ON BATEMAN'S METHOD FOR SOLVING LINEAR INTEGRAL EQUATIONS

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ON BATEMAN'S METHOD FOR SOLVING LINEAR INTEGRAL EQUATIONS

§ 1. Introduction. The integral equations which will be considered here are those of Fredholm type and second kind, that is, of the form

$$(1.1) \qquad x(s) - \lambda \int_0^1 K(s,t)\, x(t)\, dt = y(s) \qquad 0 \le s \le 1$$

where $y(s)$ and $K(s,t)$ are given, and $x(s)$ is to be found.[1] If $K(s,t)$ is a bilinear form of 2n functions $\phi_1(s), \dots, \phi_n(s)$; $\Psi_1(t), \dots, \Psi_n(t)$, it is well known that (1.1) is equivalent to an ordinary system of n linear algebraic equations [8, 11].[2] In this case the kernel is said to be of "finite rank" or "degenerate". Ordinarily a kernel is not of finite rank; but one may still hope to take advantage of the reduction to a finite system by replacing $K(s,t)$ by an approximation of finite rank. The essential idea of H. Bateman's process, which will be described in § 2, is to accomplish this systematically, the ϕ's and Ψ's being determined directly from the kernel $K(s,t)$ itself in a very simple way.

When $K(s,t)$ is replaced by, say,

$$K_n(s,t) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \phi_i(s)\, \Psi_j(t),$$

the original equation (1.1) becomes

$$(1.2) \quad x_n(s) - \lambda \int_0^1 K_n(s,t)\, x_n(t)\, dt = y(s), \qquad 0 \le s \le 1.$$

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(1) All functions considered in this paper are real.

(2) Numbers in brackets refer to the list of references at the end of the paper.

Now it is hoped that $x_n(s)$ is approximately the unknown true solution $x(s)$. Shortly after Bateman's paper [3] appeared, F. Tricomi [12] published an error bound. His result was based on the determinantal formulas of Fredholm [6] and was obtained by the use of Hadamard's inequality; but he did not assume that the kernel was of finite rank, much less of Bateman's variety. Consequently his error bound could not be expected to be particularly good for methods of the type discussed here. In fact, although his bound has been reproduced in books of Bückner [4] and Kantorovich and Krylov [9], it does not appear ever to have been applied (at least in print) to a numerical example. Tricomi's ideas seem to be more appropriate to an iterative scheme for determining the Fredholm resolvent for (1.1). An adaptation of his results has been carried out recently by T. L. Glahn [7].

A general bound for the error incurred when $K(s,t)$ is replaced by an approximate kernel of finite rank has been derived by Lonseth [10]. But this bound, although derived outside the framework of Fredholm's formulas, is still not directed toward the Bateman scheme.

In this paper the Bateman idea is analyzed in detail for the physically important case in which $K(s,t)$ is the Green's function for the self-adjoint second order linear differential operator

$$L(u) \equiv (pu')' - qu$$

with homogeneous boundary conditions. An error bound is derived [inequalities (8.1), (8.2)] which is $O(\varepsilon_n/\sqrt{n})$, where ε_n is a measure of the error in the approximation to $K(s,t)$ by $K_n(s,t)$.

A numerical example is described in § 9. Bateman himself illustrated the process in the homogeneous case $y(s) \equiv 0$ by computing five characteristic values $\lambda_1, \lambda_2, \dots, \lambda_5$ for

$$u'' = \lambda u, \qquad u(0) = u(1) = 0.$$

§ 2. The Bateman Process. We shall now discuss Bateman's method in detail.

The resolvent kernel of the integral equation

$$y(s) = x(s) - \lambda \int_0^1 K(s,t)\, x(t)\, dt \tag{2.1}$$

is a function, call it $k(s,t)$, which satisfies

$$x(s) = y(s) + \lambda \int_0^1 k(s,t)\, y(t)\, dt. \tag{2.2}$$

Assume $k(s,t)$ is known, $y(s)$ is continuous, and let $H_n(s,t)$ be defined by

$$\begin{vmatrix} K(s,t) - H_n(s,t) & f_1(s) & \cdots & f_n(s) \\ g_1(t) & A_{11} & \cdots & A_{1n} \\ g_n(t) & A_{n1} & \cdots & A_{nn} \end{vmatrix} = 0, \tag{2.3}$$

where $f_j(s)$ $(j = 1, \dots, n)$ and $g_i(t)$ $(i = 1, \dots, n)$ are any given continuous functions and A_{ij} $(i,j = 1, \dots, n)$ are arbitrary constants. Bateman proved [2] that the resolvent kernel, call it $h_n(s,t)$, associated with $H_n(s,t)$ is given by the determinantal equation

$$(2.4)\qquad \begin{vmatrix} k(s,t) - h_n(s,t) & \phi_1(s) & \cdots & \phi_n(s) \\ \Psi_1(t) & a_{11} & \cdots & a_{1n} \\ \cdots & & \cdots & \\ \Psi_n(t) & a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0,$$

where the ϕ's, Ψ's and a_{ij} $(i,j = 1, \ldots, n)$ are determined by

$$\phi_j(s) = f_j(s) + \lambda\int_0^1 k(s,t)\, f_j(t)\, dt,$$

$$(2.5)\qquad \Psi_i(t) = g_i(t) + \lambda\int_0^1 g_i(s)\, k(s,t)\, ds,$$

$$a_{ij} = A_{ij} + \lambda\int_0^1 g_i(x)\, \phi_j(x)\, dx,$$

and $k(s,t)$ satisfies (2.2). All functions are continuous throughout the intervals $(0 \leq s \leq 1)$, $(0 \leq t \leq 1)$.

If the results above are to be used to solve approximately a given integral equation, e.g.

$$(2.6)\qquad y(s) = x(s) - \lambda\int_0^1 W(s,t)\, x(t)\, dt,$$

then

a) $K(s,t)$ must be defined so that $k(s,t)$ is easily determined and

b) the elements of (2.3) must be defined so that $H_n(s,t)$ approximates $W(s,t)$.

