AN ABSTRACT OF THE THESIS OF

Lancelot G. Sylvester for the degree of Master of Science in Industrial Engineering presented on June 10, 1985

Title: A Computerized Approach To Finding The Minimum Sample Size For Single Sample Attribute Sampling Plans

Abstract approved: Redacted for Privacy
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Much has been written concerning the determination of a single sampling plan given two points on the operating characteristic (OC) curve. A computer program is presented which will provide minimum sample size for single sample attribute sampling plans. The program provides sampling plans based on the binomial distribution. It incorporates the Fibonacci search technique for discrete valued functions. The search is conducted over an interval, for the variable representing the acceptance number, known to contain the optimal solution. For each point evaluated within this interval, a Fibonacci search is used to determine the least value of the sample size. It is shown how an initial feasible solution could provide the values needed to establish the search intervals. An algorithm already developed to produce integral solutions is used to determine the initial feasible solution.

The algorithm developed in this paper is tested to determine its performance. The testing is to illustrate the accuracy of the proposed method. Use of the program also highlights accuracy problems existing with the previous minimum sample size method. Exact solutions are also determined through exhaustive searches for this purpose. A proof of optimality based on the Poisson approximation to the binomial is offered.

A Computerized Approach to Finding the Minimum Sample Size for Single Sample Attribute Sampling Plans

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

Completed June 10, 1985

Commencement June 1986

APPROVED:

Redacted for Privacy

Professor of Industrial and General Engineering in charge of major

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Dean of Graduate School

Date thesis is presented June 10, 1985

Typed by Jane Doe for Lancelot G. Sylvester

ACKNOWLEDGEMENT

I would like to extend special appreciation to Dr. Edward McDowell, my major professor, for his patience, support, valuable advice and encouragement during the writing of this thesis and during my stay.

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INTRODUCTION

Background

A problem frequently faced by the quality control professional is the determination of acceptance sampling plans that provide desired levels of protection for both producer and consumer. The sampling plan must provide a probability of at least $1-\alpha$ of accepting a lot if the lot proportion defective is at the acceptable quality level (AQL). The plan must also provide a probability of acceptance of no more than β if the lot proportion defective is at the rejectable or unacceptable quality level (LTPD). Typically, α is referred to as the "producer's risk" and β as the "consumer's risk".

Much has been written concerning the determination of a single sampling plan given p_1, p_2, α and β. Various methods have been developed to aid the quality professional in sampling plan determination. Principally, the methods have employed graphs, formulas, tables, or some combinations of these methods. The tables, however, are restricted to the more popular α and β values.

The Problem

Let n denote the sample size and c the acceptance number for a single sampling plan. The probability of acceptance, i.e. the probability of getting c or fewer defectives in the sample, considered as a function of the fraction defective, p, in the inspected lot (or process) is called the operating characteristic of the plan and is denoted by P(p). We shall consider operating characteristic computed from the binomial distribution.

Since n and c have to be integers it is usually not possible to find a plan satisfying the requirement exactly. Hald(1967) restates the problem such that the following are satisfied: $P(p_1) \geq 1-\alpha$, $P(p_2) \leq \beta$, c is as small as possible, $p_1 < p_2$ and $1-\alpha > \beta$. These modified requirements lead to a unique value of c and a range of values for n. Since there is a choice of values for n, given c, the problem now becomes one of selecting n based on some criterion which is significant to the user. If, for example, it is essential that the consumer's risk deviate as little as possible from β with not too much concern about α, then n is readily determined. Likewise, if for some reason minimum deviation from α is of primary interest, another n would be selected. In fact, Hald proves that the smallest n which satisfies these two requirements (i.e., minimum deviation from either α or β) is the n which minimizes the deviation from β. Conversely, when a minimum deviation from α is of prime concern, n is a maximum. It follows that if the size of n is critical, the choice is obvious. If, however, there is no preference for either of these two characteristics, n may be determined so that some linear combination involving the two risks α and β is minimized. In particular, Hald discusses two such combinations: the weighted sum $P(p_1)/\alpha + P(p_2)/\beta$, and $P(p_2)/P(p_1) - \beta/\alpha$. The n which minimizes the latter can be considered as a weighted average of the minimum n and maximum n previously discussed. Other combinations may be of interest and n could be determined accordingly.

Objective

One of the chief problems in starting a sampling inspection procedure is to decide what size of sample is needed. The question can be solved in various ways, the most common being:

(1) by applying one of the standard sampling inspection tables such as the Dodge and Romig Tables, the Military Standard MIL-STD-105D, or the Philips SSS Tables;

(2) by choosing numerical values for a suitable set of parameters (AOQL, Producer's or Consumer's Risk Point, etc.) and constructing a corresponding sampling plan; or

(3) on the basis of an economic theory which takes into consideration various costs.

All three methods have been applied with success. However, although methods (1) and (2) may be convenient, one can never feel certain that they will lead to what must be considered an optimum sample size (Hamaker,1958).

The main purpose of this paper is to reconsider the fundamental problem of finding an optimum (minimum) sample size in the light of recent papers on the subject. A computer program will be written and tested, that will accurately determine the minimum sample size without resorting to an exhaustive search. The program will be used to highlight accuracy problems existing with previous minimum sample size methods, in particular, Jaech's method (1980) which is based on the methodology of Mr. Larry Joe Stephens (1978) involving Borges' (1970) normal approximation to the binomial.

Approach

Cost of inspection is directly proportional to the sample size and is not dependent on the acceptance number. Since the stated problem leads to a unique value of c and a range of values for n, minimizing c does not guarantee that the sample size obtained is a minimum. The computer program presented by Mr. William A. Hailey (1980) provides accurate minimum sample size single sampling plans that meet prescribed protection levels for both producer and consumer. The routine is a modification of the search procedure developed by Guenther (1969). It, however, involves an exhaustive search procedure. Even with today's high speed computers, one is justified in asking for better procedures than this.

In order to avoid an exhaustive search the Fibonacci search technique will be used in the program. The search will be conducted over an interval C_0 for the variable c that is known to contain the optimal solution (c_{min}, n_{min}). For each point evaluated within this interval a Fibonacci search will also be needed to determine the value of n_{min}, the least value of n such that the chosen value of the consumer's risk β is not exceeded. The experimental point is considered to be feasible when both the α and β values are not exceeded. Thus, an initial interval N_0 known to contain n_{min} for each point in the interval C_0 is also needed. Since n_{min} is an increasing function of c, an initial feasible solution is all that is needed to provide the intervals N_0 and C_0. Jaech's (1980) algorithm will be used to help determine that initial feasible solution. Once the intervals are established, the least sample size for the problem can easily be determined. The optimal solution is that value of n_{min} corresponding to c_{min}.

BACKGROUND

Concepts And Terminology

The fundamental tool for analysis of a sampling plan is the operating characteristic curve. Two types of curves are recognized:

Type A. Sampling from an individual (or isolated) lot, showing probability that the lot will be accepted plotted against lot proportion defective.

Type B. Sampling from a process (such as the producer's process which produces the lot), showing proportions of lots which will be accepted plotted against process proportion defective.

The probability distributions utilized in plotting these types of OC curves are inherently different. They also depend upon the measure in which quality is expressed. They include:

Attributes. A dichotomous (two class) classification of units into defectives and nondefectives.

Counting. An enumeration of occurrences of a given characteristic per given number of units counted.

Variables. The measurement of some characteristic along a continuous scale.

The distinction is made between defect (an imperfection great enough to be counted) and defective(a unit containing one or more defects, which cuuld be rejected for any one of them).

The probability distributions appropriate for the derivation of operating characteristic curves of the two types are shown in Table I (Schilling,1982).

Table I. Probability Distributions For Operating Characteristic Curves

Characteristic	Type A	Type B
Attribute	Hypergeometric	Binomial
Count	Poisson	Poisson
Measurement	Applicable continuous distribution of measurement involved	

Choosing Quality Levels

The choice of quality levels with which to construct a sampling plan must be made considering the seriousness of the defects to which it is applied, the operating characteristics of the resulting sampling plan, economic consequences in terms of sample size, the ability of the producer to meet the levels, and the needs of the consumer which must be met. The construction of any sampling plan involves a trade-off of these items.

Specifying A Plan

Discriminating use of sampling procedures demands knowledge and specification of the characteristics of the plans to be employed. A primary consideration is the protection afforded to both the producer and consumer. Since two points may be used to characterize the OC curve, it is customary to specify:

p_1 = producer quality level

p_2 = consumer quality level

α = producer risk

β = consumer risk.

For single-sampling attributes plans, $1-\alpha$ and β can be determined directly from the distribution function of the probability distribution involved. Fig. 2-1 shows the relation of these quantities to the OC curve. Also shown are the regions of acceptance, indifference, and rejection defined by these points. Quality levels of p_1 or better

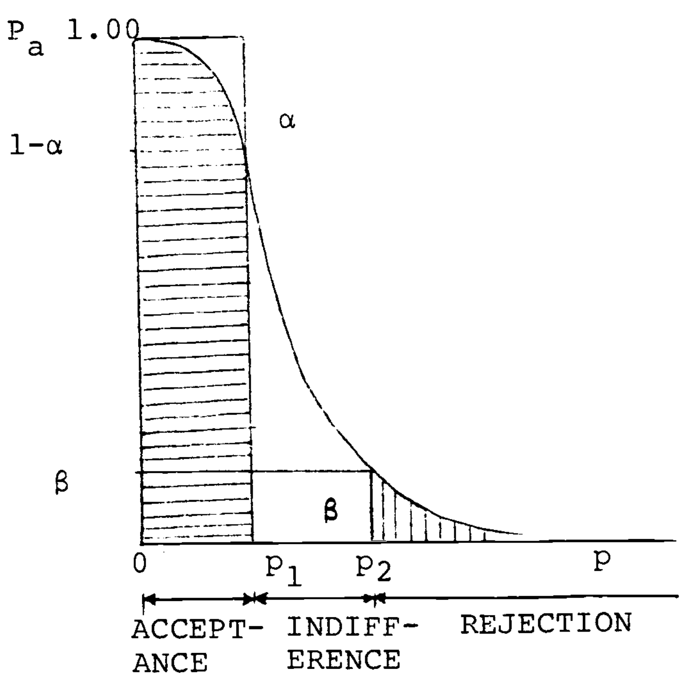

Fig. 2-1 Relation of p_1, p_2, $1-\alpha$ and β to the OC curve.

are expected to be accepted most of the time ($\geq 1-\alpha$) by the plan depicted. Quality levels of p_2 or worse are expected to be rejected most of the time ($\leq 1-\beta$) while intermediate levels will experience decreasing probability of acceptance as levels move from p_1 to p_2. Occasionally, only one set of parameters (p_1,α) or (p_2,β) is specified. A single-sampling attributes plan may be specified by any two of the following: (p_1,α), (p_2,β), n, c.

Single Sampling By Attributes

The single-sampling plan is basic to all acceptance sampling. The simplest form of such a plan is single sampling by attributes which relates to dichotomous situations, i.e., those in which inspection results can be classified into only two classes of outcomes. This includes go no-go gauging procedures as well as other classifications, such as measurements in or out of specifications. Applicable to all sampling situations, the attributes single-sampling plan has become the benchmark against which other sampling plans are judged. It is employed in inspection by counting the number of defects found in the sample (Poisson distribution) or evaluating the proportion defective from processes or large lots (binomial distribution) or from individual lots (hypergeometric distribution). Single sampling is undoubtedly the most used of any sampling procedure.

Implementation of an attribute single-sampling plan involves taking a random sample of size n from a lot of size N. The sample may be intended to represent the lot itself (Type A sampling) or the process used to produce the lot (Type B sampling). The number of defectives (or defects) d found is compared to an acceptance number c. If the number found is less than or equal to c, the lot is accepted. If the number found is greater than c, the lot is rejected.

Sampling plans are frequently used in consort to produce levels of protection not attainable by any of the component plans individually. Such combinations of plans are called sampling schemes or sampling systems. Sampling plans are the basic elements of sampling schemes, while sampling systems may be considered to involve a grouping of one or more sampling schemes (Schilling,1982).

Attribute sampling schemes include the tables of AOQL plans prepared by Dodge and Romig (1959), which resulted in a stated AOQL with minimum total inspection when used as directed with 100 percent inspection. Many schemes, however, are included in the AQL systems. AQL refers to the acceptable quality level, i.e., what has been called the producer's quality level for a single plan. These systems are intended to be applied to a stream of lots. Such plans specify an upper limit on quality, the AQL, not to be exceeded by the producer without penalty of an excessive number of rejected lots. That is, for levels of quality less than the AQL, rejections will be relatively infrequent, say less than 1 in 10, while for levels of quality in excess of the AQL, rejections will be more frequent, say more than 1 in 10. This is achieved by switching back and forth between the plans included in the system. Tighter plans are used when quality levels are shown to be poor, while looser plans involving small sample sizes are utilized when quality is shown to be good. Over a continuing supply, schemes can be devised to incorporate the best properties of the plans included as elements. Frequently, schemes are selected within a system in relation to the lot size involved.

MIL-STD-105D(1963) combines several individual sampling plans in schemes constructed to employ economic, psychological and operational means to motivate the producer to sustain acceptable quality levels. The procedure for switching between plans is essential to the system; it is so designated as to exert pressure on the producer to take corrective action when quality falls below prescribed levels and to provide rewards, in terms of reduced sample size, for quality improvement. The standard ties together sets of three attribute sampling plans, each at a different level of severity, into a unified procedure for lot acceptance through the use of its switching rules. These action rules determine the level of severity to be employed depending on the level of quality previously submitted. Thus, inspection of a succession of lots is intended to move among the specified set of tightened, normal and reduced sampling plans as quality levels degenerate or improve. Switching between tightened and normal plans is made mandatory by the standard, while the use of reduced plans is optional. The MIL-STD-105D, as such, does not allow for application of individual plans without use of the switching rules, since such an approach can lead to a serious loss of protection from that achieved when the system is properly applied. Quality levels are specified in terms of acceptable quality level (AQL) for the producer, while consumer protection is afforded by the switching rules which lead to tightened plans when quality is poor.

