AN ABSTRACT OF THE THESIS OF

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(Name) (Degree)

in Mathematics presented on May 12, 1967
(Major) (Date)

Title: GENERALIZED MEASURES OF DEFORMATION-RATES IN NON-NEWTONIAN HYDRODYNAMICS AND THEIR APPLICATIONS TO SOME FLOW PROBLEMS

Abstract approved: Redacted for Privacy
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Integral representations for the generalized measures of deformation and deformation-rates have been obtained and suitable constitutive equations using these measures have been developed for viscoelastic materials. These new constitutive equations are then applied to study some flow problems. In order to assess the comparative advantage of this approach over the existing nonlinear theories of continuous media, a brief review of the latter has been first made. In this review, special attention has been drawn to the ever-increasing complexity of the constitutive equations which involve a number of terms in powers and products of the ordinary measures of strain or strain rate and several unknown response functions of invariants of kinematic matrices. This complexity in these constitutive equations has arisen since generalized measures have not been used and consequently the order of the measures could not be fixed.

With a view to bring simplicity and at the same time to retain generality, to ensure the effectiveness of the present theory, generalized measures of deformation-rates have been suitably extended before using them in the constitutive equations for isotropic incompressible fluids. After the orders of the generalized measures have been fixed, these new constitutive equations have been found to contain only four terms in the deformation-rate tensors and four rheological constants, but no unknown functions of the invariants.

This new constitutive theory based on generalized measures has been applied in the solution of the following three types of problems:

a. rectilinear flows,

b. helical flows and

c. torsional flow.

The normal stress effects including swelling and thinning in Poiseuille flow, and climbing in Couette flows, velocity profiles, pressure variations, etc. have been studied in much greater detail and added precision than has been done in the literature so far. The phenomena of back flow between two parallel plates and helical flow in a narrow annular gap have also been studied. The results have been compared with the classical theory. To enhance the value of this investigation and to make it more useful for practical purposes, graphs of velocity profiles, pressure variations, etc. have also been drawn. A special feature of this analysis is to bring out important non-Newtonian effects in real fluids with an unparalleled precision and simplicity. All this has been accomplished because of the use of generalized measures.

Possible scope of future work, where the idea of generalized measures may be profitably exploited, has also been discussed.

Generalized Measures of Deformation-rates in Non-Newtonian Hydrodynamics and Their Applications to Some Flow Problems

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1967

APPROVED:

Redacted for Privacy

Professor of Mathematics

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Date thesis is presented May 12, 1967

Typed by Muriel Davis for Kewal Singh Sra

ACKNOWLEDGMENTS

I am forever indebted to Professor M. N. L. Narasimhan for his kind and indispensable help in the preparation of this thesis. I shall always remember him most gratefully for spending much of his valuable time, five or six hours almost daily, throughout the preceding year, in teaching me, with utmost patience and perseverance, the subject of nonlinear continuum mechanics, and also guiding me very generously in my present research.

I also owe much gratitude to Professor A. T. Lonseth, Chairman of the Mathematics Department, for his kind encouragement and financial help that he has always generously given to me.

I have been immensely benefitted by my contact with Professor P. M. Anselone, too. My thanks are due to him for the help and encouragement that he has given to me in my graduate studies here.

Financial support for the present work was provided by the U. S. Atomic Energy Commission under terms of contract (AT (45-1)-1947) "Applied Analysis".

TABLE OF CONTENTS

Chapter | Page

PART II

APPLICATIONS OF THE NEW THEORY OF CONSTITUTIVE EQUATIONS TO SOME FLOW PROBLEMS

LIST OF FIGURES

NOTATION AND DIMENSIONS

Notation

We will be using the notation of tensor analysis extensively. We therefore assume some knowledge of tensors. For our purposes, Truesdell and Toupin (1960, appendix), Eringen (1962, appendix), and Sokolnikoff (1964) have been found to be enough. We shall explain the meaning of various symbols used, as and when they arise. The following list of the frequently used symbols may, however, be referred to, whenever needed:

Symbol	Meaning
a_i	the i^{th} vector component of acceleration
a_x, a_y, a_z	physical components of acceleration in rectangular Cartesian coordinates along the x-, y- and z-axis respectively.
a_r, a_θ, a_z	physical components of acceleration in the cylindrical coordinates along the r-, θ-, and z-directions respectively.
α	dimension correcting constant
B	second deformation-rate matrix
I_B, II_B, III_B	first, second and third invariants of the second deformation-rate matrix

b_{ij}	second deformation-rate tensor
B^*	second generalized deformation-rate matrix
b^*_{ij}	second generalized deformation-rate tensor
β	dimension correcting constant
D	first deformation-rate matrix
I_D, II_D, III_D	first, second and third invariants of the first deformation-rate matrix
d_{ij}	first deformation-rate tensor
D^*	first generalized deformation-rate matrix
d^*_{ij}	first generalized deformation-rate tensor
dx^i	displacement gradient
$\frac{D}{Dt}$	material derivative
δ_{ij}	kronecker delta
g_{ij}	metric tensor
γ	dimension correcting constant
I	identity matrix
M	torque
μ	Newtonian shear coefficient of viscosity
n, n'	integers denoting orders of generalized measures
p	pressure of the fluid
Q	volumetric flow rate
q, q'	integers denoting orders of generalized measures
r, θ, z	cylindrical coordinates

ρ	density of the fluid
T	stress matrix
t_{ij}	stress tensor
t_{xx}, t_{xy}, etc.	physical components of the stress tensor in rectangular Cartesian coordinates
$t_{rr}, t_{r\theta}$, etc.	physical components of the stress tensor in cylindrical coordinates
t	time variable
u	axial velocity component
v_i	i^{th} vector component of velocity
v_x, v_y, v_z	physical components of velocity in rectangular Cartesian coordinates in the x-, y- and z-directions respectively
v_r, v_θ, v_z	physical components of velocity in cylindrical coordinates along the r-, θ- and z-directions respectively
ω	angular velocity ($\equiv \frac{d\theta}{dt}$)
x^i	spatial coordinates
x, y, z	Cartesian coordinates

Dimensions

For convenience, we give below dimensions of certain rheological constants used in our work. Here, the symbol [ ] stands for the "dimension of", M for mass, L for length and T for time;

$$[\alpha] = \frac{M}{L} ,$$

$$[\beta] = T ,$$

$$[\gamma] = \frac{MT^4}{L} .$$

GENERALIZED MEASURES OF DEFORMATION-RATES IN NON-NEWTONIAN HYDRODYNAMICS AND THEIR APPLICATIONS TO SOME FLOW PROBLEMS

PART I. THEORY OF GENERALIZED MEASURES IN CONTINUUM MECHANICS

CHAPTER 1

INTRODUCTION

1.1. Preliminary Remarks

It is a matter of common experience that when toothpaste emerges from the tube, it swells near the exit. But when water comes out of a tap, we do not notice any such swelling of the fluid stream just leaving the tap. Again, if a cylindrical cup containing paint and having a rod fixed along its axis is rotated, the paint can be seen to be climbing up the rod. If, however, we perform the same experiment with skimmed milk, we notice, on the other hand, a depression of the fluid surface near the axial rod. Similar phenomena can be noticed in solids too. For example, if we twist a rod of steel severely, it will lengthen in proportion to the square of the twist. But in a similar torsion experiment performed with a rod of lead, the length of the rod would be found to be shortened. There are many such examples to show that material bodies with the same mass and geometry when subjected to the same external forces in an identical manner, respond differently. A natural question then arises:

What is responsible for this difference in response? An obvious answer is: Internal constitution of the materials is responsible for these differences in behavior. To explain, therefore, the response of a material to the applied forces, we need to set up a relation, depending on the internal constitution of the material, between the loading to which the material is subjected and its deformation or motion. This leads to the formulation of constitutive equation of the material which is a relation between the stress tensor and the deformation (strain) or motion (strain rate). In the classical theories of elasticity and fluid mechanics, the constitutive equations relate the stress tensor to the strain or strain rate tensors linearly. These theories have enjoyed and are still enjoying a tremendous success in "explaining" and controlling the structures and the mechanisms, the winds and the tides, sailing and flying, etc.

But there remain simple mechanical phenomena, for example, swelling and climbing in fluids, lengthening and shortening of twisted bars, etc., which the classical theories of elasticity and fluid dynamics have entirely failed to explain. Owing to the rapid progress in science and technology, the number of materials which exhibit this 'anomalous' behavior is ever increasing. A few examples of such materials which we come across in our everyday life are paints, pastes, emulsions, condensed milk, concrete, glue, many oils, lubricating greases, etc. It has, therefore, become important to study the internal constitution of these materials and to construct adequate but simple constitutive equations for these materials. Many attempts have been made in this direction resulting in the birth of several nonlinear theories based on experience and experiment. In all these theories, the concept of stress is well defined. The dynamical equations of equilibrium, for example, the equations of motion and continuity are applicable to all materials alike. But the measure of strain and the stress-strain relations (also called the constitutive equations) are flexible. This is natural.

In the classical theory of elasticity, the displacements are assumed to be so small that the squares and the products of displacement gradients are neglected and the measure of strain thus becomes linear. Such a linear measure cannot lead in many cases to a satisfactory solution of problems in which the displacements are finite. The present trend to explain experimental results involving finite deformations is based on the use of a linear strain measure even though we know from experiments that the strain is nonlinear in character. Consequently, the constitutive equations have to be unnecessarily complicated.

The present state of viscoelastic and rheological problems also indicates that to explain non-Newtonian effects in real fluids, the constitutive equations have been made more and more complex in form by the introduction of many response coefficients and unknown functions of strain rate invariants. Here again, one source of all this trouble is the use of classical measures of strain rate.

1.2. Object of the Present Study

In order to avoid any further complexity of the stress-strain relations and at the same time to explain the phenomena arising out of finite deformations in the case of solids and non-Newtonian behavior in the case of fluids, there is a need for constructing and using generalized measures instead of classical measures.
Seth (1962, 1964, 1966) has already done a great deal of pioneering work in this direction. He has generalized the classical measures in elasticity and has shown how successfully the generalized measures explain the mechanical behavior of real materials. He (1966) has also suggested the generalized measure of deformation-rate to be used in fluid dynamics.

It is the object of the present investigation to extend Seth's generalization of the measure of deformation-rate to viscoelastic and rheological problems, to set up new constitutive equations using the generalized measures and to apply them to some flow problems. It will be seen that this new approach will be of great help in explaining viscoelastic and rheological phenomena in a manner more precise than the previous theories. The following problems have been selected for a detailed study with a view to demonstrate how efficiently this new theory explains real physical phenomena:

a. rectilinear flows,

b. helical flows and

c. torsional flow .

1.3. Basic Assumptions

The following assumptions will be made in the analysis of the flow problems:

a. the flow is steady and laminar;

b. the fluid is homogenous, isotropic and incompressible, and

c. the flow is isothermal.

1.4. Plan of the Present Investigation

We have divided our work into two parts. Part I is devoted to the theory of generalized measures of deformation and rates of deformation to be used in nonlinear continuum mechanics. In Chapter 2 we show how the two important kinematic tensors d_{ij} and b_{ij} arise. In Chapter 3 we state the limitations of the classical theory and make a critical review of the nonlinear theories of continuum mechanics using ordinary measures. In Chapter 4 we generalize the ordinary measures of deformation and deformation-rate and set up for incompressible, isotropic fluids constitutive equations involving generalized measures of deformation-rate.

In Part II we apply the concept of generalized measures to some flow problems and show that this new powerful tool works in bringing out the non-Newtonian effects in real fluids with remarkable ease and clarity. The problems have been divided into three categories. Rectilinear flows between a pair of parallel plates, which arise either by a relative motion of the plates, or by the application of a pressure gradient, or both, are discussed in Chapter 5. In Chapter 6 we discuss the non-Newtonian effects arising in helical flows, Poiseuille and Couette flows. In Chapter 7 torsional flow is investigated. In the above problems, the influence of generalized measures of rates of deformation on the pressure and velocity fields has been determined and their graphs are drawn. Chapter 8 contains the summary, general discussion and scope of further work.

CHAPTER 2

BASIC CONCEPTS OF NONLINEAR CONTINUUM MECHANICS

2.1. Preliminary Remarks

In Section 2 we explain what we mean by continuum approach. In Section 3 we define the important concept of material derivative and thereafter derive expressions for the first and the second deformation-rate tensors. In Section 4 the two basic equations governing continuous media, viz, the equation of continuity and the equations of motion are expressed in the Cartesian and the cylindrical coordinates for use later in the flow problems. The material of this chapter is fundamental to all our subsequent work.