Bateman shows how this may be done in the second of two papers concerned with this topic [3]. First set $K(s,t) \equiv 0$. Clearly $k(s,t) \equiv 0$. To fulfill requirement b) let

$$
\begin{aligned}
f_j(s) &= - W(s,t_j), \\
g_i(t) &= - W(s_i,t), \qquad (2.7)\\
A_{ij} &= - W(s_i,t_j) ,
\end{aligned}
$$

where $s_1, \dots , s_n; t_1, \dots , t_n$ are particular values of s and t belonging to the interval $(o,1)$. Then equation (2.3) becomes

$$
(2.8) \qquad \begin{vmatrix} H_n(s,t) & W(s,t_1) \cdots W(s,t_n) \\ W(s_1,t) & W(s_1,t_1) \cdots W(s_1,t_n) \\ \cdots\cdot & \cdots \\ W(s_n,t) & W(s_n,t_1) \cdots W(s_n,t_n) \end{vmatrix} = 0.
$$

It is necessary to require that $W_n \neq 0$, where W_n is by definition

$$
(2.9) \qquad W_n = \begin{vmatrix} W(s_1,t_1) & \cdots & W(s_1,t_n) \\ . & \cdots & . \\ W(s_n,t_1) & \cdots & W(s_n,t_n) \end{vmatrix} .
$$

or (2.8) would not define $H_n(s,t)$. To show b) is satisfied we set $t = t_j$ and subtract column $(j+1)$ from column (1) in (2.8) to give

$$
(2.10) \qquad \begin{vmatrix} H_n(s,t_j) - W(s,t_j) & W(s,t_1) & \cdots & W(s,t_n) \\ 0 & W(s_1,t_1) & \cdots & W(s_1,t_n) \\ . & . & \cdots & . \\ 0 & W(s_n,t_1) & \cdots & W(s_n,t_n) \end{vmatrix} =
$$

$$
[H_n(s,t_j) - W(s,t_j)] \cdot W_n = 0
$$

Since $W_n \neq 0$, then $[H_n(s,t_j) - W(s,t_j)] = 0$, or

$$H_n(s,t_j) = W(s,t_j) \qquad (j = 1, \ldots, n)(0 \leq s \leq 1).$$

Similarly it may be shown that

$$H_n(s_i,t) = W(s_i,t) \qquad (i = 1, \ldots, n)(0 \leq t \leq 1).$$

Therefore $H_n(s,t)$ equals $W(s,t)$ on a network of lines in the (s,t) plane over the region $(0 \leq s,\ t \leq 1)$; and consequently $h_n(s,t)$, which can be obtained now from (2.4), may be used to obtain an approximate solution $x_n(s)$:

$$(2.11) \qquad x_n(s) = y(s) + \lambda\int_0^1 h_n(s,t)\, y(t)\, dt, \qquad (0 \leq s \leq 1).$$

From (2.08) it is evident that $H_n(s,t)$ may be written

$$(2.12) \qquad H_n(s,t) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}}{W_n}\, W(s,t_j)\, W(s_i,t),$$

where a_{ij} is the cofactor of $W(s_i,t_j)$ in W_n. When $W(s,t)$ is itself "degenerate", it is possible to prove the following lemma stated by Bateman [3].

<u>Lemma I</u>. If $W(s,t)$ is of the form

$$(2.13) \qquad W(s,t) = \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij}\, \phi_j(s)\, \psi_i(t)$$

then $W(s,t) \equiv H_n(s,t)$.

Let $\gamma_i(s) = \sum_{j=1}^{n} \beta_{ij}\phi_j(s)$. Then $W(s,t) = \sum_{i=1}^{n} \gamma_i(s)\Psi_i(t)$.

Now consider

$$W(a,t_1) - H_n(a,t) = \begin{vmatrix} \sum_{i=1}^{n}\gamma_i(a)\psi_i(t) & \sum_{i=1}^{n}\gamma_i(a)\psi_i(t_1) & \cdots & \sum_{i=1}^{n}\gamma_i(a)\psi_i(t_n) \\ \sum_{i=1}^{n}\gamma_i(a_1)\psi_i(t) & \sum_{i=1}^{n}\gamma_i(a_1)\psi_i(t_1) & \cdots & \sum_{i=1}^{n}\gamma_i(a_1)\psi_i(t_n) \\ \cdot & \cdot & \cdots & \cdot \\ \sum_{i=1}^{n}\gamma_i(a_n)\psi_i(t) & \sum_{i=1}^{n}\gamma_i(a_n)\psi_i(t_1) & \cdots & \sum_{i=1}^{n}\gamma_i(a_n)\psi_i(t_n) \end{vmatrix}$$

Factoring the above determinant into the product of two (n+1) order determinants yields

$$W(a,t) - H_n(a,t) = \begin{vmatrix} \gamma_1(a) & \gamma_2(a) & \cdots & \gamma_n(a) & 0 \\ \gamma_1(a_1) & \gamma_2(a_1) & \cdots & \gamma_n(a_1) & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \gamma_1(a_n) & \gamma_2(a_n) & \cdots & \gamma_n(a_n) & 0 \end{vmatrix} \begin{vmatrix} \psi_1(t) & \psi_1(t_1) & \cdots & \psi_1(t_n) \\ \psi_2(t) & \psi_2(t_1) & \cdots & \psi_2(t_n) \\ \cdot & \cdot & \cdots & \cdot \\ \psi_n(t) & \psi_n(t_1) & \cdots & \psi_n(t_n) \\ 0 & 0 & \cdots & 0 \end{vmatrix}$$

$$\equiv 0.$$

Thus $W(s,t) \equiv H_n(s,t)$.

<u>§ 3. Some General Metric Notions.</u> Since $x(s)$ is to be approximated by another function $x_n(s)$, one must devise a scheme to "measure" them (or more frequently, their differences) to estimate how good the

approximation is. For this purpose one ordinarily uses some <u>norm</u> of $x(s)$, denoted by $||x(s)||$, a real number such that: $||x|| \geq 0$, $||ax|| = |a|\ ||x||$ if a is any real number, $||x + y|| \leq ||x|| + ||y||$. The quadratic norm is used in this paper, that is,

$$(3.1) \qquad ||x|| = \left(\int_0^1 x^2(s)\, ds \right)^{1/2}$$

It is easily shown that all the properties above are satisfied by this functional of $x(s)$. Other possibilities would be

$$(3.2) \qquad ||x|| = \max_{0 \leq s \leq 1} |x(s)|$$

$$(3.3) \qquad ||x|| = \left(\int_0^1 |x(s)|^p\, ds \right)^{1/p} \qquad p > 1$$

The last norm includes the first two since, if $p = 2$, then (3.3) becomes (3.1) and if $p \to \infty$, then (3.3) becomes (3.2).

Using operator notation one may write (2.6) as

$$(3.4) \qquad y = (I - W)x = Tx$$

where I stands for the identity transformation. It is possible to relate $||Tx||$ to $||x||$. We shall assume x is an element of a (real) normed linear vector space L, which is defined with these properties:

i) If x_1 and x_2 are in L, then $ax_1 + bx_2$ is also, where a and b are real;

ii) for every x in L, $||x||$ exists.

Moreover, T is a linear operator:

i) It is additive: $T(x_1 + x_2) = Tx_1 + Tx_2$.

ii) If $||x||$ is finite, then $||Tx||$ is also.

iii) If $||x_n - x|| \to 0$ as $n \to \infty$, then $||Tx_n - Tx|| \to 0$.

