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Many formulas for mechanial quadrature of differential equations and continuous functions have been developed but very little has been done toward irrational functions where the integrand becomes infinite or vanishes.

The object of this thesis is to develop formulas for mechanical quadrature of irrational functions of the following three types: Type A, The function $f(x) = \sqrt{\theta(x)}$ where $\theta(x)$ has a pole of the first order, Type B, The function $f(x) = \sqrt{\theta(x)}$ where $\theta(x)$ has a zero of the first order, Type C, The function $f(x) = \sqrt{\theta(x)}$ where $\theta(x)$ has two poles of the first order.

Although the Newton-Cotes formulas can be applied to these types of functions after making a transformation, the present formulas are more convenient because they are developed for these particular cases where the infinite ordinate or vertical tangent lie at either or both ends of the interval.

FORMULAS FOR MECHANICAL QUADRATURE
OF IRRATIONAL FUNCTIONS

by

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FORMULAS FOR MECHANICAL QUADRATURE OF IRRATIONAL FUNCTIONS

CHAPTER I

INTRODUCTION

The problem of mechanical quadrature or numerical integration has become one of great interest today in many fields of science on account of its practical and time saving value. Often it is desirable to find the integral of a function by numerical integration because either, the general formula can not be found; or, if found, it is very difficult to apply. The first attempt to answer the problem of numerical integration was made by Sir Isaac Newton.(3) He followed through the developments and applications of the subject to a degree of detail which has been difficult to surpass. Therefore, his work has been the basis for many of the later developments.

The formulas for mechanical quadrature are usually developed by either of the following methods: (1) the method of differences, or (2) the method of ordinates.(4) Although these methods are equal in importance, they differ greatly in appearance. The latter has an advantage in that the calculation of a difference table is not required.

There are many formulas and methods which have contributed to this specialized field of mathematics. Among those who have directed their interest toward the numerical solution of differential equations are Runge, Heun, Kutta, Nystrom, Moulton, and J. C. Adams.[2] Still others have directed their attention to quadrature formulas for continuous functions by one of the above mentioned methods. Some of the outstanding accomplishments in this branch of numerical integration are due to Newton, Cotes, Euler, Gregory and Steffensen.[1] Up to the present time, however, very little has been done toward the numerical integration of irrational functions where the integrand becomes infinite as in the case of $1/\sqrt{x}$ or vanishes as in $\sqrt{x}$ at $x = 0$. It is the purpose of this thesis to develop mechanical quadrature formulas for such functions.

CHAPTER II

DEVELOPMENT OF FORMULAS

Section I. Formulas of Newton-Cotes.

Before taking up the method of development of the formulas of this thesis, let us consider the basis for the Newton-Cotes formulas.

Let F(s) denote a given continuous function of s in the interval (a, b), which may be expressed in this manner,

$$\int_a^b F(s)ds,$$

Although Newton-Cotes formulas can be applied to any closed interval (a, b), it is convenient to make a change of variable so that the relationship between the original variable s and the new variable x is given by the expression:

(1) $$s = (b - a)\,x + a.$$

By direct substitution of the original limits (a, b) in (1) we have the new limits (0, 1). It also follows from (1) that $ds = (b - a)dx$, and $F(s) = F[(b - a)x + a]$ which we shall call f(x) for convenience.

Therefore, we have

(2) $$\int_a^b F(s)ds = (b - a)\int_0^1 F[(b - a)x + a]\,dx = (b - a)\int_0^1 f(x)dx.$$

We will now derive the formula for $\int_0^1 f(x)dx$ since $\int_a^b F(s)ds$ can be found by (2).

A change of variable does not affect the continuity of a function. Therefore, since f(x) is continuous, let us assume that it can be approximately represented by a polynomial in x of degree n, expressed as $P_n(x)$. By the rules of integration, the desired integral of the function f(x) with respect to x in the interval (0, 1) is approximately equal to the integral of the approximating polynomial with respect to x in the interval. Hence we have

$$(3) \int_0^1 f(x)dx = \int_0^1 P_n(x)dx + \text{Remainder},$$

where the right hand part of the equation may be approximately represented in this manner

$$(4) \int_0^1 P_n(x)dx + \text{Remainder} = A_0Y_0 + A_1Y_1 + - - - A_nY_n .$$

In (4), Y_i is the ordinate corresponding to x_i in the interval (0, 1) ($Y_i = f(x_i)$, where i = 1, 2 - - - n), and where A_i is the undetermined coefficient of Y_i. Also the interval (0, 1) is subdivided into n equal segments of length h, whence h = 1/n.