Most Economical Sampling Schemes

In discussions on the application of schemes, particularly those involving the choice of a producer's or consumer's risk, mention is often made of the general considerations that govern the various choices to be made-for example, the consumer's risk should be very low if the acceptance of unsatisfactory batches could cause much damage. Attempts have been made to give these considerations quantitative expression by dealing with the various costs involved.

"Decision costs" depend on what is done with rejected effective articles- whether, for example, they are scrapped or sent back to the producer- and on the damage that can result from the utilization of accepted defective articles. Generally, the larger the sample the smaller is the decision cost. Further, we need to know the sampling cost. Since the sample is defined by the scheme, the total sampling cost per batch can be calculated. These two, the decision and sampling costs are in principle, calculated for a range of schemes involving samples of different size, and the schemes for which the sum of the two costs is a minimum is the most economical scheme (Tippet,1958).

It seems that the attempts that are being made to put acceptance sampling on an economic basis are important. Control of quality by inspection is important and inspection often adds substantially to manufacturing costs; operating efficiency requires that the economically optimum degree of sampling should be adopted.

Review Of Literature

The literature on the subject of acceptance sampling is extensive, scattered, and somewhat confusing (Tippet, 1958). The contributions Professor Harold F. Dodge have been chronicled and are represented in the Dodge Memorial Issue of the Journal of Quality Technology (July, 1977). Professor Dodge, as a member of that small band of quality control pioneers at the Bell Telephone Laboratories of the

Western Electric Company, is considered by some to be the father of acceptance sampling as a statistical science. His paper, published in four parts (Dodge,1969,1970) outlines how the LTPD, the AOQL, the AQL and the CSP systems of plans and some other plans came into being over four decades, reviews the growth of concepts during that period and discusses a number of factors that influenced the development of the sampling inspection plans and tables that are in common use today.

Procedures are also available for determining so-called two-point single sampling plans for specified values of p_1, p_2, α and β. Five such procedures that relate to the derivation of plans are indicated in Table II (Schilling, 1982).

Factors for constructing of single sampling plans are available in the literature which are based on the Poisson distribution and which provide excellent approximations to the binomial sampling situation as well. These include the original approach of Peach and Littauer (1946) together with the work of Grubbs (1949) and Cameron (1952) and the tabulations by the U.S. Army, Chemical Corps Engineering Agency (1953). These so-called unity values can be easily used to construct and evaluate plans on the basis of the operating ratio ($R=p_2/p_1$). The theory of construction of unity values is explained by Duncan (1974).

Table II. Procedures For Determining Single Sampling Plans

Type Plan	Method	Use
Type B (defectives) (defects)	Table of Poisson unity values	Table for derivation of plan given operating ratio R for tabulated values of α, β, and c. Poisson approx. to binomial for defectives. May be used as exact for defects.
Type B (defectives)	Binomial nomograph	Nomograph for derivation of plan given α, β, p1, p2. Uses binomial distribution directly. Hence exact for defectives.
Type A (defectives)	f-Binomial nomograph	Uses binomial nomograph to derive Type A plans given α, β, p1, p2 through f-binomial approx. to hypergeometric distrib. Given lot size gives approximate plan for defectives.
Type B (defects) (defectives)	Thorndyke chart	Procedure for use of Thorndyke chart for Poisson distribution to derive plan given α, β, p1, p2. Exact for defects. Approximate for defectives through Poisson approx. to binomial.
Type A (defectives)	Hypergeometric tables	Iterative procedure for derivation of exact hypergeometric plan given N, α, β, p1, p2 using Lieberman-Owen tables of hypergeometric distributions.

The Larson (1966) nomograph can also be used to derive single sampling attributes plans. The nomograph can also be used to evaluate the operating characteristic curve of a plan. The Larson nomograph is based on the binomial distribution and so will allow direct evaluation of Type B plans for fraction defective. It allows derivation and evaluation of plans for values of probability of acceptance not shown in the Cameron tables. It provides a reasonable and conservative approximation (Schilling, 1982) for the derivation of plans when the hypergeometric distribution should apply and the binomial approximation to the hypergeometric distribution is appropriate. A graphical trial-and-error approach using Larson's nomograph (which is designed for solution of cases in which both p_1 and p_2, or alternately, $1-p_1$ and $1-p_2$, are smaller than 0.5) has been outlined by Ladany (1977) for the derivation of single-stage attribute sampling plans in which either the Acceptance Quality Level or the Lot Tolerance Fraction Defective is larger than 0.5.

Although somewhat more complicated than Larson's binomial nomograph, the Thorndyke (1926) chart, as given in Dodge and Romig (1959), may be used to derive a single-sampling attributes plan. Burges (1948) describes the procedure.

The operation of the MIL-STD-105D has been described in detail by Hahn and Schilling (1975). The background of

MIL-STD-105D and its development of the 105 series is given in a paper by Pabst (1973). It explains some of the intricacies of the system and its development. The theory behind its structure is well presented in a paper by Hill (1973). An extensive and informative investigation of the properties of MIL-STD-105D is presented in a paper by Stephens and Larson (1967). Scheme properties are also investigated by Schilling and Sheesley (1978) and measures of performance tabulated. A set of plans indexed by limiting quality and compatible with MIL-STD-105D (same lot size classes and sample sizes) has been proposed by Duncan, Mundel, Godfrey and Partridge (1980). The proposed table, which simplifies the selection of a limiting quality plan, can be used independently or in conjunction with MIL-STD-105D and associated standards.

Hald and Kousgaard (1967) have constructed tables to provide simple and comprehensive means for computing the binomial operating characteristic of single sampling plans or, equivalently, to find confidence limits for p in the binomial distribution. Accurate approximation formulas are available for $c > 50$. An advantage of the table as compared to other tables of the binomial distribution is that for $c \leq 50$ interpolation is only required with respect to n for determining p.

Among the basic concepts in the theory of sampling inspection the producer's and consumer's risks are the most widely used for characterizing systems of sampling plans. A comprehensive theory based on these concepts for the case of single sampling by attributes is presented in a paper by Hald (1967). The requirements defining a system of sampling plans are usually of such a nature that no explicit solution exists for the sample size and acceptance number. Hald supplements the exact (implicit) solutions by asymptotic solutions which give a better insight into the basic properties of the system.

Most single sampling plans assume that the lot size is large compared with sample size, and the calculated operating characteristic curves are strictly valid only under these conditions. In his article Hamaker (1959) describes a simple method for finding the sample size and acceptance number number appropriate to a lot of finite size, so that the resulting operating characteristic curve closely approximates to that for a given plan with an infinite lot.

A paper by Hald (1967) gives a survey of solutions to the problem of determining a single sampling plan. Solutions corresponding to Poisson, binomial, and hypergeometric operating characteristics are given, and the accuracy of the approximations is given by numerical investigations.

The administration of acceptance sampling plans has been greatly simplified by the computer. Data bases can provide an excellent source for quality history, while individual computer programs can be used to set up and evaluate sampling plans and even to sentence individual lots.

A number of computer programs useful in acceptance sampling, have been published in the literature. Among these include GRASP (Schilling, Sheesley and Nelson, 1978) that will evaluate an arbitrary single, double, or multiple sampling plan using hypergeometric, binomial, Poisson, or normal probabilities. An option is also included that will permit the calculation of the fraction defective values associated with arbitrary specified values of probability of acceptance. A program by Snyder and Storer (1972) is based on Hald's (1967) paper and considers only the Poisson distribution. Hailey's program (1980), however, provides sampling plans based on either binomial or Poisson distribution and involves the use of an exhaustive search procedure developed by Guenther (1969).

In the following two sections summaries of two approximate methods developed recently for determining sample sizes and acceptance numbers for single sample attribute sampling plans are presented.

Summary Of Stephens' Paper

In his paper, Stephens (1978) uses the normal approximation to the binomial distribution developed by Borges (1970) with error term of order 1/n to find the sample size and acceptance number in a single sample acceptance sampling plan when two points on the operating characteristic (OC) curve are specified.

Borges' approximation can be described in the following way:

$$B(k;n,p) = \sum_{i=0}^{k} {}^{n}C_{i} . p^{i} . (1-p)^{n-i} \qquad (1)$$

$$y_k = (pq)^{-1/6} . (n+1/3)^{1/2} . \int_{p}^{(k+1/6)/(n+1/3)} (s(1-s))^{-1/3} . ds \qquad (2)$$

$$\phi(x) = \int_{-\infty}^{x} (1/(2\pi)^{1/2}) . e^{-t^2/2} . dt \qquad (3)$$

Then,

$$B(k;n,p) \simeq \phi(y_{k+1/2}) \qquad (4)$$

Let $(p_1, 1-\alpha)$ and (p_2, β) be the two points specified on the OC curve. It is desired to find the sample size, n, and the acceptance number, c, such that,

$$B(c;n,p_1) = 1-\alpha \qquad (5)$$

$B(c;n,p_2) = \beta$

Suppose $\phi(\mu_1) = 1-\alpha$ and $\phi(\mu_2) = \beta$

Since $\phi(y_{c+1/2}) \simeq 1-\alpha \simeq \phi(\mu_1)$ at p_1 and

$\phi(y_{c+1/2}) \simeq \beta \simeq \phi(\mu_2)$ at p_2, it follows that

$$\mu_1 \simeq (p_1 q_1)^{-1/6}.(n+1/3)^{1/2}. \int_{p_1}^{(c+2/3)/(n+1/3)} (s(1-s))^{-1/3}.ds \qquad (7)$$

$$\mu_2 \simeq (p_2 q_2)^{-1/6}.(n+1/3)^{1/2}. \int_{p_2}^{(c+2/3)/(n+1/3)} (s(1-s))^{-1/3}.ds \qquad (8)$$

Let $g(s) = \int_0^s (t(1-t))^{-1/3}dt.$

Stephens' table (see Appendix B-2) was formed using a generalized Gauss quadrature subroutine and contains values of $g(s)$ for s, 0(0.001)1. Then expressions (7) and (8) may be written as

$$\mu_1 \simeq (p_1 q_1)^{-1/6}(n+1/3)^{1/2}\{g((c+2/3)/(n+1/3))-g(p_1)\} \qquad \ldots.(9)$$

$$\mu_2 \simeq (p_2 q_2)^{-1/6}(n+1/3)^{1/2}\{g((c+2/3)/(n+1/3))-g(p_2)\} \qquad \ldots.(10).$$

From expressions (9) and (10) the following may be obtained:

$$\mu_1(p_1 q_1)^{1/6}-\mu_2(p_2 q_2)^{1/6}=(n+1/3)^{1/2}\{g(p_2)-g(p_1)\} \qquad (11)$$

or, upon solving for n,

$$n = \{(\mu_1(p_1q_1)^{1/6}-\mu_2(p_2q_2)^{1/6}) \div (g(p_2)-g(p_1))\}^2-1/3 \qquad (12)$$

The variable n may be found from equation (12) and then c may be found from expression (9) or (10). Stephens, however, gives no guidance on how to round the resulting acceptance number to an integer and, in fact, avoids dealing with this issue by generally treating only those examples in which the calculated acceptance number is close to an integer.

Summary of Jaech's Paper

Jaech (1980) indicates in his paper how Stephens' approach may be extended to provide integral solutions using an iterative calculation procedure. The problem is as formulated by Hald (1967) and may be stated as follows: Determine (n,c) so that $P(p_1) \geq (1-\alpha)$, $P(p_2) \leq \beta$, and c is as small as possible, where $p_1 < p_2$ and $(1-\alpha) > \beta$.

In obtaining an integral solution by extending Stephens' methodology, the first step is to round up the calculated acceptance number c to the next integer, c_0. A decision is then made either:

a. to control the value for α close to its design value and choose the sample size such that the actual value for β is smaller than the design value; or

b. to control the value for β close to its design value and to choose the sample size such that the actual value for α is smaller than the design value.

Hald (1967) proves that the sample size under choice b will always be smaller than that under choice a.

The following simplifying notation is introduced:

n_0 = initial sample size using Stephen's equation (12)

$s_0 = (c+2/3)/(n_0+1/3)$

n_i = ith iteration sample size.

The sample sizes $n_1, n_2, n_3, \ldots$ will be defined to be integers. The iterative procedure to be given will stop either when n_k and n_{k+1} are the same value or when they flip-flop back and forth between two integers upon successive iterations, in which case the larger integer becomes the sample size. The iterative procedure is as follows:

1. Calculate n_1 as the smallest integer larger than or equal to

   $(c_0+2/3)s_0-1/3.$

2. Calculate $g(s_1)$ from the equation

$$g(s_1) = \frac{\mu_i \{ p_i(1-p_i)\}^{1/6}}{(n_1+1/3)^{1/2}} + g(p_i) \qquad (13)$$

   using i=1 under choice a, control of producer's risk, and i = 2 under choice b, control of consumer's risk. In (13), $g(p_i)$ is read from Stephens' table (see Appendix B-2), while μ_i is defined by $\phi(\mu_1) = 1-\alpha$ and $\phi(\mu_2) = \beta$, where $\phi(x)$ is the cumulative normal (expression (3)).

3. Find s_1 from Stephens' table, given $g(s_1)$ from step 2.
4. Calculate n_2 as the smallest integer larger than or equal to

   $(c_0+2/3)/s_1-1/3.$
5. Calculate $g(s_2)$ using equation (13) with n_1 replaced by n_2
6. Find s_2 from Stephens' table
7. Calculate n_3 as the smallest integer larger than or equal to

   $(c_0+2/3)/s_2-1/3.$

The iterative procedure indicated in these steps is continued until 'convergence' to a final sample size is obtained.

Jaech also suggests that an alternative solution reported by Hald (1967) may be more conveniently applied to find the required sample size once Stephens' acceptance number is derived and appropriately rounded. Since both Stephens' and Hald's methods are based on approximations, they do not necessarily yield the same results.

THE METHODOLOGY

Development Of The Algorithm

The problem, as formulated by Hald (1967), in the following way:

> Determine (n,c) so that $P(p_1) \geq 1-\alpha$, $P(p_2) \leq \beta$, and c is as small as possible, where $p_1 < p_2$ and $1-\alpha > \beta$,

leads to a uniquely determined value of c and an interval for values of n, all satisfying the conditions. Thus, minimizing c does not imply that n is minimized.