For this type of operator over L there exist [2, p. 54] non-negative numbers M(T) and m(T) such that

$$(3.5) \quad M(T) = \underset{||x|| \neq 0}{\text{l.u.b.}} \frac{||Tx||}{||x||}, \quad m(T) = \underset{||x|| \neq 0}{\text{g.l.b.}} \frac{||Tx||}{||x||}$$

which satisfy the inequalities

$$(3.6) \qquad m(T)\,||x|| \leq ||Tx|| \leq M(T)\,||x||.$$

If $T_1 + T_2$ and T_1T_2 are defined so that $(T_1 + T_2)x = T_1x + T_2x$ and $(T_1T_2)x = T_1(T_2x)$, then they are bounded and

$$(3.7) \quad M(T_1 + T_2) \leq M(T_1) + M(T_2); \quad M(T_1T_2) \leq M(T_1)\,M(T_2).$$

One must restrict the space L and the transformation W further to guarantee a unique solution x which satisfies (3.4) and the inequality [10, pp. 194, 195]

$$(3.8) \qquad ||x|| \leq ||y||/[1 - M(W)].$$

A normed linear vector space is a Banach space B if it is also complete: if $\{x_n\}$ is an infinite set of vectors in B, and if $||x_m - x_n|| \to 0$ as $m, n \to \infty$, then there exists an x_o in B such that $||x_n - x_o|| \to 0$ as $n \to \infty$. If $M(W) < 1$, $y \in B$ and x is found in B, then (3.8) is true. Moreover it is true if $M(W) < 1$ and W is a completely continuous transformation, i.e., one which carries a bounded set of vectors into a compact one, even though L may not be complete.

§ 4. General Bounds for $||x - x_n||$. With inequalities (3.6), (3.7), and (3.8) it is possible to develop general bounds for $||x - x_n||$ which will be adapted to Bateman's method in § 6 and refined in § 8.

Consider again

$$x - Wx = y \tag{4.1}$$

and the equation

$$x_n - H_n x_n = y. \tag{4.2}$$

By applying (3.8) to (4.1) and (4.2) respectively we have

$$||x|| \leq \frac{||y||}{1 - M(W)} \qquad \text{if} \qquad M(W) < 1, \tag{4.3}$$

and

$$||x_n|| \leq \frac{||y||}{1 - M(H_n)} \qquad \text{if} \qquad M(H_n) < 1. \tag{4.4}$$

Subtracting (4.2) from (4.1) we find that

$$x - x_n - Wx + H_n x_n = 0, \tag{4.5}$$

which may be written either as

$$(x - x_n) - W(x - x_n) = (W - H_n)x_n \tag{4.6}$$

or as

$$(x - x_n) - H(x - x_n) = (W - H_n)\,x. \tag{4.7}$$

Then if both (3.8) and (3.6) are applied to this pair of equations, we obtain

$$(4.8)\qquad ||x - x_n|| \leq \frac{M(W - H_n)\ ||x_n||}{1 - M(W)}\ ,\ \text{if}\ \ M(W) < 1,$$

and

$$(4.9)\qquad ||x - x_n|| \leq \frac{M(W - H_n)\ ||x||}{1 - M(H_n)}\ ,\ \text{if}\ \ M(H_n) < 1.$$

Now applying (4.3) to (4.9) yields

$$(4.10)\qquad ||x - x_n|| \leq \frac{M(W - H_n)\ ||y||}{[1 - M(H_n)][1 - M(W)]}\ ,\ \text{if}\ \ M(W),\ M(H_n) < 1.$$

If we write

$$W = W - H_n + H_n$$

then because of (3.7)

$$(4.11)\qquad M(W) \leq M(W - H_n) + M(H_n)$$

which, when substituted into (4.8), gives

$$(4.12)\qquad ||x - x_n|| \leq \frac{M(W - H_n)\ ||x_n||}{1 - M(W - H_n) - M(H_n)}\ ,\ \text{if}\ M(W - H_n) + M(H_n) < 1.$$

If (4.9) is altered similarly we find that

$$(4.13)\qquad ||x - x_n|| \leq \frac{M(W - H_n)\ ||x||}{1 - M(W - H_n) - M(W)}\ ,\ \text{if}\ \ M(W - H_n) + M(W) < 1.$$

From (4.10) we obtain

$$(4.14)\qquad ||x - x_n|| \leq \frac{M(W - H_n)\ ||y||}{[1 - M(H_n)][1 - M(W - H_n) - M(H_n)]}\ ,$$

$$\text{if}\ \ M(W - H_n) + M(H_n) < 1$$

if we use (4.11); and

$$(4.15)\qquad ||x - x_n|| \leq \frac{M(W - H_n)\ ||y||}{[1 - M(W)][1 - M(W - H_n) - M(W)]} ,$$

$$\text{if}\quad M(W - H_n) + M(W) < 1$$

by altering $M(H)$ in the denominator.

Each of these bounds is suitable under different circumstances. In this paper (4.8) and (4.15) will be used.

§ 5. The Error in Bateman's Approximation. If the quadratic norm (3.1) is used, and if kernel $K(s,t)$ its square integrable on $0 \leq s,\ t \leq 1$, the integral operator

$$Kx = \int_0^1 K(s,t)\ x(t)\ dt$$

is bounded, and

$$(5.1)\qquad M(K) \leq \left[\int_0^1 \int_0^1 K^2(s,t)\ ds\ dt\right]^{1/2} = N(K) .$$

This follows directly from the definitions by use of the Schwarz inequality. Hence

$$(5.2)\qquad M(W - H_n) \leq N(W - H_n) .$$

Now write

$$(5.3)\qquad W_n^{(i,j)}(s,t) = \begin{vmatrix} W(s,t) & W(s,t_1) & \cdots & W(s,t_n) \\ W(s_1,t) & W(s_1,t_1) & \cdots & W(s_1,t_n) \\ \cdot & \cdot & \cdots & \cdot \\ W(s_n,t) & W(s_n,t_1) & \cdots & W(s_n,t_n) \end{vmatrix}$$

where $s_i \le s \le s_{i+1}$; $t_j \le t \le t_{j+1}$. Then referring to (2.8) and (2.9) we find that

$$(5.4) \quad N(W - H_n) = \left[\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{s_i}^{s_{i+1}} [W(s,t) - H_n(s,t)]^2 \, ds \, ds \right]^{\frac{1}{2}}$$

$$= \left[\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{s_i}^{s_{i+1}} \left[\frac{W_n^{i,j}(s,t)}{W_n} \right]^2 ds \, dt \right]^{\frac{1}{2}}$$

