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There exists a variety of methods to solve the radiation transport equation in stochastic materials, including the Levermore-Pomraning method and the atomic mix method, each of which is designed to replicate the results given by benchmarks with materials rearranged in a randomized fashion. While the atomic mix method is very simple to implement, the results given are less than ideal due to the assumption that the materials are mixed at the atomic level, precluding the account of larger individual pieces of a material. The Levermore-Pomraning method is often used to approximate the benchmark solutions for exponentially-distributed materials, but includes a coupling term in its differential equation that is more difficult to implement and is significantly inaccurate in optically thick and highly scattering materials.

By assuming that the materials will not change within a cell, as happens in the benchmark solutions if the realizations are generated on a constant mesh, a set of differential equations can be derived that are coupled only at the cell boundaries. The analytic and then discretized equations for this method and each of the standard and benchmark methods are derived herein. Two separate methods of coupling for the mesh-based equations are shown, and their results using a variety of material parameters are compared with each of the standard and benchmark methods.

An Assessment of a Mesh-Based Method for Slab-Geometry Transport in Stochastic Materials

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Brody R. Bassett, Author

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF FIGURES (CONTINUED)

LIST OF TABLES

LIST OF TABLES (CONTINUED)

1 Introduction

1.1 Radiation Transport in a Stochastic Medium

The transport equation describes the interaction of particles with their environment, stating particle conservation over phase space. The transport equation is based on the Boltzmann equation, which itself was developed to study liquids and gases not in thermal equilibrium, or the interaction and collision of chemical constituents. In the transport equation, particles are assumed to be non-interacting, as particles like neutrons and electrons interact more strongly with the environment than with the other particles of their same species.

While in most cases, the material properties for the phase space in which the transport equation is to be solved are specifically known, in some cases the locations of the materials are known only probabilistically, which gives rise to a stochastic problem. While some statistical models assume that the materials themselves vary in some normal distribution, most utilize the immiscible material assumption [1].

The standard for transport in a stochastic medium is the brute force approach of creating a large number of realizations of the material distribution and then solving these transport problems using the standard transport equation. In the standard method of brute force calculation, the material distribution is generated by filling the phase space with alternating exponentially-distributed sections of each of the materials that are present. While realistic in theory, this approach requires significant computational resources and is impractical for any but the simplest of problems due to the huge number of realizations required to obtain solutions with sufficiently small statistical errors [2]. Two separate categories of deterministic solution methods have been proposed to approximate the brute force solution, the atomic mix model and the Levermore-Pomraning method.

The atomic mix model assumes, as its name implies, that the stochastic materials are mixed at the atomic level, which gives rise to a transport equation with homogenized physical constants using volume fractions. This equation can be solved with any of the standard numerical approaches applied to radiation transport. The atomic mix model can have large errors, as stochastic effects such as the likelihood of distinct larger pieces of each material are not considered.

To remedy the inaccuracies of the atomic mix model, Levermore and Pomraning derived a separate method, now known as the Levermore-Pomraning (LP) or the Standard model, which results in as many transport equations as there are materials, with the equation for each material containing a term coupling it to each of the other materials [3]. This method is much more accurate than than the atomic mix method for a wide variety of stochastic mixture transport problems. However, the coupling term is difficult to implement in a transport code without significant alterations to the discretized equations and the method has high errors when compared to the brute force method in optically thick slabs [4, 5, 6, 7].

The method proposed herein was originally designed to calculate the ensemble average solution to the very specific set of benchmark solutions with invariable material ratios. With some minor changes, this method can, like the LP method, also approximate the brute force solution with exponentially distributed slabs. This method uses constant mesh for the problem, which makes the coupling term disappear except at cell edges. This method has been previously implemented in three dimensions [8], but the lack of three-dimensional benchmarks makes its accuracy difficult to verify [9]. In addition, these benchmarks have been run for only

a percentage-fill case and not the exponentially-distributed slabs commonly used in this type of transport. In contrast, there are a number of brute force type benchmarks available for the one-dimensional case [10, 4, 11, 12, 13]. In addition, some analytical solutions available for the purely absorbing, one-dimensional case can help to verify a method's accuracy within those constraints [14].

All the methods used depend on the mean chord length, or average length that a randomized vector drawn in the problem would traverse in a certain material before exiting into another [15]. The chord lengths are usually assumed to be exponentially distributed, which has been shown to be a good assumption for the Cartesian geometry used [16]. For some of the methods of approximating the brute force solution, including the atomic mix model, the chord length is used only to calculate the transition probability, or the probability of transitioning from one material to another in a certain location.

1.2 Applications

In nuclear reactors, the high water content of concrete makes it an efficient neutron reflector. Concrete consists of a randomized mix of variously sized granular material held together by a binding agent. This stochastic material is often considered to be homogenized. This same issue arises in pebble bed reactors, both in the location of the pebbles within the reactor and the location of the fuel particles within the pebbles. The assumptions often used that concrete and pebbles are homogeneous and that pebbles within a reactor are constant in position are often poor and can lead to inaccurate results [6].

Randomized mixing is also common in inertial confinement fusion (ICF), which depends on thermal radiation transport. When the pellets are bombarded by radiation from lasers, the outside layer is heated to a very high temperature and begins to vaporize in a process known as ablation, which creates an inward reaction force that compresses the deuterium and tritium to initiate fusion. When the outer layer of the fusion pellet is liquified, the very different materials that make up the pellet develop Rayleigh-Taylor instabilities, which occur when a lighter fluid accelerates into a heavier one, creating exponentially growing, randomized deformation in the materials. These mixing zones can affect the radiation transport solution significantly [11, 14].

Another inhomogeneous medium with high scattering is clouds. When determining climate, the amount of radiation transferred through clouds is of high importance. Clouds are made from aerosols, which are in this case tiny particles of fluid suspended in the air, which is often mixed with various atmospheric gases. With many current approaches, radiation transfer through this stochastic medium gives only mean characteristics due to the homogeneous assumption, but doesn't take into account the randomized size and position of the water droplets and other particles within the clouds [17].

The techniques described herein are derived in a neutron transport context, but have close analogies in thermal radiation transport. All of the methods take into account stochasticity in space (randomizing the location of what could represent water rocks in concrete, the mixing materials in ICF, or water droplets in clouds), while some also include variability in the size of the material chunks (for example, the size of the rocks or water droplets). This allows accurate modeling of randomized mixtures that the homogeneous assumption doesn't allow.

2 Derivation of Equations

The Levermore-Pomraning (LP) and mesh-based equations are derived in this section. The two sets of equations have a common derivation before diverging in a term coupling the angular flux to be transported in one material to that for the other materials. An upwinding approach is also derived for the LP equations. The derivation of the analytic equations is presented for three-dimensional space.

The equations will be discretized in one spatial dimension. The transport equation is discretized using a discontinuous finite element method (DFEM) known as lumped linear discontinuous (LLD). This method is then applied to the LP and mesh-based equations.

2.1 Analytic Equations

Variable	Description	Units
E	Energy	MeV
$p_n(r)$	Probability that position r is in material n	-
$S(r,E,\Omega,t)$	Internal radiation source	particles/(m^3·MeV·s)
r	Position vector	m
t	Time	sec
$v(E)$	Particle speed	m/sec
Λ_n	Mean chord length for material n	m
$\Sigma_t(r,E,t)$	Total cross section	m^{-1}
$\Sigma_s(r,E'\rightarrow E,\Omega'\rightarrow\Omega,t)$	Scattering cross section	m^{-1}
$\phi(r,E,t)$	Scalar flux	particles/(m^2·steradian·MeV·s)
$\psi(r,E,\Omega,t)$	Angular flux	particles/(m^2·MeV·s)
Ω	Particle direction unit vector	-

Table 2.1: Analytic variables

2.1.1 Levermore-Pomraning Equations

The Levermore-Pomraning equations refer to a set of stochastic transport equations derived from the standard neutron transport equation, which in three dimensions is:

$$\frac{1}{v(E)}\frac{\partial\psi\left(r,E,\Omega,t\right)}{\partial t}+\Omega\cdot\nabla\psi\left(r,E,\Omega,t\right)+\Sigma_t\left(r,E,t\right)\psi\left(r,E,\Omega,t\right)=$$
$$\int_{4\pi}d\Omega'\int_0^\infty dE'\Sigma_s\left(r,E'\rightarrow E,\Omega'\rightarrow\Omega,t\right)\psi\left(r,E',\Omega',t\right)+S\left(r,E,\Omega,t\right). \quad (2.1)$$

Their goal is to approximate the brute force benchmark solutions with exponentially distributed chord lengths.

For simplicity, the transport equation can be written with suppressed dependencies as:

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \Omega \cdot \nabla \psi + \Sigma_t \psi = \int_{4\pi} d\Omega' \int_0^\infty dE' \Sigma_s \psi + S. \tag{2.2}$$

In order to mathematically represent the stochastic nature of the problem, define the characteristic function $\chi_n(r)$, which specifies spatially where the two or more materials are located:

$$\chi_n(r) = \begin{cases} 1, & \text{if } r \text{ is in material } n \\ 0, & \text{if } r \text{ is in material } m \neq n \end{cases} \tag{2.3}$$

As the properties for each material will be uniform, the material properties' dependence on position is equivalent to a dependence on the characteristic function at each position:

$$\chi_n(r)\,\Sigma_t(r) = \Sigma_t^n \chi_n(r), \tag{2.4}$$

$$\chi_n(r)\,\Sigma_s(r) = \Sigma_s^n \chi_n(r), \tag{2.5}$$

$$\chi_n(r)\,S(r,t) = S^n \chi_n(r). \tag{2.6}$$

Next, equation 2.2 will be multiplied by $\chi_n(r)$. Using the chain rule, the second term can be rewritten as:

$$\chi_n(\Omega \cdot \nabla \psi) = \Omega \cdot \nabla(\chi_n \psi) - \psi(\Omega \cdot \nabla \chi_n). \tag{2.7}$$

Using this replacement, the equation becomes:

$$\frac{1}{v}\frac{\partial(\chi_n \psi)}{\partial t} + \Omega \cdot \nabla(\chi_n \psi) + \Sigma_t^n \chi_n \psi = \int_{4\pi} d\Omega' \int_0^\infty dE' \Sigma_s^n \chi_n \psi + \chi_n S_i + \psi(\Omega \cdot \nabla \chi_n),\ n \in N. \tag{2.8}$$

Next, take the ensemble average of the transport equation, or the average over many different realizations of the stochastic mixture. Summing the ensemble-averaged flux for each of the materials gives the total ensemble-averaged flux:

$$\begin{aligned}\langle\psi(r,\Omega,t)\rangle &= \left\langle\psi(r,\Omega,t)\sum_n\chi_n(r)\right\rangle \\ &= \sum_n\langle\chi_n(r)\psi(r,\Omega,t)\rangle \\ &= \sum_n p_n(r)\psi^n(r,\Omega,t).\end{aligned} \tag{2.9}$$

The following ensemble quantities were used to simplify the problem:

$$\psi^n(r,\Omega,t) = \frac{\sum_{n'}\psi^{n'}(r,\Omega,t)}{N} = \frac{\langle\chi_n(r)\psi(r,\Omega,t)\rangle}{\langle\chi_n(r)\rangle}, \tag{2.10}$$

$$p_n(r)\psi^n(r,\Omega,t) = \langle\chi_n(r)\psi(r,\Omega,t)\rangle. \tag{2.11}$$

These replacements result in the initial form of the Levermore-Pomraning equations:

$$\frac{1}{v}\frac{\partial(p_n\psi^n)}{\partial t} + \Omega\cdot\nabla(p_n\psi^n) + \Sigma_t^n p_n\psi^n = \int_{4\pi}d\Omega'\int_0^\infty dE'\Sigma_s^n p_n\psi^n + p_nS^n + \langle\psi(\Omega\cdot\nabla\chi_n)\rangle,\ n\in N. \tag{2.12}$$

Following Vasques [13], the stochastic balance method can be used to compute the coupling term above. This method uses geometry to model the probability that a volume is intersected by a cell boundary, leading to the exact Levermore-Pomraning equations :

$$\begin{aligned}&\frac{1}{v}\frac{\partial(p_n\psi^n)}{\partial t} + \Omega\cdot\nabla(p_n\psi^n) + \Sigma_t^n p_n\psi^n = \\ &\qquad\int_{4\pi}d\Omega'\int_0^\infty dE'\Sigma_s^n p_n\psi^n + p_nS^n - \frac{p_n\Psi^n}{\Lambda_n} + \sum_{m\neq n}\left\{\frac{p_m\Psi^m}{\Lambda_m}\frac{\Lambda_n}{\sum_{o\neq m}\Lambda_o}\right\},\ n\in N.\end{aligned} \tag{2.13}$$

Here Ψ^n represents the ensemble average of ψ at the material interfaces where

$$\Psi^n = \lim_{\epsilon\to 0}\frac{\left\langle(\Omega\cdot u_n)\psi\left(\epsilon^2-r^2\right)\right\rangle^*_{\Omega\cdot u_n>0}}{\left\langle(\Omega\cdot u_n)\left(\epsilon^2-r^2\right)\right\rangle^*_{\Omega\cdot u_n>0}}. \tag{2.14}$$

Given the surface of an interface between material n and another material, u_n is the unit vector pointing outward from material n, and $\langle\cdot\rangle^*_{\Omega\cdot u_n>0}$ is the ensemble average over all realizations in which an interface is within ϵ of the point r and has Ω exiting the material n. Ψ^n then applies only to the points at which interfaces from material n occur, while ψ^n applies anywhere that r is in material n. The closure commonly used is to simply equate ψ^n and Ψ^n, which assumes that the angular flux exiting a material (averaged over

all realizations) will be equal to the angular flux inside of the material (also averaged over all realizations):

$$\frac{1}{v}\frac{\partial\left(p_n\psi^n\right)}{\partial t}+\Omega\cdot\nabla\left(p_n\psi^n\right)+\Sigma_t^n p_n\psi^n= \int_{4\pi}d\Omega'\int_0^\infty dE'\Sigma_s^n p_n\psi^n+p_nS^n-\frac{p_n\psi^n}{\Lambda_n}+\sum_{m\neq n}\left\{\frac{p_m\psi^m}{\Lambda_m}\frac{\Lambda_n}{\sum_{o\neq m}\Lambda_o}\right\},\ n\in N. \quad (2.15)$$

Define the following modified flux and source to simplify the equation:

$$\widetilde{\psi}^n=p_n\psi^n, \quad (2.16)$$

$$\widetilde{S}^n=p_nS^n. \quad (2.17)$$

This simplifies the Levermore-Pomraning equations to:

$$\frac{1}{v}\frac{\partial\widetilde{\psi}^n}{\partial t}+\Omega\cdot\nabla\widetilde{\psi}^n+\Sigma_t^n\widetilde{\psi}^n=\int_{4\pi}d\Omega'\int_0^\infty dE'\Sigma_s^n\widetilde{\psi}^n+\widetilde{S}^n-\frac{\widetilde{\psi}^n}{\Lambda_n}+\sum_{m\neq n}\left\{\frac{\widetilde{\psi}^m}{\Lambda_m}\frac{\Lambda_n}{\sum_{o\neq m}\Lambda_o}\right\},\ n\in N. \quad (2.18)$$

While in a purely absorbing medium this is exact, when in a highly scattering medium this is a poor assumption. Consider, for instance, a one-dimensional problem with two materials, one highly scattering and the other highly absorbing. For a certain realization, let the surface at $x-\epsilon$ (for some small ϵ) be in the highly scattering material. If the surface at $x+\epsilon$ is also in the highly scattering material, then the surface at x will receive significant amounts of backscattered radiation. If, however, the surface at $x+\epsilon$ is in the highly absorbing material, then the flux at the point x will be much lower than it would be without an interface due to the lack of backscattering from the highly absorbing material. The amount of radiation flowing in the positive direction from x will be very different depending on whether there's a boundary in its vicinity, and in this case the assumption that the flux will be equal in realizations when the surface at x is a boundary when compared to realizations when the surface at x is not a boundary is not true.

As can be seen in Equation 2.18, this form of the Levermore-Pomraning equations is comprised of a transport equation for each material coupled to that of each other material.

2.1.2 Levermore-Pomraning Equations on a constant mesh

For any benchmarks using a constant mesh and the immiscible material assumption, $(\Omega\cdot\nabla\chi_n)=0$ away from cell edges for each realization. Applying this to Equation 2.12, the coupling term disappears:

$$\frac{1}{v}\frac{\partial\left(p_n\psi^n\right)}{\partial t}+\Omega\cdot\nabla\left(p_n\psi^n\right)+\Sigma_t^n p_n\psi^n=\int_{4\pi}d\Omega'\int_0^\infty dE'\Sigma_s^n p_n\psi^n+p_nS,\ n\in N. \quad (2.19)$$

This removes the need to make the problematic assumption in deriving the LP equations that the interface ensemble averaged flux equals the material ensemble averaged flux. Using the same modified flux and source as in Equation 2.18 gives an almost identical equation to the original transport equation, which applies inside the cells (which is where the lowest level of the sweeps occurs):

$$\frac{1}{v}\frac{\partial\widetilde{\psi}^n}{\partial t}+\Omega\cdot\nabla\widetilde{\psi}^n+\Sigma_t^n\widetilde{\psi}^n=\int_{4\pi}d\Omega'\int_0^\infty dE'\Sigma_s^n\widetilde{\psi}^n+\widetilde{S}. \tag{2.20}$$

At the edge of each cell, the angular flux is summed and then redistributed according to the transition probability p_n:

$$\widetilde{\psi}^n=p_n\sum_{n'}\widetilde{\psi}^{n'}=p_n\left\langle\psi\right\rangle. \tag{2.21}$$

After the iterations have converged, the total ensemble-averaged flux is calculated using Equation 2.9:

$$\left\langle\psi\left(r,\Omega,t\right)\right\rangle=\sum_{n'}\widetilde{\psi}^{n'}(r,\Omega,t). \tag{2.22}$$

2.2 Discretized Equations

Variable	Description	Units
$b_{ik}(x)$	Basis functions	-
p_{ni}	Probability that cell i contains material n	-
$S_m(x)$	Internal radiation source	particles/(m^3·MeV·s)
$v_{L\|R,i}(x)$	Basis functions	-
x	Position	m
Δx_i	The length of cell i	m
Λ_n	Chord length for material n	m
Σ_{ti}	Neutron total cross section for cell i	m^{-1}
ψ_m	Angular flux	particles/(m^2·MeV·s)
μ_m	Cosine between direction vector and x-axis	-

Table 2.2: Discretized variables

2.2.1 Standard DFEM Equations

To simplify the following derivations, the scattering term is included in the definition of the source S. When written in one dimension and in steady state, the transport equation (Equation 2.2) simplifies to:

$$\mu \frac{\partial}{\partial x} \psi(x,\mu) + \Sigma_t(x)\,\psi(x,\mu) = S(x,\mu). \tag{2.23}$$

All further equations in this section will apply to the full set of discrete ordinates $m \in M$. With the discrete ordinates approximation for the angularly-dependent terms, the partial differential equation becomes a set of ordinary differential equations in the spatial variable:

$$\mu_m \frac{\partial}{\partial x} \psi_m(x) + \Sigma_t(x)\,\psi_m(x) = S_m(x). \tag{2.24}$$

The standard discontinuous finite element method (DFEM) equations take an integral weighted by $v_{ik}(x)$ (the weight functions) over each cell in the transport equation to give:

$$\mu_m \left[v_{L|R,i}(x)\,\psi_m(x)\right]_{x_{i-1/2}}^{x_{i+1/2}} - \mu_m \int_{x_{i-1/2}}^{x_{i+1/2}} dx \psi_m(x) \frac{dv_{L|R,i}}{dx} + \int_{x_{i-1/2}}^{x_{i+1/2}} dx v_{L|R,i}(x)\,\Sigma_t(x)\psi_m(x) = \int_{x_{i-1/2}}^{x_{i+1/2}} dx v_{L|R,i}(x)\,S_m(x). \tag{2.25}$$

In this case, two basis and weight functions will be used ($k \in (1,2)$). The closure term for these equations uses the angular flux from the boundary shared by the previous cell in the sweep:

$$\psi_m\left(x_{i\pm1/2}\right) \equiv \psi_{m,i\pm1/2} \equiv \begin{cases} \lim_{\epsilon\to 0^+} \psi_m\left(x_{i\pm1/2} - \epsilon\right), & \mu_m > 0, \\ \lim_{\epsilon\to 0^+} \psi_m\left(x_{i\pm1/2} + \epsilon\right), & \mu_m < 0. \end{cases} \tag{2.26}$$

The total angular flux for each cell is computed by summing the product of the angular flux coefficient and basis functions $b_{Li}(x)$ and $b_{Ri}(x)$ on each node:

$$\psi_m(x) = \psi_{mLi} b_{Li}(x) + \psi_{mRi} b_{Ri}(x),\ x \in \left(x_{i-1/2}, x_{i+1/2}\right). \tag{2.27}$$

2.2.2 LD Equations

The linear discontinuous (LD) method uses the DFEM equations with the following basis and weight functions ($j \in (L,R)$):

$$b_{Li} = v_{Li} = \frac{x_{i+1/2} - x}{\Delta x_i}, \tag{2.28}$$

$$b_{Ri} = v_{Ri} = \frac{x - x_{i-1/2}}{\Delta x_i}. \tag{2.29}$$

This gives for the angular flux and source:

$$\begin{aligned}\psi_m(x) &= \psi_{mLi} b_{Li}(x) + \psi_{mRi} b_{Ri}(x) \\ &= \left(\frac{x_{i+1/2} - x}{\Delta x_i}\right)\psi_{mLi} + \left(\frac{x - x_{i-1/2}}{\Delta x_i}\right)\psi_{mRi}. \end{aligned} \tag{2.30}$$

$$\begin{aligned}S_m(x) &= S_{mLi} b_{Li}(x) + S_{mRi} b_{Ri}(x) \\ &= \left(\frac{x_{i+1/2} - x}{\Delta x_i}\right)S_{mLi} + \left(\frac{x - x_{i-1/2}}{\Delta x_i}\right)S_{mRi}. \end{aligned} \tag{2.31}$$

With ψ_m and S_m from Equations 2.30 and 2.31 inserted into Equation 2.25, replace v_{ik} with v_{Li} and v_{Ri} successively.

For $\boldsymbol{v_{Li}}$:

$$\begin{aligned}&\mu_m \left[\left(\frac{x - x_{i-1/2}}{\Delta x_i}\right)\left(\left[\frac{x_{i+1/2} - x}{\Delta x_i}\right]\psi_{mLi} + \left[\frac{x - x_{i-1/2}}{\Delta x_i}\right]\psi_{mRi}\right)\right]_{x_{i-1/2}}^{x_{i+1/2}} - \\ &\qquad \mu_m \int_{x_{i-1/2}}^{x_{i+1/2}} dx \left[\left(\frac{x_{i+1/2} - x}{\Delta x_i}\right)\psi_{mLi} + \left(\frac{x - x_{i-1/2}}{\Delta x_i}\right)\psi_{mRi}\right]\left(\frac{1}{\Delta x_i}\right) + \\ &\qquad \int_{x_{i-1/2}}^{x_{i+1/2}} dx \left(\frac{x - x_{i-1/2}}{\Delta x_i}\right)\Sigma_{ti}\left[\left(\frac{x_{i+1/2} - x}{\Delta x_i}\right)\psi_{mLi} + \left(\frac{x - x_{i-1/2}}{\Delta x_i}\right)\psi_{mRi}\right] = \\ &\qquad\qquad \int_{x_{i-1/2}}^{x_{i+1/2}} dx \left(\frac{x - x_{i-1/2}}{\Delta x_i}\right)\left[\left(\frac{x_{i+1/2} - x}{\Delta x_i}\right)S_{mLi} + \left(\frac{x - x_{i-1/2}}{\Delta x_i}\right)S_{mRi}\right]. \end{aligned} \tag{2.32}$$

Performing this integration, the result simplifies to:

$$\mu_m \psi_{mRi} - \frac{\mu_m}{2}\left[\psi_{mLi} + \psi_{mRi}\right] + \frac{\Sigma_{ti}\Delta x_i}{6}\psi_{mLi} + \frac{\Sigma_{ti}\Delta x_i}{3}\psi_{mRi} = \frac{\Delta x_i}{6}S_{mLi} + \frac{\Delta x_i}{3}S_{mRi}. \tag{2.33}$$

For $\boldsymbol{v_{Ri}}$:

$$\mu_m \left[\left(\frac{x_{i+1/2} - x}{\Delta x_i} \right) \left(\psi_{mLi} \left[\frac{x_{i+1/2} - x}{\Delta x_i} \right] + \psi_{mRi} \left[\frac{x - x_{i-1/2}}{\Delta x_i} \right] \right) \right]_{x_{i-1/2}}^{x_{i+1/2}} -$$
$$\mu_m \int_{x_{i-1/2}}^{x_{i+1/2}} dx \left[\psi_{mLi} \left(\frac{x_{i+1/2} - x}{\Delta x_i} \right) + \psi_{mRi} \left(\frac{x - x_{i-1/2}}{\Delta x_i} \right) \right] \left(-\frac{1}{\Delta x_i} \right) +$$
$$\int_{x_{i-1/2}}^{x_{i+1/2}} dx \left(\frac{x_{i+1/2} - x}{\Delta x_i} \right) \Sigma_{ti} \left[\psi_{mLi} \left(\frac{x_{i+1/2} - x}{\Delta x_i} \right) + \psi_{mRi} \left(\frac{x - x_{i-1/2}}{\Delta x_i} \right) \right] =$$
$$\int_{x_{i-1/2}}^{x_{i+1/2}} dx \left(\frac{x_{i+1/2} - x}{\Delta x_i} \right) \left[S_{mLi} \left(\frac{x_{i+1/2} - x}{\Delta x_i} \right) + S_{mRi} \left(\frac{x - x_{i-1/2}}{\Delta x_i} \right) \right]. \quad (2.34)$$

Performing the integration as for v_{Li} gives:

$$\mu_m \psi_{mRi} - \frac{\mu_m}{2} \left[\psi_{mLi} + \psi_{mRi} \right] + \frac{\Sigma_{ti} \Delta x_i}{3} \psi_{mLi} + \frac{\Sigma_{ti} \Delta x_i}{6} \psi_{mRi} = \frac{\Delta x_i}{3} S_{mLi} + \frac{\Delta x_i}{6} S_{mRi}.$$

This results in the LD equations:

$$\left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Sigma_{ti} \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \right) \begin{bmatrix} \psi_{mLi} \\ \psi_{mRi} \end{bmatrix} = \mu_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{m,i+1/2} \\ \psi_{m,i-1/2} \end{bmatrix} + \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} S_{mLi} \\ S_{mRi} \end{bmatrix}. \quad (2.35)$$

with the upstream values given by:

$$\psi_{m,i+1/2} = \begin{cases} \psi_{m,inc}(x_{1/2}), & \mu_m > 0, \ i = 0, \\ \psi_{m,inc}(x_{I+1/2}), & \mu_m < 0, \ i = I, \\ \psi_{mRi}, & \mu_m > 0, \ i \geq 1, \\ \psi_{mL,i+1}, & \mu_m < 0, \ i \leq I - 1. \end{cases} \quad (2.36)$$

Using the upstream values for $\mu_m > 0$ and $\mu_m < 0$ to replace $\psi_{m,i\pm1/2}$, the directionally-dependent equations are:

$$\mu_m > 0: \left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Sigma_{ti} \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \right) \begin{bmatrix} \psi_{mLi} \\ \psi_{mRi} \end{bmatrix} =$$
$$\mu_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{mRi} \\ \psi_{mR,i-1} \end{bmatrix} + \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} S_{mLi} \\ S_{mRi} \end{bmatrix}, \quad (2.37)$$

$$\mu_m < 0: \left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Sigma_{ti} \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \right) \begin{bmatrix} \psi_{mLi} \\ \psi_{mRi} \end{bmatrix} = \mu_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{mL,i+1} \\ \psi_{mLi} \end{bmatrix} + \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} S_{mLi} \\ S_{mRi} \end{bmatrix}. \quad (2.38)$$

This gives a linear system of equations for ψ_{mLi} and ψ_{mRi}, which will be dependent on the flux exiting the previous cell and the source (as calculated from the scattering from the last iteration and the external source):

$$\mu_m > 0: \left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \Sigma_{ti} \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \right) \begin{bmatrix} \psi_{mLi} \\ \psi_{mRi} \end{bmatrix} = \mu_m \begin{bmatrix} 1 \\ 0 \end{bmatrix} \psi_{mR,i-1} + \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} S_{mLi} \\ S_{mRi} \end{bmatrix}, \quad (2.39)$$

$$\mu_m < 0: \left(\mu_m \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Sigma_{ti} \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \right) \begin{bmatrix} \psi_{mLi} \\ \psi_{mRi} \end{bmatrix} = \mu_m \begin{bmatrix} 0 \\ -1 \end{bmatrix} \psi_{mL,i+1} + \Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} S_{mLi} \\ S_{mRi} \end{bmatrix}. \quad (2.40)$$

2.2.3 Lumped LD Equations

The lumped LD (LLD) equations modify the mass matrix term in the discretized equations to prevent the angular flux from oscillating or going negative [18], which gives better results in the optically thick limit, with a reduction in accuracy from third-order globally to second-order. The mass matrix in Equation 2.35 arises in both the collision and source terms:

$$\Delta x_i \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}. \quad (2.41)$$

The lumped LD equations are identical to Equations 2.36, 2.39, and 2.40, but with the off-diagonal elements of the mass matrix added into the diagonal:

$$\Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{2.42}$$

With the lumped mass matrices, Equations 2.39 and 2.40 become, respectively:

$$\mu_m > 0: \left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \Sigma_{ti} \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \psi_{mLi} \\ \psi_{mRi} \end{bmatrix} = \mu_m \begin{bmatrix} 1 \\ 0 \end{bmatrix} \psi_{mR,i-1} + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} S_{mLi} \\ S_{mRi} \end{bmatrix}, \tag{2.43}$$

$$\mu_m < 0: \left(\mu_m \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Sigma_{ti} \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \psi_{mLi} \\ \psi_{mRi} \end{bmatrix} = \mu_m \begin{bmatrix} 0 \\ -1 \end{bmatrix} \psi_{mL,i+1} + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} S_{mLi} \\ S_{mRi} \end{bmatrix}. \tag{2.44}$$

2.3 LLD Mesh-Based Stochastic Mixture Equations

For the mesh-based method, by using Equation 2.20, the LLD equations inside of each cell will be equivalent to their standard counterparts (Equations 2.43-2.44), but the modified flux and source as given in Equations 2.16-2.17 are used to give a set of discretized transport equations for each material n:

$$\mu_m > 0: \left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \Sigma_{ti}^n \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} = \mu_m p_{ni} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \langle \psi_{mR,i-1} \rangle + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \widetilde{S}_{mLi}^n \\ \widetilde{S}_{mRi}^n \end{bmatrix}, \; n \in N, \tag{2.45}$$

$$\mu_m < 0: \left(\mu_m \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Sigma_{ti}^n \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} =$$
$$\mu_m p_{ni} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \langle \psi_{mL,i+1} \rangle + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \widetilde{S}_{mLi}^n \\ \widetilde{S}_{mRi}^n \end{bmatrix}, \; n \in N. \quad (2.46)$$

The only place the equations for the different materials are coupled is at the cell edges, where the total flux is summed and then redistributed according to the probabilities p_{ni}, as in Equation 2.21. Applying this redistribution gives the final form of the LLD mesh-based stochastic mixture equations:

$$\mu_m > 0: \left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \Sigma_{ti}^n \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} =$$
$$\mu_m p_{ni} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sum_{n'} \widetilde{\psi}_{mR,i-1}^{n'} + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \widetilde{S}_{mLi}^n \\ \widetilde{S}_{mRi}^n \end{bmatrix}, \; n \in N, \quad (2.47)$$

$$\mu_m < 0: \left(\mu_m \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Sigma_{ti}^n \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} =$$
$$\mu_m p_{ni} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sum_{n'} \widetilde{\psi}_{mL,i+1}^{n'} + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \widetilde{S}_{mLi}^n \\ \widetilde{S}_{mRi}^n \end{bmatrix}, \; n \in N. \quad (2.48)$$

The solution for these equations from one cell to another in the two-node, one-dimensional, purely absorbing case with a boundary source and no internal source is:

$$\widetilde{\psi}_{mLi}^n = \begin{cases} \left(\frac{2\mu_m(\mu_m + \Sigma_{ti}^n \Delta x_i)}{(\mu_m + \Sigma_{ti}^n \Delta x_i)^2 + (\mu_m)^2} \right) p_{ni} \sum_{n'} \widetilde{\psi}_{mR,i-1}^{n'} & \mu_m > 0, \\ \left(\frac{2(\mu_m)^2}{(-\mu_m + \Sigma_{ti}^n \Delta x_i)^2 + (\mu_m)^2} \right) p_{ni} \sum_{n'} \widetilde{\psi}_{mL,i+1}^{n'} & \mu_m < 0, \end{cases} \quad (2.49)$$

$$\widetilde{\psi}_{mRi}^n = \begin{cases} \left(\frac{2(\mu_m)^2}{(\mu_m + \Sigma_{ti}^n \Delta x_i)^2 + (\mu_m)^2} \right) p_{ni} \sum_{n'} \widetilde{\psi}_{mR,i-1}^{n'} & \mu_m > 0, \\ \left(\frac{-2\mu_m(-\mu_m + \Sigma_{ti}^n \Delta x_i)}{(-\mu_m + \Sigma_{ti}^n \Delta x_i)^2 + (\mu_m)^2} \right) p_{ni} \sum_{n'} \widetilde{\psi}_{mL,i+1}^{n'} & \mu_m < 0. \end{cases} \quad (2.50)$$

It is apparent that nowhere can the flux go negative, which is a primary benefit of the LLD method. The

results for each of the four situations are included below.

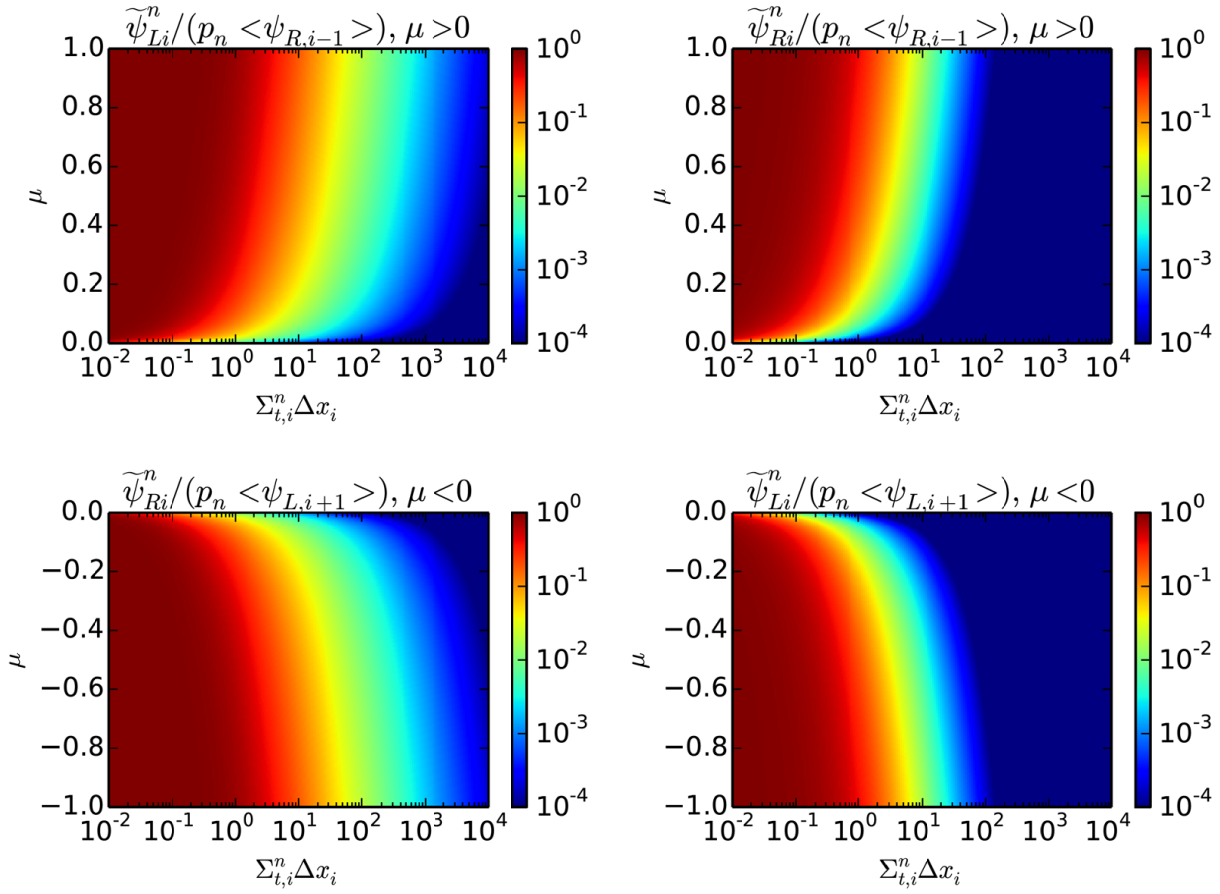


Figure 2.1: Attenuation coefficient for LLD mesh-based stochastic mixture method in pure absorber

2.4 LLD LP Equations

2.4.1 Standard LP Equations

The LLD derivation can also be applied to the LP equations. The term that couples the material-averaged angular fluxes is discretized in the same manner as the collision term or the source to give:

$$\mu_m > 0: \quad \left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \Delta x_i \left(\Sigma_{ti}^n + \frac{\mu_m}{\Lambda_n} \right) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} =$$

$$\mu_m \begin{bmatrix} 1 \\ 0 \end{bmatrix} \widetilde{\psi}_{mR,i-1}^n + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} \widetilde{S}_{mLi}^n \\ \widetilde{S}_{mRi}^n \end{bmatrix} + \sum_{o \neq n} \frac{\mu_m}{\Lambda_o} \frac{\Lambda_n}{\sum_{l \neq m} \Lambda_l} \begin{bmatrix} \widetilde{\psi}_{mLi}^o \\ \widetilde{\psi}_{mRi}^o \end{bmatrix} \right), \; n \in N, \quad (2.51)$$

$$\mu_m < 0: \quad \left(\mu_m \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Delta x_i \left(\Sigma_{ti}^n + \frac{\mu_m}{\Lambda_n} \right) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} =$$

$$\mu_m \begin{bmatrix} 0 \\ -1 \end{bmatrix} \widetilde{\psi}_{mL,i+1}^n + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} \widetilde{S}_{mLi}^n \\ \widetilde{S}_{mRi}^n \end{bmatrix} + \sum_{o \neq n} \frac{\mu_m}{\Lambda_o} \frac{\Lambda_n}{\sum_{l \neq m} \Lambda_l} \begin{bmatrix} \widetilde{\psi}_{mLi}^o \\ \widetilde{\psi}_{mRi}^o \end{bmatrix} \right), \; n \in N. \quad (2.52)$$

2.4.2 Modified LP Equations

As an alternate derivation, the approximation that the cell-centered flux of the current cell is equal to the flux exiting the previous cell's edge can be used:

$$\mu_m > 0: \quad \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(-\frac{\mu_m}{\Lambda_n} \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} + \sum_{o \neq n} \frac{\mu_m}{\Lambda_o} \begin{bmatrix} \widetilde{\psi}_{mLi}^o \\ \widetilde{\psi}_{mRi}^o \end{bmatrix} \right) \approx$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(-\frac{\mu_m}{\Lambda_n} \widetilde{\psi}_{mR,i-1}^n + \sum_{o \neq n} \frac{\mu_m}{\Lambda_o} \frac{\Lambda_n}{\sum_{l \neq m} \Lambda_l} \widetilde{\psi}_{mR,i-1}^o \right), \quad (2.53)$$

$$\mu_m < 0: \quad \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(-\frac{\mu_m}{\Lambda_n} \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} + \sum_{o \neq n} \frac{\mu_m}{\Lambda_o} \begin{bmatrix} \widetilde{\psi}_{mLi}^o \\ \widetilde{\psi}_{mRi}^o \end{bmatrix} \right) \approx$$

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \left(-\frac{\mu_m}{\Lambda_n} \widetilde{\psi}_{mL,i+1}^n + \sum_{o \neq n} \frac{\mu_m}{\Lambda_o} \frac{\Lambda_n}{\sum_{l \neq m} \Lambda_l} \widetilde{\psi}_{mL,i+1}^o \right). \quad (2.54)$$

This alters Equations 2.51-2.52 to:

$$\mu_m > 0: \quad \left(\mu_m \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \Delta x_i \Sigma_{ti}^n \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} =$$

$$\mu_m \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\left(1 - \frac{\Delta x_i}{\Lambda_n} \right) \widetilde{\psi}_{mR,i-1}^n + \sum_{o \neq n} \frac{\Delta x_i}{\Lambda_o} \widetilde{\psi}_{mR,i-1}^o \right) + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \widetilde{S}_{mLi}^n \\ \widetilde{S}_{mRi}^n \end{bmatrix}, \; n \in N, \quad (2.55)$$

$$\mu_m < 0: \quad \left(\mu_m \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \Delta x_i \Sigma_{ti}^n \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} \widetilde{\psi}_{mLi}^n \\ \widetilde{\psi}_{mRi}^n \end{bmatrix} =$$

$$\mu_m \begin{bmatrix} 0 \\ -1 \end{bmatrix} \left(\left(1 - \frac{\Delta x_i}{\Lambda_n} \right) \widetilde{\psi}_{mL,i+1}^n + \sum_{o \neq n} \frac{\Delta x_i}{\Lambda_o} \widetilde{\psi}_{mL,i+1}^o \right) + \Delta x_i \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \widetilde{S}_{mLi}^n \\ \widetilde{S}_{mRi}^n \end{bmatrix}, \; n \in N. \quad (2.56)$$

Note these equations are identical to the mesh-based equations (Equations 2.45-2.46) except for the cell edge mixing term, meaning that the LP equations are no longer coupled except at the cell edges. A solver for the mesh-based equations can then can be converted by changing a few lines of code to solve the modified LP equations.

3 Solution Methods

Method	Acronym	Transport equations	Realizations	Description
Atomic Mix	AT	Normal	One	Materials are assumed to be mixed at the atomic level, with homogenized cross sections weighted according to p_1 and p_2.
Chord	CB	Normal	Many	For each realization, material slabs are generated as exponentially distributed lengths of alternating materials
Distribution	DM	Mesh-based	One	The mesh-based transport method, but using a spatially-dependent p_1 and p_2 calculated based on exponentially distributed chord lengths.
Exponential	EB	Normal	Many	The number of cells for each realization in each material is exponentially distributed, but their position is randomized.
LP Modified	LPM	Mesh-based	One	The Levermore-Pomraning method, with the coupling term treated using an upwinding approach.
LP	LPT	LP	One	The Levermore-Pomraning method.
Mesh	MM	Mesh-based	One	The mesh-based method, using a p_1 and p_2 that aren't spatially dependent.
Mesh Skip	M#	Mesh-based	One	The mesh-based method, with the solver only applying mixing every # cells.
Percentage	PB	Normal	Many	Each realization uses the same percentage of each material, determined by p_1 and p_2, but with the order of the materials randomized.

Table 3.1: Summary of methods

A summary of the solution methods is given in Table 3.1. Those with abbreviations ending in "M" are the mesh-based methods, those ending in "T" are the other transport methods with one realization, while those ending in "B" are benchmark methods (which require many realizations). The methods most often discussed in the literature are the atomic mix (AT), chord benchmarks (CB), and the LP method (LPM).

3.1 Mesh-Based

The mesh-based method uses the equations derived in Sections 2.1.2, 2.3, and 2.4.2. Two methods of calculation for p_n were used. The first (MM) used a simple ratio of the mean chord lengths to compute a

spatially uniform p_n:

$$p_n = \frac{\Lambda_n}{\sum_n \Lambda_n}. \tag{3.1}$$

The second (DM) averages the characteristic function $\chi_n(r)$ over many realizations of a mesh that would be generated for the exponentially distributed solution (see Section 3.4). As can be seen in Equation 3.2, p_n is the ensemble average of the characteristic function, or over many realizations, the probability that material n is located at position r:

$$p_n(r) = \langle \chi_n(r) \rangle \,. \tag{3.2}$$

p_{ni} is the spatial average of this probability in cell i.

Two variations on the mixing term are given. The first applies the mixing term (Equation 2.21) only on the edges and to a less-refined mesh, skipping a predetermined number of cells each time before mixing again. The number of cells between interface mixing is included in the abbreviation for the name (i.e. M5 for mixing every 5 cells or M200 for 200 cells). The second (LPM) uses the modified derivation of the LP equations (Equations 2.55 and 2.56), and employs chord lengths instead of p_n in its calculations.

3.2 Levermore-Pomraning

The Levermore-Pomraning method (LPT) uses the equations in Sections 2.1.1 and 2.4.1. For the coupling term, the angular flux from the previous iteration is used, which once the solution is converged should be within tolerance range of the correct answer. This method is the standard non-benchmark method used, but has been shown to perform poorly in optically thick slabs [4].

3.3 Atomic Mix

The Atomic Mix (AT) method uses the standard transport equation (Equation 2.2) but with spatially homogenized cross sections:

$$\Sigma_t = \sum_n p_n \Sigma_t^n, \tag{3.3}$$

$$\Sigma_s = \sum_n p_n \Sigma_s^n. \tag{3.4}$$

While by far the simplest to implement in code, this method has also been shown to be poor at solving the stochastic problem when one material is optically thick and has short chord lengths [13].