From (3) and (4), we have by direct substitution

$$(5) \int_0^1 f(x)dx = A_0Y_0 + A_1Y_1 + A_2Y_2 + - - - - - - - + A_nY_n .$$

By letting f(x) in the above equation take on the following values 1, x, x^2, - - - - x^n, a set of n simultaneous equations in A are now obtained by direct integration,

$$(6)\quad \int_0^1 dx = 1 = A_0 + A_1 + A_2 + A_3 + \text{- - - - - - - - - -} + A_n,$$

$$\int_0^1 x\,dx = 1/2 = hA_1 + 2hA_2 + 3hA_3 + \text{- - - - - -} + nhA_n,$$

$$\int_0^1 x^2dx = 1/3 = h^2A_1 + 2^2h^2A_2 + 3^2h^2A_3 + \text{- -} + n^2h^2A_n,$$

- -

$$\int_0^1 x^ndx = \frac{1}{n+1} = h^nA_1 + 2^nh^nA_2 + 3^nh^nA_3 + \text{--} + n^nh^nA_n,$$

Before solving these simultaneous equations for the A's we will eliminate the factor h from each equation, and at the same time substitute for it, its value 1/n, We then obtain

$$(7)\quad 1 = A_0 + A_1 + A_2 + \text{- - - - - - - - - - - - - - -} + A_n,$$

$$\frac{n}{2} = \quad A_1 + 2A_2 + 3A_3 + \text{- - - - - - - - - - - - -} nA_n,$$

$$\frac{n^2}{3} = \quad A_1 + 2^2A_2 - 3^2A_3 + \text{- - - - - - - - - - - -} n^2A_n,$$

- -

$$\frac{n^n}{n+1} = \quad A_1 + 2^nA_2 + \text{- - - - - - - - - - - - -} + n^nA_n,$$

In order that the A's in these n simultaneous equations may be uniquely determined, the determinant of the coefficients of the A's must not equal zero, Let us next study this determinant,

$$(8)\quad D = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & n \\ 0 & 1^2 & 2^2 & 3^2 & \cdots & n^2 \\ 0 & 1^3 & 2^3 & 3^3 & \cdots & n^3 \\ - & - & - & - & \cdots & - \\ - & - & - & - & \cdots & - \\ 0 & 1^n & 2^n & 3^n & \cdots & n^n \end{vmatrix},$$

We will evaluate D by the first column. Since all of the elements are 0 except the first, we have

$$(9)\quad D = 1 \cdot \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1^2 & 2^2 & 3^2 & \cdots & n^2 \\ 1^3 & 2^3 & 3^3 & \cdots & n^3 \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ 1^n & 2^n & 3^n & \cdots & n^n \end{vmatrix},$$

This cofactor, D_1, is a well known determinant of Vandermonde type and can be expressed in this manner,

(10) $D_1 = (n-1)! \cdots 2!\ 1!,$

Because $D_1 \neq 0$, and therefore by (9) $D \neq 0$, the A's can be determined.

It will be convenient to write (6) in the following manner,

$$(11)\quad 1 = A_0 + A_1 + A_2 + \cdots + A_n,$$

$$\frac{1}{2} = x_0A_0 + x_1A_1 + x_2A_2 + \cdots + x_nA_n,$$

$$\frac{1}{3} = x_0^2A_0 + x_1^2A_1 + x_2^2A_2 + \cdots + x_n^2A_n,$$

$$- -$$

$$\frac{1}{n+1} = x_0^nA_0 + x_1^nA_1 + x_2^nA_2 + \cdots + x_n^nA_n,$$

where $x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, $\cdots$ $x_n = 1$, Let us assume that

$$P(x) = (x - x_0)(x - x_1)(x - x_2) ------ (x - x_n).$$

$$P_0(x) = \frac{P(x)}{x-x_0} = (x - x_1)(x - x_2) ---- (x - x_n), \text{ or}$$

$$= x^n + a_1{}^0 x^{n-1} + a_2{}^0 x^{n-2} + ------- + a_n{}^0,$$

$$P_1(x) = \frac{P(x)}{(x-x_1)} = (x - x_0)(x - x_2) --- (x - x_n), \text{ or}$$

$$= x^n + a_1{}^1 x^{n-1} + a\ {}^1 x^{n-2} --------- a_n{}^1.$$

- -

$$P_n(x) = \frac{P(x)}{(x-x_n)} = (x - x_0)(x - x_1)(x - x_2) -- (x-x_{n-1})$$

or

$$= x^n + a_1^n x^{n-1} + ------------ + a_n^n.$$

We will now multiply the left hand members of (6) by the coefficients of $P_0(x)$ in reverse order. This gives us

$$(12)\quad a_n{}^0 + a_{n-1}^0 \cdot \frac{1}{2} + a_{n-2}^0 \cdot \frac{1}{3} + ---------- + \frac{1}{n+1},$$

on the left hand side, while on the right we get

$$A_0 P_0(x_0) + A_1 P_0(x_1) + --- + A_n P_0(x_n)$$

which reduces to $A_0 P_0(x_0)$, since $P_0(x_1) = 0$, $P_0(x_1) = 0$, - -, By setting $x = x_0$ in the equation for $P_0(x)$, we have

$$P_0(x_0) = (x_0 - x_1)(x_0 - x_2) ---------- (x_0 - x_n),$$

Therefore from (12), we obtain

$$A_0 = \frac{a_n{}^0 + a_{n-1}^0 \cdot \frac{1}{2} + ----- + \frac{1}{n+1}}{(x_0 - x_1)(x_0 - x_2) --- (x_0 - x_n)}.$$

Similarly, we get

$$A_1 = \frac{a_n^{\,1} + a^1_{n-1} \cdot \frac{1}{2} + - - - - - - + \frac{1}{n+1}}{P_1(x_1)}$$

- - - - - - - - - - - - - - - - - -

$$A_n = \frac{a_n^{\,n} + a^1_{n-1} \cdot \frac{1}{2} + - - - - - - + \frac{1}{n+1}}{P_n(x_n)}$$

The exact value of the A's can now be found by substituting in the values $x_0 = 0$, $x_1 = \frac{1}{n}$, - - - -, $x_n = 1$, and the values of the a's.