2.2. The Continuum Approach

Modern physics is based on the molecular structure of matter. From the molecular point of view, solids, liquids, and gases differ in their average molecular spacing. Whereas the solids are more closely packed, liquids contain more empty spaces, called holes, and in gases the average molecular spacing becomes much larger than the corresponding average molecular diameter. Accordingly, the physicist regards matter as a discrete conglomeration of molecules.

For many practical purposes, however, we can conveniently ignore the statistical viewpoint of matter as composed of classical molecules and assume that matter is indefinitely divisible and hence has no gaps and empty spaces. A material body viewed from this angle is called a continuum or a continuous medium.

Although both the theories of matter, that is, molecular as well as continuum have their merits and demerits, each helps the understanding of the other, too. We will, however, follow here exclusively the continuum approach, also called the macroscopic or phenomenological approach.

2.3. Kinematics of Continuous Media

Material Derivative. The material derivative of a tensor $A_{\vdots\vdots\vdots}(x^1, x^2, x^3, t)$ is defined by

$$\frac{DA_{\vdots\vdots\vdots}}{Dt} = \frac{\partial A_{\vdots\vdots\vdots}}{\partial t} + A_{\vdots\vdots\vdots;\ell} \frac{\partial x^\ell}{\partial t}, \tag{2.3.1}$$

where the symbol $_{;\ell}$ denotes covariant differentiation with respect to the spatial coordinate x^ℓ. In (2.3.1) $\frac{\partial A_{\vdots\vdots\vdots}}{\partial t}$ is called the local change and $A_{\vdots\vdots\vdots;\ell} \frac{\partial x^\ell}{\partial t}$ is called the convective change.

First Deformation-rate Tensor. The first material derivative of the square of the line element is given by

$$\frac{D(ds^2)}{Dt} = 2d_{ij}\, dx^i\, dx^j, \tag{2.3.2}$$

where

$$d_{ij} = \frac{1}{2}(v_{i;j} + v_{j;i}) \qquad (2.3.3)$$

is called the first deformation-rate tensor. Here dx^i and v_i are the i^{th} displacement gradient and the i^{th} vector component of velocity respectively.

To prove (2.3.2), we write

$$\frac{D(ds^2)}{Dt} = \frac{D(g_{ij}dx^i dx^j)}{Dt}$$

$$= g_{ij}dx^j \frac{D(dx^i)}{Dt} + g_{ij}dx^i \frac{D(dx^j)}{Dt}$$

$$= g_{ij}v^i_{;m} dx^m dx^j + g_{ij}v^j_{;m} dx^m dx^i$$

$$= (v_{i;j} + v_{j;i})\, dx^i dx^j$$

$$= 2d_{ij}\, dx^i dx^j .$$

Here g_{ij} denotes the metric tensor. We may add here that the first deformation-rate tensor d_{ij} will be of special interest to us in our subsequent work.

Second Deformation-rate Tensor. The second material derivative of the square of the line element is given by

$$\frac{D^2(ds^2)}{Dt^2} = 2b_{ij}dx^i dx^j, \qquad (2.3.4)$$

where

$$b_{ij} = \frac{1}{2}[a_{i;j} + a_{j;i} + 2v_{m;i}v^m_{;j}] \qquad (2.3.5)$$

is called the second deformation-rate tensor. Here a_i is the i^{th} vector component of acceleration.

To prove (2.3.4), we write

$$\frac{D^2(ds^2)}{Dt^2} = \frac{D}{Dt}\left(\frac{D(ds^2)}{Dt}\right)$$

$$= 2\,\frac{D(d_{ij}dx^i dx^j)}{Dt}$$

$$= 2[dx^i dx^j \frac{D(d_{ij})}{Dt} + d_{ij}dx^j \frac{D(dx^i)}{Dt} + d_{ij}dx^i \frac{D(dx^j)}{Dt}]$$

$$= 2[\frac{\partial(d_{ij})}{\partial t} + d_{ij;m}v^m + d_{mj}v^m_{;i} + d_{im}v^m_{;j}]\,dx^i dx^j$$

$$= [(\frac{\partial v_i}{\partial t})_{;j} + (\frac{\partial v_j}{\partial t})_{;i} + v^m(v_{i;jm} + v_{j;im})$$

$$+ v^m_{;i}(v_{m;j} + v_{j;m}) + v^m_{;j}(v_{i;m} + v_{m;i})]\,dx^i dx^j$$

$$= [a_{i;j} + a_{j;i} + 2v_{m;i}\,v^m_{;j}]\,dx^i dx^j$$

$$= 2b_{ij}dx^i dx^j\,.$$

The second deformation-rate tensor, like the first one, will also be of significance to us in our subsequent work.

2.4. Basic Equations of Continuous Media

The principles of conservation of mass and conservation of linear momentum lead to the following two equations respectively:

Equation of continuity

$$\frac{\partial \rho}{\partial t} + (\rho v^k)_{;k} = 0, \qquad (2.4.1)$$

Equations of motion

$$\rho a^i = t^{ij}_{;j} + \rho f^i, \qquad (2.4.2)$$

where $\rho \equiv$ density of the material,

$t^{ij} \equiv$ the stress tensor, (2.4.3)

$f^i \equiv i^{th}$ component of body force per unit mass.

Since, in our flow problems we shall asume the fluid to be incompressible, homogeneous and having fixed boundaries, we can take the density ρ to be constant and also neglect the body force. Consequently the equations of continuity and of motion reduce to the following simpler forms:

$$v^k_{;k} = 0 \quad , \qquad (2.4.4)$$

$$\rho a^i = t^{ij}_{;j} \; . \qquad (2.4.5)$$

We give below, for future reference, the equations of continuity and of motion in the Cartesian as well as cylindrical coordinates.

Cartesian

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \qquad (2.4.6)$$

$$\rho a_x = \frac{\partial (t_{xx})}{\partial x} + \frac{\partial (t_{xy})}{\partial y} + \frac{\partial (t_{zx})}{\partial z},$$

$$\rho a_y = \frac{\partial (t_{xy})}{\partial x} + \frac{\partial (t_{yy})}{\partial y} + \frac{\partial (t_{yz})}{\partial z}, \qquad (2.4.7)$$

$$\rho a_z = \frac{\partial (t_{zx})}{\partial x} + \frac{\partial (t_{yz})}{\partial y} + \frac{\partial (t_{zz})}{\partial z}.$$

Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0, \qquad (2.4.8)$$

$$\rho a_r = \frac{1}{r}\frac{\partial (r t_{rr})}{\partial r} + \frac{1}{r}\frac{\partial (t_{r\theta})}{\partial \theta} + \frac{\partial (t_{zr})}{\partial z} - \frac{t_{\theta\theta}}{r},$$

$$\rho a_\theta = \frac{1}{r}\frac{\partial (r t_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial (t_{\theta\theta})}{\partial \theta} + \frac{\partial (t_{\theta z})}{\partial z} + \frac{t_{r\theta}}{r}, \qquad (2.4.9)$$

$$\rho a_z = \frac{1}{r}\frac{\partial (r t_{zr})}{\partial r} + \frac{1}{r}\frac{\partial (t_{\theta z})}{\partial \theta} + \frac{\partial (t_{zz})}{\partial z}.$$

Here

v_x, v_y, v_z ≡ physical components of velocity in rectangular Cartesian coordinates

v_r, v_θ, v_z ≡ physical components of velocity in the cylindrical coordinates

a_x, a_y, a_z ≡ physical components of acceleration in rectangular Cartesian coordinates (2.4.10)

a_r, a_θ, a_z ≡ physical components of acceleration in cylindrical coordinates

t_{xx}, t_{xy}, etc. ≡ physical components of the stress tensor in rectangular Cartesian coordinates

$t_{tt}, t_{r\theta}$, etc. ≡ physical components of the stress tensor in cylindrical coordinates.

CHAPTER 3

PREVIOUS WORK ON THE THEORY OF CONSTITUTIVE EQUATIONS OF VISCOELASTIC MATERIALS

3.1. Preliminary Remarks

In Section 2 we cite some of the limitations of the classical theory of fluid dynamics; for example, its failure to explain the normal stress effects, variable viscosity of fluids, stress relaxation, etc. Section 3 deals briefly with various nonlinear theories that have emerged from time to time since 1945 in an attempt to find suitable mathematical models which could explain the aforesaid fluid behavior. In Section 4 we offer some comments on these theories, drawing special attention to the complicated constitutive equations which these theories propose and the occurrence of a number of unknown functions.

3.2. Limitations of the Classical Theory of Hydrodynamics

In classical hydrodynamics, the constitutive equation of incompressible viscous fluids is

$$T = -pI + 2\mu D, \qquad (3.2.1)$$

with $\qquad I_D = 0 ,$

where $T \equiv ||t_{ij}||$ = strain matrix,

p = fluid pressure,

$I \equiv ||\delta_{ij}||$ = identity matrix, (3.2.2)

μ = coefficient of viscosity,

$D \equiv ||d_{ij}||$ = the first deformation-rate matrix and

$I_D \equiv$ first invariant of D.

The special features of (3.2.1), are that it is linear in D and that the viscosity is a function of temperature. Fluids whose behavior is governed by (3.2.1) are called incompressible Newtonian fluids.

There are some fluids for example, water, alcohol, etc. whose mechanical behavior could be described with a fair degree of accuracy by (3.2.1). But there are perhaps many more fluid-like, incompressible materials whose properties are not described at all by (3.2.1).

Merrington (1943), in the course of certain measurements of the discharge of rubber solutions and of oils containing metallic soaps, observed that the fluid column swelled on emerging from the tube. This swelling phenomena is known as the Merrington effect.

Garner and Nissan (1946) made the observation that when a rod is rotated in a hydrocarbon gel, the fluid climbs up the rod while in an ordinary Newtonian fluid, there is a slight depression near the stirrer. Later, Weissenberg (1947) demonstrated in a series of experiments essentially the same phenomena, which is now known as the Weissenberg effect.

Besides the Merrington and the Weissenberg effects which are also known as the normal stress effects, many real as well as industrial fluids, for example, blood, asphalts, marine glue, paint, pitch, polymer solutions, protein solutions, colloidal suspensions, have been found to exhibit predominantly one or more of the following phenomena: varying flow rates and torques in the Poiseuille and Couette flows respectively, variable viscosity, viscoelasticity, viscoplasticity, pseudoplasticity, stress relaxation, time-dependent effects etc. Furthermore, in classical theory of Newtonian fluids rectilinear flows are possible in a cylinder of any cross section. But for non-Newtonian fluids, it was discovered by Ericksen (1956) and Green and Rivlin (1956) that such flows cannot be maintained in non-circular tubes without an appropriate body-force distribution in addition to a uniform pressure gradient along the tube. In the absence of such forces, there exists a superposed flow on the steady primary flow. Such a superposed flow is well known as secondary flow. Equation (3.2.1) of classical hydrodynamics cannot furnish any explanation for any of these phenomena. Fluids characterized by this 'anomalous' behavior are called non-Newtonian fluids.

3.3. Nonlinear Theories of Hydrodynamics

Since the number of non-Newtonian fluids is increasing rapidly in modern industry and biological investigations, there is a definite need for setting up adequate mathematical models for such fluids. Motivated by this need, a lot of research has been done in recent times leading to a number of nonlinear theories, which we are now going to mention very briefly.

Reiner-Rivlin Theory. Stokes (1845, p. 287) made the following hypothesis to describe the motion of a viscous fluid:

> That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the difference of the pressure abovementioned.

This idea of Stokes led to the following form of constitutive equation for incompressible, isotropic viscous fluids:

$$T = -pI + f(D),$$
$$f(0) = 0, \qquad (3.3.1)$$

where $f(D)$ is an arbitrary function of the first deformation-rate matrix D. If we assume that $f(D)$ is a polynomial (or infinite series) in D, then the use of Cayley-Hamilton theorem (that is, every square matrix satisfies its characteristic equation) reduces

(3.3.1) to

$$T = -pI + \alpha_1 D + \alpha_2 D^2, \tag{3.3.2}$$

where α_1 and α_2 are functions of the second and the third invariants of D. The derivation of (3.3.2) is due to Reiner (1945) and Rivlin (1948).