3.4 Benchmarks

For the benchmark methods, the solution is computed for many realizations of the chosen mixing statistics, and then the ensemble-average solution is calculated as the average of these realizations.

For the chord method (CB), large sections of material made up each realization of the mesh. In slab geometry, the chunks of material of length L are taken to be a Markovian distribution, with the probability distribution for each material n given by [19]:

$$f_n(L) = \frac{1}{\Lambda_n} e^{-L/\Lambda_n}. \tag{3.5}$$

Starting at one of the two edges randomly, material chunk size is calculated from the distribution in Equation 3.5. Then a chunk size is sampled from a randomized different material and added on to the length. This continues until the desired slab length is reached. If the final slab extends beyond the a problem boundary, the realization is terminated at the predetermined slab length.

For the exponential method (EB), the same procedure is followed, but after a mesh is populated with the materials, the order of the cells is randomized, which causes many of the large slabs to be broken up into smaller pieces. This process tests whether varying magnitudes in the angular flux are due to the varying amount of each material in each problem or due to the large slabs of the lower-absorbing materials.

Finally, for the percentage method (PB), a simple percentage p_n of the cells is allocated to each material. Where I is the total number of cells, I_n (the total number of cells for material n) is:

$$I_n = p_n I. \tag{3.6}$$

The order of the cells (and thus the order of the materials) is shuffled after each realization.

4 Numerical Results

Material					Scattering			Length
Case	Σ_t^0	Λ_0	Σ_t^1	Λ_1	Case	c_0	c_1	
1	10/99	99/100	100/11	11/100	1	0.0	1.0	0.1
2	10/99	99/10	100/11	11/10	2	1.0	0.0	1.0
3	2/101	101/20	200/101	101/20	3	0.9	0.9	10.0

Table 4.1: Material parameters

Numerical results were generated for nine different methods of stochastic transport, which are summarized in Table 3.1. In all cases a S_{16} Gauss-Legendre quadrature set was used for the angular discretization. All of the results used 2,000 spatial cells and the benchmark cases were generated from 5,000 realizations of the mixing statistics. The tolerance used for $\widetilde{\phi}^n$was 10^{-8}.

False convergence was prevented by using the following scheme:

$$\left|\widetilde{\phi}_{i,it}^n - \widetilde{\phi}_{i,it-1}^n\right| < 10^{-8}\left(1 - \frac{\left|\widetilde{\phi}_{i,it}^n - \widetilde{\phi}_{i,it-1}^n\right|}{\left|\widetilde{\phi}_{i,it-1}^n - \widetilde{\phi}_{i,it-2}^n\right|}\right). \tag{4.1}$$

Here $\widetilde{\phi}_{i,it}^n$ refers to the nodal flux in each cell at iteration it, and $|\cdots|$ refers to the absolute value.

The material data and error calculations follow those of the benchmarks by Brantley and Palmer [4]. Each of the solution methods was used on each of the 27 permutations of material, scattering, and length cases listed in Table 4.1. Numerical results for the leakage and error in ϕ, ϕ_0, ϕ_1, leakage, and standard errors of these quantities are given for each case.

For the boundary source problems, reflection refers to the outgoing partial current at the left boundary:

$$\langle J_0 \rangle = \int_{-1}^{0} |\mu| \langle \psi(0,\mu) \rangle \, d\mu. \tag{4.2}$$

The leakage for the both types of problems at the right boundary is defined as:

$$\langle J_L \rangle = \int_{0}^{1} \mu \langle \psi(L,\mu) \rangle \, d\mu. \tag{4.3}$$

Given the symmetric nature of the problem, the leakage at the left boundary for the internal source problems is equal to the leakage at the right hand boundary, assuming perfect convergence for all methods and an infinite number of realizations for the benchmark methods. As such, only the right hand leakage results are presented for the internal source problems.

The error for the leakage and flux is computed as:

$$E_{\langle J_L \rangle} = \frac{\langle J_L \rangle_{method} - \langle J_L \rangle_{chord}}{\langle J_L \rangle_{chord}}. \tag{4.4}$$

$$E_{\langle \phi \rangle} = \sqrt{\frac{1}{I} \sum_{i=0}^{I-1} \left(\frac{\langle \phi_{i,method} \rangle - \langle \phi_{i,benchmark} \rangle}{\langle \phi_{i,benchmark} \rangle} \right)^2}. \tag{4.5}$$

Here the summation is over the I spatial cells. The errors in $\langle \phi_0 \rangle$ and $\langle \phi_1 \rangle$ are calculated as in Equation 4.5, except for in the atomic mix (AT) method, which doesn't calculate those quantities.

Where A is the total number of realizations for the benchmark solutions, the standard error was calculated as:

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, \tag{4.6}$$

$$\langle x \rangle = \frac{1}{A} \sum_{a=0}^{A-1} x_a, \tag{4.7}$$

$$\langle x^2 \rangle = \frac{1}{A} \sum_{a=0}^{A-1} x_a^2. \tag{4.8}$$

4.1 Boundary Source Problems

For the boundary source problems, the source used is an isotropic incident flux:

$$\psi_{m,inc} = 2, \ \mu_m > 0. \tag{4.9}$$

Tables 4.3 to 4.5 include data for the reflection and leakage results for the various methods and their corresponding errors, while Tables 4.6 to 4.8 give the error for $\langle \phi \rangle$, $\langle \phi_0 \rangle$ and $\langle \phi_1 \rangle$. All errors are computed from the chord length benchmark.

Figures 4.1 to 4.9 include plots for $\langle \phi \rangle$, $\langle \phi_0 \rangle$ and $\langle \phi_1 \rangle$ for $L = 10.0$ and each of the material and scattering cases in Table 4.1, with the internal source in Equation 4.9.

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	-0.0768	-0.0459	-0.0768	-0.0456	-0.0452	-0.0768	-0.0765
1	1	1.0	-0.3627	-0.2251	-0.3279	-0.2314	-0.2014	-0.3618	-0.3615
1	1	10.0	-0.4982	-0.4261	-0.4946	-0.4790	-0.3194	-0.4862	-0.4953
1	2	0.1	-0.0067	-0.0088	-0.0068	-0.0087	-0.0082	-0.0068	-0.0068
1	2	1.0	-0.0190	-0.0583	-0.0215	-0.0481	-0.0429	-0.0191	-0.0193
1	2	10.0	-0.0199	-0.0968	-0.0196	-0.0270	-0.0547	-0.0212	-0.0223
1	3	0.1	-0.0760	-0.0457	-0.0759	-0.0456	-0.0488	-0.0759	-0.0757
1	3	1.0	-0.3552	-0.2384	-0.3324	-0.2396	-0.2518	-0.3548	-0.3545
1	3	10.0	-0.4807	-0.4797	-0.4823	-0.4801	-0.4944	-0.4823	-0.4807
2	1	0.1	-0.0768	-0.0439	-0.0748	-0.0453	-0.0385	-0.0768	-0.0768
2	1	1.0	-0.3627	-0.1112	-0.3611	-0.1133	-0.0778	-0.3618	-0.3626
2	1	10.0	-0.4982	-0.2081	-0.4708	-0.2925	-0.1294	-0.4862	-0.4962
2	2	0.1	-0.0067	-0.0089	-0.0068	-0.0088	-0.0084	-0.0068	-0.0068
2	2	1.0	-0.0190	-0.0755	-0.0197	-0.0732	-0.0645	-0.0191	-0.0192
2	2	10.0	-0.0199	-0.3061	-0.0218	-0.1973	-0.1851	-0.0212	-0.0222
2	3	0.1	-0.0760	-0.0432	-0.0742	-0.0443	-0.0430	-0.0759	-0.0760
2	3	1.0	-0.3552	-0.1373	-0.3548	-0.1382	-0.1438	-0.3548	-0.3552
2	3	10.0	-0.4807	-0.4257	-0.4823	-0.4273	-0.4376	-0.4823	-0.4807
3	1	0.1	-0.0848	-0.0779	-0.0856	-0.0779	-0.0751	-0.0848	-0.0848
3	1	1.0	-0.4387	-0.3165	-0.4369	-0.3200	-0.2766	-0.4386	-0.4387
3	1	10.0	-0.7891	-0.6863	-0.7868	-0.7007	-0.4873	-0.7877	-0.7891
3	2	0.1	-0.0007	-0.0010	-0.0007	-0.0010	-0.0009	-0.0007	-0.0007
3	2	1.0	-0.0020	-0.0090	-0.0020	-0.0086	-0.0054	-0.0020	-0.0020
3	2	10.0	-0.0021	-0.0376	-0.0021	-0.0169	-0.0131	-0.0021	-0.0021
3	3	0.1	-0.0760	-0.0687	-0.0766	-0.0688	-0.0677	-0.0759	-0.0760
3	3	1.0	-0.3552	-0.2398	-0.3541	-0.2424	-0.2474	-0.3551	-0.3552
3	3	10.0	-0.4807	-0.4427	-0.4809	-0.4451	-0.4660	-0.4809	-0.4807

Table 4.2: Reflection, boundary source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	0.9083	0.9392	0.9083	0.9394	0.9388	0.9083	0.9086
1	1	1.0	0.4814	0.6201	0.5146	0.6127	0.6176	0.4818	0.4824
1	1	10.0	0.0047	0.0162	0.0046	0.0090	0.0148	0.0047	0.0049
1	2	0.1	0.8392	0.9098	0.8393	0.9100	0.9027	0.8393	0.8399
1	2	1.0	0.2307	0.5281	0.2810	0.5238	0.4819	0.2316	0.2322
1	2	10.0	0.0000	0.0026	0.0000	0.0013	0.0015	0.0000	0.0000
1	3	0.1	0.9074	0.9400	0.9075	0.9404	0.9425	0.9075	0.9078
1	3	1.0	0.4752	0.6332	0.5196	0.6316	0.6651	0.4761	0.4765
1	3	10.0	0.0039	0.0179	0.0044	0.0148	0.0516	0.0044	0.0040
2	1	0.1	0.9083	0.9413	0.9103	0.9399	0.9454	0.9083	0.9083
2	1	1.0	0.4814	0.7360	0.4820	0.7332	0.7531	0.4818	0.4815
2	1	10.0	0.0047	0.1114	0.0069	0.0975	0.1158	0.0047	0.0049
2	2	0.1	0.8392	0.9162	0.8432	0.9137	0.9163	0.8393	0.8392
2	2	1.0	0.2307	0.7766	0.2316	0.7744	0.7530	0.2316	0.2307
2	2	10.0	0.0000	0.2297	0.0000	0.2201	0.1592	0.0000	0.0000
2	3	0.1	0.9074	0.9419	0.9097	0.9403	0.9499	0.9075	0.9074
2	3	1.0	0.4752	0.7826	0.4761	0.7809	0.8189	0.4761	0.4752
2	3	10.0	0.0039	0.2239	0.0091	0.2199	0.2981	0.0044	0.0039
3	1	0.1	0.9162	0.9233	0.9155	0.9232	0.9256	0.9162	0.9162
3	1	1.0	0.5448	0.6672	0.5464	0.6638	0.6947	0.5449	0.5448
3	1	10.0	0.0668	0.1708	0.0666	0.1515	0.2388	0.0666	0.0668
3	2	0.1	0.8333	0.8493	0.8318	0.8493	0.8528	0.8333	0.8333
3	2	1.0	0.2206	0.4945	0.2227	0.4886	0.4809	0.2207	0.2206
3	2	10.0	0.0000	0.0891	0.0000	0.0822	0.0698	0.0000	0.0000
3	3	0.1	0.9074	0.9145	0.9066	0.9144	0.9183	0.9074	0.9074
3	3	1.0	0.4752	0.6128	0.4773	0.6089	0.6724	0.4753	0.4752
3	3	10.0	0.0039	0.1155	0.0039	0.1114	0.2455	0.0039	0.0039

Table 4.3: Leakage, boundary source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	0.6746	-	0.6736	-0.0067	-0.0150	0.6737	0.6671
1	1	1.0	0.6115	-	0.4567	0.0282	-0.1055	0.6073	0.6062
1	1	10.0	0.1692	-	0.1608	0.1242	-0.2503	0.1409	0.1623
1	2	0.1	-0.2299	-	-0.2279	-0.0088	-0.0693	-0.2296	-0.2282
1	2	1.0	-0.6735	-	-0.6308	-0.1748	-0.2641	-0.6715	-0.6691
1	2	10.0	-0.7948	-	-0.7976	-0.7213	-0.4351	-0.7809	-0.7694
1	3	0.1	0.6602	-	0.6596	-0.0030	0.0671	0.6595	0.6538
1	3	1.0	0.4899	-	0.3942	0.0051	0.0561	0.4884	0.4872
1	3	10.0	0.0021	-	0.0054	0.0008	0.0306	0.0054	0.0021
2	1	0.1	0.7515	-	0.7046	0.0330	-0.1216	0.7506	0.7514
2	1	1.0	2.2630	-	2.2484	0.0192	-0.3000	2.2546	2.2620
2	1	10.0	1.3936	-	1.2621	0.4052	-0.3781	1.3358	1.3840
2	2	0.1	-0.2411	-	-0.2325	-0.0050	-0.0587	-0.2408	-0.2408
2	2	1.0	-0.7482	-	-0.7397	-0.0304	-0.1458	-0.7467	-0.7460
2	2	10.0	-0.9351	-	-0.9287	-0.3556	-0.3952	-0.9307	-0.9275
2	3	0.1	0.7588	-	0.7181	0.0259	-0.0054	0.7581	0.7588
2	3	1.0	1.5867	-	1.5841	0.0068	0.0473	1.5841	1.5867
2	3	10.0	0.1291	-	0.1328	0.0036	0.0279	0.1329	0.1291
3	1	0.1	0.0898	-	0.0992	0.0003	-0.0353	0.0898	0.0898
3	1	1.0	0.3861	-	0.3803	0.0110	-0.1259	0.3857	0.3861
3	1	10.0	0.1497	-	0.1464	0.0209	-0.2900	0.1477	0.1496
3	2	0.1	-0.2425	-	-0.2519	-0.0002	-0.1008	-0.2424	-0.2425
3	2	1.0	-0.7800	-	-0.7753	-0.0445	-0.4001	-0.7797	-0.7796
3	2	10.0	-0.9452	-	-0.9440	-0.5508	-0.6529	-0.9444	-0.9436
3	3	0.1	0.1048	-	0.1139	0.0002	-0.0159	0.1048	0.1048
3	3	1.0	0.4814	-	0.4770	0.0108	0.0318	0.4812	0.4814
3	3	10.0	0.0860	-	0.0864	0.0056	0.0527	0.0864	0.0860

Table 4.4: Error in reflection, boundary source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	-0.0329	-	-0.0329	0.0003	-0.0004	-0.0328	-0.0325
1	1	1.0	-0.2237	-	-0.1701	-0.0120	-0.0039	-0.2229	-0.2219
1	1	10.0	-0.7080	-	-0.7170	-0.4461	-0.0883	-0.7136	-0.6961
1	2	0.1	-0.0776	-	-0.0776	0.0002	-0.0078	-0.0776	-0.0769
1	2	1.0	-0.5632	-	-0.4680	-0.0081	-0.0875	-0.5615	-0.5604
1	2	10.0	-0.9967	-	-0.9960	-0.4810	-0.4265	-0.9958	-0.9965
1	3	0.1	-0.0346	-	-0.0346	0.0005	0.0027	-0.0346	-0.0342
1	3	1.0	-0.2496	-	-0.1795	-0.0026	0.0503	-0.2481	-0.2475
1	3	10.0	-0.7843	-	-0.7557	-0.1723	1.8846	-0.7518	-0.7791
2	1	0.1	-0.0351	-	-0.0330	-0.0015	0.0043	-0.0350	-0.0351
2	1	1.0	-0.3460	-	-0.3451	-0.0038	0.0233	-0.3453	-0.3459
2	1	10.0	-0.9574	-	-0.9383	-0.1255	0.0392	-0.9582	-0.9562
2	2	0.1	-0.0840	-	-0.0796	-0.0027	0.0001	-0.0839	-0.0840
2	2	1.0	-0.7029	-	-0.7017	-0.0028	-0.0303	-0.7018	-0.7029
2	2	10.0	-1.0000	-	-0.9998	-0.0418	-0.3070	-1.0000	-1.0000
2	3	0.1	-0.0366	-	-0.0342	-0.0018	0.0085	-0.0366	-0.0366
2	3	1.0	-0.3928	-	-0.3916	-0.0021	0.0464	-0.3916	-0.3928
2	3	10.0	-0.9828	-	-0.9596	-0.0176	0.3318	-0.9802	-0.9828
3	1	0.1	-0.0076	-	-0.0084	-0.0000	0.0026	-0.0076	-0.0076
3	1	1.0	-0.1834	-	-0.1810	-0.0051	0.0412	-0.1833	-0.1834
3	1	10.0	-0.6090	-	-0.6102	-0.1132	0.3980	-0.6100	-0.6086
3	2	0.1	-0.0189	-	-0.0206	-0.0000	0.0041	-0.0189	-0.0189
3	2	1.0	-0.5539	-	-0.5496	-0.0120	-0.0275	-0.5537	-0.5539
3	2	10.0	-0.9999	-	-0.9999	-0.0772	-0.2170	-0.9999	-0.9999
3	3	0.1	-0.0077	-	-0.0086	-0.0000	0.0042	-0.0077	-0.0077
3	3	1.0	-0.2245	-	-0.2211	-0.0063	0.0973	-0.2243	-0.2245
3	3	10.0	-0.9666	-	-0.9662	-0.0352	1.1252	-0.9660	-0.9666

Table 4.5: Error in leakage, boundary source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	0.0383	-	0.0396	0.0003	0.0033	0.0382	0.0380
1	1	1.0	0.0969	-	0.0754	0.0073	0.0187	0.0967	0.0965
1	1	10.0	0.3862	-	0.4097	0.2813	0.1686	0.4022	0.3762
1	2	0.1	0.0926	-	0.0927	0.0038	0.0065	0.0924	0.0918
1	2	1.0	0.4908	-	0.4310	0.1071	0.0700	0.4890	0.4879
1	2	10.0	0.9167	-	0.9168	0.7262	0.3859	0.9115	0.9115
1	3	0.1	0.0394	-	0.0407	0.0006	0.0052	0.0393	0.0391
1	3	1.0	0.1216	-	0.0922	0.0271	0.0453	0.1210	0.1207
1	3	10.0	0.5365	-	0.5155	0.2734	0.9570	0.5054	0.5275
2	1	0.1	0.0413	-	0.0398	0.0009	0.0068	0.0413	0.0414
2	1	1.0	0.1689	-	0.1737	0.0046	0.0343	0.1687	0.1691
2	1	10.0	0.6258	-	0.6024	0.2469	0.0576	0.6332	0.6225
2	2	0.1	0.0976	-	0.0934	0.0029	0.0006	0.0974	0.0975
2	2	1.0	0.5812	-	0.5778	0.0236	0.0294	0.5797	0.5802
2	2	10.0	0.9592	-	0.9528	0.2736	0.2642	0.9573	0.9573
2	3	0.1	0.0426	-	0.0410	0.0011	0.0088	0.0425	0.0426
2	3	1.0	0.1959	-	0.1998	0.0108	0.0644	0.1954	0.1961
2	3	10.0	0.7790	-	0.7241	0.1938	0.2357	0.7697	0.7771
3	1	0.1	0.0130	-	0.0136	0.0001	0.0031	0.0130	0.0130
3	1	1.0	0.0942	-	0.0925	0.0033	0.0474	0.0941	0.0943
3	1	10.0	0.2374	-	0.2373	0.0779	0.1808	0.2397	0.2374
3	2	0.1	0.0296	-	0.0311	0.0000	0.0039	0.0295	0.0296
3	2	1.0	0.4368	-	0.4321	0.0227	0.0221	0.4366	0.4368
3	2	10.0	0.9335	-	0.9325	0.4159	0.1643	0.9332	0.9332
3	3	0.1	0.0135	-	0.0141	0.0001	0.0041	0.0135	0.0135
3	3	1.0	0.1171	-	0.1149	0.0050	0.0971	0.1170	0.1172
3	3	10.0	0.6922	-	0.6880	0.2236	0.9953	0.6903	0.6919

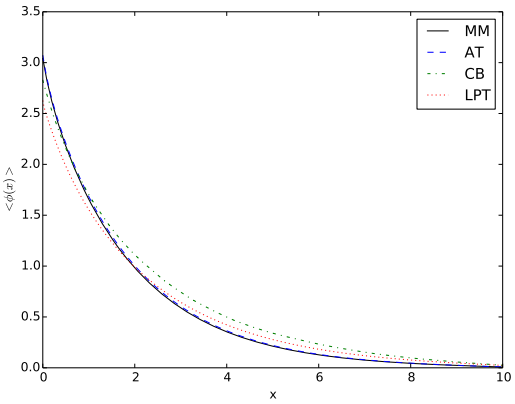
Table 4.6: Error in $\langle \phi(x) \rangle$, boundary source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	-	-	0.1121	0.0025	0.1003	0.1114	0.0558
1	1	1.0	-	-	0.1108	0.0188	0.1096	0.1329	0.0959
1	1	10.0	-	-	0.4468	0.2544	0.2353	0.4392	0.3638
1	2	0.1	-	-	0.2171	0.0074	0.1045	0.2170	0.1356
1	2	1.0	-	-	0.5044	0.1096	0.1554	0.5610	0.5153
1	2	10.0	-	-	0.9285	0.7304	0.4465	0.9236	0.9167
1	3	0.1	-	-	0.1240	0.0028	0.1000	0.1234	0.0572
1	3	1.0	-	-	0.1633	0.0321	0.0784	0.1987	0.1307
1	3	10.0	-	-	0.5705	0.2748	0.7570	0.5608	0.5399
2	1	0.1	-	-	0.1127	0.0006	0.1002	0.1158	0.0673
2	1	1.0	-	-	0.2103	0.0040	0.1021	0.2060	0.2014
2	1	10.0	-	-	0.6137	0.1783	0.1153	0.6473	0.6138
2	2	0.1	-	-	0.2237	0.0010	0.1013	0.2295	0.1507
2	2	1.0	-	-	0.6456	0.0269	0.1226	0.6475	0.6130
2	2	10.0	-	-	0.9595	0.2735	0.3284	0.9635	0.9601
2	3	0.1	-	-	0.1255	0.0006	0.1002	0.1289	0.0683
2	3	1.0	-	-	0.2781	0.0149	0.0952	0.2757	0.2297
2	3	10.0	-	-	0.7552	0.1927	0.0966	0.7975	0.7871
3	1	0.1	-	-	0.5055	0.0006	0.4999	0.5005	0.0748
3	1	1.0	-	-	0.5066	0.0081	0.4999	0.5108	0.2325
3	1	10.0	-	-	0.5443	0.0713	0.4836	0.5454	0.1912
3	2	0.1	-	-	0.5784	0.0009	0.5001	0.5736	0.1626
3	2	1.0	-	-	0.8001	0.0241	0.5100	0.8032	0.6268
3	2	10.0	-	-	0.9757	0.4286	0.5799	0.9760	0.9563
3	3	0.1	-	-	0.5146	0.0007	0.4999	0.5096	0.0763
3	3	1.0	-	-	0.5753	0.0086	0.5000	0.5796	0.2642
3	3	10.0	-	-	0.8794	0.2422	0.3777	0.8808	0.7831

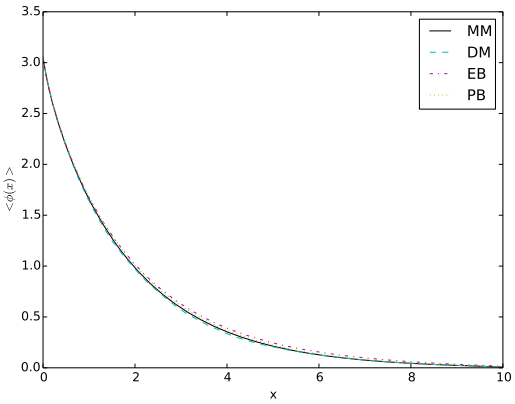
Table 4.7: Error in $\langle \phi_0(x) \rangle$, boundary source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	-	-	0.8984	0.0288	0.8997	0.8985	0.1496
1	1	1.0	-	-	0.9216	0.2121	0.9063	0.9069	0.1323
1	1	10.0	-	-	0.9434	0.4668	0.9261	0.9430	0.4623
1	2	0.1	-	-	0.8108	0.0635	0.8968	0.8114	0.9859
1	2	1.0	-	-	0.8581	0.4053	0.9011	0.8483	0.6687
1	2	10.0	-	-	0.9686	0.6841	0.9167	0.9674	0.8353
1	3	0.1	-	-	0.8870	0.0319	0.8934	0.8870	0.2197
1	3	1.0	-	-	0.8983	0.2499	0.8765	0.8817	0.2432
1	3	10.0	-	-	0.9231	0.3407	0.7686	0.9217	0.4318
2	1	0.1	-	-	0.8993	0.0058	0.8993	0.8963	0.2240
2	1	1.0	-	-	0.9042	0.1227	0.8994	0.9041	0.2234
2	1	10.0	-	-	0.9688	0.7596	0.9313	0.9656	0.6988
2	2	0.1	-	-	0.7698	0.0205	0.8999	0.7642	1.6271
2	2	1.0	-	-	0.3999	0.4837	0.8849	0.3893	6.5629
2	2	10.0	-	-	0.9176	1.2410	0.8921	0.9207	2.2488
2	3	0.1	-	-	0.8842	0.0074	0.8907	0.8808	0.3398
2	3	1.0	-	-	0.7119	0.2607	0.6877	0.7133	2.1330
2	3	10.0	-	-	0.8781	0.8634	0.5220	0.8799	1.2844
3	1	0.1	-	-	0.4935	0.0006	0.5000	0.4984	0.0476
3	1	1.0	-	-	0.4986	0.0148	0.4954	0.4962	0.0828
3	1	10.0	-	-	0.6309	0.1989	0.5806	0.6326	0.2939
3	2	0.1	-	-	0.4239	0.0009	0.4999	0.4287	0.1643
3	2	1.0	-	-	0.2196	0.0625	0.4858	0.2123	0.9406
3	2	10.0	-	-	0.8470	0.6606	0.4860	0.8484	0.8700
3	3	0.1	-	-	0.4848	0.0006	0.4977	0.4897	0.0548
3	3	1.0	-	-	0.4109	0.0202	0.3929	0.4085	0.2405
3	3	10.0	-	-	0.6198	0.4246	1.5936	0.6227	0.5376

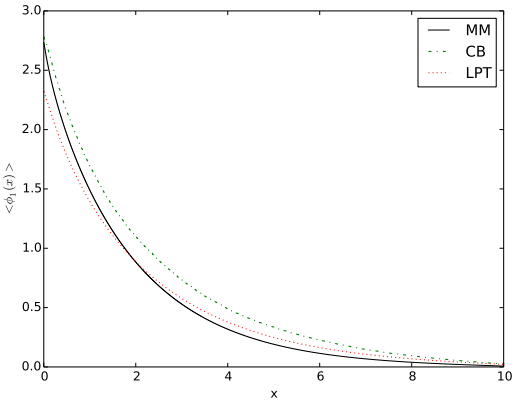
Table 4.8: Error in $\langle \phi_1(x) \rangle$, boundary source problems



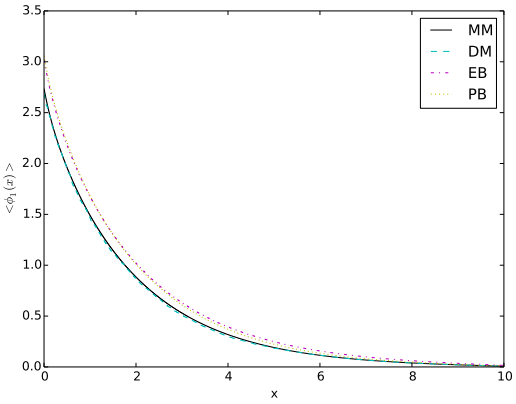
(a) $\langle \phi(x) \rangle$, standard methods



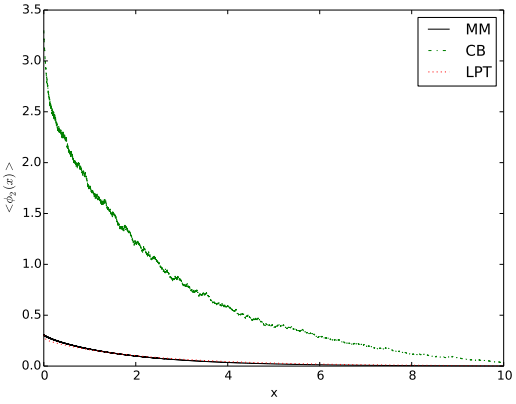
(b) $\langle \phi(x) \rangle$, comparison methods



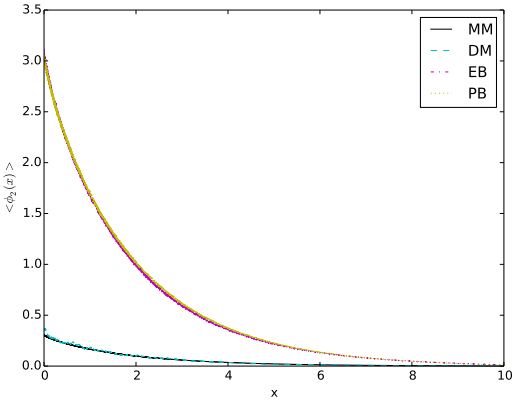
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

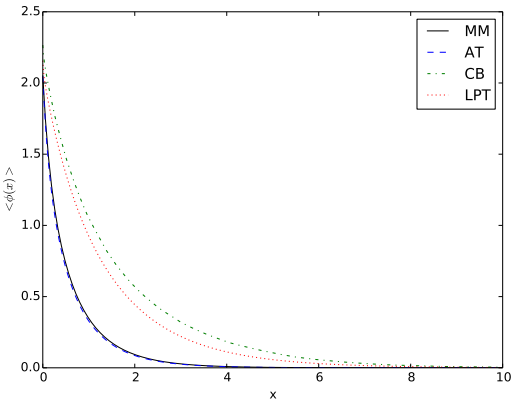


(e) $\langle \phi_1(x) \rangle$, standard methods

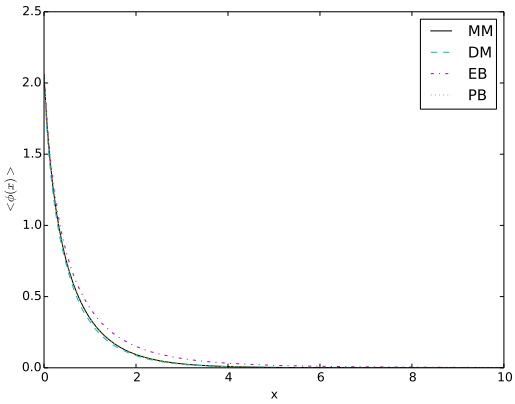


(f) $\langle \phi_1(x) \rangle$, comparison methods

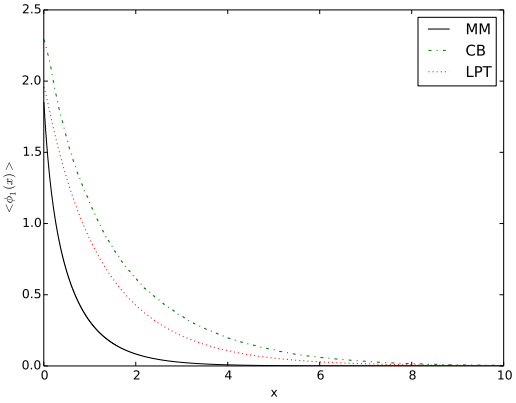
Figure 4.1: Results for M1, S1, L10.0, boundary source problems



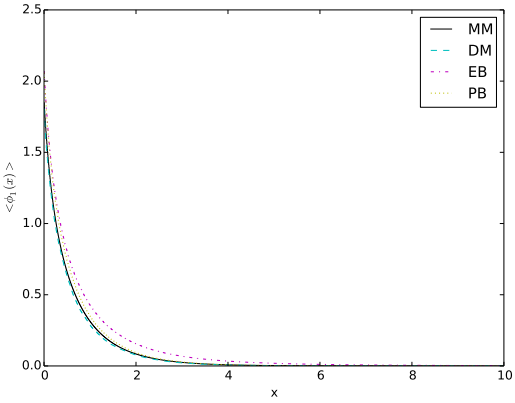
(a) $\langle \phi(x) \rangle$, standard methods



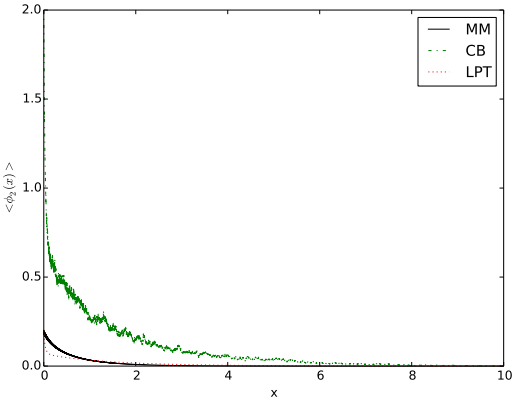
(b) $\langle \phi(x) \rangle$, comparison methods



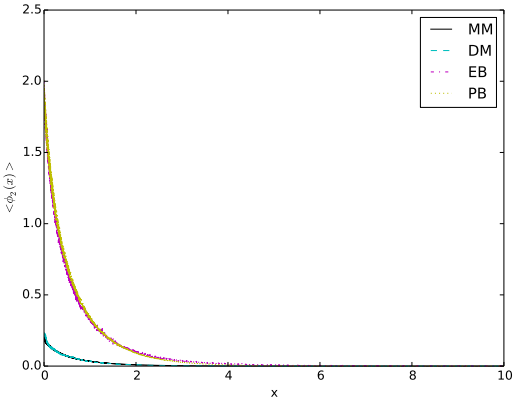
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

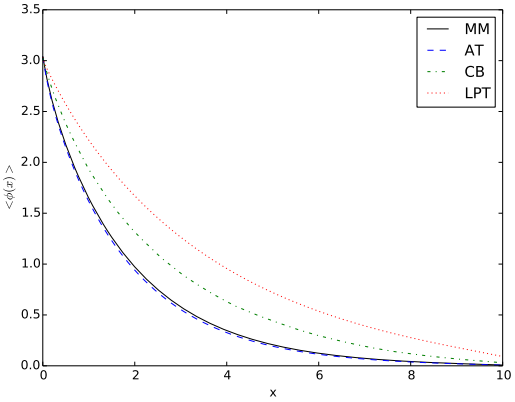


(e) $\langle \phi_1(x) \rangle$, standard methods

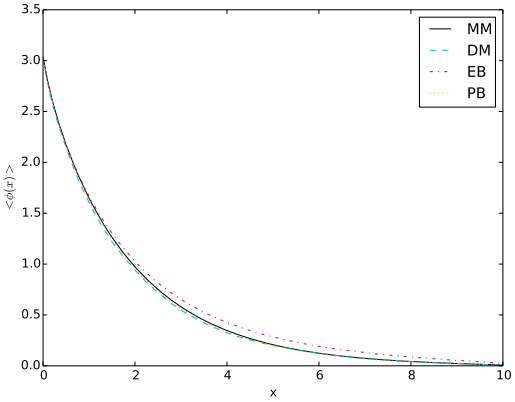


(f) $\langle \phi_1(x) \rangle$, comparison methods

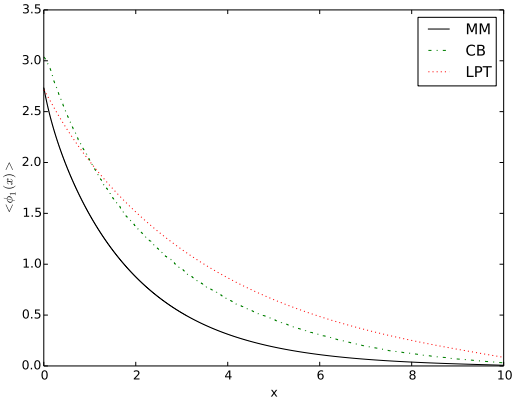
Figure 4.2: Results for M1, S2, L10.0, boundary source problems



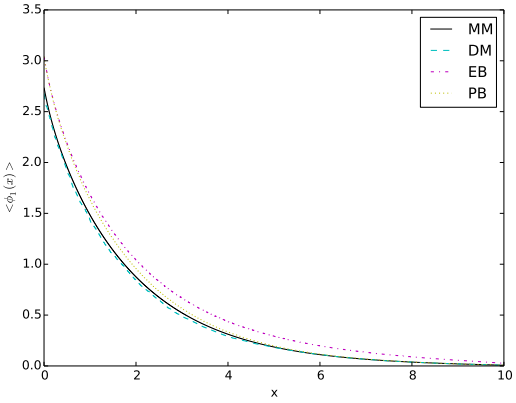
(a) $\langle \phi(x) \rangle$, standard methods



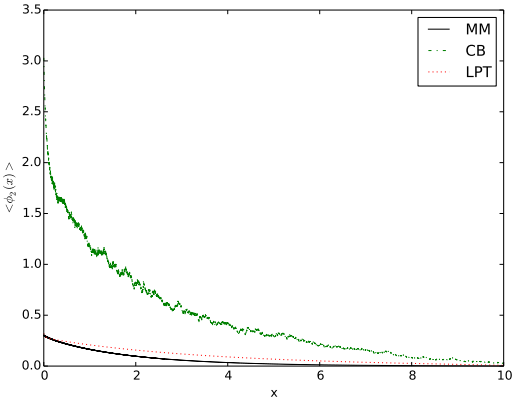
(b) $\langle \phi(x) \rangle$, comparison methods



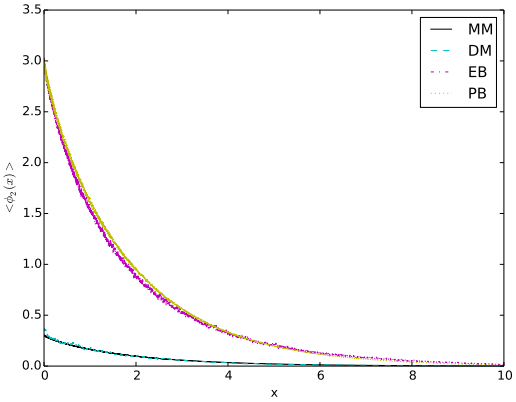
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

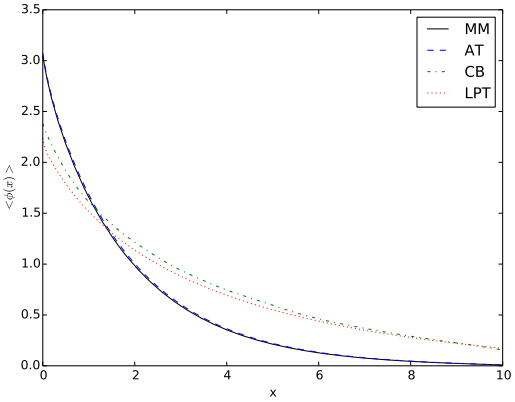


(e) $\langle \phi_1(x) \rangle$, standard methods

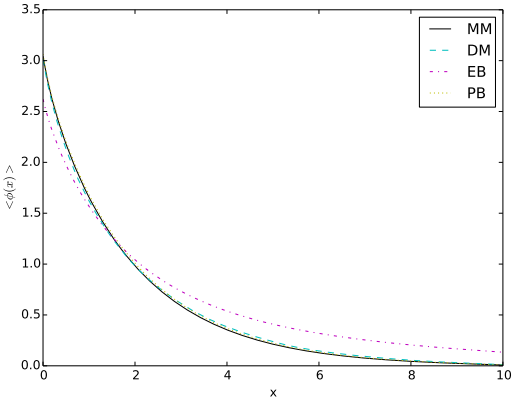


(f) $\langle \phi_1(x) \rangle$, comparison methods

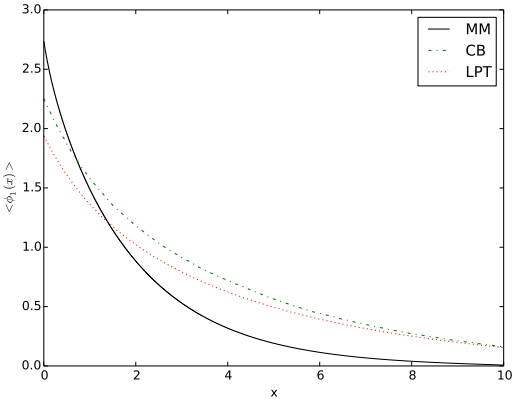
Figure 4.3: Results for M1, S3, L10.0, boundary source problems



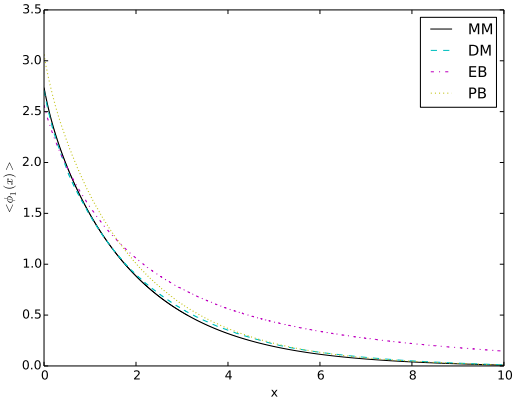
(a) $\langle \phi(x) \rangle$, standard methods



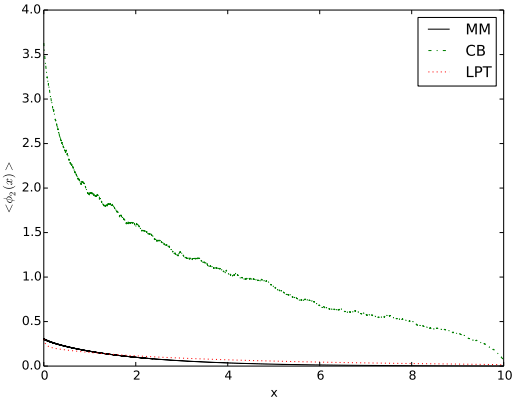
(b) $\langle \phi(x) \rangle$, comparison methods



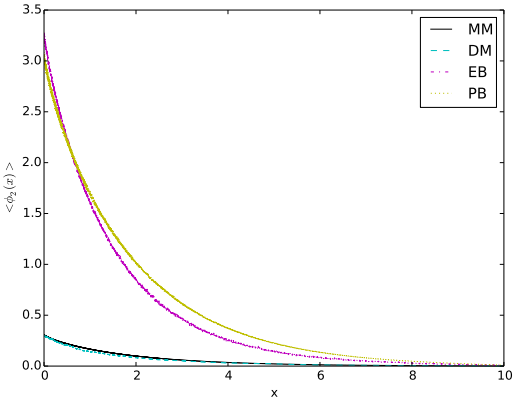
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

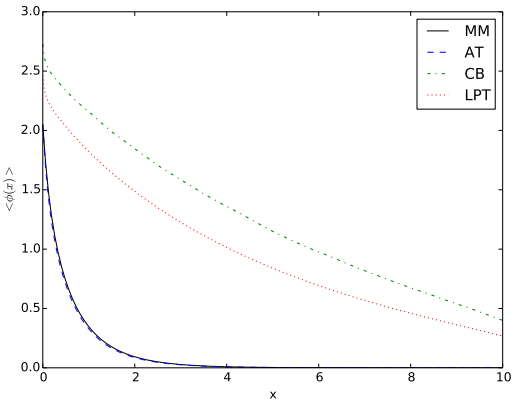


(e) $\langle \phi_1(x) \rangle$, standard methods

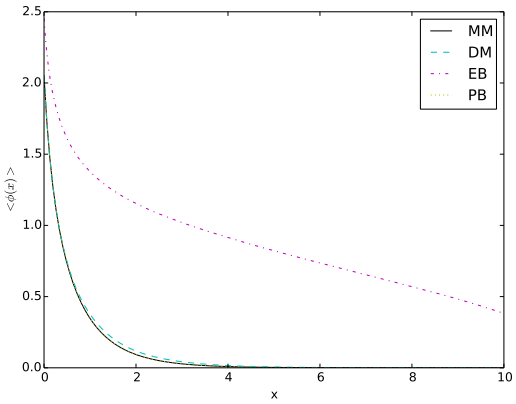


(f) $\langle \phi_1(x) \rangle$, comparison methods

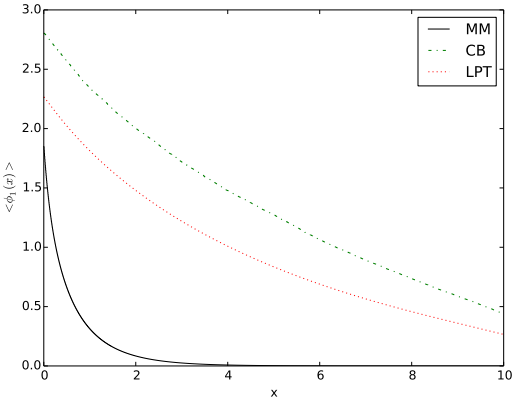
Figure 4.4: Results for M2, S1, L10.0, boundary source problems



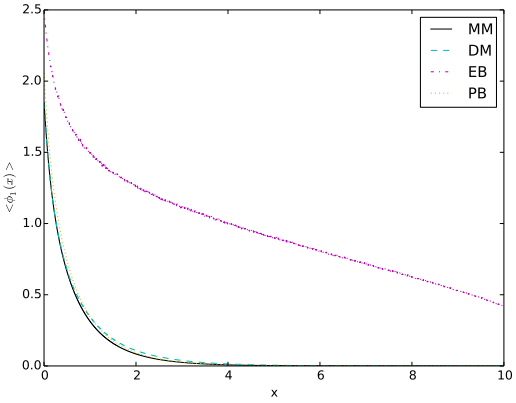
(a) $\langle \phi(x) \rangle$, standard methods