These values of the A's are then substituted in the integral formula (5), which gives the desired formula for the numerical integration of the function.

By this procedure the following Newton-Cotes formulas are developed:

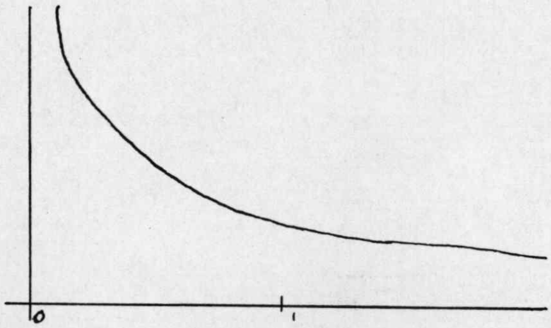
$$\int_0^{2h} f(x)dx = \frac{h}{3}(Y_0 + 4Y_1 + Y_2),$$

$$\int_0^{4h} f(x)dx = \frac{4h}{90}\,(7Y_0 + 32Y_1 + 12Y_2 + 32Y_3 + 7Y_4),$$

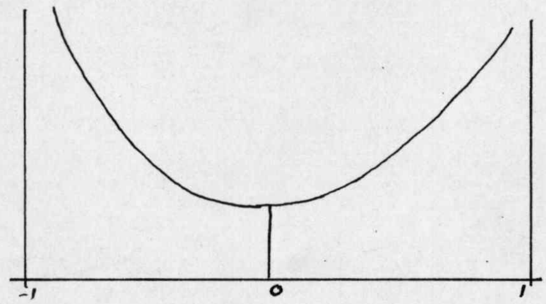
$$\int_0^{6h} f(x)dx = \frac{6h}{840}\,(41Y_0 + 216Y_1 + 27Y_2 + 272Y_3 + 27Y_4 + 216Y_5 + 41Y_6),$$

Section II. Derivation of new formulas.

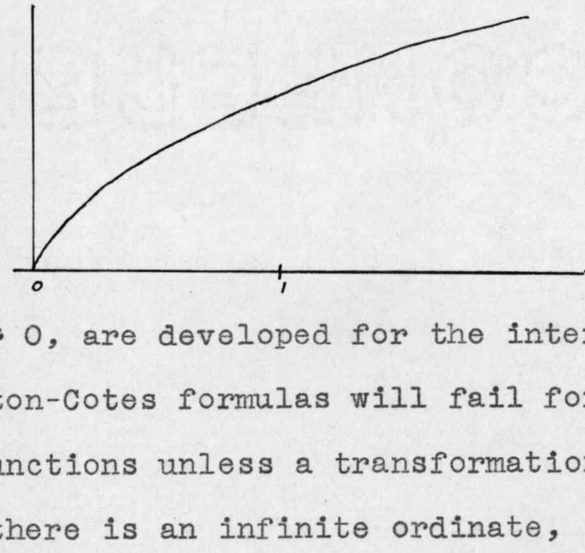
In this section, formulas for mechanical quadrature of irrational functions, where the integrand becomes infinite



as (a) in the case of $1/\sqrt{x}$ at $x = 0$, and (b) in the case



of $1/\sqrt{1-x^2}$ at $x = \pm 1$, and where the integrand vanishes as



in $\sqrt{x}$ at $x = 0$, are developed for the interval 0 nh,

The Newton-Cotes formulas will fail for these three irrational functions unless a transformation is made, because in case (a) there is an infinite ordinate, in (b) there are two infinite ordinates and in (c) one vertical tangent,

To bring out the difficulties encountered when using a Newton-Cotes formula for one of these irrational functions, we will apply it to case (a), $\int_0^1 \frac{1}{\sqrt{x}} dx$,

Expressing x as $x\,\psi(x)$, where $\psi(x)$ is a holomophic function, we have $\int_0^1 \frac{1}{\sqrt{x\,\psi(x)}} dx$. Let us now make the transformation $x = s^2$, then $dx = 2sds$. We get $\int_0^1 \frac{2sds}{\sqrt{s^2\psi(s^2)}}$ $= \int_0^1 \frac{2ds}{\sqrt{\psi(s^2)}}$, which is continuous within the closed interval because of the definition of $\psi(x)$, which becomes $\psi(s^2)$ by the transformation.

Still greater difficulties, however, arise in the calculations because x_i must be determined from s_i before Y_i in the formulas, Section I, 5 can be found.

