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Title: A NONLINEAR FREE BOUNDARY VALUE PROBLEM

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We discuss a mathematical model arising in the filtration of a fluid through a porous medium. The model leads to a free boundary value problem whose governing equation depends on the retention function. A numerical approximation by means of finite elements is used to present an existence and uniqueness theorem along with an error estimate for a linear retention function. Finally, numerical algorithms for finding approximate solutions for linear and quadratic retention functions are suggested and the results of numerical calculations are given.

A NONLINEAR FREE BOUNDARY VALUE PROBLEM

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Typed by Donna Moore for Abdelwahab Kharab

This Thesis is dedicated

to

my parents

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I. INTRODUCTION AND STATEMENT OF THE PROBLEM

1.1. Introduction.

In many applications one is frequently faced with the problem of determining both the domain where the problem is to be solved as well as the solution. Such problems are referred as free boundary value problems. A particularly important problem of this type deals with the determination of the water level underground and is the subject of this dissertation.

The one dimensional form of this problem is obtained by thinking of the soil as a large slab which consists of a homogeneous porous medium. Our problem is to describe the pressure at a given time and to determine the depth to which the fluid has penetrated.

The treatment is complicated by the fact that the governing equation of the problem that we will derive below is in general nonlinear. It is a function of the pressure and describes the water content in the soil at a given pressure. This function, the so-called retention function, is of great interest and has been intensively studied in soil science. Although considerable effort has been spent to obtain theoretical curves, they are at present mainly empirical.

Two of the most common expressions for the retention function are considered in this dissertation. The first one deals with the linear case where the retention depends linearly on the pressure and the second case where the dependence is quadratic for positive pressures.

The governing equations for the physical situation outlined above are derived in section 1.2 and initial and boundary conditions are described. A complete statement of the problem for the general case is given in section 1.4.

The second chapter deals with the case of a linear retention function. Existence and uniqueness theorems are proved in section 2.1 by use of the variational formulation of the problem. In section 2.2 the error analysis is discussed and an error estimate is derived based on the assumption that the solution is sufficiently smooth. In section 2.4 a numerical method using finite elements is described.

In section 3.1 of the third chapter, a finite element method is used to find a numerical solution of the problem for a quadratic retention function and in section 3.2 a numerical algorithm for the approximation of both linear and quadratic is given.

1.2. Mathematical Model.

We shall consider a homogeneous porous medium which is assumed to consist of a large slab so that physical properties can be determined by the single variable y, (see Fig. 1) and which is assumed to be saturated with water to a certain depth. Here we shall assume throughout that the process can be considered one dimensional even though the real problem is three dimensional. As time progresses the liquid will flow toward areas of lower pressure; in this case the water will flow downward. The flow will continue as long as a sufficiently high saturation is maintained. The level of the fluid in the ground is determined by the function $\overline{s}(\tau)$ at time τ known as the "free boundary".

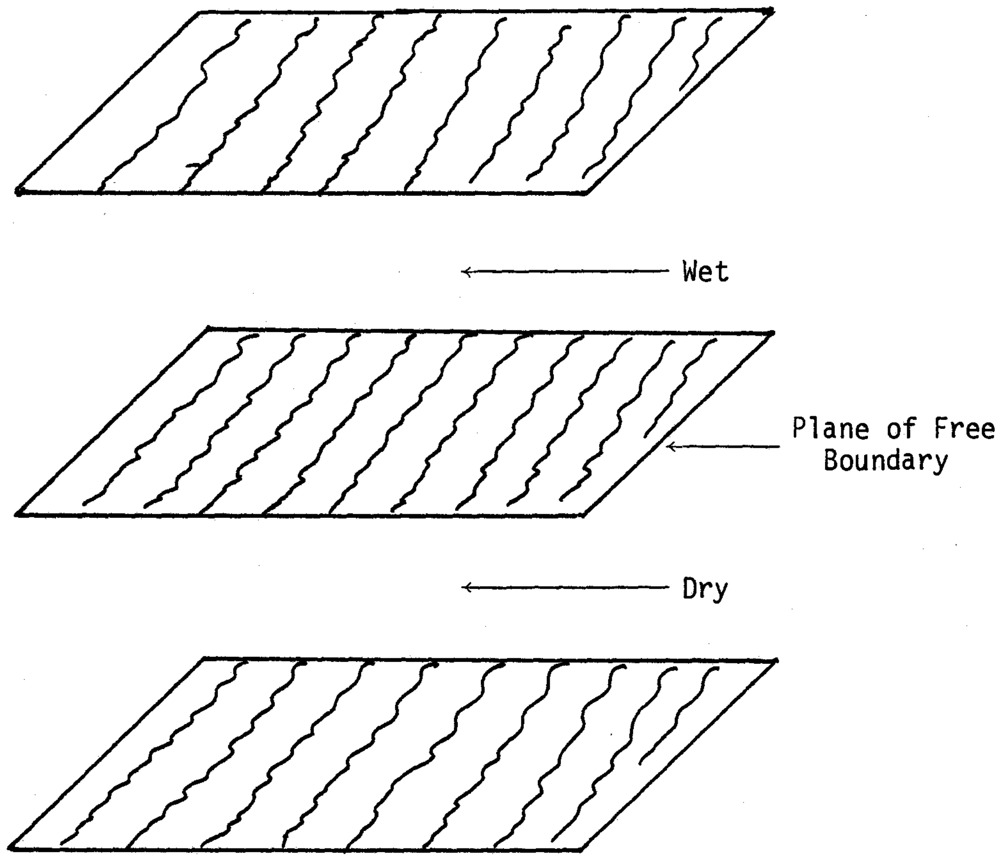

Fig. 1. The physical situation.

To derive the mathematical equations, let y denote the spatial coordinate above some fixed reference level. We take the positive direction upward. Let $U(y,\tau)$ be the pressure of the fluid depending on y and τ, and g the acceleration due to gravity. Under conditions of slow flow, the flow is governed by Darcy's law

$$v = -k \left(\frac{\partial U}{\partial y} + \rho g \right)$$

where ρ denote the density of the fluid, k the permeability coefficient, and v the seepage velocity of the fluid. The law governing the conservation of mass is given by

$$\frac{\partial}{\partial \tau} (\rho \phi S) + \frac{\partial \rho v}{\partial y} = 0$$

the so-called continuity equation. ϕ and S denote, respectively, the porosity and the saturation of the medium, both of which are functions of the pressure and satisfy the inequalitites

$$0 < \phi \leq 1 \qquad \text{and} \qquad 0 \leq S \leq 1 .$$

A precise definition of these terms as well as detailed physical discussion is given in [11] and [13].

Combining Darcy's law and the conservation of mass equation we obtain

$$\frac{R(U)}{\partial \tau} + \frac{\partial}{\partial y} \rho \left[-k \left(\frac{\partial U}{\partial y} + g\rho \right) \right] = 0$$

Without loss of generality for the developments in the sequel, the constants k, g and ρ will be set to be equal to one. Thus, the governing equation for the pressure becomes

$$\frac{\partial R(U)}{\partial \tau} = \frac{\partial^2 U}{\partial y^2}$$

where $R(U) = \phi S$. This is a nonlinear, differential equation in U of parabolic type. The product ϕS is known as the retention function.

In order to derive the boundary conditions, let us assume that the top and the bottom of the soil are located respectively at $y = 1$ and $y = 0$. The motion of the free boundary is denoted by function $\bar{s}(\tau)$ at time τ with $\bar{s}(0) = 1$. At the fixed boundary $y = 0$ the flux is given by $U_y(0,\tau) = -1$. Along the free boundary $y = \bar{s}(\tau)$ the pressure is assumed to be zero: $U(\bar{s}(\tau),\tau) = 0$. The motion of the free boundary is governed by the differential equation

$$\frac{d\bar{s}}{d\tau} = -U_y(\bar{s}(\tau),\tau)-1$$

Finally the initial pressure at time $\tau = 0$ is given by $U(y,0) = g(y)$ where $g(y)$ is a positive decreasing function in $0 \leq y \leq 1$ satisfying

$$g'(0) = -1 \quad , \quad g(1) = 0$$

and

$$g'(y) < -1 \quad , \quad 0 < y \leq 1$$

1.3. Literature Review.

There is an extensive literature on free boundary value problems dealing both with the theoretical and the numerical aspects. This literature has been rapidly expanding since the introduction of the concept of variational formulations which usually lead in the end to the use of finite element approximations. See [6]. Because of this large literature, we will discuss here only the papers related directly to our work.

The method developed in the present dissertation is based on the technique introduced by Nitsche [17] for the one phase Stefan problem. First the original free boundary problem is reduced to one with a fixed boundary by use of Landau transformation. In most cases the resulting equation is then nonlinear. In [17] the original Stefan problem splits into a nonlinear parabolic initial boundary value problem for a fixed domain and two ordinary differential equations. We shall follow a similar procedure and first transform our free boundary value problem to a problem with fixed domain. In contrast to [17], the equations in our transformed problem will be coupled which complicates the analysis substantially. After carrying out this reduction, we then derive a variational formulation of the problem.

In [12] Gastaldi follows [17] to study the numerical approximation of a free boundary value problem in one space dimension arising in the filtration of a compressible fluid through porous medium. The study of our error estimate of the solution is based on Gastaldi's paper since she also ends up with coupled differential equations in the transformed problem.

In both papers [17] and [12], the problems deal only with the particular case when the retention function is linear and the treatment is very theoretical. The physical significance of the retention function is not discussed. Dr. Lenhard [14] in his Ph.D. dissertation, O.S.U., has made an in-depth of study of retention functions for different liquids on unconsolidated porous media. Liquid retention functions obtained on samples containing different liquids are compared. Two resulting forms of the retention function are studied in our dissertation, a linear and quadratic type which are supposed to be the most common ones in applications.

We shall refer to Friedman [9] for results concerning general theory of parabolic equations. For the relationship existing between free boundary value problem and variational inequalities, see Magenes [15]. Existence and uniqueness theorems are proved for classical models and the question about the regularity of the free boundary is also discussed. For the weak solutions of parabolic variational inequalities see Brezis [5]. For the detailed proofs of existence and uniqueness theorems, we refer to the papers [3], [4], [6].

1.4. Statement of the Problem.

Following the plan described in section 1.2 the problem we want to solve now reads in one space dimension.

Problem 1. Given $T_0 > 0$ and $g(y)$ which satisfies

$$g(y) \geq 0 \quad , \; g'(y) < -1 \quad \text{for} \quad 0 < y \leq 1$$

and

$$g'(0) = -1 \quad , \quad g(1) = 0$$

Find the pair $\{U(y,\tau) \; , \; \overline{s}(\tau)\}$ such that

$$\overline{s}(\tau) > 0 \quad , \quad 0 < \tau < T_0 \quad , \quad \overline{s}(0) = 1 \tag{1.1}$$

$$U_{yy}(y,\tau) = \frac{\partial R(U)}{\partial \tau} \text{ in } \Omega = \{(y,\tau) \mid 0 < \tau \leq T_0 \, , \, 0 < y < \bar{s}(\tau)\} \tag{1.2}$$

$$U_y(0,\tau) = -1 \, , \, 0 \leq \tau \leq T_0 \tag{1.3}$$

$$U(\bar{s}(\tau),\tau) = 0 \, , \, 0 \leq \tau \leq T_0 \tag{1.4}$$

$$U(y,0) = g(y) \, , \, 0 \leq y \leq 1 \tag{1.5}$$

and in addition

$$\frac{d\bar{s}}{d\tau} + U_y(\bar{s}(\tau),\tau) = -1 \, , \, 0 \leq \tau \leq T_0 \tag{1.6}$$

where

i) $R(U)$ is the retention function which depends on the pressure U.

ii) $\bar{s}(\tau)$ is the free boundary.

iii) $g(y)$ is the initial pressure.

1.5. Variational Formulation.

A. Transformed Problem.

In order to reduce the problem to one with fixed boundaries, we introduce the new space variable

$$x = \frac{y}{\bar{s}(\tau)}$$

while the new time variable $t = t(\tau)$ will be defined as the unique solution to the ordianry differential equation

$$\frac{dt}{d\tau} = \frac{1}{\bar{s}^2(\tau)} \quad \text{with} \quad t(0) = 0$$

Let us consider first the case when R(U) is linear, i.e. let R(U) = U be given. The governing equation of Problem 1 takes the form

$$U_{yy}(y,\tau) = U(y,\tau) \tag{1.7}$$

setting now

$$\begin{cases} u(x,t) = U(y,\tau) + y - \bar{s}(\tau) \\ \bar{s}(\tau) = s(t) \end{cases} \tag{1.8}$$

and using the new variables, one obtains

$$U_y = u_x \frac{\partial x}{\partial y} - 1$$

But

$$\frac{\partial x}{\partial y} = \frac{1}{s}$$

Thus

$$U_y = \frac{1}{s} u_x - 1$$

differentiating one more time with respect to y, we get

$$U_{yy}(y,\tau) = \frac{1}{s^2} u_{xx}(x,t) \tag{1.9}$$

Taking the partial derivative of $U(y,\tau)$ with respect to τ in equation (1.8), we obtain

$$U_\tau = u_x \frac{\partial x}{\partial \tau} + u_t \frac{\partial t}{\partial \tau} - \frac{\partial y}{\partial \bar{s}} \frac{\partial \bar{s}}{\partial \tau} + \frac{\partial \bar{s}}{\partial \tau} \tag{1.10}$$

Further,

$$\frac{\partial x}{\partial \bar{s}} = - \frac{y}{\bar{s}^2(\tau)} = - \frac{x}{s(t)}$$

$$\frac{d\bar{s}}{d\tau} = - U_y(\bar{s}(\tau),\tau) -1 = -(\frac{1}{s} u_x(1,t)-1) -1 = - \frac{1}{s} u_x(,t)$$

substituting these results into (1.10) leads to

$$U_\tau = \frac{1}{s^2} xu_x(1,t)u_x(x,t) + \frac{1}{s^2} u_t(x,t) + \frac{1}{s} xu_x(x,t) - \frac{1}{s} u_x(1,t) \tag{1.11}$$

finally using (1.9) and (1.0) and making all necessary simplifications, (1.7) takes the form

$$U_{xx} - u_t = xu_x(1,t)u_x(x,t) + su_x(1,t)(x-1)$$

in

$$Q = \{(x,t) \mid 0 < x < 1 \quad , \quad 0 < t \leq T\}$$

where $t = T$ is the corresponding value of $\tau = T_0$.

For the initial and boundary conditions, we have

i) setting $t = 0$ in (1.8) and using (1.5), we get the initial condition

$$U(y,0 = u(x,0) - x + 1 = g(x)$$

Hence

$$u(x,0) = g(x) + x - 1$$

ii) we have

$$U_y(y,\tau) = \frac{1}{s} u_x(x,t) - 1$$

set $y = 0$ to get

$$U_y(0,\tau) = \frac{1}{s} u_x(0,t) - 1 = -1$$

Thus,

$$u_x(0,t) = 0$$

iii) Set $y = \overline{s}(\tau)$ in (1.8) to get

$$U(\overline{s}(\tau),\tau) = u(1,t) - \overline{s}(\tau) + \overline{s}(\tau)$$

$$= 0$$

Hence the boundary condition at $x = 1$ is

$$u(1,t) = 0$$

iv) For the equation (1.6), we have

$$\frac{d\overline{s}}{d\tau} = \frac{ds}{dt}\frac{dt}{d\tau} = \frac{ds}{dt}\left(\frac{1}{s^2}\right)$$

$$= -\left(\frac{1}{s} u_x(1,t) - 1\right) - 1$$

Consequently,

$$\frac{ds}{dt} = -su_x(1,t)$$

The transformed problem now reads

Problem 2. Find the pair $\{u(x,t) \quad , \quad s(t)\}$ such that

$$u_{xx} - u_t = xu_x(1,t)u_x(x,t) + s(t)u_x(1,t)(x-1) \tag{1.12}$$

in

$$Q = \{(x,t) \mid 0 < x < 1 \ , \ 0 < t \leq T\}$$

$$u_x(0,t) = 0 \quad , \quad 0 < t \leq T \tag{1.13}$$

$$u(1,t) = 0 \quad , \quad 0 < t \leq T \tag{1.14}$$

$$u(x,0) = g(x) + x - 1 \quad , \quad 0 \leq x \leq 1 \tag{1.15}$$

$$\frac{ds(t)}{dt} = -u_x(1,t)s(t) \ , \ s(0) = 1 \ , \ 0 < t \leq T \ , \ s(t) > 0 \tag{1.16}$$

The analogous governing equation for the one phase Stephan problem (see [ ]) is almost identical, but in the equation corresponding to (1.12) the additional term $s(t)u_x(1,t)(x-1)$ is lacking. Such a term makes (1.12) and (1.16) coupled, while the corresponding equations for the Stefan problem are uncoupled.