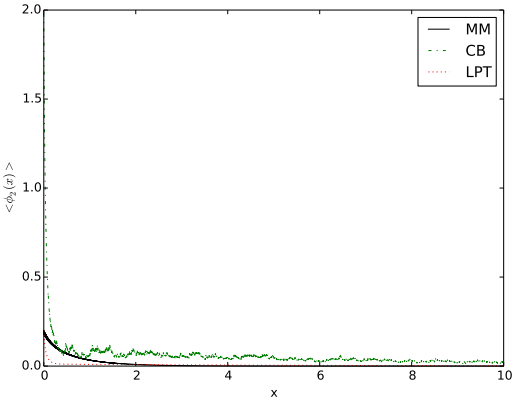
(b) $\langle \phi(x) \rangle$, comparison methods



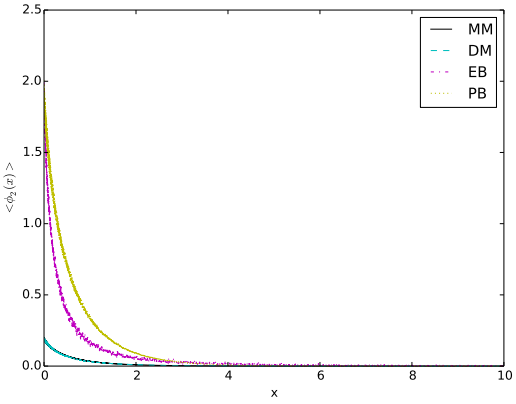
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

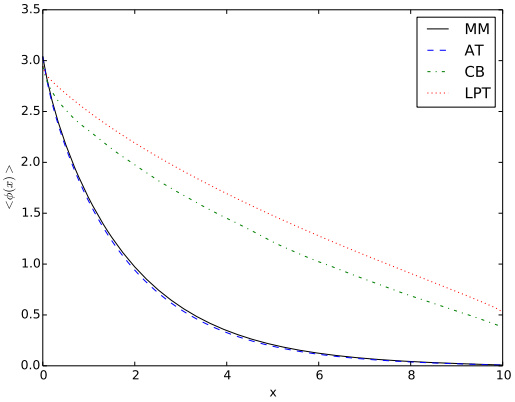


(e) $\langle \phi_1(x) \rangle$, standard methods

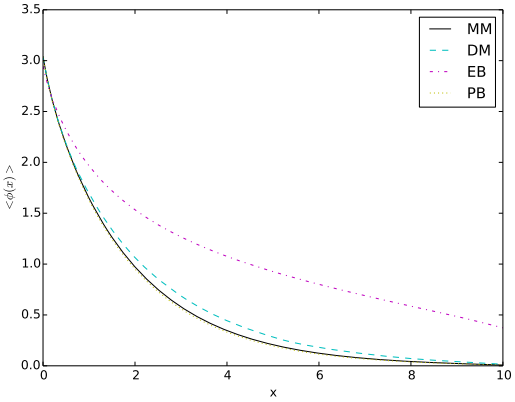


(f) $\langle \phi_1(x) \rangle$, comparison methods

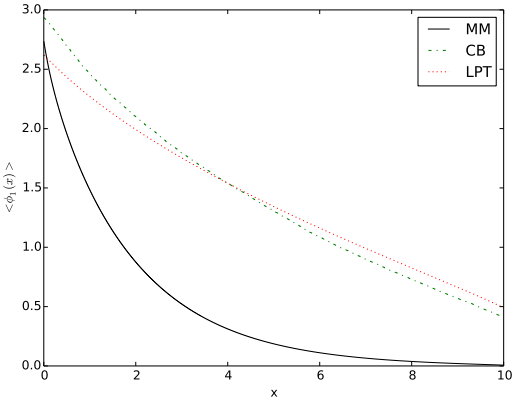
Figure 4.5: Results for M2, S2, L10.0, boundary source problems



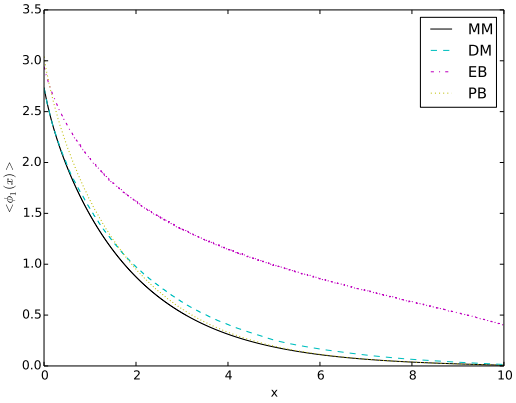
(a) $\langle \phi(x) \rangle$, standard methods



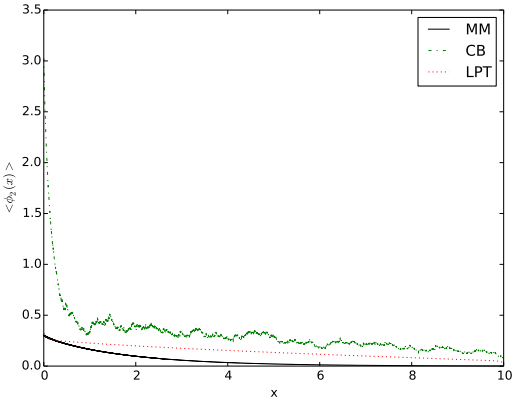
(b) $\langle \phi(x) \rangle$, comparison methods



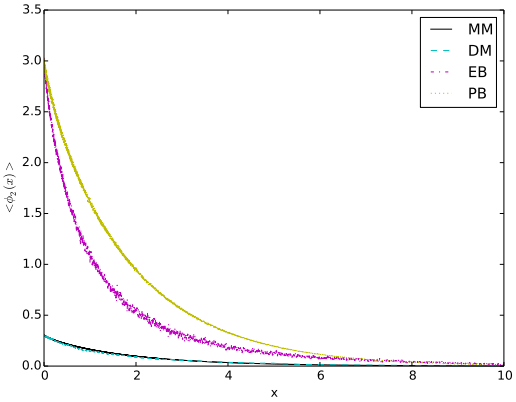
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

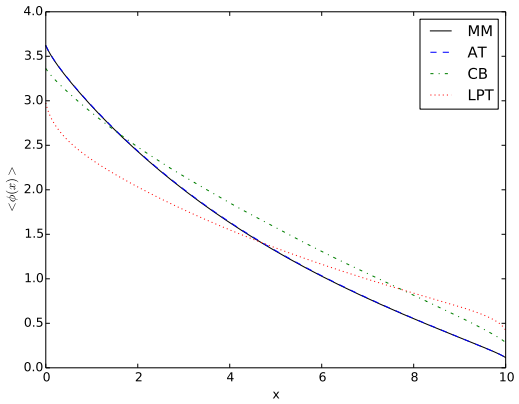


(e) $\langle \phi_1(x) \rangle$, standard methods

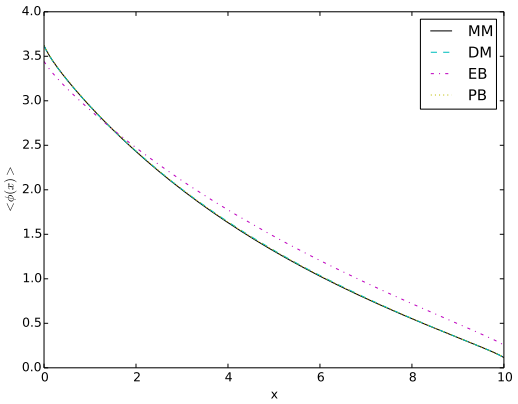


(f) $\langle \phi_1(x) \rangle$, comparison methods

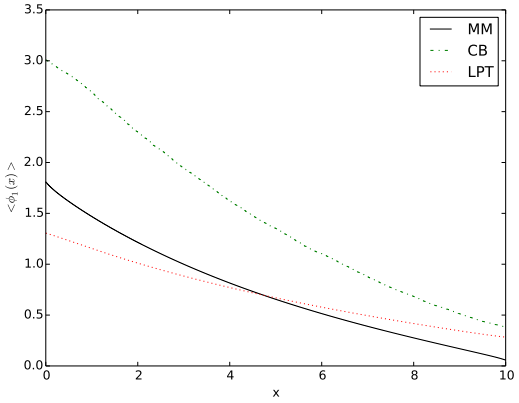
Figure 4.6: Results for M2, S3, L10.0, boundary source problems



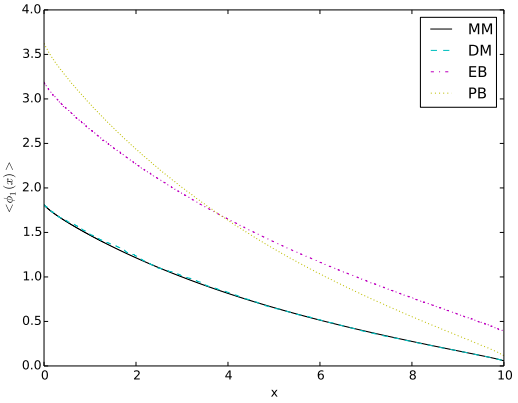
(a) $\langle \phi(x) \rangle$, standard methods



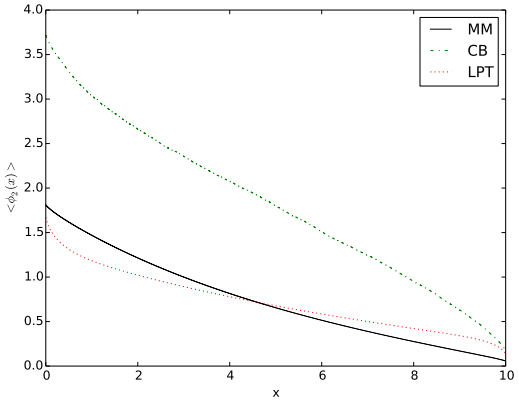
(b) $\langle \phi(x) \rangle$, comparison methods



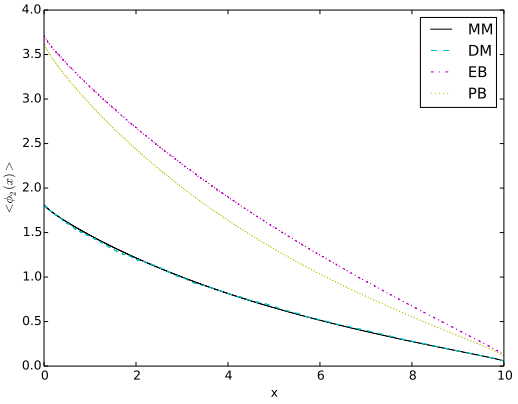
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

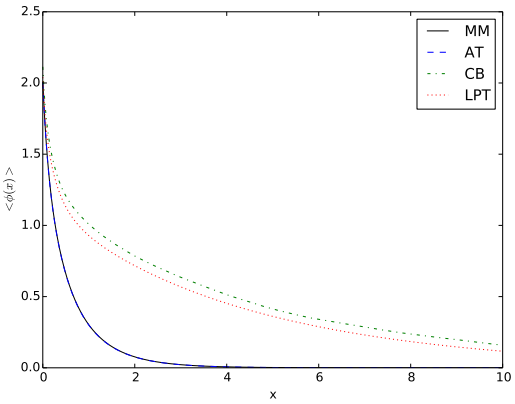


(e) $\langle \phi_1(x) \rangle$, standard methods

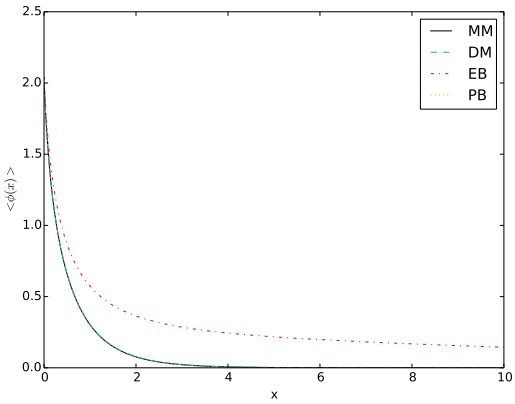


(f) $\langle \phi_1(x) \rangle$, comparison methods

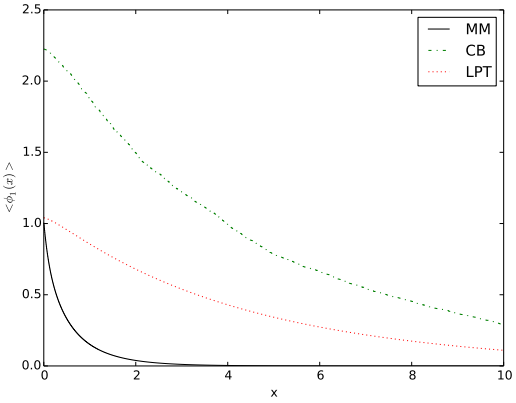
Figure 4.7: Results for M3, S1, L10.0, boundary source problems



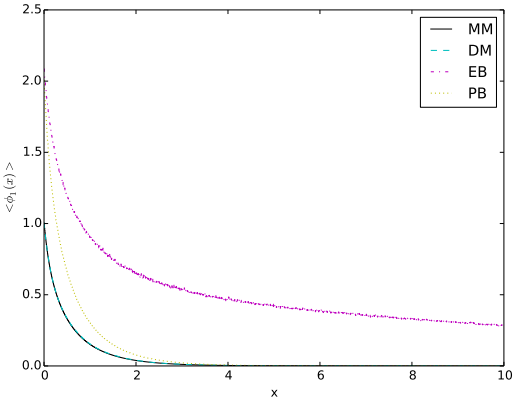
(a) $\langle \phi(x) \rangle$, standard methods



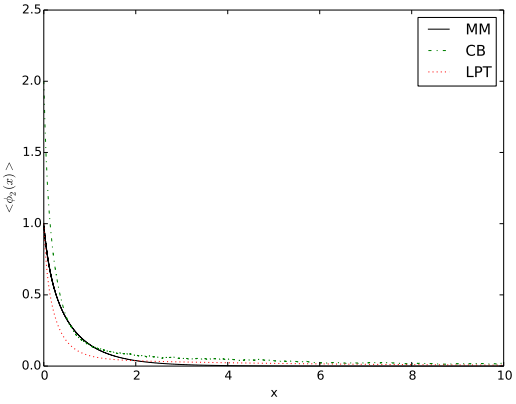
(b) $\langle \phi(x) \rangle$, comparison methods



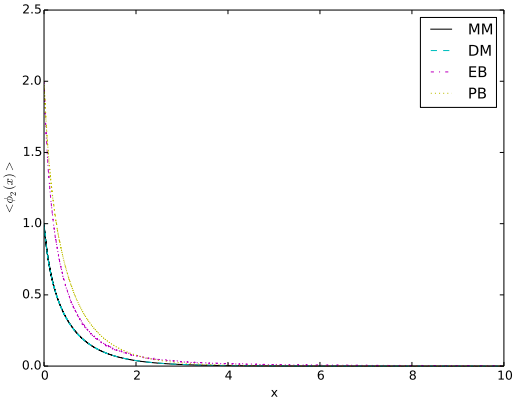
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

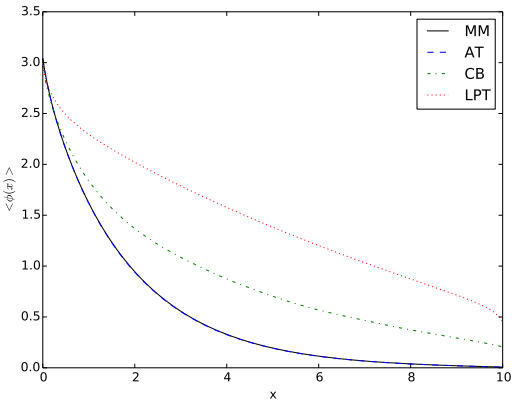


(e) $\langle \phi_1(x) \rangle$, standard methods

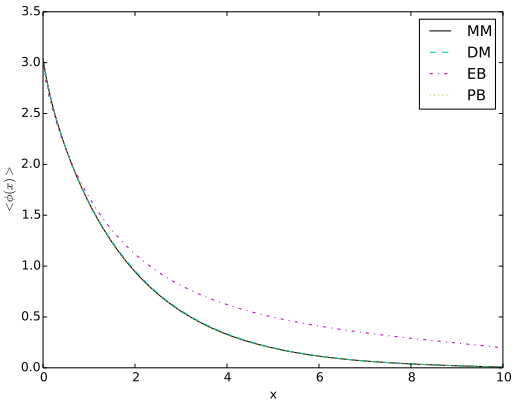


(f) $\langle \phi_1(x) \rangle$, comparison methods

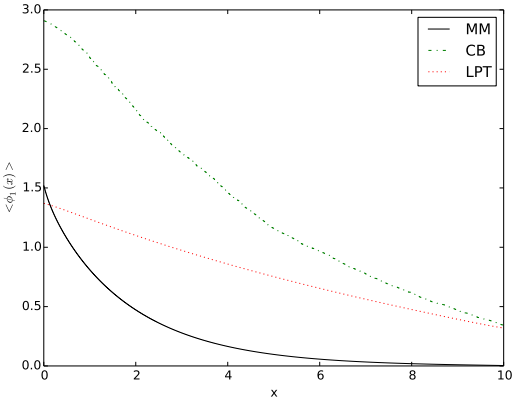
Figure 4.8: Results for M3, S2, L10.0, boundary source problems



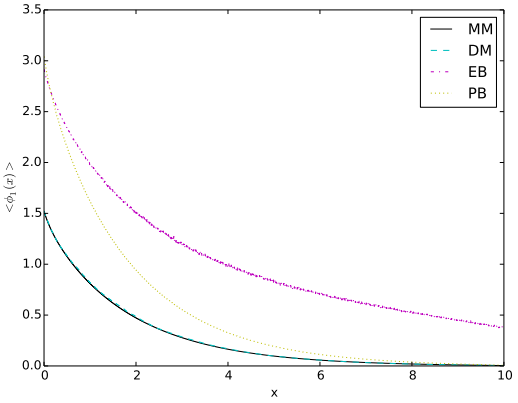
(a) $\langle \phi(x) \rangle$, standard methods



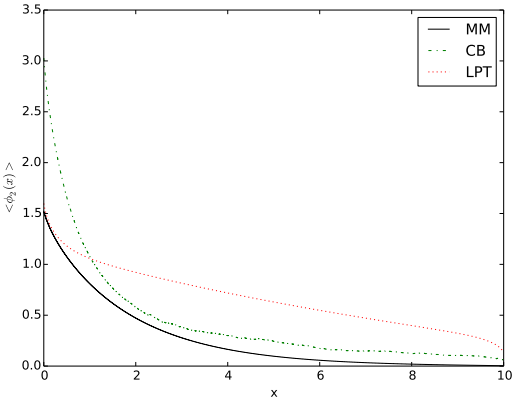
(b) $\langle \phi(x) \rangle$, comparison methods



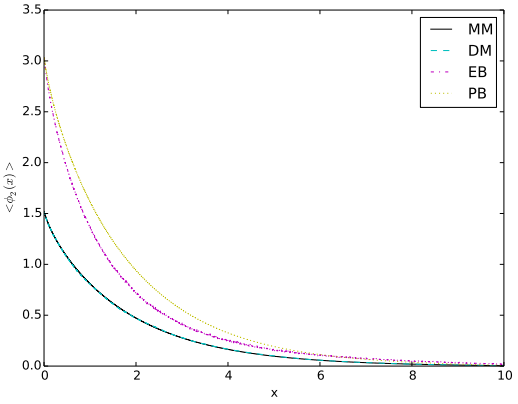
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods



(e) $\langle \phi_1(x) \rangle$, standard methods



(f) $\langle \phi_1(x) \rangle$, comparison methods

Figure 4.9: Results for M3, S3, L10.0, boundary source problems

4.2 Internal Source

For the internal source problems, the source was scaled to the length of the problem to give:

$$S_m = \frac{1}{x_{max}},\ m \in M. \tag{4.10}$$

Tables 4.9 to 4.10 include data for the reflection and leakage results for the various methods and their corresponding errors from the chord length benchmark, while Tables 4.11 to 4.13 give the error for $\langle\phi\rangle$, $\langle\phi_0\rangle$ and $\langle\phi_1\rangle$. All errors are computed from the chord length benchmark.

Figures 4.10 to 4.18 include plots for $\langle\phi\rangle$, $\langle\phi_0\rangle$ and $\langle\phi_1\rangle$ for $L = 10.0$ and each of the material and scattering cases in Table 4.1, with the internal source in Equation 4.10.

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	0.4923	0.4918	0.4921	0.4919	0.4981	0.4923	0.4922
1	1	1.0	0.4371	0.4304	0.4401	0.4325	0.4724	0.4380	0.4369
1	1	10.0	0.1375	0.1560	0.1394	0.1409	0.1712	0.1398	0.1380
1	2	0.1	0.4320	0.4682	0.4319	0.4668	0.4655	0.4321	0.4323
1	2	1.0	0.2072	0.3624	0.2312	0.3259	0.3401	0.2079	0.2081
1	2	10.0	0.0270	0.0821	0.0280	0.0344	0.0675	0.0283	0.0283
1	3	0.1	0.4915	0.4932	0.4913	0.4931	0.5014	0.4915	0.4915
1	3	1.0	0.4316	0.4526	0.4449	0.4422	0.5176	0.4330	0.4319
1	3	10.0	0.1296	0.1830	0.1374	0.1463	0.2971	0.1379	0.1310
2	1	0.1	0.4923	0.4919	0.4922	0.4920	0.4961	0.4923	0.4922
2	1	1.0	0.4371	0.4320	0.4378	0.4316	0.4310	0.4380	0.4369
2	1	10.0	0.1375	0.1914	0.1466	0.1610	0.1890	0.1398	0.1379
2	2	0.1	0.4320	0.4699	0.4337	0.4687	0.4697	0.4321	0.4320
2	2	1.0	0.2072	0.4408	0.2084	0.4312	0.4283	0.2079	0.2075
2	2	10.0	0.0270	0.3139	0.0345	0.2225	0.2428	0.0283	0.0281
2	3	0.1	0.4915	0.4928	0.4917	0.4926	0.5002	0.4915	0.4915
2	3	1.0	0.4316	0.4574	0.4325	0.4532	0.4891	0.4330	0.4316
2	3	10.0	0.1296	0.3259	0.1619	0.2612	0.4065	0.1379	0.1305
3	1	0.1	0.4991	0.4992	0.4992	0.4991	0.5019	0.4991	0.4992
3	1	1.0	0.4923	0.4914	0.4915	0.4915	0.5056	0.4924	0.4923
3	1	10.0	0.3716	0.4161	0.3715	0.3976	0.3828	0.3722	0.3717
3	2	0.1	0.4268	0.4363	0.4263	0.4362	0.4376	0.4269	0.4269
3	2	1.0	0.1971	0.3029	0.1973	0.2962	0.2965	0.1971	0.1971
3	2	10.0	0.0253	0.1348	0.0253	0.0781	0.1165	0.0254	0.0254
3	3	0.1	0.4915	0.4917	0.4914	0.4917	0.4955	0.4915	0.4915
3	3	1.0	0.4316	0.4369	0.4314	0.4353	0.4833	0.4318	0.4317
3	3	10.0	0.1296	0.2298	0.1297	0.1851	0.3852	0.1306	0.1298

Table 4.9: Leakage, internal source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	0.0008	-	0.0005	0.0002	0.0128	0.0009	0.0008
1	1	1.0	0.0154	-	0.0224	0.0049	0.0976	0.0177	0.0150
1	1	10.0	-0.1182	-	-0.1062	-0.0963	0.0976	-0.1039	-0.1149
1	2	0.1	-0.0774	-	-0.0775	-0.0032	-0.0059	-0.0772	-0.0768
1	2	1.0	-0.4284	-	-0.3621	-0.1009	-0.0616	-0.4265	-0.4257
1	2	10.0	-0.6707	-	-0.6590	-0.5811	-0.1783	-0.6559	-0.6556
1	3	0.1	-0.0035	-	-0.0038	-0.0001	0.0167	-0.0034	-0.0035
1	3	1.0	-0.0463	-	-0.0171	-0.0229	0.1437	-0.0433	-0.0457
1	3	10.0	-0.2919	-	-0.2493	-0.2009	0.6233	-0.2466	-0.2841
2	1	0.1	0.0008	-	0.0007	0.0003	0.0087	0.0009	0.0007
2	1	1.0	0.0117	-	0.0135	-0.0010	-0.0023	0.0139	0.0113
2	1	10.0	-0.2815	-	-0.2342	-0.1589	-0.0125	-0.2699	-0.2797
2	2	0.1	-0.0805	-	-0.0769	-0.0024	-0.0003	-0.0804	-0.0805
2	2	1.0	-0.5300	-	-0.5272	-0.0217	-0.0283	-0.5284	-0.5292
2	2	10.0	-0.9139	-	-0.8900	-0.2909	-0.2263	-0.9100	-0.9104
2	3	0.1	-0.0026	-	-0.0022	-0.0003	0.0150	-0.0025	-0.0026
2	3	1.0	-0.0564	-	-0.0545	-0.0091	0.0692	-0.0534	-0.0565
2	3	10.0	-0.6022	-	-0.5032	-0.1983	0.2475	-0.5768	-0.5996
3	1	0.1	-0.0000	-	-0.0000	-0.0001	0.0055	-0.0000	-0.0000
3	1	1.0	0.0017	-	0.0002	0.0001	0.0287	0.0020	0.0017
3	1	10.0	-0.1069	-	-0.1071	-0.0443	-0.0799	-0.1055	-0.1067
3	2	0.1	-0.0216	-	-0.0230	-0.0002	0.0031	-0.0216	-0.0216
3	2	1.0	-0.3493	-	-0.3486	-0.0218	-0.0211	-0.3490	-0.3490
3	2	10.0	-0.8125	-	-0.8124	-0.4206	-0.1360	-0.8116	-0.8115
3	3	0.1	-0.0005	-	-0.0007	-0.0001	0.0076	-0.0005	-0.0005
3	3	1.0	-0.0120	-	-0.0125	-0.0037	0.1062	-0.0116	-0.0119
3	3	10.0	-0.4359	-	-0.4354	-0.1943	0.6764	-0.4315	-0.4351

Table 4.10: Error in leakage, internal source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	0.0336	-	0.0350	0.0020	0.0083	0.0335	0.0334
1	1	1.0	0.0854	-	0.0821	0.0225	0.0485	0.0833	0.0851
1	1	10.0	0.0563	-	0.0548	0.0462	0.0393	0.0527	0.0539
1	2	0.1	0.1114	-	0.1136	0.0070	0.0059	0.1111	0.1106
1	2	1.0	0.5158	-	0.4618	0.1725	0.0589	0.5124	0.5118
1	2	10.0	0.7339	-	0.7129	0.6568	0.1838	0.7127	0.7114
1	3	0.1	0.0386	-	0.0402	0.0025	0.0119	0.0385	0.0384
1	3	1.0	0.1502	-	0.1296	0.0583	0.0855	0.1469	0.1489
1	3	10.0	0.2378	-	0.1865	0.1907	0.4553	0.1912	0.2275
2	1	0.1	0.0357	-	0.0344	0.0008	0.0061	0.0356	0.0357
2	1	1.0	0.1432	-	0.1464	0.0109	0.0718	0.1409	0.1429
2	1	10.0	0.0885	-	0.0779	0.0485	0.1874	0.0846	0.0876
2	2	0.1	0.1152	-	0.1106	0.0032	0.0004	0.1149	0.1151
2	2	1.0	0.5838	-	0.5841	0.0322	0.0258	0.5810	0.5816
2	2	10.0	0.9067	-	0.8816	0.3561	0.2153	0.8992	0.8992
2	3	0.1	0.0402	-	0.0385	0.0012	0.0118	0.0401	0.0402
2	3	1.0	0.1685	-	0.1709	0.0100	0.0529	0.1655	0.1677
2	3	10.0	0.4019	-	0.3002	0.1892	0.1796	0.3659	0.3957
3	1	0.1	0.0091	-	0.0095	0.0001	0.0061	0.0090	0.0091
3	1	1.0	0.1026	-	0.1019	0.0055	0.0160	0.1023	0.1026
3	1	10.0	0.0658	-	0.0659	0.0759	0.4176	0.0661	0.0662
3	2	0.1	0.0337	-	0.0354	0.0002	0.0042	0.0336	0.0337
3	2	1.0	0.4317	-	0.4285	0.0352	0.0205	0.4312	0.4314
3	2	10.0	0.8259	-	0.8245	0.5137	0.1215	0.8243	0.8244
3	3	0.1	0.0102	-	0.0107	0.0001	0.0081	0.0102	0.0102
3	3	1.0	0.1208	-	0.1192	0.0102	0.0833	0.1203	0.1207
3	3	10.0	0.2875	-	0.2834	0.1737	0.5515	0.2823	0.2867

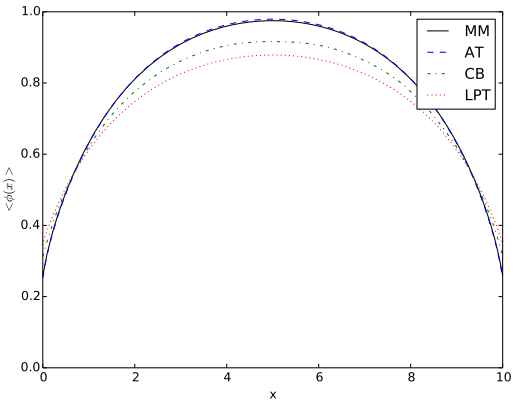
Table 4.11: Error in $\langle \phi(x) \rangle$, internal source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	-	-	0.1377	0.0054	0.0969	0.1358	0.0478
1	1	1.0	-	-	0.1658	0.0306	0.0628	0.1780	0.0967
1	1	10.0	-	-	0.0676	0.0491	0.1182	0.0685	0.0598
1	2	0.1	-	-	0.2451	0.0121	0.1033	0.2429	0.1592
1	2	1.0	-	-	0.5395	0.1821	0.1454	0.5898	0.5438
1	2	10.0	-	-	0.7599	0.6759	0.2689	0.7596	0.7318
1	3	0.1	-	-	0.1507	0.0065	0.0971	0.1487	0.0594
1	3	1.0	-	-	0.2281	0.0761	0.0435	0.2556	0.1771
1	3	10.0	-	-	0.2960	0.2146	0.2698	0.2998	0.2569
2	1	0.1	-	-	0.1331	0.0007	0.0996	0.1368	0.0505
2	1	1.0	-	-	0.1810	0.0116	0.0961	0.1773	0.1079
2	1	10.0	-	-	0.1221	0.1128	0.1405	0.1351	0.2208
2	2	0.1	-	-	0.2469	0.0014	0.1011	0.2529	0.1712
2	2	1.0	-	-	0.6579	0.0365	0.1192	0.6553	0.6177
2	2	10.0	-	-	0.9000	0.3587	0.2854	0.9162	0.9069
2	3	0.1	-	-	0.1473	0.0008	0.0998	0.1513	0.0634
2	3	1.0	-	-	0.2860	0.0183	0.0901	0.2805	0.2070
2	3	10.0	-	-	0.4000	0.2063	0.0181	0.4661	0.4353
3	1	0.1	-	-	0.5259	0.0005	0.5000	0.5210	0.0560
3	1	1.0	-	-	0.5508	0.0188	0.4954	0.5541	0.1458
3	1	10.0	-	-	0.2917	0.1938	0.5492	0.2911	0.5022
3	2	0.1	-	-	0.5959	0.0007	0.5002	0.5911	0.1846
3	2	1.0	-	-	0.8203	0.0435	0.5091	0.8224	0.6456
3	2	10.0	-	-	0.9524	0.5636	0.5637	0.9523	0.9047
3	3	0.1	-	-	0.5346	0.0006	0.5000	0.5297	0.0696
3	3	1.0	-	-	0.6145	0.0245	0.4967	0.6178	0.2481
3	3	10.0	-	-	0.7535	0.2926	0.4243	0.7527	0.5096

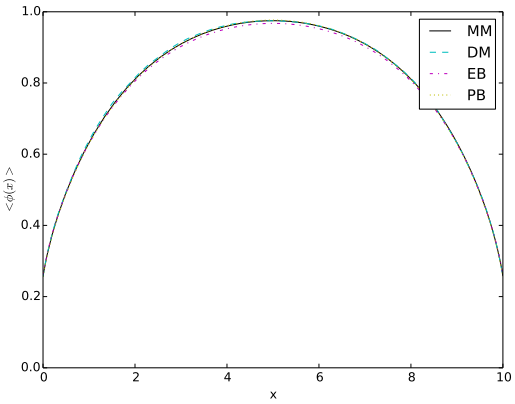
Table 4.12: Error in $\langle \phi_0(x) \rangle$, internal source problems

M	S	L	AT	CB	DM	EB	LPT	MM	PB
1	1	0.1	-	-	0.8874	0.0444	0.8929	0.8868	0.1626
1	1	1.0	-	-	0.9088	0.2759	0.8860	0.8918	0.0963
1	1	10.0	-	-	0.9038	0.0631	0.9103	0.9043	0.0703
1	2	0.1	-	-	0.7966	0.0800	0.8979	0.7957	1.0486
1	2	1.0	-	-	0.8265	0.2784	0.9006	0.8136	0.8858
1	2	10.0	-	-	0.8858	0.1781	0.8991	0.8861	0.1835
1	3	0.1	-	-	0.8751	0.0507	0.8867	0.8744	0.2741
1	3	1.0	-	-	0.8814	0.3001	0.8495	0.8621	0.3887
1	3	10.0	-	-	0.8702	0.1937	0.7856	0.8712	0.2416
2	1	0.1	-	-	0.8935	0.0058	0.8937	0.8903	0.1465
2	1	1.0	-	-	0.9447	0.0491	0.9396	0.9439	0.4644
2	1	10.0	-	-	0.9660	0.4573	0.9650	0.9590	0.5963
2	2	0.1	-	-	0.7756	0.0089	0.8998	0.7699	1.3157
2	2	1.0	-	-	0.5363	0.2086	0.8958	0.5326	3.7068
2	2	10.0	-	-	0.7784	0.5249	0.8979	0.7764	1.2715
2	3	0.1	-	-	0.8786	0.0061	0.8854	0.8750	0.2780
2	3	1.0	-	-	0.8276	0.1588	0.8053	0.8257	0.7669
2	3	10.0	-	-	0.7491	0.7111	0.5530	0.7310	1.6477
3	1	0.1	-	-	0.4775	0.0005	0.4948	0.4824	0.0447
3	1	1.0	-	-	0.5396	0.0127	0.4976	0.5366	0.0853
3	1	10.0	-	-	0.6467	0.0360	0.7747	0.6473	0.2971
3	2	0.1	-	-	0.4069	0.0006	0.5000	0.4116	0.1796
3	2	1.0	-	-	0.2425	0.0259	0.4978	0.2409	0.5187
3	2	10.0	-	-	0.3566	0.1620	0.5064	0.3566	0.2940
3	3	0.1	-	-	0.4686	0.0005	0.4929	0.4736	0.0599
3	3	1.0	-	-	0.4635	0.0153	0.4012	0.4606	0.0902
3	3	10.0	-	-	0.3228	0.1864	0.3343	0.3226	0.3866

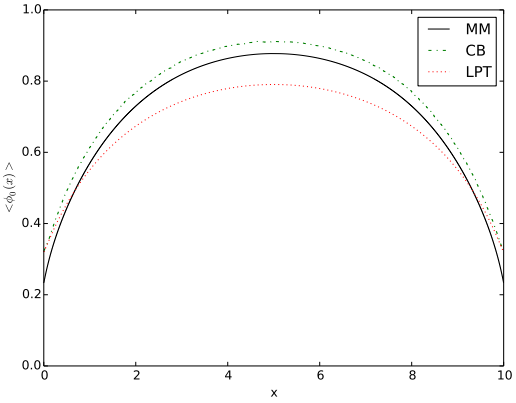
Table 4.13: Error in $\langle \phi_1(x) \rangle$, internal source problems



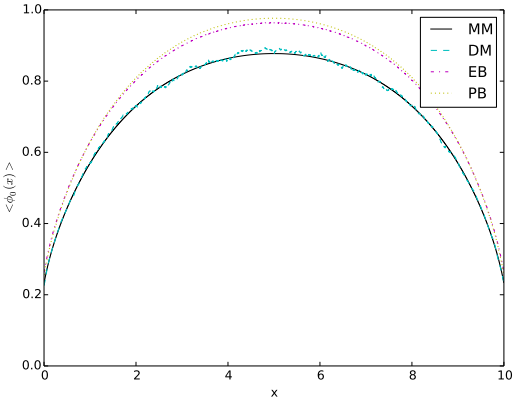
(a) $\langle\phi(x)\rangle$, standard methods



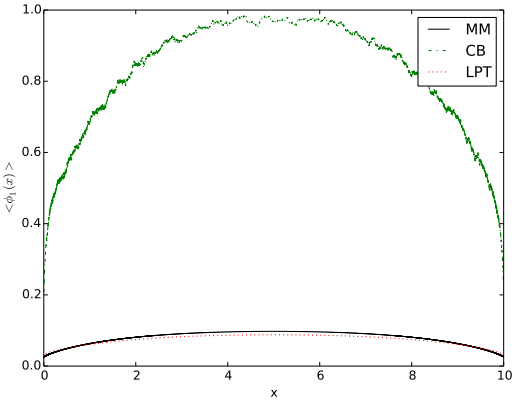
(b) $\langle\phi(x)\rangle$, comparison methods



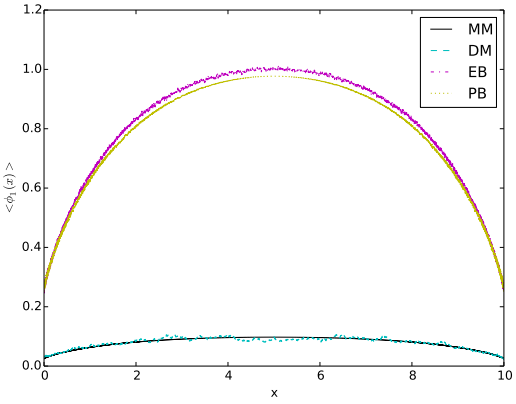
(c) $\langle\phi_0(x)\rangle$, standard methods



(d) $\langle\phi_0(x)\rangle$, comparison methods

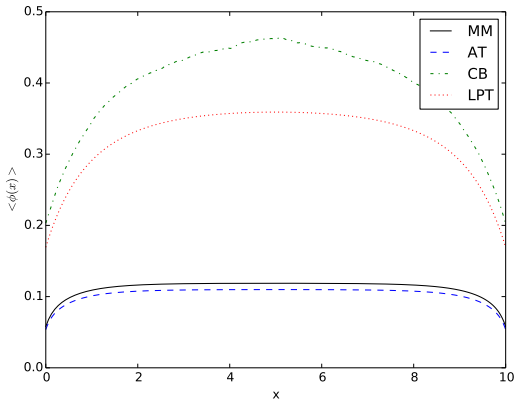


(e) $\langle\phi_1(x)\rangle$, standard methods

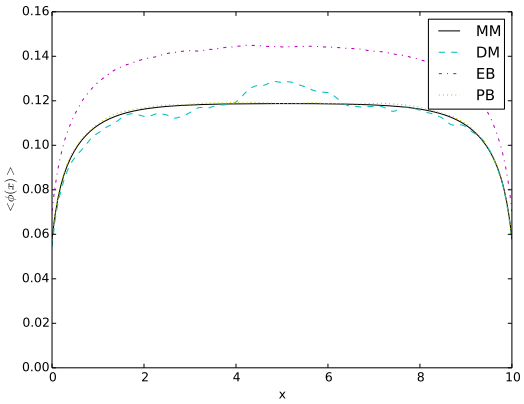


(f) $\langle\phi_1(x)\rangle$, comparison methods

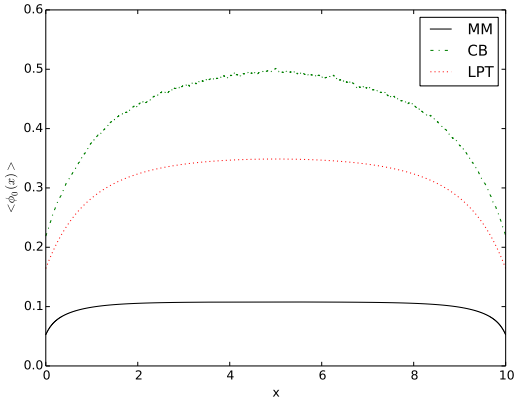
Figure 4.10: Results for M1, S1, L10.0, internal source problems



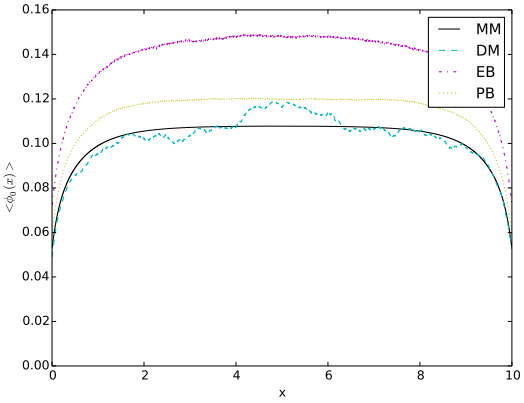
(a) $\langle\phi(x)\rangle$, standard methods



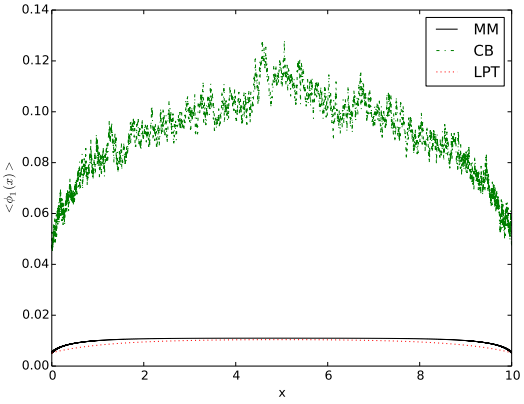
(b) $\langle\phi(x)\rangle$, comparison methods



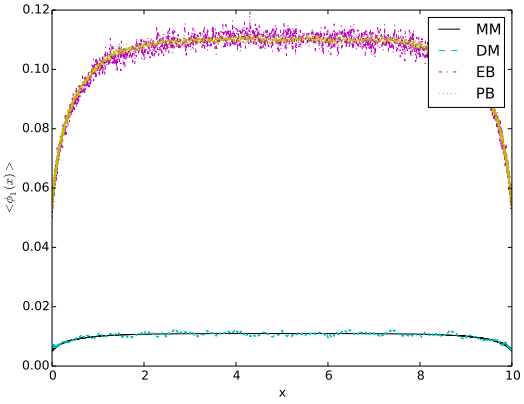
(c) $\langle\phi_0(x)\rangle$, standard methods



(d) $\langle\phi_0(x)\rangle$, comparison methods

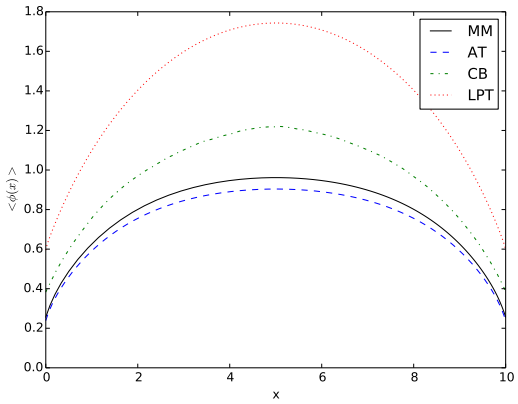


(e) $\langle\phi_1(x)\rangle$, standard methods

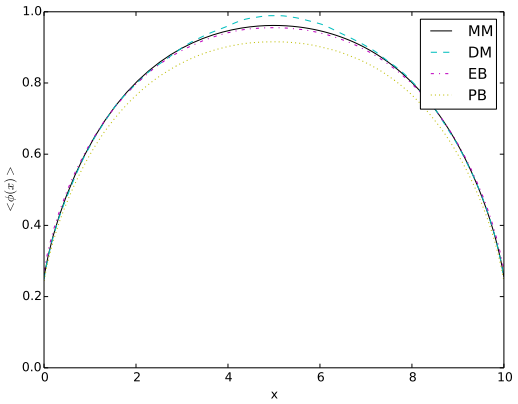


(f) $\langle\phi_1(x)\rangle$, comparison methods

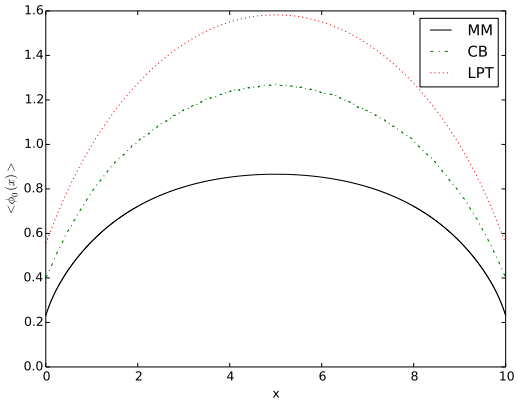
Figure 4.11: Results for M1, S2, L10.0, internal source problems