For the binomial case,

$P(p_1) = B(c;n,p_1) \geq 1-\alpha$ (designated as α-constraint henceforth)

$P(p_2) = B(c;n,p_2) \leq \beta$ (designated as β-constraint henceforth).

For a given value of c, $B(c;n,p)$ is a decreasing function of n, and it will be possible to determine the least value of n (n_{min}) satisfying the β-constraint. It will also be possible to determine the greatest value of n (n_{max}) satisfying the α-constraint if $B(c;c+1,p_1) \geq 1-\alpha$. Thus, to satisfy both constraints, n_{max} must be greater than or equal to n_{min}. Also it may be stated that n_{min} (and n_{max}) increases as c increases. Consequently, the least feasible solution (n_{min}) for the problem is the least n corresponding to c_{min}, and the problem may be restated in the following way:

$$
\begin{aligned}
&\text{minimize } n \\
&\text{subject to, } B(c;n,p_1) \geq 1-\alpha \\
&\qquad B(c;n,p_2) \leq \beta \\
&\qquad p_1 < p_2 \\
&\qquad 1-\alpha > \beta.
\end{aligned}
$$

The variation of $(n_{max}-n_{min})$ with c (see Appendix B-3) seems to suggest that c is feasible for all values of c greater than c_{min}. This has been found to be very difficult to prove mathematically, using the cumulative binomial distribution. However, in this paper, this will be shown using the Poisson approximation to the binomial as follows:

Theorem 3.1: If $c = c_{min}$ is the least feasible solution to the stated problem, then any value of $c > c_{min}$ is also feasible.

Proof:

$$B(c;n,p) \simeq P(c;np)$$

$$P(c;np) = P(\chi^2 > 2np);\ f = 2(c+1) \quad \text{.................(1)}$$

$$2np_1 \leq \chi^2_\alpha \text{ and } 2np_2 \geq \chi^2_{1-\beta};\ f = 2(c+1) \quad \text{..........(2)}$$

(Hald, 1967)

i.e. $n_{max} = \chi^2_\alpha/2p_1$ and $n_{min} = \chi^2_{1-\beta}/2p_2$; $f = 2(c+1)$...(3)

Let $c = c_1$ be a feasible solution. Then,

$$(n_{max})_1 \geq (n_{min})_1$$

From (3), it follows that

$$(\chi_\alpha^2/2p_1)_1 - (\chi_{1-\beta}^2/2p_2)_1 \geq 0$$

or $$1/p_1 - (1/p_2)(\chi_{1-\beta}^2/\chi_\alpha^2)_1 \geq 0 \qquad \ldots\ldots\ldots\ldots(4)$$

Let $c_2 = c_1 + 1$.

Since $\chi_{1-\beta}^2/\chi_\alpha^2$ is a decreasing function of c (Hald, 1967),

$$(\chi_{1-\beta}^2/\chi_\alpha^2)_2 < (\chi_{1-\beta}^2/\chi_\alpha^2)_1$$

and it follows from (4) that

$$1/p_1 - (1/p_2)(\chi_{1-\beta}^2/\chi_\alpha^2)_2 \geq 0$$

This implies that

$$(n_{max})_2 \geq (n_{min})_2.$$

Hence, $c = c_2$ is also feasible and it can easily be inferred that this proves the theorem.

The Fibonacci search technique (see Beveridge and Schechter, 1970) may be used to find the optimum value of an unconstrained objective function of a single variable, a function that is unimodal and bounded over a fixed interval L_0. It may be used to determine n_{min} for a given c as follows:

Let $\delta = 1$ if the β-constraint is satisfied

$= 0$ otherwise.

If n_0 is a value of n known to satisfy the β-constraint, the search interval is as shown in Fig. 3-1.

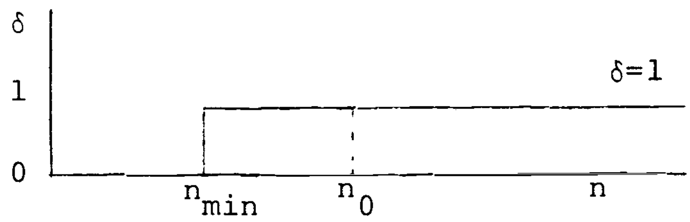

Fig. 3-1: δ, as a function of n.

δ is a step function consistent with that described in the following section (IIIc), and n_{min} can easily be determined using the Fibonacci search technique.

In order to determine c_{min}, the variable γ may be similarly defined:

Let γ = 1 if c is feasible (i.e. n_{min} satisfies the α-constraint)

= 0 otherwise.

If c_0 is a value of c that is known to be feasible, the search interval is as shown in Fig. 3-2.

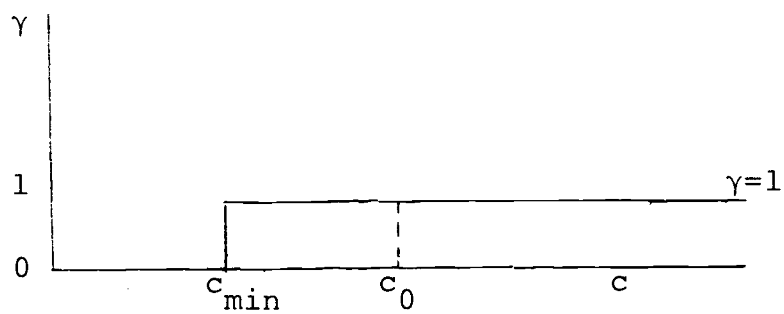

Fig. 3-2: γ, as a function of c.

For each experiment, n_{min} is first determined and then tested for feasibility (α-constraint). Since n_{min} increases with c, $n=(n_{min})_{c_0}$ may be used as the upper bound for the search intervals for determining n_{min} for all $c < c_0$. Thus, given an initial feasible solution (c_0, n_0), the optimal solution (c_{min}, n_{min}) can be easily determined using the Fibonacci search technique. Since n and c are discrete variables, it may be necessary to add fictitious locations at one end so that the interval length correspond to a Fibonacci number. The interval for determining n_{min} given (c_0, n_0) is shown in Fig. 3-3.

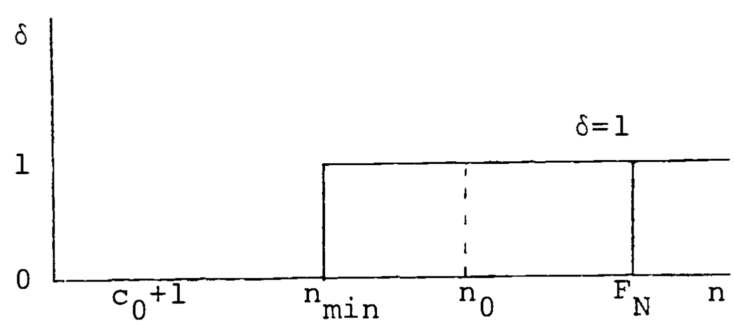

Fig. 3-3: Search interval for n_{min}, $n_0 \geq n_{min}$.

Only experiments that fall between c_0+1 and n_0 need to be evaluated since $\delta = 0$ for $n \leq c_0$ and $\delta = 1$ for $n \geq n_0$.

The search interval for determining c_{min} is shown in Fig. 3-4. In this case, $\gamma = 1$ for $c \geq c_0$ and only experiments that fall between 0 and c_0 need to be evaluated.

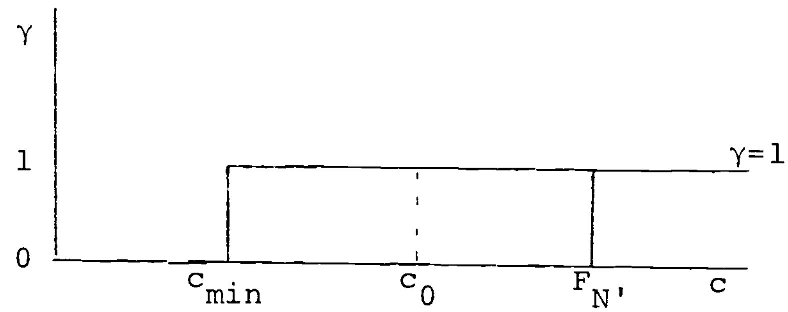

Fig.3-4: Search interval for c_{min}, $c_0 \geq c_{min}$.

Jaech's algorithm (see Sect. IIh) will be used to determine the initial solution. However, due consideration must be given to the fact that the algorithm does not guarantee a feasible solution. The value of n_0 so obtained may not satisfy the β-constraint, in which case, the next feasible n, where n is a Fibonacci number greater than n_0 is determined (see Fig. 3-5). Also, upon determining $(n_{min})_{c_0}$ it may be found that the α-constraint is not satisfied, in which case, the next feasible c, where c is a Fibonacci number greater than c_0 is determined (see Fig. 3-6).

$\delta=0$ for $n \leq n_0$,
$\delta=1$ for $n \geq F_N$.

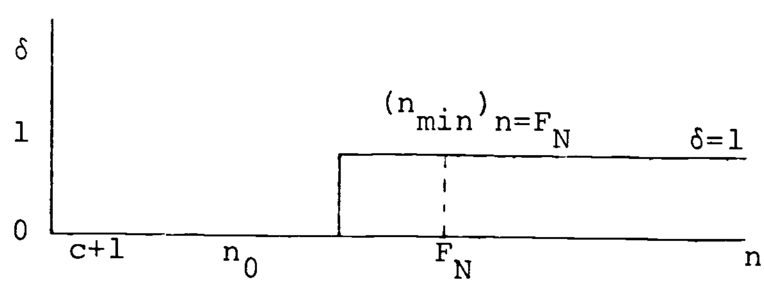

Fig. 3-5: Search interval for n_{min}, $n_0 < n_{min}$.

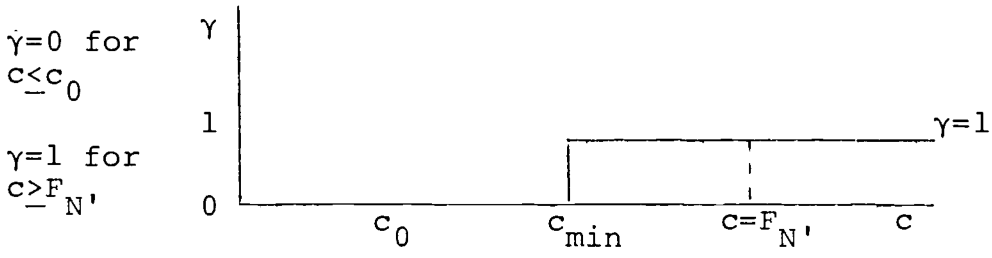

Fig. 3-6 Search interval for c_{min}, $c_0 < c_{min}$.

Interval eliminations for the Fibonacci search is considered in the following section.

Interval Elimination

Suppose we seek the location x^* where y(x) achieves its minimum value y^* in the unit interval $0 \leq x \leq 1$.

If

$$x_1 < x_2 < x^*$$
$$y_1 < y_2 < y^*$$

and if

$$x^* < x_1 < x_2$$
$$y^* > y_1 > y_2$$

y(x) may be said to be strictly unimodal. Consider two points $x_1 < x_2$. The possible outcomes are shown in Fig. 3-7: $y_1 > y_2$, $y_1 < y_2$, or $y_1 = y_2$. When $y_1 \geq y_2$, the minimum cannot lie to the left of y_1 and we can conclude that $x^* > x_1$. Similarly, $y_1 \leq y_2$ implies that $x^* < x_2$. When the two outcomes are exactly equal ($y_1 = y_2$), the minimum must lie between the points ($x_1 < x^* < x_2$).

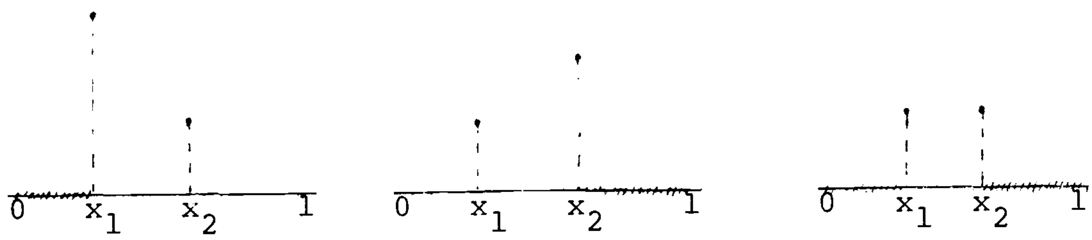

Fig.3-7: Possible outcomes of two experiments.

For the case of the step function $\delta(d)$, where

$$\delta = 0, \; d < d^*$$

$$\delta = 1, \; d \geq d^*,$$

the possible outcomes of two experiments d_1 and d_2 are shown in Fig. 3-8.

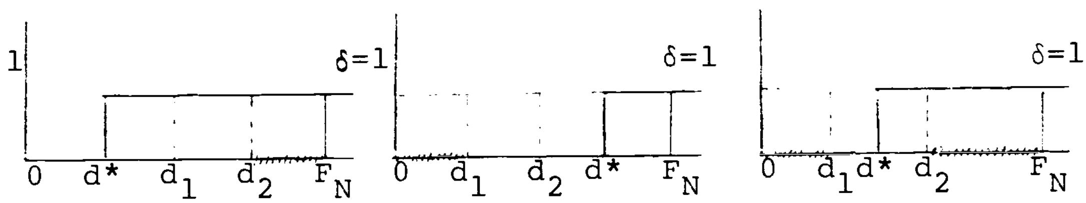

Fig. 3-8: Possible outcomes of two experiments for the step function δ.

Interval eliminations are so chosen that at least one of the experiments becomes an end point of the next interval so as to facilitate the use of the Fibonacci search technique.

The Algorithm

The algorithm to determine the minimum sample size may now be stated as follows:

Step 1. Determine (n_0, c_0) using Jaech's algorithm.