Rivlin-Ericksen Theory. Rivlin and Ericksen (1955) assumed that the stress at a point x at time t is a function of the gradients, in the spatial system, of velocity, acceleration, second acceleration and higher accelerations at the point x, measured at time t. This assumption ultimately led to the formulation of the constitutive equation

$$T = \alpha_0 I + \sum_{P=1}^{N} \alpha_P (\pi_P + \pi_P^*) \tag{3.3.3}$$

for incompressible, isotropic fluids, where α's are unknown functions of the second and the third invariants of the kinematic matrices $D^{(1)}, D^{(2)}, \cdots, D^{(n)}$ and are also expressible as functions of the traces of certain other matrix products $\tilde{\pi}_Q$ $(Q = 1, 2, \cdots, M)$; π_P are certain matrix products formed from the kinematic matrices $D^{(1)}, D^{(2)}, \cdots, D^{(n)}$, while π_P^* is the transpose of π_P. The kinematic matrix $D^{(r)} \equiv ||d_{ij}^{(r)}||$, is defined by

$$d_{ij}^{(1)} = \frac{1}{2}(v_{i;j} + v_{j;i}) ,$$

and

$$d_{ij}^{(r)} = \frac{\partial(d_{ij}^{(r-1)})}{\partial t} + v^m d_{ij;m}^{(r-1)} + d_{im}^{(r-1)} v^m_{;j} + d_{jm}^{(r-1)} v^m_{;i}$$

$$(r \geq 2) \qquad (3.3.4)$$

and is generated by taking the r^{th} material derivative of the square of the arclength.

Green-Rivlin Theory. Green and Rivlin (1957), not restricting themselves to fluids, but considering viscoelastic materials more generally, assumed that the stress t_{ij} depends on the complete deformation history of the material and expressed this assumption by taking t_{ij} to be a functional of $\frac{\partial x^p(\tau)}{\partial X^q}$ over the range $-\infty < \tau \leq t$, thus:

$$t_{ij} = \mathcal{F}_{ij}{\Big[}\frac{\partial x^p(\tau)}{\partial X^q}{\Big]}_{\tau=-\infty}^{t} .$$

where X^q denotes material coordinates.

They specialized this equation to the case when the material is a fluid, by observing, that for a fluid the only reference configuration is that at the instant of measurement of the stress. They therefore took $X^q = x^q$ and obtained

$$t_{ij} = \mathcal{F}_{ij}{\Big[}\frac{\partial x^p(\tau)}{\partial x^q}{\Big]}_{\tau=-\infty}^{t} .$$

Assuming further the stress as a continuous function of the gradients of velocity and accelerations, they finally obtained the following constitutive equation for viscoelastic fluids:

$$T = \phi_0(t)I + \sum_{N=1}^{5} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \sum_{P=0}^{R} \phi_P(t, \tau_1, \cdots, \tau_N)[\pi_P^{(N)} + \pi_P^{(N)*}]d\tau_1 \cdots d\tau_N , \tag{3.3.5}$$

where $\pi_P^{(N)}$ $(P=1, 2, \cdots, R)$ are certain matrix products formed from the matrices $G(\tau_1)$, $G(\tau_2), \cdots, G(\tau_N)$, $[G(\tau) \equiv ||g_{pq}(\tau)|| \equiv ||x^i_{,p}(\tau)x^j_{,q}(\tau)\delta_{ij}||]$ and the kinematic matrices $D^{(1)}, D^{(2)}, \cdots, D^{(n)}$, already defined in the Rivlin-Ericksen theory, and $\pi_P^{(N)}$ is multi-linear in the matrices $G(\tau)$; $\pi_P^{(N)*}$ is transpose of $\pi_P^{(N)}$; ϕ's are continuous functions of $t, \tau_1, \cdots, \tau_N$ and polynomials in expressions of the form

$$\int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \psi_P(t, \tau_1, \cdots, \tau_N) \mathrm{tr}\, \tilde{\pi}_P^{(N)} d\tau_1 \cdots d\tau_N \quad (N=0, 1, \cdots, 6) ,$$

where $\tilde{\pi}_P^{(N)}$ $(P=1, 2, \cdots)$ are certain matrix products formed from the matrices $G(\tau)$ and $D^{(r)}$, each of the matrix products $\tilde{\pi}_P^{(N)}$ being multinlinear in the matrices $G(\tau)$.

Oldroyd's Theory. The theory of viscoelasticity was first initiated by Maxwell (1867). He proposed the following constitutive equation

$$(1 + \lambda_1 \frac{\partial}{\partial t}) t_{ij}^{(e)} = 2\mu d_{ij}, \tag{3.3.6}$$

where

$$t_{ij} = -pg_{ij} + t_{ij}^{(e)},$$

and (3.3.7)

$$d_i^i = 0 \text{ (for all p).}$$

Here t_{ij} = the stress tensor,

$t_{ij}^{(e)}$ = deviatoric part of the stress tensor,

p = fluid pressure, (3.3.8)

d_{ij} = the deformation rate tensor,

μ = coefficient of viscosity,

λ_1 = relaxation time constant.

Later, Fröhlich and Sack (1946) derived essentially the following equation of state based on a structural model for a colloidal suspension in which Hookean elastic spherical particles are supposed distributed in a Newtonian viscous liquid:

$$(1 + \lambda_1 \frac{\partial}{\partial t})\, t_{ij}^{(e)} = 2\mu\, (1 + \lambda_2 \frac{\partial}{\partial t})\, d_{ij} \;, \qquad (3.3.9)$$

where λ_2 is another relaxation time constant.

Oldroyd (1950) chose (3.3.9) as the basis for generalization to nonlinear theory of viscoelasticity. His first generalization was to replace the time derivatives by convected derivatives so as to ensure invariance, thus getting

$$(1+\lambda_1 \frac{\delta}{\delta t})\, t_{ij}^{(e)} = 2\mu(1+\lambda_2 \frac{\delta}{\delta t})\, d_{ij} \;, \qquad (3.3.10)$$

where

$$\frac{\delta t_{ij}^{(e)}}{\delta t} \equiv \frac{\partial t_{ij}^{(e)}}{\partial t} + t_{ij;m}^{(e)} v^m + t_{mj} v^m_{\;;i} + t_{im} v^m_{\;;j} \;, \qquad (3.3.11)$$

and similarly for $\frac{\delta d_{ij}}{\delta t}$.

Another generalization of (3.3.9) which was also considered by Oldroyd (1950) was the following:

$$(1+\lambda_1 \frac{\delta}{\delta t}) t^{(e)ij} = 2\mu(1+\lambda_2 \frac{\delta}{\delta t})d^{ij} . \qquad (3.3.12)$$

Liquids with equations of state (3.3.7) and (3.3.10) are called liquids A, and those with equations of state (3.3.7) and (3.3.12) are called liquids B. Although (3.3.10) and (3.3.12) might appear to be trivially different generalizations of (3.3.9), Oldroyd (1950) showed that if $\lambda_1 > \lambda_2$, the liquids A and B would exhibit very different bulk properties.

The foregoing generalizations predicted normal stress effects but not the variable behavior of viscosity. To obtain the latter, Oldroyd (1951) added to (3.3.10) an arbitrary linear combination of $d_{im}d_j^m$, $d_{im}t_j^{(e)m}$, and $d_{jm}t_i^{(e)m}$ to obtain the further generalization:

$$(1+\lambda_1 \frac{\delta}{\delta t}) t_{ij}^{(e)} - 2k_1(d_{im}t_j^{(e)m} + d_{jm}t_i^{(e)m}) = 2\mu(1+\lambda_2 \frac{\delta}{\delta t})d_{ij}$$

$$- 8\mu k_2 d_{im}d_j^m , \qquad (3.3.13)$$

where k_1 and k_2 are arbitrary scalar constants. A similar modification was made in (3.3.12). More complicated equations involving contracted triple, and higher products of the stress and rate of strain tensors, give other distinct generalizations.

Oldroyd (1958) introduced another generalization of (3.3.9) by using Jaumann derivative $\frac{\mathcal{D}}{\mathcal{D}t}$ instead of convected derivative $\frac{\delta}{\delta t}$. By definition, the Jaumann derivative of a tensor $b^{\cdots i \cdots}_{\cdots j \cdots}$ is given by:

$$\frac{\mathcal{D}}{\mathcal{D}t} b^{\cdots i \cdots}_{\cdots j \cdots} = \frac{D}{Dt} b^{\cdots i \cdots}_{\cdots j \cdots} - \sum w^{m}_{j} \, b^{\cdots i \cdots}_{\cdots m \cdots} - \sum{}' w^{i}_{m} \, b^{\cdots m \cdots}_{\cdots j \cdots} \, , \qquad (3.3.14)$$

where $\frac{D}{Dt}$ is the material derivative and $\sum \left(\sum{}'\right)$ stands for summation of similar terms, one for each covariant (contravariant) index and w_{ij} is the spin tensor

$$w_{ij} = \frac{1}{2} \left(v_{i;j} - v_{j;i}\right) .$$

Walters' Analysis. Walters (1960) showed that the general linearized equations of state of an isotropic incompressible elastico-viscous liquid have the form:

$$t_{ij} = -pg_{ij} + t^{(e)}_{ij} \, ,$$

and

$$t^{(e)}_{ij}(\vec{x}, t) = 2 \int_{-\infty}^{t} \psi(t-t') \, d_{ij}(\vec{x}, t') dt' \, , \qquad (3.3.15)$$

where

$$\psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau} e^{-\frac{(t-t')}{\tau}} d\tau,$$

and $N(\tau)$ is called the relaxation spectrum. [$N(\tau)$ is defined such that $N(\tau)\,d\tau$ represents the total viscosity of the Maxwell elements with relaxation times between τ and $\tau + d\tau$]. Equation $(3.3.15)_2$ is regarded as a convected integral following the moving particle, the restrictions being small rates of shear in a stationary material element. Later on, Walters (1962) generalized further $(3.3.15)_2$ into the following forms:

$$t_{ij}^{(e)}(\vec{x}, t) = 2\int_{-\infty}^{t} \psi(t-t') \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^r}{\partial x^j} d_{mr}(\vec{x}', t')\, dt' ,$$

and (3.3.16)

$$t^{(e)ij}(\vec{x}, t) = 2\int_{-\infty}^{t} \psi(t-t') \frac{\partial x^i}{\partial x'^m} \frac{\partial x^j}{\partial x'^r} d^{mr}(\vec{x}', t')\, dt' ,$$

where $x'^i = x'^i(\vec{x}, t, t')$ is the position at time t' of the element which is instantaneously at the point x^i at time t.

The restrictive conditions of small rates of shear no longer apply to equations (3.3.16). The liquid with equations of state $(3.3.15)_1$ and $(3.3.16)_1$ is called liquid A', and that with $(3.3.15)_1$ and $(3.3.16)_2$ is called liquid B'. The liquids designated A and B by Oldroyd are the special cases of A' and B' respectively, obtained by substituting

$$N(\tau) = \mu \frac{\lambda_2}{\lambda_1} \delta(\tau) + \mu\left(\frac{\lambda_1-\lambda_2}{\lambda_1}\right)\delta(\tau-\lambda_1) \qquad (3.3.17)$$

in equations $(3.3.15)_3$ and (3.3.16). The Newtonian liquid is also a special case obtained by writing

$$N(\tau) = \mu\delta(\tau) ,$$

where δ denotes the Dirac delta function, defined in such a way that

$$\begin{aligned} \delta(x) &= 0 \quad \text{for} \quad x \neq 0 , \\ &= \infty \quad \text{for} \quad x = 0 , \qquad (3.3.18) \\ \int_{-\infty}^{\infty} \delta(x)\, dx &= \int_{0}^{\infty} \delta(x) dx = 1 . \end{aligned}$$

Further generalizations involving contracted double, triple, and higher product terms in the stress and rate of strain tensors can be obtained.

Noll's Theory. Noll (1958) assumed that the stress in an incompressible fluid at time t depends, to within a hydrostatic pressure, on the history of motion, (in particular, the past history of the relative deformation gradient) up to time t. His constitutive equation has thus the form

$$T = -pI + \mathop{\mathcal{F}}_{s=0}^{\infty} (G(s)), \qquad (3.3.19)$$

where $\mathcal{F}$ is the constitutive functional and G(s)* is the history of the relative deformation gradient.