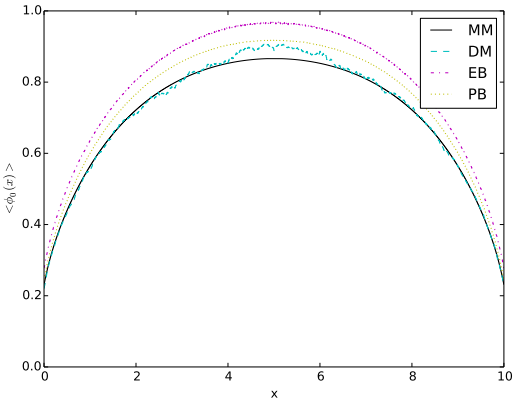
(a) $\langle \phi(x) \rangle$, standard methods



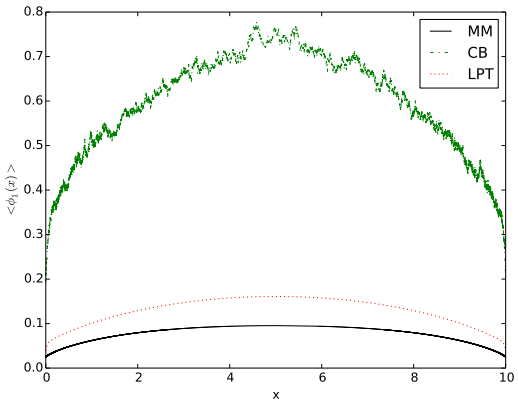
(b) $\langle \phi(x) \rangle$, comparison methods



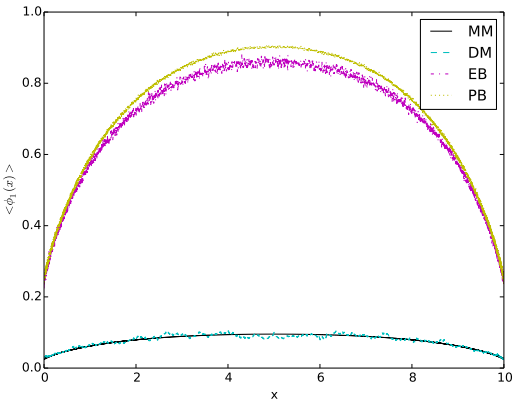
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

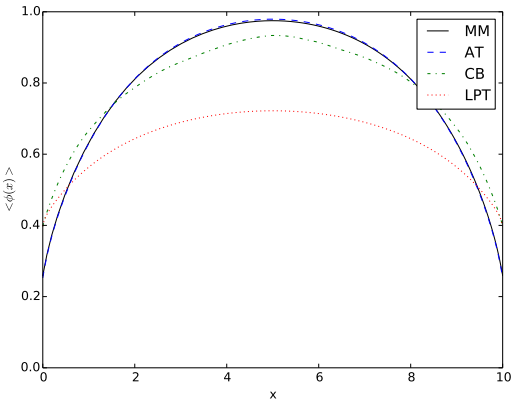


(e) $\langle \phi_1(x) \rangle$, standard methods

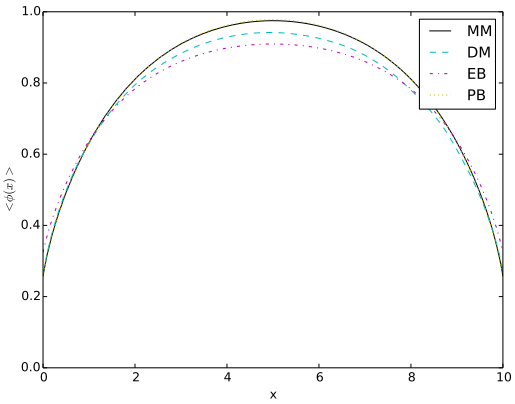


(f) $\langle \phi_1(x) \rangle$, comparison methods

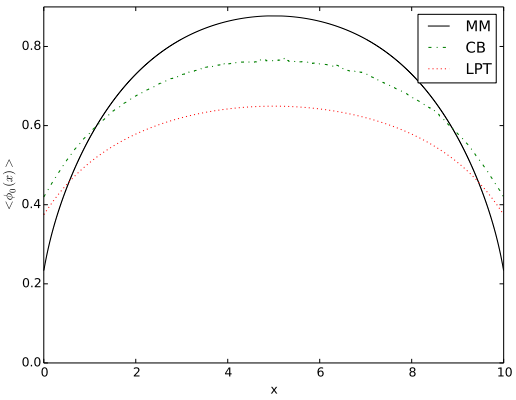
Figure 4.12: Results for M1, S3, L10.0, internal source problems



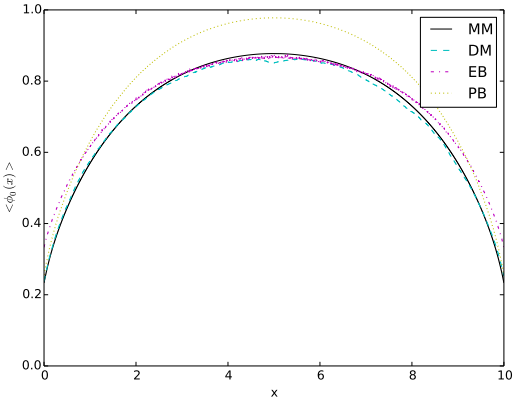
(a) $\langle \phi(x) \rangle$, standard methods



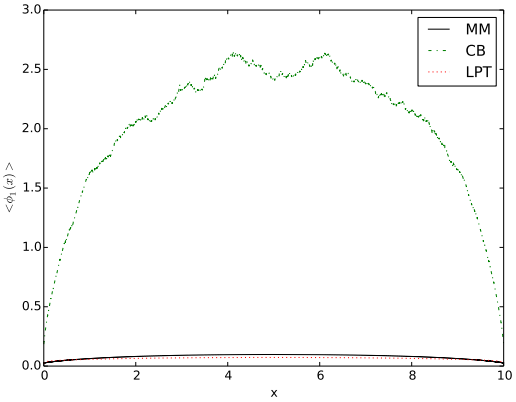
(b) $\langle \phi(x) \rangle$, comparison methods



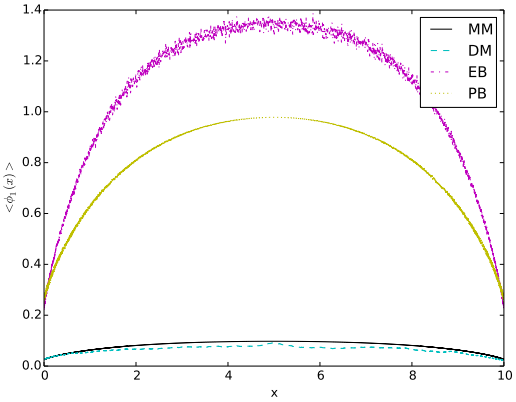
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods

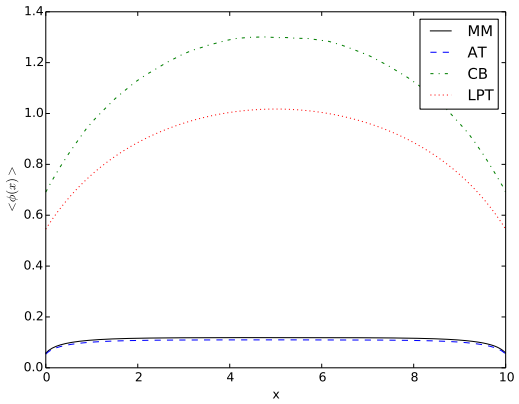


(e) $\langle \phi_1(x) \rangle$, standard methods

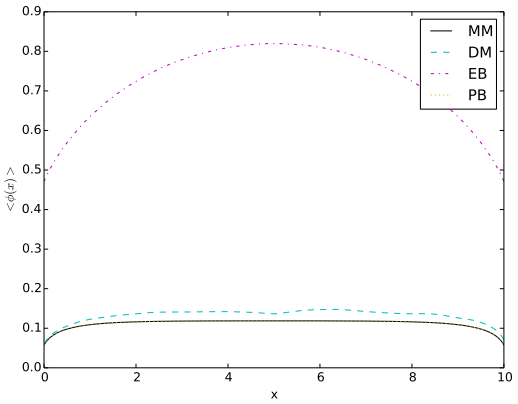


(f) $\langle \phi_1(x) \rangle$, comparison methods

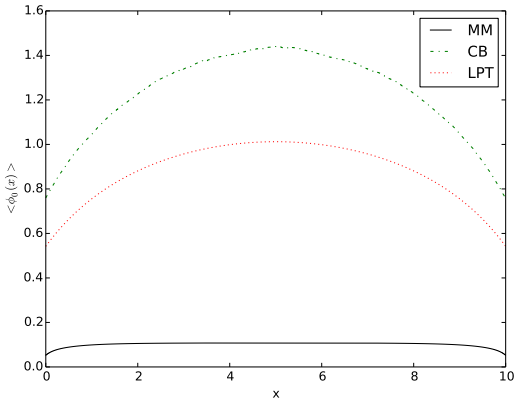
Figure 4.13: Results for M2, S1, L10.0, internal source problems



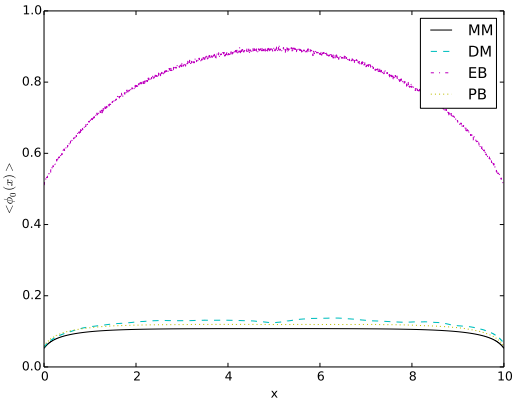
(a) $\langle\phi(x)\rangle$, standard methods



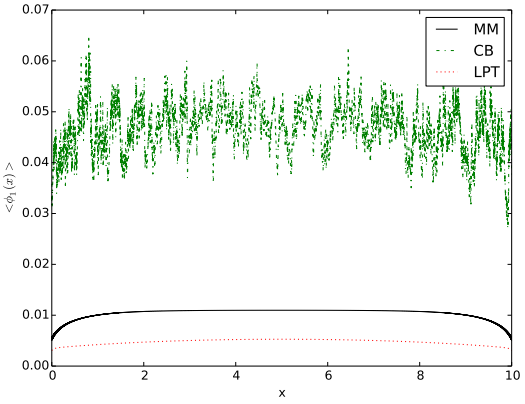
(b) $\langle\phi(x)\rangle$, comparison methods



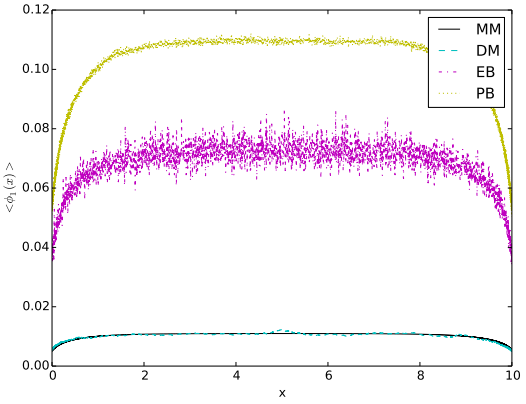
(c) $\langle\phi_0(x)\rangle$, standard methods



(d) $\langle\phi_0(x)\rangle$, comparison methods

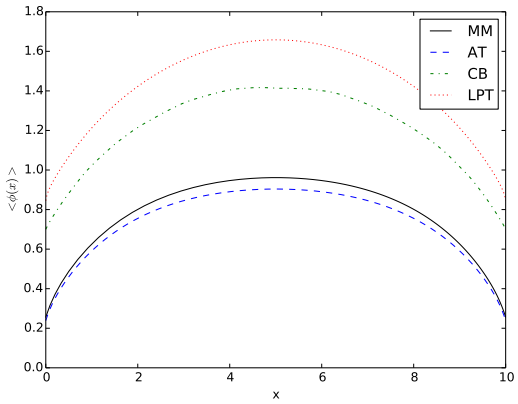


(e) $\langle\phi_1(x)\rangle$, standard methods

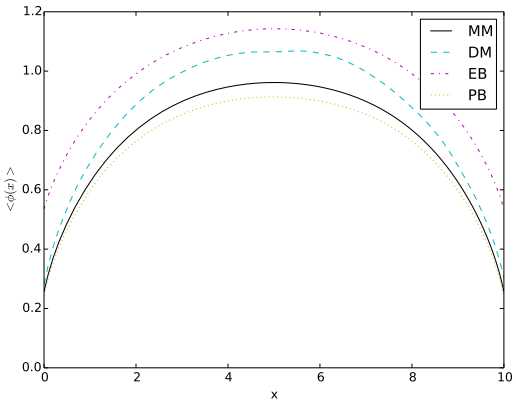


(f) $\langle\phi_1(x)\rangle$, comparison methods

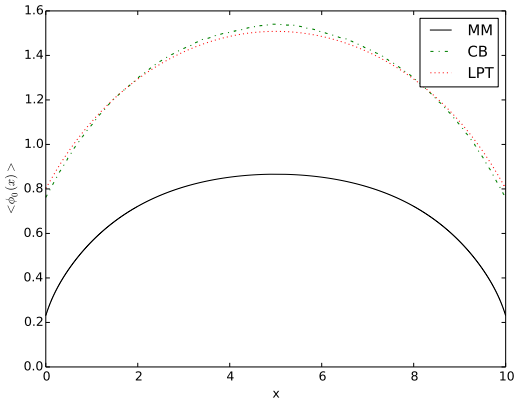
Figure 4.14: Results for M2, S2, L10.0, internal source problems



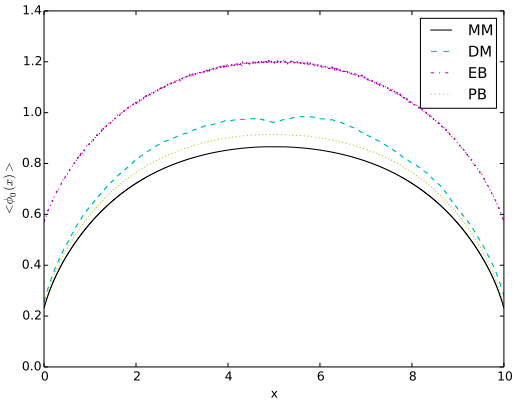
(a) $\langle\phi(x)\rangle$, standard methods



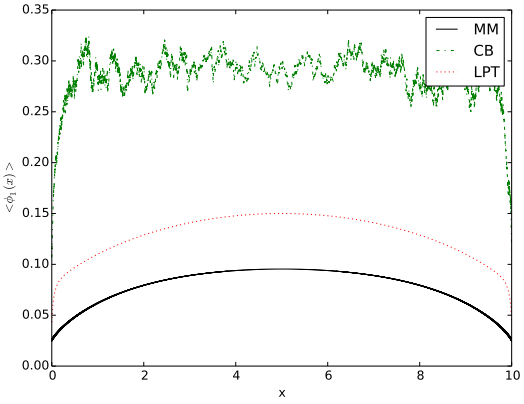
(b) $\langle\phi(x)\rangle$, comparison methods



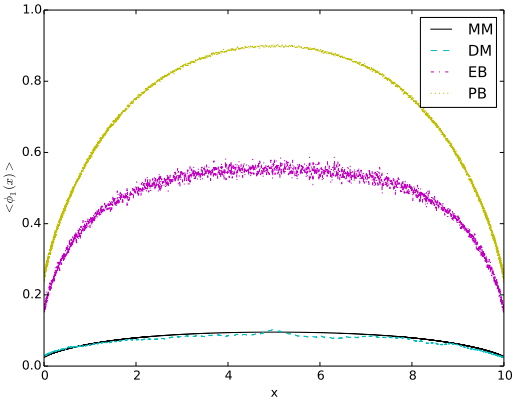
(c) $\langle\phi_0(x)\rangle$, standard methods



(d) $\langle\phi_0(x)\rangle$, comparison methods

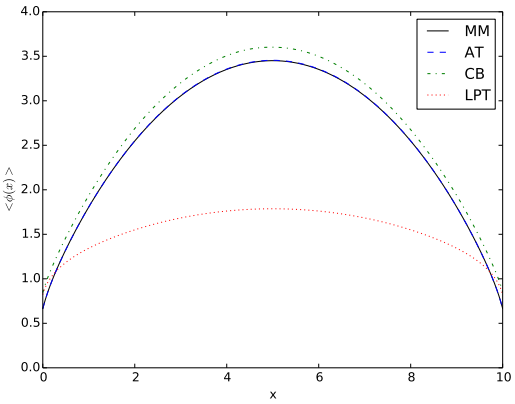


(e) $\langle\phi_1(x)\rangle$, standard methods

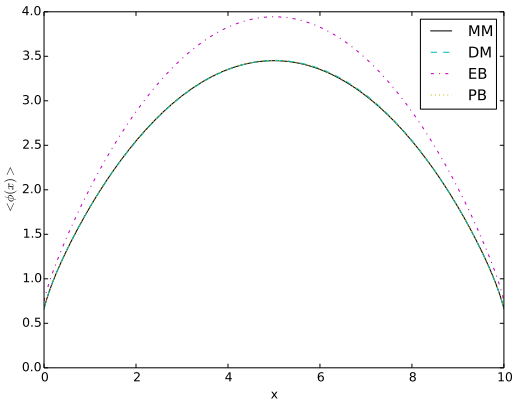


(f) $\langle\phi_1(x)\rangle$, comparison methods

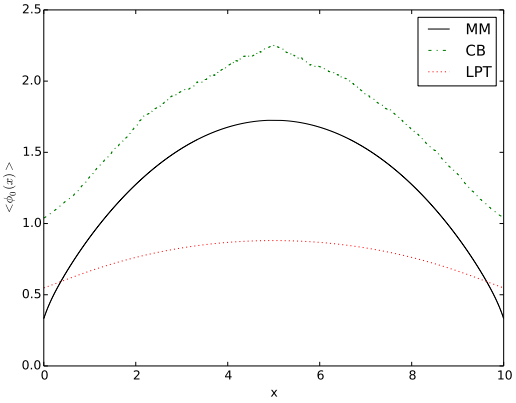
Figure 4.15: Results for M2, S3, L10.0, internal source problems



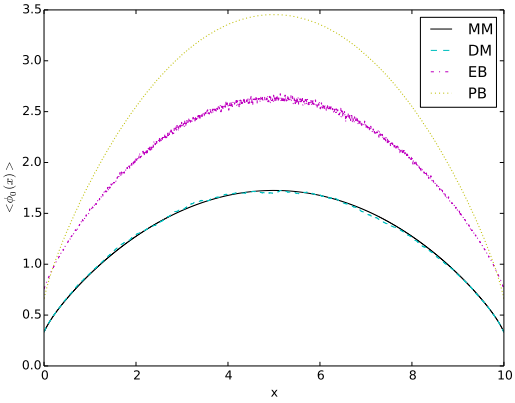
(a) $\langle\phi(x)\rangle$, standard methods



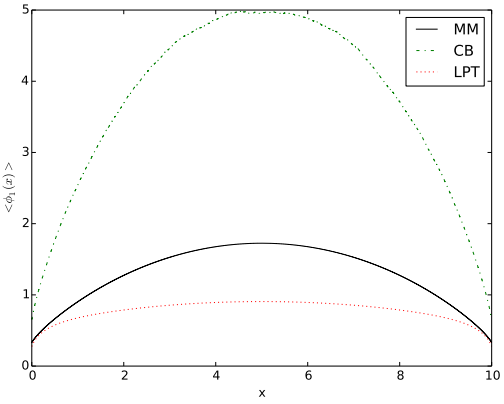
(b) $\langle\phi(x)\rangle$, comparison methods



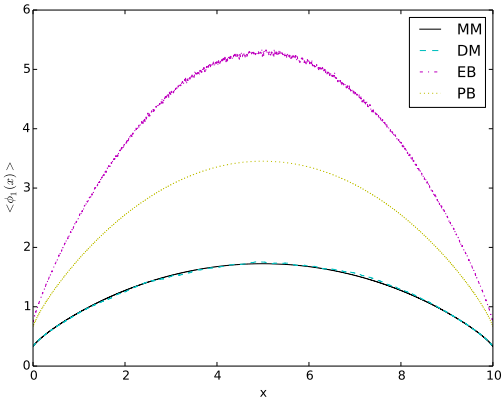
(c) $\langle\phi_0(x)\rangle$, standard methods



(d) $\langle\phi_0(x)\rangle$, comparison methods

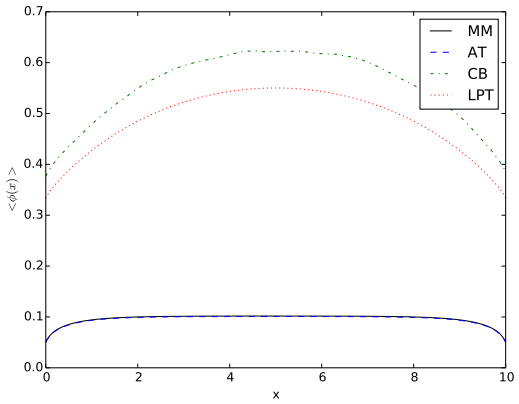


(e) $\langle\phi_1(x)\rangle$, standard methods

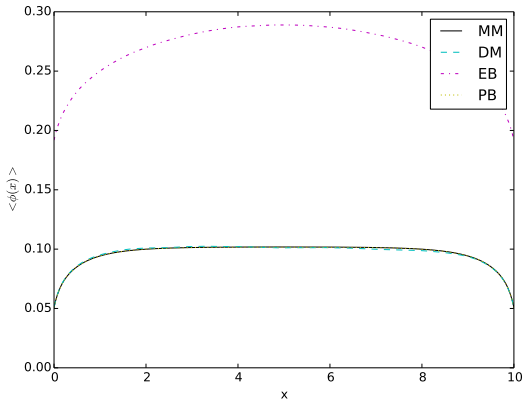


(f) $\langle\phi_1(x)\rangle$, comparison methods

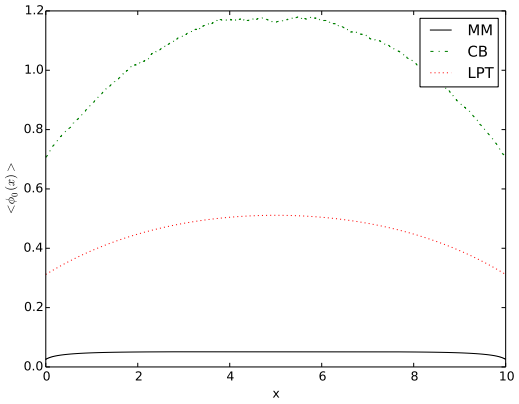
Figure 4.16: Results for M3, S1, L10.0, internal source problems



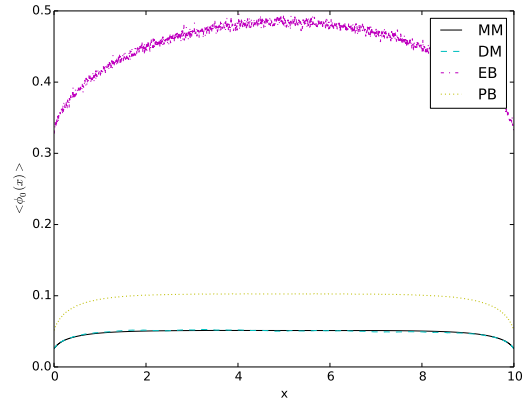
(a) $\langle\phi(x)\rangle$, standard methods



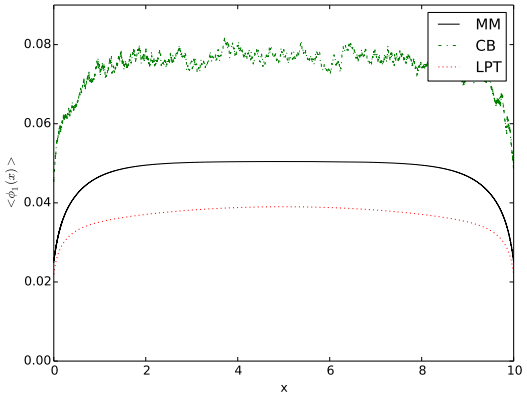
(b) $\langle\phi(x)\rangle$, comparison methods



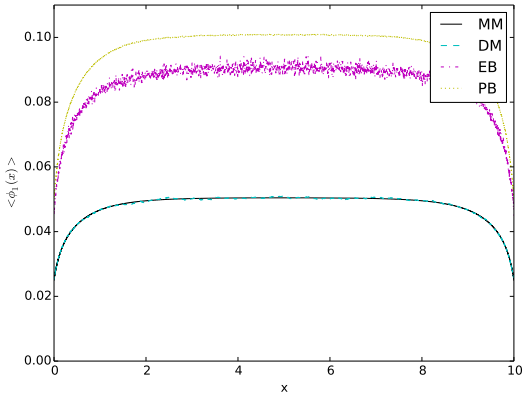
(c) $\langle\phi_0(x)\rangle$, standard methods



(d) $\langle\phi_0(x)\rangle$, comparison methods

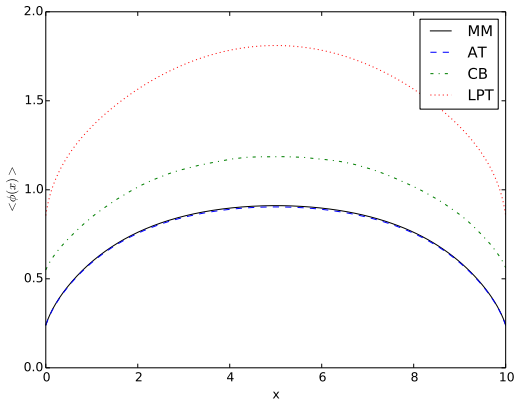


(e) $\langle\phi_1(x)\rangle$, standard methods

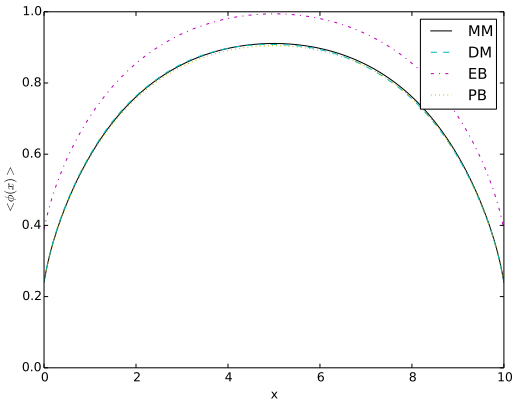


(f) $\langle\phi_1(x)\rangle$, comparison methods

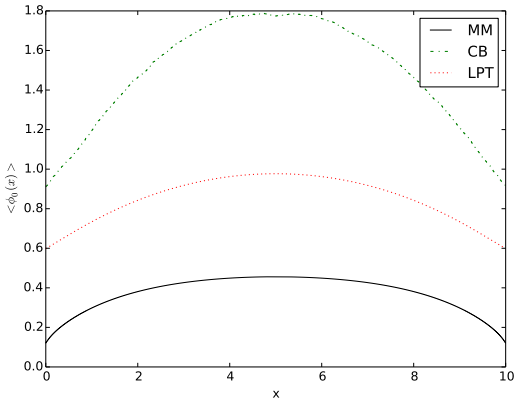
Figure 4.17: Results for M3, S2, L10.0, internal source problems



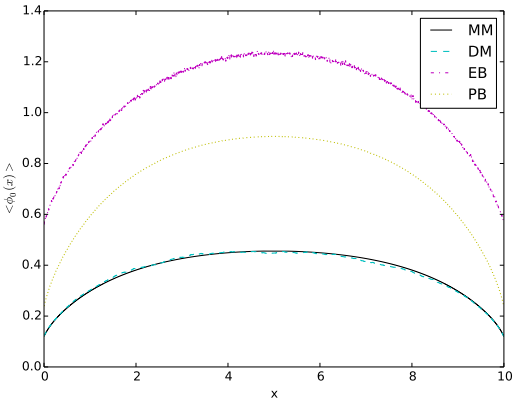
(a) $\langle \phi(x) \rangle$, standard methods



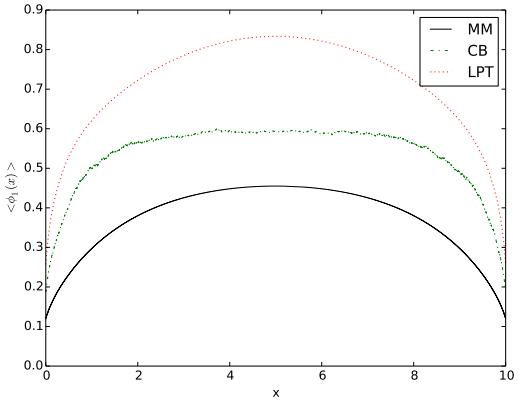
(b) $\langle \phi(x) \rangle$, comparison methods



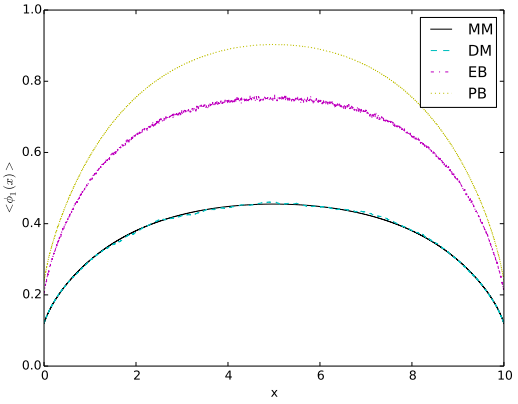
(c) $\langle \phi_0(x) \rangle$, standard methods



(d) $\langle \phi_0(x) \rangle$, comparison methods



(e) $\langle \phi_1(x) \rangle$, standard methods



(f) $\langle \phi_1(x) \rangle$, comparison methods

Figure 4.18: Results for M3, S3, L10.0, internal source problems

5 Discussion

The standard error helps to ascertain the validity of the comparisons between the benchmark methods and the deterministic methods. The standard error for the chord and exponential benchmarks is never above $\sigma = 0.018$, and is often much lower. The standard error in all cases increases with higher scattering. The standard error for the percentage benchmarks is between five and ten times lower than that for the chord benchmarks, maxing out around $\sigma = 0.004$. This makes the standard error in either case around three or four orders of magnitude smaller than the solution itself, which makes the difference in the percentage error results around a tenth or hundredth of a percent for each standard error the actual solution deviates from the solution calculated:

$$\Delta E_x = \left(\frac{1.001x_b - x_m}{1.001x_b} - \frac{x_b - x_m}{x_b} \right) 100\% \approx \left(\frac{x_m}{x_b} \right) 0.1\% \approx 0.1\% \tag{5.1}$$

This makes the error from the benchmark results accurate to at least two decimal places for the given mesh.

The two variations on the mixing term help show the utility of the mesh-based method. The modified LP method gave results at most one percent different from the standard LP method (see Tables 8.17 and 8.22), which error could be expected to decrease in the limit of a high number of cells, as the LP coupling term is upwinded (uses a value from the previous cell in the transport sweep). This means that the modified LP method could be used as a substitute for the mesh-based method in those cases where the LP method performed well and the mesh-based method did not.

The distribution method in most cases gave answers within a few percent of the mesh method. The spatially-dependent transition probability was in all cases within ten percent of the static transition probability, and more closely approached the static values with more realizations.

5.1 Comparison to Percentage Benchmarks

The percentage benchmark tests were run to show what effect having no information about the chord lengths (other than their relative values) would do to a solution. In all cases, the mesh method method very closely resembled the solution given by the atomic mix method and percentage benchmarks. The method itself predisposes it to converge to these solutions, as the amount of mixing between the materials increases linearly with the number of cells. In these test cases, the mixing occurred 20,000 times per unit length for $L = 0.1$, 2,000 times for $L = 1.0$ and 200 times for $L = 10.0$. For the boundary source problems, the atomic mix method was as accurate as the mesh-based method in predicting the leakage (see Table 4.5), while for the internal source problems (see Table 4.10) it was nearly as accurate.

For most cases, the error of the mesh method against the percentage benchmark was well under one percent. For one boundary source case with a very highly scattering optically thin region, the error was as large as eight percent, although the likely explanation is the small magnitude at the right edge of the problem, giving a very small magnitude error but a deceptively large percentage error (Material 1, Scattering 2, Length 10.0, boundary source: Figure 4.2b).

Interestingly, the percentage benchmark gave much closer results to the chord benchmark in some cases for the flux in one of the two materials than did the non-benchmark methods, including for $\langle \phi_1(x) \rangle$ in one

boundary source case (Material 1, Scattering 1, Length 10.0, internal source: Table 4.8 and Figure 4.10). For this case, the much more optically thick material was entirely scattering, which will raise the flux levels considerably if rearranged into a slab (as there will be no attenuation in these areas).

The methods that do involve the chord length and not just p_1 and p_2 come closer to approximating the chord benchmark solution.

5.2 Comparison to Chord Benchmarks

The mesh-based method occasionally gave results that were more accurate than the LP method, but this happened only in those instances when the atomic mix model also got better results than the LP method. For instance, for one problem with relatively short chord lengths but high scattering in both materials (Material 1, Scattering 3, Length 10.0, boundary source: Table 4.6 and Figure 4.3a), the error in $\langle \phi(x) \rangle$ is almost double for the LP method as opposed to the mesh-based method, but the atomic mix model also did significantly better than the LP method in that case. The same holds for a problem with slightly longer chord lengths but overall smaller cross sections (Material 3, Scattering 3, Length 10.0, boundary source: Figure 4.9a), and the corresponding internal source problems (Table 4.11). This trend seems to occur only in cases with generally smaller chord lengths and very high scattering for the material with the larger total cross section (as in Material 1, with $c_1 = 1.0$ for $\Sigma_t^1 = 100/11 \approx 90\Sigma_t^0$), where due to the large amount of scattering the LP equations aren't as accurate.

The Exponential benchmark tests were run to ascertain what effect the ordering of the cells had on the behavior of the solution. As these tests used the same amount of material in each realization as the Chord benchmark (but in a randomized order), the difference between these two methods should show the handicap of not having localized sections of a homogeneous material. These effects were more pronounced for larger sections of material ($L = 10.0$), and interestingly when the more optically thin material is highly scattering (Material 3, Scattering 2, Length 10.0, internal source: Figure 4.17d).

The comparison of Material 1 and Material 2 offers some interesting insights as to the effects of chord length on the solution. The two material cases are identical except for the chord lengths, which are ten times as large for Material 2. For the atomic mix and mesh-based methods and the percentage benchmark, no information about the chord lengths except their magnitude relative to one another is given, while the chord benchmark and LP methods use the chord length as a fundamental quantity and the distribution method uses some information about the chord length, but indirectly and not to much effect. Comparing Figures 4.12a and 4.12b to Figures 4.15a and 4.15b, the solutions for the atomic mix, mesh-based method, and percentage benchmarks do not change while those for the LP method changed slightly, but drastically overestimated the solution for a highly-scattering problem with relatively short chord lengths (Material 1, Scattering 3, Length 10.0, internal source) and the chord and exponential benchmarks changed significantly. Interestingly, the distribution method solution in this case was much closer to the chord benchmarks than was the mesh-based method, which shows that the chord length information given in the spatial distribution of p_1 and p_2 helped the solution (particularly at the center of the large slab).

The mesh-based method with mixing skipped for some cells was used to investigate the effect that a lower amount of mixing would have on the solution. As can be seen in Figures 8.1-8.9 and Tables 8.5 and 8.10, the number of cells skipped in these highly-scattering cases provides solutions that vary significantly in

magnitude. This shows that the amount of mixing (or number of cells if mixing every cell) significantly impacts the accuracy of the method. In many cases (such as Material 2, Scattering 1, Length 10.0, both boundary and internal source: Figure 8.4) the results for the mesh-based method become significantly better with certain amounts of mixing.

6 Conclusion

While the mesh-based method did achieve reasonable results when compared to the exponentially-distributed slabs, it didn't improve much on the atomic mix model. This could be due to the mixing after every spatial cell. When changing the length after which mixing occurs, the mesh-based method often got very close to correct chord-length benchmark solutions. This shows a weakness of the mesh-based method, namely that the solution will depend strongly on the number of cells due to the mixing that occurs on the cell boundaries. The mixing term in the equation would need information about the cell length and the total chord length to give better solutions in comparison to the exponential-type benchmarks. This method might achieve very good results compared to these benchmarks if a suitable approximation were developed. Notable is that the results for the percentage fill and exponential benchmarks will also change based on cell size, as larger sections of material become much more likely, while the results from the chord length method should not change based on cell size.

By using a spatially distributed transition probability with the mesh-based equations, the solution improved somewhat on problems with long chord lengths. In problems in which the transition probability is less homogeneous, this effect could be heightened. As the exponentially-distributed slabs did give almost homogeneous transition probabilities, the large difference in their solutions to the mesh-based solutions can partially be attributed to the often-large slab sizes in completely scattering materials, in which there is no attenuation of the flux as there is in the non-benchmark methods.

The mesh-based method does converge to the percentage fill benchmark in most cases, as has been shown before. With the standard mixing term, it is likely not a suitable replacement for the LP method, as the method does not contain information about the magnitude of the chord length but only the magnitudes relative to one another. This makes the method not ideal for approximating solutions given by the standard exponentially-distributed benchmarks. By using an LP-like mixing term, the mesh-based method can get solutions that are comparable in accuracy to the LP method without as heavy a sweeper modification as the LP method requires.

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8 Appendix

8.1 Data for Mesh Skip Method

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	-0.0452	-0.0768	-0.0766	-0.0765	-0.0761	-0.0749	-0.0731	-0.0696
1	1	1.0	-0.2014	-0.3618	-0.3581	-0.3535	-0.3444	-0.3189	-0.2828	-0.2289
1	1	10.0	-0.3194	-0.4862	-0.4448	-0.4047	-0.3491	-0.2546	-0.1798	-0.1201
1	2	0.1	-0.0082	-0.0068	-0.0068	-0.0068	-0.0068	-0.0069	-0.0070	-0.0072
1	2	1.0	-0.0429	-0.0191	-0.0196	-0.0202	-0.0214	-0.0250	-0.0307	-0.0397
1	2	10.0	-0.0547	-0.0212	-0.0266	-0.0334	-0.0466	-0.0827	-0.1309	-0.1925
1	3	0.1	-0.0488	-0.0759	-0.0758	-0.0756	-0.0753	-0.0744	-0.0729	-0.0700
1	3	1.0	-0.2518	-0.3548	-0.3534	-0.3515	-0.3478	-0.3358	-0.3148	-0.2759
1	3	10.0	-0.4944	-0.4823	-0.4875	-0.4925	-0.4968	-0.4916	-0.4695	-0.4373
2	1	0.1	-0.0385	-0.0768	-0.0766	-0.0765	-0.0761	-0.0749	-0.0731	-0.0696
2	1	1.0	-0.0778	-0.3618	-0.3581	-0.3535	-0.3444	-0.3189	-0.2828	-0.2289
2	1	10.0	-0.1294	-0.4862	-0.4448	-0.4047	-0.3491	-0.2546	-0.1798	-0.1201
2	2	0.1	-0.0084	-0.0068	-0.0068	-0.0068	-0.0068	-0.0069	-0.0070	-0.0072
2	2	1.0	-0.0645	-0.0191	-0.0196	-0.0202	-0.0214	-0.0250	-0.0307	-0.0397
2	2	10.0	-0.1851	-0.0212	-0.0266	-0.0334	-0.0466	-0.0827	-0.1309	-0.1925
2	3	0.1	-0.0430	-0.0759	-0.0758	-0.0756	-0.0753	-0.0744	-0.0729	-0.0700
2	3	1.0	-0.1438	-0.3548	-0.3534	-0.3515	-0.3478	-0.3358	-0.3148	-0.2759
2	3	10.0	-0.4376	-0.4823	-0.4875	-0.4925	-0.4968	-0.4916	-0.4695	-0.4373
3	1	0.1	-0.0751	-0.0848	-0.0848	-0.0848	-0.0847	-0.0846	-0.0843	-0.0838
3	1	1.0	-0.2766	-0.4386	-0.4382	-0.4376	-0.4366	-0.4335	-0.4281	-0.4173
3	1	10.0	-0.4873	-0.7877	-0.7822	-0.7755	-0.7624	-0.7277	-0.6831	-0.6174
3	2	0.1	-0.0009	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007
3	2	1.0	-0.0054	-0.0020	-0.0020	-0.0020	-0.0020	-0.0021	-0.0023	-0.0025
3	2	10.0	-0.0131	-0.0021	-0.0022	-0.0023	-0.0026	-0.0036	-0.0050	-0.0078
3	3	0.1	-0.0677	-0.0759	-0.0759	-0.0759	-0.0759	-0.0757	-0.0755	-0.0751
3	3	1.0	-0.2474	-0.3551	-0.3550	-0.3547	-0.3542	-0.3528	-0.3503	-0.3450
3	3	10.0	-0.4660	-0.4809	-0.4817	-0.4827	-0.4849	-0.4912	-0.4983	-0.5019

Table 8.1: Reflection, boundary source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.9388	0.9083	0.9085	0.9087	0.9090	0.9101	0.9118	0.9150
1	1	1.0	0.6176	0.4818	0.4837	0.4860	0.4908	0.5063	0.5341	0.5854
1	1	10.0	0.0148	0.0047	0.0046	0.0053	0.0084	0.0272	0.0650	0.1140
1	2	0.1	0.9027	0.8393	0.8396	0.8399	0.8407	0.8429	0.8464	0.8532
1	2	1.0	0.4819	0.2316	0.2352	0.2397	0.2488	0.2775	0.3277	0.4220
1	2	10.0	0.0015	0.0000	0.0000	0.0001	0.0003	0.0079	0.0527	0.1543
1	3	0.1	0.9425	0.9075	0.9076	0.9078	0.9083	0.9095	0.9115	0.9154
1	3	1.0	0.6651	0.4761	0.4798	0.4843	0.4935	0.5205	0.5621	0.6284
1	3	10.0	0.0516	0.0044	0.0074	0.0129	0.0296	0.1008	0.1991	0.2959
2	1	0.1	0.9454	0.9083	0.9085	0.9087	0.9090	0.9101	0.9118	0.9150
2	1	1.0	0.7531	0.4818	0.4837	0.4860	0.4908	0.5063	0.5341	0.5854
2	1	10.0	0.1158	0.0047	0.0046	0.0053	0.0084	0.0272	0.0650	0.1140
2	2	0.1	0.9163	0.8393	0.8396	0.8399	0.8407	0.8429	0.8464	0.8532
2	2	1.0	0.7530	0.2316	0.2352	0.2397	0.2488	0.2775	0.3277	0.4220
2	2	10.0	0.1592	0.0000	0.0000	0.0001	0.0003	0.0079	0.0527	0.1543
2	3	0.1	0.9499	0.9075	0.9076	0.9078	0.9083	0.9095	0.9115	0.9154
2	3	1.0	0.8189	0.4761	0.4798	0.4843	0.4935	0.5205	0.5621	0.6284
2	3	10.0	0.2981	0.0044	0.0074	0.0129	0.0296	0.1008	0.1991	0.2959
3	1	0.1	0.9256	0.9162	0.9162	0.9163	0.9163	0.9164	0.9167	0.9172
3	1	1.0	0.6947	0.5449	0.5452	0.5456	0.5464	0.5489	0.5531	0.5618
3	1	10.0	0.2388	0.0666	0.0660	0.0652	0.0641	0.0639	0.0727	0.1068
3	2	0.1	0.8528	0.8333	0.8333	0.8334	0.8334	0.8337	0.8342	0.8352
3	2	1.0	0.4809	0.2207	0.2211	0.2216	0.2227	0.2260	0.2317	0.2438
3	2	10.0	0.0698	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0016
3	3	0.1	0.9183	0.9074	0.9074	0.9075	0.9075	0.9077	0.9080	0.9086
3	3	1.0	0.6724	0.4753	0.4758	0.4764	0.4776	0.4813	0.4875	0.5004
3	3	10.0	0.2455	0.0039	0.0042	0.0046	0.0056	0.0096	0.0213	0.0641

