,\delta), P(0,\delta), N(1,\delta), N(0,\delta)$ at $\delta = 0$ it follows that

$$\lim_{j} \sup \binom{j}{q}\binom{i}{m} |\Delta_{i-m,j-q}\mu_{m+1,q}| = 0.$$

A symmetric argument yields the second relationship in the theorem.

Continuing we have a theorem which will yield the result that the only Quasi-Hausdorff matrices which are regular come from moment sequences.

Theorem 2.18. In order that $\{\mu_{rs}\}$ be a sequence of double moment constants it is necessary and sufficient that

$$\sup_{m,n} \sum_{r,s=m,n}^{\infty} \binom{r}{m}\binom{s}{n} |\Delta_{r-m,s-n}\mu_{m+1,n+1}| < \infty \tag{A}$$

$$\sup_{m} \sum_{r=m}^{\infty} \binom{r}{m} |\Delta_{r-m,0}\mu_{m+1,0}| < \infty \tag{A_1}$$

$$\sup_{n} \sum_{s=n}^{\infty} \binom{s}{n} |\Delta_{0,s-n}\mu_{0,n+1}| < \infty \tag{A_2}$$

Proof. Hildebrant and Schoenberg [8] have shown that $\{\mu_{rs}\}$ is a sequence of double moment constants if and only if

$$\sup_{p} \sum_{m,n=0}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{mn}| < \infty. \tag{B}$$

Ramanujan [21] has shown that $\{\mu_r\}$ is a sequence of moment constants if and only if

$$\sup_m \sum_{r=m}^{\infty} \binom{r}{m} |\Delta_{r-m}\mu_{m+1}| < \infty \tag{a}$$

while Kuttner [16] has shown this is also true if and only if

$$\sup_p \sum_{r=0}^{p} \binom{p}{r} |\Delta_{p-r}\mu_r| < \infty. \tag{b}$$

We first show that (A), (A_1), (A_2) imply (B). By a series of propositions we will show that (A) implies

$$\sup_p \sum_{m,n=1}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}| < \infty . \tag{B_3}$$

<u>Proposition 2.19</u>. If $0 \le k, \ell \le p-1$ and $M \ge 2p-k-1$, $N \ge 2p-\ell-1$ then

$$\Delta_{p-k,p-\ell}\mu_{k\ell} = \sum_{m=p-k}^{M-p+1} \binom{m-1}{p-k-1} \sum_{q=p-\ell}^{N-p+1} \binom{q-1}{p-\ell-1} \Delta_{m,q}\mu_{p,p}$$

$$+ \sum_{m=p-k}^{M-p+1} \binom{m-1}{p-k-1} \sum_{s=\ell-1}^{p-2} \binom{N-p+1}{s-\ell+1} \Delta_{m,N-s}\mu_{p,s+1} \quad +$$

$$+\sum_{r=k-1}^{p-2}\binom{M-p+1}{r-k+1}\sum_{t=p-\ell}^{N-p+1}\binom{t-1}{p-\ell-1}\Delta_{M-r,t}\mu_{r+1,p}$$

$$+\sum_{r=k-1}^{p-2}\binom{M-p+1}{r-k+1}\sum_{v=\ell-1}^{p-2}\binom{N-p+1}{v-\ell+1}\Delta_{M-r,N-v}\mu_{r+1,v+1}\,.$$

<u>Proof</u>. By induction on M and N. Call the right hand side

$$(1a) + (1b) + (1c) + (1d).$$

For $N = 2p-\ell-1$, $M = 2p-k-1$ we see that

$$(1a) = \binom{p-k-1}{p-k-1}\Delta_{p-k,p-\ell}\mu_{pp}\,.$$

$$(1b) = \sum_{s=\ell-1}^{p-2}\binom{p-k-1}{p-k-1}\binom{p-\ell}{s-\ell+1}\Delta_{p-k,2p-\ell-1-s}\mu_{p,s+1}$$

$$(1c) = \sum_{r=k-1}^{p-2}\binom{p-\ell-1}{p-\ell-1}\binom{p-k}{r-k+1}\Delta_{2p-k-1-r,p-1}\mu_{r+1,p}$$

$$(1d) = \sum_{r=k-1}^{p-2}\sum_{v=\ell-1}^{p-2}\binom{p-k}{r-k+1}\binom{p-\ell}{v-\ell+1}\Delta_{2p-k-1-r,2p-\ell-1-v}\mu_{r+1,v+1}\,.$$

Summing we find the above yields

$$\sum_{r=k-1}^{p-1} \sum_{v=\ell-1}^{p-1} \binom{p-k}{r-k+1}\binom{p-\ell}{v-\ell+1} \Delta_{2p-k-1-r,\, 2p-\ell-1-v}\, \mu_{r+1,\, v+1}$$

$$= \Delta_{p-k,\, p-\ell}\, \mu_{k\ell} .$$

Now let us assume the result is true for N and show true for $N+1$; by symmetry of M and N in the statement of the proposition this will also establish the result for M. Consider (1b):

$$\sum_{m=p-k}^{M-p+1} \sum_{v=\ell-1}^{p-2} \binom{m-1}{p-k-1}\binom{N-p+1}{v-\ell+1} \Delta_{m,\, N-v}\, \mu_{p,\, v+1}$$

$$= \sum_{m=p-k}^{M-p+1} \sum_{v=\ell-1}^{p-2} \binom{m-1}{p-k-1}\binom{N-p+1}{v-\ell+1} \Delta_{m,\, N-v+1}\, \mu_{p,\, v+1}$$

$$+ \sum_{m=p-k}^{M-p+1} \sum_{v=\ell-1}^{p-2} \binom{m-1}{p-k-1}\binom{N-p+1}{v-\ell+1} \Delta_{m,\, N-v}\, \mu_{p,\, v+2}$$

$= (1b1) + (1b2)$ say.

In (1b1) we let $T = v$, while in (1b2) $T = v+1$. With these substitutions and some rearranging of terms we find

$$(1b) = \sum_{m=p-k}^{M-p+1} \sum_{T=\ell}^{p-2} \binom{m-1}{p-k-1} \Delta_{m,N-T+1}\mu_{p,T+1} \left\{ \binom{N-p+1}{T-\ell+1} + \binom{N-p+1}{T-1} \right\}$$

$$+ \sum_{m=p-k}^{M-p+1} \binom{m-1}{p-k-1} \left\{ \binom{N-p+1}{0} \Delta_{m,N-\ell+1+1}\mu_{p,\ell-1+1} + \binom{N-p+1}{p-\ell-1} \Delta_{m,N-p+1+1}\mu_{p,p+1-1} \right\}$$

$$= \sum_{m=p-k}^{M-p+1} \sum_{T=\ell}^{p-2} \binom{m-1}{p-k-1} \Delta_{m,N-T+1}\mu_{p,T+1} \binom{N-p+2}{T-\ell+1}$$

$$+ \sum_{m=p-k}^{M-p+1} \binom{m-1}{p-k-1} \left\{ \Delta_{m,N-\ell+2}\mu_{p,\ell} + \binom{N-p+1}{p-\ell-1} \Delta_{m,N-p+2}\mu_{p,p} \right\}.$$

We call this last result (1B).

Now proceeding with (1d) we find in a similar manner

$$(1d) = \sum_{r=k-1}^{p-2} \sum_{T=\ell}^{p-2} \binom{M-p+1}{r-k+1} \binom{N-p+2}{T-\ell+1} \Delta_{M-r,N-T+1}\mu_{r+1,T+1}$$

$$+ \sum_{r=k-1}^{p-2} \binom{M-p+1}{r-k+1} \left\{ \Delta_{M-r,N-\ell+2}\mu_{r+1,\ell} + \binom{N-p+1}{p-\ell-1} \Delta_{M-r,N-p+2}\mu_{r+1,p} \right\}.$$

We call this result (1D).

Shifting to (1a) we find, upon adding a term and then subtracting it back out,

$$(1a) = \sum_{m=p-k}^{M-p+1} \sum_{q=p-\ell}^{(N+1)-p+1} \binom{m-1}{p-k-1}\binom{q-1}{p-\ell-1} \Delta_{m,q}\mu_{p,p}$$

$$- \sum_{m=p-k}^{M-p+1} \binom{m-1}{p-k-1}\binom{N+1-p+1-1}{p-\ell-1} \Delta_{m,N-p+2}\mu_{p,p} .$$

We note that the second term above is the negative of the last term in (1B).

Finally (1c) yields in an analogous fashion

$$(1c) = \sum_{r=k-1}^{p-2} \sum_{t=p-\ell}^{(N+1)-p+1} \binom{M-p+1}{r-k+1}\binom{t-1}{p-\ell-1} \Delta_{M-r,t}\mu_{r+1,p}$$

$$- \sum_{r=k-1}^{p-2} \binom{M-p+1}{r-k+1}\binom{N+1-p+1-1}{p-\ell-1} \Delta_{M-r,N-p+2}\mu_{r+1,p} .$$

Again the second term is the negative of the last term in (1D).

Hence in recombining the terms (1a, B, c, D) we find

$$\Delta_{p-k,p-\ell}\mu_{k,\ell} = \sum_{m=p-k}^{M-p+1} \sum_{q=p-\ell}^{(N+1)-p+1} \binom{m-1}{p-k-1}\binom{q-1}{p-\ell-1} \Delta_{m,q}\mu_{p,p}$$

$$+ \sum_{m=p-k}^{M-p+1} \sum_{T=\ell-1}^{p-2} \binom{m-1}{p-k-1}\binom{(N+1)-p+1}{T-\ell+1} \Delta_{m,(N+1)-T}\mu_{p,T+1} +$$

$$+\sum_{r=k-1}^{p-2}\ \sum_{s=p-\ell}^{(N+1)-p+1} \binom{M-p+1}{r-k+1}\binom{s-1}{p-\ell-1}\Delta_{M-r,s}\mu_{r+1,p}$$

$$+\sum_{r=k-1}^{p-2}\ \sum_{T=\ell-1}^{p-2} \binom{M-p+1}{r-k+1}\binom{(N+1)-p+1}{T-\ell+1}\Delta_{M-r,N+1-T}\mu_{r+1,T+1} .$$

The result obtained is precisely the statement in the lemma with N replaced by $(N+1)$. Hence the lemma is true for all M, N.

Proposition 2.20. If $\{\mu_{k\ell}\}$ is a sequence of constants then

$$\sup_n \sum_{m=n}^{\infty} \binom{m}{n}|\Delta_{m-n,z}\mu_{n+1,t}| < \infty$$

if and only if

$$\sup_R \sum_{m=0}^{R} \binom{R}{m}|\Delta_{R-m,z}\mu_{m,t}| < \infty, \quad \text{fixed } z,t .$$

Similar results hold for $|\Delta_{z,m-n}\mu_{t,n+1}|$ and $|\Delta_{z,R-m}\mu_{t,m}|$.

Proof. The proof is a straight forward modification of a proof by Kuttner [16] for a sequence of constants $\{\mu_k\}$ and will not be given. In his proof Kuttner was able to establish the result for $p \geq 1$.

$$\sum_{m=1}^{p} \binom{p}{m} |\Delta_{p-m}\mu_m| \leq \sum_{n=0}^{\infty} \binom{n+p-1}{p-1} |\Delta_n \mu_p| .$$

The modification of his proof yields for $p \geq 1$

$$\sum_{k=1}^{p} \binom{p}{k} |\Delta_{p-k,0}\mu_{k,p}| \leq \sum_{r=1}^{\infty} \binom{r+p-1}{p-1} |\Delta_{r,0}\mu_{pp}| \tag{2A}$$

and

$$\sum_{\ell=1}^{p} \binom{p}{\ell} |\Delta_{0,p-\ell}\mu_{p,\ell}| \leq \sum_{s=1}^{\infty} \binom{s+p-1}{p-1} |\Delta_{0,s}\mu_{pp}| \tag{2B}$$

<u>Proposition 2.21</u>. If $\{\mu_{mn}\}$ is a double sequence and (A) is true and the conditions of Proposition 2.19 are satisfied then

$$\sum_{m=p-k}^{M-p+1} \sum_{T=\ell-1}^{p-2} \binom{m-1}{p-k-1} \binom{N-p+1}{T-\ell+1} \Delta_{m,N-T}\mu_{p,T+1} \tag{1b}$$

and

$$\sum_{r=k-1}^{p-2} \sum_{s=p-\ell}^{N-p+1} \binom{M-p+1}{r-k+1} \binom{s-1}{p-\ell-1} \Delta_{M-r,s}\mu_{r+1,p} \tag{1c}$$

both tend to zero as M, N tend to infinity, for fixed p, k, ℓ.

<u>Proof</u>. We will only show that (1b) tends to zero as M, N tend to infinity for fixed p, k, ℓ as (1c) follows by a similar argument.

To this end we observe that since (A) is assumed true then

$$\sum_{N=s}^{\infty} \binom{N}{s} \sum_{R=z}^{\infty} \binom{R}{z} |\Delta_{R-z,\,N-s}\mu_{z+1,\,s+1}|$$

is finite. Hence

$$\lim_{N \to \infty} \binom{N}{s} \sum_{R=z}^{\infty} \binom{R}{z} |\Delta_{R-z,\,N-s}\mu_{z+1,\,s+1}|$$

is zero for fixed s, z. Therefore any finite sum

$$\sum_{s=\ell-1}^{z-1} \binom{N}{s} \sum_{R=z}^{\infty} \binom{R}{z} |\Delta_{R-z,\,N-s}\mu_{z+1,\,s+1}|$$

tends to zero as M, N tend to infinity for fixed z, ℓ.

Let $Q = z-k+1$, hence Q will be fixed when z, k are fixed. The finite sum becomes

$$\sum_{s=\ell-1}^{Q+k-2} \binom{N}{s} \sum_{R=Q+k-1}^{\infty} \binom{R}{Q+k-1} |\Delta_{R-(Q+k-1),\,N-s}\mu_{Q+k,\,s+1}|$$

which is identical, upon letting $m = R-(Q+k-1)$, to

$$\sum_{s=\ell-1}^{Q+k-2} \binom{N}{s} \sum_{m=0}^{\infty} \binom{m+Q+k-1}{Q+k-1} |\Delta_{m,\,N-s}\mu_{Q+k,\,s+1}| .$$

This double series still tends to zero as M, N tend to infinity for fixed k, ℓ, Q. But since

$$\binom{m}{Q} \leq \binom{m+Q+k-1}{Q+k-1}$$

we find

$$\sum_{s=\ell-1}^{Q+k-2} \binom{N}{s} \sum_{m=0}^{\infty} \binom{m}{Q} |\Delta_{m, N-s}\mu_{Q+k, s+1}|$$

tends to zero as M, N tend to infinity for fixed k, ℓ, Q.

Consider now (1b). Since

$$\binom{m}{p-k} \geq \binom{m-1}{p-k-1} \quad \text{and} \quad \binom{N}{T} \geq \binom{N-p+1}{T-\ell+1}$$

we see

$$|(1b)| \leq \sum_{T=\ell-1}^{p-2} \binom{N}{T} \sum_{m=p-k}^{\infty} \binom{m}{p-k} |\Delta_{m, N-T}\mu_{p, T+1}|$$

$$= \sum_{T=\ell-1}^{p-2} \binom{N}{T} \sum_{m=0}^{\infty} \binom{m}{p-k} |\Delta_{m, N-T}\mu_{p, T+1}| \; .$$

Substitute $W = p-k$, or $p = W+k$, to yield

$$= \sum_{T=\ell-1}^{W+k-2} \binom{N}{T} \sum_{m=0}^{\infty} \binom{m}{W} |\Delta_{m, N-T}\mu_{W+k, T+1}| \; .$$

As M, N tend to infinity we have previously found this double series tends to zero for fixed W, k, ℓ, hence for fixed p, k, ℓ. Thus (1b) tends to zero as M, N tend to infinity for fixed p, k, ℓ.

Proposition 2.22. If $\{\mu_{k\ell}\}$ is a double sequence and (A) is true and the conditions of Proposition 2.19 are satisfied then

$$\sum_{r=k-1}^{p-2} \binom{M-p+1}{r-k+1} \sum_{v=\ell-1}^{p-2} \binom{N-p+1}{v-\ell+1} \Delta_{M-r,N-v}\mu_{r+1,v+1} \tag{1d}$$

tends to zero as M, N tend to infinity for fixed p, k, ℓ.

Proof. Since (A) is assumed true then

$$\lim_{M,N\to\infty} \binom{N}{v}\binom{M}{r} |\Delta_{M-r,N-v}\mu_{r+1,v+1}| = 0$$

But with

$$\binom{M}{r} \geq \binom{M-p+1}{r-k+1}, \quad \binom{N}{v} \geq \binom{N-p+1}{v-l+1}$$

we see

$$|(1d)| \leq \sum_{r=k-1}^{p-2} \binom{M}{r} \sum_{v=\ell-1}^{p-2} \binom{N}{v} |\Delta_{M-r,N-v}\mu_{r+1,v+1}|$$

where the right hand side tends to zero as M, N tend to infinity for fixed k, ℓ, p. Therefore (1d) tends to zero also under those conditions.

Proposition 2.23. If $\{\mu_{k\ell}\}$ is a double sequence and (A) is true and the conditions of Proposition 2.19 are satisfied then

$$\Delta_{p-k,p-\ell}\mu_{k,\ell} = \sum_{\substack{r=p-k\\ s=p-\ell}}^{\infty} \binom{r-1}{p-k-1}\binom{s-1}{p-\ell-1}\Delta_{r,s}\mu_{p,p}$$

for fixed k,ℓ and $p \geq 1$.

Proof. Follows immediately by taking the limit as M, N tend to infinity of the result in Proposition 2.19 and applying the results of Propositions 2.21, 2.22.

Proposition 2.24. If the conditions of Proposition 2.23 are satisfied than

$$\sum_{m,n=1}^{p-1} \binom{p}{m}\binom{p}{n}|\Delta_{p-m,p-n}\mu_{m,n}| \leq \sum_{r,s=1}^{\infty} \binom{r+p-1}{p-1}\binom{s+p-1}{p-1}|\Delta_{rs}\mu_{p,p}| .$$

Proof. It follows from Proposition 2.23 that

$$\sum_{m,n=1}^{p-1} \binom{p}{m}\binom{p}{n}|\Delta_{p-m,p-n}\mu_{m,n}|$$

$$= \sum_{m,n=1}^{p-1} \binom{p}{m}\binom{p}{n}\left|\sum_{\substack{r=p-m\\ s=p-n}}^{\infty} \binom{r-1}{p-m-1}\binom{s-1}{p-n-1}\Delta_{r,s}\mu_{p,p}\right| \leq$$

$$\leq \sum_{m,n=1}^{p-1} \binom{p}{m}\binom{p}{n} \sum_{\substack{r=p-m \\ s=p-n}}^{\infty} \binom{r-1}{p-m-1}\binom{s-1}{p-n-1} |\Delta_{r,s}\mu_{p,p}|$$

$$\leq \sum_{r,s=1}^{\infty} |\Delta_{r,s}\mu_{p,p}| \sum_{\substack{m=\max\{1,p-r\} \\ n=\max\{1,p-s\}}}^{p-1} \binom{p}{m}\binom{p}{n}\binom{r-1}{p-m-1}\binom{s-1}{p-n-1}$$

$$\leq \sum_{r,s=1}^{\infty} |\Delta_{r,s}\mu_{p,p}| \sum_{\substack{m=\max\{0,p-r\} \\ n=\max\{0,p-s\}}}^{p-1} \binom{p}{m}\binom{p}{n}\binom{r-1}{p-m-1}\binom{s-1}{p-n-1}$$

$$\leq \sum_{r,s=1}^{\infty} |\Delta_{r,s}\mu_{p,p}| \binom{r+p-1}{p-1}\binom{s+p-1}{p-1} .$$

We are now in a position to finish the first part of the proof of Theorem 2. 18.

We consider the double series in (B_3):

$$\sum_{m,n=1}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}|$$

$$\leq |\mu_{p,p}| + \sum_{m,n=1}^{p-1} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}|$$

$$+ \sum_{m=1}^{p-1} \binom{p}{m} |\Delta_{p-m,0}\mu_{m,p}| + \sum_{n=1}^{p-1} \binom{p}{n} |\Delta_{0,p-n}\mu_{p,n}| .$$

It follows from (2A), (2B) that the last two terms obey

$$\sum_{m=1}^{p-1} \binom{p}{m} |\Delta_{p-m,0}\mu_{m,p}| \leq \sum_{r=1}^{\infty} \binom{r+p-1}{p-1} |\Delta_{r,0}\mu_{p,p}|$$

and

$$\sum_{n=1}^{p-1} \binom{p}{n} |\Delta_{0,p-n}\mu_{p,n}| \leq \sum_{s=1}^{\infty} \binom{s+p-1}{p-1} |\Delta_{0,s}\mu_{p,p}|$$

while by Proposition 2.24 the second term obeys

$$\sum_{m,n=1}^{p-1} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}| \leq \sum_{r,s=1}^{\infty} \binom{r+p-1}{p-1}\binom{s+p-1}{p-1} |\Delta_{r,s}\mu_{p,p}| .$$

Thus

$$\sum_{m,n=1}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}|$$

$$\leq |\mu_{p,p}| + \sum_{r,s=1}^{\infty} \binom{r+p-1}{p-1}\binom{s+p-1}{p-1} |\Delta_{r,s}\mu_{p,p}|$$

$$+ \sum_{r=1}^{\infty} \binom{r+p-1}{p-1} |\Delta_{r,0}\mu_{p,p}| + \sum_{s=1}^{\infty} \binom{s+p-1}{p-1} |\Delta_{0,s}\mu_{p,p}|$$

$$\leq \sum_{r,s=0}^{\infty} \binom{r+p-1}{p-1}\binom{s+p-1}{p-1} |\Delta_{r,s}\mu_{p,p}| .$$

Since (A) is assumed true we know there exists a constant K such that

$$\sum_{R,S=A,B}^{\infty} \binom{R}{A}\binom{S}{B} |\Delta_{R-A,S-B}\mu_{A+1,B+1}| < K.$$

Let $R-A = r$, $S-B = s$, $A = B$, $p = A+1$. Then (A) implies

$$\sum_{r,s=0}^{\infty} \binom{r+p-1}{p-1}\binom{s+p-1}{p-1} |\Delta_{r,s}\mu_{p,p}| \leq K; \quad p \geq 1.$$

Therefore we see

$$\sum_{m,n=1}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}| \leq K.$$

Thus (A) implies (B_3).