It is due to the inconvenience of the Newton-Cotes formulas, for these three mentioned types of functions that the writer of this thesis has derived the following formulas.

Only the first formula in each type will be completely derived due to similarity in development of the rest.

Type A. $f(x) = \sqrt{\theta(x)}$ where $\theta(x)$ has a pole of the first order.

1. <u>Four-point formula</u>.

Let f(x) represent a function in x where $\lim_{x \to 0} \sqrt{x}\, f(x) \to C$ in the interval 0 to h. (In case the infinite ordinate should not be at an end of the interval, it would be necessary to make a change of variable, Section I, 1, before completing the derivation).

We now assume that $\sqrt{x}\, f(x)$ may be approximately represented within the interval by a polynomial in x of degree 3, thus

$$\sqrt{x}\, f(x) = P_3(x) + \text{Remainder},$$

Then it follows by Section I, (3) and (4) that

$$(1)\quad \int_0^h f(x)dx = \int_0^h \frac{P_3(x)dx}{\sqrt{x}} + \text{Remainder}$$

$$= A_0\varphi_0 + A_1\varphi_1 + A_2\varphi_2 + A_3\varphi_3 ,$$

where $\varphi(x) = \sqrt{x}\, f(x)$ and A_i the undetermined coefficients.

Replacing $\varphi(x)$ by 1, x, x^2, x^3 respectively, in (1) we obtain by direct integration the following four simultaneous equations which must be satisfied by the A's,

$$(2)\quad \int_0^h \frac{dx}{\sqrt{x}} = 2\sqrt{h} = A_0 + A_1 + A_2 + A_3 ,$$

$$\int_0^h \frac{xdx}{\sqrt{x}} = \frac{2}{3}h\sqrt{h} = 0 + hA_1 + 2hA_2 + 3hA_3,$$

$$\int_0^h \frac{x^2dx}{\sqrt{x}} = \frac{2}{5}h^2\sqrt{h} = 0 + h^2A_1 + 4h^2A_2 + 9h^2A_3,$$

and

$$\int_0^h \frac{x^3dx}{\sqrt{x}} = \frac{2}{7}h^3\sqrt{h} = 0 + h^3A_1 + 8h^3A_2 + 27h^3A_3,$$

After the elimination of a factor h, h^2, and h^3, respectively, from the last three equations above and the common factor $\frac{\sqrt{h}}{315}$ from all four equations, it is found that the determinant of the coefficients of the A's is not equal to zero by Section I, (8) and (9). Hence, there is one solution which uniquely determines the A's, These A's can be found by solving the 4 simultaneous equations in (2) - the method used by the writer - or by the method given in Section I, (11) through (13).

$$A_3 = 22 \qquad A_1 = 360$$

$$A_2 = -108 \qquad A_0 = 356$$

It follows by the substitution of the above values of the A's in (1) that

$$\int_0^h f(x)dx = \frac{\sqrt{h}}{315}\left(356\varphi_0 + 360\varphi_1 - 108\varphi_2 + 22\varphi_3\right),$$

which is the desired four-point formula of degree 3 for the interval 0 to h,

We also have by the above method formulas where the intervals are $0 \rightarrow 2h$ and $0 \rightarrow 3h$. See Appendix, Section II, Type A, 1.

2, Five-point Formulas.

The five-point formulas for the intervals 0 to nh, where n = 1, 2, ,3, 4 are listed in the Appendix, Section II, Type A, 2.

3, Six-point Formulas.

The six-point formulas for the intervals 0 to nh, where n = 1, 2, 3, 4, 5 are listed in the Appendix, Section II, Type A, 3.

4, Seven-point Formula.

The seven-point formula for the interval 0 to 6h is given in the Appendix, Section II, Type A, 4.

5. Eight-point Formula.

The eight-point formula for the interval 0 to 7h is given in the Appendix, Section II, Type A, 5.

Type B. $f(x) = \sqrt{\theta(x)}$ where $\theta(x)$ has a zero of the first order.

1. Four-point formula.

Let f(x) represent a function in x where $\lim_{x\to 0} f(x)/\sqrt{x} \to C$ in the interval 0 to h, where the interval is the result of a change of variable. The vanishing ordinate of the function then lies at one end of the interval, 0.

We now assume that $f(x)/\sqrt{x}$ may be approximately represented within this interval by a polynomial in x of degree 3, thus

$$f(x)/\sqrt{x} = P_3(x) + \text{ Remainder},$$

and by the rules of integration

$$(1) \quad \int_0^h f(x)dx = \int_0^h \sqrt{x}\, P_3(x)dx + \text{Remainder}$$

$$= A_0\varphi_0 + A_1\varphi_1 + A_2\varphi_2 + A_3\varphi_3$$

where $\varphi(x)$ is equal to $f(x)/\sqrt{x}$.