Subtracting row (i+1) from the first row and column (j+1) from the first column of $W_n^{(i,j)}$ yields

$$W_n^{(i,j)} =$$

$$(5.5) \quad \begin{vmatrix} W(s,t) - W(s_i,t) - [W(s,t_j) - W(s_i,t_j)] & W(s,t_1) - W(s_i,t_1) & \cdots & W(s,t_n) - W(s_i,t_n) \\ W(s_1,t) - W(s_1,t_j) & W(s_1,t_1) & \cdots & W(s_1,t_n) \\ \cdot & \cdot & \cdots & \cdot \\ W(s_n,t) - W(s_n,t_j) & W(s_n,t_1) & \cdots & W(s_n,t_n) \end{vmatrix}$$

If $W_n^{(i,j)}(s,t)$ is expanded in terms of elements of the first row and column, we have

$$(5.6) \quad \frac{W_n^{(i,j)}(s,t)}{W_n} = [\{ W(s,t) - W(s_i,t) \} - \{ W(s,t_j) - W(s_i,t_j) \}]$$

$$+ \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{C_{kl}}{W_n} [W(s,t_l) - W(s_i,t_l)][W(s_k,t) - W(s_k,t_j)]$$

where C_{kl} is the negative of the cofactor of $W(s_k,t_l)$ in W_n. If $W(s,t)$ is continuous on the closed unit square, for any $\varepsilon > 0$ there exists a positive integer $K(\varepsilon)$ such that

$$|W(s,t) - W(s_i,t_j)| < \varepsilon \quad \text{for} \quad n > K(\varepsilon), \quad \begin{Bmatrix} s_i \leq s \leq s_{i+1} \\ t_j \leq t \leq t_{j+1} \end{Bmatrix} ;$$

and then

$$(5.7) \qquad \frac{W_n^{(i,j)}(s,t)}{W_n} \leq 2\varepsilon + \varepsilon^2 \sum_{k=1}^{n} \sum_{l=1}^{n} \left| \frac{C_{kl}}{W_n} \right| .$$

Applying this result to (5.4) gives

$$(5.8) \quad N(W - H_n) \leq \left[\sum_{i=o}^{n-1} \sum_{j=o}^{n-1} \int_{t_j}^{t_{j+1}} \int_{s_i}^{s_{i+1}} \left[2\varepsilon + \varepsilon^2 \sum_{k=1} \sum_{l=1} \left| \frac{C_{kl}}{W_n} \right| \right]^2 ds dt \right]^{\frac{1}{2}}$$

$$= \left[2\varepsilon + \varepsilon^2 \sum_{k=1}^{n} \sum_{l=1}^{n} \left| \frac{C_{kl}}{W_n} \right| \right] \left[\sum_{i=o}^{n-1} \sum_{j=o}^{n-1} \int_{t_j}^{t_{j+1}} \int_{s_i}^{s_{i+1}} ds\, dt \right]^{\frac{1}{2}}$$

$$= 2\varepsilon + \varepsilon^2 \sum_{k=1}^{n} \sum_{l=1}^{n} \left| \frac{C_{kl}}{W_n} \right| .$$

Since $N(W - H_n) \geq M(W - H_n)$ one may apply (5.8) to any of the results of § 4 to obtain a bound for $||x - x_n||$. If (4.8) is selected, we have

$$(5.9) \quad ||x - x_n|| \leq \left[2\varepsilon + \varepsilon^2 \sum_{k=1}^{n} \sum_{l=1}^{n} \left| \frac{C_{kl}}{W_n} \right| \right] \left[\frac{||x_n||}{1 - M(W)} \right]$$

whereas (4.15) would give

$$(5.10) \quad ||x - x_n|| \leq \left[2\varepsilon + \varepsilon^2 \sum_{k=1}^{n} \sum_{l=1}^{n} \left| \frac{C_{kl}}{W_n} \right| \right] \left[\frac{||y||}{[1 - M(W)][1 - M(W - H_n) - M(W)]} \right]$$

§ 6. Green's Functions for Second Order Differential Boundary Problems.

Here we include a sketch of the "Green's function" approach to the solution of simple boundary value problems. Further details and proofs may be found in [5] and elsewhere.

We consider first a homogeneous self-adjoint ordinary differential equation of second order

(6.1) $$L(u) \equiv (pu')' - qu = 0$$

for $u(s)$ in the interval $(0 \leq s \leq 1)$, where p, p', q are continuous functions of s and $p(s) > 0$. Then the associated nonhomogeneous expression is

(6.2) $$L(u) = -y(s)$$

where $y(s)$ is piecewise continuous. The problem is to find a solution in the given interval which will satisfy prescribed homogeneous boundary conditions, e.g., $u = 0$. With this in mind we define the Green's function, denoted by $W(s,t)$, for the differential equation (6.1): $W(s,t)$ is, for fixed t, a continuous function of s which satisfies the boundary conditions. Except at the point $s = t$, the first and second derivatives with respect to s are continuous in the interval $(0 \leq s \leq 1)$. At the point $s = t$ the first derivative has a jump discontinuity given by

(6.3) $$\left.\frac{dW(s,t)}{ds}\right|_{s=t-o}^{s=t+o} = -\frac{1}{p(t)} .$$

Moreover, considered as a function of s, $W(s,t)$ satisfies the differential equation $L(W) = 0$ in the given interval except at the point $s = t$.

The following theorem is basic.

<u>Theorem I</u>: <u>If</u> $y(s)$ <u>is a continuous or piecewise continuous function of</u> s, <u>then the function</u>

$$(6.4) \qquad u(s) = \int_0^1 W(s,t)\, y(t)\, dt$$

<u>is a solution of the differential equation</u>

$$(6.5) \qquad L(u) = -\, y(s)$$

<u>and satisfies the boundary conditions</u>. <u>Conversely</u>, <u>if the function</u> $y(s)$ <u>satisfies</u> (6.5) <u>it can be represented by</u> (6.4).

It is also possible to establish that the Green's function of a self-adjoint differential operator is a symmetric function of the parameter t and the argument s. That is,

$$W(s,t) = W(t,s).$$

We now turn to the construction of the Green's function from the differential operator $L(u)$. Consider any solution u_0 of $L(u) = 0$ which satisfies the homogeneous boundary conditions at $s = 0$. Then generally $c_0\, u_0$ is a solution of the same nature for any constant c_0. Similarly, if u_1 satisfies $L(u) = 0$ and the boundary conditions at $s = 1$, then $c_1\, u_1$ is the most general such solution. We will assume that u_0 and u_1 are linearly independent. Then

$$(6.6) \qquad u_0 u_1' - u_0' u_1 = 1/p.$$

Now if we let

$$(6.7) \qquad W(s,t) = \begin{cases} -u_o(s)\,u_1(t), & 0 \le s \le t \le 1; \\ -u_o(t)\,u_1(s), & 0 \le t \le s \le 1; \end{cases}$$

the defining properties of a Green's function are satisfied.