Table 8.2: Leakage, boundary source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	-0.0150	0.6737	0.6703	0.6661	0.6577	0.6329	0.5929	0.5176
1	1	1.0	-0.1055	0.6073	0.5908	0.5703	0.5301	0.4167	0.2563	0.0170
1	1	10.0	-0.2503	0.1409	0.0439	-0.0503	-0.1807	-0.4025	-0.5782	-0.7181
1	2	0.1	-0.0693	-0.2296	-0.2285	-0.2271	-0.2243	-0.2163	-0.2037	-0.1810
1	2	1.0	-0.2641	-0.6715	-0.6636	-0.6536	-0.6330	-0.5703	-0.4729	-0.3186
1	2	10.0	-0.4351	-0.7809	-0.7256	-0.6547	-0.5185	-0.1462	0.3519	0.9883
1	3	0.1	0.0671	0.6595	0.6567	0.6533	0.6463	0.6258	0.5926	0.5295
1	3	1.0	0.0561	0.4884	0.4823	0.4746	0.4589	0.4088	0.3206	0.1574
1	3	10.0	0.0306	0.0054	0.0163	0.0266	0.0356	0.0248	-0.0213	-0.0885
2	1	0.1	-0.1216	0.7506	0.7471	0.7426	0.7339	0.7079	0.6660	0.5873
2	1	1.0	-0.3000	2.2546	2.2212	2.1797	2.0982	1.8687	1.5439	1.0593
2	1	10.0	-0.3781	1.3357	1.1371	0.9442	0.6773	0.2231	-0.1364	-0.4229
2	2	0.1	-0.0587	-0.2408	-0.2397	-0.2384	-0.2356	-0.2277	-0.2153	-0.1930
2	2	1.0	-0.1458	-0.7467	-0.7406	-0.7328	-0.7170	-0.6687	-0.5935	-0.4745
2	2	10.0	-0.3952	-0.9307	-0.9132	-0.8908	-0.8477	-0.7299	-0.5724	-0.3710
2	3	0.1	-0.0054	0.7581	0.7551	0.7514	0.7441	0.7224	0.6871	0.6203
2	3	1.0	0.0473	1.5841	1.5735	1.5601	1.5327	1.4457	1.2927	1.0093
2	3	10.0	0.0279	0.1329	0.1451	0.1567	0.1669	0.1547	0.1028	0.0270
3	1	0.1	-0.0353	0.0898	0.0895	0.0892	0.0885	0.0866	0.0833	0.0768
3	1	1.0	-0.1259	0.3857	0.3844	0.3828	0.3795	0.3695	0.3527	0.3185
3	1	10.0	-0.2900	0.1477	0.1397	0.1299	0.1108	0.0602	-0.0048	-0.1005
3	2	0.1	-0.1008	-0.2424	-0.2421	-0.2417	-0.2408	-0.2382	-0.2338	-0.2252
3	2	1.0	-0.4001	-0.7797	-0.7786	-0.7771	-0.7743	-0.7655	-0.7505	-0.7194
3	2	10.0	-0.6529	-0.9444	-0.9415	-0.9377	-0.9298	-0.9054	-0.8662	-0.7922
3	3	0.1	-0.0159	0.1048	0.1045	0.1042	0.1036	0.1018	0.0987	0.0925
3	3	1.0	0.0318	0.4812	0.4804	0.4794	0.4774	0.4713	0.4610	0.4390
3	3	10.0	0.0527	0.0864	0.0881	0.0904	0.0953	0.1097	0.1257	0.1338

Table 8.3: Error in reflection, boundary source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	-0.0004	-0.0328	-0.0327	-0.0325	-0.0321	-0.0310	-0.0292	-0.0257
1	1	1.0	-0.0039	-0.2229	-0.2199	-0.2162	-0.2084	-0.1834	-0.1386	-0.0560
1	1	10.0	-0.0883	-0.7136	-0.7139	-0.6738	-0.4818	0.6725	2.9995	6.0151
1	2	0.1	-0.0078	-0.0776	-0.0772	-0.0768	-0.0760	-0.0736	-0.0697	-0.0622
1	2	1.0	-0.0875	-0.5615	-0.5547	-0.5462	-0.5289	-0.4745	-0.3796	-0.2010
1	2	10.0	-0.4265	-0.9958	-0.9904	-0.9748	-0.8663	2.0591	19.4528	58.8540
1	3	0.1	0.0027	-0.0346	-0.0344	-0.0342	-0.0337	-0.0324	-0.0302	-0.0262
1	3	1.0	0.0503	-0.2481	-0.2424	-0.2352	-0.2207	-0.1780	-0.1123	-0.0077
1	3	10.0	1.8846	-0.7518	-0.5860	-0.2793	0.6539	4.6337	10.1278	15.5407
2	1	0.1	0.0043	-0.0350	-0.0349	-0.0347	-0.0343	-0.0332	-0.0314	-0.0279
2	1	1.0	0.0233	-0.3453	-0.3428	-0.3396	-0.3331	-0.3121	-0.2743	-0.2047
2	1	10.0	0.0392	-0.9582	-0.9583	-0.9524	-0.9245	-0.7562	-0.4169	0.0227
2	2	0.1	0.0001	-0.0839	-0.0836	-0.0832	-0.0824	-0.0800	-0.0761	-0.0687
2	2	1.0	-0.0303	-0.7018	-0.6972	-0.6914	-0.6796	-0.6426	-0.5780	-0.4566
2	2	10.0	-0.3070	-1.0000	-0.9999	-0.9997	-0.9985	-0.9657	-0.7705	-0.3285
2	3	0.1	0.0085	-0.0366	-0.0364	-0.0362	-0.0357	-0.0344	-0.0322	-0.0282
2	3	1.0	0.0464	-0.3916	-0.3870	-0.3811	-0.3694	-0.3349	-0.2817	-0.1971
2	3	10.0	0.3318	-0.9802	-0.9669	-0.9424	-0.8678	-0.5498	-0.1107	0.3219
3	1	0.1	0.0026	-0.0076	-0.0076	-0.0076	-0.0075	-0.0074	-0.0071	-0.0066
3	1	1.0	0.0412	-0.1833	-0.1828	-0.1822	-0.1810	-0.1773	-0.1711	-0.1580
3	1	10.0	0.3980	-0.6100	-0.6138	-0.6182	-0.6248	-0.6258	-0.5743	-0.3745
3	2	0.1	0.0041	-0.0189	-0.0189	-0.0188	-0.0187	-0.0184	-0.0178	-0.0167
3	2	1.0	-0.0275	-0.5537	-0.5529	-0.5518	-0.5496	-0.5429	-0.5314	-0.5069
3	2	10.0	-0.2170	-0.9999	-0.9999	-0.9999	-0.9999	-0.9997	-0.9986	-0.9824
3	3	0.1	0.0042	-0.0077	-0.0077	-0.0076	-0.0076	-0.0074	-0.0071	-0.0065
3	3	1.0	0.0973	-0.2243	-0.2235	-0.2226	-0.2206	-0.2146	-0.2044	-0.1833
3	3	10.0	1.1252	-0.9660	-0.9634	-0.9599	-0.9518	-0.9172	-0.8154	-0.4452

Table 8.4: Error in leakage, boundary source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.0033	0.0382	0.0380	0.0377	0.0371	0.0355	0.0328	0.0278
1	1	1.0	0.0187	0.0967	0.0959	0.0948	0.0926	0.0840	0.0645	0.0285
1	1	10.0	0.1686	0.4022	0.4377	0.4373	0.3499	0.1519	0.8118	1.5443
1	2	0.1	0.0065	0.0924	0.0918	0.0910	0.0895	0.0851	0.0781	0.0657
1	2	1.0	0.0700	0.4890	0.4818	0.4725	0.4533	0.3921	0.2920	0.1359
1	2	10.0	0.3859	0.9115	0.8870	0.8440	0.6922	1.0453	8.3070	22.7098
1	3	0.1	0.0052	0.0393	0.0391	0.0388	0.0382	0.0365	0.0337	0.0286
1	3	1.0	0.0453	0.1210	0.1184	0.1151	0.1084	0.0872	0.0540	0.0284
1	3	10.0	0.9570	0.5054	0.3645	0.1473	0.3904	2.0320	3.7669	5.2412
2	1	0.1	0.0068	0.0413	0.0410	0.0408	0.0402	0.0386	0.0359	0.0309
2	1	1.0	0.0343	0.1687	0.1678	0.1667	0.1643	0.1551	0.1353	0.0975
2	1	10.0	0.0576	0.6332	0.6502	0.6516	0.6165	0.4525	0.2433	0.0588
2	2	0.1	0.0006	0.0974	0.0968	0.0960	0.0945	0.0902	0.0833	0.0710
2	2	1.0	0.0294	0.5797	0.5740	0.5667	0.5517	0.5044	0.4284	0.3123
2	2	10.0	0.2642	0.9573	0.9494	0.9379	0.9106	0.8004	0.5858	0.2768
2	3	0.1	0.0088	0.0425	0.0423	0.0420	0.0414	0.0397	0.0370	0.0319
2	3	1.0	0.0644	0.1954	0.1935	0.1911	0.1862	0.1700	0.1427	0.1017
2	3	10.0	0.2357	0.7697	0.7296	0.6732	0.5532	0.2543	0.0584	0.2230
3	1	0.1	0.0031	0.0130	0.0130	0.0129	0.0129	0.0126	0.0123	0.0115
3	1	1.0	0.0474	0.0941	0.0940	0.0939	0.0935	0.0925	0.0907	0.0859
3	1	10.0	0.1808	0.2397	0.2487	0.2595	0.2791	0.3162	0.3208	0.2554
3	2	0.1	0.0039	0.0295	0.0295	0.0293	0.0291	0.0285	0.0275	0.0254
3	2	1.0	0.0221	0.4366	0.4357	0.4346	0.4324	0.4256	0.4136	0.3873
3	2	10.0	0.1643	0.9332	0.9318	0.9300	0.9262	0.9121	0.8801	0.7836
3	3	0.1	0.0041	0.0135	0.0135	0.0134	0.0134	0.0131	0.0127	0.0119
3	3	1.0	0.0971	0.1170	0.1168	0.1165	0.1158	0.1139	0.1105	0.1032
3	3	10.0	0.9953	0.6903	0.6829	0.6732	0.6523	0.5781	0.4271	0.2164

Table 8.5: Error in $\langle \phi(x) \rangle$, boundary source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.1003	0.1114	0.1113	0.1113	0.1111	0.1106	0.1098	0.1082
1	1	1.0	0.1096	0.1329	0.1339	0.1350	0.1369	0.1390	0.1334	0.1183
1	1	10.0	0.2353	0.4392	0.4751	0.4767	0.3985	0.1522	0.7165	1.4039
1	2	0.1	0.1045	0.2170	0.2163	0.2154	0.2135	0.2082	0.1997	0.1845
1	2	1.0	0.1554	0.5610	0.5539	0.5449	0.5261	0.4667	0.3696	0.2182
1	2	10.0	0.4465	0.9236	0.9011	0.8617	0.7230	0.8808	7.5968	21.0160
1	3	0.1	0.1000	0.1234	0.1233	0.1231	0.1227	0.1216	0.1198	0.1165
1	3	1.0	0.0784	0.1987	0.1959	0.1924	0.1853	0.1636	0.1308	0.0903
1	3	10.0	0.7570	0.5608	0.4360	0.2457	0.2560	1.7184	3.2793	4.6106
2	1	0.1	0.1002	0.1158	0.1157	0.1156	0.1154	0.1148	0.1138	0.1120
2	1	1.0	0.1021	0.2060	0.2062	0.2064	0.2065	0.2038	0.1920	0.1652
2	1	10.0	0.1153	0.6473	0.6652	0.6676	0.6352	0.4809	0.2858	0.1181
2	2	0.1	0.1013	0.2295	0.2287	0.2278	0.2261	0.2208	0.2125	0.1976
2	2	1.0	0.1226	0.6475	0.6421	0.6351	0.6207	0.5756	0.5033	0.3925
2	2	10.0	0.3284	0.9635	0.9561	0.9453	0.9200	0.8182	0.6213	0.3388
2	3	0.1	0.1002	0.1289	0.1288	0.1286	0.1282	0.1270	0.1251	0.1216
2	3	1.0	0.0952	0.2757	0.2735	0.2707	0.2649	0.2467	0.2180	0.1765
2	3	10.0	0.0966	0.7975	0.7621	0.7123	0.6066	0.3421	0.1140	0.0819
3	1	0.1	0.4999	0.5005	0.5005	0.5005	0.5005	0.5006	0.5006	0.5006
3	1	1.0	0.4999	0.5108	0.5108	0.5109	0.5111	0.5116	0.5123	0.5134
3	1	10.0	0.4836	0.5454	0.5518	0.5594	0.5728	0.5987	0.6061	0.5768
3	2	0.1	0.5001	0.5736	0.5734	0.5732	0.5728	0.5715	0.5694	0.5653
3	2	1.0	0.5100	0.8032	0.8024	0.8015	0.7997	0.7942	0.7846	0.7640
3	2	10.0	0.5799	0.9760	0.9751	0.9739	0.9713	0.9621	0.9425	0.8875
3	3	0.1	0.4999	0.5096	0.5096	0.5096	0.5095	0.5094	0.5091	0.5086
3	3	1.0	0.5000	0.5796	0.5794	0.5791	0.5785	0.5767	0.5737	0.5673
3	3	10.0	0.3777	0.8808	0.8780	0.8742	0.8662	0.8375	0.7777	0.6445

Table 8.6: Error in $\langle \phi_0(x) \rangle$, boundary source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.8997	0.8985	0.8984	0.8984	0.8984	0.8983	0.8982	0.8981
1	1	1.0	0.9063	0.9069	0.9070	0.9070	0.9070	0.9071	0.9070	0.9066
1	1	10.0	0.9261	0.9430	0.9457	0.9456	0.9390	0.9093	0.8731	0.8414
1	2	0.1	0.8968	0.8114	0.8117	0.8122	0.8130	0.8156	0.8199	0.8283
1	2	1.0	0.9011	0.8483	0.8487	0.8493	0.8506	0.8554	0.8654	0.8870
1	2	10.0	0.9167	0.9674	0.9638	0.9582	0.9424	0.8668	0.9384	1.2233
1	3	0.1	0.8934	0.8870	0.8870	0.8870	0.8871	0.8872	0.8873	0.8878
1	3	1.0	0.8765	0.8817	0.8815	0.8813	0.8808	0.8796	0.8782	0.8769
1	3	10.0	0.7686	0.9217	0.9040	0.8781	0.8213	0.6835	0.5965	0.5749
2	1	0.1	0.8993	0.8963	0.8963	0.8963	0.8962	0.8961	0.8960	0.8959
2	1	1.0	0.8994	0.9041	0.9041	0.9040	0.9039	0.9036	0.9029	0.9018
2	1	10.0	0.9313	0.9656	0.9671	0.9673	0.9648	0.9540	0.9419	0.9315
2	2	0.1	0.8999	0.7642	0.7646	0.7651	0.7662	0.7693	0.7746	0.7849
2	2	1.0	0.8849	0.3893	0.3901	0.3915	0.3949	0.4125	0.4568	0.5524
2	2	10.0	0.8921	0.9207	0.9159	0.9104	0.9012	0.8850	0.8899	0.9245
2	3	0.1	0.8907	0.8808	0.8808	0.8808	0.8809	0.8810	0.8811	0.8816
2	3	1.0	0.6877	0.7133	0.7126	0.7118	0.7102	0.7060	0.7010	0.6960
2	3	10.0	0.5220	0.8799	0.8616	0.8367	0.7865	0.6718	0.5829	0.5246
3	1	0.1	0.5000	0.4984	0.4984	0.4984	0.4984	0.4984	0.4984	0.4983
3	1	1.0	0.4954	0.4962	0.4961	0.4961	0.4961	0.4960	0.4959	0.4958
3	1	10.0	0.5806	0.6326	0.6373	0.6428	0.6526	0.6711	0.6753	0.6516
3	2	0.1	0.4999	0.4287	0.4288	0.4290	0.4293	0.4303	0.4319	0.4353
3	2	1.0	0.4858	0.2123	0.2129	0.2136	0.2151	0.2198	0.2283	0.2483
3	2	10.0	0.4860	0.8484	0.8465	0.8441	0.8393	0.8245	0.8013	0.7814
3	3	0.1	0.4977	0.4897	0.4898	0.4898	0.4898	0.4899	0.4900	0.4902
3	3	1.0	0.3929	0.4085	0.4084	0.4082	0.4079	0.4069	0.4054	0.4026
3	3	10.0	1.5936	0.6227	0.6133	0.6010	0.5748	0.4856	0.3392	0.4780

Table 8.7: Error in $\langle \phi_1(x) \rangle$, boundary source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.4981	0.4923	0.4925	0.4927	0.4930	0.4941	0.4957	0.4982
1	1	1.0	0.4724	0.4380	0.4418	0.4463	0.4548	0.4742	0.4881	0.4867
1	1	10.0	0.1712	0.1398	0.1475	0.1552	0.1659	0.1819	0.1894	0.1909
1	2	0.1	0.4655	0.4321	0.4324	0.4327	0.4333	0.4352	0.4382	0.4433
1	2	1.0	0.3401	0.2079	0.2106	0.2142	0.2214	0.2432	0.2761	0.3233
1	2	10.0	0.0675	0.0283	0.0333	0.0402	0.0549	0.1005	0.1650	0.2415
1	3	0.1	0.5014	0.4915	0.4917	0.4920	0.4924	0.4937	0.4955	0.4984
1	3	1.0	0.5176	0.4330	0.4383	0.4448	0.4573	0.4882	0.5162	0.5286
1	3	10.0	0.2971	0.1379	0.1704	0.2089	0.2689	0.3588	0.3977	0.4083
2	1	0.1	0.4961	0.4923	0.4925	0.4927	0.4930	0.4941	0.4957	0.4982
2	1	1.0	0.4310	0.4380	0.4418	0.4463	0.4548	0.4742	0.4881	0.4867
2	1	10.0	0.1890	0.1398	0.1475	0.1552	0.1659	0.1819	0.1894	0.1909
2	2	0.1	0.4697	0.4321	0.4324	0.4327	0.4333	0.4352	0.4382	0.4433
2	2	1.0	0.4283	0.2079	0.2106	0.2142	0.2214	0.2432	0.2761	0.3233
2	2	10.0	0.2428	0.0283	0.0333	0.0402	0.0549	0.1005	0.1650	0.2415
2	3	0.1	0.5002	0.4915	0.4917	0.4920	0.4924	0.4937	0.4955	0.4984
2	3	1.0	0.4891	0.4330	0.4383	0.4448	0.4573	0.4882	0.5162	0.5286
2	3	10.0	0.4065	0.1379	0.1704	0.2089	0.2689	0.3588	0.3977	0.4083
3	1	0.1	0.5019	0.4991	0.4992	0.4992	0.4993	0.4994	0.4997	0.5002
3	1	1.0	0.5056	0.4924	0.4931	0.4939	0.4955	0.5003	0.5078	0.5203
3	1	10.0	0.3828	0.3722	0.3746	0.3776	0.3834	0.3979	0.4116	0.4189
3	2	0.1	0.4376	0.4269	0.4269	0.4269	0.4270	0.4273	0.4277	0.4285
3	2	1.0	0.2965	0.1971	0.1975	0.1979	0.1986	0.2011	0.2053	0.2141
3	2	10.0	0.1165	0.0254	0.0259	0.0266	0.0280	0.0327	0.0417	0.0612
3	3	0.1	0.4955	0.4915	0.4915	0.4916	0.4916	0.4918	0.4921	0.4926
3	3	1.0	0.4833	0.4318	0.4325	0.4334	0.4351	0.4403	0.4486	0.4635
3	3	10.0	0.3852	0.1306	0.1347	0.1400	0.1510	0.1849	0.2375	0.3157

Table 8.8: Leakage, internal source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.0128	0.0009	0.0013	0.0017	0.0025	0.0047	0.0079	0.0128
1	1	1.0	0.0976	0.0177	0.0264	0.0369	0.0566	0.1017	0.1341	0.1307
1	1	10.0	0.0976	-0.1039	-0.0545	-0.0051	0.0637	0.1665	0.2147	0.2243
1	2	0.1	-0.0059	-0.0772	-0.0767	-0.0760	-0.0746	-0.0705	-0.0642	-0.0532
1	2	1.0	-0.0616	-0.4265	-0.4188	-0.4091	-0.3892	-0.3289	-0.2381	-0.1079
1	2	10.0	-0.1783	-0.6559	-0.5943	-0.5098	-0.3309	0.2246	1.0098	1.9415
1	3	0.1	0.0167	-0.0034	-0.0030	-0.0025	-0.0016	0.0009	0.0047	0.0106
1	3	1.0	0.1437	-0.0433	-0.0316	-0.0172	0.0104	0.0786	0.1406	0.1679
1	3	10.0	0.6233	-0.2466	-0.0688	0.1414	0.4690	0.9604	1.1728	1.2306
2	1	0.1	0.0087	0.0009	0.0012	0.0016	0.0024	0.0046	0.0079	0.0128
2	1	1.0	-0.0023	0.0139	0.0226	0.0331	0.0527	0.0976	0.1299	0.1265
2	1	10.0	-0.0125	-0.2699	-0.2296	-0.1894	-0.1333	-0.0495	-0.0102	-0.0024
2	2	0.1	-0.0003	-0.0804	-0.0798	-0.0791	-0.0777	-0.0737	-0.0674	-0.0564
2	2	1.0	-0.0283	-0.5284	-0.5221	-0.5141	-0.4978	-0.4482	-0.3735	-0.2665
2	2	10.0	-0.2263	-0.9100	-0.8939	-0.8718	-0.8250	-0.6797	-0.4743	-0.2306
2	3	0.1	0.0150	-0.0025	-0.0021	-0.0016	-0.0007	0.0018	0.0056	0.0115
2	3	1.0	0.0692	-0.0534	-0.0418	-0.0276	-0.0002	0.0672	0.1285	0.1556
2	3	10.0	0.2475	-0.5768	-0.4769	-0.3588	-0.1748	0.1012	0.2205	0.2529
3	1	0.1	0.0055	-0.0000	0.0000	0.0001	0.0002	0.0005	0.0010	0.0020
3	1	1.0	0.0287	0.0020	0.0033	0.0050	0.0083	0.0180	0.0332	0.0586
3	1	10.0	-0.0799	-0.1055	-0.0997	-0.0925	-0.0785	-0.0438	-0.0108	0.0068
3	2	0.1	0.0031	-0.0216	-0.0215	-0.0214	-0.0212	-0.0207	-0.0197	-0.0179
3	2	1.0	-0.0211	-0.3490	-0.3480	-0.3467	-0.3441	-0.3361	-0.3222	-0.2930
3	2	10.0	-0.1360	-0.8116	-0.8078	-0.8029	-0.7925	-0.7571	-0.6907	-0.5457
3	3	0.1	0.0076	-0.0005	-0.0005	-0.0004	-0.0003	0.0001	0.0007	0.0018
3	3	1.0	0.1062	-0.0116	-0.0100	-0.0079	-0.0039	0.0079	0.0269	0.0609
3	3	10.0	0.6764	-0.4315	-0.4136	-0.3906	-0.3429	-0.1952	0.0336	0.3741

Table 8.9: Error in leakage, internal source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.0083	0.0335	0.0330	0.0323	0.0311	0.0277	0.0230	0.0163
1	1	1.0	0.0485	0.0833	0.0752	0.0660	0.0523	0.0502	0.0682	0.0620
1	1	10.0	0.0393	0.0527	0.0406	0.0268	0.0130	0.0782	0.1370	0.1791
1	2	0.1	0.0059	0.1111	0.1101	0.1088	0.1062	0.0988	0.0878	0.0700
1	2	1.0	0.0589	0.5124	0.4991	0.4824	0.4486	0.3526	0.2303	0.0949
1	2	10.0	0.1838	0.7127	0.6243	0.5097	0.3028	0.1774	0.6843	1.1964
1	3	0.1	0.0119	0.0385	0.0379	0.0372	0.0357	0.0318	0.0263	0.0184
1	3	1.0	0.0855	0.1469	0.1340	0.1185	0.0900	0.0431	0.0723	0.0934
1	3	10.0	0.4553	0.1912	0.0255	0.1793	0.4114	0.5903	0.5745	0.5279
2	1	0.1	0.0061	0.0356	0.0351	0.0344	0.0332	0.0297	0.0249	0.0178
2	1	1.0	0.0718	0.1409	0.1319	0.1210	0.1009	0.0567	0.0238	0.0143
2	1	10.0	0.1874	0.0846	0.0716	0.0604	0.0548	0.0976	0.1510	0.1911
2	2	0.1	0.0004	0.1149	0.1138	0.1125	0.1099	0.1026	0.0917	0.0739
2	2	1.0	0.0258	0.5810	0.5696	0.5552	0.5262	0.4439	0.3394	0.2237
2	2	10.0	0.2153	0.8992	0.8681	0.8278	0.7550	0.5875	0.4102	0.2320
2	3	0.1	0.0118	0.0401	0.0395	0.0387	0.0373	0.0334	0.0279	0.0198
2	3	1.0	0.0529	0.1655	0.1536	0.1394	0.1139	0.0709	0.0773	0.0849
2	3	10.0	0.1796	0.3659	0.2331	0.1162	0.1540	0.2567	0.2205	0.1673
3	1	0.1	0.0061	0.0090	0.0090	0.0089	0.0088	0.0083	0.0077	0.0066
3	1	1.0	0.0160	0.1023	0.1009	0.0991	0.0957	0.0860	0.0715	0.0489
3	1	10.0	0.4176	0.0661	0.0676	0.0696	0.0746	0.0995	0.1617	0.2779
3	2	0.1	0.0042	0.0336	0.0335	0.0333	0.0329	0.0319	0.0301	0.0269
3	2	1.0	0.0205	0.4312	0.4293	0.4269	0.4221	0.4076	0.3838	0.3379
3	2	10.0	0.1215	0.8243	0.8180	0.8098	0.7926	0.7381	0.6504	0.4985
3	3	0.1	0.0081	0.0102	0.0101	0.0100	0.0098	0.0094	0.0086	0.0074
3	3	1.0	0.0833	0.1203	0.1187	0.1166	0.1124	0.1006	0.0828	0.0569
3	3	10.0	0.5515	0.2823	0.2617	0.2356	0.1842	0.0975	0.2465	0.4373

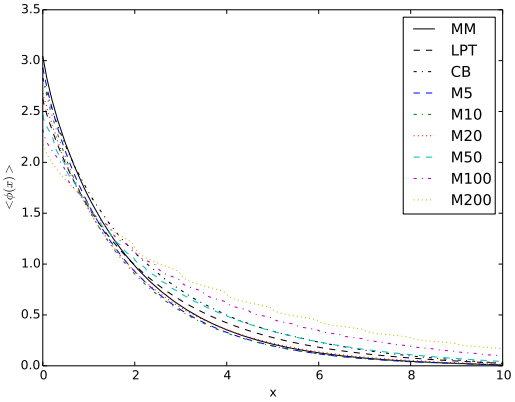
Table 8.10: Error in $\langle \phi(x) \rangle$, internal source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.0969	0.1358	0.1351	0.1342	0.1326	0.1279	0.1212	0.1114
1	1	1.0	0.0628	0.1780	0.1686	0.1573	0.1363	0.0891	0.0535	0.0498
1	1	10.0	0.1182	0.0685	0.0653	0.0692	0.0901	0.1520	0.2058	0.2445
1	2	0.1	0.1033	0.2429	0.2416	0.2399	0.2366	0.2273	0.2132	0.1899
1	2	1.0	0.1454	0.5898	0.5770	0.5607	0.5280	0.4348	0.3152	0.1812
1	2	10.0	0.2689	0.7596	0.6777	0.5714	0.3795	0.0672	0.5371	1.0136
1	3	0.1	0.0971	0.1487	0.1479	0.1470	0.1451	0.1397	0.1320	0.1204
1	3	1.0	0.0435	0.2556	0.2430	0.2276	0.1983	0.1276	0.0655	0.0369
1	3	10.0	0.2698	0.2998	0.1439	0.0285	0.2300	0.3914	0.3842	0.3512
2	1	0.1	0.0996	0.1368	0.1361	0.1352	0.1335	0.1288	0.1222	0.1123
2	1	1.0	0.0961	0.1773	0.1686	0.1583	0.1394	0.0992	0.0689	0.0535
2	1	10.0	0.1405	0.1351	0.1211	0.1057	0.0799	0.0632	0.1047	0.1446
2	2	0.1	0.1011	0.2529	0.2516	0.2500	0.2468	0.2376	0.2237	0.2007
2	2	1.0	0.1192	0.6553	0.6444	0.6308	0.6033	0.5251	0.4249	0.3128
2	2	10.0	0.2854	0.9162	0.8876	0.8505	0.7835	0.6291	0.4654	0.3006
2	3	0.1	0.0998	0.1513	0.1505	0.1495	0.1476	0.1423	0.1347	0.1231
2	3	1.0	0.0901	0.2805	0.2686	0.2540	0.2265	0.1606	0.1035	0.0731
2	3	10.0	0.0181	0.4661	0.3492	0.2285	0.1040	0.0806	0.0386	0.0224
3	1	0.1	0.5000	0.5210	0.5208	0.5206	0.5202	0.5190	0.5172	0.5138
3	1	1.0	0.4954	0.5541	0.5529	0.5515	0.5488	0.5408	0.5284	0.5082
3	1	10.0	0.5492	0.2911	0.2925	0.2942	0.2982	0.3158	0.3584	0.4419
3	2	0.1	0.5002	0.5911	0.5908	0.5904	0.5896	0.5872	0.5834	0.5760
3	2	1.0	0.5091	0.8224	0.8210	0.8193	0.8158	0.8052	0.7875	0.7530
3	2	10.0	0.5637	0.9523	0.9488	0.9443	0.9348	0.9048	0.8564	0.7721
3	3	0.1	0.5000	0.5297	0.5295	0.5293	0.5289	0.5275	0.5254	0.5215
3	3	1.0	0.4967	0.6178	0.6165	0.6149	0.6117	0.6023	0.5875	0.5618
3	3	10.0	0.4243	0.7527	0.7442	0.7333	0.7110	0.6458	0.5614	0.4772

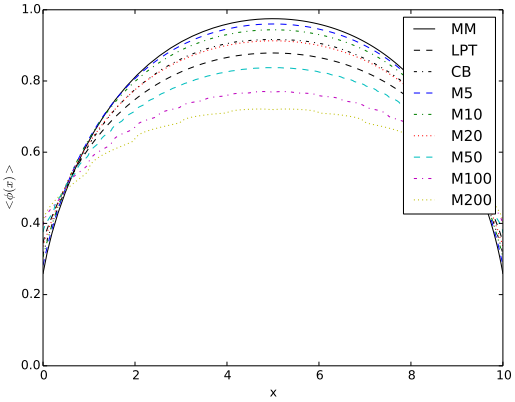
Table 8.11: Error in $\langle \phi_0(x) \rangle$, internal source problems

M	S	L	LPT	MM	M5	M10	M20	M50	M100	M200
1	1	0.1	0.8929	0.8868	0.8869	0.8872	0.8876	0.8888	0.8907	0.8937
1	1	1.0	0.8860	0.8918	0.8914	0.8908	0.8899	0.8879	0.8868	0.8870
1	1	10.0	0.9103	0.9043	0.9048	0.9057	0.9078	0.9132	0.9177	0.9209
1	2	0.1	0.8979	0.7957	0.7965	0.7975	0.7995	0.8053	0.8143	0.8303
1	2	1.0	0.9006	0.8136	0.8155	0.8178	0.8227	0.8373	0.8596	0.8910
1	2	10.0	0.8991	0.8861	0.8882	0.8910	0.8966	0.9122	0.9321	0.9536
1	3	0.1	0.8867	0.8744	0.8746	0.8749	0.8754	0.8770	0.8795	0.8834
1	3	1.0	0.8495	0.8621	0.8613	0.8603	0.8585	0.8544	0.8516	0.8510
1	3	10.0	0.7856	0.8712	0.8471	0.8215	0.7907	0.7697	0.7759	0.7857
2	1	0.1	0.8937	0.8903	0.8905	0.8907	0.8911	0.8923	0.8941	0.8970
2	1	1.0	0.9396	0.9439	0.9436	0.9433	0.9428	0.9416	0.9410	0.9411
2	1	10.0	0.9650	0.9590	0.9591	0.9594	0.9602	0.9622	0.9639	0.9652
2	2	0.1	0.8998	0.7699	0.7708	0.7719	0.7742	0.7807	0.7909	0.8089
2	2	1.0	0.8958	0.5326	0.5373	0.5433	0.5556	0.5934	0.6514	0.7329
2	2	10.0	0.8979	0.7764	0.7807	0.7864	0.7983	0.8321	0.8748	0.9155
2	3	0.1	0.8854	0.8750	0.8753	0.8756	0.8761	0.8777	0.8801	0.8840
2	3	1.0	0.8053	0.8257	0.8247	0.8234	0.8211	0.8158	0.8121	0.8112
2	3	10.0	0.5530	0.7310	0.6811	0.6285	0.5663	0.5247	0.5375	0.5579
3	1	0.1	0.4948	0.4824	0.4825	0.4826	0.4829	0.4836	0.4848	0.4871
3	1	1.0	0.4976	0.5366	0.5362	0.5357	0.5348	0.5322	0.5284	0.5231
3	1	10.0	0.7747	0.6473	0.6482	0.6495	0.6521	0.6628	0.6859	0.7272
3	2	0.1	0.5000	0.4116	0.4119	0.4123	0.4130	0.4150	0.4184	0.4251
3	2	1.0	0.4978	0.2409	0.2417	0.2428	0.2449	0.2514	0.2629	0.2881
3	2	10.0	0.5064	0.3566	0.3578	0.3595	0.3634	0.3785	0.4104	0.4733
3	3	0.1	0.4929	0.4736	0.4737	0.4738	0.4741	0.4750	0.4763	0.4790
3	3	1.0	0.4012	0.4606	0.4601	0.4595	0.4583	0.4550	0.4500	0.4424
3	3	10.0	0.3343	0.3226	0.3058	0.2846	0.2424	0.1413	0.1596	0.2801

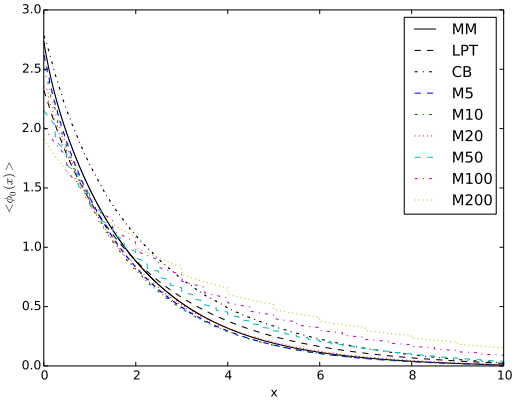
Table 8.12: Error in $\langle \phi_1(x) \rangle$, internal source problems



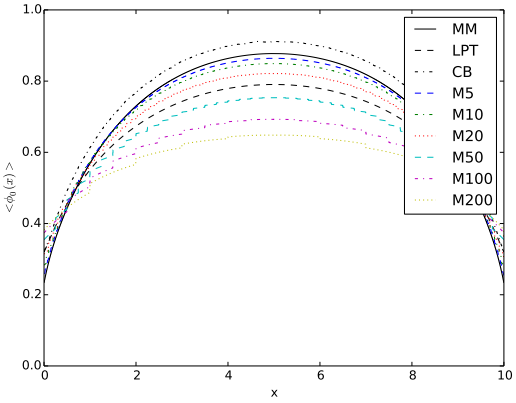
(a) $\langle \phi(x) \rangle$, boundary source



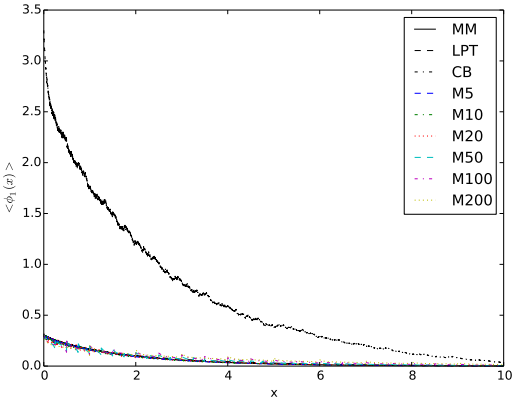
(b) $\langle \phi(x) \rangle$, internal source



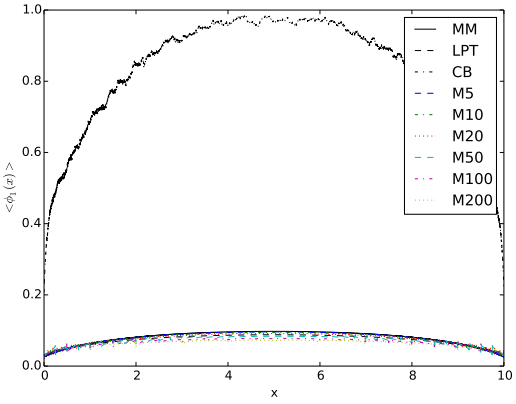
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

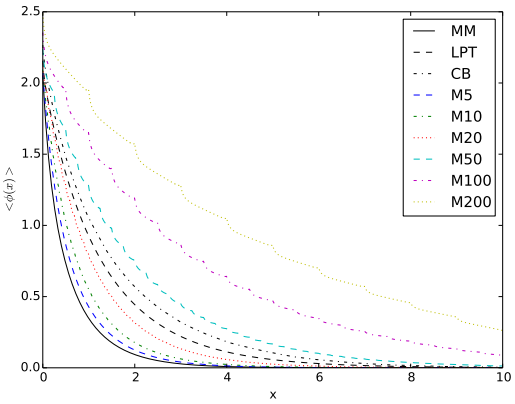


(e) $\langle \phi_1(x) \rangle$, boundary source

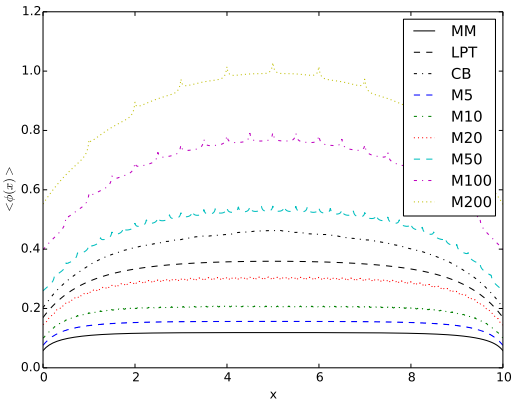


(f) $\langle \phi_1(x) \rangle$, internal source

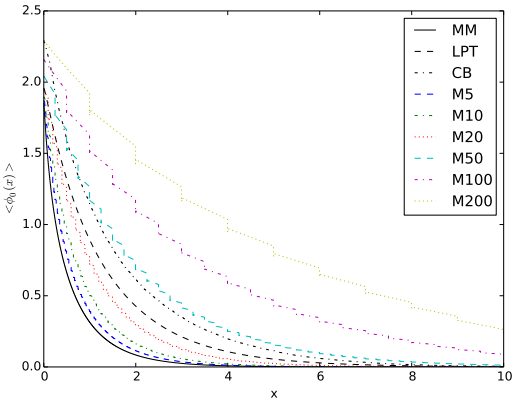
Figure 8.1: Results for M1, S1, L10.0, skip problems



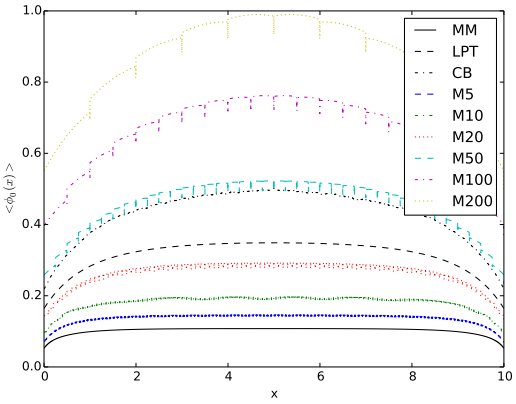
(a) $\langle\phi(x)\rangle$, boundary source



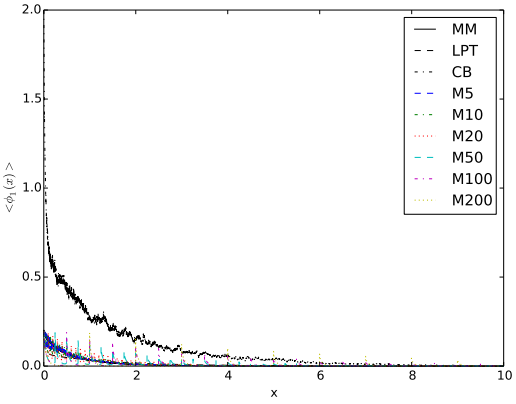
(b) $\langle\phi(x)\rangle$, internal source



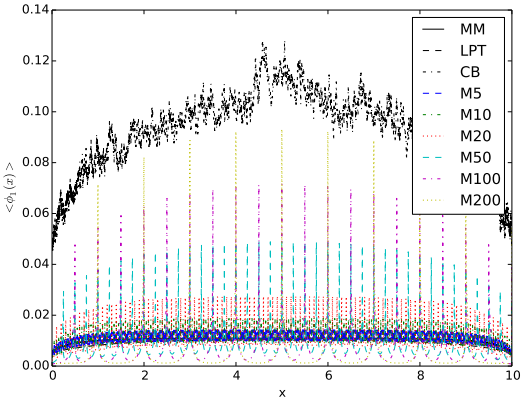
(c) $\langle\phi_0(x)\rangle$, boundary source



(d) $\langle\phi_0(x)\rangle$, internal source

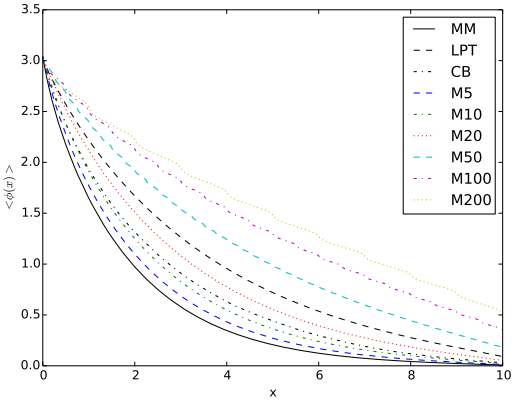


(e) $\langle\phi_1(x)\rangle$, boundary source

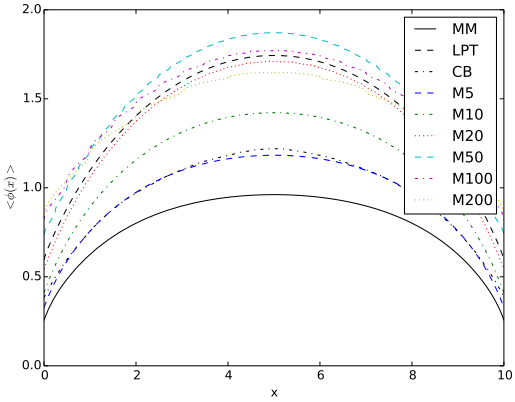


(f) $\langle\phi_1(x)\rangle$, internal source

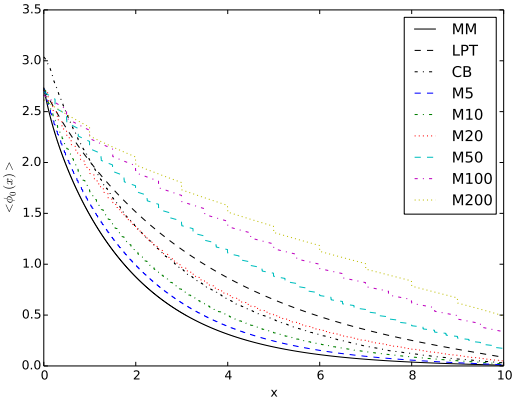
Figure 8.2: Results for M1, S2, L10.0, skip problems



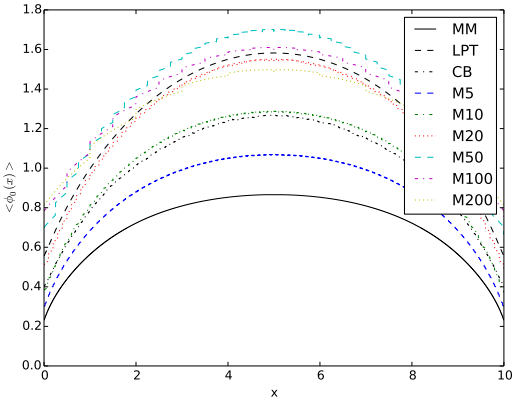
(a) $\langle \phi(x) \rangle$, boundary source



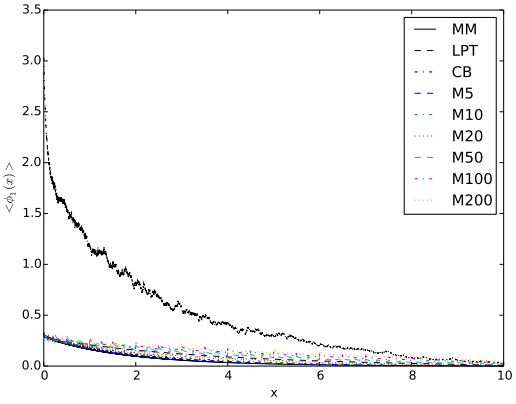
(b) $\langle \phi(x) \rangle$, internal source



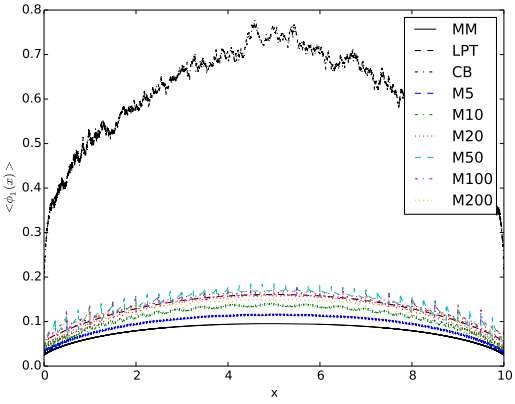
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