3.4. Present Status of the Nonlinear Theories

Although the nonlinear theories outlined in the preceding section have answered many outstanding questions regarding the non-Newtonian behavior of fluids and have also predicted the occurrence of certain phenomena, for example, the secondary flows which were experimentally confirmed later by Giesekus (1963) and Hopmann and Barnett (1964), we would like to make the following comments:

(a) The Reiner-Rivlin theory appears to be mathematically simpler than the other theories, but it has one obvious drawback, that, the rheological coefficients which are functions of the invariants of the first deformation-rate tensor are unknown and the theory, by itself, has no way of specifying them explicitly.

---

* Let $\vec{\xi}$ and $\vec{x}$ be the spatial positions of the material point $\vec{X}$, at times τ and t respectively ($\tau \leq t$). Then, if we express the dependence of $\vec{\xi}$ on $\vec{x}$, τ and t by writing

$$\vec{\xi} = \chi_t(\vec{x}, t) ,$$

the gradient of χ_t with respect to $\vec{x}$ is called the deformation gradient of the material point $\vec{X}$ at time τ, relative to time t and is denoted by $G_t(\tau)$. The tensor defined by

$$G(s) = G_t(t-s) \qquad s \geq 0$$

is called the history of the relative deformation gradient.

Moreover, experiments conducted so far have shown that the fluids characterized by this theory, commonly called Stokesian fluids, do not exist in Nature or in industry. This theory always predicts the existence of two equal normal stresses in certain steady viscometric flows; but experiments with polyisobutylene solutions by Padden and Dewitt (1954) seem to contradict such a prediction when the rate of shear becomes appreciably large. This experimental evidence was perhaps the greatest reason for the rejection of the Reiner-Rivlin equation (3.3.2) as an adequate basis for a physical theory and also a strong motive for the search for constitutive equations of greater generality.

(b) Rivlin and Ericksen, in their theory of viscoelastic fluids, have been successful in obtaining normal stresses which need not be equal but their constitutive equation has been made very complicated by the introduction of several higher order kinematic matrices and unknown functions of their invariants. Even in the simplest case, when $D^{(r)} = 0$ for $r \geq 3$, the Rivlin-Ericksen equation (3.3.3) takes the form

$$T = \alpha_0 I + \alpha_1 D^{(1)} + \alpha_2 D^{(2)} + \alpha_3 D^{(1)^2}$$
$$+ \alpha_4 D^{(2)^2} + \alpha_5 (D^{(1)} D^{(2)} + D^{(2)} D^{(1)})$$
$$+ \alpha_6 (D^{(1)^2} D^{(2)} + D^{(2)} D^{(1)^2})$$
$$+ \alpha_7 (D^{(1)} D^{(2)^2} + D^{(2)^2} D^{(1)})$$
$$+ \alpha_8 (D^{(1)^2} D^{(2)^2} + D^{(2)^2} D^{(1)^2}) . \qquad (3.4.1)$$

It contains as many as nine unknown functions of invariants. Besides, the repeated occurrence of powers and products of the kinematic matrices $D^{(1)}$ and $D^{(2)}$ is bound to make the solutions of the dynamical equations all the more difficult, especially in problems which by their physical nature are quite involved.

(c) Green and Rivlin's theory is a further generalization of Rivlin and Ericksen s theory. Besides unknown functions α's of invariants, it also contains the functions ϕ's of $t, \tau_1, \cdots, \tau_N$, etc. Therefore, the above remarks apply to this theory even more strongly than to the Rivlin-Ericksen theory. Such complicated equations as those of Green and Rivlin could become really unwieldy except possibly in a few simple cases. Although these were derived about a decade back, these do not seem to have been employed in their general form to solve any physical problem so far.

(d) The theories of Oldroyd and Walters also involve a number of material constants and the nonlinearity has been introduced in a very arbitrary fashion.

(e) Noll's Theory is somewhat similar in idea to the theory of Green and Rivlin. The solution of any problem in this theory depends on the experimental determination of the three material functions, that is, the viscosity function and the two normal stress functions.

The ever-increasing complexity of the constitutive equations of continuous media and its ad hoc generalizations aimed at obtaining simple results have also been criticized by Seth on more than one occasion. He (1964, 1966) observed that the constitutive equations have to be complicated so long as we use classical measures of strain (or strain rate) in their formulation. Cauchy measure or the linear measure of strain is being used even when strain produced in a body is large. Any such restriction placed on the strain measure results in complicating the constitutive relations. Hence strain measure to be used should be nonlinear in character. To avoid bringing unnecessary complications in the stress and strain relations and at the same time to predict results fairly compatible with the experimental investigations, he has strongly felt the need to construct generalized measures of deformation which should reduce to the known ones in special cases. The function of these measures should be to condense the nonlinear effects of the deformation into one or two terms.

It may be mentioned here that the importance of introducing generalized measures in nonlinear continuum mechanics and of referring the strains to the strained system rather than the unstrained one has been stressed by Seth as early as in 1935 in his celebrated paper on "Finite Strains in Elastic Problems". His paper was followed by a series of papers by A. Signorini, F. D. Murnaghan, C. Truesdell, M. Riener, W. M. Shephard, R. S. Rivlin, K. H. Swainger, A. E. Green, R. Kappus, D. Panov, P. M. Riz, N. V. Zovolinsky, V. V. Novozhilov, L. M. Milne-Thomson, J. L. Ericksen, Z. Karni and others.

In the next chapter we extend the generalizations of the measures of deformation made by Seth (1966) and then set up new constitutive equations involving generalized measures.

CHAPTER 4

PRESENT CONTRIBUTION TO THE CONSTITUTIVE THEORY OF VISCOELASTIC MATERIALS BASED ON GENERALIZED MEASURES

4.1. Preliminary Remarks

We saw in the previous chapter that if we adhere to the use of ordinary measures of strain or rates of strain to explain irreversible phenomena involving finite deformations or flows, the constitutive equations have to be made very complicated and unwieldy. This is because of the introduction of nonlinear terms involving ordinary measures and a number of unknown response functions. Since in the existing nonlinear theories the order of the deformation or flow is not fixed, one has no control over the nature of these response functions. They could be in general infinite series expansions in powers of the invariants of the deformation tensors. This, in turn, complicates unnecessarily the work of the theorist insofar as the solution of a practical problem employing such constitutive equations is concerned. Besides, the experimentalist who must determine a large number of unknown response functions also has to face similar difficulties.

Since our ultimate object is to find a suitable explanation for the nonlinear phenomena, it is natural to think of a generalized measure instead of an ordinary one for this purpose. As stated earlier, the idea of generalized measure has already been introduced in continuum mechanics by Seth. In this chapter we extend this idea to viscoelastic and rheological problems in such a way that it becomes an effective tool to predict nonlinear effects in these problems. Section 2 deals with generalized measures in elasticity whereas Section 3 is devoted to the generalized measures to be used in fluid dynamics. In Section 4 we set up suitable constitutive equations involving generalized measures of fluid dynamics. These new constitutive equations contain at the most four constants which the experimentalist would have to determine.

This new approach, which will be found to be very simple and effective, is a major departure from the existing trends in nonlinear continuum mechanics.

4.2. Generalized Measures in Elasticity

The various strain measures already in use in elasticity are

(a) Cauchy measure: $e^C = \dfrac{\ell - \ell_0}{\ell_0}$,

(b) Swainger measure: $e^S = \dfrac{\ell - \ell_0}{\ell}$,

(c) Hencky measure: $e^H = \log \dfrac{\ell}{\ell_0}$, (4.2.1)

(d) Almansi measure: $e^A = \frac{1}{2}[\,1-(\frac{\ell_0}{\ell})^2\,]$,

(e) Green measure: $e^G = \frac{1}{2}[\,(\frac{\ell}{\ell_0})^2 - 1]$,

where ℓ_0 and ℓ are the undeformed and deformed lengths respectively.

These measures have the following integral representations and are obtained by introducing weight functions of various orders:

$$
\begin{aligned}
e^C &= \int_{\ell_0}^{\ell} \frac{d\ell}{\ell_0} , \\
e^S &= \int_{\ell_0}^{\ell} (\frac{\ell_0}{\ell})^2 \frac{d\ell}{\ell_0} , \\
e^H &= \int_{\ell_0}^{\ell} (\frac{\ell_0}{\ell}) \frac{d\ell}{\ell_0} , \\
e^A &= \int_{\ell_0}^{\ell} (\frac{\ell_0}{\ell})^3 \frac{d\ell}{\ell_0} , \\
e^G &= \int_{\ell_0}^{\ell} (\frac{\ell_0}{\ell})^{-1} \frac{d\ell}{\ell_0} .
\end{aligned}
\tag{4.2.2}
$$

This suggests a further generalization of these measures to

$$
e^* = \frac{1}{n}[\,1 - (\frac{\ell_0}{\ell})^n\,] = \int_{\ell_0}^{\ell} (\frac{\ell_0}{\ell})^{n+1} \frac{d\ell}{\ell_0} , \tag{4.2.3}
$$

where e^* may be called the n^{th} order measure. The weight function in this case is $(\frac{\ell_0}{\ell})^{n+1}$. Putting n=-1, 1, 0, 2, -2, we can get the known measures.

If a_{ii} (not summed on i) are the lengths of the principal axes of the strain quadric and $a_{ii,0}$ their undeformed lengths, then the generalized principal strain measure in Cartesian coordinates can be expressed with the help of (4.2.1) and (4.2.3) in terms of any of the known principal measures. To illustrate this, let e^C_{ii}, e^S_{ii}, e^A_{ii}, e^G_{ii} (not summed on i) be the principal Cauchy, Swainger, Almansi and Green measures respectively. Then the generalized principal measure e^*_{ii} (not summed on i) in Cartesian coordinates may be written in the form:

$$e^*_{ii} = \int_0^{e^C_{ii}} (1+e^C_{ii})^{-n-1} \, de^C_{ii} = \frac{1}{-n} \left[(e^C_{ii} + 1)^{-n} - 1 \right] , \tag{4.2.4}$$

or

$$e^*_{ii} = \int_0^{e^S_{ii}} (1-e^S_{ii})^{n-1} \, de^S_{ii} = \frac{1}{n} \left[1 - (1-e^S_{ii})^n \right] , \tag{4.2.5}$$

or

$$e^*_{ii} = \int_0^{e^A_{ii}} (1-2e^A_{ii})^{\frac{n}{2}-1} \, de^A_{ii} = \frac{1}{n} \left[1-(1-2e^A_{ii})^{\frac{n}{2}} \right] , \tag{4.2.6}$$

or
$$e^{*}_{ii} = \int_0^{e^G_{ii}} (1+2e^G_{ii})^{-\frac{n}{2}-1} de^G_{ii}$$
$$= \frac{1}{-n} [(1+2e^G_{ii})^{-\frac{n}{2}} - 1] . \qquad (4.2.7)$$

We see at once that for $n=-1, 1, 2$, and -2, we get from (4.2.4), (4.2.5), (4.2.6) and (4.2.7) the Cauchy, Swainger, Almansi and Green Measures respectively. For rendering the above generalized forms suitable for further generalizations, these formulae are written below in a more uniform way:

$$e^{*}_{ii} = \frac{2}{n} [(e^C_{ii} + 1)^{\frac{n}{2}} - 1] ,$$

$$e^{*}_{ii} = \frac{2}{n} [1-(1-e^S_{ii})^{\frac{n}{2}}] ,$$

$$e^{*}_{ii} = \frac{1}{n} [1-(1-2e^A_{ii})^{\frac{n}{2}}] , \qquad (4.2.8)$$

$$e^{*}_{ii} = \frac{1}{n} [(1+2e^G_{ii})^{\frac{n}{2}} - 1] .$$

We next make the important observation that all the relations (4.2.8) can be further generalized to the form

$$e^{*}_{ii} = \frac{1}{mn} [1-(1-2me_{ii})^{\frac{n}{2}}] , \qquad (4.2.9)$$

where m should correspond to the appropriate type of the ordinary measure e_{ii} (not summed on i) chosen. We note that for $m = -\frac{1}{2}, \frac{1}{2}$, 1 and -1, (4.2.9) reduces to (4.2.8). We also note that for n=2, the generalized measure e^*_{ii} specializes to the ordinary measure e_{ii}. The generalized form of an ordinary measure e_{ii} is therefore

$$e^*_{ii} = \frac{1}{mn}\left[1-(1-2me_{ii})^{\frac{n}{2}}\right]$$

$$= \int_0^{e_{ii}} (1-2me_{ii})^{\frac{n}{2}-1} \, de_{ii} . \tag{4.2.10}$$

4.3. Generalized Measures in Fluid Dynamics

We are now ready to generalize the ordinary measures of deformation-rates for use in viscoelasticity, viz,

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) , \tag{4.3.1}$$

and

$$b_{ij} = \frac{1}{2}(a_{i,j}+a_{j,i}+2v_{m,i}v^m_{,j}) , \tag{4.3.2}$$

where $,i$ and $,j$ denote partial differentiation with respect to x^i and x^j respectively.