By hypothesis

$$\sup_m \sum_{k=m}^{\infty} \binom{k}{m} |\Delta_{k-m,0}\mu_{m+1,0}| < \infty \qquad (A_1)$$

$$\sup_n \sum_{\ell=n}^{\infty} \binom{\ell}{n} |\Delta_{0,\ell-n}\mu_{0,n+1}| < \infty. \qquad (A_2)$$

Using Ramanujan's result in one dimensional theory then yields that $\{\mu_{m,0}\}$ and $\{\mu_{0,n}\}$ are sequences of moment constants, i.e., there exist two functions of bounded variation $\chi_1(t)$, $\chi_2(w)$ such that

$$\mu_{m,0} = \int_{[0,1]} t^m d\chi_1(t), \quad \mu_{0,n} = \int_{[0,1]} w^n d\chi_2(w), \quad m, n = 0, 1, 2, \ldots$$

and furthermore

$$\int_{[0,1]} d\chi_1(t) = \int_{[0,1]} d\chi_2(w) = \mu_{00}.$$

Let $\chi_3(t, w)$ be any function of bounded variation in the sense of Hardy-Krause which satisfies

$$\chi_3(t, 0) = -\chi_1(t) + \chi_1(0), \quad 0 \leq t \leq 1$$

$$\chi_3(0, w) = -\chi_2(w) + \chi_2(0), \quad 0 \leq w \leq 1.$$

One such function is given by defining it to be zero for all other values in the unit square. Define a sequence of moment constants corresponding to such a function $\chi_3(t, w)$ by

$$\nu_{m,n} = \iint_{[0,1]\times[0,1]} t^m w^n d\chi_3(t, w), \quad m,n = 0,1,2,\ldots$$

Then

$$\nu_{m,0} = \mu_{m,0} \quad \text{and} \quad \nu_{0,n} = \mu_{0,n}.$$

By the theorem of Hildebrant and Schoenberg the sequence $\{\nu_{m,n}\}$ satisfies

$$\sup_p \sum_{m,n=0}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\nu_{m,n}| < \infty .$$

Hence

$$\sup_p \sum_{m=0}^{p} \binom{p}{m}\binom{p}{0} |\Delta_{p-m,p}\nu_{m,0}| < \infty$$

and

$$\sup_p \sum_{n=0}^{p} \binom{p}{0}\binom{p}{n} |\Delta_{p,p-n}\nu_{0,n}| < \infty .$$

Replacing $\nu_{m,0}$, $\nu_{0,n}$ by $\mu_{m,0}$, $\mu_{0,n}$ respectively we find that for $m = 0$

$$\sum_{n=1}^{p} \binom{p}{n} |\Delta_{p,p-n}\mu_{0,n}| \leq K$$

while for $n = 0$

$$\sum_{m=1}^{p} \binom{p}{m} |\Delta_{p-m,p}\mu_{m,0}| \leq K .$$

When both are zero we see

$$\Delta_{p,p}\mu_{0,0} = \mu_{0,0} - \sum_{m,n=1}^{p} \binom{p}{m}\binom{p}{n} \Delta_{p-m,p-n}\mu_{m,n}$$

$$- \sum_{n=1}^{p} \binom{p}{n} \Delta_{p,p-n}\mu_{0,n} - \sum_{m=1}^{p} \binom{p}{m} \Delta_{p-m,p}\mu_{m,0} .$$

Therefore

$$|\Delta_{p,p}\mu_{0,0}| \le |\mu_{0,0}| + K + K + K .$$

Hence combining these results we find

$$\sum_{m,n=0}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}|$$

$$= |\Delta_{p,p}\mu_{0,0}| + \sum_{m,n=1}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}|$$

$$+ \sum_{n=1}^{p} \binom{p}{n} |\Delta_{p,p-n}\mu_{0,n}| + \sum_{m=1}^{p} \binom{p}{m} |\Delta_{p-m,p}\mu_{m,0}|$$

$$\le 6K + |\mu_{0,0}| .$$

Taking the supremum of both sides yields

$$\sup_{p} \sum_{m,n=0}^{p} \binom{p}{m}\binom{p}{n} |\Delta_{p-m,p-n}\mu_{m,n}| < \infty. \qquad \text{(B)}$$

Thus (A), (A_1), (A_2) imply (B).

Now to show that (B) implies (A), (A_1), (A_2). From the identity

$$\Delta_{k-m,\ell-n}\mu_{m+1,n+1} = \sum_{\substack{t=m+1\\ s=n+1}}^{R-k+m+1,R-\ell+n+1} \binom{R-k}{t-m-1}\binom{R-\ell}{s-n-1}\Delta_{R+1-t,R+1-s}\mu_{t,s}$$

we find the partial sum

$$\sum_{k,\ell=m,n}^{R} \binom{k}{m}\binom{\ell}{n}\left|\Delta_{k-m,\ell-n}\mu_{m+1,n+1}\right|$$

$$\leq \sum_{k,\ell=m,n}^{R} \binom{k}{m}\binom{\ell}{n}\left\{\sum_{t=m+1,\, s=n+1}^{R-k+m+1,R-\ell+n+1} \binom{R-k}{t-m-1}\binom{R-\ell}{s-n-1}\left|\Delta_{R+1-t,R+1-s}\mu_{t,s}\right|\right\}$$

$$\leq \sum_{\substack{t=m+1\\ s=n+1}}^{R+1} \binom{R+1}{t}\binom{R+1}{s}\left|\Delta_{R+1-t,R+1-s}\mu_{t,s}\right| .$$

Since (B) is assumed true this last double series is bounded for all R. We let R tend to infinity and obtain

$$\sup_{m,n} \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n}\left|\Delta_{k-m,\ell-n}\mu_{m+1,n+1}\right| < \infty .$$

Thus (B) implies (A).

On the other hand (B) also implies that the double sequence $\{\mu_{m,n}\}$ is a moment constant sequence so that

$$\mu_{m,n} = \iint_{[0,1]\times[0,1]} u^m v^n d\chi(u,v), \quad m,n = 0,1,2,\ldots$$

and so

$$\mu_{m,0} = \iint_{[0,1]\times[0,1]} u^m d\chi(u,v) = \int_{[0,1]} u^m d(\chi(u,1)-\chi(u,0))$$

$$\mu_{0,n} = \iint_{[0,1]\times[0,1]} v^n d\chi(u,v) = \int_{[0,1]} v^n d(\chi(1,v)-\chi(0,v)) .$$

Since $\chi(u,v)$ is of bounded variation in the sense of Hardy-Krause we know that both $\chi(u,1) - \chi(u,0)$, $\chi(1,v) - \chi(0,v)$ are of bounded variation and so $\{\mu_{m,0}\}$, $\{\mu_{0,n}\}$ are single sequences of moment constants. Ramanujan's results (a) then imply

$$\sup_m \sum_{k=m}^{\infty} \binom{k}{m} |\Delta_{k-m,0}\mu_{m+1,0}| < \infty \qquad (A_1)$$

$$\sup_n \sum_{\ell=n}^{\infty} \binom{\ell}{n} |\Delta_{0,\ell-n}\mu_{0,n+1}| < \infty \qquad (A_2)$$

are both true. Thus (B) implies (A), (A_1), (A_2); Theorem 2.18 is now proved.

<u>Theorem 2.25.</u> The Quasi-Hausdorff matrix $H^*(\mu_{m,n})$ is an α_0-matrix for convergent double series with bounded partial sums if $\{\mu_{m,n}\}$ is a sequence of regular moment constants.

<u>Proof.</u> By Proposition 2.8 we see that $H^*(\mu_{m,n})$ is an α-matrix if and only if

$$G = (g_{mnij} = \sum_{r,s=0}^{m,n} h_{rsij})$$

is a γ-matrix. Thus we need to satisfy the conditions of Lemma 2.7 with out elements g_{mnij}. We find

a) conditions (i) and (vii) are true by Theorem 2.18 and Proposition 2.15,

b) conditions (ii) and (iii) are shown true by using the arguments in the proof of Proposition 2.17. We have

$$\Delta_{1,0} g_{mnij} = \sum_{r,s=0}^{m,n} \Delta_{1,0} h_{rsij}$$

where

$$h_{rsij} = \binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{r,s}$$

$$= \iint_{[0,1]\times[0,1]} \binom{i}{r}\binom{j}{s} u^r(1-u)^{i-r} v^s (1-v)^{j-s} dg(u,v) .$$

By the techniques used in Proposition 2.17, it follows that

$$0 \leq \left|\binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{r,s}\right|$$

$$\leq \iint_{[0,1]\times[0,1]} \{dP(u,v)+dN(u,v)\} + \binom{j}{s}(1-\delta)^{j-s}\mathrm{Var}\ g$$

and by continuity that

$$\lim_{j\to\infty} \sup\left|\binom{i}{r}\binom{j}{s}\Delta_{i-r,j-s}\mu_{r,s}\right| = 0 .$$

A similar argument holds for $\Delta_{0,1}g_{mnij}$.

c) conditions (iv) and (v) are trivially true since for $m > i$ or $n > j$ we have

$$\Delta_{11}g_{mnij} = 0,$$

d) condition (vi) uses Proposition 2.16 and is satisfied if $\mu_{00} = 1$.

By the regularity of $\mu_{m,n}$ we have

$$\mu_{0,0} = \iint_{[0,1]\times[0,1]} dg(u,v) = 1.$$

Hence $H^*(\mu_{m,n})$ is an α-matrix. In order to be an α_0-matrix we need G to be a γ_0-matrix. But by definition G is a γ_0-matrix if and only if

$$\lim_{i,j \to \infty} g_{mnij} = 0.$$

This is valid whenever

$$\lim_{i,j \to \infty} h_{rsij} = 0,$$

which follows again by arguments analogous to those in Proposition 2.17.

Thus for regular moment constants $\mu_{m,n}$ we do have $H^*(\mu_{m,n})$ being an α_0-matrix.

Theorem 2.26. A Quasi-Hausdorff matrix A is a T-matrix if and only if

(a) $\mu_{m,n}$ is a moment constant

(b) $$\iint_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv} = 1,$$

where $g(u,v)$ is a function which generates the sequence $\{\mu_{m,n}\}$.

Proof. We show that (a), (b) true implies the Quasi-Hausdorff matrix A is a T-matrix. Since A is Quasi-Hausdorff we know its elements have the form

$$a_{mnk\ell} = \begin{cases} \binom{k}{m}\binom{\ell}{n}\Delta_{k-m,\ell-n}\mu_{m,n}, & k \geq m,\ \ell \geq n \\ 0, & \text{otherwise,} \end{cases}$$

We first show that (a) implies

$$\sum_{k,\ell=0}^{\infty} a_{mnk\ell} = \iint_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv},$$

if it exists.

Case one. $\Delta g(u,v) \geq 0$. Then using the definition of $\Delta_{pq}\mu_{mn}$ we find

$$\sum_{k,\ell=0}^{\infty} a_{mnk\ell} = \sum_{k,\ell=0}^{\infty} \left\{ \binom{k}{m}\binom{\ell}{n} \sum_{r,s=0}^{k-m,\ell-n} (-1)^{r+s}\binom{k-m}{r}\binom{\ell-n}{s} \times \iint_{[0,1]\times[0,1]} u^{r+m}v^{s+n}dg(u,v) \right\}$$

$$= \sum_{k,\ell=0}^{\infty} \binom{k}{m}\binom{\ell}{n} \iint_{[0,1]\times[0,1]} u^m(1-u)^{k-m}v^n(1-v)^{\ell-n}dg(u,v)$$

$$= \iint_{[0,1]\times[0,1]} \sum_{k,\ell=0}^{\infty} \binom{k}{m}\binom{\ell}{n} u^m(1-u)^{k-m}v^n(1-v)^{\ell-n}dg(u,v)$$

$$= \iint_{(0,1]\times(0,1]} \left(\frac{u^m}{(1-(1-u))^{m+1}}\right)\left(\frac{v^n}{(1-(1-v))^{n+1}}\right)dg(u,v) =$$

$$= \iint_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv} .$$

Case two. $\Delta g(u,v)$ not always positive. Here we consider the two series

$$\sum_{k,\ell=0}^{\infty} \binom{k}{m}\binom{\ell}{n} \iint_{[0,1]\times[0,1]} u^m(1-u)^{k-m}v^n(1-v)^{\ell-n}dP(u,v) \qquad (13.1)$$

and

$$\sum_{k,\ell=0}^{\infty} \binom{k}{m}\binom{\ell}{n} \iint_{[0,1]\times[0,1]} u^m(1-u)^{k-m}v^n(1-v)^{\ell-n}dN(u,v) \qquad (13.2)$$

where $P(u,v)$ and $N(u,v)$ are the positive and negative variations of $g(u,v)$. Using the same techniques as in Case one we find these to be equal to

$$\iint_{(0,1]\times(0,1]} \frac{dP(u,v)}{uv} \quad \text{and} \quad \iint_{(0,1]\times(0,1]} \frac{dN(u,v)}{uv} .$$

Now these integrals both exist if and only if

$$\iint_{(0,1]\times(0,1]} \frac{|dg(u,v)|}{uv}$$

exists, which by a theorem in Lebesgue-Stieljes integration [19], will exist if and only if

$$\iint\limits_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv}$$

exists. Since (13.1), (13.2) both converge absolutely we can subtract to find

$$\sum_{k,\ell=0}^{\infty} \binom{k}{m}\binom{\ell}{n} \iint\limits_{(0,1]\times(0,1]} u^m(1-u)^{k-m}v^n(1-v)^{\ell-n}\{dP(u,v)-dN(u,v)\}$$

$$= \sum_{k,\ell=0}^{\infty} a_{mnk\ell}$$

On the other hand

$$(13.1) - (13.2) = \iint\limits_{(0,1]\times(0,1]} \frac{dP(u,v)}{uv} - \iint\limits_{(0,1]\times(0,1]} \frac{dN(u,v)}{uv}$$

$$= \iint\limits_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv}.$$

Thus again

$$\sum_{k,\ell=0}^{\infty} a_{mnk\ell} = \iint\limits_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv}.$$

if the integral is finite.

Now we show that if (b) is also true then A is a T-matrix. Considering the conditions for a T-matrix, Lemma 2.5, we see that

(i) $\lim_{m,n\to\infty} a_{mnk\ell} = 0$ is trivially satisfied;

(ii) $\lim_{m,n\to\infty} \sum_{k,\ell=0}^{\infty} a_{mnk\ell} = 1$, because (b) is assumed true and

$$\iint_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv} = \sum_{k,\ell=0}^{\infty} a_{mnk\ell} ;$$

(iii) $\sup_{m,n} \sum_{k,\ell=0}^{\infty} |a_{mnk\ell}|$ finite. Since

$$\sum_{k,\ell=0}^{\infty} |a_{mnk\ell}| \le \iint_{(0,1]\times(0,1]} \frac{|dg(u,v)|}{uv}$$

and since

$$\iint_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv} = 1$$

implies

$$\iint_{(0,1]\times(0,1]} \frac{|dg(u,v)|}{uv}$$

exists we find (iii) is satisfied;

(iv), (v)
$$\lim_{m,n\to\infty} \sum_{k=0}^{\infty} |a_{mnk\ell}| = 0 \quad \text{for each } \ell$$

$$\lim_{m,n\to\infty} \sum_{\ell=0}^{\infty} |a_{mnk\ell}| = 0 \quad \text{for each } k$$

are trivially true for elements of A.

Thus conditions (a), (b) imply that A is a T-matrix.

On the other hand if A is a Quasi-Hausdorff T-matrix then by Lemma 2.5(iii) we see

$$\sup_{m,n} \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} |\Delta_{k-m,\ell-n}\mu_{m,n}|$$

is finite. Define a sequence $\{\nu_{mn}\}$ by

$$\nu_{mn} = \begin{cases} \mu_{m-1,n-1}, & m \geq 1,\ n \geq 1 \\ 0\ , & \text{either or } n = 0. \end{cases}$$

Then $\mu_{m,n} = \nu_{m+1,n+1}$ and

$$\sup_{m,n} \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} |\Delta_{k-m,\ell-n}\nu_{m+1,n+1}| < \infty \qquad (A)$$

$$\sup_{m} \sum_{k=m}^{\infty} \binom{k}{m} |\Delta_{k-m,o}\nu_{m+1,0}| = 0 < \infty \qquad (A_1)$$

$$\sup_n \sum_{\ell=n}^{\infty} \binom{\ell}{n} |\Delta_{0,\ell-n} \nu_{0,n+1}| = 0 < \infty \qquad (A_2)$$

Thus Theorem 2. 18 states that $\{\nu_{mn}\}$ is a sequence of moment constants, and by a theorem of Hildebrant and Schoenberg [8], a function $h(u, v)$ of bounded variation in the sense of Hardy-Krause exists such that

$$\nu_{mn} = \iint_{[0, 1]\times[0, 1]} u^m v^n \, dh(u, v), \quad m, n = 0, 1, 2, \ldots$$

But then

$$\mu_{mn} = \nu_{m+1,n+1} = \iint_{[0, 1]\times[0, 1]} u^m v^n (uv \, dh(u, v)).$$

We define a function $g(u, v)$ by

$$g(u, v) = \iint_{[0,u]\times[0,v]} \xi\eta \, dh(\xi, \eta).$$

This function is of bounded variation since

$$dg(u, v) = uv \, dh(u, v)$$

and

$$\iint_{[0,1]\times[0,1]} |dg(u,v)| = \iint_{[0,1]\times[0,1]} |uv\, dh(u,v)|$$

$$\leq \iint_{[0,1]\times[0,1]} |dh(u,v)| \leq B < \infty .$$

Hence since $h(u,v)$ is of bounded variation in the sense of Hardy-Krause then so is $g(u,v)$. Then also

$$\mu_{mn} = \iint_{[0,1]\times[0,1]} u^m v^n \, dg(u,v), \quad m,n = 0,1,2,\ldots$$

and so $g(u,v)$ generates the sequence $\{\mu_{mn}\}$ which is a moment sequence. We also note that $g(u,v)$ is continuous at the axes and zero on them.

Finally Lemma 2.5(ii) states

$$\lim_{m,n\to\infty} \sum_{k,\ell=0}^{\infty} a_{mnk\ell} = 1 ,$$

where

$$a_{mnk\ell} = \binom{k}{m}\binom{\ell}{n}\Delta_{k-m,\ell-n}\mu_{mn}$$

$$= \binom{k}{m}\binom{\ell}{n}\Delta_{k-m,\ell-n}\nu_{m+1,n+1}$$

$$= \binom{k}{m}\binom{\ell}{n} \iint_{[0,1]\times[0,1]} u^{m+1}(1-u)^{k-m}v^{n+1}(1-v)^{\ell-n} dh(u,v) ,$$

when A is a Quasi-Hausdorff T-matrix. Since the integral is zero when u and/or v equals zero this integral can be taken over $(0, 1] \times (0, 1]$.

Consider the sequence of functions $\{f_{pq}\}$ where

$$f_{pq}(u, v) = \sum_{k,\ell=m,n}^{p,q} \binom{k}{m}\binom{\ell}{n} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n}, \quad 0 < u,v \leq 1.$$

Then

$$|f_{pq}(u, v)| \leq 1 \quad \text{and} \quad \lim_{p,q \to \infty} f_{pq}(u, v) = 1$$

By the Dominated Convergence theorem we have

$$\lim_{p,q\to\infty} \iint_{[0,1]\times[0,1]} f_{pq}(u, v)dh(u, v) = \iint_{[0,1]\times[0,1]} \lim_{p,q\to\infty} f_{pq}(u,v)dh(u,v)$$

or

$$\sum_{k,\ell=m,n}^{\infty} \iint_{[0,1]\times[0,1]} \binom{k}{m}\binom{\ell}{n} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n} dh(u, v)$$

$$= \iint_{(0,1]\times(0,1]} dh(u, v) .$$

Thus because A is a T-matrix,

$$1 = \lim_{m,n\to\infty} \sum_{k,\ell=m,n}^{\infty} \iint_{[0,1]\times[0,1]} \binom{k}{m}\binom{\ell}{n} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n} dh(u,v)$$

$$= \lim_{m,n\to\infty} \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} \Delta_{k-m,\ell-n}\nu_{m+1,n+1}$$

$$= \iint_{[0,1]\times[0,1]} dh(u,v) = \iint_{(0,1]\times(0,1]} \frac{dg(u,v)}{uv} ,$$

and Theorem 2.26 is proved.

Corollary 2.26. A Quasi-Hausdorff matrix is boundedness preserving if and only if μ_{mn} is a moment constant defined by a function $g(u,v)$ such that

$$\iint_{(0,1]\times(0,1]} \frac{|dg(u,v)|}{uv} < \infty.$$

Proof. It is well known that a matrix A maps bounded sequences to bounded sequences if and only if

$$\sup_{m,n} \sum_{k,\ell=0}^{\infty} |a_{mnk\ell}| \qquad (*)$$

is finite. The proof of the one dimensional case easily extends to the two dimensional case.

In the proof of Theorem 2.26 we found condition (*) to be sufficient for $\{\mu_{mn}\}$ to be a sequence of moment constants generated by a function $g(u, v)$ which satisfied the inequality

$$\iint_{(0,1]\times(0,1]} \frac{|dg(u,v)|}{uv} \leq \iint_{[0,1]\times[0,1]} |dh(u,v)| < \infty .$$

On the other hand if we assume $\{\mu_{mn}\}$ is generated by a function $g(u, v)$ such that

$$\iint_{(0,1]\times(0,1]} \frac{|dg(u,v)|}{uv} < \infty$$

then again the proof of Theorem 2.26 shows that (*) is satisfied.

In the one dimensional case Vermes [28] has established a connection between series to series transformations and sequence to sequence transformations. He proved

Lemma 2.27. Given an α_0-matrix H defining a transformation of the series Σa_n to the series Σb_m, there exists a K-matrix F which transforms the sequence $\{s_n\}$ of partial sums of Σa_n into the sequence $\{t_m\}$ of the partial sums of Σb_m and conversely.

We shall do the same for two dimensions (Proposition 2.31). First we prove an intermediate result.

Proposition 2.28. Let $F = (f_{mnk\ell})$ be a T-matrix, Σc_{mn} be a series with bounded partial sums $S_{k\ell}$, and

$$\sigma_{mn} = \sum_{k,\ell=0}^{\infty} f_{mnk\ell} S_{k\ell}, \quad m, n = 0, 1, 2, \ldots, \tag{5.1}$$

define a sequence to sequence transformation. Then the matrix $B = (b_{mnk\ell})$ where

$$b_{mnk\ell} = \sum_{r,s=k,\ell}^{\infty} f_{mnrs} \tag{5.2}$$

defines a series to sequence transformation

$$\sigma_{rs} = \sum_{k,\ell=0}^{\infty} b_{rsk\ell} c_{k\ell} \tag{5.3}$$

of the series $\Sigma c_{k\ell}$ with bounded partial sums into the sequence $\{\sigma_{mn}\}$ defined in (5.1). Furthermore the matrix B is a γ_0-matrix which by Corollary 2.8 induces an α_0-matrix $A = (a_{mnk\ell})$ defined by

$$a_{mnk\ell} = \sum_{r,s=k,\ell}^{\infty} (f_{mnrs} - f_{m,n-1,rs} - f_{m-1,nrs} + f_{m-1,n-1,rs}) \tag{5.4}$$

$m, n = 0, 1, \ldots$; negative subscripts indicate the elements are zero.

Proof. Using the definition of $S_{k\ell}$ we see

$$\sigma_{mn} = \sum_{k,\ell=0}^{\infty} f_{mnk\ell} \sum_{p,q=0}^{k,\ell} c_{pq}$$

$$= \sum_{k,\ell=0}^{\infty} b_{mnk\ell} c_{k\ell} ,$$

where $b_{mnk\ell}$ is as defined by (5.2), the interchange being valid because of the absolute convergence of $\sum_{k,\ell} f_{mnk\ell}$ and the boundedness $\{S_{k\ell}\}$. The fact that B is a γ_0-matrix follows from F being a T-matrix.