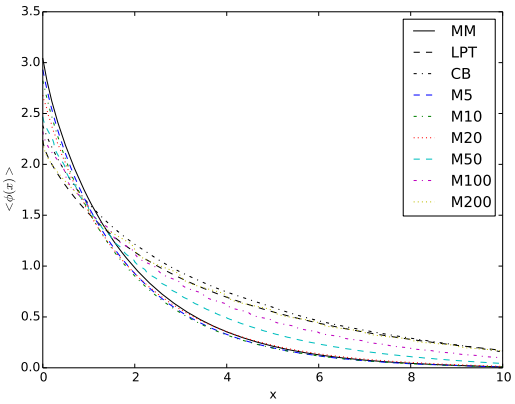


(e) $\langle \phi_1(x) \rangle$, boundary source

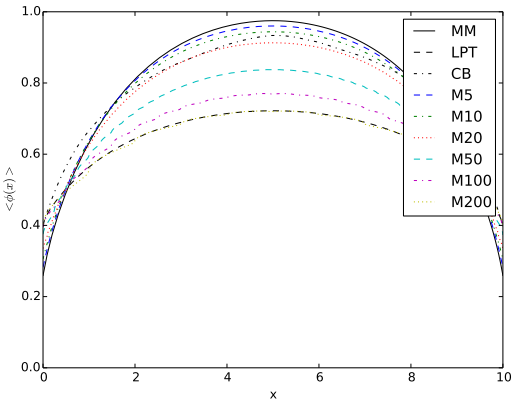


(f) $\langle \phi_1(x) \rangle$, internal source

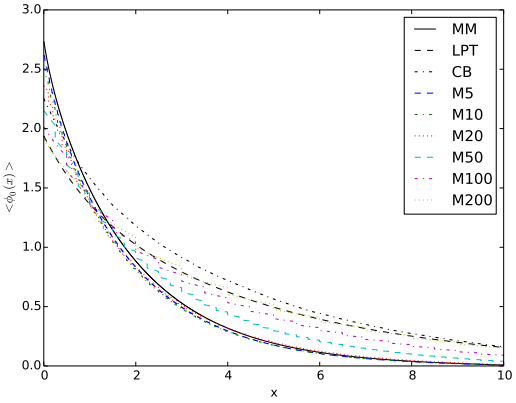
Figure 8.3: Results for M1, S3, L10.0, skip problems



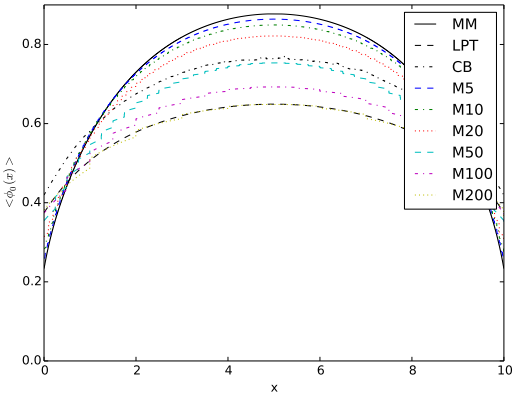
(a) $\langle \phi(x) \rangle$, boundary source



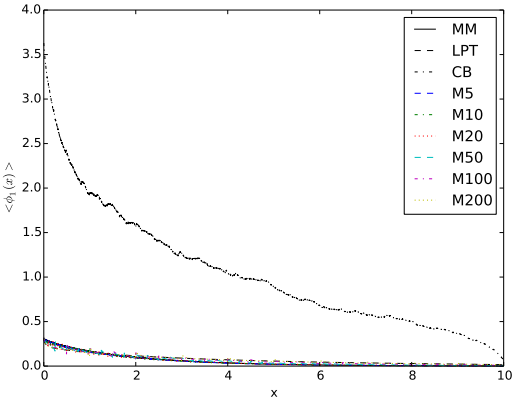
(b) $\langle \phi(x) \rangle$, internal source



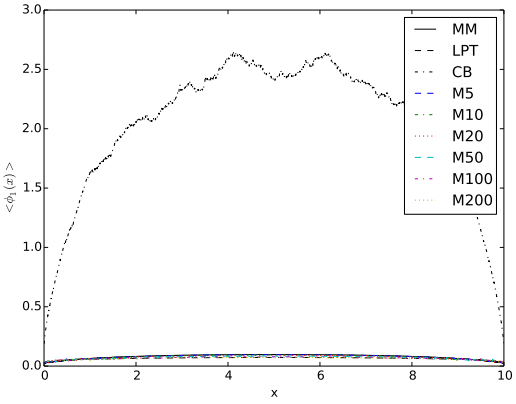
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

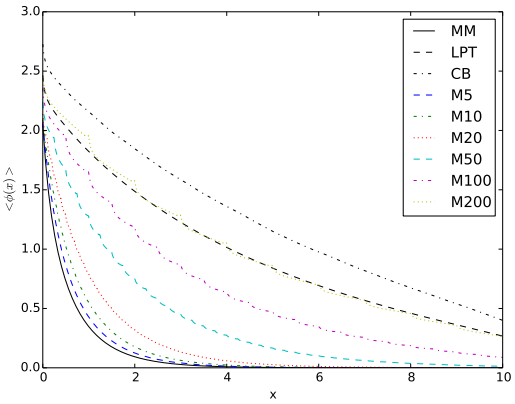


(e) $\langle \phi_1(x) \rangle$, boundary source

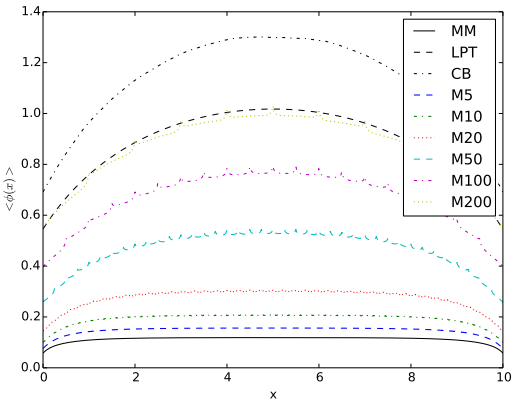


(f) $\langle \phi_1(x) \rangle$, internal source

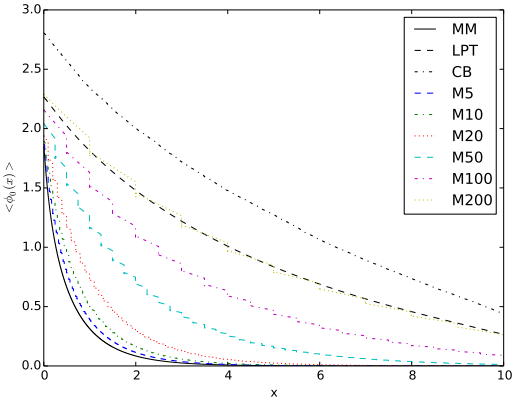
Figure 8.4: Results for M2, S1, L10.0, skip problems



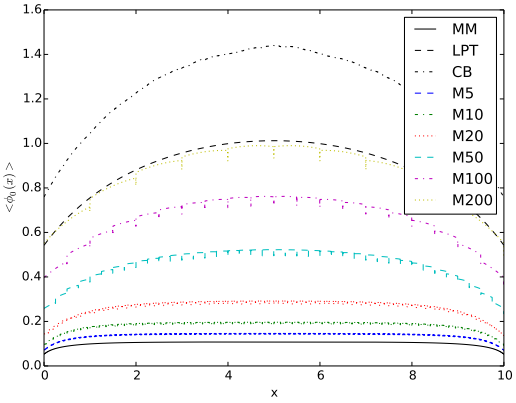
(a) $\langle \phi(x) \rangle$, boundary source



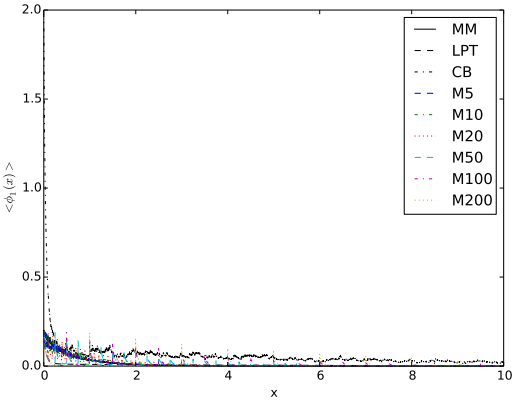
(b) $\langle \phi(x) \rangle$, internal source



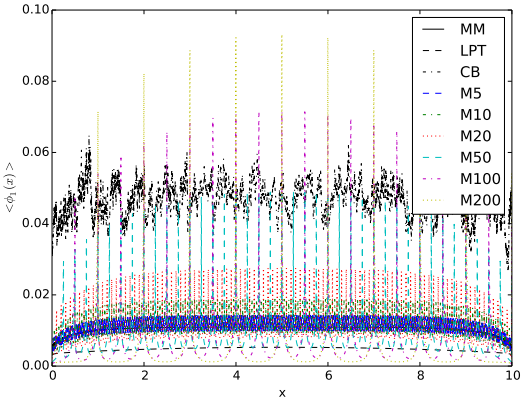
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

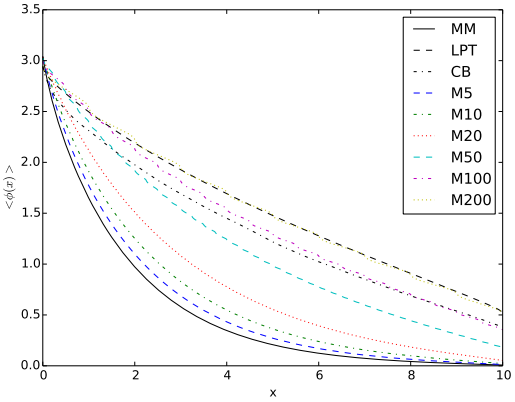


(e) $\langle \phi_1(x) \rangle$, boundary source

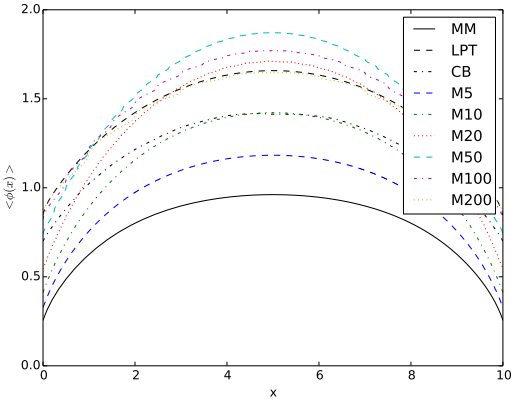


(f) $\langle \phi_1(x) \rangle$, internal source

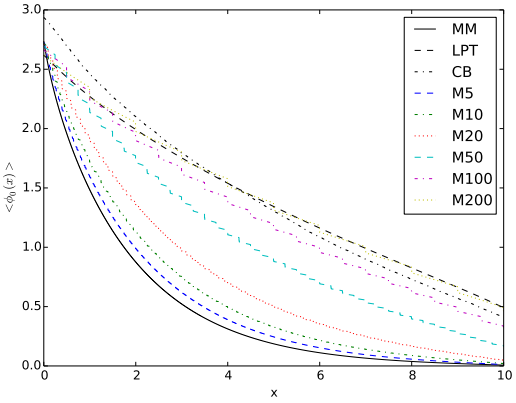
Figure 8.5: Results for M2, S2, L10.0, skip problems



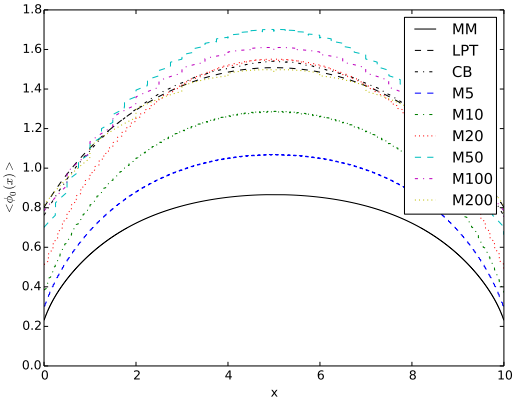
(a) $\langle \phi(x) \rangle$, boundary source



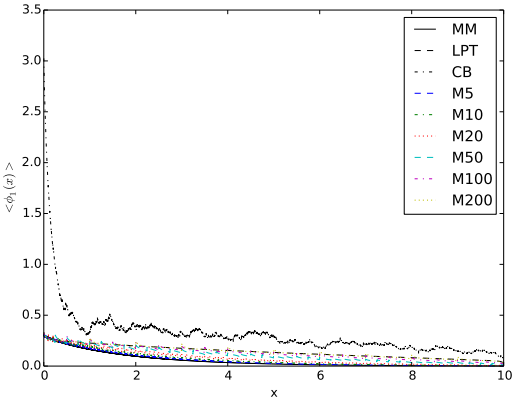
(b) $\langle \phi(x) \rangle$, internal source



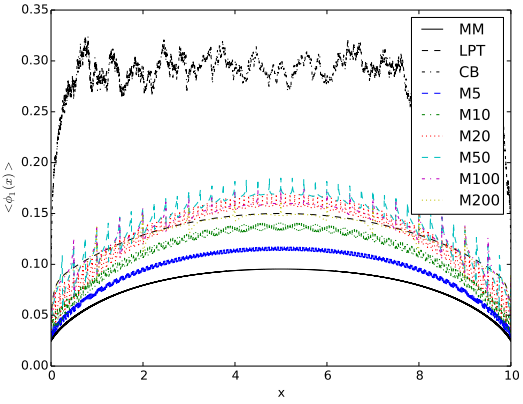
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

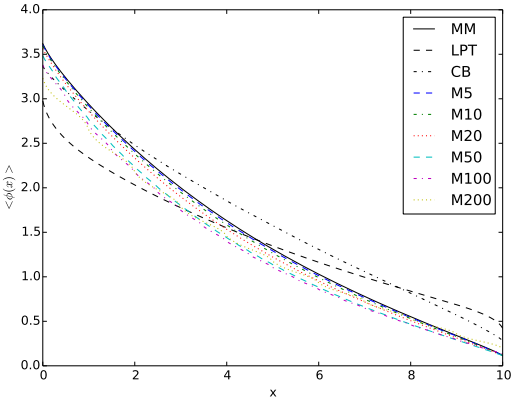


(e) $\langle \phi_1(x) \rangle$, boundary source

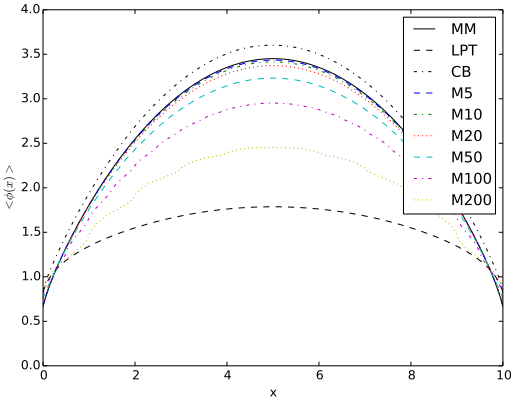


(f) $\langle \phi_1(x) \rangle$, internal source

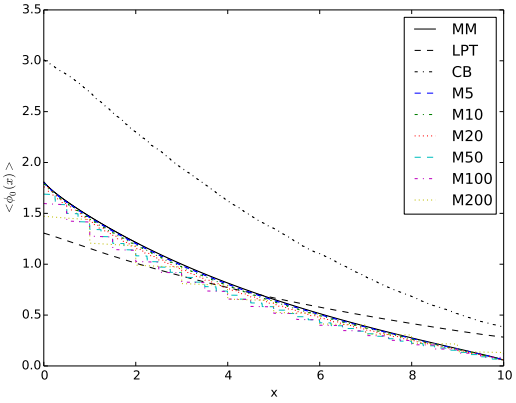
Figure 8.6: Results for M2, S3, L10.0, skip problems



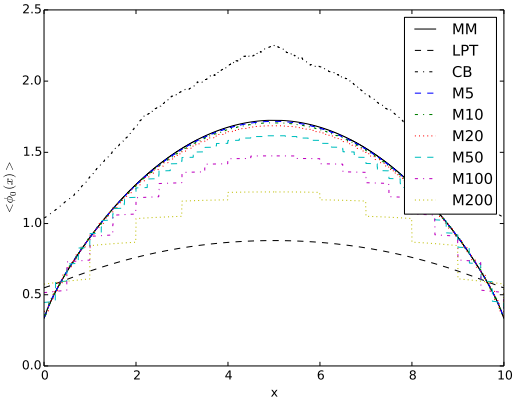
(a) $\langle \phi(x) \rangle$, boundary source



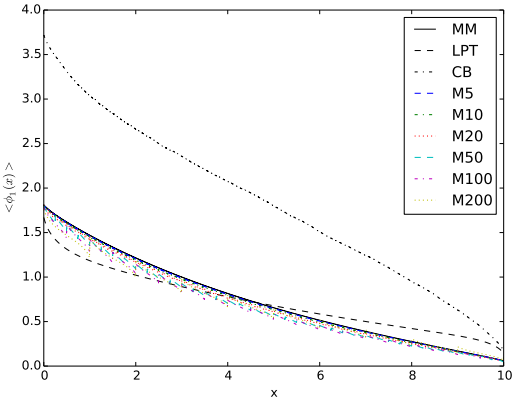
(b) $\langle \phi(x) \rangle$, internal source



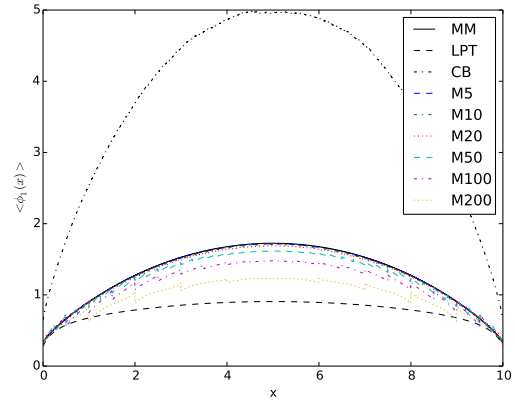
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

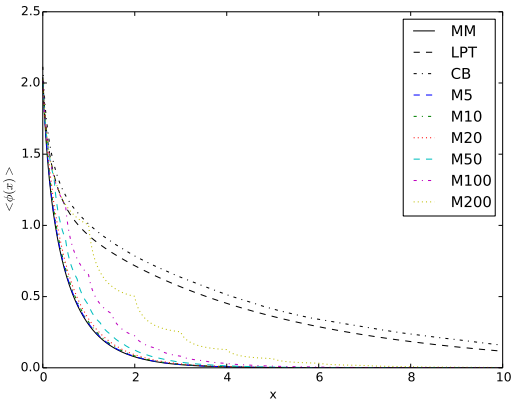


(e) $\langle \phi_1(x) \rangle$, boundary source

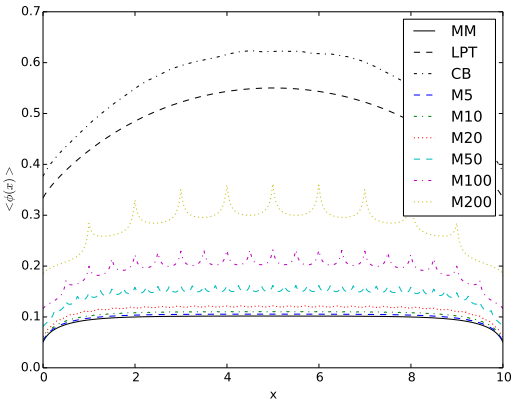


(f) $\langle \phi_1(x) \rangle$, internal source

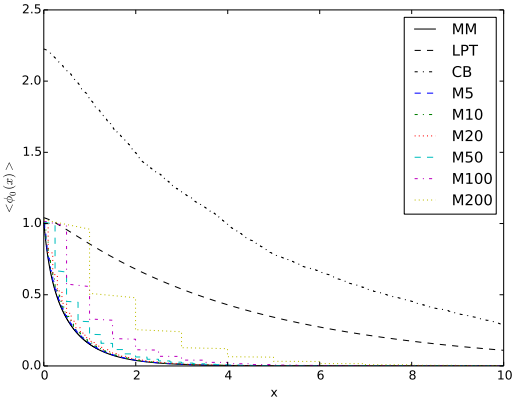
Figure 8.7: Results for M3, S1, L10.0, skip problems



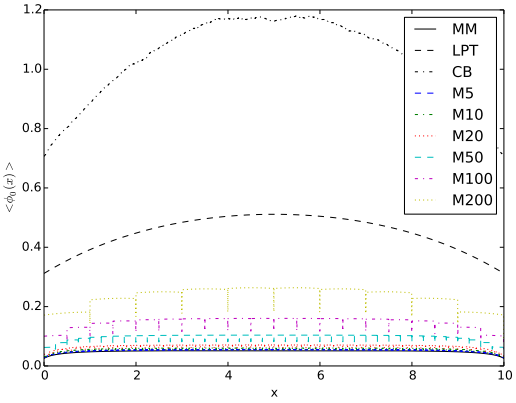
(a) $\langle \phi(x) \rangle$, boundary source



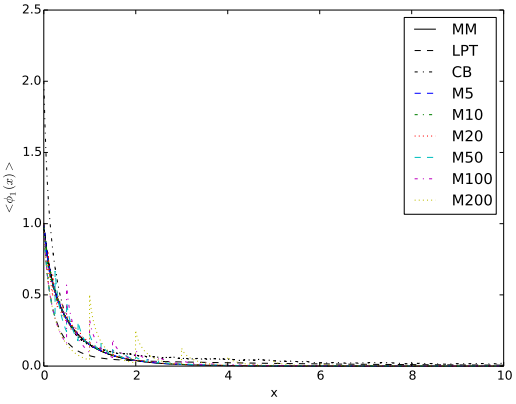
(b) $\langle \phi(x) \rangle$, internal source



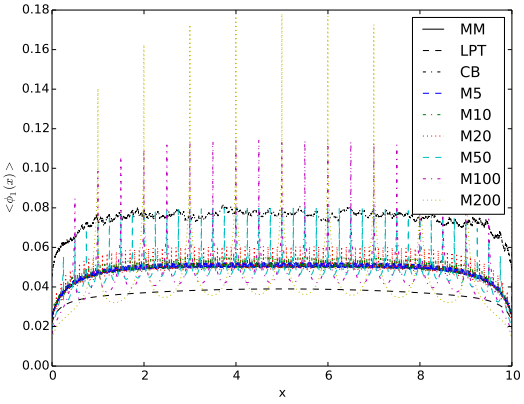
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

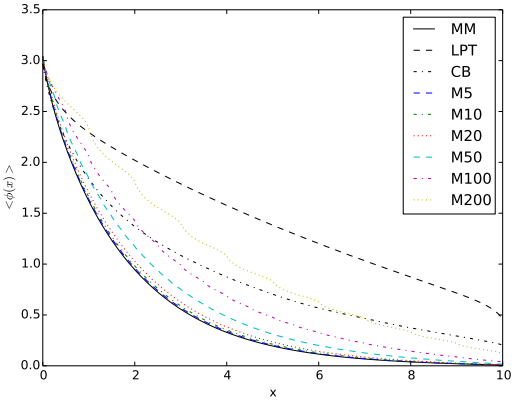


(e) $\langle \phi_1(x) \rangle$, boundary source

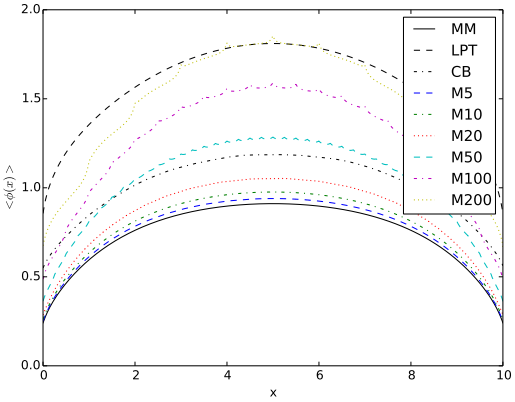


(f) $\langle \phi_1(x) \rangle$, internal source

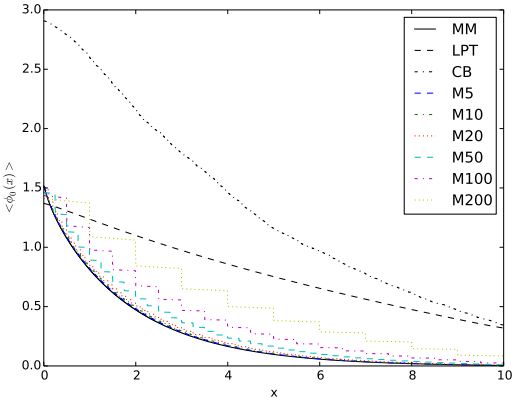
Figure 8.8: Results for M3, S2, L10.0, skip problems



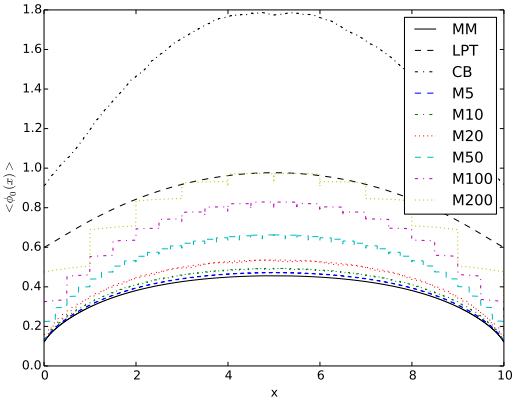
(a) $\langle \phi(x) \rangle$, boundary source



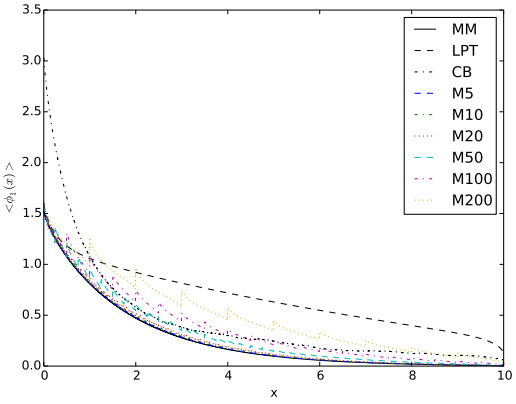
(b) $\langle \phi(x) \rangle$, internal source



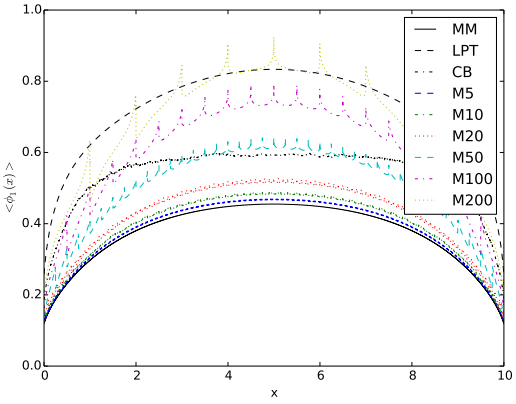
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source



(e) $\langle \phi_1(x) \rangle$, boundary source



(f) $\langle \phi_1(x) \rangle$, internal source

Figure 8.9: Results for M3, S3, L10.0, skip problems

8.2 Data for LP Method

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	-0.0452	-0.0452	-0.0768	-0.0768	-0.0459
1	1	1.0	-0.2014	-0.2015	-0.3618	-0.3627	-0.2251
1	1	10.0	-0.3194	-0.3215	-0.4862	-0.4982	-0.4261
1	2	0.1	-0.0082	-0.0082	-0.0068	-0.0067	-0.0088
1	2	1.0	-0.0429	-0.0428	-0.0191	-0.0190	-0.0583
1	2	10.0	-0.0547	-0.0540	-0.0212	-0.0199	-0.0968
1	3	0.1	-0.0488	-0.0488	-0.0759	-0.0760	-0.0457
1	3	1.0	-0.2518	-0.2519	-0.3548	-0.3552	-0.2384
1	3	10.0	-0.4944	-0.4945	-0.4823	-0.4807	-0.4797
2	1	0.1	-0.0385	-0.0385	-0.0768	-0.0768	-0.0439
2	1	1.0	-0.0778	-0.0778	-0.3618	-0.3627	-0.1112
2	1	10.0	-0.1294	-0.1296	-0.4862	-0.4982	-0.2081
2	2	0.1	-0.0084	-0.0084	-0.0068	-0.0067	-0.0089
2	2	1.0	-0.0645	-0.0645	-0.0191	-0.0190	-0.0755
2	2	10.0	-0.1851	-0.1850	-0.0212	-0.0199	-0.3061
2	3	0.1	-0.0430	-0.0430	-0.0759	-0.0760	-0.0432
2	3	1.0	-0.1438	-0.1438	-0.3548	-0.3552	-0.1373
2	3	10.0	-0.4376	-0.4377	-0.4823	-0.4807	-0.4257
3	1	0.1	-0.0751	-0.0751	-0.0848	-0.0848	-0.0779
3	1	1.0	-0.2766	-0.2766	-0.4386	-0.4387	-0.3165
3	1	10.0	-0.4873	-0.4874	-0.7877	-0.7891	-0.6863
3	2	0.1	-0.0009	-0.0009	-0.0007	-0.0007	-0.0010
3	2	1.0	-0.0054	-0.0054	-0.0020	-0.0020	-0.0090
3	2	10.0	-0.0131	-0.0131	-0.0021	-0.0021	-0.0376
3	3	0.1	-0.0677	-0.0677	-0.0759	-0.0760	-0.0687
3	3	1.0	-0.2474	-0.2474	-0.3551	-0.3552	-0.2398
3	3	10.0	-0.4660	-0.4660	-0.4809	-0.4807	-0.4427

Table 8.13: Reflection, boundary source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.9388	0.9388	0.9083	0.9083	0.9392
1	1	1.0	0.6176	0.6174	0.4818	0.4814	0.6201
1	1	10.0	0.0148	0.0144	0.0047	0.0047	0.0162
1	2	0.1	0.9027	0.9027	0.8393	0.8392	0.9098
1	2	1.0	0.4819	0.4815	0.2316	0.2307	0.5281
1	2	10.0	0.0015	0.0014	0.0000	0.0000	0.0026
1	3	0.1	0.9425	0.9425	0.9075	0.9074	0.9400
1	3	1.0	0.6651	0.6649	0.4761	0.4752	0.6332
1	3	10.0	0.0516	0.0502	0.0044	0.0039	0.0179
2	1	0.1	0.9454	0.9454	0.9083	0.9083	0.9413
2	1	1.0	0.7531	0.7531	0.4818	0.4814	0.7360
2	1	10.0	0.1158	0.1157	0.0047	0.0047	0.1114
2	2	0.1	0.9163	0.9163	0.8393	0.8392	0.9162
2	2	1.0	0.7530	0.7530	0.2316	0.2307	0.7766
2	2	10.0	0.1592	0.1588	0.0000	0.0000	0.2297
2	3	0.1	0.9499	0.9499	0.9075	0.9074	0.9419
2	3	1.0	0.8189	0.8189	0.4761	0.4752	0.7826
2	3	10.0	0.2981	0.2979	0.0044	0.0039	0.2239
3	1	0.1	0.9256	0.9256	0.9162	0.9162	0.9233
3	1	1.0	0.6947	0.6947	0.5449	0.5448	0.6672
3	1	10.0	0.2388	0.2386	0.0666	0.0668	0.1708
3	2	0.1	0.8528	0.8528	0.8333	0.8333	0.8493
3	2	1.0	0.4809	0.4809	0.2207	0.2206	0.4945
3	2	10.0	0.0698	0.0696	0.0000	0.0000	0.0891
3	3	0.1	0.9183	0.9183	0.9074	0.9074	0.9145
3	3	1.0	0.6724	0.6724	0.4753	0.4752	0.6128
3	3	10.0	0.2455	0.2453	0.0039	0.0039	0.1155

Table 8.14: Leakage, boundary source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	-0.0150	-0.0149	0.6737	0.6746	-
1	1	1.0	-0.1055	-0.1046	0.6073	0.6115	-
1	1	10.0	-0.2503	-0.2455	0.1409	0.1692	-
1	2	0.1	-0.0693	-0.0693	-0.2296	-0.2299	-
1	2	1.0	-0.2641	-0.2646	-0.6715	-0.6735	-
1	2	10.0	-0.4351	-0.4419	-0.7809	-0.7948	-
1	3	0.1	0.0671	0.0671	0.6595	0.6602	-
1	3	1.0	0.0561	0.0567	0.4884	0.4899	-
1	3	10.0	0.0306	0.0308	0.0054	0.0021	-
2	1	0.1	-0.1216	-0.1216	0.7506	0.7515	-
2	1	1.0	-0.3000	-0.3000	2.2546	2.2630	-
2	1	10.0	-0.3781	-0.3775	1.3357	1.3936	-
2	2	0.1	-0.0587	-0.0587	-0.2408	-0.2411	-
2	2	1.0	-0.1458	-0.1458	-0.7467	-0.7482	-
2	2	10.0	-0.3952	-0.3958	-0.9307	-0.9351	-
2	3	0.1	-0.0054	-0.0054	0.7581	0.7588	-
2	3	1.0	0.0473	0.0473	1.5841	1.5867	-
2	3	10.0	0.0279	0.0281	0.1329	0.1291	-
3	1	0.1	-0.0353	-0.0353	0.0898	0.0898	-
3	1	1.0	-0.1259	-0.1259	0.3857	0.3861	-
3	1	10.0	-0.2900	-0.2898	0.1477	0.1497	-
3	2	0.1	-0.1008	-0.1008	-0.2424	-0.2425	-
3	2	1.0	-0.4001	-0.4001	-0.7797	-0.7800	-
3	2	10.0	-0.6529	-0.6531	-0.9444	-0.9452	-
3	3	0.1	-0.0159	-0.0159	0.1048	0.1048	-
3	3	1.0	0.0318	0.0318	0.4812	0.4814	-
3	3	10.0	0.0527	0.0528	0.0864	0.0860	-

Table 8.15: Error in reflection, boundary source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	-0.0004	-0.0004	-0.0328	-0.0329	-
1	1	1.0	-0.0039	-0.0043	-0.2229	-0.2237	-
1	1	10.0	-0.0883	-0.1118	-0.7136	-0.7080	-
1	2	0.1	-0.0078	-0.0078	-0.0776	-0.0776	-
1	2	1.0	-0.0875	-0.0883	-0.5615	-0.5632	-
1	2	10.0	-0.4265	-0.4662	-0.9958	-0.9967	-
1	3	0.1	0.0027	0.0027	-0.0346	-0.0346	-
1	3	1.0	0.0503	0.0500	-0.2481	-0.2496	-
1	3	10.0	1.8846	1.8032	-0.7518	-0.7843	-
2	1	0.1	0.0043	0.0043	-0.0350	-0.0351	-
2	1	1.0	0.0233	0.0232	-0.3453	-0.3460	-
2	1	10.0	0.0392	0.0380	-0.9582	-0.9574	-
2	2	0.1	0.0001	0.0001	-0.0839	-0.0840	-
2	2	1.0	-0.0303	-0.0303	-0.7018	-0.7029	-
2	2	10.0	-0.3070	-0.3085	-1.0000	-1.0000	-
2	3	0.1	0.0085	0.0085	-0.0366	-0.0366	-
2	3	1.0	0.0464	0.0464	-0.3916	-0.3928	-
2	3	10.0	0.3318	0.3307	-0.9802	-0.9828	-
3	1	0.1	0.0026	0.0026	-0.0076	-0.0076	-
3	1	1.0	0.0412	0.0412	-0.1833	-0.1834	-
3	1	10.0	0.3980	0.3973	-0.6100	-0.6090	-
3	2	0.1	0.0041	0.0041	-0.0189	-0.0189	-
3	2	1.0	-0.0275	-0.0275	-0.5537	-0.5539	-
3	2	10.0	-0.2170	-0.2183	-0.9999	-0.9999	-
3	3	0.1	0.0042	0.0042	-0.0077	-0.0077	-
3	3	1.0	0.0973	0.0973	-0.2243	-0.2245	-
3	3	10.0	1.1252	1.1238	-0.9660	-0.9666	-

Table 8.16: Error in leakage, boundary source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.0033	0.0033	0.0382	0.0383	-
1	1	1.0	0.0187	0.0186	0.0967	0.0969	-
1	1	10.0	0.1686	0.1771	0.4022	0.3862	-
1	2	0.1	0.0065	0.0065	0.0924	0.0926	-
1	2	1.0	0.0700	0.0706	0.4890	0.4908	-
1	2	10.0	0.3859	0.4105	0.9115	0.9167	-
1	3	0.1	0.0052	0.0052	0.0393	0.0394	-
1	3	1.0	0.0453	0.0451	0.1210	0.1216	-
1	3	10.0	0.9570	0.9215	0.5054	0.5365	-
2	1	0.1	0.0068	0.0068	0.0413	0.0413	-
2	1	1.0	0.0343	0.0343	0.1687	0.1689	-
2	1	10.0	0.0576	0.0578	0.6332	0.6258	-
2	2	0.1	0.0006	0.0006	0.0974	0.0976	-
2	2	1.0	0.0294	0.0294	0.5797	0.5812	-
2	2	10.0	0.2642	0.2652	0.9573	0.9592	-
2	3	0.1	0.0088	0.0088	0.0425	0.0426	-
2	3	1.0	0.0644	0.0644	0.1954	0.1959	-
2	3	10.0	0.2357	0.2352	0.7697	0.7790	-
3	1	0.1	0.0031	0.0031	0.0130	0.0130	-
3	1	1.0	0.0474	0.0474	0.0941	0.0942	-
3	1	10.0	0.1808	0.1807	0.2397	0.2374	-
3	2	0.1	0.0039	0.0039	0.0295	0.0296	-
3	2	1.0	0.0221	0.0221	0.4366	0.4368	-
3	2	10.0	0.1643	0.1650	0.9332	0.9335	-
3	3	0.1	0.0041	0.0041	0.0135	0.0135	-
3	3	1.0	0.0971	0.0971	0.1170	0.1171	-
3	3	10.0	0.9953	0.9946	0.6903	0.6922	-

Table 8.17: Error in $\langle\phi(x)\rangle$, boundary source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.1003	0.1003	0.1114	-	-
1	1	1.0	0.1096	0.1096	0.1329	-	-
1	1	10.0	0.2353	0.2429	0.4392	-	-
1	2	0.1	0.1045	0.1045	0.2170	-	-
1	2	1.0	0.1554	0.1559	0.5610	-	-
1	2	10.0	0.4465	0.4687	0.9236	-	-
1	3	0.1	0.1000	0.1000	0.1234	-	-
1	3	1.0	0.0784	0.0784	0.1987	-	-
1	3	10.0	0.7570	0.7254	0.5608	-	-
2	1	0.1	0.1002	0.1002	0.1158	-	-
2	1	1.0	0.1021	0.1021	0.2060	-	-
2	1	10.0	0.1153	0.1156	0.6473	-	-
2	2	0.1	0.1013	0.1013	0.2295	-	-
2	2	1.0	0.1226	0.1226	0.6475	-	-
2	2	10.0	0.3284	0.3292	0.9635	-	-
2	3	0.1	0.1002	0.1002	0.1289	-	-
2	3	1.0	0.0952	0.0952	0.2757	-	-
2	3	10.0	0.0966	0.0962	0.7975	-	-
3	1	0.1	0.4999	0.4999	0.5005	-	-
3	1	1.0	0.4999	0.4999	0.5108	-	-
3	1	10.0	0.4836	0.4836	0.5454	-	-
3	2	0.1	0.5001	0.5001	0.5736	-	-
3	2	1.0	0.5100	0.5100	0.8032	-	-
3	2	10.0	0.5799	0.5802	0.9760	-	-
3	3	0.1	0.4999	0.4999	0.5096	-	-
3	3	1.0	0.5000	0.5000	0.5796	-	-
3	3	10.0	0.3777	0.3778	0.8808	-	-

Table 8.18: Error in $\langle \phi_0(x) \rangle$, boundary source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.8997	0.8997	0.8985	-	-
1	1	1.0	0.9063	0.9063	0.9069	-	-
1	1	10.0	0.9261	0.9267	0.9430	-	-
1	2	0.1	0.8968	0.8968	0.8114	-	-
1	2	1.0	0.9011	0.9010	0.8483	-	-
1	2	10.0	0.9167	0.9185	0.9674	-	-
1	3	0.1	0.8934	0.8934	0.8870	-	-
1	3	1.0	0.8765	0.8765	0.8817	-	-
1	3	10.0	0.7686	0.7718	0.9217	-	-
2	1	0.1	0.8993	0.8993	0.8963	-	-
2	1	1.0	0.8994	0.8994	0.9041	-	-
2	1	10.0	0.9313	0.9314	0.9656	-	-
2	2	0.1	0.8999	0.8999	0.7642	-	-
2	2	1.0	0.8849	0.8848	0.3893	-	-
2	2	10.0	0.8921	0.8921	0.9207	-	-
2	3	0.1	0.8907	0.8907	0.8808	-	-
2	3	1.0	0.6877	0.6877	0.7133	-	-
2	3	10.0	0.5220	0.5222	0.8799	-	-
3	1	0.1	0.5000	0.5000	0.4984	-	-
3	1	1.0	0.4954	0.4954	0.4962	-	-
3	1	10.0	0.5806	0.5807	0.6326	-	-
3	2	0.1	0.4999	0.4999	0.4287	-	-
3	2	1.0	0.4858	0.4858	0.2123	-	-
3	2	10.0	0.4860	0.4859	0.8484	-	-
3	3	0.1	0.4977	0.4977	0.4897	-	-
3	3	1.0	0.3929	0.3929	0.4085	-	-
3	3	10.0	1.5936	1.5928	0.6227	-	-

Table 8.19: Error in $\langle \phi_1(x) \rangle$, boundary source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.4981	0.4981	0.4923	0.4923	0.4918
1	1	1.0	0.4724	0.4725	0.4380	0.4371	0.4304
1	1	10.0	0.1712	0.1708	0.1398	0.1375	0.1560
1	2	0.1	0.4655	0.4655	0.4321	0.4320	0.4682
1	2	1.0	0.3401	0.3400	0.2079	0.2072	0.3624
1	2	10.0	0.0675	0.0666	0.0283	0.0270	0.0821
1	3	0.1	0.5014	0.5014	0.4915	0.4915	0.4932
1	3	1.0	0.5176	0.5176	0.4330	0.4316	0.4526
1	3	10.0	0.2971	0.2951	0.1379	0.1296	0.1830
2	1	0.1	0.4961	0.4961	0.4923	0.4923	0.4919
2	1	1.0	0.4310	0.4310	0.4380	0.4371	0.4320
2	1	10.0	0.1890	0.1890	0.1398	0.1375	0.1914
2	2	0.1	0.4697	0.4697	0.4321	0.4320	0.4699
2	2	1.0	0.4283	0.4283	0.2079	0.2072	0.4408
2	2	10.0	0.2428	0.2426	0.0283	0.0270	0.3139
2	3	0.1	0.5002	0.5002	0.4915	0.4915	0.4928
2	3	1.0	0.4891	0.4891	0.4330	0.4316	0.4574
2	3	10.0	0.4065	0.4065	0.1379	0.1296	0.3259
3	1	0.1	0.5019	0.5019	0.4991	0.4991	0.4992
3	1	1.0	0.5056	0.5056	0.4924	0.4923	0.4914
3	1	10.0	0.3828	0.3829	0.3722	0.3716	0.4161
3	2	0.1	0.4376	0.4376	0.4269	0.4268	0.4363
3	2	1.0	0.2965	0.2965	0.1971	0.1971	0.3029
3	2	10.0	0.1165	0.1164	0.0254	0.0253	0.1348
3	3	0.1	0.4955	0.4955	0.4915	0.4915	0.4917
3	3	1.0	0.4833	0.4833	0.4318	0.4316	0.4369
3	3	10.0	0.3852	0.3852	0.1306	0.1296	0.2298

Table 8.20: Leakage, internal source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.0128	0.0128	0.0009	0.0008	-
1	1	1.0	0.0976	0.0977	0.0177	0.0154	-
1	1	10.0	0.0976	0.0953	-0.1039	-0.1182	-
1	2	0.1	-0.0059	-0.0059	-0.0772	-0.0774	-
1	2	1.0	-0.0616	-0.0620	-0.4265	-0.4284	-
1	2	10.0	-0.1783	-0.1891	-0.6559	-0.6707	-
1	3	0.1	0.0167	0.0167	-0.0034	-0.0035	-
1	3	1.0	0.1437	0.1437	-0.0433	-0.0463	-
1	3	10.0	0.6233	0.6120	-0.2466	-0.2919	-
2	1	0.1	0.0087	0.0087	0.0009	0.0008	-
2	1	1.0	-0.0023	-0.0022	0.0139	0.0117	-
2	1	10.0	-0.0125	-0.0125	-0.2699	-0.2815	-
2	2	0.1	-0.0003	-0.0003	-0.0804	-0.0805	-
2	2	1.0	-0.0283	-0.0283	-0.5284	-0.5300	-
2	2	10.0	-0.2263	-0.2270	-0.9100	-0.9139	-
2	3	0.1	0.0150	0.0150	-0.0025	-0.0026	-
2	3	1.0	0.0692	0.0692	-0.0534	-0.0564	-
2	3	10.0	0.2475	0.2475	-0.5768	-0.6022	-
3	1	0.1	0.0055	0.0055	-0.0000	-0.0000	-
3	1	1.0	0.0287	0.0287	0.0020	0.0017	-
3	1	10.0	-0.0799	-0.0798	-0.1055	-0.1069	-
3	2	0.1	0.0031	0.0031	-0.0216	-0.0216	-
3	2	1.0	-0.0211	-0.0211	-0.3490	-0.3493	-
3	2	10.0	-0.1360	-0.1364	-0.8116	-0.8125	-
3	3	0.1	0.0076	0.0076	-0.0005	-0.0005	-
3	3	1.0	0.1062	0.1062	-0.0116	-0.0120	-
3	3	10.0	0.6764	0.6763	-0.4315	-0.4359	-