As in (4.2.10) we have

$$d^*_{ii} = \int_0^{d_{ii}} (1-2md_{ii})^{\frac{n}{2}-1} \, d(d_{ii})$$

$$= \frac{1}{mn}\left[1-(1-2md_{ii})^{\frac{n}{2}}\right] , \tag{4.3.3}$$

and
$$b^*_{ii} = \frac{1}{m'n'}\left[1-(1-2m'b_{ii})^{\frac{n'}{2}}\right], \tag{4.3.4}$$

where m and m' would help to keep the dimensions correct.

A still further generalization would be

$$d^*_{ii} = \frac{k}{m^q n^q}\left[1-(1-2md_{ii})^{\frac{n}{2}}\right]^q, \tag{4.3.5}$$

and
$$b^*_{ii} = \frac{k'}{m'^{q'} n'^{q'}}\left[1-(1-2m'b_{ii})^{\frac{n'}{2}}\right]^{q'}, \tag{4.3.6}$$

where k and k' are also dimension correcting constants.

If we expand (4.3.5) and (4.3.6) in powers of d_{ii} and b_{ii} (no summation on i) respectively, we will, after expressing all powers higher than the second in terms of d_{ii}, d^2_{ii} and b_{ii}, b^2_{ii}, by the Cayley-Hamilton theorem, get

$$d^*_{ii} = F_0 + F_1 d_{ii} + F_2 d^2_{ii}, \tag{4.3.7}$$

and
$$b^*_{ii} = F'_0 + F'_1 b_{ii} + F'_2 b^2_{ii}, \tag{4.3.8}$$

where F's and F''s contain only finite number of terms in the respective deformation-rate invariants I, II, and III.

More generally, we have

$$d^*_{ij} = F_0 \delta_{ij} + F_1 d_{ij} + F_2 d_{i\ell} d_{\ell j}, \tag{4.3.9}$$

and $$b^*_{ij} = F'_0\delta_{ij} + F'_1 b_{ij} + F'_2 b_{i\ell}b_{\ell j} \quad . \tag{4.3.10}$$

Hence we have the generalized deformation-rate measures:

$$d^*_{ij} = \frac{k}{m^q n^q} \left[\delta_{ij} - (\delta_{ij} - 2md_{ij})^{\frac{n}{2}} \right]^q \quad , \tag{4.3.11}$$

and $$b^*_{ij} = \frac{k'}{m'^{q'} n'^{q'}} \left[\delta_{ij} - (\delta_{ij} - 2m'b_{ij})^{\frac{n'}{2}} \right]^{q'} \quad . \tag{4.3.12}$$

We may add here that the relations (4.3.11) and (4.3.12) are further equivalent to

$$d^*_{ij} = \frac{k}{m^q n^q} \left[(\delta_{ij} + 2md_{ij})^{\frac{n}{2}} - \delta_{ij} \right]^q \quad , \tag{4.3.13}$$

and $$b^*_{ij} = \frac{k'}{m'^{q'} n'^{q'}} \left[(\delta_{ij} + 2m'b_{ij})^{\frac{n'}{2}} - \delta_{ij} \right]^{q'} \quad , \tag{4.3.14}$$

respectively. The former can be obtained from the latter (or vice versa) by assigning the same values to m (or m') but with opposite signs.

The measures d^*_{ij} and b^*_{ij} given by (4.3.11) and (4.3.12) or (4.3.13) and (4.3.14) are the generalized measures of deformation-rates. For $n, n'=2$, $q, q'=1$, and $k, k'=1$, these generalized measures reduce to the ordinary measures d_{ij} and b_{ij} respectively.

4.4 Consitutive Equations Involving Generalized Measures

Having generalized the measures of deformation-rates, we now proceed to use them to set up a suitable constitutive equation. For this purpose we choose the Newtonian stress strain-velocity relation (3.2.1) which uses the linear measure $D(=||d_{ij}||)$. Since that relation, as remarked earlier, is unable to explain any non-Newtonian phenomena, we expect it to predict non-Newtonian effects, if we replace in it the classical measure D by the corresponding generalized measure D^* $(=||d^*_{ij}||)$. Substitution of the generalized measure D^* from (4.3.13) in the linear relation (3.2.1) then yields

$$\begin{aligned} T &= -pI + 2\mu D^* \\ &= -pI + 2\mu \frac{k}{m^q n^q} \left[(I+2mD)^{\frac{n}{2}} - I \right]^q . \end{aligned} \qquad (4.4.1)$$

If we assign specific positive integral values to n and q, we shall get from (4.4.1) after using Cayley-Hamilton theorem, for incompressible isotropic fluids, a constitutive relation of the following form

$$T = -pI + \alpha_1 D + \alpha_2 D^2 , \qquad (4.4.2)$$

where α_1 and α_2 are known functions of the invariants of D consisting of a finite number of terms only. This is definitely an improvement on the Reiner-Rivlin relation (3.3.1). But a close examination has revealed that, like the Reiner-Rivlin relation, it also predicts two equal normal stresses in certain steady viscometric flows. To overcome this situation we add to the right side of (4.4.1) the second deformation-rate term, viz, $4\eta B^*$. Equation (4.4.1) then generalizes to

$$T = -pI + 2\mu D^* + 4\eta B^*, \tag{4.4.3}$$

where $B^* = \| b^*_{ij} \|$ and η is the dimension correcting constant. Substituting the expressions for D^* and B^* from (4.3.13) and (4.3.14) into (4.4.3) we obtain, for incompressible isotropic fluids

$$T = -pI + \alpha_1 D + \alpha_2 D^2 + \beta_1 B + \beta_2 B^2, \tag{4.4.4}$$

where, for specific values of n, q, n', q', the coefficients α_1, α_2 are known functions of the invariants of D, and β_1, β_2 are known function of the invariants of B with finite number of terms in each case.

Equation (4.4.4) is the matrix form of the constitutive equation for incompressible, isotropic fluids, which has arisen as the result of using generalized measures. It is obvious that whatever the positive integral values of n, q, n', q', the deviatoric part of the stress matrix in (4.4.4) can never contain more than four terms. Besides other things, this has a clear advantage over the general Rivlin-Ericksen constitutive equation (3.3.3) or even its simplest form (3.4.1). A glance through the other nonlinear theories of fluid dynamics, summarized earlier in this chapter, would at once show that in none of them has the order of the measures of deformation-rates been fixed with the result that one does not know in those theories how to choose the rheological coefficients, since they are, in general, infinite series of the invariants of the kinematic matrices. Further, to explain nonlinear effects adequately with the help of classical measures, the constitutive equations in those theories had to be made complicated by taking a number of nonlinear terms. On the other hand, by first generalizing the ordinary measures and then fixing the orders of the generalized measures, the nonlinear effects have been condensed, essentially into two terms, viz. $2\mu D^*$ and $4\eta B^*$ and the rheological coefficients α_1, α_2, α_3 and α_4 occurring in (4.4.4) which is the final form of our constitutive equation, are also known explicitly. A further advantage of the generalized measures is that they help to avoid the unnecessary introduction of a number of response constants, which are now being so extensively used, thus indicating the direction in which the analytical treatment may be generalized.

To illustrate how the new constitutive equation (4.4.4) predicts the non-Newtonian phenomena, we first fix the order of the generalized measure as follows:

$$n = 4$$
$$q = 2 \qquad (4.4.5)$$
$$n' = 2$$
$$q' = 3 \; .$$

In the real situation, however, the order of the measure as well as the constants k, m, k' and m' involved in (4.4.4) will have to be determined for a particular fluid from the experimental data. Using these values of n, q, n' and q' in (4.3.13) and (4.3.14) and substituting the resulting expression for D^* and B^* in (4.4.3), we obtain

$$T = -pI + \frac{2\mu k}{m^2 4^2} \left[(I + 2mD)^2 - I \right]^2 + \frac{4\eta k'}{m'^3 2^3} \left[(I + 2m'B) - I \right]^3$$

$$= -pI + 2\mu k \left(m^2 D^4 + 2mD^3 + D^2 \right) + 4\eta k' B^3 .$$

Use of Caylay-Hamilton theorem now yields

$$T = -pI + 2\mu k (2m III_D + m^2 I_D III_D) I + 2\mu k (-2m II_D - m^2 I_D II_D + m^2 III_D) D$$
$$+ 2\mu k (1 + 2m I_D + m^2 I_D^2 - m^2 II_D) D^2 + 4\eta k' III_B I - 4\eta k' II_B B$$
$$+ 4\eta k' I_B B^2 . \qquad (4.4.6)$$

Since we shall deal with incompressible fluids, we have

$$I_D = 0 \; . \qquad (4.4.7)$$

In all the specific flow problems considered here, owing to the geometry of the problem, we shall see that

$$III_D = 0,$$

$$II_B = 0, \tag{4.4.8}$$

and

$$III_B = 0,$$

and thus we are left with the invariants I_B and II_D only.

By virtue of (4.4.7) and (4.4.8), equation (4.4.6) reduces to

$$T = -pI - 4m\mu k II_D D + 2\mu k(1-m^2 II_D)D^2 + 4\eta k' I_B B^2 . \tag{4.4.9}$$

Using the tensor notation, (4.4.9) takes the form

$$t^i_j = -p\delta^i_j - 8\alpha\beta\, II_D d^i_j + 2\alpha(1-4\beta^2 II_D) d^i_\ell d^\ell_j + \gamma I_B b^i_\ell b^\ell_j , \tag{4.4.10}$$

where

$$\mu k = \alpha ,$$

$$m = 2\beta , \tag{4.4.11}$$

$$\eta k' = \frac{\gamma}{4} .$$

It will be seen from the solutions of the flow problems that in order that our results be physically meaningful, the rheological constants α and β must have the same sign. To be consistent and definite, we will assume α and β to be positive. We will, however, let γ to be either positive or negative.

PART II. APPLICATIONS OF THE NEW THEORY OF CONSTITUTIVE EQUATIONS TO SOME FLOW PROBLEMS

CHAPTER 5

RECTILINEAR FLOWS

5.1. Preliminary Remarks

In Section 2 we discuss the rectilinear flow between two infinite parallel plates, the flow being caused by the motion of one of the plates parallel to itself and a nonvanishing pressure gradient. An interesting phenomenon, namely, back flow, is also studied in this section. Section 3 deals with the case of rectilinear flow caused by pressure gradient alone, whereas the flow investigated in Section 4 is made possible by moving one of the plates parallel to itself without creating any pressure gradient. Graphs of back flow, velocity profiles and pressure difference are also drawn.

5.2. Generalized Rectilinear Flow

Formulation of the Problem

We consider a steady rectilinear flow between to infinite parallel plates at a distance h apart (Figure 5.1). We assume that one of the plates is at rest and the other is moving parallel to itself with a constant velocity V. We further assume that the pressure gradient Δp in the direction of flow does not vanish. The axes of coordinates are so chosen that the direction of flow is the positive direction of the z-axis. The equation of the plate at rest is $x=0$ and that of the moving plate is $x=h$. The velocity field is given by

$$\begin{aligned} v_x &= 0 \\ v_y &= 0 \\ v_z &= v(x) \end{aligned} \tag{5.2.1}$$

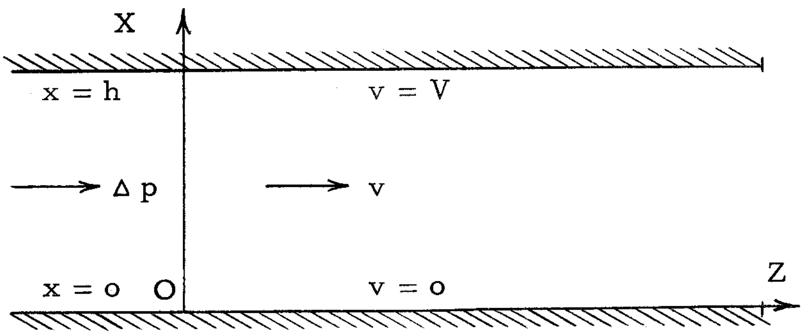


Figure 5.1. Generalized rectilinear flow.