B. Weak Formulation.

To reach a weak formulation, we make another change of unknown $v = u_x$. Because of (1.14), u can be computed if v is known, which for t fixed has to be in the space

$$\dot{H}^1 = \{w \mid w \varepsilon H^1(0,1) \text{ and } w(0) = 0\}$$

Multiplication of (1.12) by w_x with $w_x \varepsilon \dot{H}^1$ and integrating with respect to x leads to

$$\int_0^1 u_{xx}w_x dx - \int_0^1 u_t w_x dx = u_x(1,t)\int_0^1 x u_x w_x dx$$

$$+ su_x(1,t)\int_0^1 (x-1)w_x dx$$

by integrating the second terms of each side by part, we obtain

$$\int_0^1 v_x w_x dx + \int_0^1 v_t w dx = v(1,t)\int_0^1 x v w_x dx - sv(1,t)\int_0^1 w dx$$

The weak formulation of Problem 2 now reads

Problem 3. Find the pair $\{v(x,t)$ and $s(t)\}$ with

$$v(x,t) : [0,T] \longmapsto \dot{H}^1$$

such that

$$(v',w') + (\dot{v},w) = v(1)(xv,w') - sv(1)(1,w) \tag{1.17}$$

$$\text{for all } w \in \dot{H}^1$$

$$\dot{s} = -v(1)s \tag{1.18}$$

with the initial data

$$v(x,0) = g'(x) + 1 \quad , \quad 0 \le x \le 1 \tag{1.19}$$

$$s(0) = 1 \tag{1.20}$$

Here and in the following v' and $\dot{v}$ denote, respectively, differentiation with respect to x and t. The dependence on t is usually suppressed and $v(1)$ means $v(1,t)$. The L_2-product is denoted by $(.,.)$

and the norm by $||\cdot||$.

We now define a finite element method based upon the discussion above.

C. Statement of the discretized problem.

Following a procedure introduced by Ritz and Galerkin we construct a finite element solution to our boundary value problem. In this method a solution of the variational problem is sought in a finite dimensional subspace of H^1, called the finite element space and denoted here by S_h. We define S_h as follows.

Let Π_h be a subdivision of the interval $[0,1]$ into $N = 1/h$ equal parts of length h. S_h is the space of continuous functions which are piecewise polynomials of degree less than an integer r. We define $\dot{S}_h$ as

$$\dot{S}_h = \{ v_h \mid v_h \varepsilon S_h \quad \text{and} \quad v_h(0) = 0 \}$$

so that

$$\dot{S}_h \subset \dot{H}^1 \quad .$$

The discretized problem now reads.

Problem 4. Find $\{v_h(x,t) \quad , \quad s_h(t)\}$ with

$$v_h(x,t) : [0,T] \longmapsto \dot{S}_h$$

such that

$$(\dot{v}_h,\chi) + (v'_h,\chi') = v_h(1)(xv_h,\chi') - s_h v_h(1)(1,\chi) \qquad (1.21)$$

for all $\chi \varepsilon \dot{S}_h$

$$\dot{s}_h = -v_h(1)s_h \tag{1.22}$$

with the initial conditions

$$v_h(x,0) = P_h g'(x) + 1 \quad , \quad 0 \leq x \leq 1 \tag{1.23}$$

$$s_h(0) = 1 \tag{1.24}$$

where P_h is an appropriate projection in the space $\dot{S}_h$ which will be defined later.

II. SOLUTION FOR LINEAR RETENTION FUNCTION

2.1. Existence and uniqueness theorems.

In this chapter, we will be considering only the regular case of the problem. This means that we assume that the solution of the discretized problem is very smooth. Therefore the error estimate of the solution is optimal with respect to the order of powers of h.

Since $\dot{S}_h$ is finite dimensional, the solution for the problem exists then in a certain interval $(0,\bar{t})$ where $\bar{t}$ may depend only on the interval of data.

In order to write the existence theorem, we need to define the concept of local solution.

Definition 2.1. Problem 4 has a local solution if there exists a $\bar{t}$ depending on the data, i.e. g and v, such that the approximation v_h is valid for t in $(0,\bar{t})$.

We now prove

Theorem 1. Problem 4 has a local solution.

Proof. In (1.21) set $\chi = v_h$ to get

$$(\dot{v}_h,v_h) + (v_h',v_h') = v_h(1)(xv_h,v_h') - s_h v_h(1)(1,v_h) \tag{2.1}$$

The first two terms can be written in the form

$$(\dot{v}_h,v_h) = \int_0^1 \dot{v}_h v_h dx = \frac{1}{2}\int_0^1 \frac{d}{dt} v_h^2 dx = \frac{1}{2}\frac{d}{dt}\int_0^1 v_h^2 dx$$

$$(\dot{v}_h,v_h) = \frac{1}{2}\frac{d}{dt}||v_h||^2 \tag{2.2}$$

$$(v_h', v'_h) = \int_0^1 v'_h{}^2 \, dx = ||v_n'||^2 \tag{2.3}$$

We estimate the remaining terms as follows

$$v_h(1)(xv_h, v_h') = v_h(1) \int_0^1 xv_h v_h' \, dx \le |v_h(1)| \; ||v_h|| \; ||v_h'|| \tag{2.4}$$

Because $v_h(0) = 0$, we have

$$v_n^2(1) = 2 \int_0^1 v_h v_h' \, dx = 2(v_h, v_h')$$

using the Schwarz's inequality we obtain

$$v_h^2(1) \le 2||v_h|| \; ||v_h'|| \tag{2.5}$$

$$(xv_h, v_h') \le ||v_h|| \; ||v_h'|| \tag{2.6}$$

From (2.1), (2.2) and (2.3) it follows

$$\frac{1}{2}\frac{d}{dt}||v_h||^2 + ||v_h'||^2 = v_h(1)(xv_h, v_h') - s_h v_h(1)(1, v_h)$$

The estimates (2.4) to (2.6) yield the inequality

$$\frac{d}{dt}||v_h||^2 + 2||v_h'|| \le 2\sqrt{2}||v_h||^{3/2}||v_h'||^{3/2} - 2s_h v_h(1)(1, v_h) \tag{2.7}$$

The term $2s_h v_h(1)(1,v_h)$ can be dropped out of the inequality if we can show that $s_h v_h(1)(1,v_h)$ is always positive. In other words we need to show that v_h is negative in Q since $s(t) > 0$, $0 < t \le T$.

We have the following Proposition.

Proposition 1. If $\{ \bar{s}(\tau) , U(y,\tau) \}$ is a solution of Problem 1 (assumed to be very smooth), then

i) $u_x(x,t) \leq 0.$

Thus by Proposition 1, (2.7) takes the form

$$\frac{d}{dt}||v_h||^2 + 2||v_h'||^2 \leq 2\sqrt{2}||v_h'||^{3/2}||v_h||^{3/2} \tag{2.8}$$

Because of the Young's inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$$

where a, b > 0 and $\frac{1}{q} + \frac{1}{p} = 1$, we have

$$2\sqrt{2}||v_h||^{3/2}||v_h'||^{3/2} \leq 2||v_h'||^2 + c_1||v_h||^6 \tag{2.9}$$

here we took $p = \frac{4}{3}$ and $q = 4$. c_1 is a constant chosen such that (2.9) holds. With this choice of the constant c_1, by combining (2.8) and (2.9), the term $2||v_h'||^2$ drops out. Therefore

$$\frac{d}{dt}||v_h||^2 \leq c_1||v_h||^6$$

solve to obtain

$$||v_h||^2 \leq \frac{||P_h g' + 1||^2}{(1 - c_1||P_h g' + 1||^4 t)^{1/2}}$$

which holds for $t < T_1$ where

$$T_1 = \frac{1}{c_1||P_h g' + 1||^4}$$

The operator P_h to be used will be bounded uniformly

$$||P_h g'|| \leq c_2(g)$$

Then v_h exists uniformly in h for $t < (c_1 c_2(g))^{-1}$. To complete the proof of Theorem 1, we need to prove now Proposition 1.

Proof of Proposition 1.

i) Let $V = U_y$ in Problem 1.

Differentiating (1.7) and (1.5) with respect to y we obtain

$$V_{yy} = V_\tau \quad \text{in} \quad \Omega$$

$$V(0,\tau) = -1 \quad , \quad 0 < \tau \leq T_0$$

$$V(y,0) = g'(y) \quad , \quad 0 \leq y \leq 1$$

$$V(\bar{s}(\tau),\tau) = -1 - \frac{d\bar{s}}{d\tau} \quad , \quad 0 < \tau \leq T_0$$

We can apply the maximum principle (see [9]): V must take its maximum on the parabolic boundary of Ω, i.e.

$$\partial\Omega - \{(y,\tau) \; ; \; \tau = T_0\}$$

But the maximum of V cannot be greater than -1. In fact,

$$V(y,0) = g'(y) \leq -1 \quad , \quad 0 \leq y \leq 1$$

$$V(\bar{s}(\tau),\tau) = -1 - \frac{d\bar{s}}{d\tau} \leq -1 \; , \; 0 < \tau \leq T_0$$

$$V(0,\tau) = -1 \ , \ 0 < \tau \leq T_0$$

Thus

$$V(y,\tau) \leq -1$$

Because

$$V(y,\tau) = \frac{1}{s} u_x - 1$$

Consequently

$$u_x(x,t) = v(x,t) \leq 0$$

Theorem 2. For any $K > 0$ fixed, there is at most one solution v_h of problem 4, with

$$v_h(x,t) \in B = \{ w \mid w \in L_\infty(0,1), \ ||w||_{L_\infty} \leq K \}$$

almost everywhere in $t \in (0,\bar{t})$.

Proof. Suppose there exist two pairs of solutions

$$\{ v_h^{(1)} \ , \ s_h^{(1)} \} \quad \text{and} \quad \{ v_h^{(2)} \ , \ s_h^{(2)} \}$$

Let

$$w(x,t) = v_h^{(1)} - v_h^{(2)} \quad , \quad \text{with } w(x,0) = 0$$

$$f(t) = s_h^{(1)} - s_h^{(2)} \quad , \quad \text{with } f(0) = 0$$

substitute both solutions into (1.21) and subtract the resulting equations to get

$$(\dot{w},\chi) + (w',\chi') = v_h(1)(xv_h^{(1)},\chi') - s_h^{(1)} v_h^{(1)}(1)(1,\chi)$$

$$- v_h^{(2)}(1)(xv_h^{(2)},\chi') + s_h^{(2)} v_h^{(2)}(1)(1,\chi)$$

In order to group terms together we add and subtract to the left-hand side of the last equation the terms

$$v_h^{(2)}(1)(xv_h^{(1)},\chi') \quad \text{and} \quad s_h^{(1)}v_h^{(2)}(1)(1,\chi)$$

to get

$$(\dot{w},\chi) + (w',\chi') = w(1)(xv_h^{(1)},\chi') + v_h^{(2)}(1)(xw,\chi') - s_h^{(1)}w(1)(1,\chi) - v_h^{(2)}(1)f(t)(1,\chi) \tag{2.10}$$

In (2.10) the choice $\chi = w$ gives

$$(\dot{w},w) + (w',w') = w(1)(xv_h^{(1)},w') + v_h^{(2)}(1)(xw,w') - s_h^{(1)}w(1)(1,w) - v_h^{(2)}(1)f(t)(1,w) \tag{2.11}$$

Because $v_h \varepsilon B$, we have the estimates

$$w(1)(xv_h^{(1)},w') \leq K|w(1)|\;||w'||$$

$$v_h^2(1)(xw,w') \leq K||w||\;||w'||$$

Further since $||v||_\infty < K$, from (1.22) we have

$$s_h = \exp\left[-\int_0^t v(1,z)dz\right] \leq e^{T||v||_\infty} = c_1$$

i.e. that s_h is bounded. This implies that

$$s_h^{(1)}w(1)(1,w) \leq c_1\;|w(1)|\;||w||$$

$$v_h^{(2)}(1)f(t)(1,w) \le K|f(t)|\ ||w||$$

where

$$c_1 = \sup_{t\varepsilon(0,\bar{t})} |s_h^{(1)}|$$

Using (2.11) and the above estimates we obtain

$$\frac{1}{2}\frac{d}{dt}||w||^2 + ||w'||^2 \le K_1|w(1)|\ ||w'|| + K_1||w||\ ||w'|| + K_1|w(1)|\ ||w|| + K_1|f(t)|\ ||w|| \tag{2.12}$$

where

$$K_1 = \max(c_1,K)$$

To get an estimate on f(t) note that $f(t) = s_h^{(1)} - s_h^{(2)}$ solves

$$\dot{f} = -v_h^{(1)}(1)s_h^{(1)} + v_h^{(2)}(1)s_h^{(2)} + v_h^{(1)}(1)s_h^{(2)} - v_h^{(1)}(1)s_h^{(2)}$$

or equivalently solves

$$\dot{f} + v_h^{(1)}(1)f = -\ s_h^{(2)}w(1) \quad \text{with} \quad f(0) = 0$$

solving the differential equation we obtain

$$f(t) = -e^{-a(t)} \int_0^t e^{a(z)} s_h^{(2)}(z)w(1,z)dz$$

where

$$a(z) = \int_0^z v_h^{(1)}(1,t)dt$$

Since v_h and s_h are bounded it follows

$$f(t) \leq K(\bar{t})|w(1)| \tag{2.13}$$

Consequently from (2.5), (2.12) and (2.13) we have

$$\frac{1}{2}\frac{d}{dt}||w||^2 + ||w'||^2 \leq \sqrt{2}\, K_1||w'||^{3/2}||w||^{1/2} + K_1||w||\ ||w'||$$

$$+ \sqrt{2}\, K_1||w||^{3/2}||w'||^{1/2} + \sqrt{2}\, K_1K(\bar{t})||w||^{3/2}||w'||^{1/2}$$

Applying the Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\ ||w||^2 + ||w'||^2 \leq \frac{1}{3}\ ||w'||^2 + K_2||w||^2 + \frac{1}{3}\ ||w'||^2$$

$$+ K_3||w||^2 + \frac{1}{3}\ ||w'||^2 + K_4||w||^2$$

For appropriately chosen constants K_2, K_3 and K_4. Hence we have

$$\frac{1}{2}\frac{d}{dt}\ ||w||^2 \leq K_5||w||^2 \quad \text{with} \quad w(x,0) = 0$$

where

$$K_5 = K_2 + K_3 + K_4$$

which leads to

$$w(x,t) \equiv 0$$

and consequently to

$$f(t) \equiv 0$$

This completes the proof.