Table 8.21: Error in leakage, internal source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.0083	0.0083	0.0335	0.0336	-
1	1	1.0	0.0485	0.0486	0.0833	0.0854	-
1	1	10.0	0.0393	0.0377	0.0527	0.0563	-
1	2	0.1	0.0059	0.0059	0.1111	0.1114	-
1	2	1.0	0.0589	0.0593	0.5124	0.5158	-
1	2	10.0	0.1838	0.1935	0.7127	0.7339	-
1	3	0.1	0.0119	0.0119	0.0385	0.0386	-
1	3	1.0	0.0855	0.0855	0.1469	0.1502	-
1	3	10.0	0.4553	0.4504	0.1912	0.2378	-
2	1	0.1	0.0061	0.0061	0.0356	0.0357	-
2	1	1.0	0.0718	0.0718	0.1409	0.1432	-
2	1	10.0	0.1874	0.1873	0.0846	0.0885	-
2	2	0.1	0.0004	0.0004	0.1149	0.1152	-
2	2	1.0	0.0258	0.0258	0.5810	0.5838	-
2	2	10.0	0.2153	0.2158	0.8992	0.9067	-
2	3	0.1	0.0118	0.0118	0.0401	0.0402	-
2	3	1.0	0.0529	0.0529	0.1655	0.1685	-
2	3	10.0	0.1796	0.1798	0.3659	0.4019	-
3	1	0.1	0.0061	0.0061	0.0090	0.0091	-
3	1	1.0	0.0160	0.0160	0.1023	0.1026	-
3	1	10.0	0.4176	0.4176	0.0661	0.0658	-
3	2	0.1	0.0042	0.0042	0.0336	0.0337	-
3	2	1.0	0.0205	0.0205	0.4312	0.4317	-
3	2	10.0	0.1215	0.1218	0.8243	0.8259	-
3	3	0.1	0.0081	0.0081	0.0102	0.0102	-
3	3	1.0	0.0833	0.0833	0.1203	0.1208	-
3	3	10.0	0.5515	0.5515	0.2823	0.2875	-

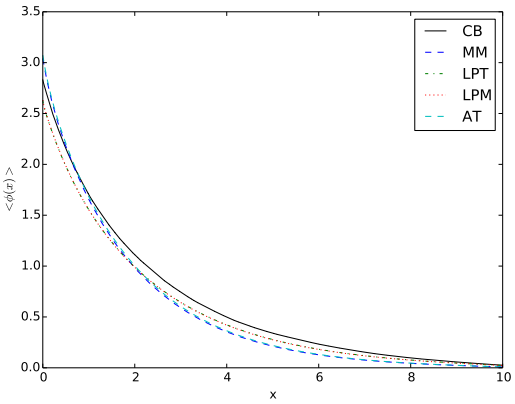
Table 8.22: Error in $\langle \phi(x) \rangle$, internal source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.0969	0.0969	0.1358	-	-
1	1	1.0	0.0628	0.0627	0.1780	-	-
1	1	10.0	0.1182	0.1168	0.0685	-	-
1	2	0.1	0.1033	0.1033	0.2429	-	-
1	2	1.0	0.1454	0.1458	0.5898	-	-
1	2	10.0	0.2689	0.2779	0.7596	-	-
1	3	0.1	0.0971	0.0971	0.1487	-	-
1	3	1.0	0.0435	0.0435	0.2556	-	-
1	3	10.0	0.2698	0.2654	0.2998	-	-
2	1	0.1	0.0996	0.0996	0.1368	-	-
2	1	1.0	0.0961	0.0961	0.1773	-	-
2	1	10.0	0.1405	0.1405	0.1351	-	-
2	2	0.1	0.1011	0.1011	0.2529	-	-
2	2	1.0	0.1192	0.1192	0.6553	-	-
2	2	10.0	0.2854	0.2859	0.9162	-	-
2	3	0.1	0.0998	0.0998	0.1513	-	-
2	3	1.0	0.0901	0.0901	0.2805	-	-
2	3	10.0	0.0181	0.0181	0.4661	-	-
3	1	0.1	0.5000	0.5000	0.5210	-	-
3	1	1.0	0.4954	0.4954	0.5541	-	-
3	1	10.0	0.5492	0.5492	0.2911	-	-
3	2	0.1	0.5002	0.5002	0.5911	-	-
3	2	1.0	0.5091	0.5091	0.8224	-	-
3	2	10.0	0.5637	0.5638	0.9523	-	-
3	3	0.1	0.5000	0.5000	0.5297	-	-
3	3	1.0	0.4967	0.4967	0.6178	-	-
3	3	10.0	0.4243	0.4243	0.7527	-	-

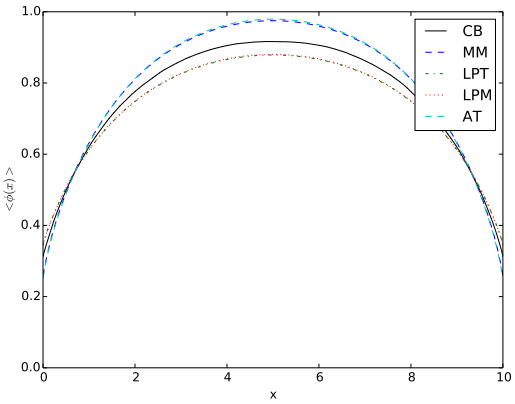
Table 8.23: Error in $\langle \phi_0(x) \rangle$, internal source problems

M	S	L	LPT	LPM	MM	AT	CB
1	1	0.1	0.8929	0.8929	0.8868	-	-
1	1	1.0	0.8860	0.8860	0.8918	-	-
1	1	10.0	0.9103	0.9102	0.9043	-	-
1	2	0.1	0.8979	0.8979	0.7957	-	-
1	2	1.0	0.9006	0.9005	0.8136	-	-
1	2	10.0	0.8991	0.8988	0.8861	-	-
1	3	0.1	0.8867	0.8867	0.8744	-	-
1	3	1.0	0.8495	0.8495	0.8621	-	-
1	3	10.0	0.7856	0.7862	0.8712	-	-
2	1	0.1	0.8937	0.8937	0.8903	-	-
2	1	1.0	0.9396	0.9396	0.9439	-	-
2	1	10.0	0.9650	0.9650	0.9590	-	-
2	2	0.1	0.8998	0.8998	0.7699	-	-
2	2	1.0	0.8958	0.8958	0.5326	-	-
2	2	10.0	0.8979	0.8978	0.7764	-	-
2	3	0.1	0.8854	0.8854	0.8750	-	-
2	3	1.0	0.8053	0.8053	0.8257	-	-
2	3	10.0	0.5530	0.5530	0.7310	-	-
3	1	0.1	0.4948	0.4948	0.4824	-	-
3	1	1.0	0.4976	0.4976	0.5366	-	-
3	1	10.0	0.7747	0.7747	0.6473	-	-
3	2	0.1	0.5000	0.5000	0.4116	-	-
3	2	1.0	0.4978	0.4978	0.2409	-	-
3	2	10.0	0.5064	0.5064	0.3566	-	-
3	3	0.1	0.4929	0.4929	0.4736	-	-
3	3	1.0	0.4012	0.4012	0.4606	-	-
3	3	10.0	0.3343	0.3343	0.3226	-	-

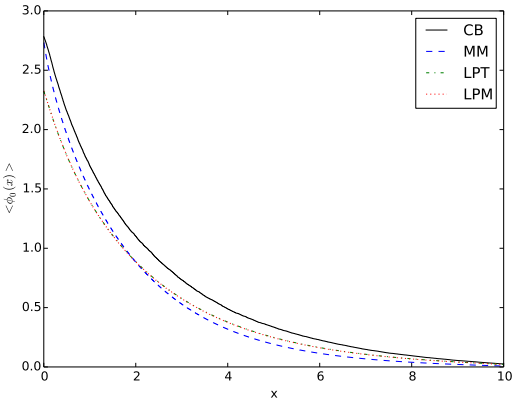
Table 8.24: Error in $\langle \phi_1(x) \rangle$, internal source problems



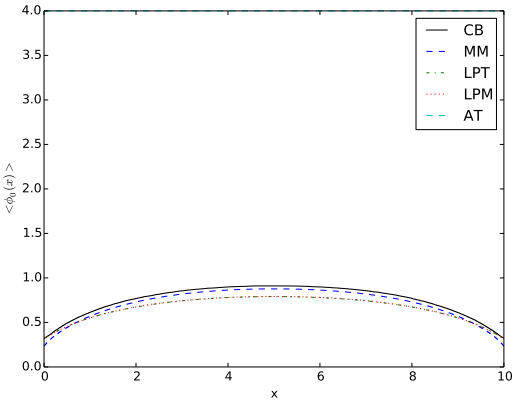
(a) $\langle \phi(x) \rangle$, boundary source



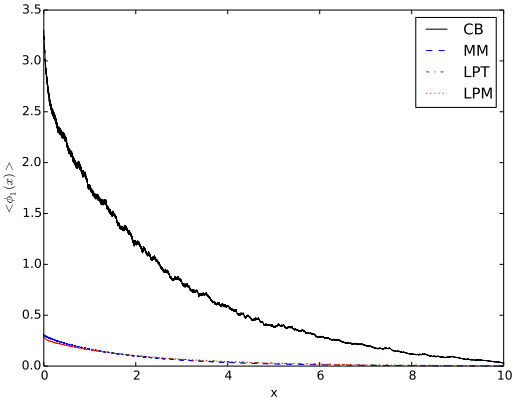
(b) $\langle \phi(x) \rangle$, internal source



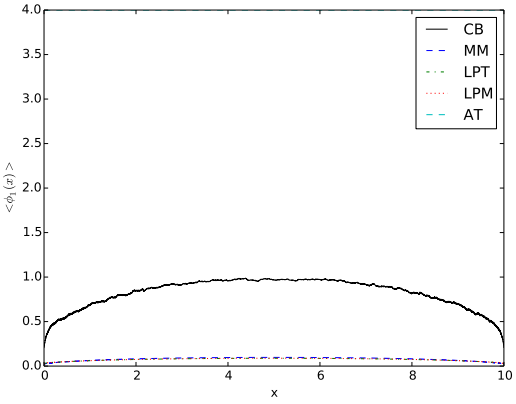
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

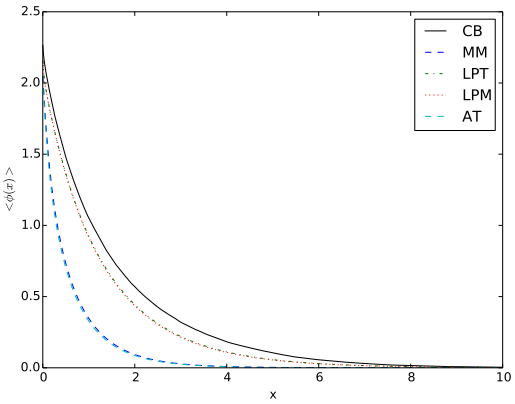


(e) $\langle \phi_1(x) \rangle$, boundary source

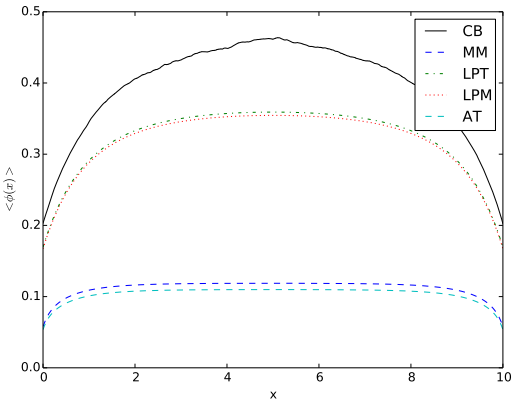


(f) $\langle \phi_1(x) \rangle$, internal source

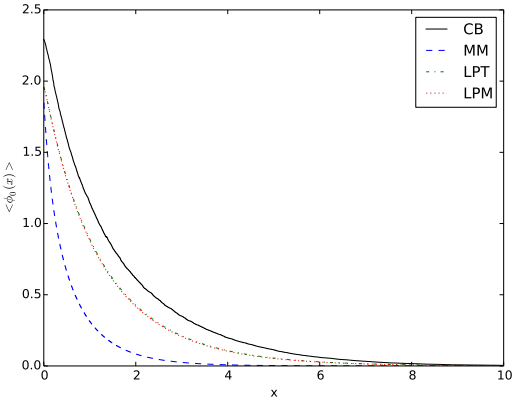
Figure 8.10: Results for M1, S1, L10.0, lp problems



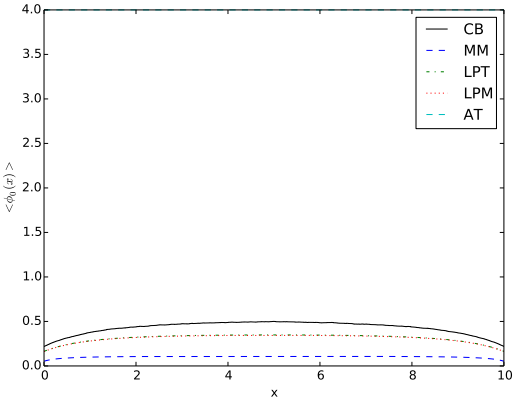
(a) $\langle \phi(x) \rangle$, boundary source



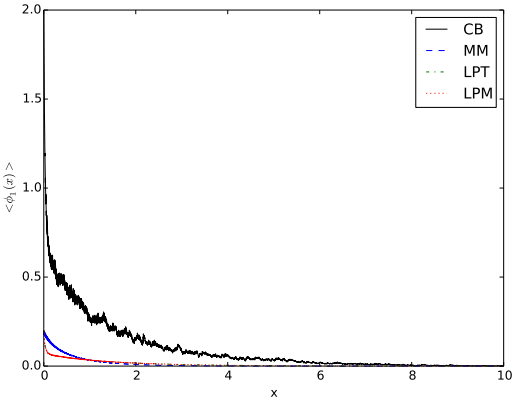
(b) $\langle \phi(x) \rangle$, internal source



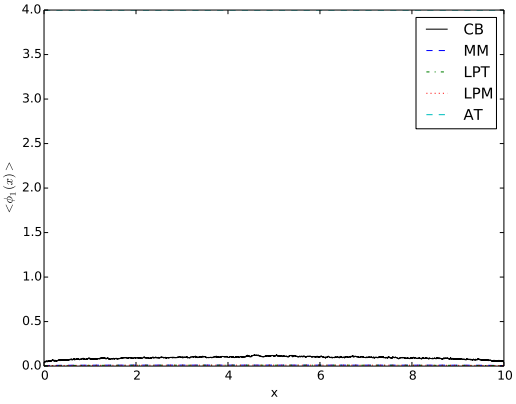
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

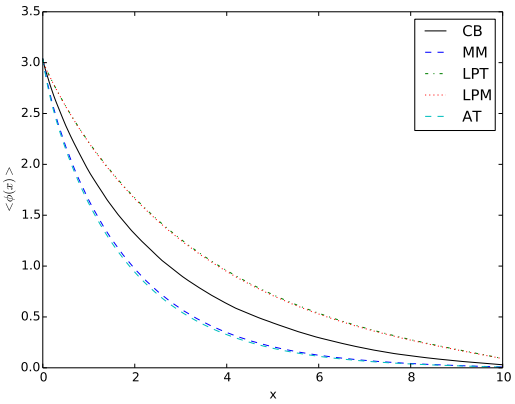


(e) $\langle \phi_1(x) \rangle$, boundary source

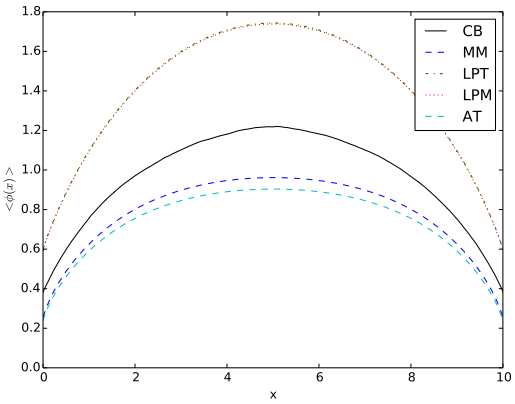


(f) $\langle \phi_1(x) \rangle$, internal source

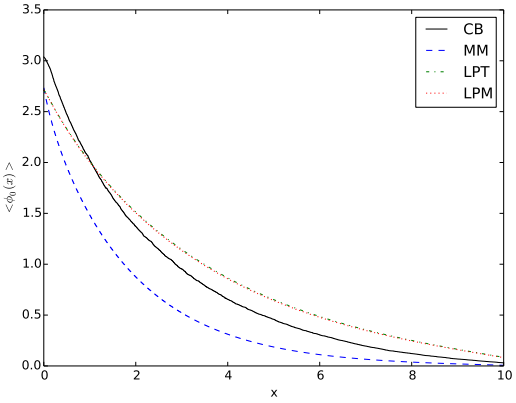
Figure 8.11: Results for M1, S2, L10.0, lp problems



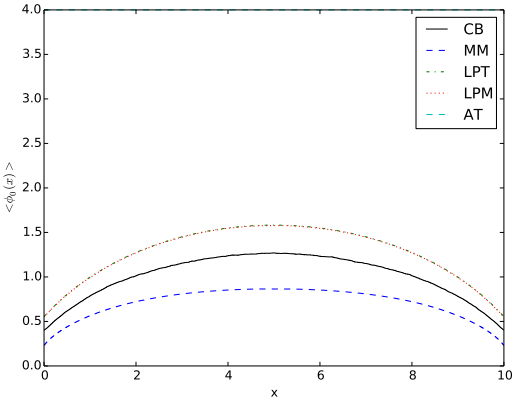
(a) $\langle \phi(x) \rangle$, boundary source



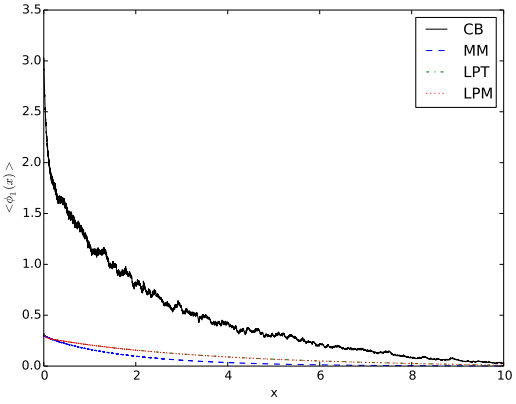
(b) $\langle \phi(x) \rangle$, internal source



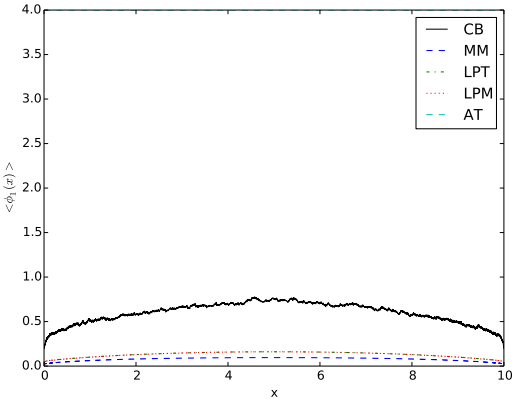
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

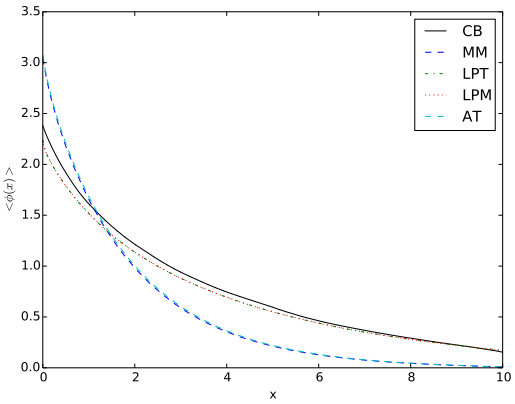


(e) $\langle \phi_1(x) \rangle$, boundary source

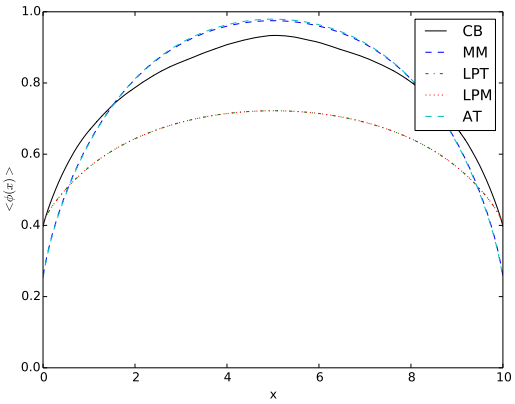


(f) $\langle \phi_1(x) \rangle$, internal source

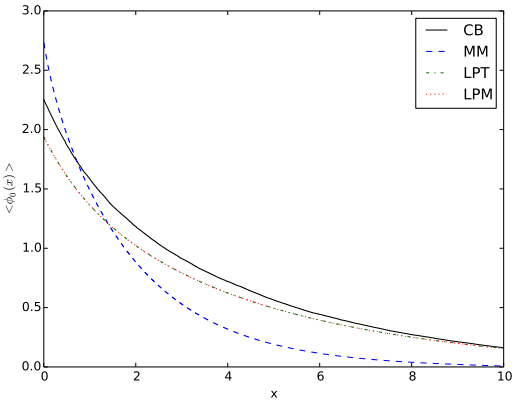
Figure 8.12: Results for M1, S3, L10.0, lp problems



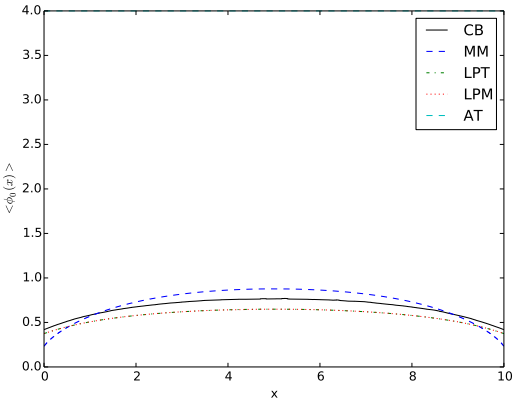
(a) $\langle \phi(x) \rangle$, boundary source



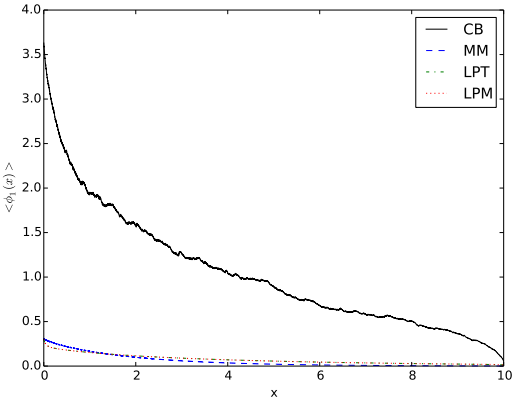
(b) $\langle \phi(x) \rangle$, internal source



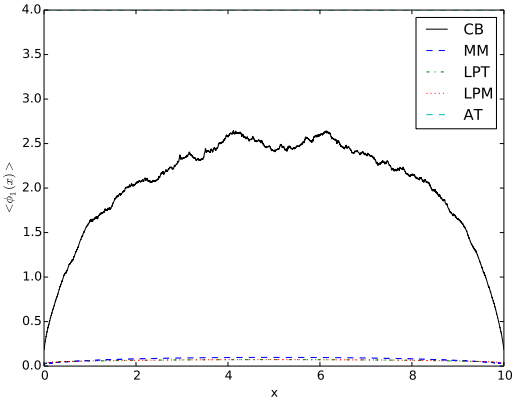
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

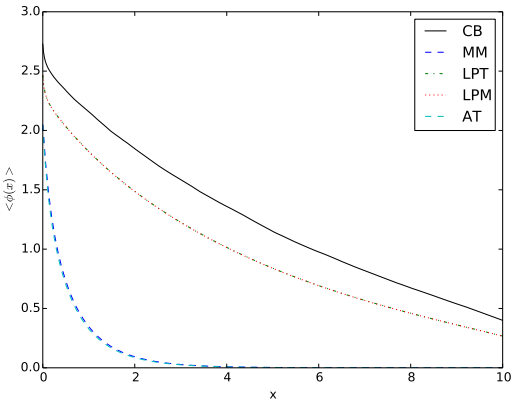


(e) $\langle \phi_1(x) \rangle$, boundary source

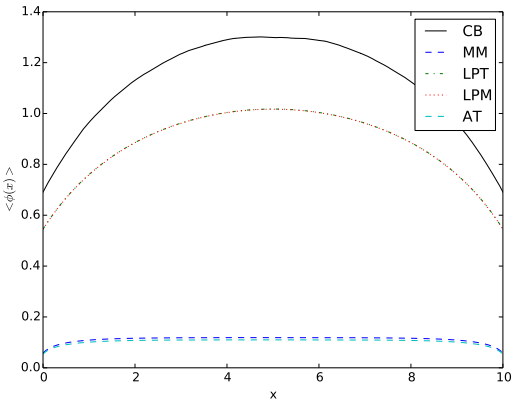


(f) $\langle \phi_1(x) \rangle$, internal source

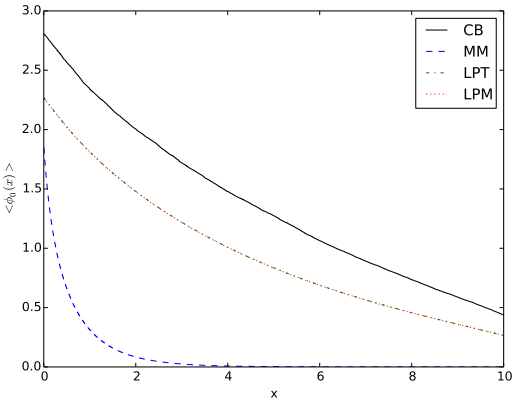
Figure 8.13: Results for M2, S1, L10.0, lp problems



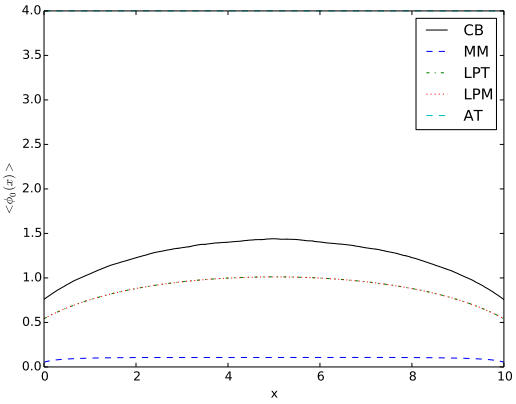
(a) $\langle \phi(x) \rangle$, boundary source



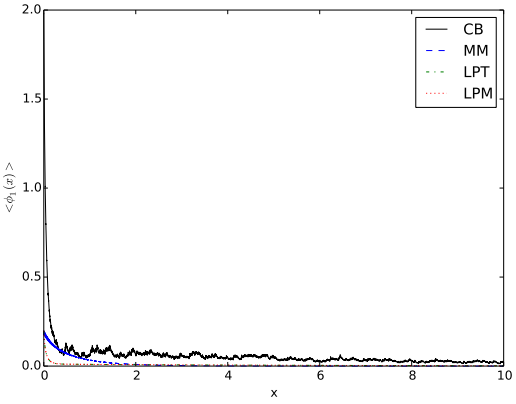
(b) $\langle \phi(x) \rangle$, internal source



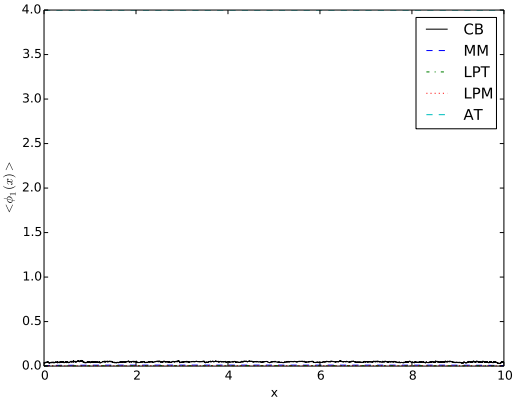
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

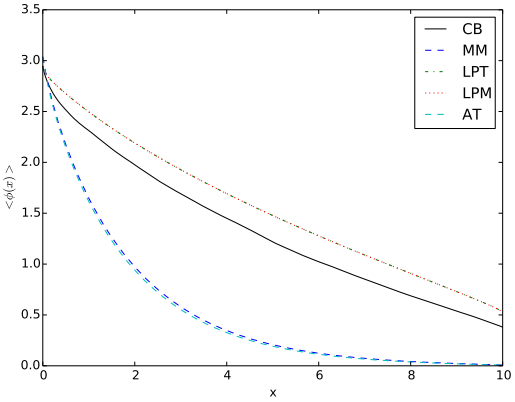


(e) $\langle \phi_1(x) \rangle$, boundary source

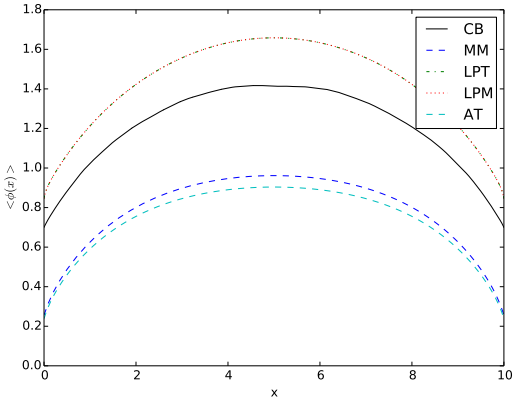


(f) $\langle \phi_1(x) \rangle$, internal source

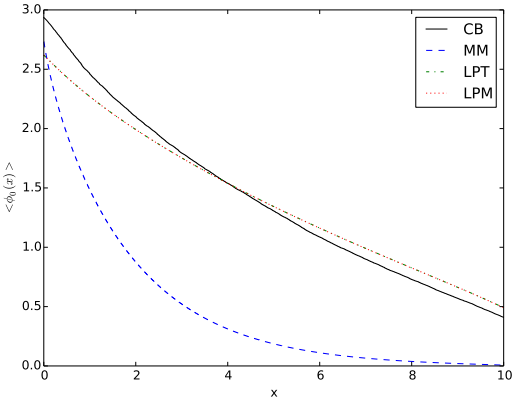
Figure 8.14: Results for M2, S2, L10.0, lp problems



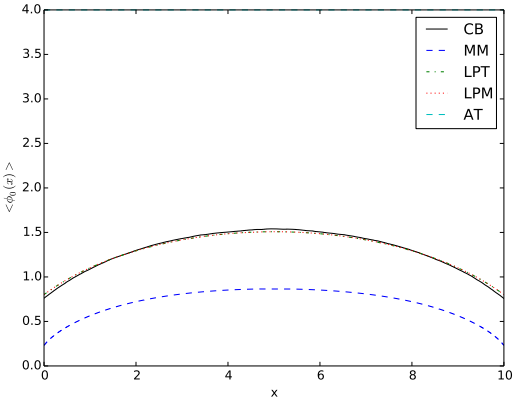
(a) $\langle \phi(x) \rangle$, boundary source



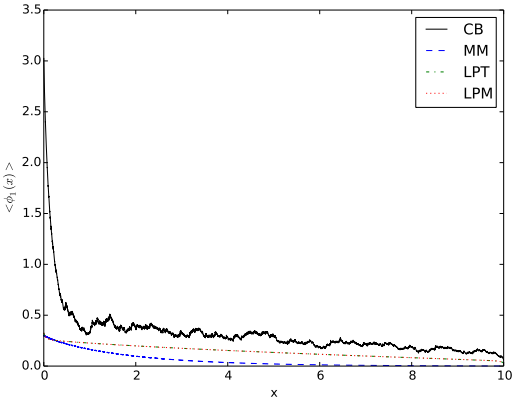
(b) $\langle \phi(x) \rangle$, internal source



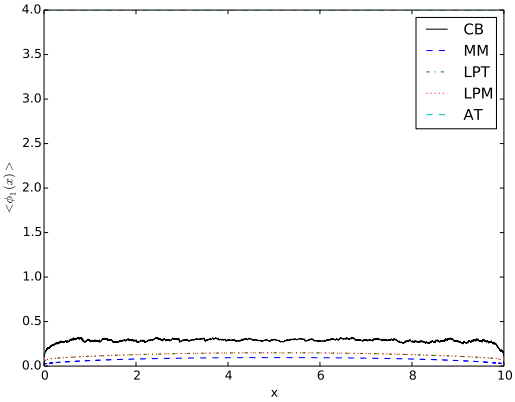
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

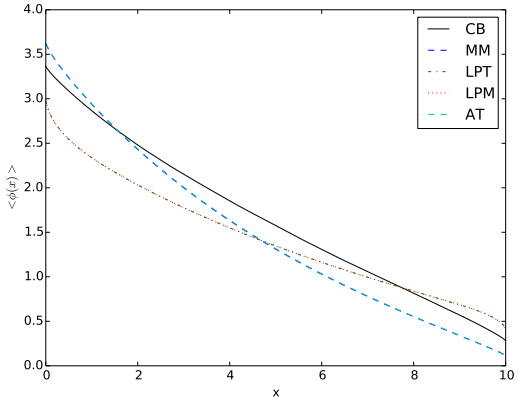


(e) $\langle \phi_1(x) \rangle$, boundary source

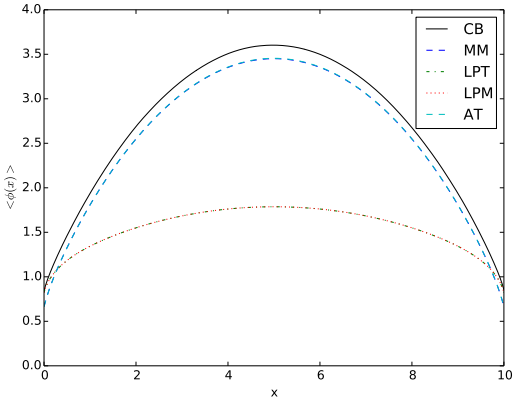


(f) $\langle \phi_1(x) \rangle$, internal source

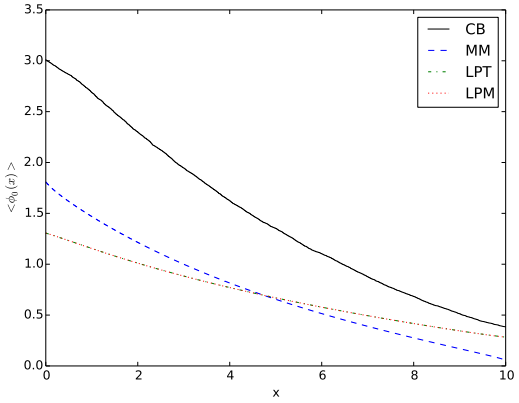
Figure 8.15: Results for M2, S3, L10.0, lp problems



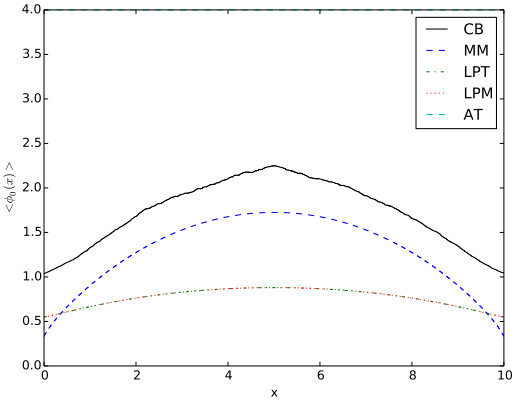
(a) $\langle \phi(x) \rangle$, boundary source



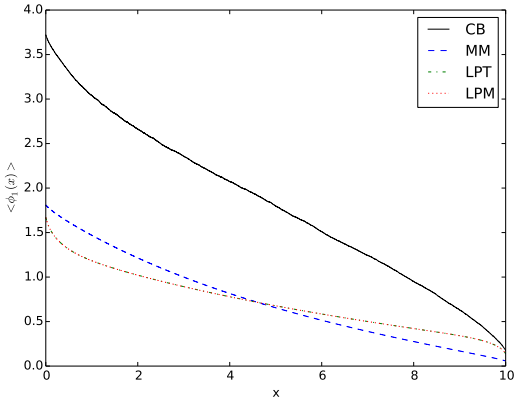
(b) $\langle \phi(x) \rangle$, internal source



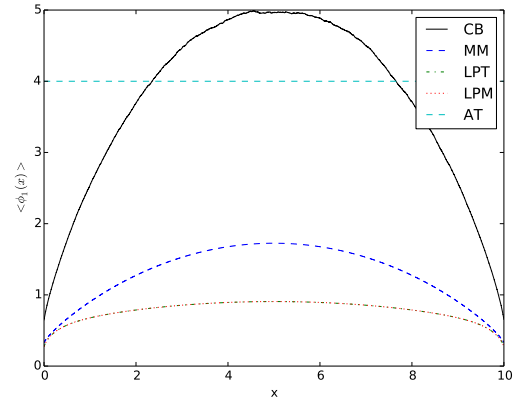
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source

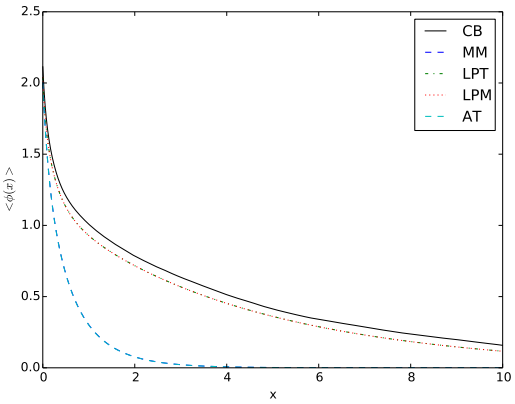


(e) $\langle \phi_1(x) \rangle$, boundary source

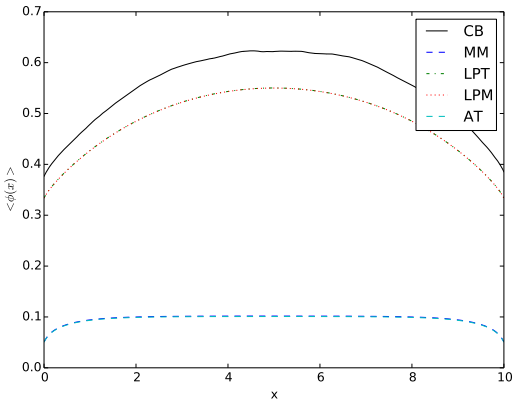


(f) $\langle \phi_1(x) \rangle$, internal source

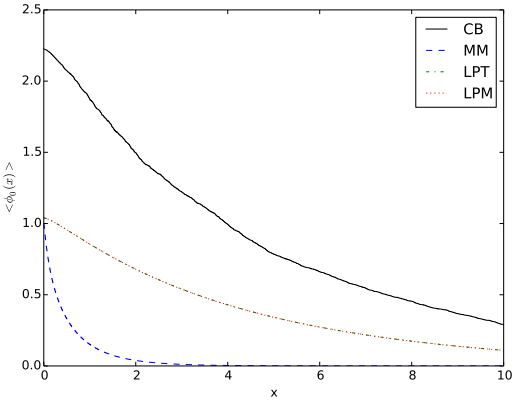
Figure 8.16: Results for M3, S1, L10.0, lp problems



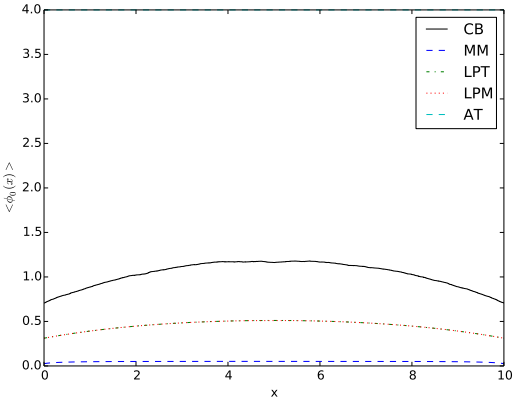
(a) $\langle\phi(x)\rangle$, boundary source



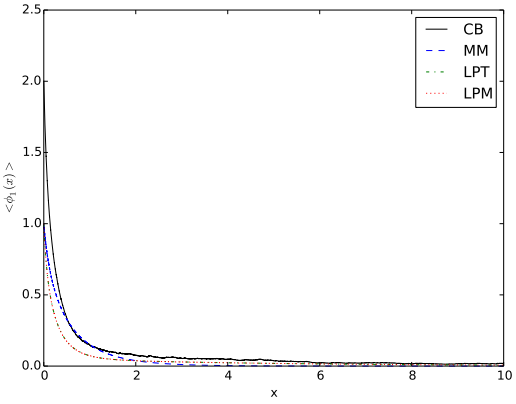
(b) $\langle\phi(x)\rangle$, internal source



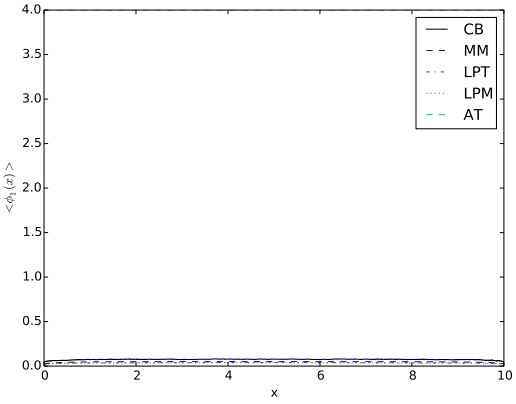
(c) $\langle\phi_0(x)\rangle$, boundary source



(d) $\langle\phi_0(x)\rangle$, internal source

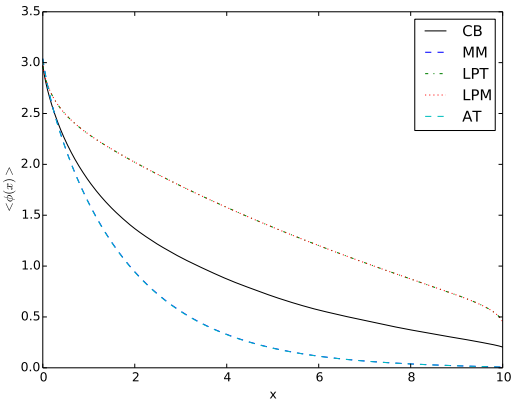


(e) $\langle\phi_1(x)\rangle$, boundary source

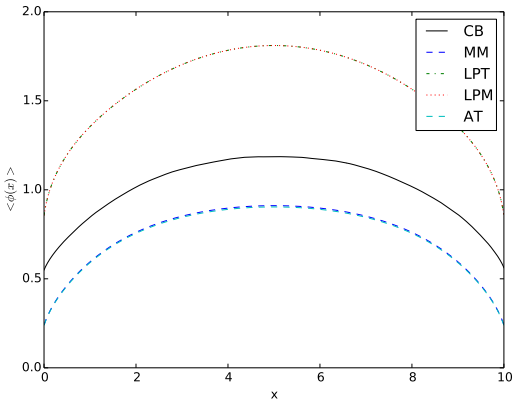


(f) $\langle\phi_1(x)\rangle$, internal source

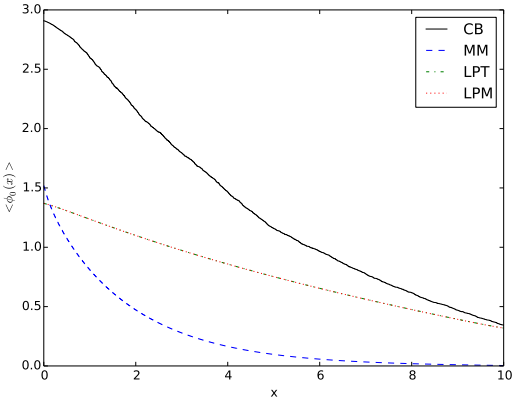
Figure 8.17: Results for M3, S2, L10.0, lp problems



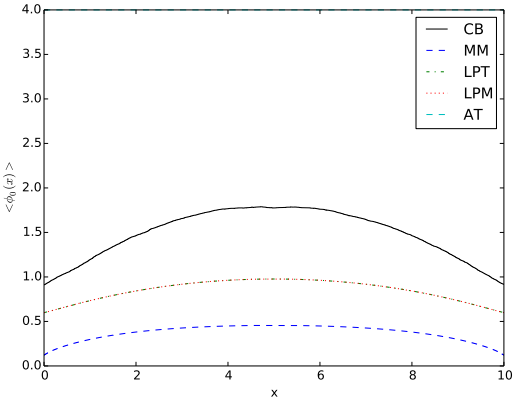
(a) $\langle \phi(x) \rangle$, boundary source



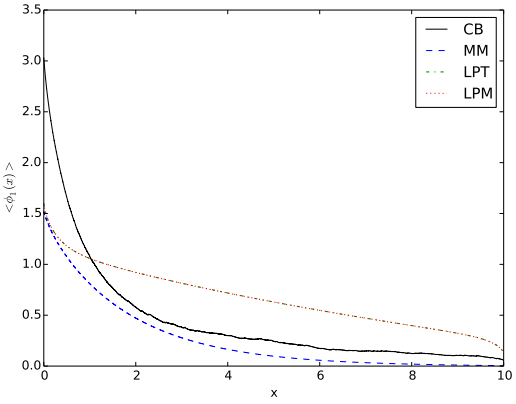
(b) $\langle \phi(x) \rangle$, internal source



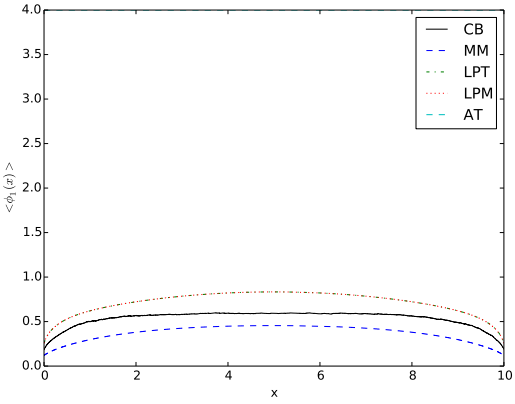
(c) $\langle \phi_0(x) \rangle$, boundary source