To satisfy the conditions of Lemma 2.7 for a γ-matrix we use (5.2) to compute $\Delta_{11} b_{mnk\ell}$ and find

$$\Delta_{11} b_{mnk\ell} = f_{mnk\ell} . \tag{5.5}$$

Since F is a T-matrix we have

a) $\sum_{i,j=0}^{\infty} |f_{mnij}| < \infty$; hence Lemma 2.7(i), (vii) are satisfied.

b) $\sum_{i,j=0}^{\infty} |f_{mnij}| < \infty$ implies

$$\sum_{i=0}^{\infty} \sum_{j=s}^{\infty} |f_{mnij}| < \infty \quad \text{implies}$$

$$\sum_{j=s}^{\infty} |f_{mnij}| < \infty \quad \text{implies}$$

$$\lim_{s \to \infty} \sum_{j=s}^{\infty} f_{mnij} = 0 \quad \text{for each } i$$

and similarly

$$\lim_{w \to \infty} \sum_{i=w}^{\infty} f_{mnij} = 0, \quad \text{for each } j .$$

Thus Lemma 2.7(ii), (iii) are satisfied.

c) $$\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |f_{mnk\ell}| = 0 \quad \text{for each } \ell$$

$$\lim_{m,n \to \infty} \sum_{\ell=0}^{\infty} |f_{mnk\ell}| = 0 \quad \text{for each } k ,$$

and again this implies Lemma 2.7(iv), (v) are true for the matrix B.

d) The last condition needed to be satisfied is Lemma 2.7(vi);

$$\lim_{m,n\to\infty} b_{mnij} = 1 = \lim_{m,n\to\infty} \sum_{r,s=i,j}^{\infty} f_{mnrs} .$$

Now from Lemma 2.5(ii) we have

$$\lim_{m,n\to\infty} \sum_{r,s=0}^{\infty} f_{mnrs} = 1 ,$$

and by Lemma 2.5(iii) the series converges absolutely. So we can write

$$\sum_{r,s=i,j}^{\infty} f_{mnrs} = \sum_{r,s=0}^{\infty} f_{mnrs} - \sum_{r=0}^{i-1} \sum_{s=j}^{\infty} f_{mnrs}$$

$$- \sum_{r=i}^{\infty} \sum_{s=0}^{j-1} f_{mnrs} - \sum_{r,s=0}^{i-1,j-1} f_{mnrs} .$$

We show that as m, n tend to infinity the last three series tend to zero. By repeated use of Lemma 2.5(v) we find

$$\sum_{r=0}^{i-1} \sum_{s=j}^{\infty} f_{mnrs} \to 0 ,$$

by repeated use of Lemma 2.5(iv) and the concept of absolute convergence we have

$$\sum_{s=0}^{j-1} \sum_{r=i}^{\infty} f_{mnrs} \to 0 ,$$

by use of Lemma 2.5(i) we have

$$\sum_{r=0}^{i-1} \sum_{s=0}^{j-1} f_{mnrs} \to 0 .$$

Thus the last condition for B to become a γ-matrix is now satisfied when F is a T-matrix.

Using Proposition 2.8 we know that B generates a α-matrix which we will term A. In addition Corollary 2.8 states that if B is a γ_0-matrix then A is an α_0-matrix. But since F is a T-matrix we know

$$\sum_{r,s=0}^{\infty} |f_{mnrs}| < K \qquad \text{for all } m,n \text{ and therefore}$$

$$\left| \sum_{r,s=k,\ell}^{\infty} f_{mnrs} \right| \leq \sum_{r,s=k,\ell}^{\infty} |f_{mnrs}| \to 0$$

as k,ℓ tend to infinity for all m,n. Hence

$$\lim_{k,\ell\to\infty} b_{mnk\ell} = \lim_{k,\ell\to\infty} \sum_{r,s=k\ell}^{\infty} f_{mnrs} = 0$$

and so B is in fact a γ_0-matrix when F is a T-matrix. We also have that a T-matrix induces an α_0-matrix under the correspondence (5.4), (4.5).

Definition 2.29. The matrix B, respectively A, defined by (5.2), respectively (5.4), is called the γ_0-matrix, respectively α_0-matrix, corresponding to the T-matrix F.

We consider now whether conversely a T-matrix can be derived from an α_0-matrix.

Proposition 2.30. Let $A = (a_{mnk\ell})$ be an α_0-matrix for the series Σa_{mn} with bounded partial sums, and let $F = (f_{mnk\ell})$ be defined by

$$f_{mnk\ell} = \sum_{p,q=0}^{m,n} \Delta_{11} a_{pqk\ell}, \quad m,n = 0,1,2,\ldots \tag{5.6}$$

Assume that A satisfies

(aa) $$\lim_{m,n\to\infty} \sum_{k,\ell=0}^{\infty} \sum_{p,q=0}^{m,n} \Delta_{11} a_{pqk\ell} = 1$$

(bb) $$\sup_{m,n} \sum_{k,\ell=0}^{\infty} \left| \sum_{p,q=0}^{m,n} \Delta_{11} a_{pqk\ell} \right| < \infty .$$

Then F is a T-matrix, and if A transforms Σa_{mn} to Σb_{rs}, both series having bounded partial sums, then F transforms the sequence of partial sums of Σa_{mn} to the sequence of partial sums of Σb_{rs}.

Proof. Let B be the γ_0-matrix generated by A as detailed in Proposition 2.8 and Corollary 2.8. Using (5.6) and (4.3) we see

$$f_{mnk\ell} = \Delta_{11} b_{m,k\ell} \tag{5.7}$$

We use this last relationship and the assumption that A is an α_0-matrix, or B is a γ_0-matrix to verify that F is a T-matrix.

(a) Lemma 2.7(vi) yields

$$\lim_{m,n\to\infty} b_{mnk\ell} = 1 ,$$

so by using this with (5.7) we have

$$\lim_{m,n\to\infty} f_{mnk\ell} = 0 \quad \text{for all } k,\ell .$$

Thus Lemma 2.5(i) satisfied with $\beta_{k\ell} = 0$.

(b) condition (ii) of Lemma 2.5 is satisfied as it is condition (aa),

(c) condition (iii) of Lemma 2.5 is satisfied as it is condition (bb),

(d) Lemma 2.7(iv), (v) and (5.7) yield

$$\lim_{m,n\to\infty} \sum_{k=0}^{\infty} |f_{mnk\ell}| = 0 \quad \text{for each} \quad \ell$$

$$\lim_{m,n\to\infty} \sum_{\ell=0}^{\infty} |f_{mnk\ell}| = 0 \quad \text{for each} \quad k.$$

Thus conditions (iv), (v) of Lemma 2.5 are satisfied.

Hence F is a T-matrix.

The last assertion of the lemma is easily verified.

Combining Proposition 2.28 with Proposition 2.30 we have a result analogous to Lemma 2.27.

<u>Proposition 2.31</u>. Given an α_0-matrix A, with (aa), (bb) satisfied, defining a transformation of the series $\sum_{m,n} a_{m,n}$ to the series $\sum_{m,n} b_{m,n}$, both series having bounded partial sums, there exists a T-matrix F which transforms the sequence of the partial sums of $\Sigma a_{m,n}$ into the sequence of partial sums of $\Sigma b_{m,n}$ and conversely.

Definition 2.32. The sequence to sequence Quasi-Hausdorff transformation corresponding to the real double sequence $\{\mu_{mn}\}$ is given by means of the equation

$$h^*_{mn} = \sum_{k,\ell=m,n}^{\infty} f_{mnk\ell} s_{k\ell} ,$$

where

$$f_{mnk\ell} = \binom{k}{m}\binom{\ell}{n} \Delta_{k-m,\ell-n} \mu_{m+1,n+1} .$$

If

$$\mu_{mn} = \iint_{[0,1]\times[0,1]} u^m v^n dg(u,v),$$

is a moment sequence defined by a function $g(u,v)$ of bounded variation in the sense of Hardy-Krause, then the transformation matrix will be denoted by

$$H^*(\mu_{m+1,n+1}) .$$

This definition is analogous to the one dimensional case. It is motivated by

Theorem 2.33. Let the Quasi-Hausdorff matrix $H^*(\mu_{mn})$, with μ_{mn} regular, define a transformation of the series Σc_{mn} to the series Σd_{rs}, both series having bounded partial sums. Then the Quasi-Hausdorff matrix $H^*(\mu_{m+1,n+1})$ is a T-matrix transforming the sequence of partial sums of Σc_{mn} into the sequence of partial sums of Σd_{rs}.

Proof. By Theorem 2.25 $H^*(\mu_{mn})$ is an α_0-matrix, and for this matrix there corresponds a series to sequence matrix B which is γ_0. The sequence to sequence matrix F which corresponds to $H^*(\mu_{mn})$ is a T-matrix if and only if Proposition 2.31 is satisfied, i.e., conditions (aa), (bb) need to be valid when

$$a_{mnk\ell} = \binom{k}{m}\binom{\ell}{n}\Delta_{k-m,\ell-n}\mu_{m,n}\,.$$

We note that (aa) can be rewritten as

$$\lim_{m,n\to\infty}\left\{\lim_{p,q\to\infty}(b_{mn,0,q+1}+b_{mn,p+1,0})\right\} = 0$$

since

$$\sum_{k,\ell=0}^{\infty}\Delta_{11}b_{mnk\ell}$$

$$= \lim_{p,q\to\infty}(b_{mn,p+1,q+1}-b_{mn,0,q+1}-b_{mn,p+1,0}+b_{mn,0,0})$$

and the first and last terms tend to 0 and 1 respectively because B is γ_0. Then by using

$$b_{mnk\ell} = \sum_{r,s=0}^{m,n} a_{rsk\ell}$$

we can write (aa) as

$$\lim_{m,n\to\infty} \sum_{r,s=0}^{m,n} \lim_{p,q\to\infty} (a_{rs,0,q+1} + a_{rs,p+1,0}) = 0.$$

By arguments analogous to those used in Proposition 2. 17 we find

$$\limsup_{k\to\infty} |a_{mnk,0}| = 0$$

$$\limsup_{\ell\to\infty} |a_{mn,0,\ell}| = 0 \quad \text{when} \quad \mu_{m,n} \quad \text{is regular.}$$

This then implies condition (aa) is satisfied.

Condition (bb) can be written for this case, after consulting Proposition 2. 15, as

$$\sup_{m,n} \sum_{k,\ell=m,n}^{\infty} \left| \binom{k}{m}\binom{\ell}{n} \Delta_{k-m,\ell-n} \mu_{m+1,n+1} \right|$$

is finite. Theorem 2. 18 states this to be true whenever $\mu_{m,n}$ is a regular moment constant. Hence F is a T-matrix corresponding to the α_0 matrix $H^*(\mu_{m,n})$.

We note that the elements of F are precisely the terms inside the absolute value signs,

$$f_{mnk\ell} = \binom{k}{m}\binom{\ell}{n}\Delta_{k-m,\ell-n}\mu_{m+1,n+1} .$$

Thus F is, by Definition 2.32, the Quasi-Hausdorff matrix $H^*(\mu_{m+1,n+1})$.

Corollary 2.31. If $\{\mu_{mn}\}$ is a moment sequence then the Quasi-Hausdorff matrix $H^*(\mu_{m+1,n+1})$ is boundedness preserving.

Proof. Using the results in the proof of Corollary 2.26 we see that we need to show that

$$\sup_{m,n} \sum_{k,\ell=0}^{\infty} |h^*_{mnk\ell}|$$

is finite where

$$h^*_{mnk\ell} = \binom{k}{m}\binom{\ell}{n}\Delta_{k-m,\ell-n}\mu_{m+1,n+1} .$$

But using Theorem 2.18 we find that $\{\mu_{mn}\}$ being a sequence of moment constants yields this result.

III. GIBBS' PHENOMENON AND QUASI-HAUSDORFF TRANSFORMATION

Throughout this chapter we assume that $f(x,y)$ is periodic with period 2π in each variable, and of bounded variation in the Hardy-Krause sense in the period cell.

Lemma 3.1. If $f(x,y)$ is periodic with period 2π in each variable, then the Fourier series corresponding to it converges at (x,y), interior to the cell $[-\pi,\pi]\times[-\pi,\pi]$, to the value

$$1/4\{f(x^+,y^+)+f(x^+,y^-)+f(x^-,y^+)+f(x^-,y^-)\},$$

provided that the function

$$\sum_{s,t} f(x\pm s, y\pm t)$$

is bounded and can be expressed as the difference of two functions, each of which is monotone non-decreasing (or monotone non-increasing) with respect to s and t in some cross neighborhood of the point (x,y) [9].

Remark. Hobson [9] states that a cross neighborhood of the point (x,y) is the set $\{(s,t)\}$ such that (s,t) belongs to $[-\pi,\pi]\times[-\pi,\pi]$, and for some $\delta>0$, at least one of the conditions

(i) $|s-x| \leq \delta$

or (ii) $|t-y| \leq \delta$

is satisfied.

We also note that a periodic function which is of bounded variation in the sense of Hardy-Krause satisfies the conditions of Lemma 3.1 everywhere.

Corollary 3.1. If, in addition to the assumptions already on $f(x,y)$, we also assume that $f(x,y)$ is normalized, then the partial sums of the Fourier series of $f(x,y)$ converge to $f(x,y)$ everywhere. In particular, the partial sums of the Fourier series of $f(x,y)$ converge to $f(x,y)$ at every point of continuity of $f(x,y)$ [26].

Lemma 3.2. Let (s,t) be a point of continuity of $f(x,y)$ and let $s_{k\ell}$ be the $k\ell^{th}$ partial sum of the Fourier series of $f(x,y)$. Then for every $\epsilon > 0$, there exists a positive integer $p(\epsilon)$ and an open neighborhood of (s,t), say $U(\epsilon)$, such that

$$|s_{k\ell}(\bar{x}) - f(\bar{x})| < \epsilon, \quad \bar{x} = (x,y)$$

for all $\bar{x}$ in $U(\epsilon)$ and $k,\ell > p(\epsilon)$ [26].

Corollary 3.2. If D is a closed set of points in the period cell such that $f(x,y)$ is continuous at each point of D, then the Fourier series of $f(x,y)$ converges uniformly on D to $f(x,y)$ [26].

Corollary 3.2A. The partial sums of the Fourier series of $f(x,y)$ do not exhibit the Gibbs' phenomenon at any point of continuity of $f(x,y)$ [26].

Lemma 3.3. Let $f(x,y)$ be a normalized, periodic function of bounded variation in the period cell. The partial sums of the Fourier series of $f(x,y)$ will exhibit the Gibbs' phenomenon at every point of discontinuity of $f(x,y)$ and only there [26].

As an immediate consequence of these results we have

Proposition 3.4. If $f(x,y)$ is periodic with period 2π in each variable, and is summable in the period cell, then the regular Quasi-Hausdorff means of the partial sums of the Fourier series corresponding to $f(x,y)$ converges at (x,y), interior to the cell $[-\pi,\pi]\times[-\pi,\pi]$, to the value

$$1/4\{f(x^+,y^+)+f(x^+,y^-)+f(x^-,y^+)+f(x^-,y^-)\}$$

provided that the function

$$\sum_{s,t} f(x\pm s, y\pm t)$$

is bounded and can be expressed as the difference of two functions, each of which is monotone non-decreasing (non-increasing) with respect to both s and t in some cross-neighborhood of the point (x,y).

Corollary 3.4. If, in addition to the previous assumptions on $f(x,y)$, we assume $f(x,y)$ to be normalized, then the regular Quasi-Hausdorff means of the partial sums of the Fourier series of $f(x,y)$ converges everywhere to $f(x,y)$, in particular, at every point of continuity of $f(x,y)$.

Proposition 3.5. Let (s,t) be a point of continuity of $f(x,y)$, and let $t_{k\ell}[f;x,y]$ be the $k\ell^{th}$ (regular) Quasi-Hausdorff transform of the partial sums of the Fourier series of $f(x,y)$. Then for every $\epsilon > 0$, there exists a positive integer $R(\epsilon)$, and an open neighborhood of (s,t), call it $U(\epsilon)$, such that

$$|t_{k\ell}[f;x,y]-f(x,y)| < \epsilon$$

for all (x,y) in $U(\epsilon)$, and $k,\ell > R(\epsilon)$.

Proof. We first note that for the regular Quasi-Hausdorff transformation

(i) $$\lim_{m,n\to\infty} \sum_{k,\ell=0}^{\infty} a_{mnk\ell} = 1 \quad \text{and}$$

(ii) there exists $M > 0$ such that

$$\sum_{k,\ell=0} |a_{mnk\ell}| < M, \quad m,n = 0,1,2,\ldots$$

Second, we see that $f(x,y)$, being of bounded variation, is bounded at the point (x,y) of $U(\epsilon)$, say

$$|f(x,y)| < T.$$

Now consider

$$|t_{k\ell}[f;x,y] - f(x,y)| = \left| \sum_{m,n=k,\ell}^{\infty} a_{k\ell mn} s_{mn} - f(x,y) \right|$$

$$= \left| \sum_{m,n=k,\ell}^{\infty} a_{k\ell mn}\{s_{mn}(x,y) - f(x,y) + f(x,y)\} - f(x,y) \right|$$

$$= \left| \sum_{m,n=k,\ell}^{\infty} a_{k\ell mn}\{s_{mn}(x,y) - f(x,y)\} + f(x,y) \sum_{m,n=k,\ell}^{\infty} a_{k\ell mn} - f(x,y) \right|$$

$$\leq \sum_{m,n=k,\ell}^{\infty} |a_{k\ell mn}|\,|s_{mn}(x,y) - f(x,y)| + |f(x,y)| \left| \sum_{m,n=k,\ell}^{\infty} a_{k\ell mn} - 1 \right| .$$

By Lemma 3.2 and (ii) above the first term can be made smaller than $\epsilon/2$ for all (x,y) in $U(\epsilon)$, an open neighborhood of (s,t) and for all $m,n > P(\epsilon)$. By (i) above and the fact that $f(x,y)$ is of bounded variation we can make the second term smaller than $\epsilon/2$ for all $m, n > Q(\epsilon)$ and for all (x, y) in $U(\epsilon)$. Letting

$$R(\epsilon) = \max\{P(\epsilon), Q(\epsilon)\}$$

we see that the result in the theorem is established.

Corollary 3.5. If D is a closed set of points in the period cell such that $f(x, y)$ is continuous at each point of D, then the regular Quasi-Hausdorff means of the partial sums of the Fourier series of $f(x, y)$ converge to $f(x, y)$ uniformly on D.

Proof. Let $\epsilon > 0$ be given. With each point (s, t) of D, associate a neighborhood $U(\epsilon; s, t)$, and a number $P(\epsilon; s, t)$ such that

$$|t_{k\ell}[f; x, y] - f(x, y)| < \epsilon/2$$

whenever $k, \ell > P(\epsilon; s, t)$ and (x, y) belongs to $U(\epsilon; s, t)$. Then the family

$$\{U(\epsilon; s, t) | (s, t) \text{ in } D\}$$

is an open cover of a closed, compact set D and thus contains a finite subcover

$$\{U(\epsilon; s_i, t_j) | (s_i, t_j) \text{ in } D; i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}.$$

Let $P(\epsilon)$ be the largest of the numbers which are associated with the sets of the subcover. Then

$$|t_{k\ell}[f;x,y]-f(x,y)| < \epsilon$$

whenever (x,y) in D and $k,\ell > P(\epsilon)$.

Corollary 3.5A. The regular Quasi-Hausdorff means of the partial sums of the Fourier series of $f(x,y)$ do not exhibit the Gibbs' phenomenon at any point of continuity of $f(x,y)$.

Using the above results it is clear that a function $f(x,y)$, of bounded variation having no removable discontinuities, will always exhibit the Gibbs' phenomenon at a point of discontinuity. If we express $f(x,y)$ by its Fourier series, form the sequence of partial sums $\{S_{mn}(f;x,y)\}$, apply a summability method to this sequence, then it is natural to ask if the transformed sequence will also exhibit the Gibbs' phenomenon. For the two dimensional case this question has been answered by Cheng for the circular Riesz means and by Ustina for the Hausdorff means.

We will extablish some results pertaining to the study of the Gibbs' phenomenon for the Quasi-Hausdorff means of the double series. The work is a natural extension of Ramanujan, Ishiguro, and Kuttner for the one dimensional case. We use their results and manner of approach freely.

As has been shown by many, to investigate the Gibbs' phenomenon for the arbitrary normalized function of bounded variation in the one dimensional case, it is sufficient to determine the Gibbs' phenomenon of the function

$$X(x) = \begin{cases} 1/2\ (\pi - x) & 0 < x < 2\pi \\ 0 & x = 0 \\ X(x+2k\pi) & k = \pm 1, \pm 2, \pm 3, \ldots \end{cases} \tag{21.1}$$

so that

$$X(x) \sim \sum_{k=1}^{\infty} \frac{\sin kx}{k},$$

$$S_m(X; x) = \sum_{k=1}^{m} \frac{\sin kx}{k}.$$

Since $X(x)$, $S_m(X; x)$ are odd, periodic functions of period 2π we need only to investigate the function in the interval $0 \leq x \leq \pi$.

For the Gibbs' phenomenon for the Quasi-Hausdorff means, of one dimension, of the Fourier series of $X(x)$, Ishiguro and Kuttner [16] obtained the following:

Lemma 3.6. For the regular Quasi-Hausdorff means of the Fourier series of the function defined by (21.1) we have

$$\lim_{n\to\infty} h_n^*(t_n) = \int_{[0,1]} dg(u) \int_{[0,\tau]} \frac{\sin y/u}{y}\, dy ,$$

provided that the weight function $g(u)$ is continuous at $u = 0$,

$$nt_n \to \tau \leq \infty \text{ and } nt_n^2 \to 0 .$$

Using the results of Ustina [26] we know that to study the Gibbs' phenomenon for an arbitrary normalized function of bounded variation in the Hardy-Krause sense in the two dimensional case, it is sufficient to study it for the functions $X(x)$, $X(y)$, and $\phi(x, y) = X(x)X(y)$, where $X(t)$ is as defined by (21. 1). As before it is also sufficient to investigate on the domain $0 \leq x, y < \pi$. We henceforth assume this restriction.

<u>Proposition 3.7</u>. If $h^*_{mn}(X;x)$, $h^*_{mn}(X;y)$ denote the mn^{th} regular transform Quasi-Hausdorff transforms of the functions $X(x)$, respectively $X(y)$, corresponding to the weight function $g(u, v)$, then

$$h^*_{mn}(X;x) = h^*_m(X;x)$$

$$h^*_{mn}(X;y) = h^*_n(X;y),$$

where the right hand sides denote the m^{th} (n^{th}) regular one dimensional Quasi-Hausdorff transforms corresponding to the weight functions $g(u, 1)$, respectively $g(1, v)$.