An important relation between integral and differential equations holds. Consider a linear family of differential equations

$$(6.8) \qquad L(u) + \lambda u = \Psi(s)$$

depending on the parameter λ ; here $\Psi(s)$ is piecewise continuous and u satisfies the boundary conditions. If the Green's function for $L(u)$ exists, then by setting

$$\phi(s) = \Psi(s) - \lambda u(s)$$

we get

$$(6.9) \qquad u(s) = \lambda \int_o^1 W(s,t)\,u(t)\,dt + g(s),$$

where

$$g(s) = -\int_o^1 W(s,t)\,\Psi(t)\,dt$$

is a known function.

§ 7. A Bound for $M(W - H_n)$ When W is a Green's Function. One notes at least two disadvantages in using $N(W - H_n)$ as a bound for $M(W - H_n)$. One is that it is not apparent that $N(W - H_n) \to 0$ as $n \to \infty$; for, although each individual $|W(s,t) - W(s_i,t_j)|$ decreases in the $(i,j)^{th}$

subrectangle as n becomes large, it is quite possible that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \left| \frac{C_{kl}}{W_n} \right|$$

will increase. Even at best the calculation of such a quantity is prohibitive when n is larger than five or six and only a desk calculation is available. In this section these difficulties will be resolved by restricting $W(s,t)$ to the Green's functions discussed in § 6. To fix the ideas, we assume such boundary conditions that $W(s,t) = 0$ on the perimeter of the unit square.

Let a mesh in the (s,t) plane be constructed such that $0 = s_0 < s_1 < s_2 \cdots < s_n < s_{n+1} = 1$; $0 = t_0 < t_1 < t_2 \cdots < t_n < t_{n+1} = 1$, where $s_i = t_i$ $(i = 0, \ldots, n+1)$. Further let (2.8) be evaluated at n^2 points (s_i, t_j) $(i, j = 1, \ldots, n)$. If (2.8) were evaluated at the points (s_i, t_j) $(i, j = 0, n+1)$ then $W_n = 0$ and H_n would not be defined. Let

$$(7.1) \qquad \begin{aligned} a(s) &= -u_0(s) \\ a_i &= -u_0(s_i) \\ b(t) &= u_1(t) \\ b_j &= u_1(t_j) \,. \end{aligned}$$

Then

$$(7.2) \qquad W_n = \begin{vmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_1b_2 & a_2b_2 & \cdots & a_2b_n \\ \cdot & \cdot & \cdots & \cdot \\ a_1b_n & a_2b_n & & a_nb_n \end{vmatrix}$$

<u>Theorem II</u>: $W_n = a_1 b_n \prod_{i=1}^{n-1} [a_{i+1} b_i - a_i b_{i+1}]$

The proof is by induction. For $n = 2$

$$W_2 = \begin{vmatrix} a_1 b_1 & a_1 b_2 \\ a_1 b_2 & a_2 b_2 \end{vmatrix} = a_1 b_2 [a_2 b_1 - a_1 b_2] .$$

Assuming

$$W_k = a_1 b_k \prod_{i=1}^{k-1} [a_{i+1} b_i - a_i b_{i+1}] \tag{7.3}$$

is true, we must prove

$$W_{k+1} = a_1 b_{k+1} \prod_{i=1}^{k} [a_{i+1} b_i - a_i b_{i+1}] . \tag{7.4}$$

By definition

$$W_{k+1} = \begin{vmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_k & a_1 b_{k+1} \\ a_1 b_2 & a_2 b_2 & \cdots & a_2 b_k & a_2 b_{k+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1 b_k & a_2 b_k & \cdots & a_k b_k & a_k b_{k+1} \\ a_1 b_{k+1} & a_2 b_{k+1} & \cdots & a_k b_{k+1} & a_{k+1} b_{k+1} \end{vmatrix} \tag{7.5}$$

It has been required previously that $W_k \neq 0$; therefore $b_k \neq 0$. Hence by multiplying the k-th column by b_{k+1}/b_k and subtracting from (k+1)st column we obtain

$$(7.6) \quad W_{k+1} = \begin{vmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_k & 0 \\ a_1b_2 & a_2b_2 & \cdots & a_2b_k & 0 \\ \cdot & \cdot & \cdots & \cdot & \\ a_1b_k & a_2b_k & \cdots & a_kb_k & 0 \\ a_1b_{k+1} & a_2b_{k+1} & \cdots & a_kb_{k+1} & a_{k+1}b_{k+1} - \dfrac{a_kb_{k+1}^2}{b_k} \end{vmatrix}$$

$$W_{k+1} = \frac{b_{k+1}}{b_k}\,[a_{k+1}b_k - a_kb_{k+1}]\,W_k = a_1b_{k+1}\prod_{i=1}^{k}[a_{i+1}b_i - a_ib_{i+1}]$$

and the theorem is proved.

Theorem III may be stated as simply as Theorem II.

<u>Theorem III</u>: <u>If</u> $i \neq j$, <u>then</u> $W_n^{(i,j)} = 0$.

If $i < j$, with the new notation

$$(7.7) \quad W_n^{(i,j)}(a,t) =$$

$$\begin{vmatrix} a(a)b(t) & a_1b(a) & \cdots & a_ib(a) & a(a)b_{i+1} & \cdots & a(a)b_j & a(a)b_{j+1} & \cdots & a(a)b_{n-1} & a(a)b_n \\ a_1b(t) & a_1b_1 & \cdots & a_1b_i & a_1b_{i+1} & \cdots & a_1b_j & a_1b_{j+1} & \cdots & a_1b_{n-1} & a_1b_n \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_ib(t) & a_1b_i & \cdots & a_ib_i & a_ib_{i+1} & \cdots & a_ib_j & a_ib_{j+1} & \cdots & a_ib_{n-1} & a_ib_n \\ a_{i+1}b(t) & a_1b_{i+1} & \cdots & a_ib_{i+1} & a_{i+1}b_{i+1} & \cdots & a_{i+1}b_j & a_{i+1}b_{j+1} & \cdots & a_{i+1}b_{n-1} & a_{i+1}b_n \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_jb(t) & a_1b_j & \cdots & a_ib_j & a_{i+1}b_j & \cdots & a_jb_j & a_jb_{j+1} & \cdots & a_jb_{n-1} & a_jb_n \\ a(t)b_{j+1} & a_1b_{j+1} & \cdots & a_ib_{j+1} & a_{i+1}b_{j+1} & \cdots & a_jb_{j+1} & a_{j+1}b_{j+1} & \cdots & a_{j+1}b_{n-1} & a_{j+1}b_n \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ a(t)b_{n-1} & a_1b_{n-1} & \cdots & a_ib_{n-1} & a_{i+1}b_{n-1} & \cdots & a_jb_{n-1} & a_{j+1}b_{n-1} & \cdots & a_{n-1}b_{n-1} & a_{n-1}b_{n-1} \\ a(t)b_n & a_1b_n & \cdots & a_ib_n & a_{i+1}b_n & \cdots & a_jb_n & a_{j+1}b_n & & a_{n-1}b_n & a_nb_n \end{vmatrix}$$

By multiplying column n by $[b_n/b_{n-1}]$ and subtracting it from the last column (column $n+1$) we reduce every element to zero except the bottom one, which is $[b_n/b_{n-1}][a_n b_{n-1} - a_{n-1}b_n]$.