(d) $\langle \phi_0(x) \rangle$, internal source



(e) $\langle \phi_1(x) \rangle$, boundary source



(f) $\langle \phi_1(x) \rangle$, internal source

Figure 8.18: Results for M3, S3, L10.0, lp problems

8.3 Data for Standard Error

M	S	L	CB	EB	PB
1	1	0.1	0.0017	0.0017	0.0000
1	1	1.0	0.0033	0.0033	0.0000
1	1	10.0	0.0025	0.0012	0.0004
1	2	0.1	0.0000	0.0000	0.0000
1	2	1.0	0.0004	0.0005	0.0000
1	2	10.0	0.0012	0.0003	0.0001
1	3	0.1	0.0013	0.0013	0.0000
1	3	1.0	0.0022	0.0022	0.0000
1	3	10.0	0.0001	0.0000	0.0000
2	1	0.1	0.0018	0.0018	0.0000
2	1	1.0	0.0038	0.0039	0.0000
2	1	10.0	0.0040	0.0038	0.0004
2	2	0.1	0.0000	0.0000	0.0000
2	2	1.0	0.0004	0.0004	0.0000
2	2	10.0	0.0023	0.0029	0.0001
2	3	0.1	0.0014	0.0014	0.0000
2	3	1.0	0.0020	0.0020	0.0000
2	3	10.0	0.0008	0.0008	0.0000
3	1	0.1	0.0011	0.0011	0.0000
3	1	1.0	0.0041	0.0041	0.0000
3	1	10.0	0.0039	0.0036	0.0001
3	2	0.1	0.0000	0.0000	0.0000
3	2	1.0	0.0001	0.0001	0.0000
3	2	10.0	0.0007	0.0006	0.0000
3	3	0.1	0.0009	0.0009	0.0000
3	3	1.0	0.0028	0.0028	0.0000
3	3	10.0	0.0014	0.0013	0.0000

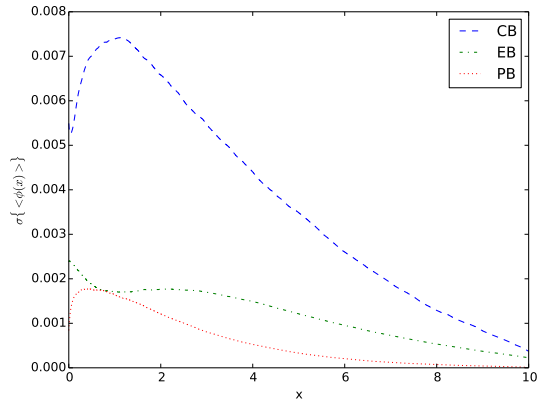
Table 8.25: Standard deviation in reflection, boundary source problems

M	S	L	CB	EB	PB
1	1	0.1	0.0016	0.0016	0.0000
1	1	1.0	0.0030	0.0031	0.0000
1	1	10.0	0.0003	0.0001	0.0000
1	2	0.1	0.0030	0.0030	0.0000
1	2	1.0	0.0052	0.0051	0.0000
1	2	10.0	0.0003	0.0001	0.0000
1	3	0.1	0.0019	0.0019	0.0000
1	3	1.0	0.0040	0.0039	0.0000
1	3	10.0	0.0005	0.0004	0.0000
2	1	0.1	0.0017	0.0017	0.0000
2	1	1.0	0.0032	0.0033	0.0000
2	1	10.0	0.0013	0.0014	0.0000
2	2	0.1	0.0032	0.0031	0.0000
2	2	1.0	0.0045	0.0047	0.0000
2	2	10.0	0.0037	0.0037	0.0000
2	3	0.1	0.0021	0.0020	0.0000
2	3	1.0	0.0041	0.0042	0.0000
2	3	10.0	0.0031	0.0031	0.0000
3	1	0.1	0.0010	0.0010	0.0000
3	1	1.0	0.0038	0.0038	0.0000
3	1	10.0	0.0027	0.0025	0.0000
3	2	0.1	0.0021	0.0021	0.0000
3	2	1.0	0.0062	0.0062	0.0000
3	2	10.0	0.0035	0.0032	0.0000
3	3	0.1	0.0012	0.0012	0.0000
3	3	1.0	0.0048	0.0048	0.0000
3	3	10.0	0.0035	0.0033	0.0000

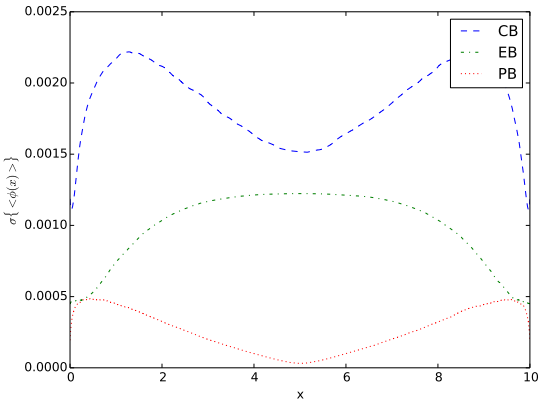
Table 8.26: Standard deviation in leakage, boundary source problems

M	S	L	CB	EB	PB
1	1	0.1	0.0003	0.0000	0.0000
1	1	1.0	0.0011	0.0001	0.0001
1	1	10.0	0.0005	0.0002	0.0001
1	2	0.1	0.0012	0.0012	0.0000
1	2	1.0	0.0022	0.0023	0.0001
1	2	10.0	0.0010	0.0003	0.0001
1	3	0.1	0.0004	0.0002	0.0000
1	3	1.0	0.0014	0.0009	0.0001
1	3	10.0	0.0014	0.0007	0.0001
2	1	0.1	0.0001	0.0000	0.0000
2	1	1.0	0.0007	0.0003	0.0001
2	1	10.0	0.0006	0.0005	0.0001
2	2	0.1	0.0012	0.0012	0.0000
2	2	1.0	0.0021	0.0023	0.0001
2	2	10.0	0.0027	0.0032	0.0001
2	3	0.1	0.0003	0.0003	0.0000
2	3	1.0	0.0015	0.0014	0.0001
2	3	10.0	0.0021	0.0022	0.0001
3	1	0.1	0.0000	0.0000	0.0000
3	1	1.0	0.0005	0.0001	0.0000
3	1	10.0	0.0015	0.0006	0.0001
3	2	0.1	0.0009	0.0009	0.0000
3	2	1.0	0.0026	0.0026	0.0000
3	2	10.0	0.0023	0.0018	0.0000
3	3	0.1	0.0001	0.0001	0.0000
3	3	1.0	0.0010	0.0008	0.0000
3	3	10.0	0.0025	0.0018	0.0001

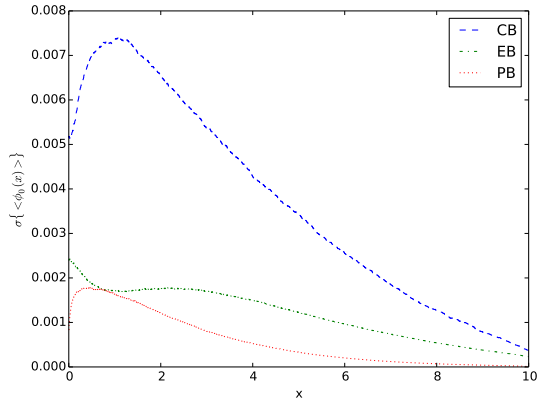
Table 8.27: Standard deviation in leakage, internal source problems



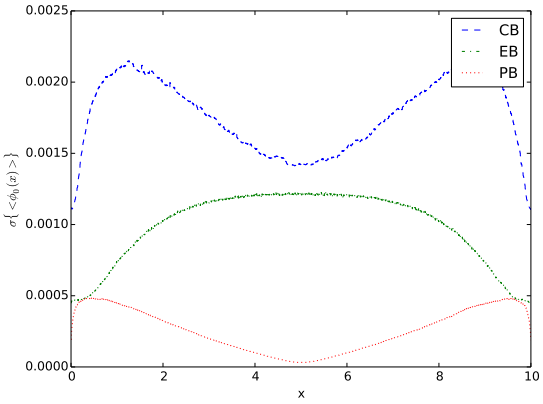
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



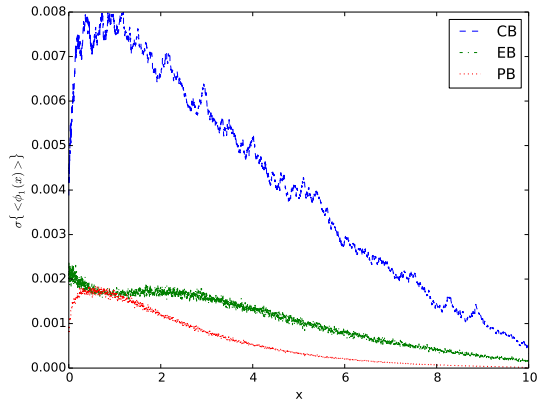
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



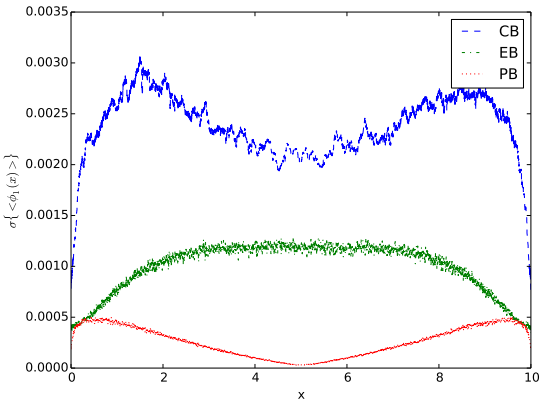
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source

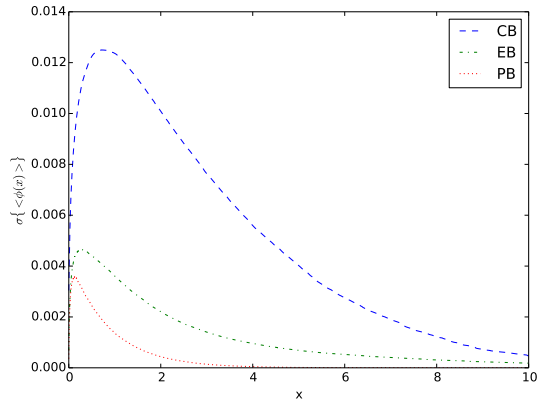


(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source

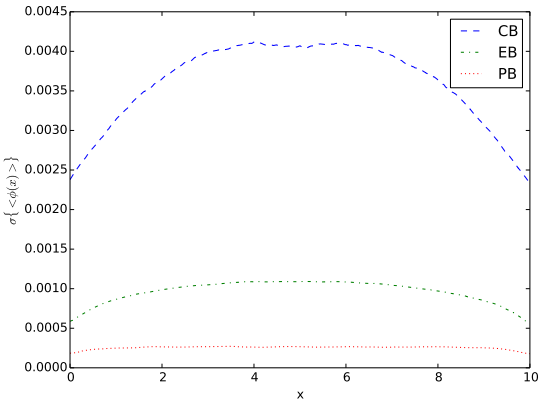


(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

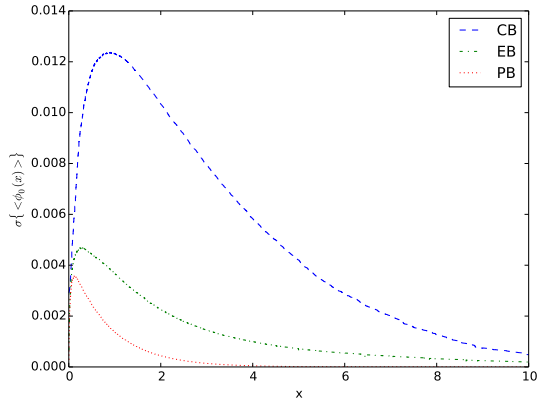
Figure 8.19: Standard deviation for M1, S1, L10.0



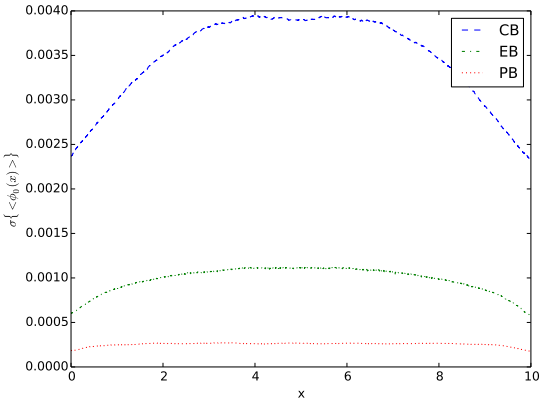
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



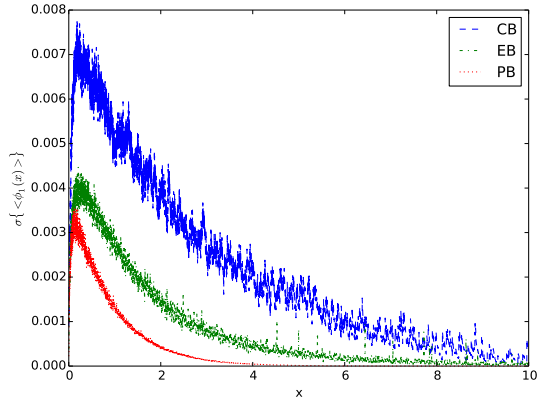
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



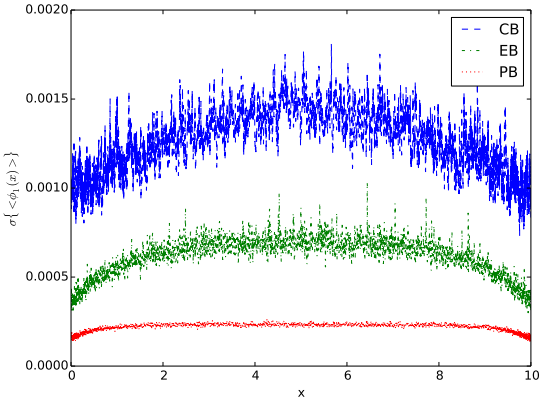
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source

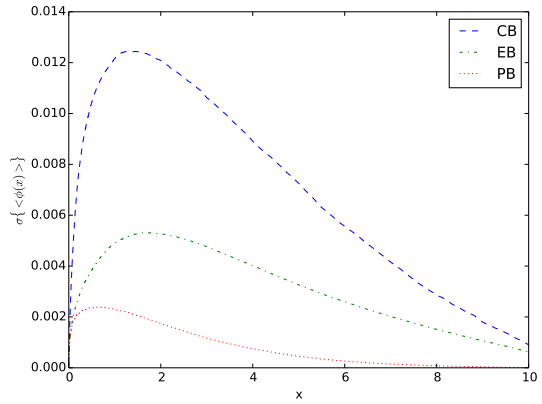


(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source

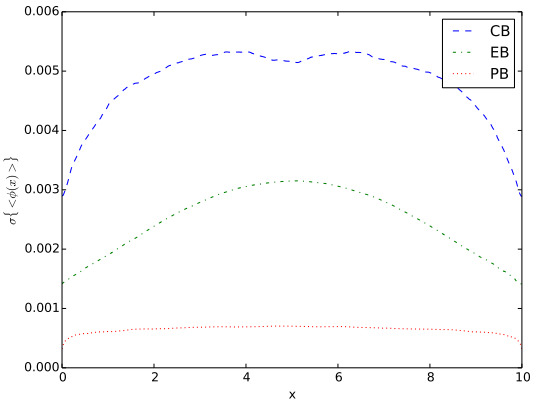


(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

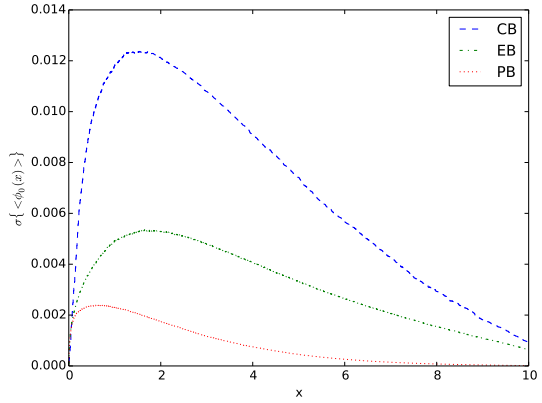
Figure 8.20: Standard deviation for M1, S2, L10.0



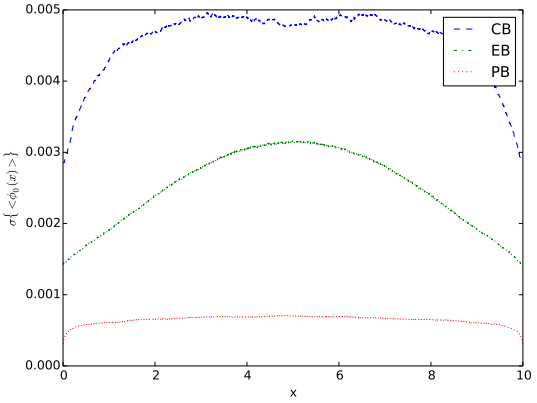
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



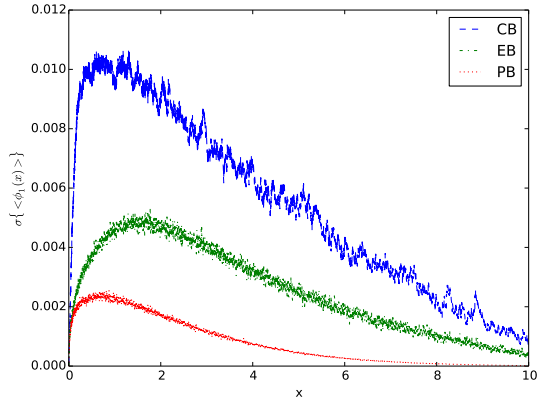
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



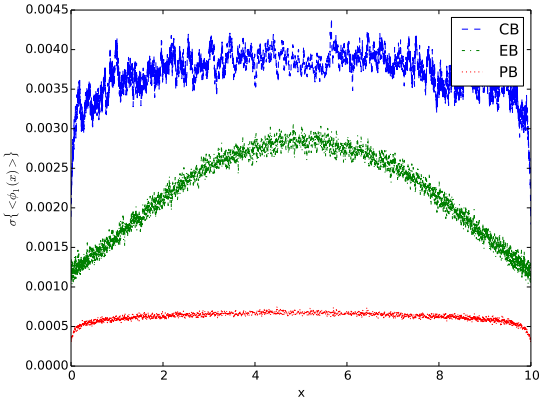
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source

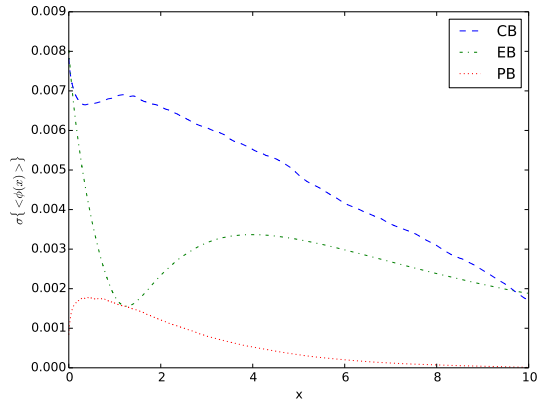


(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source

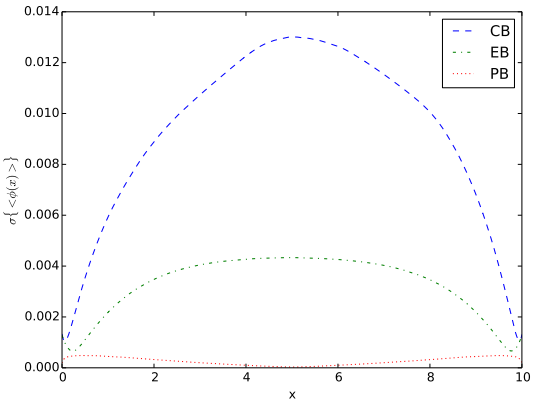


(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

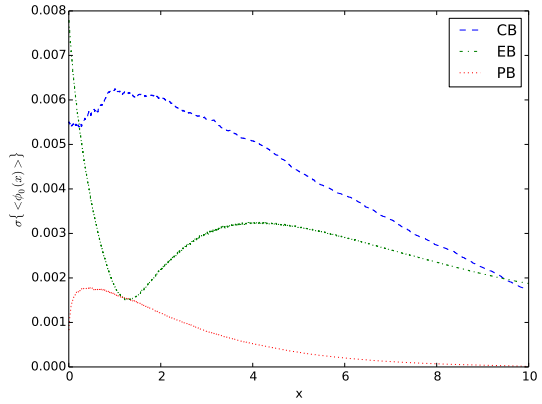
Figure 8.21: Standard deviation for M1, S3, L10.0



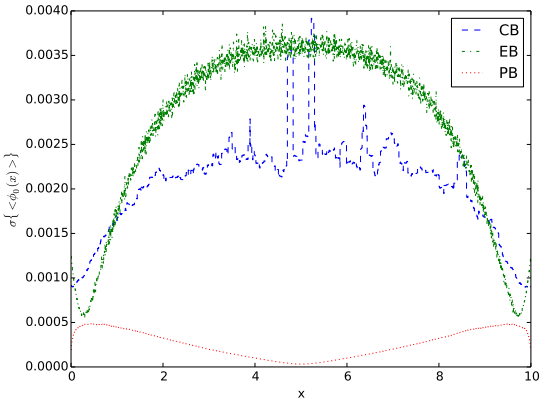
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



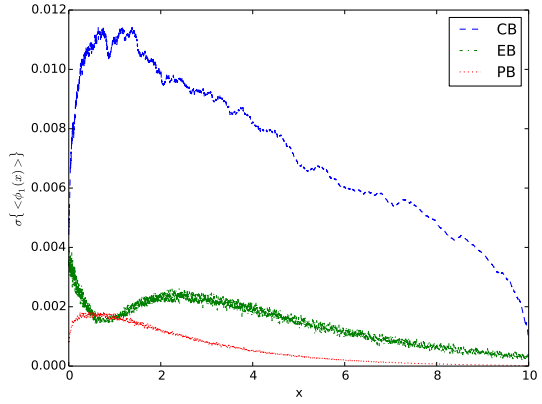
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



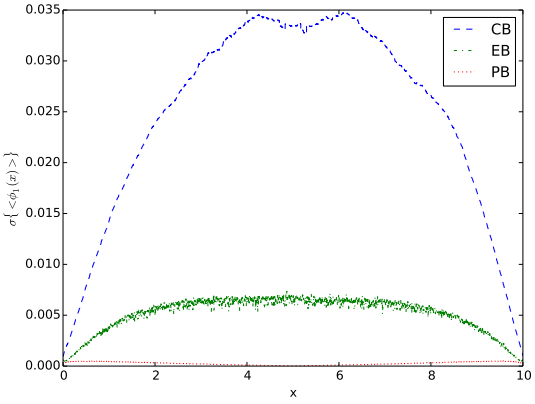
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source

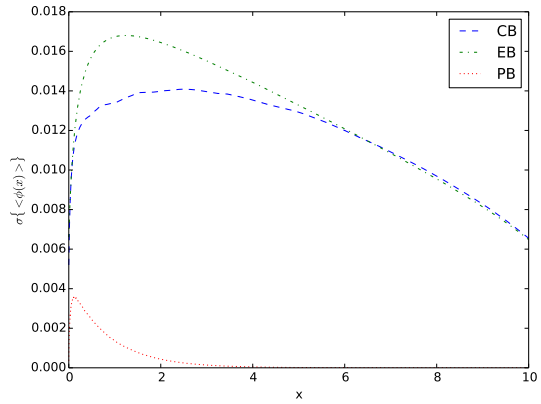


(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source

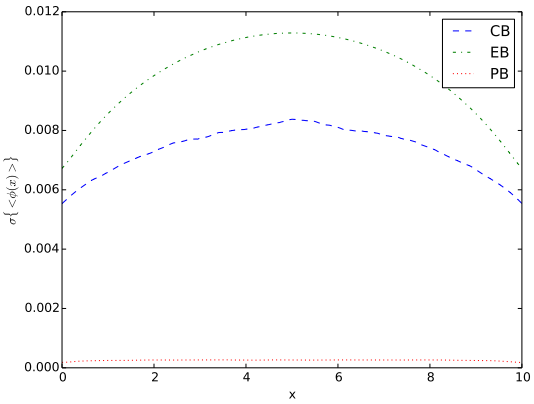


(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

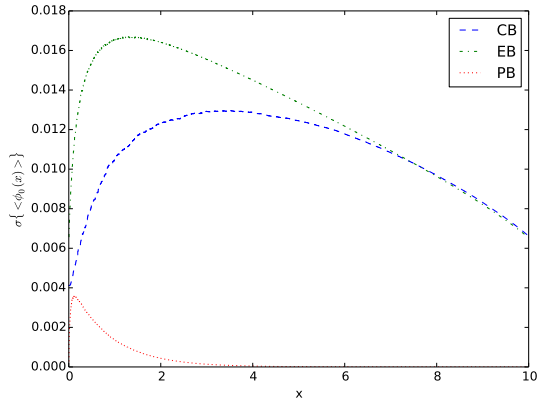
Figure 8.22: Standard deviation for M2, S1, L10.0



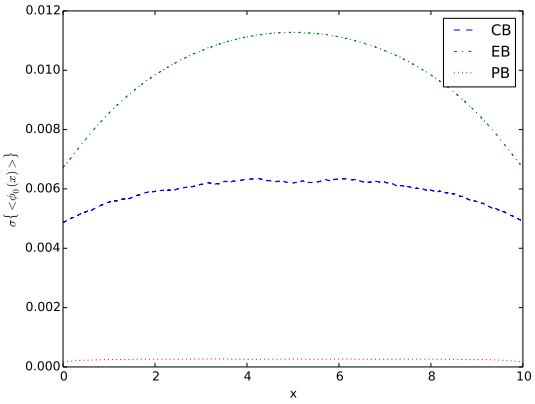
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



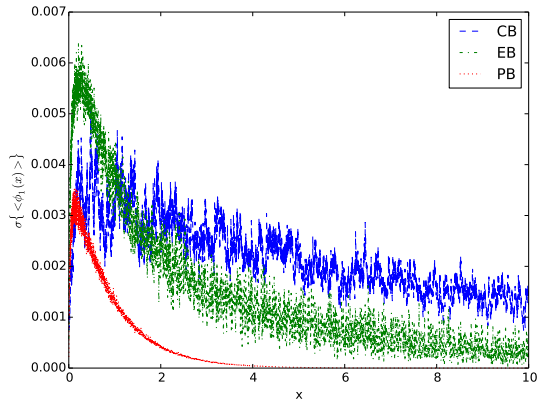
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



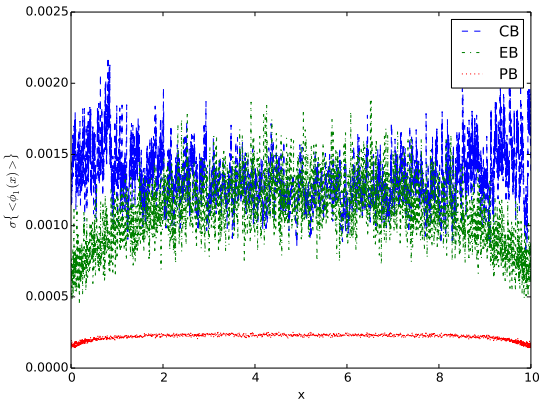
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source

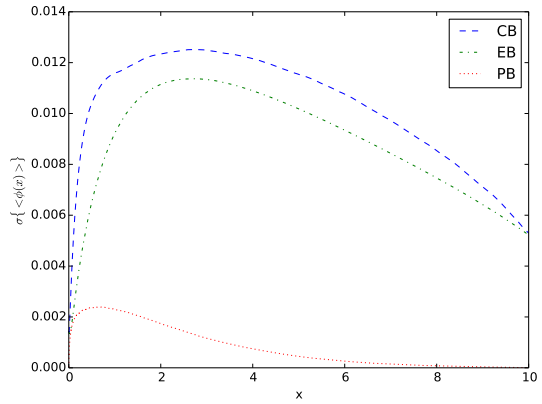


(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source

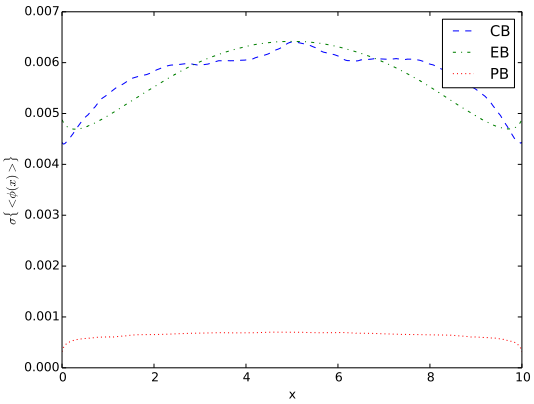


(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

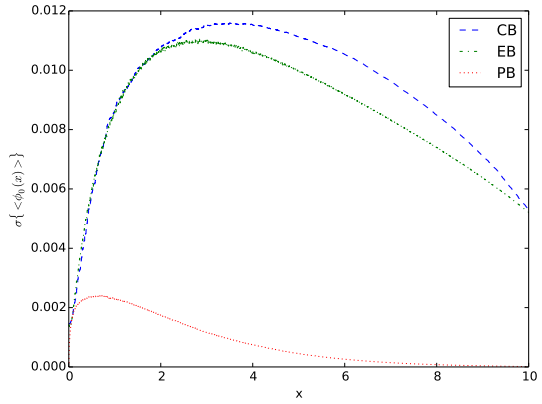
Figure 8.23: Standard deviation for M2, S2, L10.0



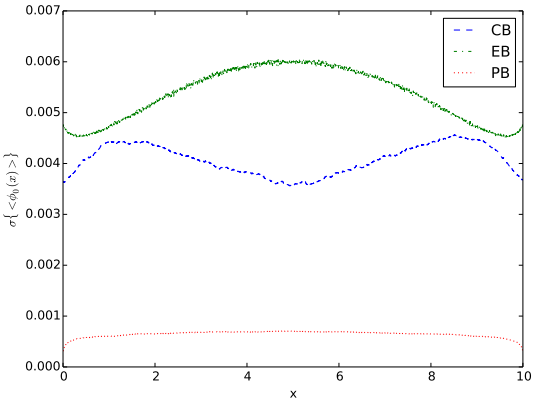
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



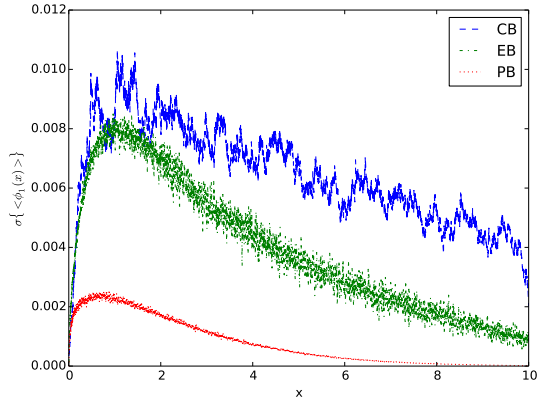
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



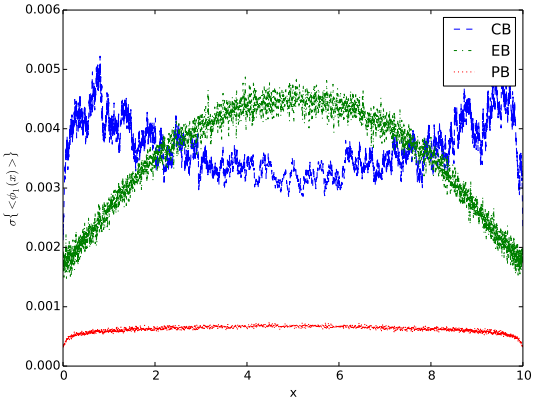
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source

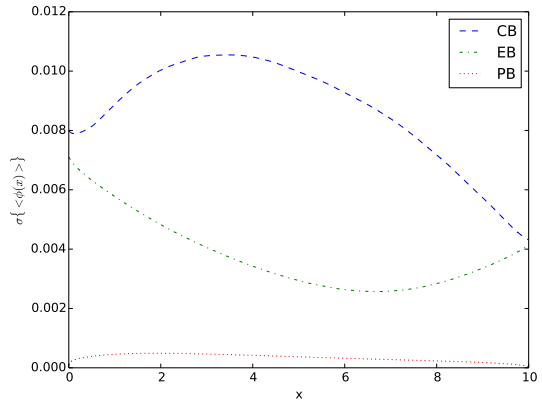


(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source

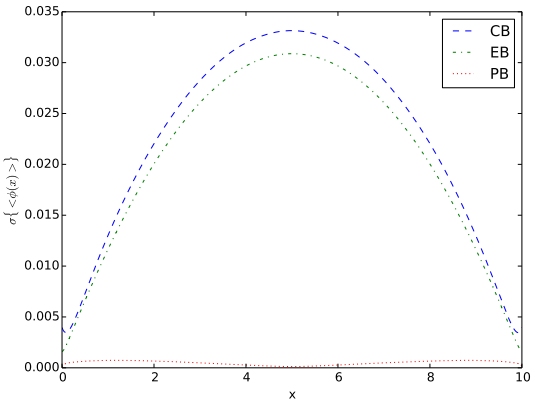


(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

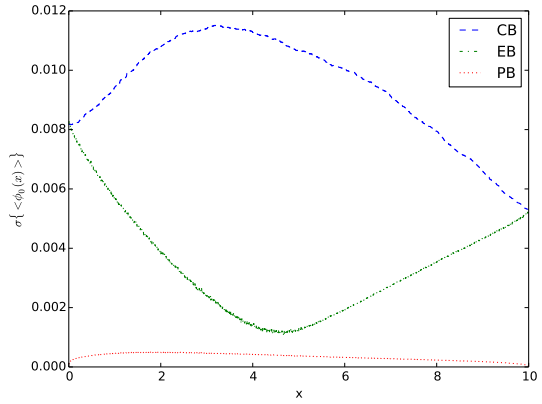
Figure 8.24: Standard deviation for M2, S3, L10.0



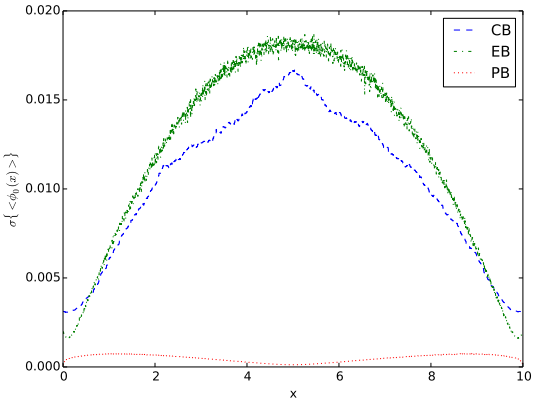
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



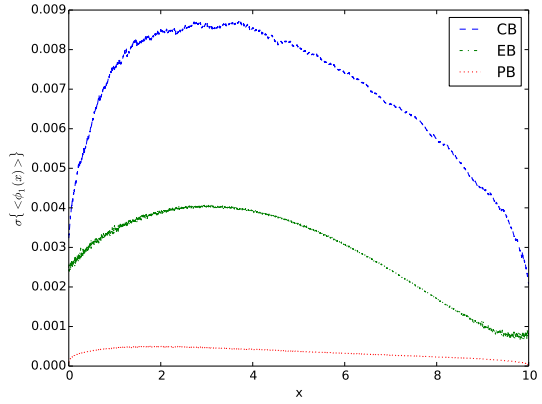
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



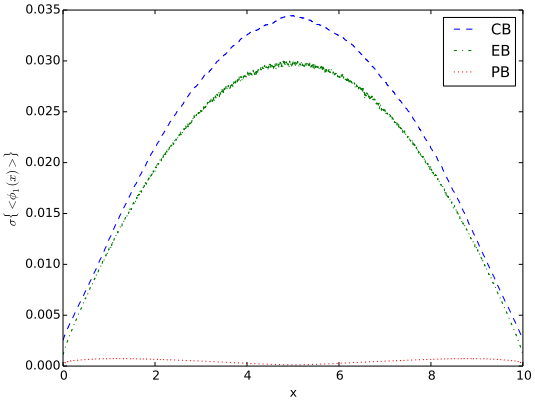
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source

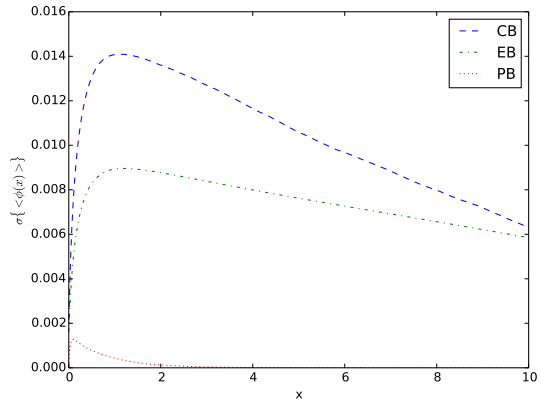


(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source

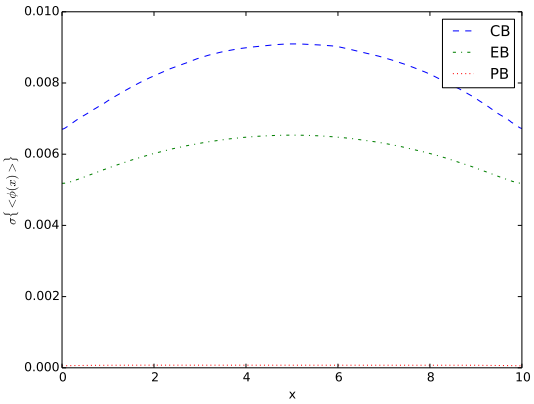


(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

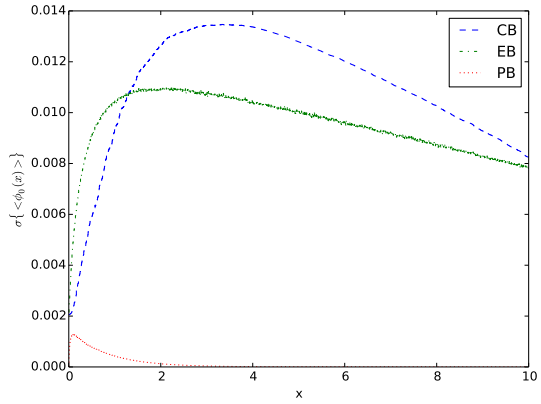
Figure 8.25: Standard deviation for M3, S1, L10.0



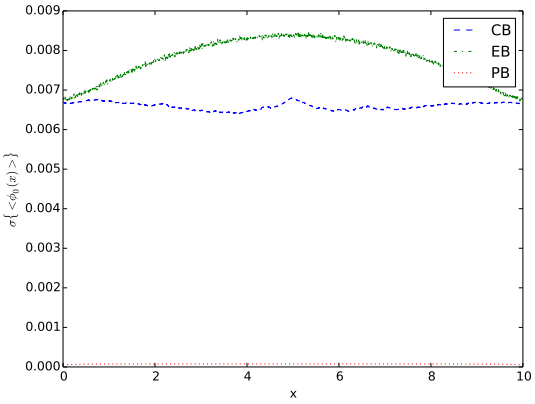
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



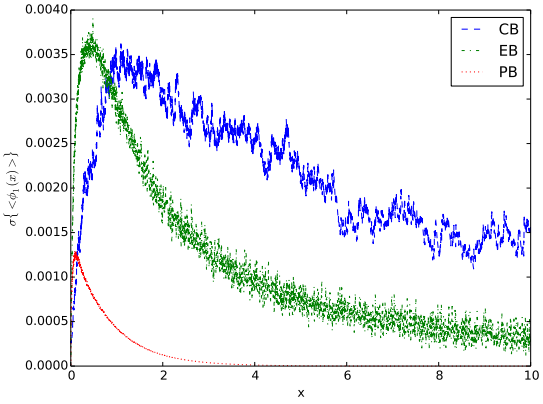
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



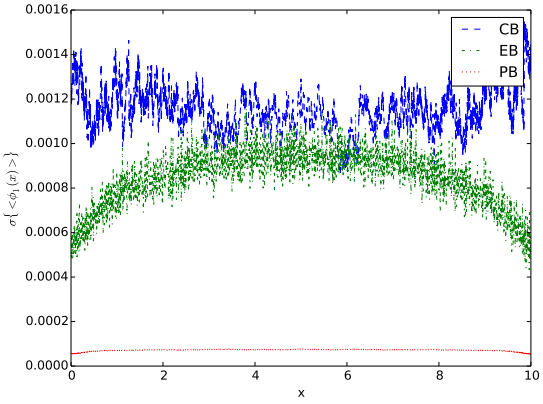
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source

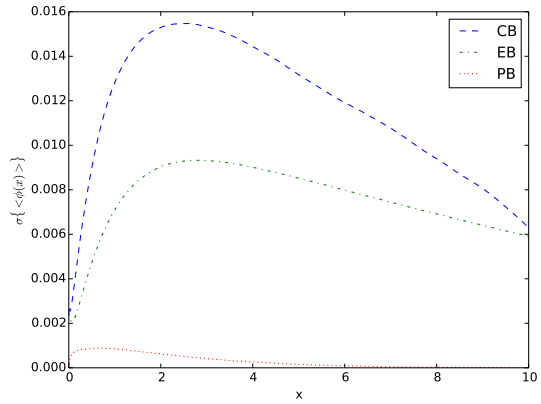


(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source

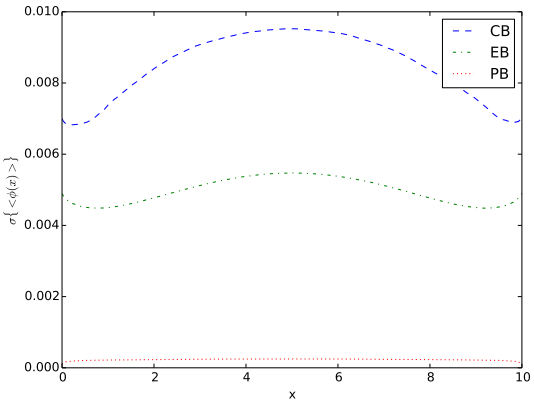


(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

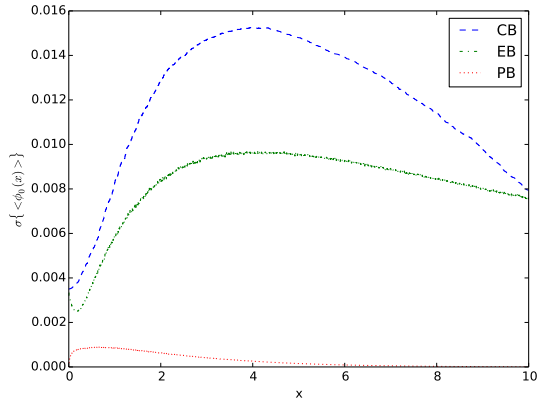
Figure 8.26: Standard deviation for M3, S2, L10.0



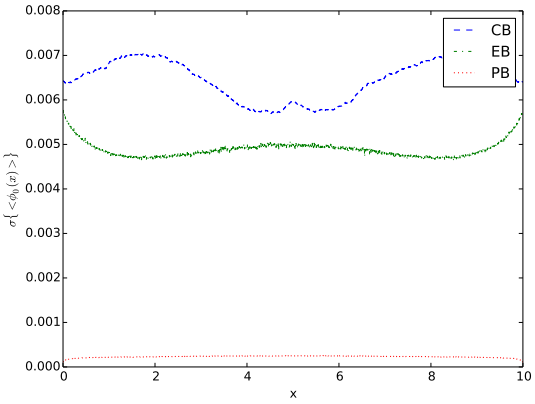
(a) $\sigma\{\langle\phi(x)\rangle\}$, boundary source



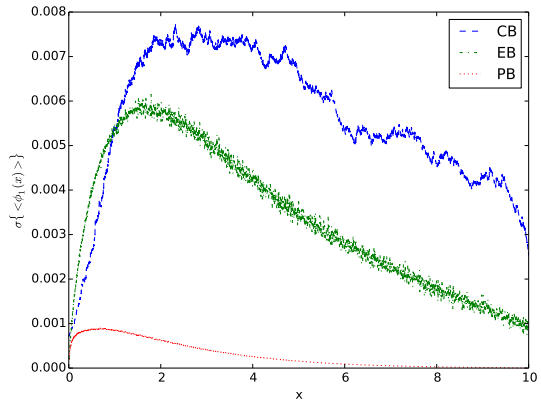
(b) $\sigma\{\langle\phi(x)\rangle\}$, internal source



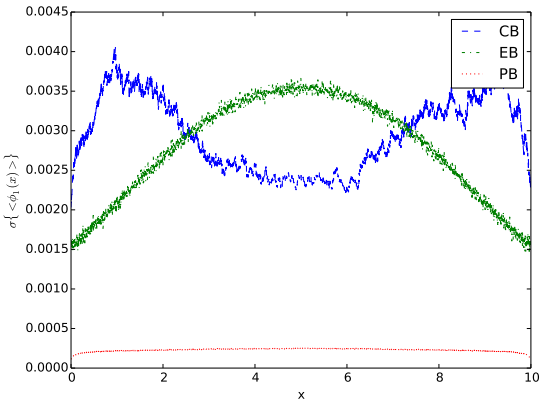
(c) $\sigma\{\langle\phi_0(x)\rangle\}$, boundary source



(d) $\sigma\{\langle\phi_0(x)\rangle\}$, internal source



(e) $\sigma\{\langle\phi_1(x)\rangle\}$, boundary source



(f) $\sigma\{\langle\phi_1(x)\rangle\}$, internal source

Figure 8.27: Standard deviation for M3, S3, L10.0