Proof. From the Fourier series representation of $X(x)$ we find

$$S_m(X;x) = \sum_{k=1}^{m} \frac{\sin kx}{k}$$

$$= -(1/2)x + \int_{[0,x]} \frac{\sin(m+1/2)s}{2\sin(1/2)s}\, ds .$$

Then

$$h^*_{mn}(X;x) = \sum_{k,\ell=m,n}^{\infty} h_{mnk\ell}\, s_k(X;x)$$

$$= \sum_{k,\ell=m,n}^{\infty} h_{mnk\ell} \left\{ -(1/2)x + \int_{[0,x]} \frac{\sin(k+1/2)s}{2\sin(1/2)s}\, ds \right\}$$

$$= -(1/2)x + (1/2) \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} \left\{ \int_{[0,x]} \frac{\sin(k+1/2)s}{\sin(1/2)s}\, ds \right.$$

$$\left. \times \iint_{[0,1]\times[0,1]} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n}\, dg(u,v) \right\} ,$$

by using the definition of the transform and its regularity. Since $h^*_{mn}(X;x)$ converges absolutely and uniformly in x,

$$h^*_{mn}(X;x) = -(1/2)x + (1/2) \sum_{k=m}^{\infty} \binom{k}{m} \left\{ \int_{[0,x]} \frac{\sin(k+1/2)s}{\sin(1/2)s}\, ds \right.$$

$$\left. \times \iint_{[0,1]\times[0,1]} u^{m+1}(1-u)^{k-m} \sum_{\ell=n}^{\infty} \binom{\ell}{n} v^{n+1}(1-v)^{\ell-n} dg(u,v) \right\}$$

$$= -(1/2)x + (1/2) \sum_{k=m}^{\infty} \binom{k}{m} \left\{ \int_{[0,x]} \frac{\sin(k+1/2)s}{\sin(1/2)s}\, ds \right.$$

$$\left. \times \iint_{[0,1]\times[0,1]} u^{m+1}(1-u)^{k-m}(1) dg(u,v) \right\}$$

$$= -(1/2)x + (1/2) \sum_{k=m}^{\infty} \binom{k}{m} \left\{ \int_{[0,1]} \frac{\sin(k+1/2)s}{\sin(1/2)s}\, ds \right.$$

$$\left. \times \int_{[0,1]} u^{m+1}(1-u)^{k-m}[dg(u,1)-dg(u,0)] \right\}$$

$$= -(1/2)x + (1/2) \sum_{k=m}^{\infty} \binom{k}{m} \int_{[0,x]} \frac{\sin(k+1/2)s}{\sin(1/2)s}\, ds$$

$$\times \int_{[0,1]} u^{m+1}(1-u)^{k-m} dg(u,1) ,$$

since $dg(u,0) = 0$,

$$= h^*_m(X;x) .$$

We used the fact that the transform was regular, so that $dg(u, 0) = 0$ and $g(u, 1)$ is a weight function for the regular one dimensional Quasi-Hausdorff transform.

Finally the other half of the theorem is proved similarly.

We now turn to examine the Quasi-Hausdorff transforms of the sequence of partial sums of the Fourier series representations of the function $\phi(x, y) = X(x)X(y)$. We have then

$$S_{m,n}(\phi; x; y) = \sum_{k,\ell=1}^{m,n} \frac{\sin kx}{k} \frac{\sin \ell y}{\ell}$$

Definition 3.8. The circle means of the Fourier series of

$$\sum_{k=1}^{\infty} \frac{\sin kt}{k}$$

are given by

$$\sigma^*_{n,r}(t) = \sigma^*_n(t) = \sum_{\nu=n}^{\infty} \binom{\nu}{n} r^{n+1} (1-r)^{\nu-n} s_\nu(t) ,$$

where

$$s_\nu(t) = \sum_{k=1}^{\nu} \frac{\sin kt}{k} \qquad \text{(Reference [10]).}$$

Now the two dimensional analog of Lemma 3.6 becomes

Proposition 3.9. For the two dimensional regular Quasi-Hausdorff means of the function $\phi(x, y)$ we have

$$\lim_{m,n\to\infty} h^*_{mn}(\phi; x_m, y_n)$$

$$= \iint_{[0,1]\times[0,1]} \left\{ \int_{[0,\tau]} \frac{\sin(y/u)}{y} dy \right\} \left\{ \int_{[0,\hat{\tau}]} \frac{\sin(y/v)}{y} dy \right\} dg(u,v)$$

provided $g(u, v)$ is continuous at the axes,

$$mx_m \to \tau \le \infty, \quad ny_n \to \hat{\tau} \le \infty$$

$$mx_m^2 \to 0, \quad ny_n^2 \to 0 \quad \text{as} \quad m, n \to \infty.$$

Proof. Consider the partial sums of the function $\phi(x, y)$. If either m or n is zero we shall call the partial sum zero, while otherwise

$$S_{m,n}(\phi; x, y) = S_m(X; x)S_n(X, y) .$$

We see that the Quasi-Hausdorff transform becomes

$$h^*_{m,n}(\phi;x,y) = \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} S_{k\ell} \iint_{[0,1]\times[0,1]} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n} dg(u,v).$$

We break the integration into three parts,

(i) over the cell $[0, 1] \times [0, \delta)$,

(ii) over the cell $[\delta, 1] \times [\delta, 1]$,

(iii) over the cell $[0, \delta) \times [\delta, 1]$, for $0 < \delta < 1$.

We note that all three sums created by integrating over these cells exists since $\{S_{k\ell}\}$ is bounded, H^* is a T-matrix, and thus each sum is in fact absolutely convergent. We let the bound on $\{S_{k\ell}\}$ be denoted by M.

Considering the first integration we see

$$\text{(i)} \quad \left| \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} S_{k\ell} \iint_{[0,1]\times[0,\delta)} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n} dg(u,v) \right|$$

$$\leq M \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} \iint_{[0,1]\times[0,\delta)} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n} |dg(u,v)|$$

$$\leq M \iint_{[0,1]\times[0,\delta)} \{dP(u,v) + dN(u,v)\},$$

where again $P(u, v)$, $N(u, v)$ are the positive and negative variations of $g(u, v)$. By continuity arguments similar to the work in Proposition 2.17 we find that the integration over the first cell can be made arbitrarily small by proper choice of δ. Doing the same for cell (iii) we find an analogous result.

Thus we are left with only cell (ii) to consider. We find

$$\sum_{k,\ell=m,n}^{\infty} \iint_{[\delta,1]\times[\delta,1]} \binom{k}{m}\binom{\ell}{n} S_{k\ell} u^{m+1}(1-u)^{k-m} v^{n+1}(1-v)^{\ell-n} dg(u,v)$$

$$= \iint_{[\delta,1]\times[\delta,1]} \left\{ \sum_{k=m}^{\infty} \binom{k}{m} S_k(X;x) u^{m+1}(1-u)^{k-m} \right.$$

$$\left. \times \sum_{\ell=n}^{\infty} \binom{\ell}{n} S_\ell(X;y) v^{n+1}(1-v)^{\ell-n} \right\} dg(u,v)$$

by the use of the absolute convergence of the double series. If we now use the definition of the circle means of one variable we further find that this summation is precisely equal to

$$\iint_{[\delta,1]\times[\delta,1]} \{\sigma_m^*(u;x)\sigma_n^*(v;y)\} dg(u,v) .$$

Ishiguro [10] shows that

$$\sigma_k^*(r,t_k) \to \int_{[0,\tau/r]} \frac{\sin y}{y} dy$$

as $k \to \infty$, uniformly in r for $\delta \leq r \leq 1$, with

$$\lim_{k\to\infty} kt_k = \tau \leq \infty \quad \text{and} \quad \lim_{k\to\infty} kt_k^2 = 0.$$

Thus we have our second integral tending to

$$\iint\limits_{[\delta,1]\times[\delta,1]} \left\{ \int\limits_{[0,\tau/u]} \frac{\sin y}{y}\, dy \right\} \left\{ \int\limits_{[0,\hat{\tau}/v]} \frac{\sin y}{y}\, dy \right\} dg(u,v)$$

as $m, n \to \infty$, where

$$mx_m \to \tau \le \infty, \quad mx_m^2 \to 0\,,$$

$$ny_n \to \hat{\tau} \le \infty, \quad ny_n^2 \to 0 \quad \text{as} \quad m, n \to \infty.$$

Hence

$$\overline{\lim_{m,n}}\, h^*_{mn}(\phi; x_m, y_n)$$

$$= \iint\limits_{[\delta,1]\times[\delta,1]} \left\{ \int\limits_{[0,\tau/u]} \frac{\sin y}{y}\, dy \int\limits_{[0,\hat{\tau}/v]} \frac{\sin y}{y}\, dy \right\} dg(u,v) + O(\delta)$$

and (*)

$$\underline{\lim}_{m,n}\, h^*_{mn}(\phi; x_m, y_n)$$

$$= \iint\limits_{[\delta,1]\times[\delta,1]} \left\{ \int\limits_{[0,\tau/u]} \frac{\sin y}{y}\, dy \int\limits_{[0,\hat{\tau}/v]} \frac{\sin y}{y}\, dy \right\} dg(u,v) + O(\delta)$$

Except for the bottom limit on the double integral this is essentially the desired result. If we replace the bottom limit by zero, the desired limit, we make an error which is given by the value of

$$\iint\limits_{[0,\delta)\times[0,\delta)} \left\{ \int\limits_{[0,\tau/u]} \frac{\sin y}{y}\,dy \int\limits_{[0,\hat{\tau}/v]} \frac{\sin y}{y}\,dy \right\} dg(u,v)$$

$$+ \iint\limits_{[\delta,1]\times[0,\delta)} \left\{ \int\limits_{[0,\tau/u]} \frac{\sin y}{y}\,dy \int\limits_{[0,\hat{\tau}/v]} \frac{\sin y}{y}\,dy \right\} dg(u,v)$$

$$+ \iint\limits_{[0,\delta)\times[\delta,1]} \left\{ \int\limits_{[0,\tau/u]} \frac{\sin y}{y}\,dy \int\limits_{[0,\hat{\tau}/v]} \frac{\sin y}{y}\,dy \right\} dg(u,v) .$$

Since for all Y we have

$$\left| \int\limits_{[0,Y]} \frac{\sin y}{y}\,dy \right| \leq N ,$$

say, we find the above error to be less than

$$N^2 \iint\limits_{[0,\delta)\times[0,\delta)} dg(u,v) + N^2 \iint\limits_{[\delta,1]\times[0,\delta)} dg(u,v) + N^2 \iint\limits_{[0,\delta)\times[\delta,1]} dg(u,v).$$

Again by the continuity conditions we can make this error arbitrarily small by making δ small. Thus in (*) the lower limit in the double integral may be replaced by zero and the resulting $O(\delta)$ error incorporated in the $O(\delta)$ term already appearing in (*). Then (*) implies the desired result.

IV. THE LEBESGUE CONSTANT FOR THE QUASI-HAUSDORFF TRANSFORMATION

In this chapter we shall investigate the Lebesgue constants for the two dimensional sequence to sequence Quasi-Hausdorff transformation corresponding to a regular moment sequence. More precisely this chapter developes the two dimensional analogues of the results in Ishiguro [14].

Definition 4.1. The Lebesgue constants for the double Fourier series are given by

$$L(m,n) = \frac{4}{\pi^2} \iint_{[0,\pi]\times[0,\pi]} \left| \frac{\sin(m+1/2)s}{2\sin(s/2)} \frac{\sin(n+1/2)t}{2\sin(t/2)} \right| ds\,dt$$

$$= \frac{4}{\pi^2} \iint_{[0,\pi]\times[0,\pi]} |D_{mn}(s,t)|\, ds\,dt .$$

If the sequence $\{D_{mn}(s,t)\}$ is transformed by some summability method and we denote this transformed sequence by $\{K_{mn}(s,t)\}$, then the sequence of constants

$$\frac{4}{\pi^2} \iint_{[0,\pi]\times[0,\pi]} |K_{mn}(s,t)|\, ds\,dt$$

are said to be the Lebesgue constants for that summability method.

Without loss of generality we may assume the integration is over the set $(0,\pi]\times(0,\pi]$ because the points deleted form a set of measure zero.

We investigate these constants for the Quasi-Hausdorff sequence to sequence transformation given in Theorem 2.33, when $g(u,v)$ is continuous at the axes.

Definition 4.2. For the Lebesgue constant $L^*(m,n;g)$ of the Quasi-Hausdorff sequence to sequence transformation $H^*(\mu_{m+1,n+1})$ we have

$$K_{mn}(s,t) = \sum_{k,\ell=m,n}^{\infty} \binom{k}{m}\binom{\ell}{n} D_{k\ell}(s,t) \iint_{[0,1]\times[0,1]} u^{m+1} \times (1-u)^{k-m} v^{n+1} (1-v)^{\ell-n} dg(u,v),$$

where

$$D_{k,\ell}(s,t) = \frac{\sin(k+1/2)s}{2\sin(s/2)} \frac{\sin(\ell+1/2)t}{2\sin(t/2)} .$$

We now consider this term $K_{m,n}(s,t)$ just defined. Since the sine function is the imaginary component of the complex exponential function we see that $D_{k,\ell}(s,t)$ can be written

$$D_{k,\ell}(s,t) = \mathrm{Im}\{\frac{e^{i(k+1/2)s}}{2\sin(s/2)}\}\ \mathrm{Im}\{\frac{e^{i(\ell+1/2)t}}{2\sin(t/2)}\}$$

$$= \mathrm{Im}\{\frac{e^{i(k-m)s+i(m+1/2)s}}{2\sin(s/2)}\}\ \mathrm{Im}\{\frac{e^{i(\ell-n)t+i(n+1/2)t}}{2\sin(t/2)}\}.$$

Thus

$$(1-u)^{k-m}(1-v)^{\ell-n}D_{k,\ell}(s,t) = \mathrm{Im}\{\frac{e^{i(m+1/2)s}}{2\sin(s/2)}((1-u)e^{is})^{k-m}\}$$

$$\times \mathrm{Im}\{\frac{e^{i(n+1/2)t}}{2\sin(t/2)}((1-v)e^{it})^{\ell-n}\}.$$

Hence

$K_{m,n}(s,t)$

$$= \sum_{k,\ell=m,n}^{\infty} \iint_{[0,1]\times[0,1]} \binom{k}{m}\binom{\ell}{n} u^{m+1}v^{n+1}[(1-u)^{k-m}(1-v)^{\ell-n}D_{k,\ell}(s,t)]dg(u,v)$$

$$= \sum_{k,\ell=m,n} \iint_{[0,1]\times[[0,1]} u^{m+1}v^{n+1}\binom{k}{m}\mathrm{Im}\{\frac{e^{i(m+1/2)s}}{2\sin(s/2)}((i-u)e^{is})^{k-m}\}$$

$$\times \binom{\ell}{n}\mathrm{Im}\{\frac{e^{i(n+1/2)t}}{2\sin(t/2)}((1-v)e^{it})^{\ell-n}\}dg(u,v).$$

Now let

$$f_{pq}(u,v) = \sum_{k,\ell=m,n}^{p,q} \mathrm{Im}\{\binom{k}{m}((1-u)e^{is})^{k-m}\}u^{m+1}$$

$$\times \mathrm{Im}\{\binom{\ell}{n}((1-v)e^{it})^{\ell-n}\}v^{n+1}$$

for (u,v) belonging to $[0,1]\times[0,1]$.

Then

$$|f_{pq}(u,v)| \leq \sum_{k,\ell=m,n}^{p,q} \binom{k}{m}\binom{\ell}{n}(1-u)^{k-m}(1-v)^{\ell-n}u^{m+1}v^{n+1} \leq 1$$

for all p, q, u, v.

Furthermore

$$f_{pq} \to \operatorname{Im}\left\{\frac{u^{m+1}}{(1-(1-u)e^{is})^{m+1}}\right\} \operatorname{Im}\left\{\frac{v^{n+1}}{(1-(1-v)e^{it})^{n+1}}\right\}$$

$$= f(u,v) \qquad \text{for all } (u,v) \text{ in } [0,1]\times[0,1].$$

Therefore by the Bounded Convergence theorem we have

$$\lim_{p,q\to\infty} \iint_{[0,1]\times[0,1]} f_{pq}dg = \iint_{[0,1]\times[0,1]} fdg .$$

Thus we may write

$$K_{mn}(s,t) = \iint_{[0,1]\times[0,1]} \frac{u^{m+1}v^{n+1}}{2\sin(s/2)2\sin(t/2)}$$

$$\times \operatorname{Im}\left\{e^{i(m+1/2)s}\frac{1}{(1-(1-u)e^{is})^{m+1}}\right\}$$

$$\times \operatorname{Im}\left\{e^{i(n+1/2)t}\frac{1}{(1-(1-v)e^{it})^{n+1}}\right\} dg(u,v).$$

Now let $s = 2z$, $t = 2w$, so that

$$K_{mn}(s,t) = \iint_{[0,1]\times[0,1]} \frac{u^{m+1}v^{n+1}}{2\sin z\; 2\sin w}\, \mathrm{Im}\left\{\frac{e^{i(2m+1)z}}{(1-(1-u)e^{i2z})^{m+1}}\right\}$$

$$\times\, \mathrm{Im}\left\{\frac{e^{i(2n+1)w}}{(1-(1-v)e^{i2w})^{n+1}}\right\} dg(u,v).$$

<u>Definition 4.3</u>. The functions p_1, p_2, q_1, q_2 are given by

$$p_1(z,u)e^{iq_1(z,u)} = \frac{1}{(1-(1-u)e^{2iz})}$$

$$p_2(w,v)e^{iq_2(w,v)} = \frac{1}{(1-(1-v)e^{2iw})}$$

when $0 \leq z, w \leq \pi/2$.

We note that the functions q_1, q_2 have values in $[0, \pi/2]$, and also if

$$0 \leq u \leq 1, \quad \text{or} \quad 0 \leq v \leq 1$$

then

$$0 \leq up_1 \leq 1, \quad \text{or} \quad 0 \leq vp_2 \leq 1.$$

This follows by considering the real and imaginary parts of the above definition. For

$$1-(1-t)e^{2iw} = \frac{e^{-iq}}{p}$$

implies

$$1 - \cos 2w + t \cos 2w = \frac{\cos q}{p} \tag{a}$$

$$\sin 2w - t \sin 2w = \frac{\sin q}{p} \ . \tag{b}$$

Squaring and adding yields

$$(tp)^2 = \frac{t^2}{t^2 + 4(1-t)\sin^2 w} = \frac{1}{1 + \frac{4(1-t)}{t^2}\sin^2 w} \ .$$

The right hand side is easily seen to be not larger than one. Hence, since $p > 0$ and $t > 0$, we have

$$0 \leq tp \leq 1.$$

The equation for $(tp)^2$ above implies that

$$up_1 = 1 \quad \text{if and only if} \quad u = 1 \quad \text{or} \quad z = 0,$$

$$vp_2 = 1 \quad \text{if and only if} \quad v = 1 \quad \text{or} \quad w = 0.$$

Using the above definition in $K_{mn}(s,t)$, we can now write

$$K_{mn}(s,t) = \iint_{[0,1]\times[0,1]} \frac{1}{4 \sin z \sin w}$$

$$\times (up_1)^{m+1}[\sin((2m-1)z + (m+1)q_1(z,u))]$$

$$\times (vp_2)^{n+1}[\sin((2n+1)w + (n+1)q_2(w,v))]dg(u,v).$$

Definition 4.4. The function $\phi(M, z, u, p_1, q_1)$ is given by

$$\phi(M, z, u, p_1, q_1) = (up_1)^M[(\sin(2Mz+Mq_1))(\cot z - 1/z) - \cos(2Mz+Mq_1)]$$

M being a positive integer, $0 \le z \le \pi/2$, $0 \le u \le 1$, p_1 and q_1 as defined in Definition 4.3.

We note that $\phi(N, w, v, p_2, q_2)$ is then given by

$$(vp_2)^N[(\sin(2Nw+Nq_2))(\cot w - 1/w) - \cos(2Nw+Nq_2)]$$

where p_2, q_2 are as defined in Definition 4.3, N be a positive integer, $0 \le w \le \pi/2$, $0 \le v \le 1$.

Since $(\cot s - 1/s)$ is bounded on $[0, \pi/2]$ this function ϕ is also bounded for all values in its domain.

Proposition 4.5.

$$\iint_{[0,1)\times[0,1)} \frac{1}{\sin z \sin w} (up_1)^{m+1}\{\sin((2m+1)z+(m+1)q_1(z,u))\}$$

$$\times (vp_2)^{n+1}\{\sin((2n+1)w+(n+1)q_2(w,v))\}dg(u,v)$$

$$= \iint_{[0,1)\times[0,1)} (up_1)^{m+1}(vp_2)^{n+1}\{\sin(2(m+1)z+(m+1)q_1(z,u))\}$$

$$\times \{\sin(2(n+1)w+(n+1)q_2(w,v))\}\{\frac{1}{zw}\}dg(u,v) \quad +$$

$$+ \iint_{[0,1)\times[0,1)} \phi(m+1, z, u, p_1, q_1)\phi(n+1, w, v, p_2, q_2)dg(u, v)$$

$$+ \iint_{[0,1)\times[0,1)} (up_1)^{m+1} \frac{\sin(2(m+1)z+(m+1)q_1(z,u))}{z}$$

$$\times \phi(n+1, w, v, p_2, q_2)dg(u, v)$$

$$+ \iint_{[0,1)\times[0,1)} (vp_2)^{n+1} \frac{\sin(2(n+1)w+(n+1)q_2(w,v))}{w}$$

$$\times \phi(m+1, z, u, p_1, q_1)dg(u, v) .$$

Proof. Using the expansion for $\sin(A-B)$ we see

$$\sin[(2m+1)z+(m+1)q_1]\sin[(2n+1)w+(n+1)q_2]$$

$$= \{\sin[2(m+1)z+(m+1)q_1]\cos z - \sin z \cos[2(m+1)z+(m+1)q_1]\}$$

$$\times \{\sin[2(n+1)w+(n+1)q_2]\cos w - \sin w \cos[2(n+1)w+(n+1)q_2]\} ,$$

(let $2(m+1)z + (m+1)q_1 = M$, and $2(n+1)w + (n+1)q_2 = N$).

Thus

$$\frac{\sin((2m+1)z+(m+1)q_1)\sin((2n+1)w+(n+1)q_2)}{\sin z \sin w}$$

$$= \sin(M)\sin(N)\cot z \cot w - \cos(M)\sin(N)\cot w - \sin(M)\cos(N)\cot z$$

$$+ \cos(M)\cos(N).$$

Call these four terms A, B, C, D.

Observe that

$$\cot z \cot w = \{1/zw\} + \{\cot z - 1/z\}\{\cot w - 1/w\} + \{\cot z - 1/z\}\{1/w\}$$
$$+ \{\cot w - 1/w\}\{1/z\} .$$

Making this substitution in A yields four new terms, labeled A_1, A_2, A_3, A_4 .

We now work with the term B, adding and subtracting $1/w$ from $\cot w$, yielding

$$-\cot w = \{-\cot w + 1/w\} - \{1/w\} .$$

This substitution in B yields two new terms, denoted by B_1 and B_2.

In the same manner we work on C to yield two new terms C_1 and C_2 upon substituting

$$-\cot z = \{-\cot z + 1/z\} - \{1/z\} .$$

If we multiply the terms A_2, B_1, C_1, and D by

$$(up_1)^{m+1}(vp_2)^{n+1}$$

and integrate over the half open cell $[0,1) \times [0,1)$ we find

$$\iint_{[0,1)\times[0,1)} (up_1)^{m+1}(vp_2)^{n+1}\{\sin(M)\sin(N)[\cot z - 1/z][\cot w - 1/w]$$

$$+ \cos(M)\sin(N)[-\cot w + 1/w]$$

$$+ \sin(M)\cos(N)[-\cot z + 1/z]$$

$$+ \cos(M)\cos(N)\}dg(u,v)$$

$$= \iint_{[0,1)\times[0,1)} \phi(m+1, z, u, p_1, q_1)\phi(n+1, w, v, p_2, q_2)dg(u,v).$$

Denote this integral by E_1.