Step 2. Test β-constraint for (n_0, c_0).
If β-constraint is satisfied, go to Step 4.

Step 3. Determine least N such that the Fibonacci number $F_N > n_0$ satisfies the β-constraint. Go to Step 5.

Step 4. Determine least N such that $F_N \geq n_0$.

Step 5. Determine n_{min} (Fibonacci search)

Step 6. Test α-constraint for (n_{min}, c_0).
If α-constraintis not satisfied, go to Step 11.

Step 7. Determine least NC such that $F_{NC} \geq c_0$.

Step 8. Determine (n_{min}, c_{min}) using Fibonacci search given initial feasible solution (n_{min}, c_0) from Step 6 or $(n_{min}, c_{F_{NC}})$ from Step 17.

Step 9. If $c_{min} \neq 1$, the optimal solution is (n_{min}, c_{min})

Step 10. If $c_{min}=1$, determine $(n_{min}, 0)$ given $(n_{min}, 1)$ using Fibonacci search. If $(n_{min}, 0)$ is a feasible solution, it is also optimal. Otherwise, optimal solution is as obtained in Step 9.

Step 11. Determine least NC such that $F_{NC} > c_0$.

Step 12. Determine $n_{F_{NC}}$, $c=F_{NC}$ using Jaech's algorithm.

Step 13. Test β-constraint for $(n_{F_{NC}}, c=F_{NC})$.
If β-constraint is satisfied, go to Step 15.

Step 14. Determine least N such that $F_N > n_{F_{NC}}$ satisfies the β-constraint. Go to Step 16.

Step 15. Determine least N such that $F_N \geq n_{F_{NC}}$.

Step 16. Determine n_{min}, $c=F_{NC}$.

Step 17. Test α-constraint for $(n_{min}, c_{F_{NC}})$

If α- constraint is satisfied, go to Step 8.

Step 18. Set $F_{NC}=F_{NC+1}$. go to Step 12.

Gauss's quadrature formula (see Engels, 1980) will be used to determine the values of the integrals (those corresponding to Stephens' table and the standard normal tables) needed to be evaluated when using Jaech's algorithm. The abcissas and weights (David and Rabinowitz, 1958) are listed in Appendix B-1. Inverses of these integrals can be determined using the Fibonacci search technique involving 16 experiments (F_N=1597). The search interval for the x-values in Stephens' table (Appendix B-2) is 0 to 1 and the region of uncertainty is $1/F_N$ = 0.000626. The search interval for determining the standard deviates μ_1 and μ_2 is taken to be 0 to 3.5 and its region of uncertainty is $3.5/F_N$= 0.00219.

EXPERIMENTAL ANALYSIS

The FORTRAN program "Aime" (see Appendix A-1) incorporates the algorithm developed in the preceding chapter. However, since the proof of optimality given in Chapter III is based on the Poisson approximation, it cannot be stated with absolute certainty that the program will always produce accurate solutions. Thus, in order to verify the accuracy of "Aime", an exhaustive search routine is also needed. A computer program to obtain minimum sample size in such a manner may be based on the following routine:

1. The user supplies the desired $1-\alpha = P(\text{acceptance}|p_1)$ and $\beta = P(\text{acceptance}|p_2)$.
2. Set n = 1 and c = 0, where n = sample size and c = acceptance number (# defectives).
3. Calculate $P(\text{acceptance}|p_2, n, c)$.
4. If $P(\text{acceptance}|p_2, n, c) \leq \beta$, go to step # 5. Otherwise, increase n by 1 and return to step # 3.
5. Calculate $P(\text{acceptance}|p_1, n, c)$.
6. If $P(\text{acceptance}|p_1, n, c) \geq 1-\alpha$, the minimum sample size plan (c_{min}, n_{min}) has been found. The plan n = current value of n and c = current value of c satisfies the requirements.
7. If $P(\text{acceptance}|p_1, n, c) \leq 1-\alpha$, increase c by 1 and go to step # 3.

The routine just described may be modified such that n_{max} is also determined for a given c. Then, the optimal solution is obtained when $n_{max} \geq n_{min}$. Five examples to illustrate the use of such a routine is given in Appendix B-3.

For the experimental analysis twenty-five test problems (see Table III) were chosen. Some of these were taken from available literature, viz., those of Stephens'(1978), Jaech's (1980) and Hald's (1967). Most previous papers on the problem have provided tables and appropriate solutions with special regard to the conventional values of α and β, viz. $\alpha = 0.05$ and $\beta = .10$. It will be noted that most of the chosen values of α, β, p_1 and p_2 are not usually the kind of values that are chosen in practice. Values that are greater than 0.5 are also included in the set of test problems. However, they all do satisfy the constraints $p_1 < p_2$ and $1-\alpha > \beta$ (see Sect. IIIa), and will be retained for illustrative purposes.

Table III also lists the solutions obtained using the methodologies of Stephens (see Sect. IIg) and Jaech (see Sect. IIh). These solutions were obtained using a pocket calculator and the required tables. Stephens gives no guidance on how to round the resulting acceptance number (and sample size) to an integer. The solutions listed in the table were obtained by rounding to the nearest integer.

Jaech's method involves the use of Stephens' approach. However, he suggests rounding the calculated acceptance number to the next integer. This rounding procedure is also adopted in the calculation of the sample size. If the

Table III. Solutions To Test Problems.

Test Prob. No.	Problem Parameters α	β	p_1	p_2	Stephen's Solution c,n	Jaech's Solution (using tables) c,n	Initial Solution of prog. Aime c,n	Optimal Solution of prog. Aime c,n
1	.1403	.0947	.015	.210	0,11	1,18	1,18	0,10
2	.0582	.0965	.040	.340	1,10	2,15	2,15	1,10
3	.1400	.1041	.010	.140	0,16	1,22	1,27	1,26
4	.0529	.0935	.025	.240	1,15	2,19	2,21	2,21
5	.0439	.0913	.040	.250	2,20	3,24	3,26	2,20
6	.0471	.0908	.070	.310	3,20	3,25	4,25	4,25
7	.0760	.0980	.040	.200	2,25	3,35	3,32	3,32
8	.0250	.0980	.100	.270	9,50	9,47	9,51	9,50
9	.0490	.0810	.040	.160	4,51	5,53	5,58	4,50
10	.2000	.6000	.200	.450	0,1	0,2	0,1	0,1
11	.6000	.2000	.500	.750	0,1	0,2	0,2	1,3
12	.3000	.4000	.450	.800	0,1	1,3	1,2	1,2
13	.0500	.1000	.100	.200	15,104	16,110	16,109	16,109
14	.0500	.0500	.100	.400	4,19	4,21	4,21	5,24
15	.0500	.0500	.100	.500	3,12	3,14	3,13	3,13
16	.0500	.0500	.100	.900	1,2	2,5	1,3	1,3
17	.5000	.0100	.500	.900	2,5	2,6	2,5	2,5
18	.3000	.1000	.400	.495	37,89	38,91	38,91	40,95
19	.0300	.0850	.040	.140	6,76	7,84	7,84	7,84
20	.0100	.0010	.010	.100	4,146	5,159	5,160	5,159
21	.0483	.0870	.075	.600	1,5	2,8	2,8	2,8
22	.0961	.0916	.020	.380	0,5	1,9	1,9	1,9
23	.0500	.0500	.100	.490	3,12	3,12	3,14	3,14
24	.0050	.0100	.010	.060	7,265	8,286	8,285	8,286
25	.0900	.0500	.100	.200	15,111	15,112	15,112	16,118

rounding procedure adopted for both methods were the same, the acceptance numbers obtained would have also been the same. Since this is not the case, one can expect the acceptance number obtained by Jaech's method to be greater than or equal to that obtained by Stephens' method. In fact, for the 25 test problems, Jaech's method produced higher acceptance numbers for 16 of them (64%).

Jaech's iterative procedure will stop when n_k and n_{k+1} are the same value or when they flip-flop back and forth between two integers upon successive iterations, in which case the larger integer becomes the sample size. Computational experience in determining the solutions using tables has uncovered one more possibility, i.e., n_k and n_{k+3} might also be the same. This possibility was also added to the set of stopping rules when the FORTRAN program "Aime" was written.

The output of "Aime" lists the initial solutions that are obtained using Jaech's method as well as the optimal solutions. These are listed in the last two columns of Table III, respectively. One would expect the initial solutions of "Aime" to be the same as those obtained using Jaech's method and the necessary tables. However, the initial solutions of "Aime" are dependent on the number of abcissas and weights of the Gaussian quadrature subroutine as well as on the number of experiments in the Fibonacci search subroutine for determining the inverses of the integrals (see Sect. IIIc). Thus, even though the initial solutions of "Aime" were obtained using Jaech's algorithm, they differ slightly from the values obtained using a pocket calculator and the required tables. These discrepancies may be reduced by increasing the number of abcissas and weights in the Gaussian quadrature subroutine and/or by increasing the number of experiments in the Fibonacci search subroutine. This would, however, give rise to considerable increases in computational times.

Table IV lists the number of exact solutions, the number of feasible solutions and the number of feasible c values (i.e. $c \geq c_{min}$) for the methodologies of Stephens and Jaech (using tables) as well as for the initial solutions of program "Aime". Exhaustive searches confirm that the optimal solutions (see Table III) obtained by program "Aime" are all accurate. As can be seen, the number of

Table IV. Number of Feasible and Exact Solutions

	Stephens' Method	Jaech's Method (Using tables)	Initial Solution of Prog. "Aime"
No. of Feasible Solutions:	7/25(28%)	15/25(60%)	18/25(72%)
No. of Feasible c Values: ($c \geq c_{min}$)	8/25(32%)	20/25(80%)	21/25(84%)
No. of Exact Solutions:	4/25(16%)	5/25(20%)	13/25(52%)

exact solutions obtained using approximate methods are few in number- only four (16%) using Stephens method and five

(20%) using Jaech's method. Program "Aime" did considerably better in producing initial solutions that are exact (13/25 or 52%). The program also produced a greater number of feasible solutions (72%) compared to Jaech's method (60%) and consequently, increases in the number of abcissas and weights of the Gaussian quadrature subroutine and/or in the number of experiments for the Fibonacci search subroutine are not warranted. It may be noted that the acceptance numbers of the initial solutions of program "Aime" do not differ significantly from those obtained using tables (see Table III). The same values are obtained for all but two test problems (nos. 6 and 16). Thus, the discrepancies are mostly in the values of the sample sizes obtained.

In order to compare a solution obtained using one of the approximate methods with the corresponding optimal solution of program "Aime", the decrease in sample size, Δn, needed to make the approximate solution optimal may be computed. The numerical value of Δn is positive only if the approximate solution is feasible. Also, in order to highlight the inaccuracies in using the approximate methods, the absolute value of Δn, viz. $|\Delta n|$, may also be computed. Appendices B-4 and B-5 list the values of Δn obtained when comparing the solutions of Stephens and Jaech, respectively, with the optimal solutions of program "Aime". The values of Δn have also been computed for the initial solutions of "Aime" and are listed in Appendix B-6. Values of the average reduction in sample size $\overline{\Delta n}$, the average absolute difference between sample size $|\overline{\Delta n}|$, the average percentage reduction in sample size $\overline{\Delta n/n}$, and the absolute value of the average percentage difference between sample size $|\overline{\Delta n/n}|$ have also been computed. Corresponding variances are also listed in the appendices. A similar analysis of the acceptance numbers can be carried out by determining the reduction in acceptance number, Δc, in a similar fashion. The values of Δc obtained for the three sets of approximate solutions are listed in Appendix B-7.

A two-tailed test of hypothesis is conducted to determine whether the average reduction in sample size is significantly different from zero. The null hypothesis, therefore is H_0: $\delta n = 0$, and the alternate hypothesis is H_1: $\delta n \neq 0$, where δn is the average reduction in sample size of the population. A level of significance $\alpha = 0.05$ is chosen. Similar tests are also conducted for the other means that are considered in this chapter, viz., $|\delta n|$, $\delta n/n$, $|\delta n/n|$, and δn (the corresponding population means).

For Stephens' method, the average reduction in sample size obtained is $\Delta n = -3.92$ ($|\Delta n| = 4.08$). The value is negative because the number of feasible solutions obtained were few in number (7/25). The two-tailed test of hypothesis (see Table V) indicates that the mean sample size reduction of the population, δn, is significantly different from zero (level of significance $\alpha = .05$). Two-tailed tests of hypotheses of the various means for Stephens' method are summarized in Table V. The tests indicate that all the means considered are significantly different from zero. It must

Table V. Two-Tailed Tests of Hypotheses (α =.05 Level of Siginificance)- "Aime vs. Stephens' Method

H_0	H_1	Test Statistic T	$\pm t_{\alpha/2,n-1}$ $\pm t_{.025,n-1}$	Reject H_0?
$\delta n=0$	$\delta n\neq 0$	-3.94025	-2.064	Yes
$\lvert\delta n\rvert=0$	$\lvert\delta n\rvert\neq 0$	3.948	2.064	Yes
$\delta n/n=0$	$\delta n/n\neq 0$	-3.158	-2.064	Yes
$\lvert\delta n/n\rvert=0$	$\lvert\delta n/n\rvert\neq 0$	3.2969	2.064	Yes
$\Delta c=0$	$\Delta c\neq 0$	-4.884	-2.064	Yes
Considering only positive reductions: (n = 7)				
$\delta n=0$	$\delta n\neq 0$	1.5490	2.447	No
$\delta n/n=0$	$\delta n/n\neq 0$	1.2355	2.447	No

be noted that the average decrease in acceptance number Δc is also negative (Δc = -.72). From the results one can say that the discrepancies between the solutions of Stephens and "Aime" (exact solutions) are not readily explainable by chance, within the chosen significance level α = .05. If only positive reductions are considered, then Δn = .286 (n = 7) and the two-tailed test indicates that the null hypothesis should not be rejected, i.e. the sample result is compatible with the null hypothesis value δn = 0. This is also true in the case of the population percentage reduction in sample size $\delta n/n$, i.e. the null hypothesis H_0: $\delta n/n = 0$ is found to be tenable.