Equations of Motion

With the velocity of field given by (5.2.1) the first and the second deformation-rate tensors take the following forms:

$$\left\| d^i_j \right\| = \left\| \begin{matrix} 0 & 0 & d^1_3 \\ 0 & 0 & 0 \\ d^3_1 & 0 & 0 \end{matrix} \right\| = \left\| \begin{matrix} 0 & 0 & \frac{1}{2}v' \\ 0 & 0 & 0 \\ \frac{1}{2}v' & 0 & 0 \end{matrix} \right\| \tag{5.2.2}$$

$$\left\| b^i_j \right\| = \left\| \begin{matrix} b^1_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

$$= \left\| \begin{matrix} v'^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| , \qquad (5.2.3)$$

where $v' \equiv \dfrac{\partial v}{\partial x}$.

Using $(5.2.2)_1$ and $(5.2.3)_1$ in (4.4.10), we get as our constitutive equation,

$$t^i_j = -p\,\delta^i_j + 8\alpha\beta\, d^1_3\, d^3_1\, d^i_j + 2\alpha(1+4\beta^2 d^1_3 d^3_1) d^i_\ell\, d^\ell_j + \gamma\, b^1_1 b^i_\ell b^\ell_j \,. \qquad (5.2.4)$$

Next, with the help of $(5.2.2)_2$ and $(5.2.3)_2$, we get the stress components:

$$\begin{aligned} t_{xx} &= -p + \frac{\alpha}{2}(1+\beta^2 v'^2)v'^2 + \gamma v'^6 , \\ t_{yy} &= -p , \\ t_{zz} &= -p + \frac{\alpha}{2}(1+\beta^2 v'^2)v'^2 , \\ t_{zx} &= \alpha\beta v'^3 , \\ t_{yz} &= 0 , \\ t_{xy} &= 0 . \end{aligned} \qquad (5.2.5)$$

The equations of motion (2.4.7) then become

$$\frac{\partial(t_{xx})}{\partial x} = 0 ,$$

$$-\frac{\partial p}{\partial y} = 0 . \qquad (5.2.6)$$

$$\frac{\partial(t_{zx})}{\partial x} - \frac{\partial p}{\partial z} = 0 .$$

Boundary Conditions

Assuming the no-slip condition on the plates we have

$$v(0) = 0 ,$$
$$v(h) = V \neq 0 . \qquad (5.2.7)$$

Solution of the Equations

Pressure Gradient. We first calculate the pressure gradient $\Delta p = \frac{\partial p}{\partial z}$. From $(5.2.6)_1$ it follows that t_{xx} does not depend on x, so that we can write

$$t_{xx} = \phi(y, z) \qquad (5.2.8)$$

or

$$(t_{xx} + p) - p = \phi(y, z) . \qquad (5.2.9)$$

Since from $(5.2.5)_1$ $t_{xx}+p$ is independent of y, we get, after differentiating (5.2.9) with respect to y,

$$-\frac{\partial p}{\partial y} = \frac{\partial \phi(y, z)}{\partial y} . \qquad (5.2.10)$$

By virtue of $(5.2.6)_2$ we have from (5.2.10)

$$\frac{\partial\phi(y,z)}{\partial y} = 0$$

and hence ϕ is independent of y. Equation (5.2.9) then becomes

$$(t_{xx}+p) - p = \phi(z) . \qquad (5.2.11)$$

Since from $(5.2.5)_1$, $(t_{xx}+p)$ is independent of z also, we get after differentiating (5.2.11) with respect to z,

$$-\frac{\partial p}{\partial z} = \frac{\partial\phi(z)}{\partial z} . \qquad (5.2.12)$$

From $(5.2.6)_3$ and (5.2.12) we obtain

$$-\frac{\partial(t_{zx})}{\partial x} = \frac{\partial\phi}{\partial z} . \qquad (5.2.13)$$

Since from $(5.2.5)_4$ we can say that $\frac{\partial(t_{zx})}{\partial x}$ is independent of z, $\frac{\partial\phi}{\partial z}$ is independent of z from (5.2.13). Consequently from (5.2.12), $\frac{\partial p}{\partial z}$ is independent of z as well as of x and y. Thus $\frac{\partial p}{\partial z} \equiv \Delta p$ is a constant. This constant is nonzero by hypothesis. We note that Δp is negative.

<u>Velocity Field.</u> From $(5.2.6)_3$ we now get

$$\frac{\partial(t_{zx})}{\partial x} = \Delta p . \qquad (5.2.14)$$

Integrating (5.2.14) and then using $(5.2.5)_4$ we obtain

$$\alpha\beta v'^3 = (\Delta p)x + k_1 , \tag{5.2.15}$$

where k_1 is the constant of integration.

From (5.2.15) we get

$$v' = \left[\frac{\Delta p}{\alpha\beta} x + \frac{k_1}{\alpha\beta} \right]^{1/3} . \tag{5.2.16}$$

Integration of (5.2.16) yields

$$v = \frac{3\alpha\beta}{4\Delta p} \left[\frac{\Delta p}{\alpha\beta} x + \frac{k_1}{\alpha\beta} \right]^{4/3} + k_2 , \tag{5.2.17}$$

where k_2 is another constant of integration. Equation (5.2.17) gives the velocity field and the constants k_1, k_2 occurring in this equation have to be determined from the boundary conditions (5.2.7). Before we apply the boundary conditions, we would like to put (5.2.17) in nondimensional form.

Dividing both sides of (5.2.17) by V we get

$$\frac{v}{V} = \left(\frac{27h^4 \Delta p}{64\alpha\beta V^3} \right)^{1/3} \left[\frac{x}{h} + \frac{k_1}{h\,\Delta p} \right]^{4/3} + \frac{k_2}{V}$$

or

$$\tilde{v} = P(\tilde{x} + \tilde{k}_1)^{4/3} + \tilde{k}_2 , \tag{5.2.18}$$

where

$$\tilde{v} = \frac{v}{V} \quad ,$$

$$P = \left(\frac{27h^4 \Delta p}{64\alpha\beta V^3}\right)^{1/3} \quad ,$$

$$\tilde{x} = \frac{x}{h} \quad , \qquad (5.2.19)$$

$$\tilde{k}_1 = \frac{k_1}{h\Delta p} \quad ,$$

$$\tilde{k}_2 = \frac{k_2}{V} \quad ,$$

are the dimensionless quantities.

The nondimensional forms of the boundary conditions (5.2.7) are

$$\tilde{v}(0) = 0 \ ,$$
$$\tilde{v}(1) = 1 \ . \qquad (5.2.20)$$

Using the boundary conditions (5.2.20) in (5.2.18) we have

$$\tilde{k}_2^3 = -P^3 \tilde{k}_1^4$$

and

$$(\tilde{k}_2 - 1)^3 = -P^3(\tilde{k}_1 + 1)^4. \qquad (5.2.21)$$

Case One

First let $\Delta p < 0$; then from $(5.2.19)_2$, we conclude that

$$P < 0 .$$

In particular, let $P = -1$ (say). Then (5.2.21) reduces to

$$\tilde{k}_2^3 = \tilde{k}_1^4 , \qquad (5.2.22)$$

and

$$(\tilde{k}_2 - 1)^3 = (\tilde{k}_1 + 1)^4 .$$

The solution of the above equations is readily found to be

$$\tilde{k}_1 = -1 \text{ and } \tilde{k}_2 = 1.$$

Then (5.2.18) becomes

$$\tilde{v} = -(\tilde{x} - 1)^{4/3} + 1 . \qquad (5.2.23)$$

Again, let $P = -2$ (say). Then (5.2.21) reduces to

$$\tilde{k}_2^3 = 8\tilde{k}_1^4 ,$$

and (5.2.24)

$$(\tilde{k}_2 - 1)^3 = 8(\tilde{k}_1 + 1)^4 .$$

It is found that $\tilde{k}_1 = -.74$ and $\tilde{k}_2 = 1.34$ is the approximate solution of the equations (5.2.24). Then equation (5.2.18) becomes

$$\tilde{v} = -2(x - .74)^{4/3} + 1.34. \qquad (5.2.25)$$

Case Two

Next let $\Delta p > 0$; then from $(5.2.19)_2$ we conclude that

$$P > 0 .$$

In particular let $P = 1$ (say). Then (5.2.21) reduces to

$$\tilde{k}_2^3 = -\tilde{k}_1^4 ,$$

and (5.2.26)

$$(\tilde{k}_2 - 1)^3 = -(\tilde{k}_1 + 1)^4 .$$

Obviously $\tilde{k}_1 = \tilde{k}_2 = 0$ is the solution of the equations (5.2.26). Then equation (5.2.18) becomes

$$\tilde{v} = \tilde{x}^{4/3} . \qquad (5.2.27)$$

Again, let $P = 2$. Then (5.2.21) reduces to

$$\tilde{k}_2^3 = -8\tilde{k}_1^4 ,$$

and (5.2.28)

$$(\tilde{k}_2 - 1)^3 = -8(\tilde{k}_1 + 1)^4 .$$

It is found that $\tilde{k}_1 = -.26$ and $\tilde{k}_2 = -.34$ is the approximate solution of the equations (5.2.28). Then equation (5.2.18) becomes

$$\tilde{v} = 2(\tilde{x} - .26)^{4/3} - .34 . \qquad (5.2.29)$$

As we shall see later, the special cases (5.2.23), (5.2.25), (5.2.27) and (5.2.29) of equation (5.2.18) will reveal an interesting phenomenon.

Pressure Field and Stress Distribution. From (5. 2. 12) we obtain after integration

$$\phi(z) = -\Delta pz + k_3 , \tag{5.2.30}$$

where k_3 is a constant of integration.

Substituting the value of $\phi(z)$ from (5. 2. 11) in (5. 2. 30), we get

$$t_{xx} = -\Delta pz + k_3 . \tag{5.2.31}$$

Using (5. 2. 16) and (5. 2. 31) in $(5.2.5)_1$ we get

$$p = \Delta pz - k_3 + \frac{\alpha^{1/3}}{2\beta^{2/3}} [(\Delta px + k_1)^{2/3} + \frac{\beta^{4/3}}{\alpha^{2/3}} (\Delta px + k_1)^{4/3}]$$

$$+ \frac{\gamma}{\alpha^2\beta^2} (\Delta px + k_1)^2 ,$$

or

$$p(x,z) = p(0,z) + \frac{\alpha^{1/3}}{2\beta^{2/3}} [(\Delta px + k_1)^{2/3} - k_1^{2/3}$$

$$+ \frac{\beta^{4/3}}{\alpha^{2/3}} \{(\Delta px + k_1)^{4/3} - k_1^{4/3}\}]$$

$$+ \frac{\gamma}{\alpha^2\beta^2} [(\Delta px + k_1)^2 - k_1^2] . \tag{5.2.32}$$

Equation (5. 2. 32) gives the pressure field. [We assume that $p(0,z)$ is already known.]

Using (5. 2. 16) and (5. 2. 32) we obtain from (5. 2. 5), the following stress distribution:

$$t_{xx} = - p(x,z) + \frac{\alpha^{1/3}}{2\beta^{2/3}} [(\Delta px + k_1)^{2/3} + \frac{\beta^{4/3}}{\alpha^{2/3}} (\Delta px + k_1)^{4/3}]$$

$$+ \frac{\gamma}{\alpha^2 \beta^2} (\Delta px + k_1)^2 ,$$

$$t_{yy} = -p(x,z) ,$$

$$t_{zz} = -p(x,z) + \frac{\alpha^{1/3}}{2\beta^{2/3}} [(\Delta px+k_1)^{2/3} + \frac{\beta^{4/3}}{\alpha^{2/3}} (\Delta px+k_1)^{4/3}] ,$$

$$t_{zx} = \Delta px + k_1, \qquad (5.2.33)$$

$$t_{yz} = 0 ,$$

$$t_{xy} = 0 ,$$

where $p(x,z)$ is given by (5.2.32).