In the same manner working with the terms A_4 and C_2 yields the integral

$$E_2 = \iint_{[0,1)\times[0,1)} (up_1)^{m+1} \frac{\sin(M)}{z} \phi(n+1, w, v, p_2, q_2)dg(u,v)$$

while A_3 and B_2 yield

$$E_3 = \iint_{[0,1)\times[0,1)} (vp_2)^{n+1} \frac{\sin(N)}{w} \phi(m+1, z, u, p_1, q_1)dg(u,v) .$$

Hence

$$A + B + C + D = A_1 + E_1 + E_2 + E_3 ,$$

which is the desired result.

Using Proposition 4.5 we can now write $L^*(m, n; g)$ as

$$L^*(m,n;g) = \frac{4}{\pi^2} \iint_{[0,\pi/2]\times[0,\pi/2]}$$

$$\Bigg| \iint_{[0,1)\times[0,1)} \Bigg\{ (up_1)^{m+1} (vp_2)^{n+1} \frac{\sin(M)\sin(N)}{zw}$$

$$+ \phi(m+1, z, u, p_1, q_1)\phi(n+1, w, v, p_2, q_2)$$

$$+ (up_1)^{m+1} \frac{\sin(M)}{z} \phi(n+1, w, v, p_2, q_2)$$

$$+ (vp_2)^{n+1} \frac{\sin(N)}{w} \phi(m+1, z, u, p_1, q_1) \Bigg\} dg(u,v)$$

$$+ \int_{[0,1)} (vp_2)^{n+1} \frac{\sin(2m+1)z}{\sin z} \frac{\sin((2n+1)w+(n+1)q_2)}{\sin w}$$

$$\times \{dg(1,v)-dg(1^-,v)\}$$

$$+ \int_{[0,1)} (up_1)^{m+1} \frac{\sin((2m+1)z+(m+1)q_1}{\sin z} \frac{\sin(2n+1)w}{\sin w}$$

$$\times \{dg(u,1)-dg(u,1^-)\}$$

$$+ \iint_{\{1\}\times\{1\}} \frac{\sin(2m+1)z}{\sin z} \frac{\sin(2n+1)w}{\sin w} dg(u,v) \Bigg| dzdw \qquad (4\text{-}30)$$

We note that the variables u, v were restricted to the cell $[0, 1] \times [0, 1]$. If we further restrict the variables to the cell $[\delta, 1] \times [\delta, 1]$ then $L^*(m, n; g)$ is transformed into the quantity that we shall term $L^*_\delta(m, n; g)$. Thus $L^*(m, n; g) = L^*_0(m, n; g)$.

We shall estimate $L^*_\delta(m, n; g)$. To work with these estimations we introduce two lemmas by Ishiguro [11].

Lemma 4.6. If $1 < \mathcal{M} < e$, then

$$r^2 p_j^2(u, r) \leq \mathcal{M}^{-8(1-r)u^2/\pi^2 r^2}$$

for sufficiently small $0 \leq u$ and $0 \leq r \leq 1$.

We note here that equality would hold if either $r = 0$, $r = 1$, or $u = 0$. Thus

$$(up_1)^2 \leq \mathcal{M}^{-8(1-u)z^2/\pi^2 u^2},$$

for sufficiently small $z \geq 0$, $1 \geq u \geq 0$

and

$$(vp_2)^2 \leq \mathcal{M}^{-8(1-v)w^2/\pi^2 v^2},$$

for sufficiently small $w \geq 0$, $1 \geq v \geq 0$.

Lemma 4.7. For small $u \geq 0$,

$$q_j(u, r) = 2((1-r)/r)u + 0(u^3),$$

uniformly if $0 < \delta \leq r \leq 1$, fixed δ.

Here $0 \leq q_j \leq \pi/2$ if $0 < r \leq 1$, and again $r = 1$ or $u = 0$ imply $q = 0$ by the comments following Definition 4.3. Thus for our needs

$$q_1(z,u) = 2((1-u)/u)z + O(z^3), \quad 0 < u \leq 1, \quad \text{small } z \geq 0, \qquad (4\text{-}31)$$

$$q_2(w,v) = 2((1-v)/v)w + O(w^3), \quad 0 < v \leq 1, \quad \text{small } w \geq 0. \qquad (4\text{-}32)$$

For convenience in Propositions 4.8-4.13 we will often replace $m+1$ by M, $n+1$ by N.

Proposition 4.8. If $I = [0,1) \times [0,1)$ or even $[\delta,1) \times [\delta,1)$, $0 < \delta < 1$, then

$$\iint_{[0,\pi/2]\times[0,\pi/2]} \left| \iint_I \phi(M,q,u,p_1,q_1)\phi(N,w,v,p_2,q_2)dg(u,v) \right| dzdw$$

$= o(\ln M \ln N)$.

Proof. We have already seen that the function ϕ is bounded for all values in its domain, thus the integral is also bounded.

Proposition 4.9. If $0 < \delta < 1$, then

$$\iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \left| \iint_{[\delta,1)\times[\delta,1)} (up_1)^M (vp_2)^N \left\{ \frac{\sin(2Mz+Mq_1)\sin(2Nw+Nq_2)}{zw} \right.\right.$$

$$\left.\left. - \frac{\sin(2Mz/u)\sin(2Nw/v)}{zw} \right\} dg(u,v) \right|$$

$$= \iint_{[0,\pi/2]\times[0,\pi/2]} |A - A_1| dzdw$$

$= o(1)$

Remark. The notation $|A-A_1|$ and any similar notation is for future reference.

Proof. Choosing a σ so that $0 \leq z, w \leq \sigma$ implies Lemma 4.6 holds and a δ so that Lemma 4.7 holds we break the integration over $[0, \pi/2] \times [0, \pi/2]$ into integration over the four subcells $[0, \sigma] \times [0, \sigma]$, $[0, \sigma] \times [\sigma, \pi/2]$, $[\sigma, \pi/2] \times [0, \sigma]$ and $[\sigma, \pi/2] \times [\sigma, \pi/2]$. We term these subcells C_1, C_2, C_3, C_4.

Let us denote the quantities $2Mz + Mq_1$, $2Nw + Nq_2$ by α and β respectively. Then

$$\left\{ \frac{\sin \alpha}{z} \frac{\sin \beta}{w} - \frac{\sin(2Mz/u)\sin(2Nw/v)}{zw} \right\}$$

$$= \left\{ \frac{\sin \beta}{w} \right\} \left\{ \frac{\sin \alpha}{z} - \frac{\sin(2Mz/u)}{z} \right\} + \left\{ \frac{\sin(2Mz/u)}{z} \right\} \left\{ \frac{\sin \beta}{w} - \frac{\sin(2Nw/v)}{w} \right\}.$$

We make the above substitutions for the integrations over the first three subcells, calling the resulting integrals T_1, T_2, and T_3.

Consider first T_1.

$$T_1 \leq \iint_{C_1} dzdw \left| \iint_{[\delta, 1)\times[\delta, 1)} (up_1)^M (vp_2)^N \left(\frac{\sin \beta}{w}\right)\left(\frac{\sin \alpha}{z} - \frac{\sin(2Mz/u)}{z}\right) \times dg(u, v) \right|$$

$$+ \iint_{C_1} dzdw \left| \iint_{[\delta, 1)\times[\delta, 1)} (up_1)^M (vp_2)^N \left(\frac{\sin(2Mz/u)}{z}\right) \times \right.$$

$$\times \left(\frac{\sin}{w} - \frac{\sin(2Nw/v)}{w}\right)dg(u,v) \Bigg|$$

$$= T_{11} + T_{12} .$$

We use Lemma 4.6 for a bound on the quantities (up_1) and (vp_2), and since $0 \leq z, w \leq \sigma$ we can use Lemma 4.7 to write

$$\alpha = 2Mz + M\{2(1/u)z - 2z + O(z^3)\}$$

$$= 2M\{(z/u) + O(z^3)\}$$

and

$$\beta = 2N\{(w/v) + O(w^3)\} .$$

Hence

$$\left|\frac{\sin\beta}{w}\right| \leq \frac{2N}{v}\{1+O(w^2)\} \leq \frac{2N}{\delta}\{1+O(W^2)\}, \qquad \delta \leq v < 1$$

and

$$\left|\frac{1}{z}\{\sin\alpha - \sin(2Mz/u)\}\right| \leq \frac{1}{z}\{(M)O(z^3)\} = (M)O(z^2) \quad \text{as} \quad z \to 0.$$

Therefore

$$T_{11} \leq \iint_{C_1} dzdw \iint_{I} \Big\{ \mathcal{M}^{-4(1-u)(M)z^2/\pi^2u^2} \mathcal{M}^{-4(1-v)(N)w^2/\pi^2v^2}$$

$$\times \left|\frac{\sin\beta}{w}\right| \left|\frac{\sin\alpha}{z} - \frac{\sin(2Mz/u)}{z}\right| \Big\} |dg(u,v)|$$

$$\leq \iint_{C_1} dzdw \iint_{I} \mathcal{M}^{-4(1-u)(M)z^2/\pi^2u^2} \{(M)^3 O(z^2)\}$$

$$\times \mathcal{M}^{-4(1-v)(N)w^2/\pi^2v^2} \{\frac{2(N)}{\delta} 1+O(w^2)\}|dg(u,v)| .$$

Now $\mathcal{M}^{-4(1-u)(M)a^2/\pi^2u^2}\{(M)^3O(z^2)\}$ tends to zero as M tends to infinity for all z not zero, while at z equal zero the product is zero;

$$\mathcal{M}^{-4(1-v)(N)w^2/\pi^2v^2}\{\frac{2(N)}{\delta}(1+O(w^2))\}$$

tends to zero as N tends to infinity for all w not zero.

Using Lebesgue's theorem on dominated convergence we find

$$T_{11} \to 0 \quad \text{as} \quad M, N \to \infty$$

Similar reasoning yields

$$T_{12} \to 0 \quad \text{as} \quad M, N \to \infty$$

and therefore

$$T_1 = o(1) .$$

Considering the integration over the second subcell C_2 we find

$$T_2 \leq \iint_{C_2} dzdw \left| \iint_{[\delta,1)\times[\delta,1)} (up_1)^M(vp_2)^N\{\frac{\sin\beta}{w}\}\{\frac{\sin\alpha}{z} - \frac{\sin(2Mz/u)}{z}\}dg(u,v)\right|$$

$$+\iint_{C_2} dzdw \left| \iint_{[\delta,1)\times[\delta,1)} (up_1)^M(vp_2)^N\{\frac{\sin(2Mz/u)}{z}\} \times \{\frac{\sin\beta}{w} - \frac{\sin(2Nw/v)}{w}\}dg(u,v)\right|$$

$$= T_{21} + T_{22} .$$

Using the same estimates as previously we easily see that

$$T_{21} \leq \iint_{C_2} dzdw \iint_{I} \mathcal{M}^{\frac{-4}{\pi^2}\{\frac{(1-u)(M)z^2}{u^2} + \frac{(1-v)(N)w^2}{v^2}\}}$$

$$\times \{\frac{1}{\sigma}\}(M)O(z^2)|dg(u,v)|$$

and

$$T_{22} \leq \iint_{C_2} dzdw \iint_{I} \mathcal{M}^{\frac{-4}{\pi^2}\{\frac{(1-u)(M)z^2}{u^2} + \frac{(1-v)(N)w^2}{v^2}\}}$$

$$\times \{2(M)\frac{1}{\delta}\}\{\frac{2}{\sigma}\}|dg(u,v)| \ .$$

Since by the Dominated Convergence Theorem both of the integrals on the right hand sides tend to zero as M, N tend to infinity we see that the second integration is also $o(1)$,

$$T_2 = o(1).$$

In the same manner the integration on the third subcell C_3 satisfies

$$T_3 = o(1).$$

For the fourth and last subcell C_4 we find

$$T_4 \leq \iint_{C_4} dzdw \left| \iint_{[\delta,1)\times[\delta,1)} (up_1)^M (vp_2)^N \{\frac{\sin\alpha\sin\beta}{zw} - \frac{\sin(2Mz/u)\sin(2Nw/v)}{zw}\} \right.$$

$$\left. \times \ dg(u,v) \right|$$

$$T_4 \leq \iint_{C_4} dzdw \iint_{[\delta,1)\times[\delta,1)} (up_1)^M (vp_2)^N \{\frac{2}{\sigma^2}\} |dg(u,v)| \ ,$$

so $T_4 \to 0$ as $M, N \to \infty$, by bounded convergence. Thus all four integrals are $o(1)$ and the result is established.

Proposition 4.10.

$$\iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \Big| \iint_{[\delta,1)\times[\delta,1)} (up_1)^M \phi(N,w,v,p_2,q_2) \times \{\frac{\sin(2Mz+Mq_1}{z}\} dg(u,v) \Big|$$

$$= \iint_{[0,\pi/2]\times[0,\pi/2]} |C_1| dzdw$$

$$= o(\ln M \ln N)$$

and

$$\iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \Big| \iint_{[\delta,1)\times[\delta,1)} (vp_2)^N \phi(M,z,u,p_1,q_1) \times \{\frac{\sin(2Nw+Nq_2}{w}\} dg(u,v) \Big|$$

$$= \iint_{[0,\pi/2]\times[0,\pi/2]} |D_1| dzdw$$

$$= o(\ln M \ln N) \ .$$

Proof. We will prove the first assertion, the second is proved in a similar manner. The function ϕ is always bounded and tends to zero as N tends to infinity for w positive, while

$$\left| \frac{\sin(2Mz+Mq_1)}{z} \right| \leq \frac{2M}{\delta} + O(z^2)$$

for small z, whereas for $z > \sigma$ the bound is $1/\sigma$.

Thus choose a δ so that Lemma 4.7 holds for (up_1), and a σ such that if $0 \leq z \leq \sigma$ then Lemma 4.6 holds.

Then

$$\iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \left| \iint_{[\delta,1)\times[\delta,1)} (up_1)^M \phi(N,w,v,p_2,q_2) \times \left\{\frac{\sin(2Mz+Mq_1}{z}\right\} dg(u,v) \right|$$

$$\leq \iint_{[0,\sigma]\times[0,\pi/2]} \left\{ \iint_{[\delta,1)\times[\delta,1)} \mathcal{M}^{-4(1-u)z^2(M)/\pi^2u^2} \times O(1)\left\{\frac{2M}{\delta} + O(z^2)\right\} |dg(u,v)| \right\} dzdw$$

$$+ \iint_{[\sigma,\pi/2]\times[0,\pi/2]} \left\{ \iint_{[\delta,1)\times[\delta,1)} O(1)\left\{\frac{1}{\sigma}\right\} |dg(u,v)| \right\} dzdw$$

$= o(\ln M \ln N)$.

Proposition 4.11.

$$\iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \left| \iint_{[\delta,1)\times[\delta,1)} \left\{ \frac{\sin((2m+1)z+(M)q_1}{\sin z} \frac{\sin((2n+1)w)}{\sin w} - \frac{\sin(2(M)z/u)}{z} \frac{\sin(2(N)w)}{w} \right\} (up_1)^M dg(u,v) \right|$$

$$= \iint_{[0,\pi/2]\times[0,\pi/2]} |E_1 - E_2| dzdw$$

$$= o(\ln M \ln N)$$

and

$$\iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \left| \iint_{\{1\}\times[\delta,1)} \left\{ \frac{\sin((2n+1)w+(N)q_2)}{\sin w} \frac{\sin((2m+1)z)}{\sin z} - \frac{\sin(2(N)w/v)}{w} \frac{\sin(2(M)z)}{z} \right\} (vp_2)^N dg(u,v) \right|$$

$$= \iint_{[0,\pi/2]\times[0,\pi/2]} |F_1 - F_2| dzdw$$

$$= o(\ln M \ln N)$$

Proof. Consider that with $\alpha = (2m+1)z + Mq_1$, $\beta = 2Mz + Mq_1$, we have

$$\left|\frac{\sin\alpha}{\sin z} - \frac{\sin\beta}{z}\right| = \left|\frac{\sin\alpha}{\sin z} - \frac{\sin\alpha}{z} + \frac{\sin\alpha}{z} - \frac{\sin\beta}{z}\right|$$

$$\leq |\sin\alpha| \left|\frac{1}{\sin z} - \frac{1}{z}\right| + \frac{1}{z}|\sin\alpha - \sin\beta| \leq$$

$$\leq O(1) + \frac{1}{z} O(1) |\sin((\alpha-\beta)/2)|$$

$$\leq O(1) + O(1)|\alpha-\beta|/z$$

$$\leq O(1) + O(1/z)z = O(1) .$$

Hence

$$\frac{\sin \alpha}{\sin z} = \frac{\sin \beta}{z} + O(1), \quad \text{uniformly in } z, w.$$

Also

$$\left| \frac{\sin((2n+1)w)}{\sin w} - \frac{\sin 2Nw}{w} \right|$$

$$= \left| \frac{\sin((2n+1)w)}{\sin w} - \frac{\sin((2n+1)w)}{w} + \frac{\sin((2n+1)w)}{w} - \frac{\sin 2Nw}{w} \right|$$

$$\leq |\sin(2n+1)w| \left| \frac{1}{\sin w} - \frac{1}{w} \right| + \frac{1}{w} |\sin(2n+1)w - \sin 2Nw|$$

$$\leq O(1) + \frac{1}{w} O(1) |\sin(w/2)| = O(1) .$$

Thus

$$\frac{\sin((2n+1)w)}{\sin w} = \frac{\sin 2Nw}{w} + O(1), \quad \text{for all } z, w \tag{A}$$

Choose σ, δ so that Lemmas 4.6 and 4.7 are true. If $0 \leq z \leq \sigma$ then $q_1 = 2(z/u) - 2z + O(z^3)$, $\delta \leq u \leq 1$, and

$$\left| \frac{\sin \beta}{z} - \frac{\sin(2(M)z/u)}{z} \right| \leq O(1) \left| \sin\left(\frac{\beta - 2(M)z/u}{2} \right) \right| \frac{1}{z}$$

$$\leq O(1/z)(M)O(z^3)$$

$$\leq (M)O(z^2) .$$

Thus

$$\frac{\sin \alpha}{\sin z} = \frac{\sin(2Mz/u)}{z} + (M)O(z^2) + O(1), \quad \text{for all} \quad w. \tag{B}$$

If $\sigma \leq z \leq \pi/2$ then

$$\left| \frac{\sin \alpha}{\sin z} - \frac{\sin(2(M)z/u)}{z} \right| \quad \text{is bounded}$$

or

$$\frac{\sin \alpha}{\sin z} = \frac{\sin(2Mz/u)}{z} + O(1) \quad \text{for all} \quad w. \tag{C}$$

For $0 \leq z \leq \sigma$, $0 \leq w \leq \pi/2$ we find by using (A), (B) that

$$\frac{\sin \alpha}{\sin z} \frac{\sin (2n+1)w}{\sin w}$$

$$= \left\{ \frac{\sin(2Mz/u)}{z} + (M)O(z^2) + O(1) \right\} \left\{ \frac{\sin 2Nw}{w} + O(1) \right\}$$

$$= \frac{\sin(2Mz/u)}{z} \frac{\sin 2Nw}{w} + O(1) \frac{\sin(2Mz/u)}{z} + (M)O(z^2) \frac{\sin 2Nw}{w}$$

$$+ (M)O(z^2) + O(1) \frac{\sin 2Nw}{w} + O(1), \tag{D}$$

while for $\sigma \leq z \leq \pi/2$, $0 \leq w \leq \pi/2$ we have

$$\frac{\sin \alpha}{\sin z} \frac{\sin(2n+1)w}{\sin w} = \left\{ \frac{\sin \alpha}{\sin z} \right\} \left\{ \frac{\sin 2Nw}{w} + O(1) \right\}$$

$$= \frac{\sin \alpha}{\sin z} \frac{\sin 2Nw}{w} + O(1) \frac{\sin \alpha}{\sin z}. \tag{E}$$

Thus

$$\iint_{[0,\pi/2]\times[0,\pi/2]} |E_1-E_2|\,dzdw$$

$$= \iint_{[0,\sigma]\times[0,\pi/2]} |E_1-E_2|\,dzdw + \iint_{[\sigma,\pi/2]\times[0,\pi/2]} |E_1-E_2|\,dzdw$$

$$\leq \iint_{[0,\sigma]\times[0,\pi/2]} dzdw \Bigg\{ \iint_{[\delta,1)\times\{1\}} O(1)\,\frac{|\sin(2Mz/u)|}{z}\,(up_1)^M|dg(u,v)|$$

$$+ \iint_{[\delta,1)\times\{1\}} (M)O(z^2)\,\frac{|\sin 2Nw|}{w}\,(up_1)^M|dg(u,v)|$$

$$+ \iint_{[\delta,1)\times\{1\}} (M)O(z^2)(up_1)^M|dg(u,v)|$$

$$+ \iint_{[\delta,1)\times\{1\}} O(1)\,\frac{|\sin 2Nw|}{w}\,(up_1)^M|dg(u,v)|$$

$$+ \iint_{[\delta,1)\times\{1\}} O(1)(up_1)^M|dg(u,v)|\Bigg\}$$

$$+ \iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \Bigg\{ \iint_{[\delta,1)\times\{1\}} \Big|\frac{\sin a}{\sin z}\,\frac{\sin 2Nw}{w} + O(1)\,\frac{\sin a}{\sin z}$$

$$- \frac{\sin 2Nw}{w}\,\frac{\sin(2Mz/u)}{z}\Big|\,(up_1)^M|dg(u,v)|\Bigg\}$$

$$\leq \iint_{[0,\sigma]\times[0,\pi/2]} dzdw \Bigg\{ \iint_{[\delta,1)\times\{1\}} O(1)\,\frac{M}{\delta}(up_1)^M|dg(u,v)| \qquad +$$

$$+ \int_{[0,1]} \frac{|\sin t|}{t}\,dt + \int_{[1,N\pi]} \frac{1}{t}\,dt \Big\} \int_{[0,\sigma]} O(z^2)$$

$$\times \left\{ \iint_{[\delta,1)\times\{1\}} (M)(up_1)^M |dg(u,v)| \right\} dz$$

$$+ \iint_{[0,\sigma]\times[0,\pi/2]} O(z^2) \left\{ \iint_{[\delta,1)\times\{1\}} (M)(up_1)^M |dg(u,v)| \right\} dzdw$$

$$+ \left\{ \int_{[0,1]} \frac{|\sin t|}{t}\,dt + \int_{[1,N\pi]} \frac{1}{t}\,dt \right\} \int_{[0,\sigma]} O(1)\mathrm{Var}(g)dz + O(1)$$

$$+ \iint_{[\sigma,\pi/2]\times[0,\pi/2]} \left\{ \iint_{[\delta,1)\times\{1\}} (up_1)^M \left\{ \left|\frac{\sin 2Nw}{w}\right| \left| \frac{\sin \alpha}{\sin z} - \frac{\sin(2Mz/u)}{z} \right| + O(1)\frac{1}{\sin \sigma} \right\} |dg(u,v)| \right\} dzdw,$$

(where $t = 2Nw$ has been used)

$$\leq o(1) + O(\ln N) + o(1) + O(\ln N) + \left\{ \int_{[0,1]} \frac{|\sin t|}{t}\,dt + \int_{[1,N\pi]} \frac{1}{t}\,dt \right\}$$

$$\times \iint_{[\sigma,\pi/2]} O(1) \left\{ \iint_{[\delta,1)\times\{1\}} (up_1)^M |dg(u,v)| \right\} dz$$

$$+ O(1) \iint_{[\sigma,\pi/2]\times[0,\pi/2]} dzdw$$

$\leq o(\ln M \ln N)$.