Then

$$(7.8) \qquad W_n^{(i,j)} = [b_n/b_{n-1}][a_n b_{n-1} - a_{n-1}b_n]\, W_{n-1}^{(i,j)}(s,t)\ .$$

Continuing in the same manner we have

$$(7.9) \quad W_n^{(i,j)}(s,t) = [b_n/b_{n-2}][a_n b_{n-1} - a_{n-1}b_n]$$

$$[a_{n-1}b_{n-2} - a_{n-2}b_{n-1}]\, W_{n-2}^{(i,j)}(s,t)$$

$$= [b_n/b_{i+1}] \prod_{k=i+1}^{n-1} [a_{k+1}b_k - a_k b_{k+1}]\, W_{i+1}^{(i,j)}(s,t)\ .$$

However,

$$(7.10) \quad W_{i+1}^{(i,j)} = \begin{vmatrix} a(s)b(t) & a_1b(s) & \cdots & a_ib(s) & a(s)b_{i+1} \\ a_1b(t) & a_1b_1 & \cdots & a_1b_i & a_1b_{i+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_ib(t) & a_1b_i & \cdots & a_ib_i & a_ib_{i+1} \\ a_{i+1}b(t) & a_1b_{i+1} & \cdots & a_ib_{i+1} & a_{i+1}b_{i+1} \end{vmatrix}$$

and since the first and last rows are linearly dependent, $W_{i+1}^{(i,j)} = 0$; consequently $W_n^{(i,j)}(s,t) = 0$. A similar proof is possible if $i > j$.

With theorems II and III it is possible to simplify $N(W - H_n)$;

$$(7.11)\quad N(W - H_n) = \left(\sum_{i=o}^{n}\sum_{j=o}^{n}\int_{t_j}^{t_{j+1}}\int_{s_i}^{s_{i+1}} [W(s,t)-H_n(s,t)]^2\, ds\, dt\right)^{\frac{1}{2}}$$

$$= \left(\sum_{i\neq j}^{n}\int_{t_j}^{t_{j+1}}\int_{s_i}^{s_{i+1}}\left(\frac{W_n^{(i,j)}(s,t)}{W_n}\right)^2 ds\, dt\right.$$

$$\left.+\sum_{i=o}^{n}\int_{t_i}^{t_{i+1}}\int_{s_i}^{s_{i+1}}\left(\frac{W_n^{(i,i)}(s,t)}{W_n}\right)^2 ds\, dt\right)^{\frac{1}{2}}$$

If $i \neq j$, then $W_n^{(i,j)}(s,t) = 0$, and

$$(7.12)\quad N(W - H_n) = \left(\sum_{i=o}^{n}\int_{t_i}^{t_{i+1}}\int_{s_i}^{s_{i+1}}\left(\frac{W_n^{(i,i)}(s,t)}{W_n}\right)^2 ds\, dt\right)^{\frac{1}{2}}$$

$$= \left(\sum_{i=o}^{n}\left[\int_{t_i}^{t_{i+1}}\int_{s_i}^{t}\left(\frac{W_n^{(i,i)}(s,t)}{W_n}\right)^2 ds\, dt\right.\right.$$

$$\left.\left.+\int_{t_i}^{t_{i+1}}\int_{t}^{s_{i+1}}\left(\frac{W_n^{(i,i)}(s,t)}{W_n}\right)^2 ds\, dt\right]\right)^{\frac{1}{2}}$$

With this arrangement it is possible to simplify $\dfrac{W_n^{(i,i)}(s,t)}{W_n}$.

Consider first $i = 0$ and $0 \leq s \leq t \leq s_1$. Then

$$(7.13)\quad \frac{W_n^{(o,o)}(s,t)}{W_n} = \frac{1}{W_n}\begin{vmatrix} a(s)b(t) & a(s)b_1 & a(s)b_2 & \cdots & a(s)b_n \\ a(t)b_1 & a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a(t)b_2 & a_1b_2 & a_2b_2 & \cdots & a_2b_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a(t)b_n & a_1b_n & a_2b_n & \cdots & a_nb_n \end{vmatrix}$$

By multiplying the second row by $a(s)/a_1$ and subtracting from the first we obtain

$$(7.14)\quad \frac{W_n^{(o,o)}(s,t)}{W_n} = \frac{1}{W_n}\begin{vmatrix} \frac{a(s)}{a_1}[a_1b(t)-a(t)b_1] & 0 & 0 & \cdots & 0 \\ a(t)b_1 & a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a(t)b_2 & a_1b_2 & a_2b_2 & \cdots & a_2b_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a(t)b_n & a_1b_n & a_2b_n & \cdots & a_nb_n \end{vmatrix}$$

$$= \frac{a(s)}{a_1}[a_1b(t) - a(t)b_1] .$$

Similarly for $0 \leq t \leq s \leq s_1$ it may be shown that

$$(7.15)\quad \frac{W_n^{(o,o)}(s,t)}{W_n} = \frac{a(t)}{a_1}[a_1b(s) - a(s)b_1] .$$

If $i = n$ and $s_n \leq s \leq t \leq 1$, then

$$(7.16)\quad \frac{W_n^{(n,n)}(s,t)}{W_n} = \frac{b(t)}{b_n}[a(s)b_n - a_nb(s)] ;$$

while if $i = n$ and $s_n \leq t \leq s \leq 1$, then

$$(7.17)\quad \frac{W_n^{(n,n)}(s,t)}{W_n} = \frac{b(s)}{b_n}[a(t)b_n - a_nb(t)] .$$