Discussion of the Results

We have worked out expressions for the dimensionless velocity for four distinct values of the dimensionless pressure gradient P defined by $(5.2.19)_2$. From the graph (Figure 5.2, p. 55) of the velocity field for these four cases, we see that the shape of the velocity profile is determined by P. When the nondimensional pressure gradient $P < 0$, (and that happens whenever the fluid pressure p decreases in the direction of motion of the moving plate) the fluid always possesses a positive velocity throughout the

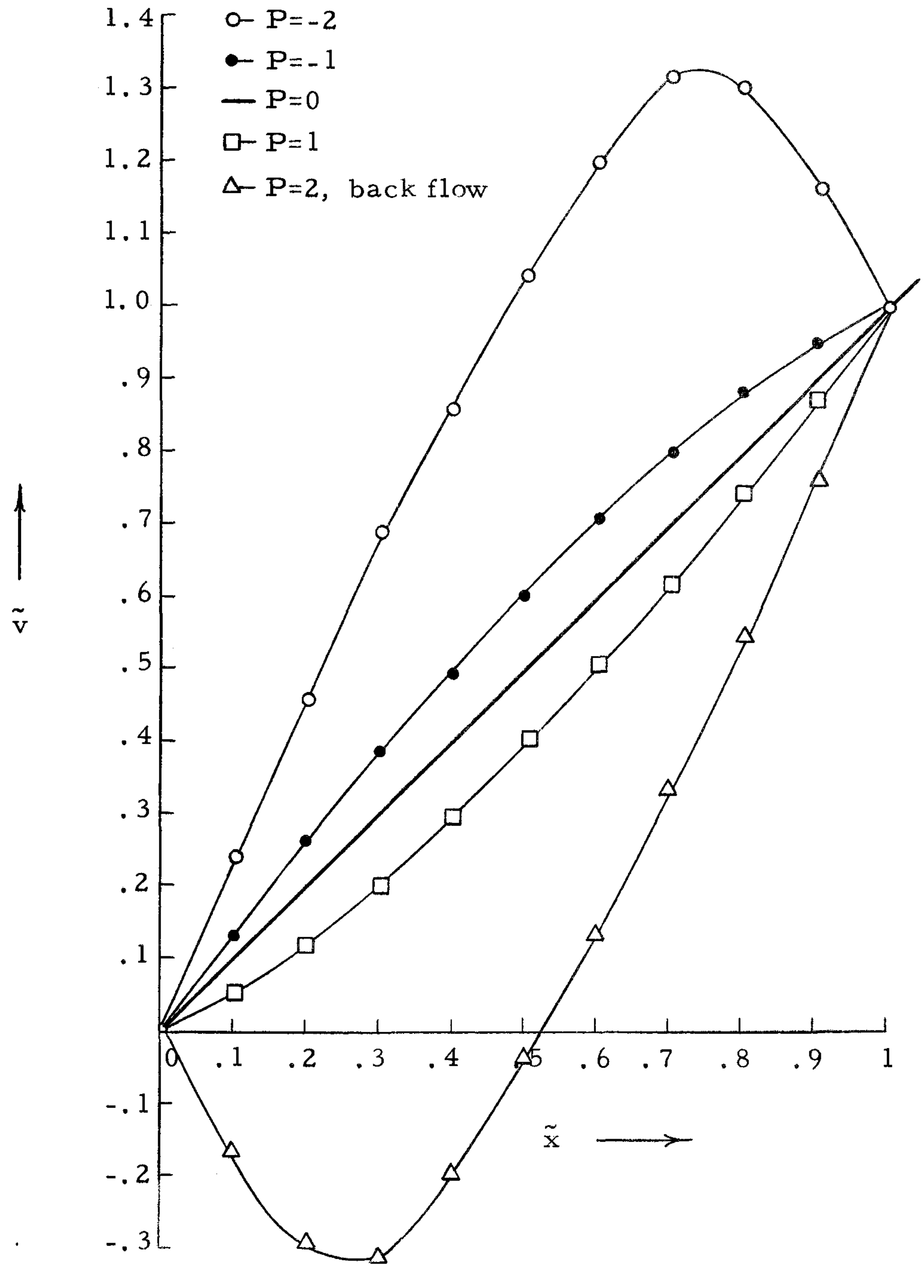


Figure 5.2. Velocity profile in generalized rectilinear flow.
$P < o$, pressure decrease in direction of plate motion;
$P = o$, zero pressure gradient;
$P > o$, pressure increase in direction of plate motion.

channel; that is, the entire flow is in the direction of z increasing. But when $P > 0$, (and that happens when the pressure p increases in the direction of motion of the moving plate) the fluid continues to possess positive velocity throughout the channel but only up to a certain positive value of P. As the graph shows when $P = 1$, the fluid velocity is positive throughout the channel. But when $P = 2$ or ≥ 2, the velocity is negative over a portion of the channel on the side of the stationary plate whereas it is positive over the rest of the channel which is on the side of the moving plate. Physically, negative velocity means back flow. This situation arises because the dragging action of the fluid layers sliding in the direction of the moving plate is not enough to overcome the influence of the adverse pressure gradient near the stationary plate.

We also note that when $P > 0$, the maximum positive velocity of the fluid is that of the moving plate. But when $P < 0$, the maximum velocity may exceed the velocity of the moving plate.

The behavior of the non-Newtonian fluid in the present problem is found to be similar to that of the classical fluid discussed by Schlichting (1960).

This type of flow finds application in the lubrication theory of fluids. The flow in the narrow space between the journal and the bearing is very much comparable with the rectilinear shear flow with a nonvanishing pressure gradient.

5.3. Channel Flow

Formulation of the Problem

This is a steady rectilinear flow between two infinite parallel plates at a distance 2d apart (Figure 5.3). We assume, as before, that the pressure gradient Δp in the direction of flow does not vanish, but both the plates are assumed to be at rest. The coordinate axes are so chosen that the origin is midway between the two plates; the positive direction of the z-axis is the direction of flow and the x-axis is perpendicular to (both) the plates. The velocity field is given by

$$\begin{aligned} v_x &= 0 \;, \\ v_y &= 0 \;, \\ v_z &= v(x) \;. \end{aligned} \tag{5.3.1}$$

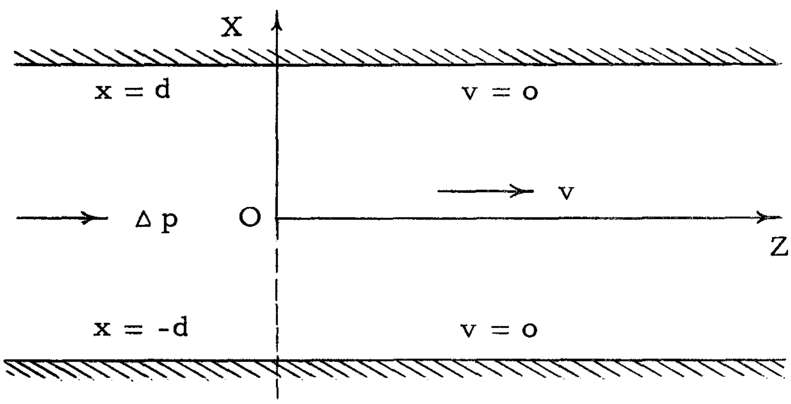


Figure 5.3. Channel flow ($\Delta p \neq o$)

Equations of Motion

The equations of motion in this case are exactly the same as in the case of generalized rectilinear flow.

Boundary Conditions

Since the fluid adheres to the plates, we have

$$v(\pm d) = 0 . \qquad (5.3.2)$$

Solution of the Equations

Pressure Gradient. As in the previous section, the pressure gradient $\Delta p \equiv \frac{\partial p}{\partial z}$ is a constant which is non-zero by hypothesis.

Velocity Field. Again as before,

$$v = \frac{3\alpha\beta}{4\Delta p}\left[\frac{\Delta p}{\alpha\beta}x + \frac{k_1}{\alpha\beta}\right]^{4/3} + k_2 . \qquad (5.3.3)$$

Changing (5.3.3) to the nondimensional form we obtain

$$\tilde{v} = P(\tilde{x} + \tilde{k}_1)^{4/3} + \tilde{k}_2 , \qquad (5.3.4)$$

where

$$\tilde{v} = \frac{v}{v_{max}} ,$$

$$P = \left(\frac{27d^4 \Delta p}{64\alpha\beta v_{max}^3}\right)^{1/3} ,$$

$$\tilde{x} = \frac{x}{d} , \tag{5.3.5}$$

$$\tilde{k}_1 = \frac{k_1}{d\,\Delta p} ,$$

$$\tilde{k}_2 = \frac{k_2}{v_{max}} ,$$

are the dimensionless quantities and v_{max} is the maximum velocity in the channel.

The nondimensional forms of the boundary conditions (5. 3. 2) are

$$\tilde{v}(\pm 1) = 0 \tag{5.3.6}$$

Using the boundary conditions (5. 3. 6) in (5. 3. 4) we have

$$(\tilde{k}_1 + 1)^4 = (\tilde{k}_1 - 1)^4 , \tag{5.3.7}$$

which means that $\tilde{k}_1$ must be zero. Consequently (5. 3. 4) reduces to

$$\tilde{v} = P\tilde{x}^{4/3} + \tilde{k}_2 . \tag{5.3.8}$$

The use of boundary conditions once again gives

$$\tilde{k}_2 = -P . \tag{5.3.9}$$

Substituting the value of $\tilde{k}_2$ from (5. 3. 9) in (5. 3. 8) we get

$$\tilde{v} = P(\tilde{x}^{4/3} - 1) . \tag{5.3.10}$$

Since P is negative, from (5. 3. 10) we have

$$\tilde{v}_{max} = -P. \qquad (5.3.11)$$

In other words

$$\frac{v_{max}}{v_{max}} = -P, \qquad (5.3.12)$$

which means $P = -1$. Hence (5. 3. 10) becomes

$$\tilde{v} = 1 - \tilde{x}^{4/3}. \qquad (5.3.13)$$

Equation (5. 3. 13) gives the velocity field.

Putting $P = -1$ in $(5.3.5)_2$ we get the maximum velocity v_{max} of the flow:

$$v_{max} = -\left(\frac{27 d^4 \Delta p}{64\,\alpha\beta}\right)^{1/3}. \qquad (5.3.14)$$

Pressure Field and Stress Distribution. Setting $k_1 = 0$ in (5. 2. 32) and (5. 2. 33) we get

$$p(x,z) = p(0,z) + \frac{\alpha^{1/3}}{2\beta^{2/3}}\left[(\Delta px)^{2/3} + \frac{\beta^{4/3}}{\alpha^{2/3}}(\Delta px)^{4/3}\right] + \frac{\gamma}{\alpha^2\beta^2}(\Delta px)^2,$$

$$t_{xx} = -p(x,z) + \frac{\alpha^{1/3}}{2\beta^{2/3}}\left[(\Delta px)^{2/3} + \frac{\beta^{4/3}}{\alpha^{2/3}}(\Delta px)^{4/3}\right]$$

$$+ \frac{\gamma}{\alpha^2\beta^2}(\Delta px)^2,$$

$$t_{yy} = -p(x,z), \qquad (5.3.15)$$

$$t_{zz} = -p(x, z) + \frac{\alpha^{1/3}}{2\beta^{2/3}} [(\Delta px)^{2/3} + \frac{\beta^{4/3}}{\alpha^{2/3}} (\Delta px)^{4/3}],$$

$$t_{zx} = \Delta px,$$

$$t_{yz} = 0,$$

$$t_{xy} = 0,$$

where $p(x, z)$ is given by $(5.3.15)_1$.

Volumetric Flow Rate. The volumetric flow rate through a cross section of the channel of width ℓ is given by

$$Q = \ell \int_{-d}^{d} v(x)dx$$

$$= \frac{3\ell}{4} (\frac{\Delta p}{\alpha\beta})^{1/3} \int_{-d}^{d} (x^{4/3} - d^{4/3})dx$$

$$= \frac{-6\ell}{7} (\frac{\Delta p}{\alpha\beta})^{1/3} d^{7/3}. \qquad (5.3.16)$$

Discussion of the Results

It is well known that for a steady channel flow of a Newtonian fluid between two infinite parallel plates at a distance $2d$ apart and with a nonvanishing pressure gradient Δp in the direction of flow the velocity profile is given by

$$v = -\frac{\Delta p}{2\mu} (d^2 - x^2), \qquad (5.3.17)$$

where μ is the Newtonian viscosity. This is obviously parabolic.

But for a non-Newtonian fluid we see from (5. 3. 13) that the velocity profile need not be parabolic.

The velocity curves (Figure 5.4, p. 63) of

$$\tilde{v} = 1-\tilde{x}^2 , \quad \text{(Newtonian)} \tag{5.3.18}$$

$$\tilde{v} = 1-\tilde{x}^{4/3} , \quad \text{(non-Newtonian)} \tag{5.3.19}$$

where $$\tilde{v} = \frac{v}{v_{max}} ,$$

$$\tilde{x} = \frac{x}{d} , \tag{5.3.20}$$

show that the non-Newtonian character of the fluid results in damping its velocity. This phenomena is due to the viscoelastic nature of the fluid.