Replacing $\varphi(x)$ by 1, x^1, x^2, x^3 respectively, we obtain by direct integration the following four simultaneous equations which must be satisfied by the A's.

$$(2) \quad \int_0^h \sqrt{x}dx = \frac{2h\sqrt{h}}{3} = A_0 + A_1 + A_2 + A_3$$

$$\int_0^h x\sqrt{x}dx = \frac{2h\sqrt{h}}{5} = 0 + hA_1 + 2hA_2 + 3hA_3$$

$$\int_0^h x^2\sqrt{x}dx = \frac{2h^3\sqrt{h}}{7} = 0 + h^2A_1 + 4h^2A_2 + 9h^2A_3$$

$$\int_0^h x^3\sqrt{x}dx = \frac{2h^4\sqrt{h}}{9} = 0 + h^3A_1 + 8h^3A_2 + 27h^3A_3,$$

From the last three equations above the factor h, h^2, and h^3, respectively, are removed and a common factor $\frac{2h\sqrt{h}}{945}$ from all four equations, The determinant of the coefficients of the A's is not equal to zero since it is of the Vandermonde type, Section I, (9), Hence the A's are uniquely determined by either method mentioned in the development of previous formula,

$A_3 = 13$, $A_2 = -66$, $A_1 = 282$, $A_0 = 86$, These values substituted in equation (1), gives

$$\int_0^h f(x)dx = \frac{2h\sqrt{h}}{945}(86\varphi_0 + 282\varphi_1 - 66\varphi_2 + 13\varphi_3),$$

which is the desired four-point formula for the interval 0 to h,

In a similar manner the complete list of formulas for this type are developed, See Appendix, Section II, Type B, 1, for the four-point formulas for the interval 0 to nh, where n = 1, 2, 3,

2, <u>Five-point Formulas</u>,

The five-point formulas for the intervals 0 to nh where n = 1, 2, 3, 4 are listed in the Appendix, Section II, Type B, 2,

3. Six-point Formulas.

The six-point formulas for the interval 0 to nh where n = 1, 2, 3, 4, 5 are listed in the Appendix, Section II, Type B, 3.

4. Seven-point Formula.

The seven-point formula for the interval 0 to 6h is listed in the Appendix, Section II, Type B, 4.

Type C. $f(x) = \sqrt{\theta(x)}$ where $\theta(x)$ has two poles of the first order.

Let us represent f(x) by a function in x where $f(x) = \frac{\varphi(x)}{\sqrt{1-x^2}}$ in the interval (-1, 1). In case the infinite ordinates do not lie at the ends of the interval, a change of variable must be made.

1. Three-point formula.

Let us assume that f(x) may be approximately represented by a polynomial of degree 2 in x,

$$\frac{\varphi(x)}{\sqrt{1-x^2}} = P_2(x) + \text{Remainder}.$$

We will now take the integral between (-1, 1), which gives

$$\int_{-1}^{1}\frac{\varphi(x)dx}{\sqrt{1-x^2}} = \int_{-1}^{1} P_2(x)dx + \text{Remainder}$$

$$(1) \qquad = A_{-1}\varphi(-1) + A_0\varphi(0) + A_1\varphi(1), \text{ where}$$

the φ's are the values of $\varphi(x)$ at the middle and ends of the interval, and the A's the undetermined coefficients.

Replacing $\varphi(x)$ by 1, x^1, x^2 respectively we obtain by direct integration the following three simultaneous equations which must be satisfied by the A's,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \pi = A_{-1} + A_0 + A_1$$

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\, dx = 0 = -A_{-1} + 0 + A_1$$

$$\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2} = A_{-1} + 0 + A_1$$

That the determinant of the coefficients of the A's is not equal to zero, can be easily shown by expanding the determinant by the element in the first row and second column. Therefore, the A's can be uniquely determined by the method of elimination which gives us

$$A_{-1} = \frac{\pi}{4}\,, \quad A_0 = \frac{\pi}{2}\,, \quad A_1 = \frac{\pi}{4}\;.$$

By substituting these values of the A's in (1) we get

$$\int_{-1}^{1} \frac{\varphi(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{4}\left[\varphi(-1) + 2\varphi(0) + \varphi(1)\right].$$

The development of the five-point and nine-point formulas is similar to the above steps. These formulas are listed in the Appendix, Type C, 2 and 3.

CHAPTER III

COMPARISON OF RESULTS OBTAINED BY FORMULA AND BY INTEGRATION FOR CERTAIN EXAMPLES

To illustrate the value of these formulas, let us apply them to examples of each type considered in Chapter II, Section II.

In all of the examples except the last we will apply the Newton-Cotes five-point formula for that part of the interval which is closed because we know that it is accurate to the seventh decimal place for these examples, and the new formulas for the ends of the interval where the infinite or vertical tangents occur.

Type A.

Example I. To find $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$, where there is an infinite ordinate at each end of the interval.