In the case of Jaech's method (using tables), the average reduction in sample size was found to be $\overline{\Delta n} = .52$ ($|\overline{\Delta n}| = 2.2$). The two-tailed test of hypothesis ($\alpha = .05$) indicates that the reduction is not significant. Table VI summarizes the two-tailed tests of hypotheses concerning the various means obtained by comparing Jaech's solutions with the exact solutions. A test also shows that the perc-

Table VI. Two-Tailed Tests of Hypotheses (α = .05 Level of Significance)- "Aime vs. Jaech's Method (Using Tables)

H_0	H_1	Test Statistic T	$\pm t_{\alpha/2,n-1}$ $\pm t_{.025,n-1}$	Reject H_0?
$\delta n=0$	$\delta n\neq 0$	.8584	2.064	No
$\|\delta n\|=0$	$\|\delta n\|\neq 0$	5.234	2.064	Yes
$\delta n/n=0$	$\delta n/n\neq 0$	1.2049	2.064	No
$\|\delta n/n\|=0$	$\|\delta n/n\|\neq 0$	4.589	2.064	Yes
$\delta c=0$	$\delta c\neq 0$	-.2721	-2.064	No
Considering only positive reductions: (n=17)				
$\delta n=0$	$\delta n\neq 0$	3.2731	2.120	Yes
$\delta n/n=0$	$\delta n/n\neq 0$	3.4762	2.120	Yes

entage reduction in sample size is also not significant. However, solutions obtained by using Jaech's method are significantly different from those of "Aime" (exact solutions) because the null hypothesis for the mean of the absolute value of the difference in sample size ($H_0: |\delta n|=0$) and the absolute value of the mean percentage difference in sample size ($H_0: |\delta n/n|=0$) are both untenable. If one were to consider only positive reductions, that would eliminate 8 of the 25 test problems having negative reductions from consideration. In this case the mean reductions δn and $\delta n/n$ of the population are found to be significantly different from zero. However, unlike in the case of Stephens method, the average reduction in acceptance number ($\overline{\Delta c}$ = -.04) was found to be insignificant.

Table VII summarizes the two-tailed tests of hypotheses concerning means for the initial solutions of program "Aime".

Table VII. Two-Tailed Tests of Hypotheses (α = .05 Level of Significance)- Optimal Solutions of "Aime" vs. Initial Solutions of "Aime"

H_0:	H_1:	Test Statistic T	$\pm t_{\alpha/2,n-1}$ $\pm t_{.025,n-1}$	Reject H_0?
$\delta n=0$	$\delta n\neq 0$	.9369	2.064	No
$\|\delta n\|=0$	$\|\delta n\|\neq 0$	3.3425	2.064	Yes
$\delta n/n=0$	$\delta n/n\neq 0$	.5659	2.064	No
$\|\delta n/n\|=0$	$\|\delta n/n\|\neq 0$	2.6923	2.064	Yes
$\Delta c=0$	$\Delta c\neq 0$	-.2959	-2.064	No
Considering only positive reductions: (n = 20)				
$\delta n=0$	$\delta n=/0$	2.4117	2.093	Yes
$\delta n/n=0$	$\delta n/n\neq 0$	2.1204	2.093	Yes

Since the solutions and values of the means obtained are close to the values obtained by using Jaech's method (using tables), one might expect similar results for the statistical tests. Comparing Tables VI and VII this is found to be true. The average reduction in sample size obtained in this case is $\overline{\Delta n}$ = .6 ($|\overline{\Delta n}|$ = 1.8). This sample result is compatible with the null hypothesis value of δn = 0. Again, the reduction is significant when considering only positive reductions in sample sizes (20 test problems). The null hypotheses are also rejected when considering the mean value of the absolute difference in sample size of the population, $|\delta n|$, and the mean value of the absolute percentage reduction in sample size of the population. The average reduction in acceptance number, $\overline{\Delta c}$, is identical to that obtained for Jaech's method ($\overline{\Delta c}$ = -.04) and it is also insignificant.

SUMMARY- CONCLUSIONS- RECOMMENDATIONS

Summary

Several methods have previously been suggested for determination of the sample size and acceptance number for attribute sampling plans. If two points (such as (AQL,α) and (LTPD,β)) are specified for the OC curve, the sampling plan must provide a probability of at least 1-α of accepting a lot if the lot proportion defective is at the acceptable quality level (AQL). The plan must also provide a probability of acceptance of no more than β if the lot proportion defective is at the rejectable quality level (RQL).

The methodologies of Stephens (1978) and Jaech (1980) use the Borges normal approximation to the binomial distribution. Though the procedures to find the acceptance number and sample size are quite straightforward, one can never feel certain that they will lead to what must be considered an optimum (minimum) sample size.

A computer program to obtain minimum sample size single sampling plans based on the binomial distribution was presented here. It was shown how the Fibonacci search technique could be used to determine the optimal solution. Jaech's algorithm was utilized to help determine the initial feasible solution needed to establish the search intervals for n and c. A proof of optimality was furnished that utilized the Poisson approximation to the binomial distribution.

The proposed method was tested to determine its accuracy by obtaining optimal solutions through exhaustive searches. The inaccuracies of both Stephens' and Jaech's methods were also highlighted.

Conclusions

The performance of the algorithm developed for determining minimum sample size single sampling plans for the binomial distribution was found to be very good. The solutions that were obtained for the test problems were confirmed to be accurate without exception, through exhaustive searches. Encouraging as it might be, it cannot be stated with complete confidence that the algorithm would always provide optimal solutions because the basis of the proof of optimality provided is the Poisson approximation to the binomial distribution. A proof using the binomial distribution was found to be exceedingly difficult to obtain.

As expected, the methodologies of Jaech and Stephens often yield inaccurate solutions. Jaech's method, however, did provide a fair number of feasible solutions, which were needed to establish the search intervals. In the FORTRAN program, if the initial solution obtained is infeasible, the next feasible Fibonacci number greater than the acceptance number or sample size (depending on which is infeasible) obtained is determined.

The use of Jaech's method in the program required a

Gaussian quadrature subroutine with 96 abcissas and weights and a Fibonacci search subroutine utilizing 16 experiments. Thus, considerable amounts of time for computation was needed to furnish the initial feasible solution. The efficiency of the algorithm could be improved considerably if a quick method, such as the Poisson solution, was used to determine the initial solution.

Tests of significance (α =.5) in comparing approximate solutions with exact solutions indicate that the mean reductions in sample sizes obtained by using exact solutions instead of approximate solutions were significant (though negative) only in the case of Stephens' method. However, when considering just positive reductions, only Stephens' method had mean reductions that were insignificant. The tests also revealed that Jaech's method (using tables) and the initial solutions of program "Aime" did not have mean values of reductions in acceptance numbers that were significantly different from zero. However, all three sets of approximate solutions had mean values of the absolute difference in sample sizes, when compared with exact solutions, that were significantly different from zero.

Recommendations

For further research on this topic it is obvious that a proof of optimality based on the binomial distribution is needed and must be the prime consideration. Without such a proof it cannot be stated with complete confidence that the solutions obtained using the algorithm that was developed in this paper are exact.

Once such a proof is obtained, along with its limitations, if any, this method might prove to be quite attractive if a quick and easy method was used to determine the initial feasible solution. Its attractiveness may be revealed in comparisons of computational times with other known methods, particularly the exhaustive search technique-the only method known to guarantee exact solutions.