The second result is established in a similar manner.

Proposition 4.12.

$$\iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \left| \iint_{\{1\}\times\{1\}} \left\{ \frac{\sin(2m+1)z}{\sin z} \frac{\sin(2n+1)w}{\sin w} - \frac{\sin 2Mz}{z} \frac{\sin 2Nw}{w} \right\} dg(u,v) \right|$$

$$= \iint_{[0,\pi/2]\times[0,\pi/2]} |G_1 - G|\, dzdw$$

$$= o(\ln M \ln N).$$

Proof. Since

$$\left| \frac{\sin(2k+1)z}{\sin z} - \frac{\sin 2(k+1)z}{z} \right| = O(1) \text{ uniformly for all } k, z$$

we can write

$$\frac{\sin(2m+1)z}{\sin z} \frac{\sin(2n+1)w}{\sin w} = \left\{\frac{\sin 2Mz}{z} + O(1)\right\}\left\{\frac{\sin 2Nw}{w} + O(1)\right\}$$

$$= \frac{\sin 2Mz}{z} \frac{\sin 2Nw}{w}$$

$$+ O(1)\left\{\frac{\sin 2Nw}{w} + \frac{\sin 2Mz}{z}\right\} + O(1).$$

Thus

$$\iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \left| \iint_{\{1\}\times\{1\}} \left\{ \frac{\sin(2m+1)z \sin(2n+1)w}{\sin z \sin w} - \frac{\sin 2Mz}{z} \frac{\sin 2Nw}{w} \right\} dg(u,v) \right| \le$$

$$\leq \iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \left| \iint_{\{1\}\times\{1\}} O(1)\{ \frac{\sin 2Nw}{w} + \frac{\sin 2Mz}{z} + 1\} dg(u,v) \right|$$

$$\leq O(1) \iint_{[0,\pi/2]\times[0,\pi/2]} \{ \frac{|\sin 2Nw|}{w} + \frac{|\sin 2Mz|}{z} + 1\} dzdw$$

$$\leq O(1) \left\{ \int_{[0,\pi/2]} \frac{|\sin 2Nw|}{w} dw + \int_{[0,\pi/2]} \frac{|\sin 2Mz|}{z} dz + 1 \right\}$$

(making the substitutions $w = t/2N$, $z = t/2M$ we have)

$$\leq O(1) \left\{ \int_{[0,1]} \frac{|\sin t|}{t} dt + \int_{[1,N\pi]} \frac{|\sin t|}{t} dt + \int_{[0,1]} \frac{|\sin t|}{t} dt \right.$$

$$\left. + \int_{[1,M\pi]} \frac{|\sin t|}{t} dt + 1 \right\}$$

$$\leq O(1)\{O(1) + \ln(N\pi) + O(1) + \ln(M\pi) + 1\}$$

$$\leq o(\ln M \ln N) .$$

<u>Proposition 4.13</u>. With p_1, q_1, p_2, q_2, and $g(u,v)$ as previously defined, we have

$$L_\delta^*(m,n;g) = \frac{4}{\pi^2} \iint_{[0,\pi/2]\times[0,\pi/2]} dzdw \left| \iint_{[\delta,1)\times[\delta,1)} (up_1)^{m+1} (vp_2)^{n+1} \right.$$

$$\times \frac{\sin(2(m+1)z/u)}{z} \frac{\sin(2(n+1)w/v)}{w}$$

$$\left. \times\, dg(u,v) \right| + o(\ln M \ln N).$$

Proof. From the definition of $L^*_\delta(m, n; g)$ we have

$$\frac{4}{\pi^2} \iint_{[0,\pi/2]\times[0,\pi/2]} \Bigg| \Bigg| \iint_{[\delta,1)\times[\delta,1)} (up_1)^{m+1}(vp_2)^{n+1} \frac{\sin(2(m+1)z+(m+1)q_1)}{z}$$

$$\times \frac{\sin(2(n+1)w+(n+1)q_2)}{w} dg(u,v)$$

$$+ \iint_{[\delta,1)\times[\delta,1)} \phi(m+1, z, u, p_1, q_1)\phi(n+1, w, v, p_2, q_2) dg(u,v)$$

$$+ \iint_{[\delta,1)\times[\delta,1)} (up_1)^{m+1} \frac{\sin(2(m+1)z+(m+1)q_1)}{z} \phi(n+1, w, v, p_2, q_2) dg(u,v)$$

$$+ \iint_{[\delta,1)\times[\delta,1)} (vp_2)^{n+1} \frac{\sin(2(n+1)w+(n+1)q_2)}{w} \phi(m+1, z, u, p_1, q_1) dg(u,v)$$

$$+ \iint_{\{1\}\times[\delta,1)} \frac{\sin(2m+1)z}{\sin z} (vp_2)^{n+1} \frac{\sin((2n+1)w+(n+1)q_2)}{\sin w} dg(u,v)$$

$$+ \iint_{[\delta,1)\times\{1\}} (up_1)^{m+1} \frac{\sin((2m+1)z+(m+1)q_1)}{\sin z} \frac{\sin(2n+1)w}{\sin w} dg(u,v)$$

$$+ \iint_{\{1\}\times\{1\}} \frac{\sin(2m+1)z}{\sin z} \frac{\sin(2n+1)w}{\sin w} dg(u,v) \Bigg|$$

$$- \Bigg| \iint_{[\delta,1]\times[\delta,1]} (up_1)^{m+1}(vp_2)^{n+1} \frac{\sin(2(m+1)z/u)}{z} \frac{\sin(2(n+1)w/v)}{w}$$

$$\times dg(u,v) \Bigg| \Bigg| dzdw \quad \leq$$

$$\leq \frac{4}{\pi^2} \iint_{[0,\pi/2]\times[0,\pi/2]} \left\{ \left| \iint_{[\delta,1)\times[\delta,1)} \left\{ (up_1)^{m+1}(vp_2)^{n+1} \right.\right.\right.$$

$$\times \frac{\sin(2(m+1)z+(m+1)q_1)}{z} \frac{\sin(2(n+1)w+(n+1)q_2)}{w}$$

$$- (up_1)^{m+1}(vp_2)^{n+1} \frac{\sin(2(m+1)z/u)}{z}$$

$$\left.\left. \times \frac{\sin(2(n+1)w/v)}{w} \right\} dg(u,v) \right|$$

$$+ \left| \iint_{[\delta,1)\times[\delta,1)} \phi(m+1,z,u,p_1,q_1)\phi(n+1,w,v,p_2,q_2)dg(u,v) \right|$$

$$+ \left| \iint_{[\delta,1)\times[\delta,1)} (up_1)^{m+1} \frac{\sin(2(m+1)z+(m+1)q_1}{z} \phi(n+1,w,v,p_2,q_2)dg(u,v) \right|$$

$$+ \left| \iint_{[\delta,1)\times[\delta,1)} (vp_2)^{n+1} \frac{\sin(2(n+1)w+(n+1)q_2}{w} \phi(m+1,z,u,p_1,q_1)dg(u,v) \right|$$

$$+ \left| \iint_{[\delta,1)\times\{1\}} (up_1)^{m+1} \left\{ \frac{\sin((2m+1)z+(m+1)q_1)}{\sin z} \frac{\sin((2n+1)w)}{\sin w} \right.\right.$$

$$\left.\left. - \frac{\sin(2(m+1)z/u)}{z} \frac{\sin(2(n+1)w)}{w} \right\} dg(u,v) \right|$$

$$+ \left| \iint_{\{1\}\times[\delta,1)} (vp_2)^{n+1} \left\{ \frac{\sin((2n+1)w+(n+1)q_2}{\sin w} \frac{\sin(2m+1)z}{\sin z} \right.\right.$$

$$\left.\left. - \frac{\sin(2(n+1)w/v)}{w} \frac{\sin(2(m+1)z)}{z} \right\} dg(u,v) \right|$$

$$\left. + \left| \iint_{\{1\}\times\{1\}} \left[\frac{\sin(2m+1)z}{\sin z} \frac{\sin(2n+1)w}{\sin w} - \frac{\sin(2(m+1)z}{z} \frac{\sin(2(n+1)w}{w} \right] dg(u,v) \right| \right\} dzdw$$

$$\leq$$

$$\leq \frac{4}{\pi^2} \iint_{[0,\pi/2]\times[0,\pi/2]} \{|A-A_1|+|B|+|C_1|+|D_1|+|E_1-E_2|+F_1-F_2| + |G-G_1|\} dz dw$$

$$\leq o(\ln M \ln N)$$

by Propositions 4.8, 4.9, 4.10, 4.11, and 4.12. (Note: $M = m+1$, $N = n+1$).

We shall now use this representation of $L^*_\delta(m, n; g)$ to develop the result

$$L^*_\delta(M, N; g) = \frac{4}{\pi^2} \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \left| \iint_{[\delta, 1]\times[\delta, 1]} \sin(z/u) \sin(w/v) dg(u, v) \right| \times (1/zw) dz dw + \frac{1}{\pi^4} \left| \iint_{\{1\}\times\{1\}} dg(u, v) \right|$$

$$+ \frac{4}{\pi^3} \left\{ \ln M \int_{[1, 2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta, 1]} \frac{\sin(y/v)}{y} dg(u, v) \right| dy + \ln N \iint_{[1,2\epsilon\delta^*\sqrt{M}]} \left| \iint_{[\delta, 1]\times\{1\}} \frac{\sin(y/u)}{y} dg(u, v) \right| dy \right\}$$

$$+ o(\ln M \ln N).$$

To this end we consider the representation of $L^*(m, n; g)$ given in Proposition 4.13 and replace $m+1$ by M, $n+1$ by N. Then

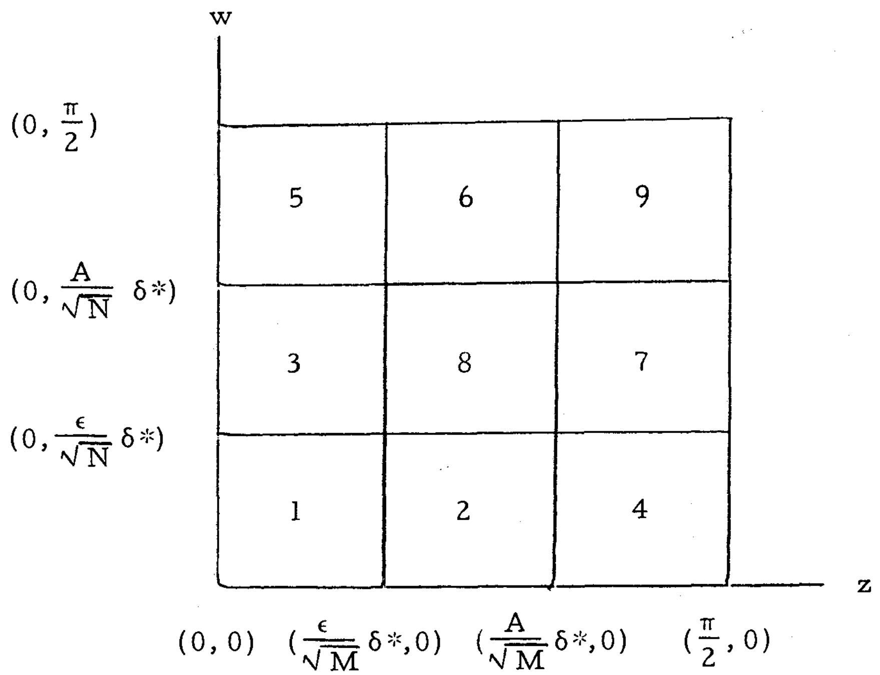
$$L_{\delta}^{*}(m, n; g) = L_{\delta}^{*}(M-1, N-1; g).$$

Straight forward estimates show that

$$|L_{\delta}^{*}(M-1, N-1; g) - L_{\delta}^{*}(M, N; g)| = o(\ln M \ln N).$$

Because of this equivalence for large M, N we shall work with $L_{\delta}^{*}(M-1, N-1; g)$ in order to simplify some of the computations.

We next divide $[0, \pi/2] \times [0, \pi/2]$ into the nine subcells given in the following diagram:



where $\delta^{*} = \frac{\delta}{\sqrt{2(1-\delta)}}$, $0 < \delta < 1$ and $0 < \epsilon < 1 < A$. We further restrict δ^{*} to satisfy $0 < \delta^{*} < \min(1/2\epsilon, \pi\sqrt{M}/2A, \pi\sqrt{N}/2A)$, $2A\delta^{*} > \max(1/\sqrt{M}, 1/\sqrt{N})$, in order that the estimates in the following propositions are valid for large M, N. The estimates made below hold for M, N sufficiently large. For convenience the quantifier "for M, N sufficiently large" will often be suppressed.

We will estimate $L^*_\delta(M-1, N-1; g)$ on each of the nine subcells. We also denote the portion of the integral representation of L^*_δ given in Proposition 4.13 over the subcell (i) by the symbol I_i and the points of the subcell by S_i.

Proposition 4.14.

$$I_1 = \frac{4}{\pi^2} \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \left| \iint_{[\delta,1]\times[\delta,1]} \sin(z/u)\sin(w/v)\,dg(u,v) \right| \frac{dz}{z}\frac{dw}{w}$$

$$+ o(\ln M \ln N) + \frac{2}{\pi^2} \mathrm{Var}(g)\epsilon^2 \ln M \ln N.$$

Proof. We first define $H_{a,b}$ by the following

$$H_{M-1,N-1}(z,w) = \iint_{[\delta,1]\times[\delta,1]} (up_1)^M (vp_2)^N \sin(2Mz/u)\sin(2Nw/v)\,dg(u,v).$$

From work at the beginning of Chapter 4 we note that

$$(up_1)^M = \left(\frac{1}{1 + \frac{4(1-u)\sin^2 z}{u^2}} \right)^{M/2}$$

$$(vp_2)^N = \left(\frac{1}{1 + \frac{4(1-v)\sin^2 w}{v^2}} \right)^{N/2}$$

For (z, w) in subcell S_1 and $M, N > 4$ we have

$$1 \geq (up_1)^M \geq 1 - \epsilon^2 > 0$$

$$1 \geq (vp_2)^N \geq 1 - \epsilon^2 > 0 .$$

Thus

$$|(up_1)^M - 1| \leq \epsilon^2, \quad |(vp_2)^N - 1| \leq \epsilon^2, \quad \text{and} \quad |(up_1)^M (vp_2)^N - 1| \leq 2\epsilon^2 .$$

Hence

$$\left| \, |H_{M-1, N-1}(z, w)| - \left| \iint_{[\delta, 1]\times[\delta, 1]} \sin(2Mz/u)\sin(2Nw/v)dg(u, v) \right| \, \right|$$

$$\leq \left| \iint_{[\delta, 1]\times[\delta, 1]} \{(up_1)^M (vp_2)^N - 1\} \sin(2Mz/u)\sin(2Nw/v)dg(u, v) \right|$$

$$\leq \iint_{[\delta, 1]\times[\delta, 1]} |(up_1)^M (vp_2)^N - 1| \, |dg(u, v)|$$

$$\leq 2\epsilon^2 \, \mathrm{Var}(g),$$

or if we replace the $\sin(2Mz/u)$ by $(2Mz/u)$ the bound becomes

$$2\epsilon^2 \, \mathrm{Var}(g)(2Mz/\delta) .$$

Also for all $z, w \geq 0$ we have

$$\left| \, |H_{M-1,N-1}(z,w)| - \left| \iint_{[\delta,1]\times[\delta,1]} \sin(2Mz/u)\sin(2Nw/v)dg(u,v) \right| \, \right|$$

$$\leq \iint_{[\delta,1]\times[\delta,1]} |(up_1)^M(vp_2)^N - 1| \; |(2Mz/u)| \; |(2Nw/v)| \; |dg(u,v)|$$

$$\leq 2 \iint_{[\delta,1]\times[\delta,1]} (2Mz/u)(2Nw/v)|dg(u,v)|$$

$$\leq \frac{8MNzw}{\delta^2} \mathrm{Var}(g).$$

So

$$I_1 = \frac{4}{\pi^2} \iint_{S_1} |H_{M-1,N-1}(z,w)| \frac{dz}{z}\frac{dw}{w}$$

$$= \frac{4}{\pi^2} \iint_{S_1} \left| \iint_{[\delta,1]\times[\delta,1]} \sin(2Mz/u)\sin(2Nw/v)dg(u,v) \right| \frac{dz}{z}\frac{dw}{w} + E_0$$

where

$$|E_0| \leq \frac{4}{\pi^2} \iint_{S_{11}} (8MNzw/\delta^2)\mathrm{Var}(g) \frac{dz}{z}\frac{dw}{w} + \frac{4}{\pi^2} \iint_{S_{12}} 2\epsilon^2 \,\mathrm{Var}(g) \frac{dz}{z}\frac{dw}{w}$$

$$+ \frac{4}{\pi^2} \iint_{S_{13}} 2\epsilon^2 \,\mathrm{Var}(g)(2Nw/\delta) \frac{dz}{z}\frac{dw}{w} + \frac{4}{\pi^2} \iint_{S_{14}} 2\epsilon^2 \mathrm{Var}(g)(2Mz/\delta) \frac{dz}{z}\frac{dw}{w},$$

where

$$S_{11} = [0, \epsilon\delta^*/M] \times [0, \epsilon\delta^*/N],$$

$$S_{12} = [\epsilon\delta^*/M, \epsilon\delta^*/\sqrt{M}] \times [\epsilon\delta^*/N, \epsilon\delta^*/\sqrt{N}],$$

$$S_{13} = [\epsilon\delta^*/M, \epsilon\delta^*/\sqrt{M}] \times [0, \epsilon\delta^*/N],$$

$$S_{14} = [0, \epsilon\delta^*/M] \times [\epsilon\delta^*/N, \epsilon\delta^*/\sqrt{N}]$$

or

$$\begin{aligned} |E_0| \le \frac{4}{\pi^2} \Big\{ & (8MN/\delta^2)\, \mathrm{Var}(g)[\frac{\epsilon\delta^*\epsilon\delta^*}{MN}] \\ & + 2\epsilon^2\, \mathrm{Var}(g)[\ln M^{1/2} \ln N^{1/2}] \\ & + 2\epsilon^2\, \mathrm{Var}(g)(2n/\delta)[\ln M^{1/2}][\frac{\epsilon\delta^*}{N}] \\ & + 2\epsilon^2\, \mathrm{Var}(g)(2M/\delta)[\frac{\epsilon\delta^*}{M})[\ln N^{1/2}] \Big\} \\ \le \frac{2\epsilon^2}{\pi^2} & \mathrm{Var}(g) \ln M \ln N + o(\ln M \ln N). \end{aligned}$$

Next observe that

$$\iint_{[\epsilon\delta^*/\sqrt{M},\, 1/2\sqrt{M}] \times [\epsilon\delta^*/\sqrt{N},\, 1/2\sqrt{N}]} \left| \iint_{[\delta, 1] \times [\delta, 1]} \sin(2Mz/u)\sin(2Nw/v) \times dg(u, v) \right| \frac{dz}{z}\frac{dw}{w}$$

$$\le \mathrm{Var}(g)[\ln 2\epsilon\delta^*]^2 .$$

Making the change of variables $Z = 2Mz$, $W = 2Nw$, we then see the following:

$$\iint_{[2\sqrt{M}\epsilon\delta^*, \sqrt{M}]\times[2\sqrt{N}\epsilon\delta^*, \sqrt{N}]} \left| \iint_{[\delta, 1]\times[\delta, 1]} \sin(Z/u)\sin(W/v)dg(u, v) \right| \times \frac{dZ}{Z}\frac{dW}{W}$$

$$\leq \operatorname{Var}(g)[\ln 2\epsilon\delta^*]^2.$$

For convenience let

$$\Delta_{MN}(Z, W) = \iint_{[\delta, 1]\times[\delta, 1]} \sin(Z/u)\sin(W/v)dg(u, v).$$

Then we can write

$$I_1 = \frac{4}{\pi^2} \iint_{[0, 2\sqrt{M}\epsilon\delta^*]\times[0, 2\sqrt{N}\epsilon\delta^*]} |\Delta_{MN}(Z, W)| \frac{dZ}{Z}\frac{dW}{W} + E_0 .$$

For large M, N and fixed ϵ, δ^* we see that

$$\frac{4}{\pi^2} \iint_{[1, 2\sqrt{M}\epsilon\delta^*]\times[1, 2\sqrt{N}\epsilon\delta^*]} |\Delta_{MN}(Z, W)| \frac{dZ}{Z}\frac{dW}{W}$$

$$\leq \frac{4}{\pi^2} \iint_{[0, 2\sqrt{M}\epsilon\delta^*]\times[0, 2\sqrt{N}\epsilon\delta^*]} |\Delta_{MN}(Z, W)| \frac{dZ}{Z}\frac{dW}{W}$$

We now integrate $\Delta_{MN}(Z, W)$ over $[1, \sqrt{M}]\times[1, \sqrt{N}]$ by the following decomposition, where we let "..." stand for the quantities behind the integral signs.

$$\frac{4}{\pi^2} \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \cdots$$

$$= \frac{4}{\pi^2} \iint_{[1,\, 2\sqrt{M}\epsilon\,\delta^*]\times[1,\, 2\sqrt{N}\epsilon\,\delta^*]} \cdots + \frac{4}{\pi^2} \iint_{[1,\, 2\sqrt{M}\epsilon\,\delta^*]\times[2\sqrt{N}\epsilon\,\delta^*,\, \sqrt{N}]} \cdots$$

$$+ \frac{4}{\pi^2} \iint_{[2\sqrt{M}\epsilon\,\delta^*,\, \sqrt{M}]\times[1,\, 2\sqrt{N}\epsilon\,\delta^*]} \cdots + \frac{4}{\pi^2} \iint_{[2\sqrt{M}\epsilon\,\delta^*,\, \sqrt{M}]\times[2\sqrt{N}\epsilon\,\delta^*,\, \sqrt{N}]} \cdots$$

and we shall call these integrals

$$A_0 = A_1 + A_2 + A_3 + A_4 .$$

Estimating A_2, A_3, A_4 we find

$$|A_2| \leq \frac{4}{\pi^2} \iint_{[1,\, 2\sqrt{M}\epsilon\,\delta^*]\times[2\sqrt{N}\epsilon\,\delta^*,\, \sqrt{N}]} (1)(1)\, \mathrm{Var}(g) \frac{dZ}{Z} \frac{dW}{W}$$

$$\leq O(1) \ln M \ln(1/2\epsilon\,\delta^*) = o(\ln M \ln N) ,$$

while in a similar fashion

$$|A_3| = o(\ln M \ln N)$$

and

$$|A_4| \leq \frac{4}{\pi^2} \mathrm{Var}(g)[\ln(2\epsilon\,\delta^*)]^2 = o(\ln M \ln N),$$

as was just shown after making the change of variables.