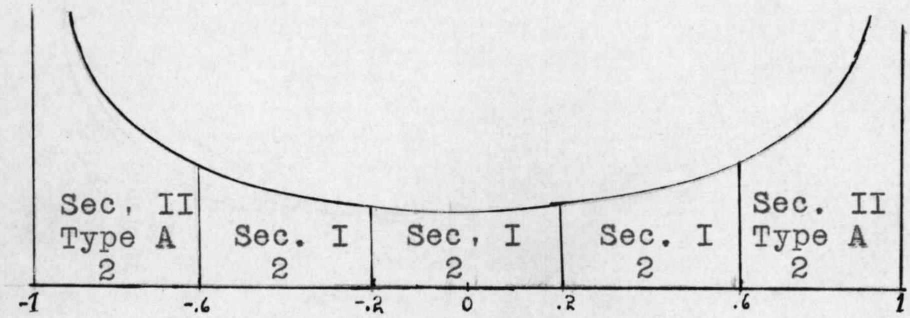
1, By direct integration, we get

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \text{arc sin } x \Big]_{-1}^{1} = \pi = 3.1415926$$

2. By formula, where $f(x) = \frac{1}{\sqrt{1-x^2}}$,

$$\varphi(x) = \frac{\sqrt{1-x}}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1+x}}$$, and the interval is divided

in this manner we get, by using the formulas



indicated in each subdivision of the interval above, the following result:

x_i	φ's	Multipliers	
.1	.7071068	250	
.9	.7254763	416	
.8	.7453560	24	
.7	.7669649	224	
.6	.7905694	31	
			$\times \frac{2\sqrt{.4}}{945} = .9272945$
	f(x)		$\times 2$
.6	1.2500000	7	
.5	1.1547005	32	
.4	1.0910894	12	
.3	1.0482848	32	
.2	1.0206207	7	
			$\times \frac{.4}{90} = .442146432$
	f(x)		$\times 2$
.2	1.0206207	7	
.1	1.0050378	32	
0.	1.	12	
-.1	1.0050378	32	
-.2	1.0206207	7	
			$\times \frac{.4}{90} = .40271068$
			Total = 3.1415925

The first two intervals are multiplied by 2 because the function is symmetrical. The result found by these two methods differs by .0000001.

Example II. Find the value of the elliptic integral

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \text{, where } k^2 = \frac{1}{2}$$

1. From Die elliptischen Funktionen von Jacobi, Milne-Thompson, we have

$$K = 1.8540747$$

2. By formula, where $f(x) = \dfrac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$,

$\varphi(x) = \dfrac{1}{\sqrt{(1+x)(1-k^2x^2)}}$ and the interval is subdivided so that from 0 → .4, we will use Newton-Cotes formula, 2, Section I, and from .4 → 1., we will use formula 4, Type A, Section II in reverse order, since the infinite ordinate is at the upper limit.

x_1	f(x)	Multipliers
0	1.	7
.1	1.0075518	32
.2	1.0309826	12
.3	1.0726983	32
.4	1.1375393	7

$\times \frac{.4}{90} = .417340582$

	φ(x)	
.4	.8811343	4,764
.5	.8728716	36,936
.6	.8730378	-11,610
.7	.8826775	79,440
.8	.9038769	-23,895
.9	.9405128	93,096
1.	1.	46,494

$\times \frac{2\sqrt{.6}}{225,225} = 1.436737284$

Total 1.854077866

The results found by these two methods differ by .00000316.

Type B.

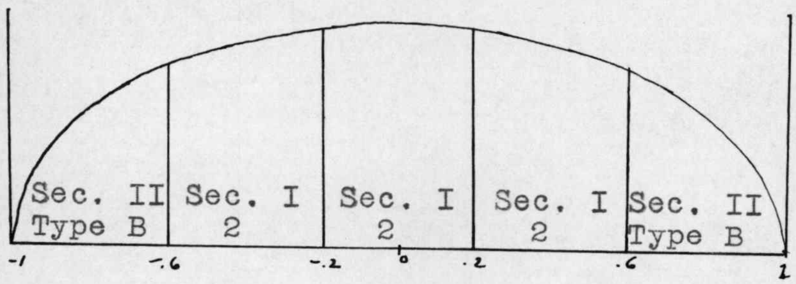
Example I. To find $\int_{-1}^{1} \sqrt{1-x^2}\, dx$, where there is a vertical tangent at each end of the interval.

1. By integration, we obtain

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\text{arc sin } x \Big]_{-1}^{1} = \pi$$

$$= 1.57079633$$

2. By formula, where $f(x) = \sqrt{1-x^2}$, $\varphi(x) = \sqrt{1+x}$ and the interval is divided in this manner



we get, by using the indicated formulas in each subdivision of the interval the following result.

	φ(x)	Multipliers	Product
1.	1.41421356	70	
.9	1.37840488	864	
.8	1.34164079	552	
.7	1.30384048	1568	
.6	1.26491106	411	
		$\times \frac{.8\sqrt{.4}}{10,395}$	=.22364840597
			x 2
	f(x)		
.6	.80000000	7	
.5	.86602540	32	
.4	.91651514	12	
.3	.95393920	32	
.2	.97979590	7	
		$\times \frac{.4}{90}$	= .363088014
			x 2
	f(x)		
.2	.97979590	7	
.1	.99498744	32	
0.	1.	12	
.1	.99498744	32	
.2	.97979590	7	
		$\times \frac{.4}{90}$	= .397313088
		Total	1.57078593

The first two intervals are multiplied by 2 because the function is symmetrical.