The other values of i may be treated collectively by considering (7.7) with $j = i$. We may, as before, reduce $W_n^{(i,i)}(s,t)$

$$(7.18)\quad W_n^{(i,i)}(s,t) = \frac{b_n}{b_{i+1}} \prod_{k=i+1}^{n-1} [a_{k+1}b_k - a_kb_{k+1}]\, W_{i+1}^{(i,i)}(s,t) ,$$

where $s < t$; but by the same steps W_n can be factored so that

(7.19) $$W_n = \frac{b_n}{b_{i+1}} \prod_{k=i+1}^{n-1} [a_{k+1}b_k - a_k b_{k+1}]\, W_{i+1} \,.$$

Then

(7.20) $$\frac{W_n^{(i,i)}(s,t)}{W_n} = \frac{W_{i+1}^{(i,i)}(s,t)}{W_{i+1}} \,.$$

According to theorem II we may write

(7.21) $$W_{i+1} = a_1 b_{i+1} \prod_{k=1}^{i} [a_{k+1}b_k - a_k b_{k+1}] \,.$$

However, it is also possible to simplify $W_{i+1}^{(i,i)}(s,t)$. We have

(7.22) $$W_{i+1}^{(i,i)}(s,t) = \begin{vmatrix} a(s)b(t) & a_1 b(s) & \cdots & a_i b(s) & a(s)b_{i+1} \\ a_1 b(t) & a_1 b_1 & \cdots & a_1 b_i & a_1 b_{i+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_i b(t) & a_1 b_i & \cdots & a_i b_i & a_i b_{i+1} \\ a(t)b_{i+1} & a_1 b_{i+1} & \cdots & a_i b_{i+1} & a_{i+1} b_{i+1} \end{vmatrix}$$

Multiplying the last column by $b(t)/b_{i+1}$ and subtracting the product from the first we have

(7.23) $$W_{i+1}^{(i,i)}(s,t) = \begin{vmatrix} 0 & a_1 b(s) & \cdots & a_i b(s) & a(s)b_{i+1} \\ 0 & a_1 b_1 & \cdots & a_i b_i & a_1 b_{i+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & a_1 b_i & \cdots & a_i b_i & a_i b_{i+1} \\ a(t)b_{i+1} - a_{i+1}b(t) & a_1 b_{i+1} & & a_i b_{i+1} & a_{i+1} b_{i+1} \end{vmatrix}$$

Similarly multiplying the next to last row by $b(s)/b_i$ and subtracting the product from the first yields

$$(7.24)\quad W_{i+1}^{(i,i)}(s,t) =$$

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & \frac{b_{i+1}}{b_i}[a(s)b_i - a_i b(s)] \\ 0 & a_1 b_1 & \cdots & a_1 b_i & a_1 b_{i+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & a_i b_1 & \cdots & a_i b_i & a_i b_{i+1} \\ [a(t)b_{i+1} - a_{i+1}b(t)] & a_1 b_{i+1} & \cdots & a_i b_{i+1} & a_{i+1}b_{i+1} \end{vmatrix}$$

so

$$(7.25)\quad W_{i+1}^{(i,i)}(s,t) = \frac{b_{i+1}}{b_i}[a(s)b_i - a_i b(s)][a(t)b_{i+1} - a_{i+1}b(t)]\, W_i$$

from Laplace's expansion. Then, if $s \leq t$, remembering (7.20), (7.21), and (7.25) we have

$$(7.26)\quad \frac{W_n^{(i,i)}(s,t)}{W_n} = \frac{\frac{b_{i+1}}{b_i}[a(s)b_i - a_i b(s)][a(t)b_{i+1} - a_{i+1}b(t)]\, W_i}{W_{i+1}}$$

$$= \frac{\frac{b_{i+1}}{b_i}[a(s)b_i - a_i b(s)][a(t)b_{i+1} - a_{i+1}b(t)]a_1 b_i \prod_{k=1}^{i-1}[a_{k+1}b_k - a_k b_{k+1}]}{a_1 b_{i+1} \prod_{k=1}^{i}[a_{k+1}b_k - a_k b_{k+1}]}$$

$$= \frac{[a(s)b_i - a_i b(s)]\,[a(t)b_{i+1} - a_{i+1}b(t)]}{[a_{i+1}b_i - a_i b_{i+1}]} \quad .$$

If $s > t$ by a similar proof we find that

$$(7.27)\quad \frac{W_n^{(i,i)}(s,t)}{W_n} = \frac{[a(t)b_i - a_i b(t)][a(s)b_{i+1} - a_{i+1}b(s)]}{[a_{i+1}b_i - a_i b_{i+1}]}$$

Moreover, it is evident from the above that $W_n^{(i,i)}(s,t)$ for $s < t$ is equal to $W_n^{(i,i)}(t,s)$, $t < s$.

Summarizing we state

__Theorem IV: If W is a Green's function belonging to the self-adjoint differential operator__

$$L(u) = (pu')' - qu,$$

__and H_n is the associated kernel derived from Bateman's formula, then__

$$(7.28)\quad M(W - H_n) \leq N(W - H_n)$$

$$= \left(\sum_{i=o}^{n} \left[\int_{t_i}^{t_{i+1}} \int_{s_i}^{t} [C_n^{(i)}(s,t)]^2 \, ds \, dt \right.\right.$$

$$\left.\left. + \int_{t_i}^{t_{i+1}} \int_{t}^{s_{i+1}} [C_n^{i}(t,s)]^2 \, ds \, dt \right] \right)^{\frac{1}{2}}$$

where

$$C_n^{(o)}(s,t) = \frac{a(s)}{a_1} [a_1 b(t) - a(t)b_1] ,$$

$$C_n^{(n)}(s,t) = \frac{b(t)}{b_n} [a(s)b_n - a_n b(s)] ,$$

$$C_n^{(i)}(s,t) = \frac{[a(s)b_i - a_i b(s)][a(t)b_{i+1} - a_{i+1}b(t)]}{[a_{i+1}b_i - a_i b_{i+1}]}$$

$$i = 1, \ldots, n-1,$$

The bounds for $||x - x_n||$ will be stated in terms of this simple expression for $N(W - H_n)$; however, using (7.28) one may obtain a grosser bound for $N(W - H_n)$ and at the same time show that $N(W - H_n) \to 0$ as $n \to \infty$.