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APPENDICES

Appendix A-1.
A FORTRAN program "Aime"

```
C     PROGRAM NAME: AIME
C     AUTHOR: LANCELOT SYLVESTER
C
C *** VARIABLE LIST  ***
C     ALPHA=PRODUCER'S RISK
C     BETA=CONSUMER'S RISK
C     P1=AQL
C     P2=LTPD
C     F=FIBONACCI NUMBERS
C     GP1=G(P1)
C     GP2=G(P2)
C     NO=INITIAL SAMPLE SIZE, USING STEPHEN'S EQUATION
C     GE=G(NO)
C     GINV1=G(INVERSE)
C     A=ABCISSAS
C     B=WEIGHTS
C     NINI=N,INITIAL
C     CINI=C,INITIAL
C     NMIN=N(MINIMUM)
C     CMIN=C(MINIMUM)
C     NIN=UPPER BOUND OF SEARCH INTERVAL FOR NMIN
C     CIN=UPPER BOUND OF SEARCH INTERVAL FOR CMIN
C     DELTA=1 IF NINI IS FEASIBLE
C           =0 OTHERWISE
C     GAMMA=1 IF CINI IS FEASIBLE
C           =0 OTHERWISE
C     TAU=1 IF ALPHA-CONSTRAINT IS ALSO SATISFIED
C         =0 OTHERWISE
C     SOL1,SOL2=INITIAL SOLUTION (NINI AND CINI)
C
      PROGRAM AIME(INPUT,OUTPUT,SORS,SORS2,TAPE1=SORS,TAPE2=SORS2)
      INTEGER  IDENT,NINI,CINI,CMIN,NMIN,DELT,DELTA,GAMMA,
     1TAU,N,NC.SOL1.SOL2,CIN,NIN
      REAL ALPHA,BETA,ZALPHA,ZBETA.F(0:16),P1,P2,
     1GP1,GP2,NO,GE,GINV1
      DOUBLE PRECISION A(96),B(96)
      DATA F/17*1.0/
      NC=0
C       GENERATE FIBONACCI NUMBERS
      DO 10 I=2,16
      F(I)=F(I-1)+F(I-2)
   10 CONTINUE
C     OUTPUT TABULAR HEADINGS
      WRITE(2,50)
      WRITE(2.60)
      WRITE(2,70)
      WRITE(2,75)
C      ABCISSAS AND WEIGHTS
      DO 20 I=1,48
      READ(1,*)A(I),B(I)
      J=96-I+1
      A(J)=-A(I)
C
```

```
C
      B(J)=B(I)
   20 CONTINUE
C     READ PROBLEM PARAMETERS
   25 READ(1,*)ALPHA,BETA,P1,P2
      IF(ALPHA.EQ.0) GO TO 90
C     DETERMINE INITIAL VALUES(JAECH'S ALGORITHM)
C     DETERMINE ZALPHA AND ZBETA
      GAMMA=1
      IDENT =1
      IF(ALPHA.LE.0.5) THEN
       CALL CATA(1.-ALPHA,F,ZALPHA,IDENT,A,B)
      ELSE
       CALL CATA(ALPHA,F,ZALPHA,IDENT,A,B)
       ZALPHA=-ZALPHA
      END IF
      IF(BETA.LE.0.5)THEN
       CALL CATA(1.-BETA,F,ZBETA,IDENT,A,B)
       ZBETA=-ZBETA
      ELSE
       CALL CATA(BETA,F,ZBETA,IDENT,A,B)
      END IF
C     DETERMINE GP1 AND GP2
      IDENT=0
      CALL CAT1(P1,GP1,IDENT,A,B)
      CALL CAT1(P2,GP2,IDENT,A,B)
C     CALCULATE INITIAL SAMPLE SIZE NO, USING STEPHEN'S EQUATION
      NO=((ZALPHA*(((1.-P1)*P1)**(1./6.))-ZBETA*(((1.-P2)*P2)
     1**(1./6.)))/(GP2-GP1))**2
C     CALCULATE INITIAL ACCEPTANCE NO.(CINI)
      GE=GP2+(ZBETA*((P2*(1.-P2))**(1./6.)))/SQRT(NO)
      CALL CATA(GE,F,GINV1,IDENT,A,B)
      CINI=NINT(GINV1*NO-1./6.)
      CIN=CINI
C     DETERMINE NMIN(JAECH'S METHOD): SET NINI=NMIN
      CALL ANEWN(NINI,CIN,GINV1,F,ZBETA,IDENT,A,B,P2,GP2)
      SOL1=NINI
      SOL2=CINI
C     WE NOW HAVE THE INITIAL VALUES OBTAINED BY JAECH'S METHOD
C
C     FIBONACCI SEARCH FOR OPTIMAL N AND C
C
C     TEST BETA CONSTRAINT
   30 CALL CONST1(CIN,NINI,BETA,P2,DELTA)
      IF(DELTA.EQ.1)THEN
C      BETA CONSTRAINT IS SATISFIED:DETERMINE LEAST N SUCH THAT F(N)>=NINI
       CALL FIBN(NINI,F,N)
       NIN=F(N)
      ELSE
C      DET. LEAST N SUCH THAT F(N)>NINI
   40  CALL FIBN(NINI+1,F,N)
       NIN=F(N)
C
```

```
C
C      TEST BETA CONSTRAINT
       CALL CONST1(CIN,NIN,BETA,P2,DELT)
       IF(DELT.EQ.0)THEN
        NINI=NIN
        GO TO 40
       END IF
      END IF
C     SEARCH FOR NMIN
      CALL AMINN(CIN,NINI,NIN,DELTA,NMIN,F,BETA,P2,N)
      NINI=NMIN
C     TEST ALPHA CONSTRAINT
      CALL CONST2(CIN,NINI,ALPHA,P1,TAU)
      IF(TAU.EQ.1)THEN
       IF(GAMMA.EQ.1)THEN
C       ALPHA CONSTRAINT IS SATISFIED:DET. LEAST NC SUCH THAT F(NC)>=CINI
        CALL FIBN(CINI,F,NC)
        CIN=F(NC)
       END IF
      ELSE
C      DET. LEAST NC SUCH THAT F(NC)>CINI
       GAMMA=0
       IF(NC.GT.0)THEN
        CINI=CIN
       END IF
       CALL FIBN(CINI+1,F,NC)
C
C
       CIN=F(NC)
       CALL ANEWN(NINI,CIN,GINV1,F,ZBETA,IDENT,A,B,P2,GP2)
       GO TO 30
      END IF
C     GIVEN INITIAL FEASIBLE SOLUTION,DETERMINE EXACT SOLUTION
      DELTA=1
      CALL AMINC(CINI,CIN,CMIN,GAMMA,ALPHA,P1,NIN.NINI,DELTA,
     1NMIN,F,BETA,P2,N,NC)
C     OUTPUT RESULTS
      WRITE(2,85)
      WRITE(2,80)ALPHA,BETA,P1,P2,SOL2,SOL1,CMIN,NMIN
      GO TO  25
   50 FORMAT(33(" "),"JAECH'S METHOD",15(" "),"PROGRAM AIME")
   60 FORMAT(3(" "),"PROBLEM PARAMETERS",11(" "),"(INITIAL VALUES)",
     1 12(" "),"(EXACT SOLUTION)")
   70 FORMAT(3(" "),7("*")," ",10("*"),23(" "),"MINIMUM",22(" "),
     1 "MINIMUM")
   75 FORMAT(" ALPHA","  BETA", "    P1","     P2 ","  ACCEPTANCE NO.",
     5 "  SAMPLE SIZE","  ACCEPTANCE NO.  SAMPLE SIZE")
   80 FORMAT(F6.5,3F6.4,8(" "),I2,12(" "),I3,11(" "),I2,13(" "),I3)
   85 FORMAT(" ")
   90 END
C
```

```
C
C
C     SUBROUTINE CAT1
C     GAUSSIAN QUADRATURE SUBROUTINE
C *** VARIABLE LIST ***
C     X=UPPER LIMIT OF INTEGRAL
C     Y=VALUE OF INTEGRAL
C     A=ABCISSAS
C     B=WEIGHTS
C
      SUBROUTINE CAT1(X,Y,IDENT,A,B)
      DOUBLE PRECISION A(96),B(96)
      REAL X,Y
      INTEGER IDENT
      IF(IDENT.EQ.1)THEN
      Y=0.5
      DO 100 I=1,96
      Y=Y+B(I)*X*(1./(2.*SQRT(2.*3.1415927)))*EXP(-((A(I)*X+X)**2)/8.)
  100 CONTINUE
      ELSE
      Y=0.
      DO 110 I=1,96
      Y=Y+B(I)*(X/2.)*((((A(I)*X+X)/2.)*(1.-((A(I)*X+X)/2.)))**(-1./3.))
  110 CONTINUE
      END IF
      END
C
C
C     SUBROUTINE CATA
C     DETERMINES ZALPHA,ZBETA, OR GINV1
C *** VARIABLE LIST ***
C     SL=DISTANCE OF END POINT TO THE NEAREST POINT THAT IS EVALUATED
C     E(1),E(2)=END POINTS
C     POINT(1),POINT(2)=POINTS EVALUATED
C     BL=CURRENT(INTERVAL LENGTH-SL)
C     G=TRUE VALUE OF INTEGRAL
C     VAL(1)=VALUE OF INTEGRAL FOR POINT(1)
C     VAL(2)=VALUE OF INTEGRAL FOR POINT(2)
C     DIF1=ABSOLUTE VALUE OF (G-VAL(1))
C     DIF2=ABSOLUTE VALUE OF (G-VAL(2))
C     A=ABCISSAS
C     B=WEIGHTS
C     WHEN IDENT=1 DETERMINE ZALPHA OR ZBETA
C                =0 DETERMINE GINV1
C
      SUBROUTINE CATA(G,F,GINV,IDENT,A,B)
      INTEGER IDENT,K
      REAL G,GINV,BL(14),SL(14),E(2),DIF1,DIF2,F(0:16),VAL(2),POINT(2)
      DOUBLE PRECISION A(96),B(96)
      BL(1)=1.0
      E(1)=0.0
      E(2)=1.0
      K=1
C
```

```
C
      SL(K)=(F(14-K)/F(16-K))*BL(K)
      POINT(1)=E(1)+SL(K)
      POINT(2)=E(2)-SL(K)
      IF(IDENT.EQ.1)THEN
       CALL CAT1(POINT(1)*3.5,VAL(1),IDENT,A,B)
       CALL CAT1(POINT(2)*3.5,VAL(2),IDENT,A,B)
      ELSE
       CALL CAT1(POINT(1),VAL(1),IDENT,A,B)
       CALL CAT1(POINT(2),VAL(2),IDENT,A,B)
      END IF
  200 DIF1=ABS(G-VAL(1))
      DIF2=ABS(G-VAL(2))
      IF(DIF1.LT.DIF2)THEN
       E(2)=POINT(2)
       POINT(2)=POINT(1)
       VAL(2)=VAL(1)
       K=K+1
       BL(K)=BL(K-1)-SL(K-1)
       SL(K)=(F(14-K)/F(16-K))*BL(K)
       POINT(1)=E(1)+SL(K)
       IF(IDENT.EQ.1)THEN
        CALL CAT1(POINT(1)*3.5,VAL(1),IDENT,A,B)
       ELSE
        CALL CAT1(POINT(1),VAL(1),IDENT,A,B)
       END IF
       IF(K.EQ.14) GO TO 210
       GO TO 200
      ELSE IF(DIF1.GT.DIF2)THEN
       E(1)=POINT(1)
       POINT(1)=POINT(2)
       VAL(1)=VAL(2)
       K=K+1
       BL(K)=BL(K-1)-SL(K-1)
       SL(K)=(F(14-K)/F(16-K))*BL(K)
       POINT(2)=E(2)-SL(K)
       IF(IDENT.EQ.1)THEN
        CALL CAT1(POINT(2)*3.5,VAL(2),IDENT,A,B)
       ELSE
        CALL CAT1(POINT(2),VAL(2),IDENT,A,B)
       END IF
       IF(K.EQ.14)GO TO 210
       GO TO 200
      ELSE
       IF(K.LT.12)THEN
        E(1)=POINT(1)
        E(2)=POINT(2)
        K=K+3
        BL(K)=E(2)-E(1)
        SL(K)=(F(14-K)/F(16-K))*BL(K)
C
```

```
C
        POINT(1)=E(1)+SL(K)
        POINT(2)=E(2)-SL(K)
        IF(IDENT.EQ.1)THEN
         CALL CAT1(POINT(1)*3.5,VAL(1),IDENT,A,B)
         CALL CAT1(POINT(2)*3.5,VAL(2),IDENT,A,B)
        ELSE
         CALL CAT1(POINT(1),VAL(1),IDENT,A,B)
         CALL CAT1(POINT(2),VAL(2),IDENT,A,B)
        END IF
        IF(K.EQ.14) GO TO 210
        GO TO 200
       ELSE
        POINT(1)=(POINT(2)-POINT(1))/2.
       END IF
      END IF
  210 IF(IDENT.EQ.1)THEN
       GINV=3.5*POINT(1)
      ELSE
       GINV=POINT(1)
      END IF
      END
C
C
C     SUBROUTINE ANEWN
C     JAECH'S ITERATIVE PROCEDURE FOR DETERMINING NMIN
C *** VARIABLE LIST ***
C     NINII=NINI
C     GIN=GINV1, OR INVERSE OF GE2 SUBSEQUENTLY
C     DN(4)=CURRENT VALUE OF SAMPLE SIZE
C     DN(3),DN(2),DN(1)=THE 3 PREVIOUS VALUES OBTAINED
C
      SUBROUTINE ANEWN(NINII,CIN,GIN,F,ZBETA,IDENT,A,B,P2,GP2)
      REAL GIN,GE2,ZBETA,F(0:16),DNJ,RCIN,P2,GP2
      INTEGER NINII,CIN,DN(4),IDENT,J
      DOUBLE PRECISION A(96),B(96)
      J=4
      DN(1)=0
      DN(2)=0
      DN(3)=0
      DN(4)=0
      NINII=0
      RCIN=CIN+(2./3.)
  300 DN(J)=NINT((RCIN/GIN)+1./6.)
      IF(DN(J).EQ.DN(J-1))NINII=DN(J)
      IF(DN(J).EQ.DN(J-2))NINII=MAX(DN(J),DN(J-1),DN(J-2))
      IF(DN(J).EQ.DN(J-3))NINII=MAX(DN(J),DN(J-1),DN(J-2),DN(J-3))
      IF(NINII.EQ.0)THEN
       DNJ=REAL(DN(J))
       GE2=(ZBETA*((P2*(1.-P2))**(1./6.)))/SQRT(DNJ+1./3.)+GP2
       DO 310 I=1,3
       DN(I)=DN(I+1)
  310  CONTINUE
C
```

```
C
        CALL CATA(GE2,F,GIN,IDENT,A,B)
        GO TO 300
      END IF
      END
C
C
C     SUBROUTINE CONST1
C     TEST BETA-CONSTRAINT
C     IF CONSTRAINT IS SATISFIED DELTAA=1
C     ELSE, DELTAA=0
C     CINN=ACCEPTANCE NO.
C     NINN=CORRESPONDING SAMPLE SIZE
C
      SUBROUTINE CONST1(CINN,NINN,BETA,P2,DELTAA)
      INTEGER CINN,NINN,DELTAA
      DOUBLE PRECISION T2,G2
      REAL BETA,P2
      G2=(1.-P2)**NINN
      T2=G2
      IF(CINN.EQ.0)GO TO 410
      DO 400 I=1,CINN
      G2=(G2/I)*(NINN-I+1)*(P2/(1.-P2))
      T2=T2+G2
  400 CONTINUE
  410 IF(T2.LE.BETA)THEN
      DELTAA=1
      ELSE
      DELTAA=0
      END IF
      END
C
C
C     SUBROUTINE CONST2
C     TEST ALPHA-CONSTRAINT
C     IF CONSTRAINT IS SATISFIED TAUU=1
C     ELSE,TAUU=0
C     CINT=ACCEPTANCE NO.
C     NTT=CORRESPONDING SAMPLE SIZE
C
      SUBROUTINE CONST2(CINT,NTT,ALPHA,P1,TAUU)
      INTEGER CINT,NTT,TAUU
      REAL ALPHA,P1
      DOUBLE PRECISION T1,G1
      G1=(1.-P1)**NTT
      T1=G1
      IF(CINT.EQ.0)GO TO 430
      DO 420 I=1,CINT
      G1=(G1/I)*(NTT-I+1)*(P1/(1.-P1))
      T1=T1+G1
  420 CONTINUE
C
```

```
C
  430 IF(T1.GE.(1.-ALPHA))THEN
      TAUU=1
      ELSE
      TAUU=0
      END IF
      END
C
C
C     SUBROUTINE FIBN
C     DETERMINE LEAST ULIM SUCH THAT F(ULIM)>=NUMB
      SUBROUTINE FIBN(NUMB,F,ULIM)
      INTEGER NUMB,ULIM
      REAL F(0:16)
      I=1
  500 IF(NUMB.LE.F(I))THEN
       ULIM=I
      ELSE
       I=I+1
       GO TO 500
      END IF
      END
C
C
C     SUBROUTIN AMINN
C     DETERMINES NMIN(FIBONACCI SEARCH)
C     FOR A GIVEN VALUE OF C
C *** VARIABLE LIST ***
C     CINN=ACCEPTANCE NO.
C     NINI=N,INITIAL
C     NIN=UPPER BOUND OF INTERVAL(=F(N))
C     NMINN=NMIN
C     END(1),END(2)=END POINTS
C     Q(1),Q(2)=POINTS WITHIN INTERVAL
C     R(1)=1 IF Q(1) SATISFIES BETA CONSTRAINT
C         =0 OTHERWISE
C     R(2)=1 IF Q(2) SATISFIES BETA CONSTRAINT
C         =0 OTHERWISE
C
      SUBROUTINE AMINN(CINN,NINI,NIN,DELTA,NMINN,F,BETA,P2,N)
      INTEGER CINN,NINI,NIN,DELTA,NMINN,N,Q(2),END(2),R(2)
      REAL BETA,P2,F(0:16)
      END(1)=0
      END(2)=NIN
      J=1
      IF(NIN.EQ.1)THEN
       NMINN=1
       GO TO 640
      ELSE IF(NIN.EQ.2.AND.NINI.EQ.1)THEN
       NMINN=2
       GO TO 640
      END IF
C
```

```
C
  600 Q(1)=END(1)+F(N-J-1)
      CALL CONST1(CINN,Q(1),BETA,P2,R(1))
      Q(2)=END(2)-F(N-J-1)
      IF(Q(1).EQ.Q(2))THEN
       IF(R(1).EQ.1)THEN
        NMINN=Q(1)
       ELSE
        NMINN=END(2)
       END IF
       GO TO 640
      END IF
  625 IF(DELTA.EQ.1)THEN
       IF(Q(1).GE.NINI)THEN
        R(1)=1
        R(2)=1
        GO TO 630
       ELSE IF(Q(2).GE.NINI)THEN
        R(2)=1
        IF(Q(1).LE.CINN)THEN
         R(1)=0
        ELSE
         CALL CONST1(CINN,Q(1),BETA,P2,R(1))
        END IF
        GO TO 630
       ELSE
        IF(Q(2).LE.CINN)THEN
         R(1)=0
         R(2)=0
        ELSE IF(Q(1).LE.CINN)THEN
         R(1)=0
         CALL CONST1(CINN,Q(2),BETA,P2,R(2))
        ELSE
         CALL CONST1(CINN,Q(1),BETA,P2,R(1))
         CALL CONST1(CINN,Q(2),BETA,P2,R(2))
        END IF
        GO TO 630
       END IF
      ELSE
       IF(Q(1).GE.NINI)THEN
        CALL CONST1(CINN,Q(1),BETA,P2,R(1))
        CALL CONST1(CINN,Q(2),BETA,P2,R(2))
       ELSE IF(Q(2).GE.NINI)THEN
        CALL CONST1(CINN,Q(2),BETA,P2,R(2))
        R(1)=0
       ELSE
        R(1)=0
        R(2)=0
       END IF
      END IF
  630 IF((R(1).EQ.0).AND.(R(2).EQ.0))THEN
       END(1)=Q(1)
C
```

```
C
      Q(1)=Q(2)
      J=J+1
      Q(2)=END(2)-F(N-J-1)
      IF(Q(1).EQ.Q(2))THEN
       IF(R(1).EQ.1)THEN
        NMINN=Q(1)
       ELSE
        NMINN=END(2)
       END IF
       GO TO 640
      ELSE
       GO TO 625
      END IF
     ELSE IF((R(1).EQ.1).AND.(R(2).EQ.1))THEN
      END(2)=Q(2)
      Q(2)=Q(1)
      J=J+1
      Q(1)=END(1)+F(N-J-1)
      IF(Q(1).EQ.Q(2))THEN
       IF(R(2).EQ.1)THEN
        NMINN=Q(1)
       ELSE
        NMINN=END(2)
       END IF
       GO TO 640
      ELSE
       GO TO 625
      END IF
     ELSE
      END(1)=Q(1)
      END(2)=Q(2)
      J=J+3
      IF((N-J-1).GE.0)GO TO 600
      NMINN=Q(2)
     END IF
 640 END
C
C
C     SUBROUTINE AMINC
C     DETERMINES CMIN ( AND NMIN ) GIVEN INITIAL FEASIBLE SOLUTION
C *** VARIABLE LIST ***
C     CEND(1),CEND(2)=END POINTS OF SEARCH INTERVAL
C     QS(1),QS(2)=POINTS WITHIN INTERVAL
C     QSV(1)=MINIMUM N FOR QS(1)
C     QSV(2)=MINIMUM N FOR QS(2)
C     RS(1)=1 IF QS(1) IS FEASIBLE
C          =0 OTHERWISE
C     RS(2)=1 IF QS(2) IS FEASIBLE
C          =0 OTHERWISE
C     CENDV(1)=MINIMUM N FOR END(1)
C     CENDV(2)=MINIMUM N FOR END(2)
C
```

```
C
C     CENDF(1)=1 IF CENDV(1) IS FEASIBLE
C               =0 OTHERWISE
C     CENDF(2)=1 IF CENDV(2) IS FEASIBLE
C               =0 OTHERWISE
C
      SUBROUTINE AMINC(CINI,CIN,CMIN,GAMMA,ALPHA,P1,NIN,NINI,
     1DELTA,NMIN,F,BETA,P2,N,NC)
      INTEGER CINI,CIN,CMIN,GAMMA,NIN,NINI,DELTA,NMIN,
     1N,NC,QS(2),QSV(2),RS(2),CEND(2),CENDF(2),CENDV(2)
      REAL F(0:16)
      L=1
      CEND(1)=0
      CEND(2)=CIN
      CENDV(2)=NINI
      IF(CIN.LT.2)GO TO 750
       QS(1)=CEND(1)+F(NC-L-1)
       QS(2)=CEND(2)-F(NC-L-1)
       IF(GAMMA.EQ.1)THEN
        IF(QS(1).GE.CINI)THEN
         RS(1)=1
         RS(2)=1
        ELSE IF(QS(2).GE.CINI)THEN
         RS(2)=1
         CALL AMINN(QS(1),NINI,NIN,DELTA,QSV(1),F,BETA,P2,N)
         CALL CONST2(QS(1),QSV(1),ALPHA,P1,RS(1))
        ELSE
         CALL AMINN(QS(1),NINI,NIN,DELTA,QSV(1),F,BETA,P2,N)
         CALL CONST2(QS(1),QSV(1),ALPHA,P1,RS(1))
         CALL AMINN(QS(2),NINI,NIN,DELTA,QSV(2),F,BETA,P2,N)
         CALL CONST2(QS(2),QSV(2),ALPHA,P1,RS(2))
        END IF
       ELSE
        IF(QS(2).LE.CINI)THEN
         RS(1)=0
         RS(2)=0
        ELSE IF(QS(1).LE.CINI)THEN
         RS(1)=0
         CALL AMINN(QS(2),NINI,NIN,DELTA,QSV(2),F,BETA,P2,N)
         CALL CONST2(QS(2),QSV(2),ALPHA,P1,RS(2))
        ELSE
         CALL AMINN(QS(1),NINI,NIN,DELTA,QSV(1),F,BETA,P2,N)
         CALL CONST2(QS(1),QSV(1),ALPHA,P1,RS(1))
         CALL AMINN(QS(2),NINI,NIN,DELTA,QSV(2),F,BETA,P2,N)
         CALL CONST2(QS(2),QSV(2),ALPHA,P1,RS(2))
        END IF
       END IF
  700  IF(QS(1).EQ.QS(2))THEN
        IF(RS(1).EQ.0)THEN
         NMIN=CENDV(2)
         CMIN=CEND(2)
        ELSE IF(CEND(1).NE.0)THEN
C
```

```
C
         NMIN=QSV(2)
         GO TO 800
        END IF
       END IF
  750 IF(CINI.EQ.1)THEN
       CALL AMINN(CEND(1),NINI,NIN,DELTA,CENDV(1),F,BETA,P2,N)
       CALL CONST2(CEND(1),CENDV(1),ALPHA,P1,CENDF(1))
       IF(CENDF(1).EQ.1)THEN
        CMIN=0
        NMIN=CENDV(1)
       ELSE
        CMIN=1
        NmIn=NINI
       END IF
      ELSE IF(GAMMA.EQ.1)THEN
       CMIN=0
       NMIN=NINI
      ELSE
       CMIN=1
       NMIN=NINI
      END IF
  800 END
```