The results of the two methods differ by .0000104.

Type C.

Example I. Find $\int_{-1}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, when $k^2 = \frac{1}{2}$

1. By Milne-Thompson's table, we get

$$\int_{-1}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = 2K = 3.7081494$$

2. By formula 3, Type C, Section II where

$$\varphi(x) = \frac{1}{\sqrt{(1-k^2x^2)}}$$, we obtain this result,

x	$\varphi(x)$	Multipliers
-1.	1.4142135	69
- .75	1.1795356	176
- .5	1.0690449	-126
- .25	1.0160010	336
0.	1.0000000	-280
.25	1.0160010	336
.5	1.0690449	-126
.75	1.1795356	176
1.	1.4142135	69

$\times \frac{\pi}{630}$

Total = 3.70863184

These results differ by .00048244

By the comparison of the results from the two methods we see that the results are very similar.

The amount of work required in using these formulas may be greatly simplified by using a calculating machine and by tabulating the results in a manner similar to that above.

APPENDIX

TABLE OF FORMULAS

Section I

Newton-Cotes Formulas

1. Three-point Formula

$$\int_0^{2h} f(x)dx = \frac{h}{3} (Y_0 + 4Y_1 + Y_2).$$

2. Five-point Formula

$$\int_0^{4h} f(x)dx = \frac{4h}{90} (7Y_0 + 32Y_1 + 12Y_2 + 32Y_3 + 7Y_4).$$

3. Seven-point Formula

$$\int_0^{6h} f(x)dx = \frac{6h}{840} (41Y_0 + 216Y_1 + 27Y_2 + 272Y_3 + 27Y_4 + 216Y_5 + 41Y_6).$$

Section II

Derived Formulas

Type A, $f(x) = \sqrt{\theta(x)}$, where $\theta(x)$ has a pole of the first order,

1, Four-point Formulas,

$$\int_0^h f(x)dx = \frac{\sqrt{h}}{315}(356\varphi_0 + 360\varphi_1 - 108\varphi_2 + 22\varphi_3),$$

$$\int_0^{2h} f(x)dx = \frac{\sqrt{2h}}{315}(244\varphi_0 + 360\varphi_1 + 18\varphi_2 + 8\varphi_3),$$

$$\int_0^{3h} f(x)dx = 2\frac{\sqrt{3h}}{105}(34\varphi_0 + 45\varphi_1 + 18\varphi_2 + 8\varphi_3),$$

2, Five-point Formulas,

$$\int_0^h f(x)dx = \frac{\sqrt{h}}{945}(1025\varphi_0 + 1252\varphi_1 - 582\varphi_2 + 238\varphi_3 - 43\varphi_4),$$

$$\int_0^{2h} f(x)dx = 2\frac{\sqrt{2h}}{945}(355\varphi_0 + 584\varphi_1 - 39\varphi_2 + 56\varphi_3 - 11\varphi_4),$$

$$\int_0^{3h} f(x)dx = \frac{\sqrt{3h}}{315}(195\varphi_0 + 306\varphi_1 + 54\varphi_2 + 84\varphi - 9\varphi_4).$$

$$\int_0^{4h} f(x)dx = 2\frac{\sqrt{4h}}{945}(250\varphi_0 + 416\varphi_1 + 24\varphi_2 + 224\varphi_3 + 31\varphi_4).$$

3. Six-point Formulas.

$$\int_0^h f(x)dx = \frac{\sqrt{h}}{10,395}(10,932\varphi_0 + 15,487\varphi_1 - 9,832\varphi_2 + 6,048\varphi_3 - 2,188\varphi_4 + 343\varphi_5),$$

$$\int_0^{2h} f(x)dx = \frac{2\sqrt{2h}}{10,395}(3,807\varphi_0 + 6,914\varphi_1 - 1,409\varphi_2 + 1,596\varphi_3 - 611\varphi_4 + 98\varphi_5),$$

$$\int_0^{3h} f(x)dx = \frac{\sqrt{3h}}{3,465}(2,082\varphi_0 + 3,681\varphi_1 - 36\varphi_2 + 1,554\varphi_3 - 414\varphi_4 + 63\varphi_5),$$

$$\int_0^{4h} f(x)dx = \frac{2\sqrt{4h}}{10,395}(2,694\varphi_0 + 4,856\varphi_1 - 296\varphi_2 + 3,024\varphi_3 + 61\varphi_4 + 56\varphi_5),$$

$$\int_0^{5h} f(x)dx = \frac{\sqrt{5h}}{2,079}(972\varphi_0 + 1,685\varphi_1 + 40\varphi_2 + 840\varphi_3 + 460\varphi_4 + 161\varphi_5).$$

4. Seven-point Formula.

$$\int_0^{6h} f(x)dx = \frac{2\sqrt{6h}}{225,225}(46,494\varphi_0 + 93,096\varphi_1 - 23,895\varphi_2 + 79,440\varphi_3 - 11,610\varphi_4 + 36,936\varphi_5 + 4,764\varphi_6),$$