Thus we can write

$$I_1 = \frac{4}{\pi^2} \iint_{[0,1]\times[0,1]} \cdots + \frac{4}{\pi^2} \iint_{[0,1)\times[1,2\sqrt{N}\epsilon\delta*]} \cdots$$

$$+ \frac{4}{\pi^2} \iint_{[1,2\sqrt{M}\epsilon\delta*]\times[0,1]} \cdots + A_1 + E_0 .$$

If we denote the first three integrals as B_1, B_2, B_3 then symbolically

$$I_1 = B_1 + B_2 + B_3 + A_0 - A_2 - A_3 - A_4 + E_0$$

$$= A_0 + E_5 ,$$

where

$$|E_5| \leq |E_0| + |B_1| + |B_2| + |B_3| + |A_2| + |A_3| + |A_4| .$$

We already have bounds for E_0, A_2, A_3, A_4. Thus we now find

$$|B_1| \leq \frac{4}{\pi^2} \iint_{[0,1]\times[0,1]} \frac{Z}{\delta}\frac{W}{\delta} \operatorname{Var}(g) \frac{dZ}{Z}\frac{dW}{W}$$

$$\leq \frac{4}{\pi^2}\frac{1}{\delta^2} \operatorname{Var}(g) = o(\ln M \ln N) ,$$

$$|B_2| \le \frac{4}{\pi^2} \iint_{[0,1]\times[1,2\sqrt{N}\epsilon\delta^*]} \frac{Z}{\delta}(1)\,\mathrm{Var}(g)\,\frac{dZ}{Z}\frac{dW}{W}$$

$$\le \frac{4}{\pi^2}\frac{1}{\delta}\,\mathrm{Var}(g)O(1)\ln N = o(\ln M \ln N),$$

and similarly

$$|B_3| \le \frac{4}{\pi^2}\frac{1}{\delta}\,\mathrm{Var}(g)O(1)\ln M = o(\ln M \ln N).$$

Combining these estimates yields

$$I_1 = \frac{4}{\pi^2} \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \left| \iint_{[\delta,1]\times[\delta,1]} \sin\frac{Z}{u}\,\sin\frac{W}{v}\,dg(u,v)\right| \frac{dZ}{Z}\frac{dW}{W} + E_5$$

where

$$|E_5| \le \mathrm{Var}(g)\,\frac{2\epsilon^2}{\pi^2}\ln M \ln N + o(\ln M \ln N).$$

This is the desired result.

Proposition 4.15.

$$I_2 = o(\ln M \ln N), \quad \text{and} \quad I_3 = o(\ln M \ln N)$$

Proof.

$$I_2 = \frac{4}{\pi^2} \iint_{[\epsilon\delta^*/\sqrt{M},\,A\delta^*/\sqrt{M}]\times[0,\,\epsilon\delta^*/\sqrt{N}]} |H_{M-1,N-1}(z,w)|(1/zw)\,dz\,dw =$$

$$= \frac{4}{\pi^2} \iint_{[\epsilon\delta^*/\sqrt{M},\, A\delta^*/\sqrt{M}]\times[0,\, \epsilon\delta^*/N]} \cdots$$

$$+ \iint_{[\epsilon\delta^*/\sqrt{M},\, A\delta^*/\sqrt{M}]\times[\epsilon\delta^*/N,\, \epsilon\delta^*/\sqrt{N}]} \cdots$$

$$\leq \frac{4}{\pi^2} \iint_{[\epsilon\delta^*/\sqrt{M},\, A\delta^*/\sqrt{M}]\times[0,\, \epsilon\delta^*/N]} (2Nw/\delta zw)\ \mathrm{Var}(g)dzdw$$

$$+ \iint_{[\epsilon\delta^*/\sqrt{M},\, A\delta^*/\sqrt{M}]\times[\epsilon\delta^*/N,\, \epsilon\delta^*/\sqrt{N}]} (1/zw)\ \mathrm{Var}(g)dzdw$$

$$\leq \frac{4}{\pi^2} \mathrm{Var}(g)\{ \frac{2N}{\delta} (\frac{\epsilon\delta^*}{N}) \ln(A/\epsilon) + (1/2) \ln(A/\epsilon) \ln N$$

$$= o(\ln M \ln N).$$

$I_3 = o(\ln M \ln N)$ follows by symmetry.

Definition 4. 16. We define the functions $\phi_1(A)$, $\phi_2(A)$, $E(t)$, $\psi_1(A)$, $\psi_2(A)$ by

$$E(t) = \exp(-4(1-t)(a\delta^*)^2/t^2\pi^2)$$

$$\phi_1(A) = \iint_{[\delta,\, 1)\times[\delta,\, 1]} E(u)\,|dg(u, v)|\ ,$$

$$\phi_2(A) = \iint_{[\delta,\, 1]\times[\delta,\, 1)} E(v)\,|dg(u, v)|\ ,$$

$$\psi_1(A) = \iint_{\{1\}\times[\delta,1)} E(v)|dg(u,v)| ,$$

$$\psi_2(A) = \iint_{[\delta,1)\times\{1\}} E(u)|dg(u,v)| ,$$

where $g(u,v)$ is usually defined.

Note that $\phi_i(A) \to 0$ as $A \to \infty$, $\psi_i(A) \to 0$ as $A \to \infty$.

Proposition 4.17.

$$I_4 = \frac{4}{\pi^3} \ln M \int_{[1, 2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,1]} \sin(y/v)dg(u,v) \right| (1/y)dy$$

$$+ O(\ln M \ln N)\{\phi_1(A)+\epsilon^2\} + o(\ln M \ln N).$$

$$I_5 = \frac{4}{\pi^3} \ln N \int_{[1, 2\epsilon\delta^*\sqrt{M}]} \left| \iint_{[\delta,1]\times\{1\}} \sin(y/u)dg(u,v) \right| (1/y)dy$$

$$+ O(\ln M \ln N)\{\phi_2(A)+\epsilon^2\} + o(\ln M \ln N) .$$

Proof. For sufficiently large M and N and for (z,w) in S_4 we have $(up_1)^M \le E(u)$ and $|(vp_2)^N - 1| < \epsilon^2$. (a)

For convenience we will "ignore" the factor $\frac{4}{\pi^2}$ in I_4. Then

$$\iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{[\delta,1]\times[\delta,1]} (up_1)^M (vp_2)^N \frac{\sin(2Mz/u)}{z} \times \frac{\sin(2Nw/v)}{w}\, dg(u,v) \right| dzdw$$

$$= \iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{[\delta,1)\times[\delta,1)} \cdots + \iint_{\{1\}\times[\delta,1]} \cdots + \iint_{[\delta,1)\times\{1\}} \cdots \right| dzdw$$

$$= \iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,1]} \cdots \right| dzdw + P_0 ,$$

where

$$|P_0| \le \iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{[\delta,1)\times[\delta,1)} \cdots \right| dzdw + \iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{[\delta,1)\times\{1\}} \cdots \right| dzdw$$

$$\le E_1 + E_2 .$$

Now

$$E_2 = \iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{[\delta,1)\times\{1\}} (up_1)^M \frac{\sin(2Mz/u)}{z} \frac{\sin(2Nw)}{w} \times dg(u,v) \right| dzdw$$

since $(vp_2) = 1$ if $v = 1$. Letting $y = 2Nw$ and by using (a) we find

$$\leq \int_{[0,\, 2\epsilon\delta^*\sqrt{N}]} \frac{|\sin y|}{y}\, dy \int_{[A\delta^*/\sqrt{M},\, \pi/2]}$$

$$\times \left\{ \iint_{[\delta,\, 1)\times\{1\}} E(u) \frac{|\sin(2Mz/u)|}{z} |dg(u,v)| \right\} dz$$

$$\leq \{1 + \ln 2\epsilon\delta^*\sqrt{N}\} \int_{[A\delta^*/\sqrt{M},\, \pi/2]} \frac{dz}{z} \iint_{[\delta,\, 1)\times[\delta,\, 1]} E(u)|dg(u,v)|$$

$$\leq \{1 + \ln 2\epsilon\delta^*\sqrt{N}\} \ln(\pi\sqrt{M}/2A\delta^*)\phi_1(A)$$

$$= o(\ln M \ln N) + 1/4\, \phi_1(A) \ln M \ln N .$$

Now consider that letting $y = 2Nw$, $x = 2Mz$ we find

$$E_1 \leq \iint_{[A\delta^*/\sqrt{M},\, \pi/2]\times[0,\, \epsilon\delta^*/\sqrt{N}]} \left\{ \iint_{[\delta,\, 1)\times[\delta,\, 1]} E(u) \frac{|\sin(2Mz/u)|}{z} \right.$$

$$\left. \times \frac{|\sin(2Nw/v)|}{w} |dg(u,v)| \right\} dz dw$$

$$\leq \left\{ \int_{[0,1]} + \int_{[1,\, 2\epsilon\delta^*\sqrt{N}]} \right\} \int_{[2A\delta^*\sqrt{M},\, \pi M]}$$

$$\times \left\{ \iint_{[\delta,\, 1)\times[\delta,\, 1]} E(u) \frac{|\sin(y/v)|}{y} \frac{|\sin(x/u)|}{x} |dg(u,v)| \right\} dx dy .$$

Now for $0 \le y \le 1$ and $\delta \le v \le 1$ replace $(\frac{\sin(y/v)}{y})$ by $1/\delta$, while for $1 \le y \le 2\epsilon\delta^*\sqrt{N}$ replace $\sin(y/v)$ by 1. Finally replace $\sin(x/u)$ by 1. Then

$$E_1 \le \{\frac{1}{\delta} + \ln 2\epsilon\,\delta^*\sqrt{N}\}\{\ln(\pi\sqrt{M}/2A\delta^*)\ \phi_1(A)\}$$

$$= o(\ln M \ln N) + 1/4\ \phi_1(A) \ln M \ln N\ .$$

Thus

$$|P_0| \le o(\ln M \ln N) + 1/2\ \phi_1(A) \ln M \ln N$$

$$= o(\ln M \ln N) + \phi_1(A)\ O(\ln M \ln N).$$

Finally

$$\iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,1]} (up_1)^M (vp_2)^N \frac{\sin(2Mz/u)}{z} \times \frac{\sin(2Nw/v)}{w} dg(u,v)\right| dzdw$$

$$= \iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,1]} (vp_2)^N \frac{\sin(2Mz)}{z} \frac{\sin(2Nw/v)}{w} \times dg(u,v)\right| dzdw$$

$$= \iint_{[A\delta^*/\sqrt{M},\pi/2]\times[0,\epsilon\delta^*/\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,1]} \frac{\sin(2Mz)}{z} \frac{\sin(2Nw/v)}{w} dg(u,v)\right| \times dzdw + P_1$$

where

$$|P_1| \leq \iint_{[A\delta^*/\sqrt{M},\, \pi/2]\times[0,\, \epsilon\delta^*/\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,\, 1]} [(vp_2)^N - 1] \frac{\sin(2Mz)}{z} \right.$$

$$\left. \times \frac{\sin(2Nw/v)}{w} \, dg(u, v) \right| dzdw$$

$$= \left\{ \int_{[0,\, \epsilon\delta^*/N]} + \int_{[\epsilon\delta^*/N,\, \epsilon\delta^*/\sqrt{N}]} \right\} \int_{[A\delta^*/\sqrt{M},\, \pi/2]} |\ldots| dzdw$$

$$= \iint_{[A\delta^*/\sqrt{M},\, \pi/2]\times[0,\, \epsilon\delta^*/N]} |\ldots| dzdw$$

$$+ \iint_{[A\delta^*/M,\, \pi/2]\times[\epsilon\delta^*/N,\, \epsilon\delta^*/\sqrt{N}]} |\ldots| dzdw$$

$$= E_3 + E_4 .$$

Now for E_3 we find by making some obvious replacements

$$E_3 \leq \iint_{[A\delta^*/\sqrt{M},\, \pi/2]\times[0,\, \epsilon\delta^*/N]} \epsilon^2 (1/z)(2Nw/\delta w) \operatorname{Var}(g) dzdw$$

$$\leq \frac{\epsilon^2}{\delta} N \operatorname{Var}(g) \left\{ \frac{\epsilon\delta^*}{N} \right\} \ln(\pi\sqrt{M}/2A\delta^*) = o(\ln M \ln N)$$

while for E_4 we find

$$E_4 \le \iint_{[A\delta^*/\sqrt{M},\, \pi/2]\times[\epsilon\delta^*/N,\, \epsilon\delta^*/\sqrt{N}]} \epsilon^2\, \mathrm{Var}(g)(1/z)(1/w)\,dz\,dw$$

$$\le \epsilon^2\, \mathrm{Var}(g)(\tfrac{1}{2}) \ln N \ln(\pi\sqrt{M}/2A\delta^*) = \epsilon^2 O(\ln M \ln N).$$

Thus

$$|P_1| \le o(\ln M \ln N) + \epsilon^2 O(\ln M \ln N), \quad \text{as} \quad M, N \to \infty.$$

Thus we now find

$$I_4 = \iint_{[A\delta^*/\sqrt{M},\, \pi/2]\times[0,\, \epsilon\delta^*/\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,\, 1]} \frac{\sin(2Mz)}{z} \frac{\sin(2Nw/v)}{w} dg(u,v) \right| dz\,dw$$

$$+ o(\ln M \ln N) + O(\ln M \ln N)\{\phi_1(A) + \epsilon^2\}.$$

On this integral we again make a change of variables $y = 2Nw$, $x = 2Mz$, to yield

$$\iint_{[2A\delta^*\sqrt{M},\, M\pi]\times[0,\, 2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,\, 1]} \frac{\sin x}{x} \frac{\sin(y/v)}{y}\, dg(u,v) \right| dx\,dy$$

$$= \int_{[2A\delta^*\sqrt{M},\, \pi M]} \frac{|\sin x|}{x}\, dx \left\{ \int_{[0,1]} + \int_{[1,\, 2\epsilon\delta^*\sqrt{N}]} \right\}$$

$$\times \left| \iint_{\{1\}\times[\delta,\, 1]} \frac{\sin(y/v)}{y}\, dg(u,v) \right| dy \quad =$$

$$= \int_{[2A\delta^*\sqrt{M},\, M\pi]} \frac{|\sin x|}{x}\,dx \int_{[0,\,1]} \left| \iint_{\{1\}\times[\delta,\,1]} \frac{\sin(y/v)}{y}\,dg(u,v) \right| dy$$

$$+ \int_{[2A\delta^*\sqrt{M},\, M\pi]} \frac{|\sin x| - (2/\pi)}{x}\,dx \int_{[1,\, 2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,\,1]} \cdots \right| dy$$

$$+ \frac{2}{\pi} \int_{[2A\delta^*\sqrt{M},\, M\pi]} \frac{1}{x}\,dx \int_{[1,\, 2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,\,1]} \cdots \right| dy$$

$$= E_5 + E_6 + E_7 .$$

Now $E_5 = O(\ln M)$, for obvious reasons, while because

$$\sup_{V > U \geq 1} \left| \int_U^V \frac{|\sin x| - (2/\pi)}{x}\,dx \right| < \infty$$

we have

$$E_6 = O(\ln N).$$

Hence

$$E_5 + E_6 = o(\ln M \ln N).$$

Finally

$$E_7 = \frac{2}{\pi} \{\ln M\pi - \ln 2A\delta^*\sqrt{M}\} \int_{[1,\, 2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,\,1]} \cdots \right| dy$$

$$= o(\ln M \ln N) + \frac{1}{\pi} \ln M \int_{[1,\, 2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,\,1]} \frac{\sin(y/v)}{y}\,dg(u,v) \right| dy.$$

Hence, collecting results and multiplying by $4/\pi^2$, we find

$$I_4 = \frac{4}{\pi^3} \ln M \int_{[1,\, 2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,\, 1]} \frac{\sin(y/v)}{y} dg(u, v) \right| dy$$

$$+ O(\ln M \ln N)\{\phi_1(A) + \epsilon^2\} + o(\ln M \ln N).$$

The results for I_5 follow by symmetry.

Proposition 4.18.

$$I_6 = o(\ln M \ln N)$$

$$I_7 = o(\ln M \ln N)$$

$$I_8 = o(\ln M \ln N).$$

Proof. On all of the subcells S_6, S_7, S_8 we have

$$|H_{M-1,N-1}(z, w)| \le \mathrm{Var}(g).$$

Thus

$$I_7 \le \mathrm{Var}(g) \frac{4}{\pi^2} \iint_{S_7} \frac{dz}{z} \frac{dw}{w}$$

$$\le \mathrm{Var}(g) \frac{4}{\pi^2} \{\ln(\pi\sqrt{M}/2A\delta^*) \ln(A/\epsilon)\}$$

$$\le o(\ln M \ln N).$$

By symmetry

$$I_6 \le \mathrm{Var}(g) \frac{4}{\pi^2} \ln(\pi\sqrt{N}/2A\delta^*) \ln(A/\epsilon) = o(\ln M \ln N).$$

On the other hand

$$I_8 \le \mathrm{Var}(g)\,\frac{4}{\pi^2}\iint_{S_8}\frac{dz}{z}\frac{dw}{w} = \frac{4}{\pi^2}\mathrm{Var}(g)\{\ln(A/\epsilon)\}^2$$

$$= o(\ln M \ln N).$$

Proposition 4.19.

$$I_9 = \left|\iint_{\{1\}\times\{1\}} dg(u,v)\right|\frac{4}{\pi^2}\ \ln M \ln N + E_{15}$$

where

$$|E_{15}| \le \ln M \ln N\ \Phi(A) + o(\ln M \ln N)$$

and

$\Phi(A)$ is a function which tends to zero as A tends to infinity.

Proof. Here $S_9 = [A\delta^*/\sqrt{M}, \pi/2] \times [A\delta^*/\sqrt{N}, \pi/2]$. Consider then

$$\iint_{S_9}\left|\left|\iint_{[\delta,1]\times[\delta,1]}(up_1)^M(vp_2)^N\sin(2Mz/u)\,\sin(2Nw/v)dg(u,v)\right| - \left|\sin(2Mz)\,\sin(2Nw)\iint_{\{1\}\times\{1\}}dg(u,v)\right|\right|(1/zw)dzdw$$

$$\le \iint_{S_9}\left|\iint_{[\delta,1]\times[\delta,1]}(up_1)^M(vp_2)^N\sin(2Mz/u)\,\sin(2Nw/v)dg(u,v) - \sin(2Mz)\,\sin(2Nw)\iint_{\{1\}\times\{1\}}dg(u,v)\right|(1/zw)dzdw \quad =$$

$$= \iint_{S_9} \Bigg| \iint_{[\delta, 1]\times[\delta, 1]} (up_1)^M (vp_2)^N \sin(2Mz/u)\ \sin(2Nw/v) dg(u, v)$$

$$+ \iint_{\{1\}\times[\delta, 1]} \cdots + \iint_{[\delta, 1]\times\{1\}} \cdots$$

$$+ \iint_{\{1\}\times\{1\}} [(up_1)^M (vp_2)^N \sin(2Mz/u)\ \sin(2Nw/v)$$

$$- \sin(2Mz)\ \sin(2Nw)] dg(u, v) \Bigg| (1/zw) dz dw$$

$$\leq \iint_{S_9} \left\{ \iint_{[\delta, 1]\times[\delta, 1]} |\ldots| \right\} (1/zw) dz dw + \iint_{S_9} \left\{ \iint_{\{1\}\times[\delta, 1]} |\ldots| \right\} (1/zw) dz dw$$

$$+ \iint_{S_9} \left\{ \iint_{[\delta, 1]\times\{1\}} |\ldots| \right\} (1/zw) dz dw$$

$$+ \iint_{S_9} \left\{ \iint_{\{1\}\times\{1\}} |\ldots| \right\} (1/zw) dz dw$$

$$= D_1 + D_2 + D_3 = D_4 .$$

Now

$$D_1 \leq \iint_{S_9} \left\{ \iint_{[\delta, 1]\times[\delta, 1]} E(u) |dg(u, v)| \right\} (1/zw) dz dw$$

$$= \iint_{S_9} \phi_1(A)(1/zw) dz dw = \iint_{S_9} \phi_1(A)(1/zw) dz dw$$

$$D_2 \le \iint_{S_9} \left\{ \iint_{\{1\}\times[\delta,1]} E(v)\,|dg(u,v)| \right\} (1/zw)\,dz\,dw$$

$$= \psi_1(A) \iint_{S_9} (1/zw)\,dz\,dw\ .$$

Similarly for D_3,

$$D_3 \le \psi_2(A) \iint_{S_9} (1/zw)\,dz\,dw\ ,$$

while

$$D_4 \equiv 0.$$

Therefore

$$D_1 + D_2 + D_3 + D_4 = \{\phi_1(A)+\psi_1(A)+\psi_2(A)\} \iint_{S_9} (1/zw)\,dz\,dw$$

$$= \Phi(A)\ \ln(\pi\sqrt{N}/2A\delta^*)\ \ln(\pi\sqrt{M}/2A\delta^*)$$

$$= (1/4)\Phi(A)\ \ln M \ln N + o(\ln M \ln N).$$

Thus

$$I_9 = \frac{4}{\pi^2} \iint_{S_9} \frac{|\sin(2Mz)\ \sin(2Nw)|}{zw}\,dz\,dw \left| \iint_{\{1\}\times\{1\}} dg(u,v) \right| + E_{10}$$

where

$$|E_{10}| \le (1/4)\Phi(A)\ \ln M \ln N + o(\ln M \ln N) = O(\ln M \ln N)\Phi(A).$$

Considering the integral in this last representation of I_9 we see

$$\left| \iint_{\{1\}\times\{1\}} dg(u,v) \right| \frac{4}{\pi^2} \iint_{S_9} \frac{|\sin(2Mz)\ \sin(2Nw)|}{zw} dzdw$$

$$= \left| \iint_{\{1\}\times\{1\}} dg(u,v) \right| \frac{4}{\pi^2} \iint_{S_9} \frac{\begin{array}{l} |\sin(2Mz)\ \sin(2Nw)| - (2/\pi)|\sin(2Nw)| \\ + (2/\pi)|\sin(2Nw)| - (4/\pi^2) \end{array}}{zw} dzdw$$

$$+ \left| \iint_{\{1\}\times\{1\}} dg(u,v) \right| \frac{16}{\pi^4} \iint_{S_9} (1/zw) dzdw$$

$$= F_1 + F_2 .$$

It follows that

$$F_2 = \left| \iint_{\{1\}\times\{1\}} dg(u,v) \right| \frac{16}{\pi^4} \ln(\pi\sqrt{N}/2A\delta^*)\ \ln(\pi\sqrt{M}/2A\delta^*)$$

$$= \left| \iint_{\{1\}\times\{1\}} dg(u,v) \right| \frac{4}{\pi^4} \ln M \ln N + o(\ln M \ln N)$$

while

$$F_1 = \frac{4}{\pi^2} \iint_{S_9} \frac{|\sin(2Nw)|}{w} \left\{ \frac{|\sin(2Mz)| - (2/\pi)}{z} \right\} dzdw$$

$$+ \frac{4}{\pi^2} \iint_{S_9} (2/\pi) \left\{ \frac{|\sin(2Nw)| - (2/\pi)}{w} \right\} \frac{dz}{z} dw \quad \leq$$

$$\leq \frac{4}{\pi^2} \{O(1)(\ln N)C_1 + O(1)(\ln M)C_2\}$$

where

$$C_i = \sup_{V_i > U_i \geq 1} \left| \int_{U_i}^{V_i} \frac{|\sin t| - (2/3)}{t} \, dt \right|$$

is finite. Thus

$$F_1 = o(\ln M \ln N).$$

Collecting the results we find

$$I_9 = \frac{4}{\pi^4} \ln M \ln N \left| \iint_{\{1\}\times\{1\}} dg(u, v) \right| + E_{15}$$

where

$$|E_{15}| \leq O(1)\Phi(A) \ln M \ln N + o(\ln M \ln N).$$

This was the desired statement.