2.2 Error estimate.

We shall study the error of both $v - v_h$ and $s - s_h$ in the L_∞ - norm. Let

$$e(x,t) = v(x,t) - v_h(x,t) \tag{2.14}$$

and

$$\sigma(t) = s(t) - s_h(t) \tag{2.15}$$

subtracting (1.17) and 2.18) from (1.21) and (1.22), we get

$$\begin{aligned}(\dot{e},\chi) + (e',\chi') &= v(1)(xv,\chi') - sv(1)(1,\chi) \\ &\quad - v_h(1)(sv_h,\chi') + s_h v_h(1)(1,\chi)\end{aligned} \tag{2.16}$$

and

$$\dot{\sigma} = -v(1)s + v_h(1)s_h \tag{2.17}$$

By adding first the term $v_h(1)(xv,\chi') + sv_h(1)(1,\chi)$ in both sides of (2.16) and then replacing $v_h(1)$ by $v(1) - e(1)$, one obtains,

$$\begin{aligned}(\dot{e},\chi) + (e',\chi') &= e(1)(xv,\chi') - se(1)(1,\chi) + v(1)(xe,\chi') \\ &\quad - e(1)(xe,\chi') - \sigma v(1)(1,\chi) + \sigma e(1)(1,\chi)\end{aligned} \tag{2.18}$$

$$\text{for all } \chi \in \dot{S}_h$$

Similarly as in (2.18), by adding first the term $v_h(1)s$ in both sides of (2.17) and then replacing v_h by $v(1) - e(1)$, one gets

$$\dot{\sigma} + (v(1) - e(1))\sigma = -se(1) \quad \text{with} \quad \sigma(0) = 0 \tag{2.19}$$

Note that (2.18) and 2.19) are still coupled. In order to eliminate σ from (2.18) we solve the linear equation (2.19) to obtain

$$\sigma = s(1 - \exp[\int_0^t e(1,z)dz]) \tag{2.20}$$

We now introduce the bilinear form b on $\dot{H}^1 \times \dot{H}^1$ defined by

$$b(\zeta,\eta) = (\zeta',\eta') - v(1)(x\zeta,\eta') - \zeta(1)(xv,\eta') + s\zeta(1)(1,\eta) \tag{2.21}$$

$$\text{for all } \zeta,\ \eta \in \dot{H}^1$$

We prove the following lemma.

Lemma 1. The bilinear form $b(\zeta,\eta)$ satisfies

i) There exists M such that

$$|b(\zeta,\eta)| \leq M||\zeta'||\ ||\eta'|| \quad , \text{ for all } \zeta,\ \eta \in \dot{H}^1$$

ii) There exist m, Λ such that

$$b(\zeta,\zeta) \geq m||\zeta'||^2 - \Lambda\, ||\zeta||^2 \quad , \text{ for all } \zeta \in \dot{H}^1$$

where M, m, Λ are positive constants depending only on

$$||v||_{L_\infty(L_\infty)} \quad \text{and} \quad \sup_{0<t\leq\bar{t}} |s(t)| \ .$$

Remark: Part (i) on the lemma simply states that b is a bounded bilinear form and that part (ii) is coercive.

Proof. We have

$$|\zeta(1)|^2 = 2\int_0^1 \zeta'(x)\zeta(x)dx \leq 2||\zeta'|| \; ||\zeta|| \tag{2.22}$$

and

$$\zeta(x) = \int_0^x \zeta'(z)dz \quad , \quad \text{for all } \zeta \varepsilon \dot{H}^1$$

The last equation implies that

$$|\zeta| \leq \int_0^x |\zeta'|dz \leq \int_0^1 |\zeta'|dx \leq (\int_0^1 \zeta'^2 dz)^{1/2}$$

Hence

$$||\zeta|| \leq ||\zeta'|| \tag{2.23}$$

Using the estimate (2.22) and the fact that v and s are bounded, we obtain

$$|b(\zeta,\eta)| \leq ||\zeta'|| \; ||\eta'|| + c_1||\zeta'|| \; ||\eta'||$$
$$+ c_2||\zeta'||^{1/2}||\zeta||^{1/2}||\eta'|| + c_3||\zeta'|| \; ||\eta||$$

where c_1, c_2, c_3 are positive constants.

Now according to (2.23), we have for all ζ, $\eta \varepsilon \dot{H}^1$

$$|b(\zeta,\eta)| \leq ||\zeta'|| \; ||\eta'|| + c_1||\zeta'|| \; ||\eta'||$$
$$+ c_2||\zeta'|| \; ||\eta'|| + c_3||\zeta'|| \; ||\eta'||$$

Set $1 + c_1 + c_2 + c_3 = M$ to get (i).

ii) To show the coercitivity, consider (2.21) and set $\eta = \zeta$ to get

$$b(\zeta,\zeta) = ||\zeta'||^2 - v(1)(x\zeta,\zeta') - \zeta(1)(xv,\zeta') + s\zeta(1)(1,\zeta)$$

This implies that

$$b(\zeta,\zeta) \geq ||\zeta'||^2 - c_4||\zeta||\ ||\zeta'|| - c_5||\zeta||^{1/2}||\zeta'||^{3/2}$$

$$- c_6||\zeta'||^{1/2}||\zeta||^{3/2}$$

where c_4, c_5 and c_6 are constants.

According to Young's inequality, the last inequality yields

$$b(\zeta,\zeta) \geq ||\zeta'||^2 - \frac{1}{2} c_4^2||\zeta||^2 - \frac{1}{2}||\zeta'||^2 - \frac{1}{4} c_5^4\delta^4||\zeta||^2$$

$$- \frac{3}{4}\frac{1}{\delta^{4/3}}||\zeta'||^2 - \frac{3}{4} c_6^{4/3}||\zeta||^2 - \frac{1}{4}||\zeta'||^2 .$$

where δ is a positive real number appropriately chosen.

$$b(\zeta,\zeta) \geq (\frac{1}{4} - \frac{3}{4}\frac{1}{\delta^{4/3}})||\zeta'||^2 - (\frac{1}{2} c_4^2 + \frac{1}{4} c_5^4\ \delta^4 + \frac{3}{4} c_6^{4/3})||\zeta||^2$$

to get (ii) we need to choose δ such that

$$\frac{1}{4} - \frac{3}{4}\frac{1}{\delta^{4/3}} > 0$$

i.e.

$$\delta^{4/3} > 3$$

with the choice of δ, we set

$$\frac{1}{4} - \frac{3}{4}\frac{1}{\delta^{4/3}} = m$$

and

$$\frac{1}{2}c_4^2 + \frac{1}{4}c_5^4\delta^4 + \frac{3}{4}c_6^{4/3} = \Lambda$$

to get the result (ii).

Hence, from Lemma 1. we conclude that

$$b_\wedge(\zeta,\eta) = b(\zeta,\eta) + \Lambda(\zeta,\eta) \tag{2.24}$$

is coercive on $\dot{H}^1 \times \dot{H}^1$.

Now we can write (2.18) in the form

$$\begin{aligned}(\dot{e},\chi) + b_\wedge(e,\chi) &= \Lambda(e,\chi) + \sigma e(1)(1,\chi) \\ &\quad - v(1)\sigma(1,\chi) - e(1)(xe,\chi')\end{aligned} \tag{2.25}$$

for all $\chi \varepsilon \dot{S}_h$.

Putting (2.20) into (2.25), one gets

$$\begin{aligned}(\dot{e},\chi) + b_\wedge(e,\chi) &= \Lambda(e,\chi) - v(1)s(1-\exp[\int_0^t e(1,z)dz])(1,\chi) \\ &\quad - e(1)(xe,\chi') + e(1)s(1-\exp[\int_0^t e(1,z)dz])(1,\chi)\end{aligned} \tag{2.26}$$

for all $\chi \varepsilon \dot{S}_h$

Because of the last three terms, equation (2.26) is nonlinear. In order to get an error estimate, we consider (2.26) with e(1,t) replaced by the bounded and measurable function E(t) which we take to be known. We take the bounded to be one in L_∞ norm. The idea is to show that the map which associates $e(1,t)$ to every $E(t)$ is a contraction in a ball of radius R small enough, so we can apply contraction-mapping theorem and obtain that the finite element solution v_h remains in a neighborhood of v. Consequently, a bound for the error e will be also obtained.

Inserting $E(t)$ into the last two terms of (2.26), we get

$$(\dot{e},\chi) + b_{\wedge}(e,\chi) = \Lambda(e,\chi) - v(1)s(1-\exp[\int_0^t e(1,z)dz])(1,\chi)$$

$$- E(t)(xe,\chi') + e(1)s(1-\exp[\int_0^t E(z)dz])(1,\chi) \qquad (2.27)$$

$$\text{for all } \chi\varepsilon\dot{S}_h$$

Note that equation (2.27) is still nonlinear because of the term $v(1)s(1-\exp[\int_0^t e(1,z)dz])(1,\chi)$. So we cannot get the optimal convergence rate for the L_∞ - norm of e. To get around this difficulty we write e in the form

$$e(x,t) = (v(x,t) - P_h v) - (v_h(x,t) - P_h v) = \varepsilon - \phi \qquad (2.28)$$

where $P_h v$ is an orthogonal projection of v on $\dot{S}_h$ and ϕ is the correction term in $\dot{S}_h$.

Replacing e by (2.28) in the nonlinear term of (2.27) gives

$$v(1)s(1-\exp[\int_0^t \varepsilon(1,z)dz] \cdot \exp[\int_0^t -\phi(1,z)dz])(1,\chi)$$

Now we define $P_h v$ as the solution of the Dirichlet problem

$$\begin{cases} P_h v \in S_h \quad \text{such that} \\ b_\wedge(v - P_h v, \chi) = 0 \quad , \quad \text{for all } \chi \in \dot{S}_h \text{ and } \chi(1) = 0 \\ P_h v(1,t) = v(1,t) \end{cases} \tag{2.29}$$

This implies that

$$\varepsilon(1,t) = 0$$

almost everywhere in t.

Since $b_\wedge$ is coercive and continuous, we know from the Lax-Milgram lemma that the above Dirichlet problem has a unique solution. By substituting (2.28) into (2.27), we obtain

$$\begin{aligned} (\dot{\phi},\chi) + b_\wedge(\phi,\chi) &= \Lambda(\phi,\chi) + (\dot{\varepsilon},\chi) + b_\wedge(\varepsilon,\chi) - \Lambda(\varepsilon,\chi) \\ &+ v(1)s(1-\exp[-\int_0^t \phi(1)dz])(1,\chi) \\ &+ \phi(1)s(1-\exp[\int_0^t E(z)dz])(1,\chi) \\ &+ E(x\varepsilon,\chi') - E(x\phi,\chi') \\ &\qquad \text{for all } \chi \in \dot{S}_h \end{aligned} \tag{2.30}$$

with

$$\phi(x,0) = 0$$

Thus $\phi \in \dot{S}_h$ is the solution of a system of nonlinear ordinary differential equations. From (2.28) we have the following result

$$||e||_{L_\infty(L_2)} \leq ||\varepsilon||_{L_\infty(L_2)} + ||\phi||_{L_\infty(L_2)} \quad (2.31)$$

So in order to get an estimate on e, we need to find estmates on ε and ϕ.

The estimate on ε is given by the following Lemma (see [17], [7], [19]).

Lemma 2. Assume $r \geq 3$ and that the exact solution v of Problem 4 be sufficiently smooth. Then for any time t fixed

$$||\varepsilon||_{H^k} + ||\dot{\varepsilon}||_{H^k} \leq ch^{r-k} \quad , \quad -1 \leq k \leq 1 \quad (2.32)$$

$$||\varepsilon||_{L_\infty} \leq ch^r \quad (2.33)$$

where

$$||\varepsilon||_{H^{-1}} = \sup\{(\varepsilon,\eta)|\eta \in \dot{H}^1 \text{ and } ||\eta'|| \leq 1\} \quad (2.24)$$

Now to find an estimate on ϕ, we write ϕ in the form,

$$\phi(x,t) = \phi_0(x,t) + z(x,t) \quad , \quad t \in (0,\overline{t}) \quad (2.35)$$

where

$$z(x,t) = \phi(1,t)x$$

and

$$\phi_0 \in \dot{S}_h \quad \text{with} \quad \phi_0(1,t) = 0$$

Inserting (2.35) into (2.30) and setting $\chi = \phi$, we obtain

$$(\dot{\phi},\phi) + b_\wedge(\phi,\phi) = \Lambda(\phi,\phi) + (\dot{\varepsilon},\phi) + b_\wedge(\varepsilon,z) - \Lambda(\varepsilon,\phi)$$
$$+ v(1)s(1-\exp[-\int_0^t \phi(1)dz])(1,\phi)$$
$$- \phi(1)s(1-\exp[\int_0^t E(z)dz])(1,\phi) \tag{2.36}$$
$$+ E(x\varepsilon,\phi') - E(x\phi,\phi')$$

Let us first study the term $b_\wedge(\varepsilon,z)$. We have

$$b_\wedge(\varepsilon,z) = (\varepsilon',z') - v(1)(x\varepsilon,z') - \varepsilon(1)(xv,z') + s\varepsilon(1)(1,z) + \Lambda(\varepsilon,z).$$

Since $\varepsilon(1,t) = 0$, $b_\wedge$ takes the form

$$b_\wedge(\varepsilon,z) = (\varepsilon',z') - v(1)(x\varepsilon,z') + \Lambda(\varepsilon,z) \tag{2.37}$$

Now let us find an estimate on each term of the right-hand side of (2.37). We have

a. $$(\varepsilon',z') = \int_0^1 \varepsilon'(x,t)z'(x,t)dx = \phi(1)\int_0^1 \varepsilon'(x,t)dx$$
$$= \phi(1)[\varepsilon(1,t) - \varepsilon(0,t)] = 0$$

By (2.28) and (2.29).

b. $$v(1)(x\varepsilon,z') \leq |v(1)|\ |(x\varepsilon,z')| = |v(1)|\ |\int_0^1 x\varepsilon(x,t)z'(x,t)dx|$$
$$\leq |\phi(1)|\ ||\varepsilon||_{-1}||v||_{L_\infty(L_\infty)}$$

c. $$\Lambda(\varepsilon,z) \leq \Lambda|(\varepsilon,z)|$$

Note that $z(x,t) \in \dot{H}^1$ since $z(0,t) = 0$. It follows from (2.34) and (2.35) that

$$\Lambda|(\varepsilon,z)| \le \Lambda||\varepsilon||_{-1}||z'|| = \Lambda||\varepsilon||_{-1}|\phi(1)|$$

Similarly, we now find an estimate on each term of the right-hand side of (2.36)

a_1. $\Lambda(\phi,\phi) \le \Lambda||\phi||^2$

a_2. $(\dot{\varepsilon},\phi) \le ||\dot{\varepsilon}||_{-1}||\phi'||$

a_3. $\Lambda(\varepsilon,\phi) \le \Lambda||\varepsilon||_{-1}||\phi'||$

a_4. $E(x\varepsilon,\phi') \le |E|\ ||\varepsilon||\ ||\phi'||$

Because $||E||_\infty \le 1$, we have the following two estimates

a_5. $E(x\phi,\phi') \le |E|\ ||\phi||\ ||\phi'|| \le ||\phi||\ ||\phi'||$

a_6. Note that $(1 - \exp[\int_0^t E(z)dz]) \le c_2$

where c_2 is constant. Thus

$$\phi(1)s(1-\exp[\int_0^t E(z)dz])(1,\phi) \le c_2|\phi(1)|\ ||\phi||\ .$$

a_7. For the nonlinear term, we have

$$1-\exp[-\int_0^t \phi(1)dz] = \int_0^t \phi(1)dz - \frac{1}{2}(\int_0^t \phi(1)dz)^2\exp[-\int_0^{t^*} \phi(1)dz]$$

where t^* is suitably chosen. We also have the relation

$$\phi = \varepsilon - e = \varepsilon - v + v_h$$

since ε, v, v_h are bounded, we conclude that

$$\exp[-\int_0^t \phi(1)dz] \leq K = \text{constant}$$

Further, from (2.22)

$$|\phi(1)| \leq \sqrt{2}||\phi||^{1/2}||\phi'||^{1/2}$$

This implies that

$$v(1)s(1-\exp[-\int_0^t \phi(1)dz])(1,\phi) = v(1)s(\int_0^1 \phi(1)dz)(1,\phi)$$

$$- \frac{1}{2} v(1)s(\int_0^t \phi(1)dz)^2 \exp[-\int_0^{t^*} \phi(1)dz](1,\phi)$$

$$\leq c_3||\phi||\int_0^t |\phi(1)|dz + c_4||\phi||(\int_0^t |\phi(1)|dz)^2$$

$$\leq \sqrt{2}\, c_3||\phi||^{1/2}||\phi'||^{1/2}\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz$$

$$+ 2c_4||\phi||^{1/2}||\phi'||^{1/2}(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz)^2$$

$$= \frac{\sqrt{2}}{2} c_3 \frac{d}{dt} (\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz)^2$$

$$+ \frac{2}{3} c_4 \frac{d}{dt} (\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz)^3$$

where c_3 and c_4 are positive constants.