5, Eight-point Formula

$$\int_0^{7h} f(x)dx = \frac{2\sqrt{7h}}{28,378,350} (87,867,656\varphi_0 - 540,027,726\varphi_1$$

$$+ 1,571,254,440\varphi_2 - 2,483,221,615\varphi_3$$

$$+ 2,360,483,160\varphi_4 - 1,334,874,786\varphi_5$$

$$+ 422,439,416\varphi_6 - 55,542,195\varphi_7),$$

Type B, $f(x) = \sqrt{\theta(x)}$ where $\theta(x)$ has a zero of first order,

1, Four-point Formulas,

$$\int_0^{h} f(x)dx = \frac{2h\sqrt{h}}{945} (86\varphi_0 + 282\varphi_1 - 66\varphi_2 + 13\varphi_3),$$

$$\int_0^{2h} f(x)dx = \frac{2h\sqrt{2h}}{945} (44\varphi_0 + 408\varphi_1 + 186\varphi_2 - 8\varphi_3),$$

$$\int_0^{3h} f(x)dx = \frac{2h\sqrt{3h}}{35} (2\varphi_0 + 9\varphi_1 + 18\varphi_2 + 6\varphi_3),$$

2, Five-point Formulas,

$$\int_0^{h} f(x)dx = \frac{h\sqrt{h}}{10,395} (1,715\varphi_0 + 6,912\varphi_1 - 2,514\varphi_2$$

$$+ 994\varphi_3 - 177\varphi_4),$$

$$\int_0^{2h} f(x)dx = \frac{2h\sqrt{2h}}{10,395} (490\varphi_0 + 4,464\varphi_1 + 2,082\varphi_2$$

$$- 112\varphi_3 + 6\varphi_4),$$

$$\int_0^{3h} f(x)dx = \frac{h\sqrt{3h}}{1155} (105\varphi_0 + 702\varphi_1 + 1,026\varphi_2 + 504\varphi_3$$

$$- 27\varphi_4),$$

$$\int_0^{4h} f(x)dx = \frac{8h\sqrt{4h}}{10,395}\,(70\varphi_0 + 864\varphi_1 + 552\varphi_2 + 1,568\varphi_3 + 411\varphi_4),$$

3, Six-point Formulas,

$$\int_0^{h} f(x)dx = \frac{h\sqrt{h}}{675,675}\,(103,436\varphi_0 + 489,475\varphi_1 - 243,800\varphi_2 + 145,000\varphi_3 - 51,700\varphi_4 + 8,039\varphi_5),$$

$$\int_0^{2h} f(x)dx = \frac{4h\sqrt{2h}}{675,675}\,(15,611\varphi_0 + 146,650\varphi_1 + 64,525\varphi_2 - 500\varphi_3 - 1,375\varphi_4 + 314\varphi_5),$$

$$\int_0^{3h} f(x)dx = \frac{h\sqrt{3h}}{225,225}\,(18,558\varphi_0 + 146,475\varphi_1 + 180,900\varphi_2 + 117,450\varphi_3 - 14,850\varphi_4 + 1,917\varphi_5$$

$$\int_0^{4h} f(x)dx = \frac{8h\sqrt{4h}}{675,675}\,(5,422\varphi_0 + 51,800\varphi_1 + 44,600\varphi_2 + 93,200\varphi_3 + 31,075\varphi_4 - 872\varphi_5),$$

$$\int_0^{5h} f(x)dx = \frac{h\sqrt{5h}}{27,027} (2,036\varphi_0 + 11,725\varphi_1 + 21,400\varphi_2 + 12,400\varphi_3 + 34,100\varphi_4 + 8,429\varphi_5).$$

4. Seven-point Formula.

$$\int_0^{6h} f(x)dx = \frac{12h\sqrt{6h}}{225,225} (798\varphi_0 + 10,728\varphi_1 + 4005\varphi_2 + 23,760\varphi_3 + 4,230\varphi_4 + 25,992\varphi_5 + 5,562\varphi_6).$$

Type C. $f(x) = \sqrt{\Theta(x)}$, where $\Theta(x)$ has two poles of the first order.

1. Three-point Formula.

$$\int_{-1}^{1} \frac{\varphi(x)dx}{\sqrt{1-x^2}} = \frac{\pi}{4} \left[\varphi(-1) + 2\varphi(0) + \varphi(1)\right].$$

2. Five-point Formula

$$\int_{-1}^{1} \frac{\varphi(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{6} \left[\varphi(-1) + 2\varphi(-.5) + 2\varphi(.5) + \varphi(1)\right].$$

3. Nine-point Formula.

$$\int_{-1}^{1} \frac{\varphi(x)dx}{\sqrt{1-x^2}} = \frac{\pi}{630} \left[69\varphi(-1) + 176\varphi(-.75) - 126\varphi(-.5) + 336\varphi(-.25) - 280\varphi(0) + 336\varphi(.25) - 126\varphi(.5) + 176\varphi(.75) + 69\varphi(1)\right].$$

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