We are now in a position to prove the result stated on page 128.

Proposition 4.20. Under the same conditions imposed in Propositions 4.15, 4.16, 4.17, 4.18 and 4.19 we find

$L^*_\delta(M, N; g)$

$$= (4/\pi^2) \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \left| \iint_{[\delta,1]\times[\delta,1]} \sin(z/u)\sin(w/v)\,dg(u,v) \right| (1/zw)\,dz\,dw$$

$$+ (4/\pi^3) \left\{ \ln M \int_{[1,2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,1]} \sin(y/v)\,dg(u,v) \right| (1/y)\,dy \right.$$

$$\left. + \ln N \int_{[1,2\epsilon\delta^*\sqrt{M}]} \left| \iint_{[\delta,1]\times\{1\}} \sin(y/u)\,dg(u,v) \right| (1/y)\,dy \right\}$$

$$+ (4/\pi^4) \left| \iint_{\{1\}\times\{1\}} dg(u,v) \right| + o(\ln M \ln N).$$

<u>Proof</u>. We combine the results of Propositions 4.15, 4.16, 4.17, 4.18 and 4.19. Therefore

$L^*_\delta(M, N; g)$

$$= (4/\pi^2) \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \left| \iint_{[\delta,1]\times[\delta,1]} \sin(z/u)\sin(w/v)\,dg(u,v) \right| (1/zw)\,dz\,dw$$

$$+ (4/\pi^4 \ln M \ln N \left| \iint_{\{1\}\times\{1\}} dg(u,v) \right|$$

$$+ (4/\pi^3) \left\{ \ln M \int_{[1,2\epsilon\delta^*\sqrt{N}]} \left| \iint_{\{1\}\times[\delta,1]} \sin(y/v)\,dg(u,v) \right| (1/y)\,dy \right.$$

$$\left. + \ln N \int_{[1,2\epsilon\delta^*\sqrt{M}]} \left| \iint_{[\delta,1]\times\{1\}} \sin(y/u)\,dg(u,v) \right| (1/y)\,dy \right\} +$$

$$+ \{(2/\pi^2)\ \mathrm{Var}(g)\epsilon^2 + \phi_1(A) + \phi_2(A) + 2\epsilon^2\} O(\ln M \ln N) + o(\ln M \ln N)$$

$$+ E_{15}\ .$$

Thus

$$\limsup_{M,N\to\infty} \Bigg| \frac{L^*_\delta(M,N;g)}{\ln M \ln N} - \frac{(4/\pi^2)}{\ln M \ln N} \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]}$$

$$\times \Bigg| \iint_{[\delta,1]\times[\delta,1]} \sin(\frac{z}{u})\sin(\frac{w}{v}) dg(u,v) \Bigg| (1/zw) dz dw$$

$$- (4/\pi^4) \Bigg| \iint_{\{1\}\times\{1\}} dg(u,v) \Bigg|$$

$$- (4/\pi^3) \Bigg\{ \ln M \int_{[1,2\epsilon\delta^*\sqrt{N}]} \Bigg| \iint_{\{1\}\times[\delta,1]} \sin(\frac{y}{v}) dg(u,v) \Bigg| (1/y) dy$$

$$+ \ln N \int_{[1,2\epsilon\delta^*\sqrt{M}]} \Bigg| \iint_{[\delta,1]\times\{1\}} \sin(y/u) dg(u,v) \Bigg| (1/y) dy \Bigg\} \Bigg|$$

$$\leq \limsup_{M,N\to\infty} \Bigg| \{(2/\pi^2\ \mathrm{Var}(g)\epsilon^2 + \phi_1(A) + \phi_2(A) + 2\epsilon^2\} O(1) + o(1) + \frac{E_{15}}{\ln M \ln N} \Bigg|$$

$$\leq \left| O(1)\{\epsilon^2 + \phi_1(A) + \phi_2(A) + \Phi(A)\} \right| .$$

Thus this last bound is independent of M, N. We therefore let $\epsilon \to 0$, $A \to \infty$ on this bound to arrive at the desired result.

Proposition 4.21. Let

$$F(z, w) = \left| \iint_{[\delta, 1]\times[\delta, 1]} \sin(z/u) \sin(w/v) dg(u, v) \right|$$

and assume $\mathcal{M}\{F\}$ exists then

$$\iint_{[1, \sqrt{M}]\times[1, \sqrt{N}]} F(z, w) \frac{dz}{z} \frac{dw}{w} - (1/4)\mathcal{M}\{F\} \ln M \ln N$$

$= o(\ln M \ln N)$.

<u>Proof</u>. Let

$$f(p, q) = \iint_{[0, p]\times[0, q]} F(z, w) dz dw \tag{a}$$

Then

$$\lim_{p, q \to \infty} \frac{f(p, q)}{pq} = \mathcal{M}\{F\},$$

which exists by hypothesis, and so we can write

$$f(p, q) = pq\mathcal{M}\{F\} + o(pq), \quad p, q \to \infty. \tag{b}$$

Also note from (a) that

$$0 \le \frac{f(p, q)}{pq} \le (1/pq) \iint_{[0, p]\times[0, q]} \mathrm{Var}(g) dz dw = \mathrm{Var}(g). \tag{c}$$

Now from (a) we have $f_{pq} = F(p, q) = f_{qp}$, where $f_w = \frac{\partial f}{\partial w}$.

Now repeated integration-by-parts yields

$$\left| \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} [F(z,w) - \mathcal{M}\{F\}] \frac{dz}{z}\frac{dw}{w} \right|$$

$$= \left| \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \{f_{zw} - \mathcal{M}\{F\}\} \frac{dz}{z}\frac{dw}{w} \right|$$

$$= \left| \int_{[1,\sqrt{N}]} (1/\sqrt{M}) \frac{f_w(\sqrt{M}, w)}{w} - \frac{f_w(1,w)}{w} \quad dw \right.$$

$$\left. + \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \frac{f_w(z,w)}{z^2 w} dz dw - (1/4)\mathcal{M}\{F\} \ln M \ln N \right|$$

$$= \left| \frac{f(\sqrt{M},\sqrt{N})}{\sqrt{MN}} - \frac{f(\sqrt{M},1)}{\sqrt{M}} + \int_{[1,\sqrt{N}]} \frac{f(\sqrt{M},w)}{(\sqrt{M}w)w} dw - \frac{f(1,\sqrt{N})}{\sqrt{N}} + f(1,1) \right.$$

$$- \int_{[1,\sqrt{N}]} \frac{f(1,w)}{ww} dw + \int_{[1,\sqrt{M}]} \frac{f(z,\sqrt{N})}{(z\sqrt{N})z} dz - \int_{[1,\sqrt{M}]} \frac{f(z,1)}{zz} dz$$

$$\left. + \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \frac{f(z,w)}{(zw)(zw)} dz dw - (1/4)\mathcal{M}\{F\} \ln M \ln N \right|$$

$$\leq \left| \frac{f(\sqrt{M},\sqrt{N})}{\sqrt{MN}} - \frac{f(\sqrt{M},1)}{\sqrt{M}} + f(1,1) - \frac{f(1,\sqrt{N})}{\sqrt{N}} \right| + 2 \left| \int_{[1,\sqrt{N}]} \frac{Var(g)}{w} dw \right|$$

$$+ 2 \left| \int_{[1,\sqrt{M}]} \frac{Var(g)}{z} dz \right| + \left| \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \left\{ \frac{f(z,w)}{zw} - \mathcal{M}\{F\} \right\} \frac{dz}{z}\frac{dw}{w} \right|, \text{ by (c)}$$

$$\leq$$

$$\leq o(\ln M \ln N) + O(\ln N) + O(\ln M)$$

$$+ \left| \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \{\mathcal{M}\{F\} + h(z,w) - \mathcal{M}\{F\}\} \frac{dz}{z}\frac{dw}{w} \right| , \text{ by using (b),}$$

where $h(p, q)$ is a function which tends to zero as p, q tends to infinity. Also $|h(p, q)| = |\frac{f(p, q)}{pq} - \mathcal{M}\{F\}|$ which is less than or equal to twice the variation of $g(u, v)$, and given $\epsilon > 0$ there exists $T_1 > 0$ such that if $p, q > T_1$ then $|h(p, q)| < \epsilon$. Here T_1 depends upon ϵ.

$\leq o(\ln M \ln N) + R$, where R is the double integral above.

Considering just this last integral R we break the integration into three parts:

$$|R| \leq \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} \frac{|h(z,w)|}{zw} dzdw$$

$$= \iint_{[1,T_1]\times[1,\sqrt{N}]} \cdots + \iint_{[T_1,\sqrt{M}]\times[1,T_1]} \cdots + \iint_{[T_1,\sqrt{M}]\times[T_1,\sqrt{N}]} \cdots$$

$$\leq \iint_{[1,T_1]\times[1,\sqrt{N}]} \frac{2\,Var(g)}{zw} dzdw + \iint_{[1,\sqrt{M}]\times[1,T_1]} \frac{2\,Var(g)}{zw} dzdw$$

$$+ \iint_{[T_1,\sqrt{M}]\times[T_1,\sqrt{N}]} \epsilon \frac{dzdw}{zw} \leq$$

$$\leq \operatorname{Var}(g) \ln T_1(\epsilon)[\ln N + \ln M] + (\epsilon/4) \ln M \ln N.$$

Thus, given any $\epsilon > 0$, it is possible to make

$$\left| \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} F(z,w) \frac{dzdw}{zw} - (1/4)\mathcal{M}\{F\} \ln M \ln N \right|$$

$$\leq o(\ln M \ln N) + \operatorname{Var}(g) \ln T_1(\epsilon)[\ln N + \ln M] + (\epsilon/4) \ln M \ln N$$

or

$$\frac{1}{\ln M \ln N}\left| \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} F(z,w) \frac{dzdw}{zw} - (1/4)\mathcal{M}\{F\} \ln M \ln N \right| < \epsilon,$$

if M, N are sufficiently large.

<u>Proposition 4.22</u>. If $g(u,v)$ is a bounded variation in the sense of Hardy-Krause and (u_i, v_j) is a point of discontinuity of $g(u,v)$ which is not on the axes, if

$$f(z,w) = \left| \sum_{i,j} \sin(z/u_i) \sin(w/v_j)\{g(u_i^+, v_j^+) - g(u_i^+, v_j^-) - g(u_i^-, v_j^+) + g(u_i^-, v_j^-)\} \right|$$

then

$$\mathcal{M}(f) = \lim_{p,q\to\infty} (1/pq) \iint_{[0,p]\times[0,q]} f(z,w)dzdw \quad \text{exists.}$$

Proof. Let

$$g(u_i^+, v_j^+; u_i^-, v_j^-) = g(u_i^+, v_j^+) - g(u_i^+, v_j^-) - g(u_i^-, v_j^+) + g(u_i^-, v_j^-) .$$

Set

$$f_{k\ell}(z, w) = \left| \sum_{i,j=1}^{k,\ell} \sin(z/u_i) \sin(w/v_j) g(u_i^+, v_j^+; u_i^-, v_j^-) \right|$$

and

$$\epsilon_{k\ell}(z, w) = f(z, w) - f_{k\ell}(z, w) .$$

Let

$$\epsilon_{k\ell} = \sup_{z, w} |\epsilon_{k\ell}(z, w)| .$$

Since $g(u, v)$ is of bounded variation we find $f(z, w)$ exists for all $z, w,$ and in fact

$$0 \leq f(z, w) \leq \mathrm{Var}(g).$$

The sum representation of $f(z, w)$ converges both absolutely and uniformly in (z, w) since

$$f(z, w) \leq \sum_{i,j} |\sin(z/u_i) \sin(w/v_j)| \, |g(u_i^+, v_j^+; u_i^-, v_j^-)|,$$

call this sum A,

$$\leq \sum_{i,j} |g(u_i^+, v_j^+; u_i^-, v_j^-)| \leq \mathrm{Var}(g).$$

Note that if $A_{k\ell}$ is the $k\ell$th partial sum of A then $A_{k\ell}$ tends to A uniformly in z, w.

Consider now

$$|\epsilon_{k\ell}(z,w)| = \left| \left| \sum_{i,j=1}^{\infty} \sin(z/u_i)\sin(w/v_u) g(u_i^+, v_j^+; u_i^-, v_j^-) \right| - \left| \sum_{i,j=1}^{k,\ell} \sin(z/u_i)\sin(w/v_j) g(u_i^+, v_j^+; u_i^-, v_j^-) \right| \right|$$

$$\leq \left| \sum_{i,j=k+1,\ell+1}^{\infty} \sin(z/u_i)\sin(w/v_j) g(u_i^+ . v_j^+; u_i^-, v_j^-) + \sum_{j=1}^{\ell} \sum_{i=k+1}^{\infty} \cdots + \sum_{j=\ell+1}^{\infty} \sum_{i=1}^{k} \cdots \right|$$

$$\leq \sum_{i,j=k+1,\ell+1}^{\infty} |\ldots| + \sum_{j=1}^{\ell} \sum_{i=k+1}^{\infty} |\ldots| + \sum_{j=\ell+1}^{\infty} \sum_{i=1}^{k} |\ldots|$$

$$= \sum_{i,j=1}^{\infty} |\ldots| - \sum_{i,j=1}^{k,\ell} |\ldots| = A - A_{k\ell} .$$

By the uniform convergence of $A_{k\ell}$ to A we have

$$\lim_{k,\ell \to \infty} \epsilon_{k\ell} = \lim_{k,\ell \to \infty} \{\sup_{z,w} |\epsilon_{k\ell}(z,w)|\} = 0$$

Then

$$f_{k\ell}(z,w) - \epsilon_{k\ell} \leq f(z,w) \leq f_{k\ell}(Z,w) + \epsilon_{k\ell}$$

so that

$$(1/pq) \iint_{[0,p]\times[0,q]} f_{k\ell}(z,w)dzdw - \epsilon_{k\ell} \leq (1/pq) \iint_{[0,p]\times[0,q]} f(z,w)dzdw$$

$$\leq (1/pq) \iint_{[0,p]\times[0,q]} f_{k\ell}(z,w) + \epsilon_{k\ell} \qquad \text{(B)}$$

holds. For each k,ℓ we have $f_{k\ell}(z,w)$ being a finite sum of continuous periodic functions, hence $f_{k\ell}(z,w)$ is almost periodic, and so its mean value exists [5], that is

$$\mathcal{M}\{f_{k\ell}\} = \lim_{p,q\to\infty} (1/pq) \iint_{[0,p]\times[0,q]} f_{k\ell}(z,w)dzdw$$

exists.

Consider now for fixed $T, S \geq 0$ the difference

$$\left| \mathcal{M}\{f_{k+T,\ell+S}(z,w)\} - \mathcal{M}\{f_{k,\ell}(z,w)\} \right|$$

$$= \left| \lim_{p,q\to\infty} (1/pq) \iint_{[0,p]\times[0,q]} \{f_{k+T,\ell+S} - f_{k,\ell}\}dzdw \right|$$

$$\leq \limsup_{p,q\to\infty} (1/pq) \iint_{[0,p]\times[0,q]} |f_{k+T,\ell+S} - f_{k,\ell}|dzdw .$$

We already know $\{f_{k,\ell}(z,w)\}$ is convergent uniformly in (z,w) so for any $\eta > 0$ we have

$$|f_{k+T,\ell+S}(z,w) - f_{k,\ell}(z,w)| < \eta$$

for all z, w when k, ℓ are sufficiently large. Thus

$$\limsup_{p,q\to\infty} (1/pq) \iint_{[0,p]\times[0,q]} |f_{k+T,\ell+S}(z,w) - f_{k,\ell}(z,w)|\,dzdw < \eta$$

for k, ℓ sufficiently large, or $\{\mathcal{M}\{f_{k,\ell}\}\}$ is a Cauchy sequence, hence $\lim_{k,\ell\to\infty} \mathcal{M}\{f_{k,\ell}\}$ exists.

Now from (B) we have

$$\limsup_{p,q\to\infty} \left[(1/pq) \iint_{[0,p]\times[0,q]} f(z,w)\,dzdw\right] \leq \mathcal{M}\{f_{k,\ell}\} + \epsilon_{k\ell}$$

and also

$$\mathcal{M}\{f_{k\ell}\} - \epsilon_{k\ell} \leq \liminf_{p,q\to\infty} \left[(1/pq) \iint_{[0,p]\times[0,q]} f(z,w)\,dzdw\right].$$

Letting k, ℓ tend to infinity yields

$$\lim_{k,\ell\to\infty} \mathcal{M}\{f_{k\ell}\} \leq \liminf_{p,q\to\infty} \left[(1/pq) \iint_{[0,p]\times[0,q]} f(z,w)\,dzdw\right] \leq$$

$$\leq \limsup_{p,q \to \infty} \left[(1/pq) \iint_{[0,p]\times[0,q]} f(z,w)dzdw \right]$$

$$\leq \lim_{k,\ell \to \infty} \mathcal{M}\{f_{k\ell}\}$$

Hence, there exists

$$\mathcal{M}(f) = \lim_{p,q \to \infty} \left[(1/pq) \iint_{[0,p]\times[0,q]} f(z,w)dzdw \right] = \lim_{k,\ell \to \infty} \mathcal{M}\{f_{k,\ell}\} .$$

As Corollary 4.23 shows, the next theorem is essentially the two dimensional analogue of Ishiguro's result in [14].

Theorem 4.23. If the weight function $g(u,v)$ which generates the regular Quasi-Hausdorff matrix associated with the Lebesgue constant $L^*(M,N;g)$ is a function which is continuous and zero on a cross neighborhood $\{(x,y) \mid 0 \leq x \leq \delta \text{ or } 0 \leq y \leq \delta\}$ for some δ, then

$$L^*(M,N;g) = C^*(g) \ln M \ln N + o(\ln M \ln N),$$

$$M,N \to \infty,$$

where

$$C^*(g) = (4/\pi^2)\left| \iint_{\{1\}\times\{1\}} dg(u,v) \right| + (2/\pi^3)\{\mathcal{M}\{f_1\}+\mathcal{M}\{f_2\}+(\pi/2)\mathcal{M}\{f_3\}\},$$

where

$$f_1(w) = \left| \iint_{\{1\}\times[\delta, 1)} \sin(w/v)dg(u,v) \right|$$

$$f_2(z) = \left| \iint_{[\delta, 1)\times\{1\}} \sin(z/u)dg(u,v) \right|$$

$$f_3(z,w) = \left| \iint_{[\delta, 1)\times[\delta, 1)} \sin(z/u)\ \sin(w/v)dg(u,v) \right|$$

and it is assumed that these mean values exist.

Proof. Since $g(u,v)$ is zero on the axes and continuous there, there exists by hypothesis, a δ such that the integration over $[0,1]\times[0,1]$ is identical with that over $[\delta,1]\times[\delta,1]$ for the measure generated by $g(u,v)$. Thus $L^*_\delta(M,N;g) = L^*(M,N;g)$.

By Proposition 4.21 the first term in $L^*_\delta(M,N;g)$ of Proposition 4.20 can be written

$$(4/\pi^2) \iint_{[1,\sqrt{M}]\times[1,\sqrt{N}]} f_3(z,w)(1/zw)dzdw$$

$$= (1/\pi^2)\mathcal{M}\{f_3\} \ln M \ln N + o(\ln M \ln N).$$

In a like manner, the one dimensional analogue of Proposition 4.21 implies

$$(4/\pi^3)\ln M \int_{[1,\, 2\epsilon\delta^*\sqrt{N}]} f_1(w)(1/w)dw = (2/\pi^3)\mathcal{M}\{f_1\} \ln M \ln N + o(\ln M \ln N),$$

$$(4/\pi^3)\ln N \int_{[1,\, 2\epsilon\delta^*\sqrt{M}]} f_2(z)(1/z)dz = (2/\pi^3)\mathcal{M}\{f_2\} \ln M \ln N + o(\ln M \ln N).$$

Combining these results we arrive at the desired result.

We now state the two dimensional analogue of Ishiguro's result in [14].

Corollary 4.23. If the weight function $g(u, v)$ which generates the regular Quasi-Hausdorff matrix which is associated with the Lebesgue constant $L^*(M, N; g)$ is a countable linear combination of two dimensional interval functions with mass points bounded away from the axes, then

$$L^*(M, N; g) = C^*(g) \ln M \ln N + o(\ln M \ln\ ), \qquad M, N \to \infty$$

where

$$C^*(g) = (4/\pi^2)\left| \int_{\{1\}\times\{1\}} dg(u, v) \right|$$

$$+ (2/\pi^3)\mathcal{M}\left\{\left| \sum_i \sin(y/r_i) g(r_i^+, 1; r_i^-, 1^-) \right|\right\}$$

$$+ (2/\pi^3)\mathcal{M}\left\{\left| \sum_j \sin(y/s_j) g(1, s_j^+; 1^-, s_j^-) \right|\right\} +$$

$$+ (1/\pi^2)\, \mathcal{M}\left\{\left|\sum_{i,j} \sin(z/u_i)\sin(w/v_j)g(u_i^+, v_j^+; u_i^-, v_j^-)\right|\right\}.$$

Here (u_i, v_j) is the (ith, jth) point of discontinuity of $g(u,v)$ on $[0,1)\times[0,1)$, $(r_i, 1)$ denotes those along $[0,1)\times\{1\}$, while $(1, s_j)$ are those along $\{1\}\times[0,1)$. The summations extend over all such (possible countably infinite) values.

Proof. We note by the conditions on the mass points that u_i, v_j are not zero so that $\sin(z/u_i)$, $\sin(w/v_j)$ are well defined. Then

$$f_3(z,w) = \left|\iint_{[\delta,1)\times[\delta,1)} \sin(z/u)\sin(w/v)dg(u,v)\right|$$

$$= \left|\sum_{i,j}\sin(z/u_i)\sin(w/v_j)g(u_i^+, v_j^+; u_i^-, v_j^-)\right|.$$

By Proposition 4.22 the $\mathcal{M}\{f_3\}$ exists. Also

$$f_1(w) = \left|\iint_{\{1\}\times[\delta,1)} \sin(w/v)dg(u,v)\right| = \left|\sum_j \sin(w/s_j)g(1, s_j^+; 1^-, s_j^-)\right|$$

$$f_2(z) = \left|\iint_{[\delta,1)\times\{1\}} \sin(z/u)dg(u,v)\right| = \left|\sum_i \sin(z/u_i)g(r_i^+, 1; r_i^-, 1^-)\right|.$$

Using the same techniques as in the proof of Proposition 4.22 we can show that both

$$\mathcal{M}\{f_1\}, \mathcal{M}\{f_2\} \quad \text{exist.}$$

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