We obtain from the above results

$$\frac{1}{2}\frac{d}{dt}||\phi||^2 + m||\phi'||^2 \le \Lambda||\phi||^2 + ||\dot{\varepsilon}||_{-1}||\phi'||$$

$$+ c_5||\varepsilon||_{-1}|\phi(1)| + \Lambda||\varepsilon||_{-1}|\phi(1)| + \Lambda||\varepsilon||_{-1}||\phi'||$$

$$+ \frac{\sqrt{2}}{2} c_3 \frac{d}{dt}\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz\right)^2 \qquad (2.38)$$

$$+ \frac{2}{3} c_4 \frac{d}{dt}\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz\right)^3 + c_2|\phi(1)|\ ||\phi||$$

$$+ ||\phi||\ ||\phi'|| + |E|\ ||\varepsilon||\ ||\phi'|| .$$

by applying (2.22) to ϕ, it follows

$$c_5||\varepsilon||_{-1}|\phi(1)| + \Lambda||\varepsilon||_{-1}|\phi(1)| \le \sqrt{2}(c_5+\Lambda)||\varepsilon||_{-1}||\phi||^{1/2}||\phi'||^{1/2}$$

Similarly,

$$c_2|\phi(1)|\ ||\phi|| \le \sqrt{2}\, c_2||\phi||^{3/2}||\phi'||^{1/2}$$

Consequently the inequality (2.38) takes the form

$$\frac{d}{dt}||\phi||^2 + 2m||\phi'||^2 \le 2\Lambda||\phi||^2 + 2||\dot{\varepsilon}||_{-1}||\phi'||$$

$$+ 2\sqrt{2}(c_5+\Lambda)||\varepsilon||_{-1}||\phi||^{1/2}||\phi'||^{1/2} + 2\Lambda||\varepsilon||_{-1}||\phi'|| \qquad (2.39)$$

$$+ \sqrt{2}\, c_3 \frac{d}{dt}\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dt\right)^2 + \frac{4}{3} c_4 \frac{d}{dt}\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dt\right)^3$$

$$+ 2\sqrt{2}\, c_2||\phi||^{3/2}||\phi'||^{1/2} + 2||\phi||\ ||\phi'|| + 2|E|\ ||\varepsilon||\ ||\phi'||$$

Taking into account that $\phi(x,0) = 0$, we integrate (2.39) over the interval $(0,t)$ to get

$$||\phi||^2 + 2m\int_0^t ||\phi'||^2 dz \leq 2\Lambda\int_0^t ||\phi||^2 dz + 2\int_0^t ||\dot{\varepsilon}||_{-1}||\phi'||dz$$

$$+ 2\sqrt{2}\,(c_5 + \Lambda)\int_0^t ||\varepsilon||_{-1}||\phi||^{1/2}||\phi'||^{1/2}dz$$

$$+ 2\Lambda\int_0^t ||\varepsilon||_{-1}||\phi'||dz + \sqrt{2}\,c_3\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz\right)^2 \tag{2.40}$$

$$+ \frac{4}{3}c_4\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz\right)^3 + 2\sqrt{2}\,c_2\int_0^t ||\phi||^{3/2}||\phi'||^{1/2}dz$$

$$+ 2\int_0^t ||\phi||\;||\phi'||dz + 2\int_0^t |E|\;||\varepsilon||\;||\phi'||dz$$

In order to apply Gronwall's Lemma, we now reduce the following terms by use of Young's and Schwarz's inequalities

b_1. $\sqrt{2}\,c_2||\phi||^{3/2}||\phi'||^{1/2} \leq k_1||\phi||^2 + \delta||\phi'||^2$

b_2. $2||\phi||\;||\phi'|| \leq ||\phi||^2 + ||\phi'||^2$

b_3. $2|E|\;||\varepsilon||\;||\phi'|| \leq k_2|E|^2||\varepsilon||^2 + \delta||\phi'||^2$

b_4. $2||\dot{\varepsilon}||_{-1}||\phi'|| \leq k_3||\dot{\varepsilon}||_{-1}^2 + \delta||\phi'||^2$

b_5. $||\phi|| \leq ||\phi'||$ implies that

$$2\sqrt{2}\,c_1||\varepsilon||_{-1}||\phi||^{1/2}||\phi'||^{1/2} \leq 2\sqrt{2}\,c_1||\varepsilon||_{-1}||\phi'||$$

$$\leq k_4||\varepsilon||_{-1}^2 + \delta||\phi'||^2$$

b_6. $\quad 2\Lambda||\varepsilon||_{-1}||\phi'|| \leq k_5||\varepsilon||_{-1}^2 + \delta||\phi'||^2$

b_7. $\quad \sqrt{2}\, c_3\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz\right)^2 \leq \sqrt{2}\, c_3\left(\int_0^t ||\phi'||\ ||\phi||dz\right)\left(\int_0^t dz\right)$

$$\leq \sqrt{2}\, c_3\ \bar{t}\int_0^t ||\phi'||\ ||\phi||dz \leq k_6\int_0^t ||\phi||^2 dz + \delta\int_0^t ||\phi'||^2 dz$$

b_8. $\quad \frac{4}{3}\, c_4\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}dz\right)^3$

$$\leq \frac{4}{3}\, c_4\left[\left(\int_0^t ||\phi||dz\right)^{1/2}\left(\int_0^t ||\phi'||dz\right)^{1/2}\right]^3$$

$$= \frac{4}{3}\, c_4\left(\int_0^t ||\phi'||dz\right)^{3/2}\left(\int_0^t ||\phi||dz\right)^{3/2}$$

Next an application of the Young's inequality yields

$$\leq \frac{1}{c_5}\,\delta\left(\int_0^t ||\phi'||dz\right)^2 + c_6\left(\int_0^t ||\phi||dz\right)^6 ,$$

where we have chosen $p = \frac{4}{3}$ and $q = 4$.

Applying the Schwarz inequality again, gives finally the estimates

$$\frac{4}{3}\, c_4\left(\int_0^t ||\phi||^{1/2}||\phi'||^{1/2}\right)^3 \leq k_7\int_0^t ||\phi||^6 dz + \delta\int_0^t ||\phi'||^2 dz$$

Here $k_1,\ldots,k_7$ and δ are positive constants.

By introducing the estimates b_1 to b_8 in (2.40), we obtain

$$||\phi||^2 + 2m\int_0^t ||\phi'||^2 dz \leq 2\Lambda \int_0^t ||\phi||^2 dz + k_3 \int_0^t ||\dot{\varepsilon}||_{-1}^2 dz$$

$$+ \delta \int_0^t ||\phi'||^2 dz + k_4 \int_0^t ||\varepsilon||_{-1}^2 dz + \delta \int_0^t ||\phi'||^2 dz$$

$$+ k_5 \int_0^t ||\varepsilon||_{-1}^2 dz + \delta \int_0^t ||\phi'||^2 dz + k_6 \int_0^t ||\phi||^2 dz$$

$$+ \delta \int_0^t ||\phi'||^2 dz + k_7 \int_0^t ||\phi||^6 dz + \delta \int_0^t ||\phi'||^2 dz$$

$$+ k_1 \int_0^t ||\phi||^2 dz + \delta \int_0^t ||\phi'||^2 dz + \int_0^t ||\phi||^2 dz$$

$$+ \int_0^t ||\phi'||^2 dz + k_2 \int_0^t ||\varepsilon||^2 |E|^2 dt + \delta \int_0^t ||\phi'|| dz$$

simplify to get

$$||\phi||^2 + 2m \int_0^t ||\phi'|| dz \leq (2\Lambda + k_6 + k_1 + 1) \int_0^t ||\phi||^2 dz$$

$$+ k_3 \int_0^t ||\dot{\varepsilon}||_{-1}^2 dz + (k_5 + k_4) \int_0^t ||\varepsilon||_{-1}^2 dz + k_2 \int_0^t ||\varepsilon||^2 |E|^2 dz$$

$$+ (7\delta + 1) \int_0^t ||\phi'||^2 dz + k_7 \int_0^t ||\phi||^6 dz$$

Since δ is arbitrary, we now choose δ such that

$$2m = 7\delta + 1$$

$$\delta = \frac{2m - 1}{7}$$

with the choice of δ, the term $\int_0^t ||\phi'||^2 dz$ drops out of the inequality. Hence we have

$$||\phi||^2 \le k_8 \int_0^t ||\phi||^2 dz + k_3 \int_0^t ||\dot{\varepsilon}||_{-1}^2 dz + K_9 \int_0^t ||\varepsilon||_{-1}^2 dz$$

$$+ k_2 ||E||_{L_\infty} \int_0^t ||\varepsilon||^2 dz + k_7 \int_0^t ||\phi||^6 dz$$

recalling that $||\phi||$ is uniformly bounded, we then have the estimate

$$||\phi||^6 \le c||\phi||^2$$

It follows that

$$||\phi||^2 \le k_{10} \int_0^t ||\phi||^2 dz + k_3 \int_0^t ||\dot{\varepsilon}||_{-1}^2 dz + k_9 \int_0^t ||\varepsilon||_{-1}^2 dz$$

$$+ k_2 ||E||_{L_\infty} \int_0^t ||E||^2 dz$$

where $k_{10} = k_8 + ck_7$

Now by applying Gronwall's Lemma, we finally get the estimate

$$||\phi(x,t)||_{L_2(0,1)}^2 \le K' \{ ||\dot{\varepsilon}||_{L_2(0,\bar{t};H^{-1}(0,1))}^2 + ||\varepsilon||_{L_2(0,\bar{t};H^{-1}(0,1))}^2$$

$$+ ||E||_{L_\infty(0,\bar{t})}^2 ||\varepsilon||_{L_2(0,\bar{t};L_2(0,1))}^2 \}$$

where K is a constant.

Thus we have proven the following Lemma.

Lemma 3. Let E(t) be fixed, measurable with

$$||E||_{L_\infty(0,\bar{t})} \leq 1$$

then

$$||\phi||_{L_\infty(L_2)} \leq K \{ ||\dot{\varepsilon}||_{L_2(H^{-1})} + ||\varepsilon||_{L_2(H^{-1})} + ||E||_{L_\infty} ||\varepsilon||_{L_2(L_2)} \} \tag{2.41}$$

From (2.31) we have

$$||e||_{L_\infty(L_2)} \leq ||\phi||_{L_\infty(L_2)} + ||\varepsilon||_{L_\infty(L_2)} \tag{2.42}$$

Further, using Lemma 2 and Lemma 3 we obtain

$$||e||_{L_\infty(L_2)} \leq K \{ h^{r+1} + h^r ||E||_{L_\infty} \} + Kh^r$$

Because $h \leq 1$, this implies

$$||e||_{L_\infty(L_2)} \leq Kh^r \{ 2 + ||E||_{L_\infty} \} \tag{2.43}$$

we have the inverse property

$$||\chi||_{L_\infty} \leq ch^{-1/2} ||\chi||_{L_2} \quad , \quad \text{for all } \chi \in \dot{S}_h$$

since $\phi \in \dot{S}_h$ it follows that

$$\begin{aligned} ||e||_{L_\infty(L_\infty)} &\leq ||\varepsilon||_{L_\infty(L_\infty)} + ch^{-1/2} ||\phi||_{L_\infty(L_2)} \\ &\leq Kh^r + ch^{-1/2} K \{ h^{r+1} + h^r ||E||_{L_\infty} \} \\ &\leq K_1 \{ h^r + ||E||_{L_\infty} h^{r-1/2} \} \end{aligned}$$

Consequently since $e \in H^1(0,1)$, we have the bound on $e(1,t)$ give by

$$||e(1,t)||_{L_\infty(0,\bar{t})} \leq ||e||_{L_\infty(L_\infty)} \leq K_1 \{ h^r + ||E||_{L_\infty} h^{r-1/2} \}$$

This proves that the application which associates $e(1,t)$ to every $E(t)$ is a contraction in

$$B_1 = \{ w(t) | \; ||w||_{L_\infty(0,\bar{t})} \leq 1 \}$$

since the image $e(1,t)$ is contained in B_1 for $h \leq \bar{h}$, with $\bar{h}$ suitably small.

Thus using the contraction-mapping theorem, we have proven the existence of a function $E(t)$ with $E(t) = e(1,t)$. Consequently, we have proved the following theorem.

Theorem 3. Let $\bar{t} > 0$ be chosen properly and assume that the solution of Problem 4 is sufficiently smooth in $Q = \{ (x,t) | 0 \leq x \leq 1, \; 0 \leq t \leq \bar{t} \}$. Then there exists exactly one finite element solution v_h in the neighborhood of v with

$$||v - v_h||_{L_\infty(0,\bar{t};L_\infty)} = o(h^r)$$

2.3. Error estimate for the original unknowns.

What we have found so far is an error estimate for $v(x,t)$. We need to go back to the solution of problem 1 to get the estimate for the initial unknown $U(y,\tau)$.

We have the inverse transformation introduced at the beginning,

$$u_h(x,t) = - \int_x^1 v_h(z,t)dz \tag{2.44}$$

which is the approximation of the solution of problem 2. Furthermore,

$$y_h = s_h(t)x$$

$$\tau_h = \tau_h(t)$$

where s_h and τ_h are given by

$$\frac{ds_h(t)}{d\tau} = -v_h(1,t)s_h(t) \quad , \quad s_h(0) = 1$$
$$\frac{d\tau_h(t)}{dt} = s_h^2(t) \quad , \quad \tau_h(0) = 0 \tag{2.45}$$

Finally, the approximation of $U(y,\tau)$ is given by

$$U_h(y_h,\tau_h) = u_h(x,t) + s_h(t)(1-x) \tag{2.46}$$

Theorem 2 and (2.45) give the errors

$$\sup_{t\varepsilon(0,\bar{t})} |s(t) - s_h(t)| = o(h^r)$$
$$\sup_{t\varepsilon(0,\bar{t})} |\tau(t) - \tau_h(t)| = o(h^r) \tag{2.47}$$

From (2.44) and (2.46) we have

$$U(y(x,t),\tau(t)) - U_h(y_h(x,t),\tau_h(t)) =$$

$$-\int_x^t (v(z,t) - v_h(z,t))dz + (1-x)(s(t) - s_h(t))$$

Theorem 2, (2.44) and (2.47) imply that

$$||U(y(x,t),\tau(t)) - U_h(y_h(x,t),\tau_h(t))||_{L_\infty(L_\infty)} = o(h^r)$$

$$\left\|\frac{\partial}{\partial x} U(y(x,t),\tau(t)) - \frac{\partial}{\partial x} U_h(y_h(x,t),\tau_h(t)\right\|_{L_\infty(L_\infty)} = o(h^r)$$

Thus we have the following corollary

Corollary 1. Under the assumptions of Theorem 4, for some positive $\bar{t}$ the errors

$$\sup_{t\varepsilon(0,\bar{t})} | s(t) - s_h(t) |$$

$$\sup_{t\varepsilon(0,\bar{t})} | \tau(t) - \tau_h(t) |$$

$$\sup_{t\varepsilon(0,\bar{t})} \sup_{x\varepsilon(0,1)} \{ | U(y(x,t),\tau(t)) - U_h(y_h(x,t),\tau_h(t)) |$$

$$+ \left| \frac{\partial}{\partial x} U(y(x,t),\tau(t)) - \frac{\partial}{\partial x} U_h(y_h(x,t),\tau_h(t)) \right\}$$

are of order h^r.