Appendix B-1. Abcissas and Weights

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} A_i f(x_i)$$

The error term

$$E_n f = \frac{(n!)^4 . 2^{2n+1} . f^{(2n)}(t)}{(2n!)^3 (2n+1)}$$

$x_i = -x_{n-i+1}$	$A_i = A_{n-i+1}$
.999689503883230766828	.000796792065552012429
.998364375863181677724	.001853960788946921732
.995981842987209290650	.002910731817934946408
.992543900323762624572	.003964554338444686674
.988054126329623799481	.005014202742927517693
.982517263563014677477	.006058545504235961683
.975939174585136466453	.007096470791153865269
.968326828463264212174	.008126876925698759217
.959688291448742539300	.009148671230783386633
.950032717784437635756	.010160770535008415758
.939370339752755216932	.011162102099838498591
.927712456722308690965	.012151604671088319635
.915071423120898074206	.013128229566961572637
.901460635315852341319	.014090941772314860916
.886894517402420416057	.015038721026994938006
.871388505909296502874	.015970562902562291381
.854959033434601455463	.016885479864245172450
.837623511228187121494	.017782502316045260838
.819400310737931675539	.018660679627411467385
.800308744139140817229	.019519081140145022410
.780369043867433217604	.020356797154333324595
.759602341176647498703	.021172939892191298988

$x_i = -x_{n-i+1}$ (contd.)	$A_i = A_{n-i+1}$ (contd.)
.73803064374440013285l	.02196664438744349195
.71567681234896762622 5	.02273706965832937400l
.69256453664217156134 4	.02348339908592621984 2
.66871831004391615395 3	.02420484179236469128 2
.64416340378496710679 8	.02490063322248361028 8
.61892584012546857038 6	.02557003600534936149 9
.59303236477757208068 4	.02621234073567241391 3
.56651041856139716840 4	.02682686672559176219 8
.53938810832435743622 7	.02741296272602924282 3
.51169417715466767358 6	.02797000761684833444 0
.48345797392059635976 8	.02849741106508538564 6
.45470942216774300863 6	.02899461415055523654 3
.42547898840730054536 5	.02946108995816790597 0
.39579764982890860328 5	.02989634113632838598 4
.36569686147231363503 1	.03029991542082759379 4
.33520852289262542261 6	.03067137612366914901 4
.30436494435449635302 4	.03101033258631383742 3
.27319881259104914148 7	.03131642559686135581 3
.24174315616384001232 8	.03158933077072716855 8
.21003131046056720360 3	.03182875889441100653 5
.17809688236761860275 9	.03203445623199266321 8
.14597371465489694198 9	.03220620479403025066 9
.11369585011066592091 1	.03234382256857592842 9
.08129749546442555899 4	.03244716371406426936 4

$x_i = -x_{n-i+1}$	$A_i = A_{n-i+1}$
(contd.)	(contd.)
.048812985136049731112	.032516118713868835987
.016276744849602969579	.032550614492363166242

Appendix B-2.
Stephens Table $g(x)=\int_0^x (t(1-t))^{-1/3}\,dt$

x	0.000	001	.002	.003	.004	.005	.006	.007	.008	.009
.000	0.0000	.0150	.0238	.0312	.0378	.0439	.0496	.0549	.0601	.0650
.010	.0697	.0743	.0787	.0831	.0873	.0914	.0954	.0994	.1033	.1071
.020	.1108	.1145	.1181	.1217	.1252	.1287	.1321	.1355	.1388	.1421
.030	.1454	.1486	.1518	.1550	.1581	.1613	.1643	.1674	.1704	.1734
.040	.1764	.1793	.1823	.1852	.1881	.1909	.1938	.1966	.1994	.2022
.050	.2050	.2077	.2104	.2132	.2159	.2186	.2212	.2239	.2265	.2292
.060	.2318	.2344	.2370	.2395	.2421	.2446	.2472	.2497	.2522	.2547
.070	.2572	.2597	.2622	.2646	.2671	.2695	.2719	.2744	.2768	.2792
.080	.2816	.2839	.2863	.2887	.2910	.2934	.2957	.2981	.3004	.3027
.090	.3050	.3073	.3096	.3119	.3141	.3164	.3187	.3209	.3232	.3254
.100	.3277	.3299	.3321	.3343	.3365	.3387	.3409	.3431	.3453	.3475
.110	.3497	.3518	.3540	.3561	.3583	.3604	.3626	.3647	.3668	.3690
.120	.3711	.3732	.3753	.3774	.3795	.3816	.3837	.3858	.3878	.3899
.130	.3920	.3941	.3961	.3982	.4002	.4023	.4043	.4064	.4084	.4104
.140	.4125	.4145	.4165	.4185	.4205	.4225	.4245	.4265	.4285	.4305
.150	.4325	.4345	.4365	.4385	.4404	.4424	.4444	.4463	.4483	.4502
.160	.4522	.4542	.4561	.4580	.4600	.4619	.4639	.4658	.4677	.4696
.170	.4716	.4735	.4754	.4773	.4792	.4811	.4830	.4849	.4868	.4887
.180	.4906	.4925	.4944	.4963	.4982	.5001	.5019	.5038	.5057	.5075
.190	.5094	.5113	.5131	.5150	.5169	.5187	.5206	.5224	.5243	.5261
.200	.5280	.5298	.5316	.5335	.5353	.5371	.5390	.5408	.5426	.5444
.210	.5463	.5481	.5499	.5517	.5535	.5553	.5571	.5590	.5608	.5626
.220	.5644	.5662	.5680	.5697	.5715	.5733	.5751	.5769	.5787	.5805
.230	.5823	.5840	.5858	.5876	.5894	.5911	.5929	.5947	.5964	.5982
.240	.6000	.6017	.6035	.6053	.6070	.6088	.6105	.6123	.6140	.6158
.250	.6175	.6193	.6210	.6228	.6245	.6262	.6280	.6297	.6315	.6332
.260	.6349	.6367	.6384	.6401	.6418	.6436	.6453	.6470	.6487	.6505
.270	.6522	.6539	.6556	.6573	.6590	.6608	.6625	.6642	.6659	.6676
.280	.6693	.6710	.6727	.6744	.6761	.6778	.6795	.6812	.6829	.6846
.290	.6863	.6880	.6897	.6914	.6931	.6947	.6964	.6981	.6998	.7015
.300	.7032	.7048	.7065	.7082	.7099	.7116	.7132	.7149	.7166	.7183
.310	.7199	.7216	.7233	.7250	.7266	.7283	.7300	.7316	.7333	.7349
.320	.7366	.7383	.7399	.7416	.7433	.7449	.7466	.7482	.7499	.7515
.330	.7532	.7548	.7565	.7582	.7598	.7615	.7631	.7647	.7664	.7680
.340	.7697	.7713	.7730	.7746	.7763	.7779	.7795	.7812	.7828	.7845
.350	.7861	.7877	.7894	.7910	.7927	.7943	.7959	.7976	.7992	.8008
.360	.8025	.8041	.8057	.8073	.8090	.8106	.8122	.8139	.8155	.8171
.370	.8187	.8204	.8220	.8236	.8252	.8269	.8285	.8301	.8317	.8333
.380	.8350	.8366	.8382	.8398	.8414	.8430	.8447	.8463	.8479	.8495
.390	.8511	.8527	.8543	.8560	.8576	.8592	.8608	.8624	.8640	.8656
.400	.8672	.8688	.8704	.8721	.8737	.8753	.8769	.8785	.8801	.8817
.410	.8833	.8849	.8865	.8881	.8897	.8913	.8929	.8945	.8961	.8977
.420	.8993	.9009	.9025	.9041	.9057	.9073	.9089	.9105	.9121	.9137
.430	.9153	.9169	.9185	.9201	.9217	.9233	.9249	.9265	.9281	.9297
.440	.9313	.9329	.9345	.9361	.9377	.9393	.9409	.9425	.9440	.9456
.450	.9472	.9488	.9504	.9520	.9536	.9552	.9568	.9584	.9600	.9616
.460	.9631	.9647	.9663	.9679	.9695	.9711	.9727	.9743	.9759	.9775
.470	.9790	.9806	.9822	.9838	.9854	.9870	.9886	.9902	.9918	.9933
.480	.9949	.9965	.9981	.9997	1.0013	1.0029	1.0045	1.0061	1.0076	1.0092
.490	1.0108	1.0124	1.0140	1.0156	1.0172	1.0188	1.0203	1.0219	1.0235	1.0251