Since $a(s)$ and $b(t)$ are continuous, as the mesh is made smaller with (s,t) in the $(0,s_1)$ rectangle, $\left|\frac{a(s)}{a_1}\right| \to 1$; $[a_1 b(t) - a(t)b_1 \to 0$; and therefore

$$C_n^{(o)}(s,t) = \frac{a(s)}{a_1}[a_1 b(t) - a(t)b_1] \to 0 .$$

By similar reasoning, $|C_n^{(n)}(s,t)| \to 0$. Moreover in the i-th interior rectangle

$$\left|\frac{a(s)b_i - a_i b(s)}{a_{i+1}b_i - a_i b_{i+1}}\right| \to 1, \quad |a(t)b_{i+1} - a_{i+1}b(t)| \to 0 ,$$

so that $|C_n^{(i)}(s,t)| \to 0$, $i = 1, \ldots, n-1$. Let

$$\underset{i}{\text{Max}}|C_n^{(i)}(s,t)| = \epsilon_n, \; i = 0, \ldots, n-1 \; (s_i \le s,t \le s_{i+1})$$

Then

$$(7.29) \; N(W - H_n) \le \left(\sum_{i=o}^{n}\left[\int_{t_i}^{t_{i+1}}\int_{s_i}^{t} \epsilon_n^2 \, ds \, dt + \int_{t_i}^{t_{i+1}}\int_{t}^{s_{i+1}} \epsilon_n^2 \, ds dt\right]\right)^{\frac{1}{2}}$$

$$\le \left(\sum_{i=o}^{n}\left[\int_{t_i}^{t_{i+1}}\int_{s_i}^{t} ds \, dt + \int_{t_i}^{t_{i+1}}\int_{t}^{s_{i+1}} ds \, dt\right]\right)^{\frac{1}{2}}$$

$$\leq \varepsilon_n \left(\sum_{i=o}^{n} \int_{t_i}^{t_{i+1}} \int_{s_i}^{s_{i+1}} ds\, dt \right)^{\frac{1}{2}}$$

$$\leq \varepsilon_n \left(\sum_{i=o}^{n} (t_{i+1} - t_i)(s_{i+1} - s_i) \right)^{\frac{1}{2}}$$

Now provided equal subdivisions have been made, we have

$$(7.30) \qquad N(W - H_n) \leq \varepsilon_n \left(\sum_{i=o}^{n} \left[\frac{1}{n+1} \right]^2 \right)^{\frac{1}{2}}$$

$$= \varepsilon_n / \sqrt{n+1}$$

Clearly $N(W - H_n) \to 0$ as $n \to \infty$.

__§ 8. The Final Bound for $||x - x_n||$.__ It is evident that (7.28) is a marked improvement over (5.8). Calculations which would have been prohibitive with large n have now been made feasible with the later formula. Using (7.28), (4.8) and (4.15) we shall now state the final result.

__Theorem IV: Suppose in the Fredholm equation of second kind__

$$(i) \qquad x(s) - \lambda \int_o^1 W(s,t)\, x(t)\, dt = y(s)$$

$x(s)$ __is unknown,__ $y(s)$ __given, and__ $W(s,t)$ __a specified Green's function associated with the differential operator__

$$L(u) = (pu')' - qu.$$

Then if (i) is solved approximately by solving

$$\text{(ii)} \qquad x_n(s) - \lambda \int_0^1 H_n(s,t)\, x_n(t)\, dt = y(s),$$

where $H_n(s,t)$ is the kernel derived from Bateman's formulas, with the resolving kernel $h_n(s,t)$, also given by Bateman, then the error involved may be bounded either by

$$\text{(8.1)} \qquad ||x - x_n|| \leq N(W - H_n) \left[\frac{||x||}{1 - M(W)}\right], \quad \text{if } M(W) < 1,$$

where

$$N(W - H_n) = \left(\sum_{i=0}^{n} \left[\int_{t_i}^{t_{i+1}} \left\{ \int_{s_i}^{t} [C_n^{(i)}(s,t)]^2\, ds + \int_{t}^{s_{i+1}} [C_n^{(i)}(t,s)]^2\, ds \right\} dt \right] \right)^{\frac{1}{2}}$$

$$C_n^{(o)}(s,t) = \frac{a(s)}{a_1} [a_1 b(t) - a(t) b_1] ,$$

$$C_n^{(n)}(s,t) = \frac{b(t)}{b_n} [a(s) b_n - a_n b(s)] ,$$

$$C_n^{(i)}(s,t) = \frac{[a(s)b_i - a_i b(s)][a(t)b_{i+1} - a_{i+1} b(t)]}{[a_{i+1} b_i - a_i b_{i+1}]} ,$$

$$i = 1, \ldots , n - 1,$$

or by

$$\text{(8.2)} \quad ||x - x_n|| \leq N(W - H_n) \left[\frac{||y||}{[1 - M(W)][1 - N(W - H_n) - M(W)]}\right]$$

$$\text{if } N(W - H_n) + M(W) < 1.$$

Using the grosser bound for $M(W - H_n)$ we have

$$(8.3) \qquad ||x - x_n|| \leq \frac{\varepsilon_n}{\sqrt{n+1}} \left[\frac{||x_n||}{1 - M(W)}\right] \quad \text{if} \quad M(W) < 1,$$

or

$$(8.4) \qquad ||x - x_n|| \leq \frac{\varepsilon_n}{\sqrt{n+1}} \left[\frac{||y||}{[1 - M(W)] \left(1 - \frac{\varepsilon_n}{\sqrt{n+1}} - M(W)\right)}\right],$$

$$\text{if} \quad \frac{\varepsilon_n}{\sqrt{n+1}} + M(W) < 1.$$

§ 9. <u>An Illustrative Example</u>. For a numerical example we consider the Green's function associated with the differential operator $L(u) = u''$ and bounding condition $u(0) = u(1) = 0$, namely

$$(9.1) \qquad W(s,t) = \begin{cases} s(1 - t) & 0 \leq s \leq t \leq 1 \\ t(1 - s) & 0 \leq t \leq s \leq 1 \end{cases}$$

Then

$$(9.2) \qquad M(W) = \left[\int_0^1 \int_0^1 W^2(s,t)\, ds\, dt\right]^{\frac{1}{2}} = 0.10541.$$

The integral equation

$$(9.3) \qquad x(s) - \int_0^1 W(s,t)\, x(t) = s^2$$

was solved with Bateman's formulas with $n = 1, 2$. In each case

$$||x_n|| < \frac{1}{2} \qquad n = 1, 2.$$

From (7.28) it is possible to calculate several values of $N(W - H_n)$ with $n = 1, 2, 3, 4$, which are listed below.

n	$N(W - H_n)$
1	0.037267
2	0.020286
3	0.013176
4	0.009428

In this case

(9.4) $\quad ||y|| = 0.44213.$

From (8.1) — modified in accordance with (4.8) — we find that

$$||x - x_1|| \leq 0.02083,$$

$$||x - x_2|| \leq 0.01134.$$

Using (8.2) we obtain

$$||x - x_1|| \leq 0.02148,$$
$$||x - x_2|| \leq 0.01147,$$
$$||x - x_3|| \leq 0.00625,$$
$$||x - x_4|| \leq 0.00526.$$

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