2.4. Numerical results.

We now develop a numerical scheme for the case of a linear retention function as follows:

The governing equation of the discretized problem is given by

$$(\dot{v}_h,\chi) + (v'_h,\chi') = v_h(1)(xv_h,\chi') - s_h v_h(1)(1,\chi) \qquad (2.48)$$

$$\text{for all } \chi \varepsilon \dot{S}_h$$

with

$$\dot{s} = -v_h(1)s_h \quad , \quad s_h(0) = 1 \qquad (2.49)$$

$$\dot{\tau}_h = s_h^2 \quad , \quad \tau_h(0) = 0 \qquad (2.50)$$

We choose $r = 4$ which means that $\dot{S}_h$ is the space of continuous functions which are polynomials of third degree. Let

$$h = 1/N \quad , \quad \text{for a fixed integer } N \geq 1.$$

$$k = T/M \quad , \quad \text{for a fixed integer } M \geq 1.$$

k stands here for the time increment and h the length of the subintervals of [0,1]. We denote

$$t_n = nk \quad , \quad n = 0,1,\ldots,M$$

We discretize the time derivative $\dot{v}_h$ at the fixed time $t = t_{n+1}$ using the backward difference

$$\dot{v}(x,t) = \frac{v_h^{n+1}(x) - v_h^n(x)}{k} \tag{2.51}$$

where

$$v_h^n(x) = v_h(x,nk)$$

To linearize (2.48), we evaluate the nonlinear terms at the previous time steps.

We now rewrite (2.48) as follows:

$$\frac{1}{k}\,[(v_h^{n+1},\chi) - (v_h^n,\chi)] + (v_{h,x}^{n+1},\chi_x) + s_h^n v_h^{n+1}(1)(1,\chi) = v_h^n(1)(xv_h^n,\chi_x) \ ,$$

$$\text{for all } \chi \in \dot{S}_h$$

where

$$s_h^n = s_h(nk)$$

$$v_{h,x}^{n+1} = \frac{\partial}{\partial x} v_h^{n+1}(x)$$

$$\chi_x = \frac{\partial}{\partial x} \chi(x)$$

Similarly using the backward difference for the time derivatives $\dot{s}_h$ and $\dot{\tau}_h$ at the fixed time $t = t_{n+1}$, we obtain according to (2.49) and (2.50)

$$\frac{1}{k}(s_h^{n+1} - s_h^n) = \frac{1}{2} s_h^n [v_h^{n+1}(1) + v_h^n(1) - k v_h^{n+1}(1) v_h^n(1)]$$

$$\frac{1}{k}(\tau_h^{n+1} - \tau_h^n) = \frac{1}{2}[(s_h^{n+1})^2 + (s_h^n)^2]$$

Note that in the last two equations we used the approximations

$$v_h(1) = \frac{1}{2}[v_h^{n+1}(1) + v_h^n(1) - k v_h^{n+1}(1) v_h^n(1)]$$

and

$$s_h^2 = \frac{1}{2}[s_h^{n+1})^2 + (s_h^n)^2]$$

For the initial values, we have

$$v_h^o(x) = g'(x)+1 \quad , \quad x = ih \quad , \quad i = 0,\ldots,N$$

$$s_h^o = 1 \quad , \quad \tau_h^o = 0$$

We now replace the discritized problem by the difference scheme:

Find $\{v_h^{n+1}, s_h^{n+1}, \tau_h^{n+1}\}$ such that

$$(v_h^{n+1},\chi) + k(v_{h,x}^{n+1},\chi_x) + ks_n^n v_n^{n+1}(1)(1,\chi) \tag{2.52}$$
$$= kv_h^n(1)(xv_h^n,\chi_x) + (v_n^n,\chi) ,$$

for all $\chi \in \dot{S}_h$

$$s_h^{n+1} = s_h^n [\, 1 - \frac{k}{2}\,(v_h^{n+1}(1) + v_h^n(1) - kv_h^{n+1}(1)v_h^n(1))\,] \tag{2.53}$$

$$\tau_h^{n+1} = \tau_h^n + \frac{k}{2}[\,(s_h^{n+1})^2 + (s_h^n)^2\,] \tag{2.54}$$

According to (2.52), (2.53) and (2.54), if the state on t = nk is known, we can proceed to the time level t = (n+1)k to find first v_n^{n+1}, then s_n^{n+1} and finally τ_h^{n+1}.

The approximate solution v_h^{n+1} is obtained by the Ritz-Galerkin's method defined as follows:

we choose, in the fpace $\dot{S}_h$, a basis

$$\psi_0, \psi_1, \ldots, \psi_N.$$

where the $\psi_j(x)$'s are defined by the cubic splines sketched in Fig. 2 with equally spaced knots at x = jh, j = 0,1,...,N and satisfying the condition

$$\psi_j(0) = 0 \quad , \quad j = 0,1,\ldots,N.$$

See [18] for more detail on spline functions.

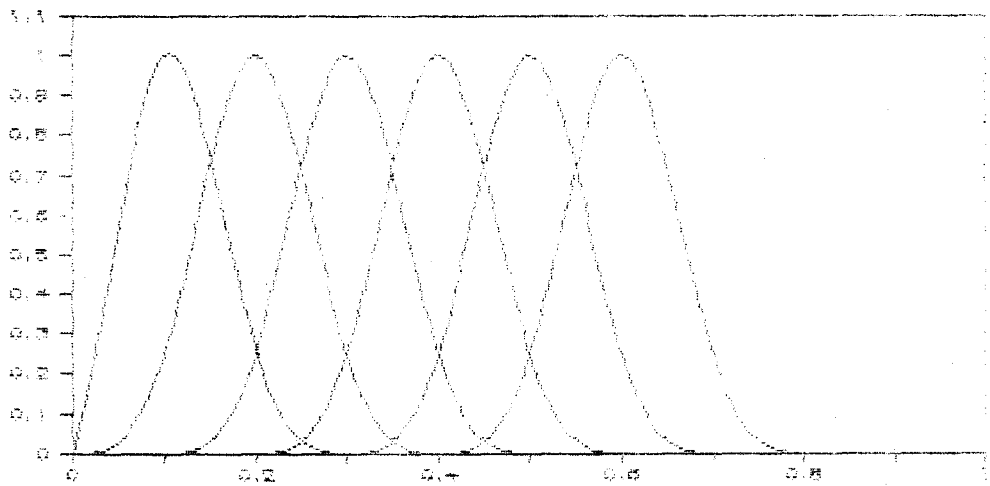

Fig. 2. Cubic splines.

We shall assume now that the "finite element" approximation v_h^{n+1} of v^{n+1} has the form

$$v_h^{n+1}(x) = \sum_{j=0}^{N} \alpha_j^{n+1} \psi_j(x) \tag{2.55}$$

where the coefficeints $\alpha_0^{n+1},\ldots,\alpha_N^{n+1}$ are to be determined. Inserting (2.55) into (2.52) and setting $\chi = \psi_i(x)$, one obtains

$$\begin{aligned}(\sum_{j=0}^{N} \alpha_j^{n+1} \psi_j, \psi_i) + k(\sum_{j=0}^{N} \alpha_j^{n+1} \psi_j', \psi_i') + ks_h^n v_h^{n+1}(1)(1,\psi_i) \\ = kv_h^n(1)(xv_h^n, \psi_i')\end{aligned} \tag{2.56}$$

Note that from (2.55), one can get

$$v_h^{n+1}(1) = \sum_{j=0}^{N} \alpha_j^{n+1} \psi_j(1) = \frac{1}{4} \alpha_{N-1}^{n+1} + \alpha_N^{n+1}$$

By the linearity of the scalar product, (2.56) takes the form

$$\sum_{j=0}^{N-2} \alpha_j^{n+1} \left[\int_0^1 (\psi_i \psi_j + k\psi_i' \psi_j')dx \right]$$

$$+ \alpha_{N-1}^{n+1} \left[\int_0^1 (\psi_i \psi_{N-1} + k\psi_i' \psi_{N-1}' + \frac{k}{4} s_h^n \psi_i)dx \right] \tag{2.57}$$

$$+ \alpha_N^{n+1} \left[\int_0^1 (\psi_i \psi_N + k\psi_i' \psi_N' + ks_h^n \psi_i)dx \right]$$

$$= \int_0^1 \left[k(\frac{1}{4} \alpha_{N-1}^n + \alpha_N^n) x v_h^n \psi_i' + v_h^n \psi_i \right] dx$$

$$i = 0,1,\ldots,N$$

These equations represent a system (the so-called Ritz system) of (N+1) linear equations for (N+1) unknowns $\alpha_0^{n+1},\ldots,\alpha_N^{n+1}$ of the type

$$A\alpha = b$$

where A is (N+1)x(N+1) matrix and b is the known coefficient vector. By solving the system $A\alpha = b$, we get the values of v_h^{n+1} at each point (ih,(n+1)k) and at the same time we calculate s_h^{n+1} and τ_h^{n+1}.

Having obtained the values of $v_h(x,t)$ in the points (ih,(n+1)k) and the values of s_h and τ_h at $t = (n+1)k$, we now integrate $v_h(x,t)$ over the interval (x,1) to obtain $u_h(x,t)$. This can be done by using an exact formula since v_h is piecewise cubic.

Finally the values of $U_h(y_h,\tau_h)$ are obtained in the points $y_h = ihs_h^{n+1}$ $(i = 0,1,\ldots,N)$ and $\tau_h = \tau_h^{n+1}$ using the relation

$$U_h(y_h,\tau_h) = u_h(ih,(n+1)k) + s_h^{n+1} - ihs_h^{n+1} ,$$

$$i = 0,1,\ldots,N$$

EXAMPLE 1.

$$U_\tau - U_{yy} = 0 \quad , \quad 0 < y < \bar{s}(\tau) \quad , \quad 0 < \tau \le 1.822$$

$$U_y(0,\tau) = -1 \quad , \quad 0 < \tau \le 1.822$$

$$U(y,0) = -\frac{1}{2}x^3 - \frac{1}{2}x^2 - x + 2 \quad , \quad 0 \le y \le 1$$

$$U(\bar{s}(\tau),\tau) = 0 \quad , \quad 0 < \tau \le 1.822$$

$$U_y(\bar{s}(\tau),\tau) = -\frac{d\bar{s}}{d\tau} - 1 \quad , \quad 0 < \tau \le 1.822$$

$$\bar{s}(0) = 1$$

with

$h = .2$ and $k = \Delta t = .1$,

$N = 5$ and $M = 10$

III. NUMERICAL SOLUTION FOR QUADRATIC RETENTION FUNCTION

3.1. Finite element method.

In section 1.5 we reduced Problem 1 to one with a fixed boundary for the case of a linear retention function. In this section we will follow the same procedure as in section 1.5 for a quadratic retention function, i.e. let

$$R(U) = U^2 \quad , \quad U \geq 0 \text{ be given.}$$

Let us consider the governing equation of Problem 1

$$U_{yy} = \frac{\partial R(U)}{\partial \tau} \tag{3.1}$$

setting

$$\begin{cases} u(x,t) = U(y,\tau) + y \\ s(t) = \bar{s}(\tau) \end{cases} \tag{3.2}$$

and using the transformation defined in section 1.4, we obtain with the assumption that R(U) is continuously differentiable.

$$U_y = \frac{1}{s} u_x - 1 \tag{3.3}$$

$$U_{yy} = \frac{1}{s^2} u_{xx} \tag{3.4}$$

$$\frac{\partial R(U)}{\partial \tau} = \frac{\partial R(U)}{\partial t}\frac{\partial t}{\partial \tau} = \left[\frac{\partial R(u-sx)}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial R(u-sx)}{\partial t}\right]\frac{1}{s^2}$$

we have

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial \tau}\frac{\partial \tau}{\partial t}$$

$$\frac{\partial x}{\partial \tau} = \frac{\partial x}{\partial s}\frac{\partial s}{\partial \tau} = -\frac{x}{s}[\, -U_y(\bar{s},\tau) - 1\,]$$

$$= -\frac{x}{s}[\,(-\frac{1}{s}u_x(1,t) + 1) - 1\,] = \frac{x}{s^2}u_x(1,t)$$

i.e.

$$\frac{\partial x}{\partial t} = \frac{x}{s^2}u_x(1,t)s^2 = xu_x(1,t)$$

Thus

$$\frac{\partial R(U)}{\partial \tau} = \frac{1}{s^2}[\, xu_x(1,t)R_x(u_1) + R_t(u_1)\,] \qquad (3.5)$$

where

$$u_1 = u - sx$$

$$R_x = \frac{\partial R}{\partial x} \quad , \quad R_t = \frac{\partial R}{\partial t}$$

Inserting (3.4) and (3.5) into (3.1), we get

$$u_{xx} - R_t(u_1) = xu_x(1,t)R_x(u_1)$$

using (3.2), we obtain for the initial and boundary condition

a. setting $\tau = 0$ in (3.2) gives

$$U(y,0) = u(x,0) - xs(0) = g(\frac{x}{s(0)})$$

i.e.

$$u(x,0) = g(x) + x$$

b. Set $y = 0$ in (3.3) to get

$$\frac{1}{s} u_x(0,t) - 1 = -1$$

whence

$$u_x(0,t) = 0$$

c. set $y = \bar{s}(\tau)$ in (3.2) to obtain

$$u(1,t) = U(\bar{s},\tau) + \bar{s} = \bar{s}$$

Thus

$$u(1,t) = s(t)$$

d. we have

$$\frac{d\bar{s}}{d\tau} = - U_y(\bar{s},\tau) - 1$$

which can be written in the form

$$\frac{d\bar{s}}{dt}\frac{dt}{d\tau} = - U_y(\bar{s},\tau) - 1$$

since

$$\frac{dt}{d\tau} = \frac{1}{s^2} \quad \text{and} \quad U_y(\bar{s},\tau) = \frac{1}{s} u_x(1,t) - 1$$

Then

$$\frac{ds}{dt} = - su_x(1,t)$$

The transformed problem now reads

Problem (TP). Find $\{u(x,t),s(t)\}$ such that

$$u_{xx} - R_t(u_1) = xu_x(1,t)R_x(u_1) \tag{3.6}$$

$$\text{in } Q = \{(x,t) \mid 0 < x < 1 \quad , \quad 0 < t \leq T\}$$

$$u_x(0,t) = 0 \quad , \quad 0 < t \leq T \tag{3.7}$$

$$u(1,t) = s(t) \quad , \quad 0 < t \leq T \tag{3.8}$$

$$u(x,0) = g(x)+x \quad , \quad 0 \leq x \leq 1 \tag{3.9}$$

$$\frac{ds}{dt} = - su_x(1,t) \quad , \quad s(0) = 1 \quad , \quad 0 < t \leq T \quad , \quad s(t) > 0 \tag{3.10}$$

where

$$u_1 = u - sx \quad , \quad R_x = \frac{\partial R}{\partial x} \quad \text{and} \quad R_t = \frac{\partial R}{\partial t}$$

To find a weak formulation of Problem (TP), we follow the same steps as in section 1.4.