$$g(x) = \int_0^x (t(1-t))^{-1/3}\,dt$$

x	0.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.500	1.0267	1.0283	1.0299	1.0315	1.0330	1.0346	1.0362	1.0378	1.0394	1.0410
.510	1.0426	1.0442	1.0457	1.0473	1.0489	1.0505	1.0521	1.0537	1.0553	1.0569
.520	1.0584	1.0600	1.0616	1.0632	1.0648	1.0664	1.0680	1.0696	1.0712	1.0727
.530	1.0743	1.0759	1.0775	1.0791	1.0807	1.0823	1.0839	1.0855	1.0870	1.0886
.540	1.0902	1.0918	1.0934	1.0950	1.0966	1.0982	1.0998	1.1014	1.1030	1.1046
.550	1.1061	1.1077	1.1093	1.1109	1.1125	1.1141	1.1157	1.1173	1.1189	1.1205
.560	1.1221	1.1237	1.1253	1.1269	1.1285	1.1301	1.1317	1.1333	1.1349	1.1365
.570	1.1381	1.1397	1.1412	1.1428	1.1444	1.1460	1.1476	1.1492	1.1508	1.1524
.580	1.1540	1.1556	1.1573	1.1589	1.1605	1.1621	1.1637	1.1653	1.1669	1.1685
.590	1.1701	1.1717	1.1733	1.1749	1.1765	1.1781	1.1797	1.1813	1.1829	1.1845
.600	1.1861	1.1878	1.1894	1.1910	1.1926	1.1942	1.1958	1.1974	1.1990	1.2006
.610	1.2023	1.2039	1.2055	1.2071	1.2087	1.2103	1.2120	1.2136	1.2152	1.2168
.620	1.2184	1.2200	1.2217	1.2233	1.2249	1.2265	1.2282	1.2298	1.2314	1.2330
.630	1.2346	1.2363	1.2379	1.2395	1.2412	1.2428	1.2444	1.2460	1.2477	1.2493
.640	1.2509	1.2526	1.2542	1.2558	1.2575	1.2591	1.2607	1.2624	1.2640	1.2656
.650	1.2673	1.2689	1.2705	1.2722	1.2738	1.2755	1.2771	1.2788	1.2804	1.2820
.660	1.2837	1.2853	1.2870	1.2886	1.2903	1.2919	1.2936	1.2952	1.2969	1.2985
.670	1.3002	1.3018	1.3035	1.3052	1.3068	1.3085	1.3101	1.3118	1.3134	1.3151
.680	1.3168	1.3184	1.3201	1.3218	1.3234	1.3251	1.3268	1.3284	1.3301	1.3318
.690	1.3334	1.3351	1.3368	1.3385	1.3401	1.3418	1.3435	1.3452	1.3468	1.3485
.700	1.3502	1.3519	1.3536	1.3553	1.3570	1.3586	1.3603	1.3620	1.3637	1.3654
.710	1.3671	1.3688	1.3705	1.3722	1.3739	1.3756	1.3773	1.3790	1.3807	1.3824
.720	1.3841	1.3858	1.3875	1.3892	1.3909	1.3926	1.3943	1.3961	1.3978	1.3995
.730	1.4012	1.4029	1.4046	1.4064	1.4081	1.4098	1.4115	1.4133	1.4150	1.4167
.740	1.4185	1.4202	1.4219	1.4237	1.4254	1.4271	1.4289	1.4306	1.4324	1.4341
.750	1.4359	1.4376	1.4393	1.4411	1.4429	1.4446	1.4464	1.4481	1.4499	1.4516
.760	1.4534	1.4552	1.4569	1.4587	1.4605	1.4622	1.4640	1.4658	1.4676	1.4693
.770	1.4711	1.4729	1.4747	1.4765	1.4783	1.4800	1.4818	1.4836	1.4854	1.4872
.780	1.4890	1.4908	1.4926	1.4944	1.4962	1.4980	1.4999	1.5017	1.5035	1.5053
.790	1.5071	1.5089	1.5108	1.5126	1.5144	1.5162	1.5181	1.5199	1.5217	1.5236
.800	1.5254	1.5273	1.5291	1.5310	1.5328	1.5347	1.5365	1.5384	1.5402	1.5421
.810	1.5440	1.5458	1.5477	1.5496	1.5514	1.5533	1.5552	1.5571	1.5590	1.5609
.820	1.5628	1.5646	1.5665	1.5684	1.5703	1.5722	1.5742	1.5761	1.5780	1.5799
.830	1.5818	1.5837	1.5857	1.5876	1.5895	1.5915	1.5934	1.5953	1.5973	1.5992
.840	1.6012	1.6031	1.6051	1.6071	1.6090	1.6110	1.6130	1.6149	1.6169	1.6189
.850	1.6209	1.6229	1.6249	1.6268	1.6288	1.6309	1.6329	1.6349	1.6369	1.6389
.860	1.6409	1.6430	1.6450	1.6470	1.6491	1.6511	1.6532	1.6552	1.6573	1.6593
.870	1.6614	1.6635	1.6655	1.6676	1.6697	1.6718	1.6739	1.6760	1.6781	1.6802
.880	1.6823	1.6844	1.6865	1.6887	1.6908	1.6929	1.6951	1.6972	1.6994	1.7016
.890	1.7037	1.7059	1.7081	1.7103	1.7125	1.7146	1.7169	1.7191	1.7213	1.7235
.900	1.7257	1.7280	1.7302	1.7325	1.7347	1.7370	1.7392	1.7415	1.7438	1.7461
.910	1.7484	1.7507	1.7530	1.7553	1.7577	1.7600	1.7624	1.7647	1.7671	1.7694
.920	1.7718	1.7742	1.7766	1.7790	1.7814	1.7839	1.7863	1.7888	1.7912	1.7937
.930	1.7962	1.7987	1.8012	1.8037	1.8062	1.8088	1.8113	1.8139	1.8164	1.8190
.940	1.8216	1.8243	1.8269	1.8295	1.8322	1.8349	1.8376	1.8403	1.8430	1.8457
.950	1.8485	1.8513	1.8540	1.8569	1.8597	1.8625	1.8653	1.8682	1.8711	1.8740
.960	1.8770	1.8800	1.8830	1.8860	1.8891	1.8921	1.8952	1.8984	1.9016	1.9048
.970	1.9080	1.9113	1.9146	1.9179	1.9213	1.9247	1.9282	1.9318	1.9353	1.9389
.980	1.9426	1.9463	1.9501	1.9540	1.9580	1.9620	1.9661	1.9703	1.9747	1.9791
.990	1.9837	1.9884	1.9933	1.9985	2.0038	2.0095	2.0156	2.0222	2.0296	2.0384

Appendix B-3 Five Examples: Exhaustive Search

Problem Parameters	Acceptance Number, c	n_{max} (=0, if n=c+1 is infeasible	n_{min}	$n_{max}-n_{min}$
α=.0961	0	5	6	-1
β=.0916	1 ✓	26	9 ✓	17
p_1=.02	2	54	13	41
p_2=.38	3	86	16	70
	4	120	20	100
	5	156	23	133
α=.0483	0	0	3	-3
β=.0870	1	5	6	-1
p_1=.075	2 ✓	11	8 ✓	3
p_2=.60	3	18	10	8
	4	27	12	15
	5	35	14	21
	6	44	16	28
	7	54	18	36
α=.1403	0 ✓	10	10 ✓	0
β=.0947	1	45	18	25
p_1=.015	2	86	24	62
p_2=.210	3	132	31	101
	4	182	37	145
α=.054	0	0	4	-4
β=.0996	1	4	8	-4
p_1=.090	2	9	10	-1
p_2=.450	3 ✓	16	13 ✓	3
	4	23	16	7
	5	30	19	11
	6	38	21	17
	7	46	24	22
α=.010	0	1	66	-65
β=.001	1	15	89	-74
p_1=.01	2	44	108	-64
p_2=.10	3	83	126	-43
	4	129	143	-14
	5 ✓	180	159 ✓	21
	6	234	175	59
	7	292	190	102
	8	353	205	148
	9	415	220	194

✓: optimal solution n_{min} and corresponding value of c

Appendix B-4. Analysis of Stephens' Method
Table VIII

Test Prob. No.	α	β	α_{cal} *	β_{cal}	Δn_1
1	.1403	.0947	.1532	.0748	1
2	.0582	.0965	.0582	.0965	0
3	.1400	.1041	.1485	.0895	-10
4	.0529	.0935	.0529	.0936	-6
5	.0439	.0913	.0439	.0913	0
6	.0471	.0908	.0472	.0908	-5
7	.0760	.0980	.0765	.0982	-7
8	.0250	.0980	.0245	.0979	0
9	.0490	.0810	.0526	.0728	1
10	.2000	.6000	.2000	.5500	0
11	.6000	.2000	.5000	.2500	-2
12	.3000	.4000	.4500	.2000	-1
13	.0500	.1000	.0541	.0935	-5
14	.0500	.0500	.0352	.0696	-5
15	.0500	.0500	.0256	.0730	-1
16	.0500	.0500	.0100	.1900	-1
17	.5000	.0100	.5000	.0086	0
18	.3000	.1000	.3384	.0821	-6
19	.0300	.0850	.0324	.0786	-5
20	.0100	.0010	.0162	.0008	-13
21	.0483	.0870	.0483	.0871	-3
22	.0961	.0916	.0961	.0917	-4
23	.0500	.0500	.0256	.0832	-2
24	.0050	.0100	.0057	.0090	-21
25	.0900	.0500	.1100	.0354	-3

Average reduction in sample size $\overline{\Delta n}_1 = -3.92$
corresponding variance = $(4.9743)^2$

Average % reduction in sample size $(\Delta n/n)_1 = -28.8$
corresponding variance = $(45.598)^2$

Average % change in sample size $|\Delta n/n|_1 = 29.69$
corresponding variance = $(45.00)^2$

* α_{cal} and β_{cal} are the calculated values of α and β.

$$|\overline{\Delta n}| = 4.08$$

corresponding varience $= (4.9743)^2$

$$\overline{\Delta c} = -.72$$

corresponding varience $= (.7371)$

Appendix B-5.
Table IX

Analysis Of Jaech's Method
(Using Tables)

Test Prob. No.	α	β	α_{cal}*	β_{cal}	Δn_2
1	.1403	.0947	.0294	.0831	8
2	.0582	.0965	.0203	.0718	5
3	.1400	.1041	.0202	.1659	-4
4	.0529	.0935	.0112	.1308	-2
5	.0439	.0913	.0143	.1150	4
6	.0471	.0908	.0936	.0263	0
7	.0760	.0980	.0500	.0605	3
8	.0250	.0980	.0163	.1465	-3
9	.0490	.0810	.0188	.1288	3
10	.2000	.6000	.3600	.3025	1
11	.6000	.2000	.7500	.0625	-1
12	.3000	.4000	.4253	.1040	1
13	.0500	.1000	.0465	.0915	1
14	.0500	.0500	.0522	.0370	-3
15	.0500	.0500	.0441	.0211	1
16	.0500	.0500	.0086	.0086	2
17	.5000	.0100	.6563	.0013	1
18	.3000	.1000	.3247	.0848	-4
19	.0300	.0850	.0193	.0840	0
20	.0100	.0010	.0056	.0010	0
21	.0483	.0870	.0177	.0498	0
22	.0961	.0916	.0131	.0882	0
23	.0500	.0500	.0256	.0832	-2
24	.0050	.0100	.0026	.0098	0
25	.0900	.0500	.0922	.0467	-6

Average reduction in sample size $\overline{\Delta n}_2 = 0.52$
corresponding variance $= (3.029)^2$

Average % reduction in sample size $(\Delta n/n)_2 = 5.34$
corresponding variance $= (22.16)^2$

Average % change in sample size $|\Delta n/n|_2 = 15.30$
corresponding variance $= (16.67)^2$

* α_{cal} and β_{cal} are the calculated values of α and β.

$|\overline{\Delta n}| = 2.2$
corresponding varience $= (2.1016)^2$

$\overline{\Delta c} = -.04$
corresponding varience $= (.7348)^2$

Appendix B-6. Table X — Analysis Of Initial Solution Of Program Aime (Obtained Using Jaech's Method)

Test Prob. No.	α	β	α_{cal}*	β_{cal}	Δn_3
1	.1403	.0947	.0294	.0831	8
2	.0582	.0965	.0203	.0719	5
3	.1400	.1041	.0298	.1660	1
4	.0529	.0935	.0148	.0898	0
5	.0439	.0913	.0189	.0802	6
6	.0471	.0908	.0275	.0746	0
7	.0760	.0980	.0377	.0931	0
8	.0250	.0980	.0279	.0851	1
9	.0490	.0810	.0281	.0808	8
10	.2000	.6000	.2000	.5500	0
11	.6000	.2000	.7500	.0625	-1
12	.3000	.4000	.2025	.3600	0
13	.0500	.1000	.0432	.0991	0
14	.0500	.0500	.0522	.0370	-3
15	.0500	.0500	.0342	.0461	0
16	.0500	.0500	.0280	.0280	0
17	.5000	.0100	.5000	.0086	0
18	.3000	.1000	.3247	.0848	-4
19	.0300	.0850	.0193	.0840	0
20	.0100	.0010	.0057	.0009	1
21	.0483	.0870	.0177	.0498	0
22	.0961	.0916	.0131	.0882	0
23	.0500	.0500	.0441	.0339	0
24	.0050	.0100	.0026	.0101	-1
25	.0900	.0500	.0922	.0467	-6

Average reduction in sample size $\overline{\Delta n}_3 = 0.6$
corresponding variance $= (3.202)^2$

Average % reduction in sample size $(\Delta n/n)_3 = 1.862$
corresponding variance $= (16.45)^2$

Average % change in sample size $|\Delta n/n|_3 = 7.813$
corresponding variance $= (14.51)^2$

* α_{cal} and β_{cal} are the calculated values of α and β.

$|\Delta n| = 1.8$

corresponding varience $= (2.6926)^2$

$\Delta c = -.04$

corresponding varience $= (.6758)^2$

Appendix B-7. Table XI

Reductions in Acceptance Numbers, Δc

Test Prob. No.	Stephens' Method Δc	Jaech's Method Δc	Initial Solution Of Prog. "Aime" Δc
1	0	1	1
2	0	1	1
3	-1	0	0
4	-1	0	0
5	0	1	1
6	-1	-1	0
7	-1	0	0
8	0	0	0
9	0	1	1
10	0	0	0
11	-1	-1	-1
12	-1	0	0
13	-1	0	0
14	-1	-1	-1
15	0	0	0
16	0	1	0
17	0	0	0
18	-3	-2	-2
19	-1	0	0
20	-1	0	0
21	-1	0	0
22	-1	0	0
23	0	0	0
24	-1	0	0
25	-1	-1	-1
Average Reduction $\overline{\Delta c}$	-.68	-.04	-.04
variance	$(.69041)^2$	$(.73485)^2$	$(.67577)^2$