Let

$$v = u_x$$

For t fixed we define v in the space

$$\dot{H}^1 = \{ w \mid w \in H^1(0,1) \ \text{and} \ w(0) = 0 \}$$

Taking into account (3.8), we obtain the relation

$$u = - \int_x^1 v dx + s(t) \tag{3.11}$$

Let us choose an arbitrary function $w_x \in \dot{H}^1$, multiply equation (3.6) by this function, and integrate over (0,1) to obtain

$$\int_0^1 u_{xx} w_x - \int_0^1 R_t(u_1) w_x dx = \int_0^1 x u_x(1) R_x(u_1) w_x dx \tag{3.12}$$

carrying out the summation for the second term in the left-hand side of (3.12), we obtain

$$\int_0^1 R_t(u_1)w_x dx = R_t(u_1)w\Big]_0^1 - \int_0^1 R_{tx}(u_1)wdx$$

$$= \frac{\partial R(u_1)}{\partial u_1}\frac{\partial u_1}{\partial t}\, w\Big]_0^1 - \int_0^1 R_{xt}(u_1)wdx$$

Note that

$$\frac{\partial u_1}{\partial t}\Big|_{x=1} = \frac{\partial u}{\partial t} - x\frac{ds}{dt}\Big|_{x=1} = \frac{ds}{dt} - \frac{ds}{dt} = 0$$

and

$$w(0) = 0$$

Thus

$$\int_0^1 R_t(u_1)w_x dx = -\int_0^1 R_{xt}(u_1)wdx$$

which yields

$$\int_0^1 v_x w_x dx + \int_0^1 R_{xt}(u_1)wdx = v(1)\int_0^1 xR_x(u_1)w_x dx$$

$$(v_x,w_x) + (R_{xt}(u_1),w) = v(1)(xR_x(u_1),w_x)$$

$$\text{for all } w \in \dot{H}^1 .$$

The weak formulation of Problem (TP) now reads

Problem (VP). Find $\{v(x,t),s(t)\}$ such that

$$(v_x, w_x) + (R_{xt}(u_1), w) = v(1)(xR_x(u_1), w_x) \tag{3.13}$$

for all $w \in \dot{H}^1$

and

$$\frac{ds}{dt} = -sv(1,t) \tag{3.14}$$

with the initial conditions

$$v(x,0) = g'(x) + 1$$

$$s(0) = 1 \tag{3.15}$$

3.2. Statement of the discritized problem.

Using Ritz-Galerkin's method and the same finite element space introduced in section 1.4 with

$$\dot{S}_h = \{ w \mid w \in S_h \quad \text{and } w(0) = 0 \} \quad , \quad \dot{S}_h \quad \dot{H}^1$$

The discritized problem for a quadratic retention function reads.

Problem (DP). Find $\{ v_h(x,t), s_h(t) \}$ such that

$$(v_{h,x}, w_x) + (R_{xt}(u_{h,1}), w) = v_h(1)(xR_x(u_{h,1}), w_x) \tag{3.16}$$

for all $w \in \dot{S}_h$

where

$$u_{h,1} = u_h - s_h x$$

$$\frac{ds_h}{dt} = - s_h v_h(1) \tag{3.17}$$

with the initial conditions

$$v_h(x,0) = P_h g'(x) + 1 \quad , \quad 0 \le x \le 1$$
$$s_h(0) = 1 \tag{3.18}$$

where P_h is an appropriate projection in the space $\dot{S}_h$.

3.3. Numerical scheme.

As developed in Chapter II, the method of discretization in time consists in the following:

Let $\dot{S}_n$ be the space of continuous functions which are polynomial of third degree. Divide the interval [0,T] by the points

$$t_n = nk \quad , \quad n = 0,\dots,M$$

into M subintervals of lengths

$$k = \frac{T}{n}$$

The derivative $\frac{\partial R_x}{\partial t}$ in (3.16) is replaced, for $t = t_h^{n+1}$, $n = 0,\dots,M$ by the difference quotient

$$R_{xt}(u_{h,1}) = \frac{R_x(u_{h,1}^{n+1}) - R_x(u_{h,1}^{n})}{k} \tag{3.19}$$

where

$$u_{h,1}^{n} = u_{h,1}(x,nk)$$

Consequently, equation (3.16) takes the form

$$\frac{1}{k}\,[(R_x(u_{h,1}^{n+1}),w) - (R_x(u_{h,1}^{n}),w)] + (v_{h,x}^{n+1},w_x) = v_h^n(1)(xR_x(u_{h,1}^{n}),w_x)$$

i.e.

$$(R_x(u_{h,1}^{n+1}),w) + k(v_{h,x}^{n+1},w_x) = kv_h^n(1)(xR_x(u_{h,1}^n),w_x) + (R_x(u_{h,1}^n),w) \qquad (3.20)$$

for all $w \varepsilon \dot{S}_h$

similarly, s_h^{n+1} and τ_h^{n+1} are given by (2.53) and (2.54).

$$s_h^{n+1} = s_h^n[1 - \frac{k}{2}(v_h^{n+1}(1) + v_h^n(1) - kv_h^{n+1}(1)v_h^n(1)] \qquad (3.21)$$

$$\tau_h^{n+1} = \tau_h^n + \frac{k}{2}[(s_h^{n+1})^2 + (s_n^n)^2] \qquad (3.22)$$

using the basis $\{\psi_i\}_{j=0}^{N}$ defined in section 2.4. Let us consider the discritized problem and let us solve it approximately, assuming it approximates solution v_h in the form

$$v_h^{n+1}(x) = \sum_{j=0}^{N} \alpha_j^{n+1} \psi_j(x) \qquad (3.23)$$

where the coefficient $\alpha_o,\dots,\alpha_N$ are to be determined. In equation (3.20) note that

$$R_x(u_{h,1}^{n+1}) = \frac{\partial R(u_{h,1}^{n+1})}{\partial u_{h,1}^{n+1}} \frac{\partial u_{h,1}^{n+1}}{\partial x}$$

It follows that (3.20) can be written in the form

$$\left(\frac{\partial R(u_{h,1}^{n+1})}{\partial u_{h,1}^{n+1}}[v_h^{n+1} - s_h^n],w\right) + k(v_x^{n+1},w_x)$$

$$= kv_h^n(1)\left(x\,\frac{\partial R(u_{h,1}^n)}{\partial u_{h,1}^n}[v_h^n - s_h^n],w_x\right) \tag{3.24}$$

$$+ \left(\frac{\partial R(u_{h,1}^n)}{\partial u_{h,1}^n}\frac{\partial u_{h,1}^n}{\partial x},w\right)$$

$$\text{for all } w \in \dot{S}_h$$

where in the first term of the left-hand side of (3.24), we evaluated s_h in the previous time steps in order to linearize the equation.

Putting (3.23) into (3.24) and setting $w = \psi_i(x)$, we get

$$\sum_{j=0}^{N} \alpha_j^{n+1} \int_0^1 [\psi_i(x)\psi_j(x)\,\frac{\partial R(u_{h,1}^{n+1})}{\partial u_{h,1}^{n+1}} + k\psi_i'(x)\psi_j'(x)]\,dx$$

$$= kv_h^n(1)\int_0^1 x\psi_i'(x)\,\frac{\partial R(u_{h,1}^n)}{\partial u_{h,1}^n}\,(v_h^n - s_h^n)dx \tag{3.25}$$

$$+ \int_0^1 \psi_i(x)\,\frac{\partial R(u_{h,1}^n)}{\partial u_{h,1}^n}\,(v_h^n - s_h^n)dx + s_h^n\int_0^1 \frac{\partial R(u_{h,1}^{n+1})}{\partial u_{h,1}^{n+1}}\,\psi_i(x)dx\ ,$$

$$n = 0,1,\ldots,M \quad \text{and} \quad i = 0,1,\ldots,N$$

Because of the term $\partial R(u_{n,1}^{n+1})/\partial u_{h,1}^{n+1}$, (3.25) is a nonlinear equation. In order to linearize (3.25), we will use a predictor-corrector method to compute first $\partial R(u_{h,1}^{n+1})/\partial u_{n,1}^{n+1}$ and then with $\partial R(u_{h,1}^{n+1})/\partial u_{h,1}^{n+1}$

known we will solve the linear system (3.25) for the unknowns $\alpha_o^{n+1},\ldots,\alpha_N^{n+1}$. The new $u_{h,1}^{n+1}$ will be used again to predict the non-linear term $\partial R(u_{h,1}^{n+1})/\partial u_{h,1}^{n+1}$, and the process continues until the change in u_h^{n+1} is sufficiently small.

We now describe the predictor - corrector method as follows: In (3.6) replace $R_t(u_{1,h})$ by the backward difference at $t = (n+1)k$

$$R_t(u_{1,h}) = \frac{R(u_{1,h}^{n+1}) - R(u_{1,h}^{n})}{k}$$

and evaluate the rest of the terms of the equation at the previous time step $t = nk$ to get

$$R(u_{h,1}^{n+1}) = k(u_{h,xx}^{n} - xu_{h,x}^{n}(1)R_x(u_{h,1}^{n})) + R(u_{h,1}^{n}) \qquad (3.26)$$

we have

$$R(u) = u^2$$

This implies that

$$R(u_{h,1}^{n}) = (u_h^n - xs_h^n)^2$$

Further

$$R_x(u_{h,1}^{n}) = \frac{\partial R(u_{h,1}^{n})}{\partial u_{h,1}^{n}}\frac{\partial u_{h,1}^{n}}{\partial x}$$

$$= 2u_{h,1}^{n}(u_{h,x}^{n} - s_h^n)$$

$$= 2(u_h^n - xs_h^n)(u_{h,x}^{n} - s_h^n)$$

let us denote the right-hand side of (3.26) by C which according to (3.11) and (3.23), takes the form

$$C = k\sum_{j=0}^{N}\alpha_j^n\psi_j'(x) + 2\,kx(\tfrac{1}{4}\,\alpha_{N-1}^n + \alpha_N^n)\,\cdot$$

$$[\sum_{j=0}^{N}\alpha_j^n\int_x^1\psi_j(x)dx + s_h^n(x-1)](\sum_{j=0}^{N}\alpha_j^n\psi_j(x) - s_h^n) \quad (3.27)$$

$$+\,[\sum_{j=0}^{N}\alpha_j^n\int_x^1\psi_j(x)dx + s_h^n(x-1)]^2$$

$$\text{for } n = 1,2,\ldots,M$$

For $n = 0$ i.e. $t = 0$, C is given by

$$C = kg''(x) - 2xk(g'(1) + 1)g(x)g'(x) + (g(x))^2 \quad (3.28)$$

Given C we can compute now the term

$$\frac{\partial R(u_{h,1}^{n+1})}{\partial u_{h,1}^{n+1}}\;.$$

From (3.26) we have

$$R(u_{h,1}^{n+1}) = C$$

or

$$(u_{h,1}^{n+1})^2 = C$$

i.e.

$$u_{h,1}^{n+1} = \sqrt{C}$$

Further,

$$\frac{\partial R(u_{h,1}^{n+1})}{\partial u_{h,1}^{n+1}} = 2\,u_{h,1}^{n+1}$$

This implies that

$$\frac{\partial R(u_{h,1}^{n+1})}{\partial u_{h,1}^{n+1}} = 2\sqrt{C} \quad , \qquad n = 0,1,\ldots,M$$

when C is given by (3.26) and (3.28).

By solving the system (3.25) we will obtain the values of v_h^{n+1} at each point $(ih,(n+1)k)$, $(i = 0,1,\ldots,N)$ and at the same time we calculate s_h^{n+1} and τ_h^{n+1}. Having obtained the values of $v_h(x,t)$ in the points $(ih,(n+1)k)$, $(i = 0,1,\ldots,N)$ and the values of s_h and τ_h in the points $(n+1)k$, we use (3.11) to obtain $u_h(x,t)$.

Finally, the values of $U_h(y_h,\tau_h)$ are obtained in the points $y_h = ihs_h^{n+1}$, $(i = 0,1,\ldots,N)$ and $\tau_h = \tau_h^{n+1}$ using the relation.

$$U_h(y_h,\tau_h) = u_h(ih,(n+1)k) - ihs_h^{n+1}$$

$$i = 0,1,\ldots,N$$

EXAMPLE 2

$$U_{yy} - \frac{\partial U^2}{\partial \tau} = 0 \quad , \quad 0 < y < \bar{s}(\tau) \quad , \quad 0 < \tau \leq 2.71$$

$$U_y(0,\tau) = -1 \quad , \quad 0 < \tau \leq 2.71$$

$$U(y,0) = -\frac{1}{2}x^3 - \frac{1}{2}x^2 - x + 2 \quad , \quad 0 \leq y \leq 1$$

$$U(\bar{s}(\tau),\tau) = 0 \quad , \quad 0 < \tau \leq 2.71$$

$$U_y(\bar{s}(\tau),\tau) = -\frac{d\bar{s}}{d\tau} - 1 \quad , \quad 0 < \tau \leq 2.71$$

$$\bar{s}(0) = 1$$

with

$h = .2$ and $k = \Delta t = .1$,

$N = 5$ and $M = 10$

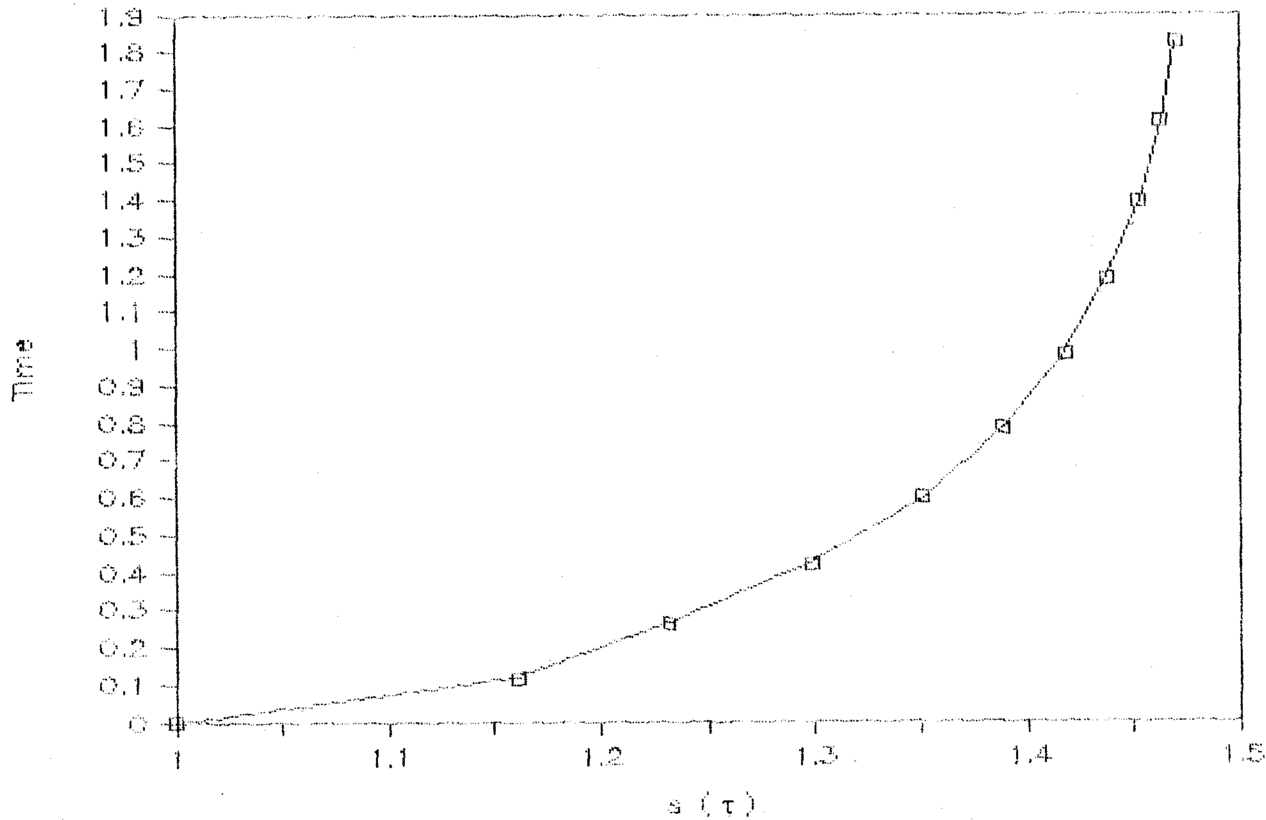

Fig. 3. Free boundary curve for example 1.

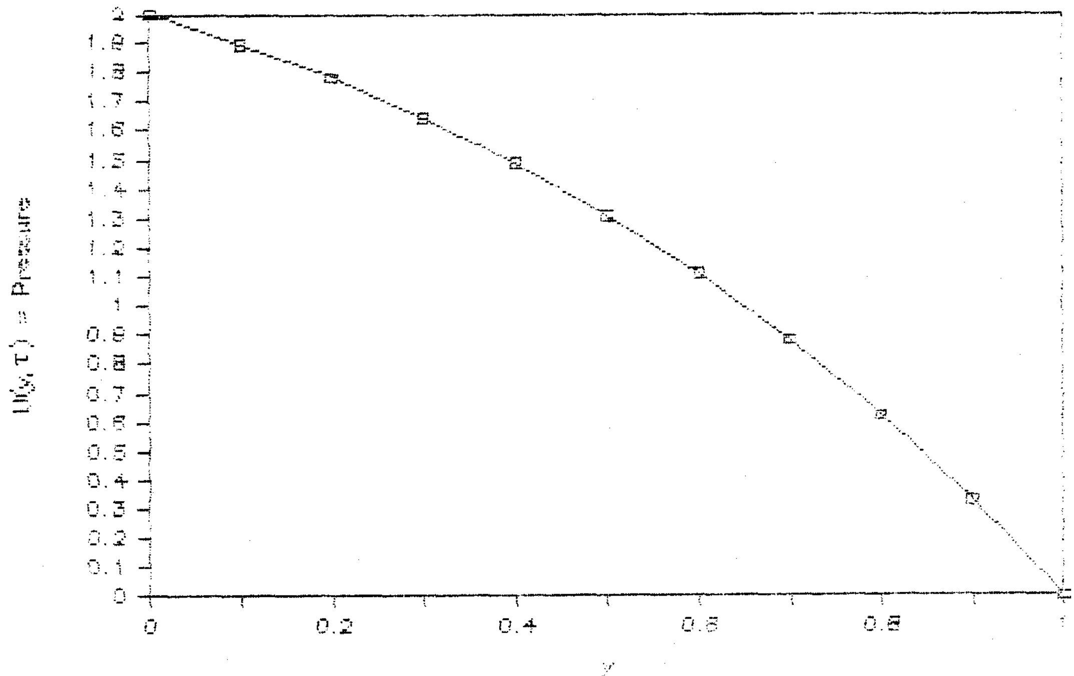

Fig. 4. Pressure when $\tau = 0.0$ for example 1.

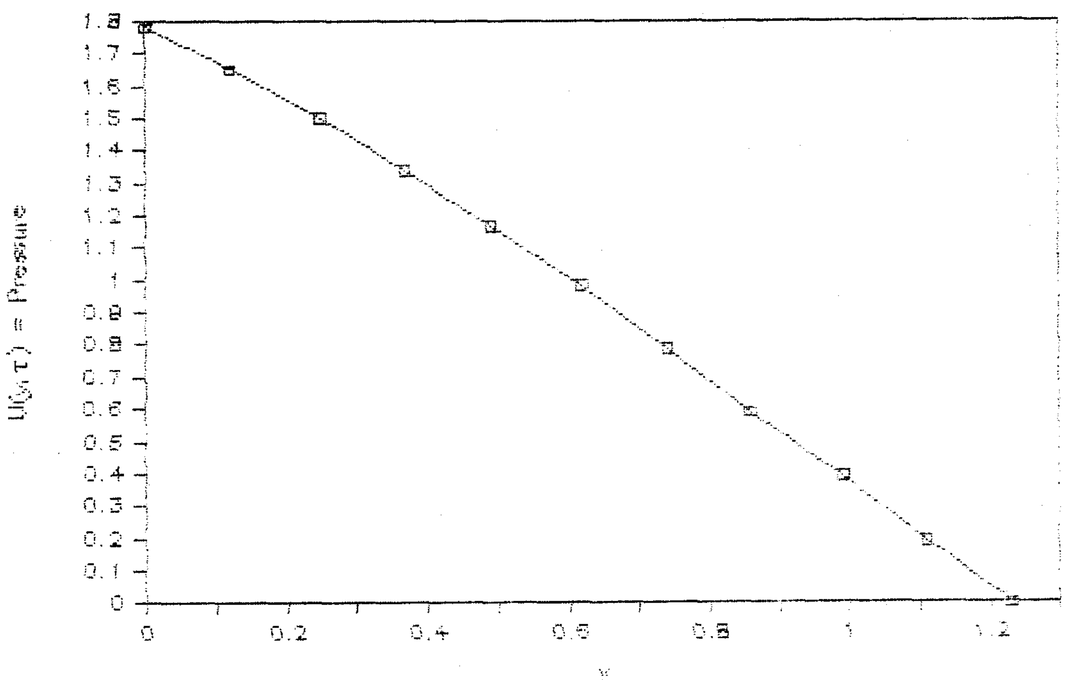

Fig. 5. Pressure when $\tau = 0.26$ for example 1.

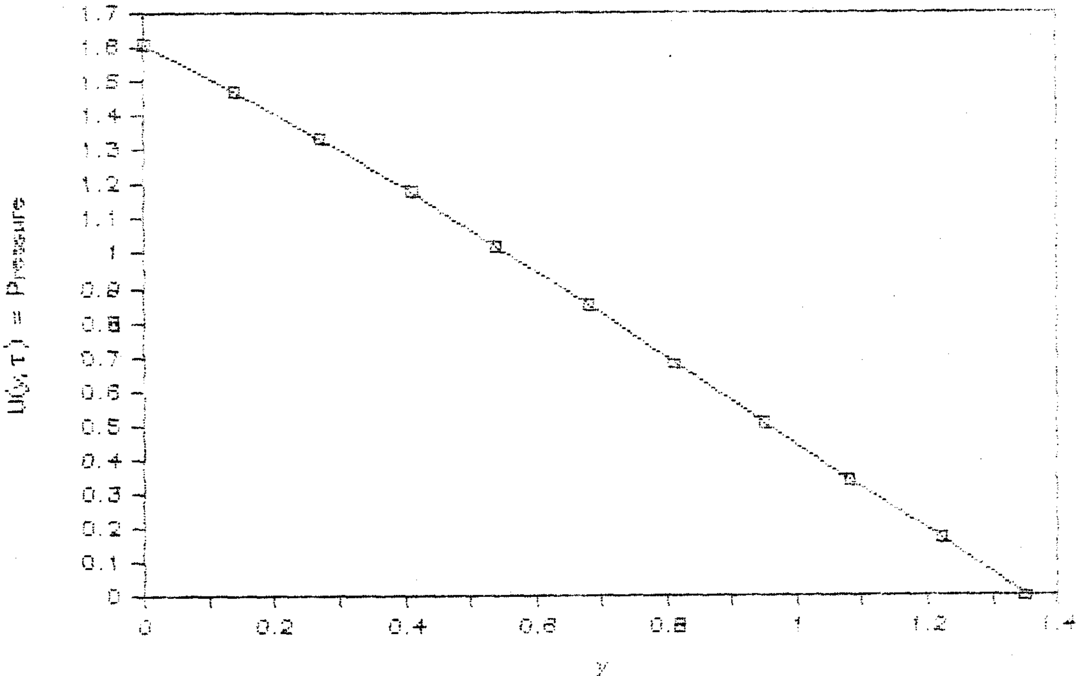

Fig. 6. Pressure when τ = 0.60 for example 1

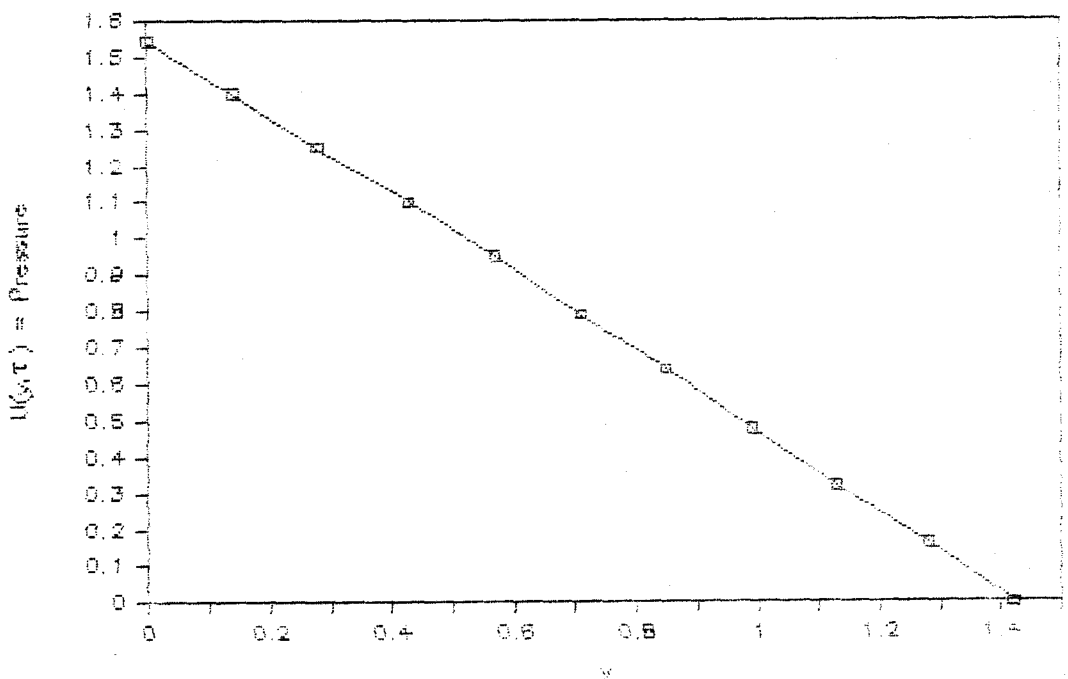

Fig. 7. Pressure when τ = 0.98 for example 1.

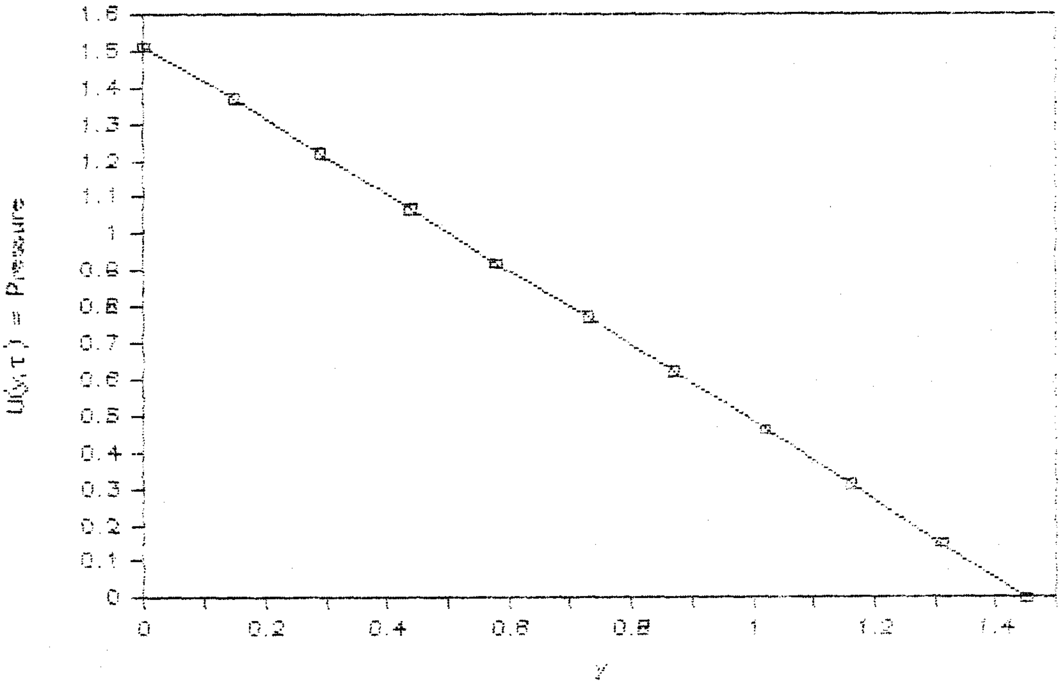

Fig. 8. Pressure when τ = 1.39 for example 1

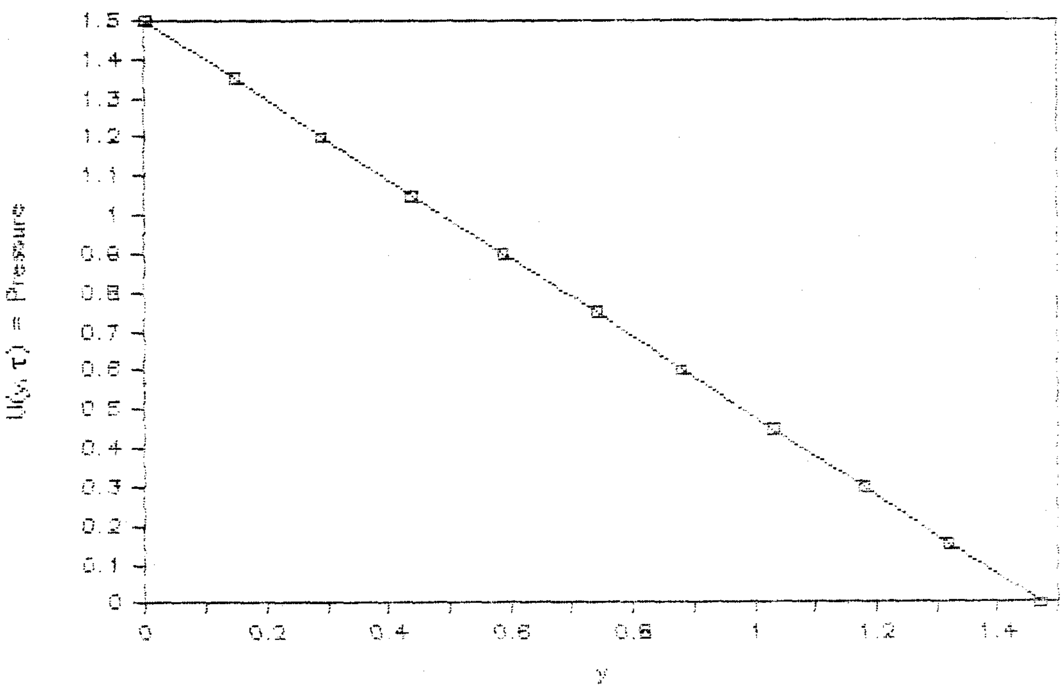

Fig. 9. Pressure when τ = 1.82 for example 1.

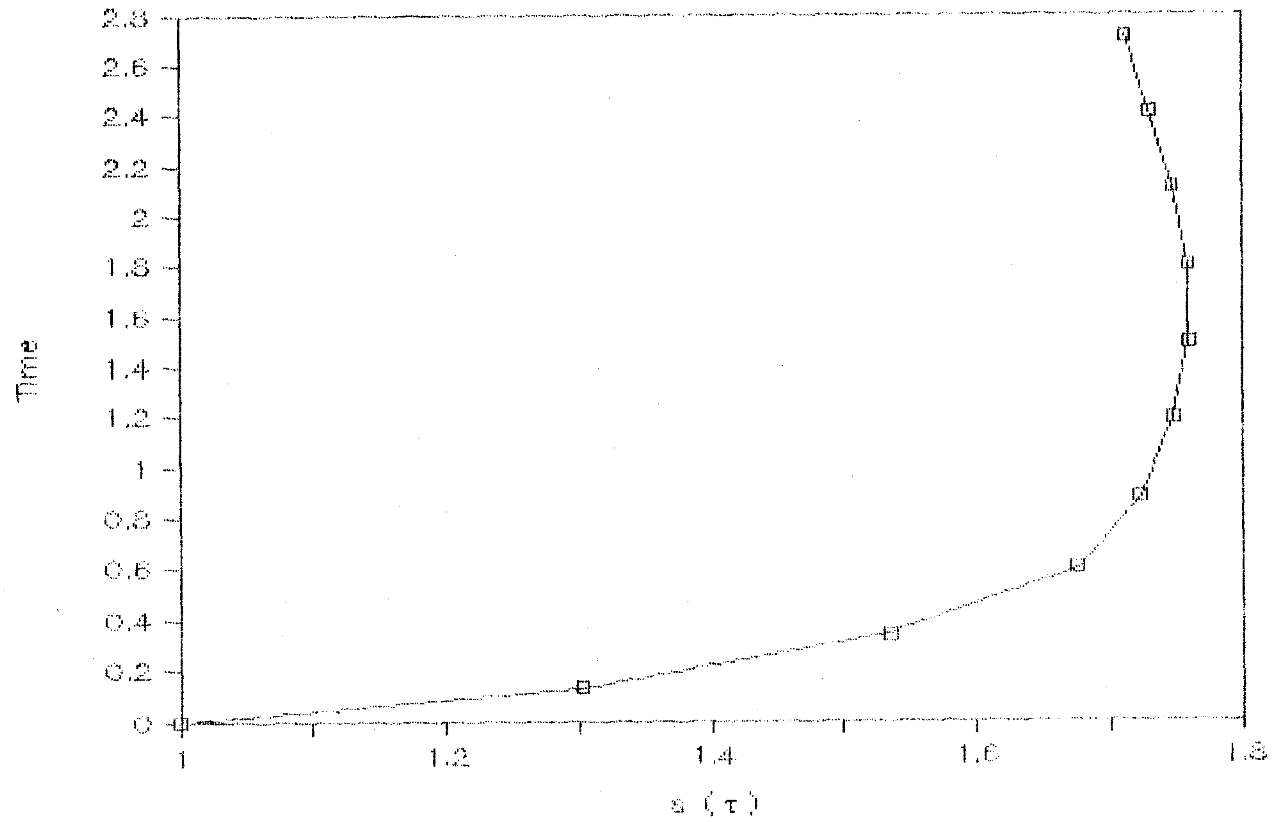

Fig. 10. Free boundary curve for example 2.

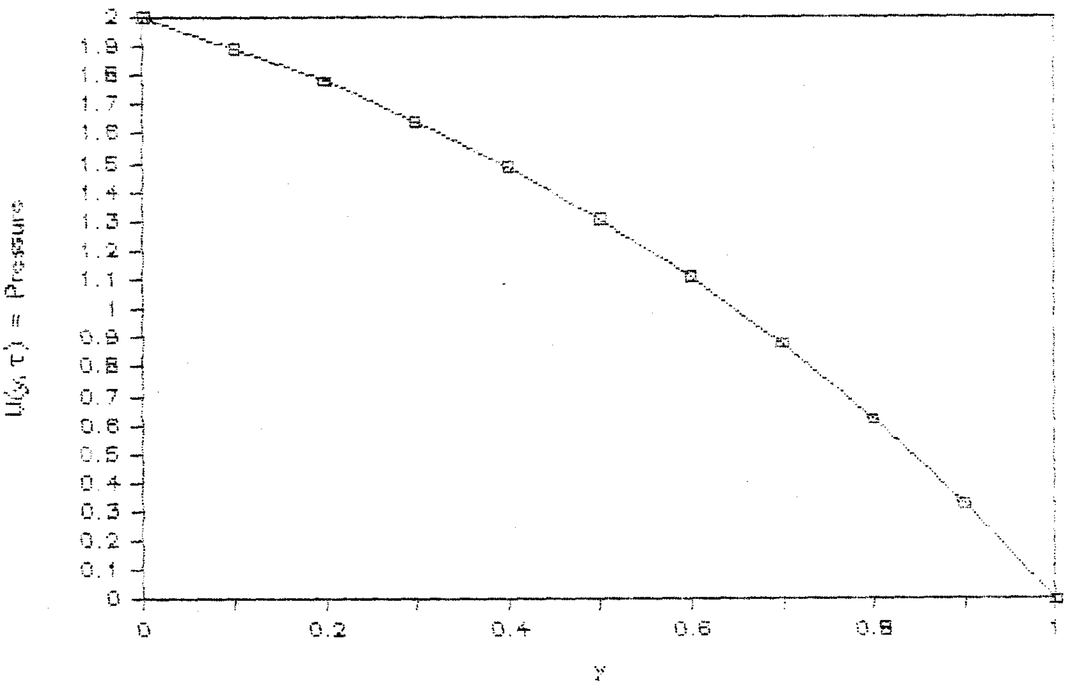

Fig. 11. Pressure when τ = 0.0 for example 2.

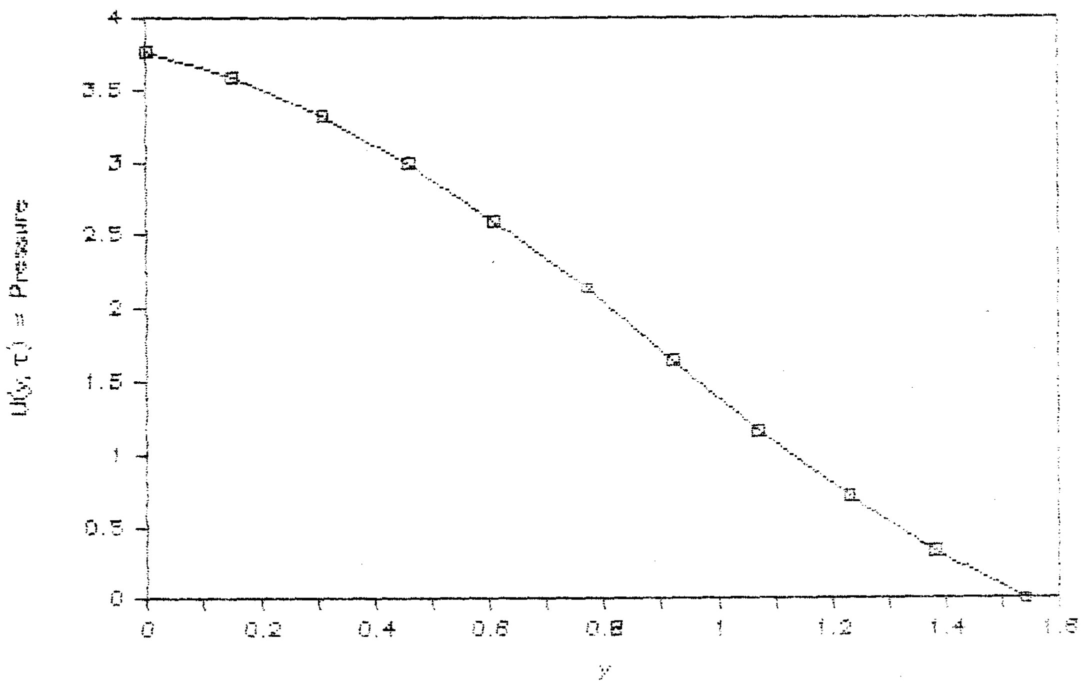

Fig. 12. Pressure when τ = 0.34 for example 2.

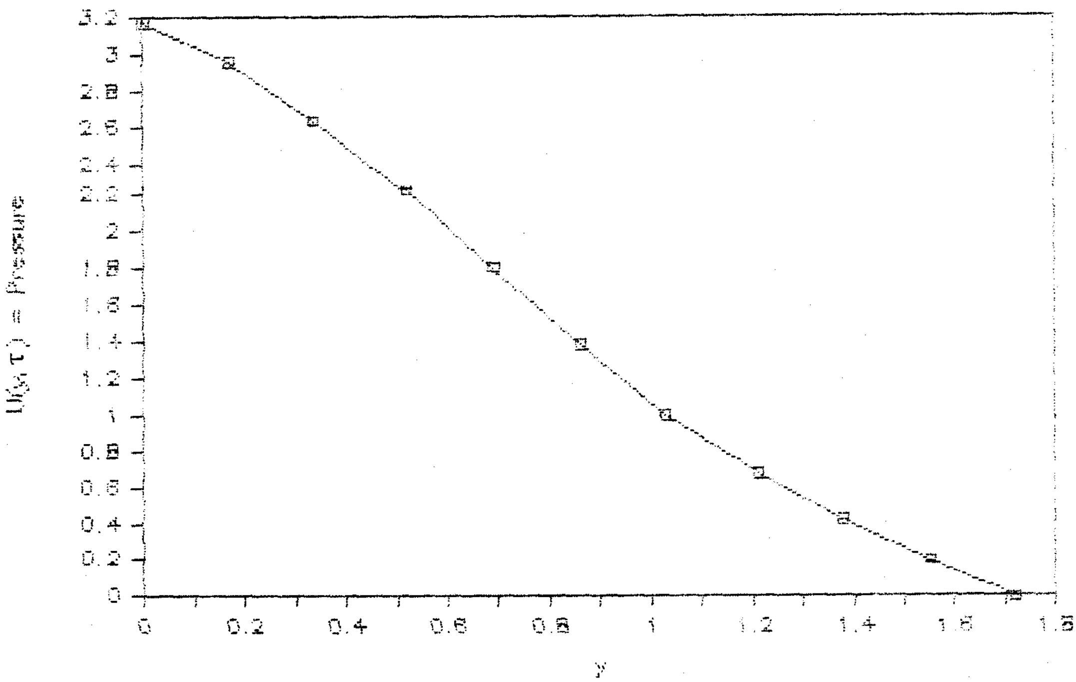

Fig. 13. Pressure when τ = 0.88 for example 2.

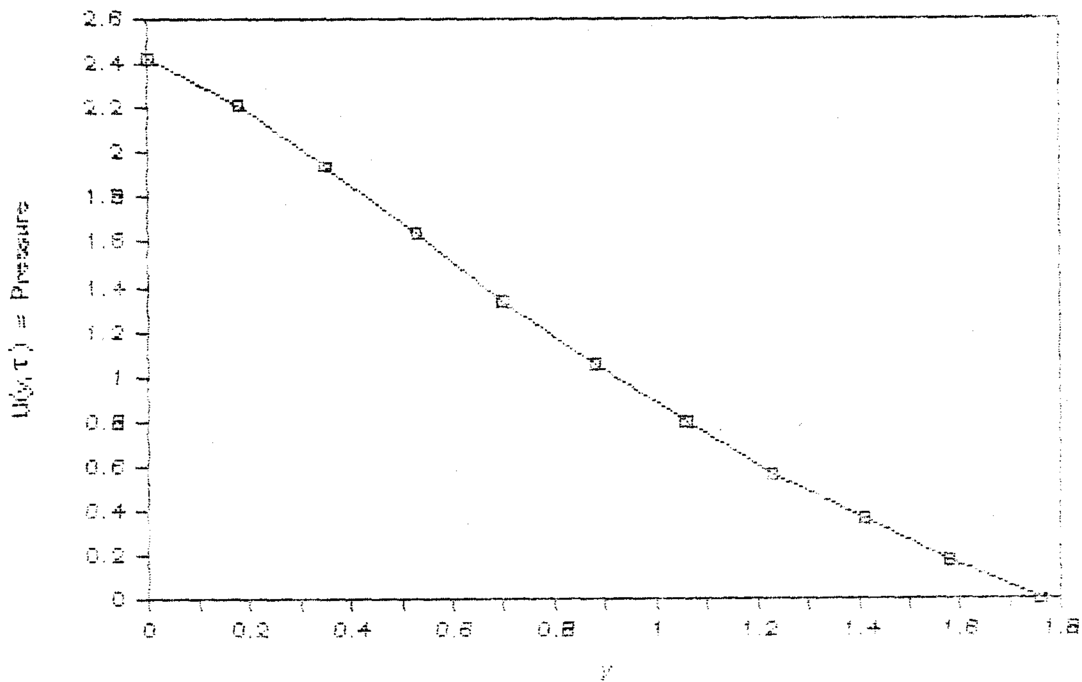

Fig. 14. Pressure when τ = 1.49 for example 2.

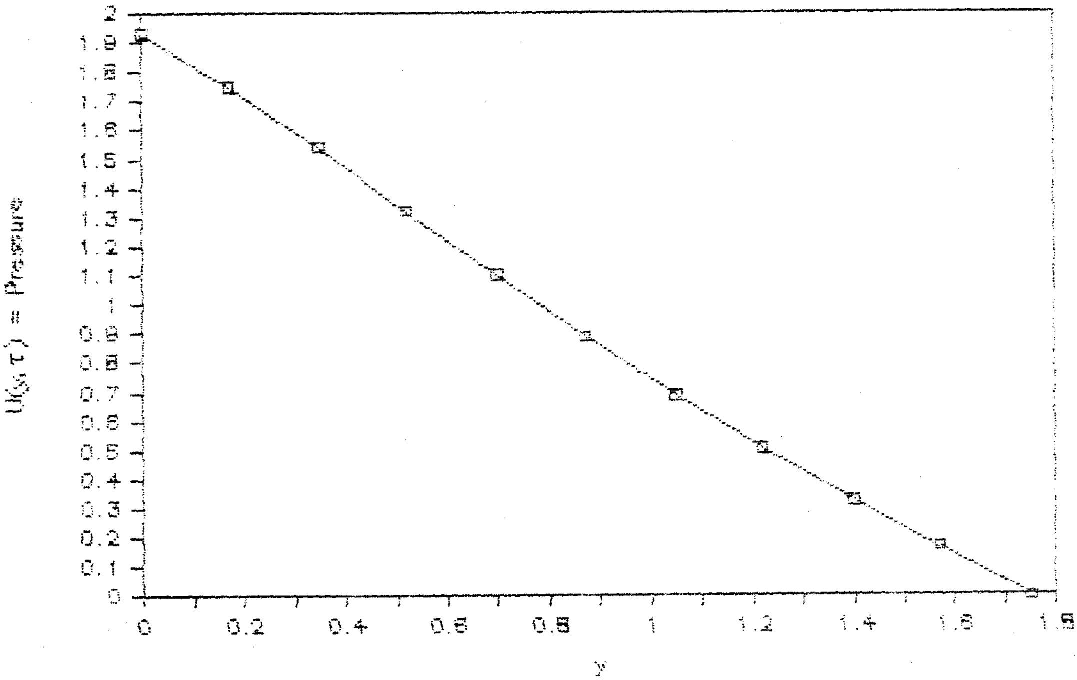

Fig. 15. Pressure when τ = 2.11 for example 2.

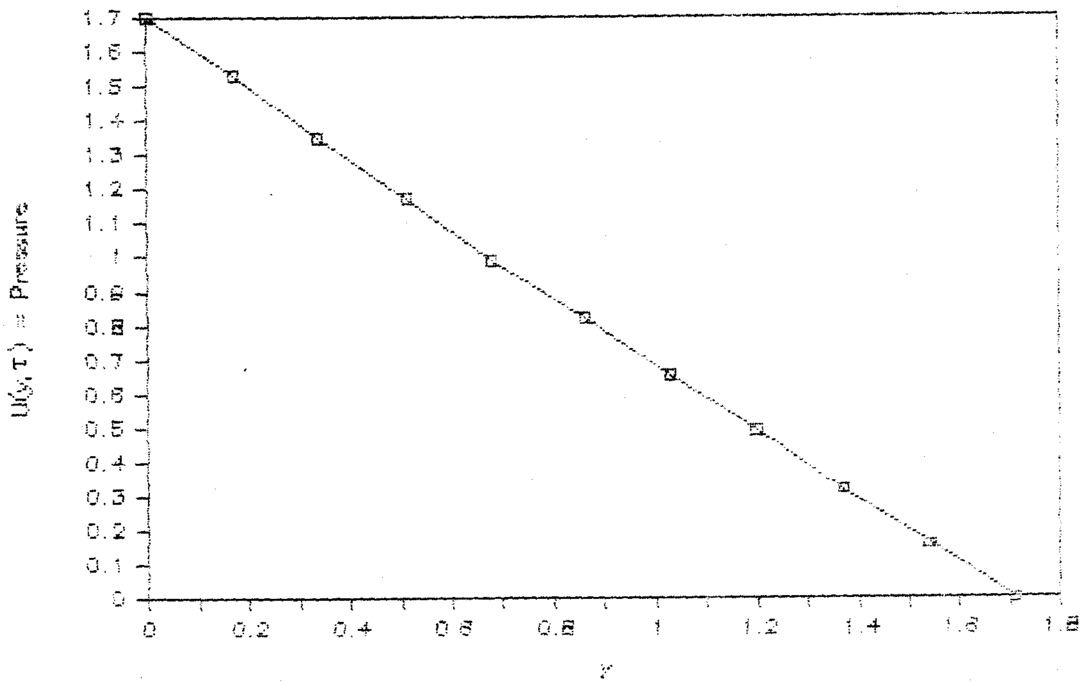

Fig. 16. Pressure when τ = 2.71 for example 2.

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