We see from $(5.3.15)_5$ that to calculate the shearing stress at a point in the flow region, we need to know only the pressure gradient along the z-axis. In spite of the fact that the velocity field is given by a nonlinear function of x, the shearing stress is linear in x.

Unlike the Newtonian case, the volumetric flow rate is no longer proportional to the cube of the channel width, so that the classical formula for the flow rate of Newtonian fluids does not hold in the case of non-Newtonian fluids.

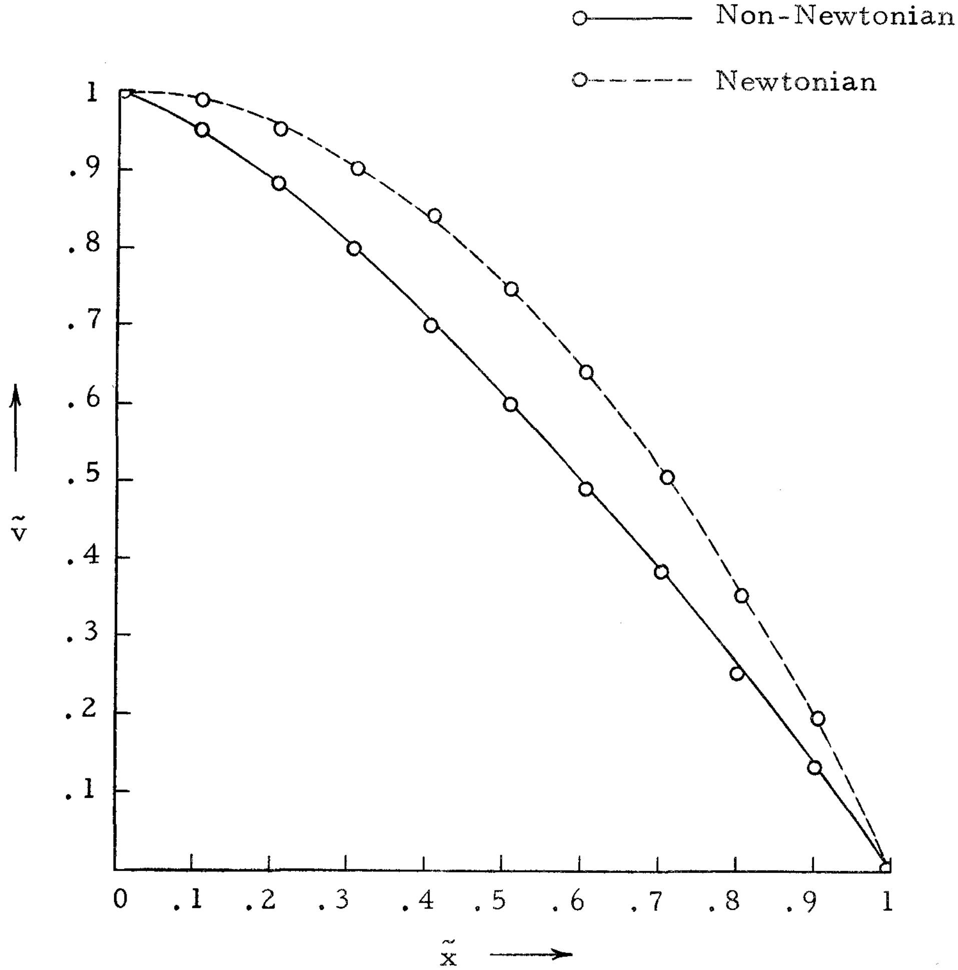


Figure 5.4. Velocity profiles in channel flow of the Newtonian and the non-Newtonian fluids.

To study the pressure difference across the channel width, we write from $(5.3.15)_1$

$$\tilde{p} = \tilde{x}^{2/3} + \tilde{m}\tilde{x}^{4/3} + \tilde{n}\tilde{x}^{2}, \tag{5.3.21}$$

where

$$\tilde{p} = \frac{p(x, z) - p(0, z)}{\frac{\alpha}{2}\left(\frac{d\Delta p}{\alpha\beta}\right)^{2/3}},$$

$$\tilde{m} = \beta^{2}\left(\frac{d\Delta p}{\alpha\beta}\right)^{2/3}, \tag{5.3.22}$$

$$\tilde{n} = \frac{2\gamma}{\alpha}\left(\frac{d\Delta p}{\alpha\beta}\right)^{4/3},$$

and $$\tilde{x} = \frac{x}{d}$$

are the dimensionless quantities.

We have plotted the graph of the radial pressure variations for various values of $\tilde{m}$ and $\tilde{n}$ to show the dependence of the normal stress effects on the rheological constants α, β and γ (Figure 5.5, p. 65). A close inspection of the graph reveals that owing to the pressure variation across the width of the channel, there would result a flow normal to the plates. But no such flow actually occurs, since the plates are held in position by the application of external normal forces. Further examination of the graph shows some more interesting features of the pressure variation which occurs in the absence of external normal forces. For positive values of the

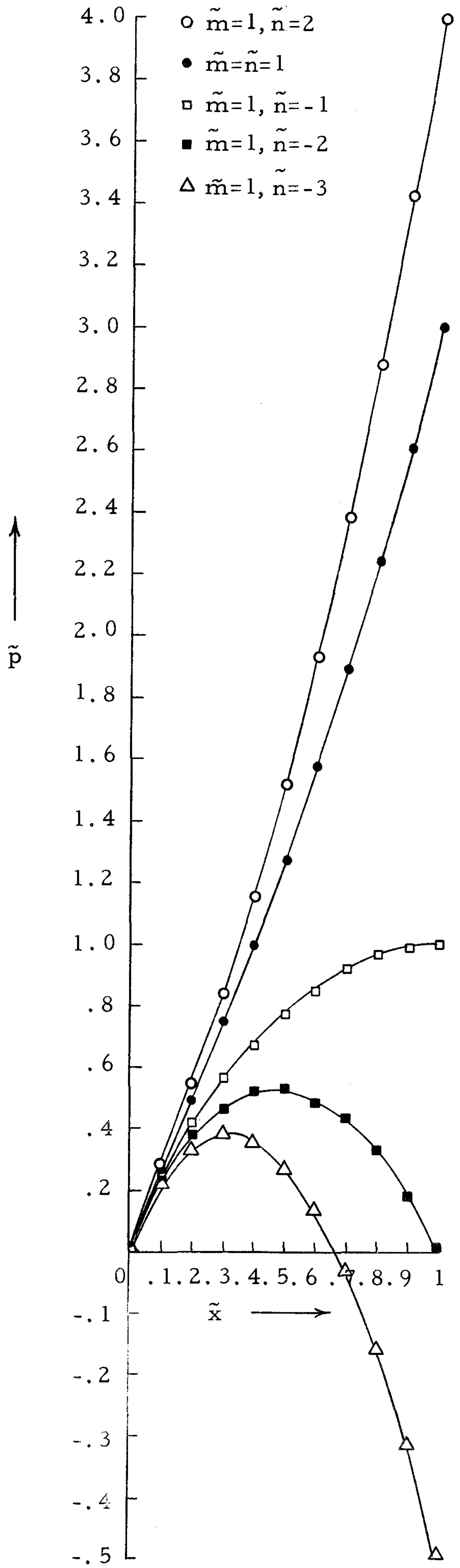


Figure 5.5 Pressure variation in channel flow.

dimensionless parameters $\tilde{m}$ and $\tilde{n}$, the pressure increases gradually as we proceed outwards from the center of the channel. The situation is, however, different when we take $\tilde{n}$ negative. For example, for $\tilde{m} = 1$ and $\tilde{n} = -1$, we again have a gradual pressure increase as before. But for $\tilde{m}=1$ and $\tilde{n}=-2$, the pressure first increases for part of the channel width and then decreases for the remaining part in such a way that it becomes the same on the channel walls as at the center of the channel. Again, for $\tilde{m}=1$ and $\tilde{n}=-3$, the pressure first increases for part of the channel width and then decreases as we proceed away from the center, so that it becomes less on the walls than at the center of the channel. We thus see that if we keep the dimensionless parameter $\tilde{m}$ fixed and assign different positive and negative values to the other parameter, that is $\tilde{n}$, we notice marked fluctuations in the behavior of pressure across the channel width. This in turn shows how different values of the rheological constants affect the variation in pressure in the channel flow. These phenomena are characteristic of non-Newtonian fluids only and do not occur in the Newtonian case.

5.4. Simple Shearing Flow

Formulation of the Problem

This is a steady rectilinear flow between two infinite parallel plates at a distance h apart, one of which is at rest and the other is moving in a direction parallel to itself with a constant velocity V (Figure 5.6). We now assume that there is no pressure gradient in the direction of flow. The axes of coordinates are so chosen that the origin lies on the plate at rest; the positive direction of the z-axis is the direction of flow, and the equation of the plate at rest is $x=0$, and that of the moving plate is $x=h$. The velocity field is given by

$$\begin{aligned} v_x &= 0 \quad , \\ v_y &= 0 \quad , \\ v_z &= v(x) \quad . \end{aligned} \tag{5.4.1}$$

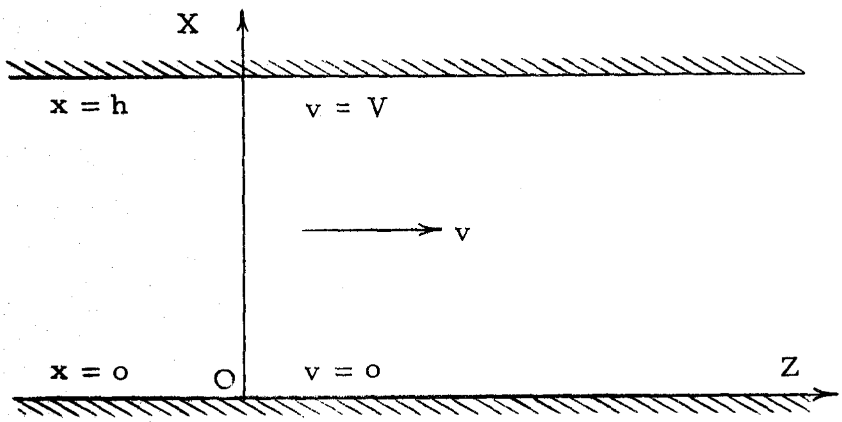


Figure 5.6. Simple shearing flow.

Equations of Motion

The equations of motion in this case are the same as in the two previous cases of flow.

Boundary Conditions

Assuming that the fluid adheres to the plates we have

$$\begin{aligned} v(0) &= 0 \\ v(h) &= V \neq 0 . \end{aligned} \tag{5.4.2}$$

Solution of the Equations

Velocity Field. From (5.2.16) we obtain after setting $\Delta p = 0$,

$$v' = \left(\frac{k_1}{\alpha\beta}\right)^{1/3} . \tag{5.4.3}$$

Integrating we get

$$v = \left(\frac{k_1}{\alpha\beta}\right)^{1/3} x + k_2 , \tag{5.4.4}$$

where k_2 is a constant of integration.

Using the boundary conditions (5.4.2) we get

$$v = \frac{V}{h} x . \tag{5.4.5}$$

Pressure Field and Stress Distribution. Since $\Delta p=0$, we see from (5.2.32) that the pressure p is constant everywhere in the flow.

From (5.4.5),

$$v' = \frac{V}{h} = \text{constant} . \qquad (5.4.6)$$

Substituting the value of v' from (5.4.6) in (5.2.5) we obtain

$$t_{xx} = -p + \frac{\alpha}{2}\left(1 + \beta^2 \frac{V^2}{h^2}\right)\frac{V^2}{h^2} + \gamma \frac{V^6}{h^6} ,$$

$$t_{xy} = -p ,$$

$$t_{zz} = -p + \frac{\alpha}{2}\left(1 + \beta^2 \frac{V^2}{h^2}\right)\frac{V^2}{h^2} ,$$

$$t_{zx} = \alpha\beta \frac{V^3}{h^3} , \qquad (5.4.7)$$

$$t_{yz} = 0 ,$$

$$t_{xy} = 0 .$$

Discussion of the Results

We already know that in simple shearing flows of both the Newtonian and the non-Newtonian fluids, the velocity profiles are linear. This is also confirmed by our investigation. A graph of the velocity profile is plotted in Figure 5.7, p. 70 . We further note from (5.4.6) that the shear rate v' remains constant.

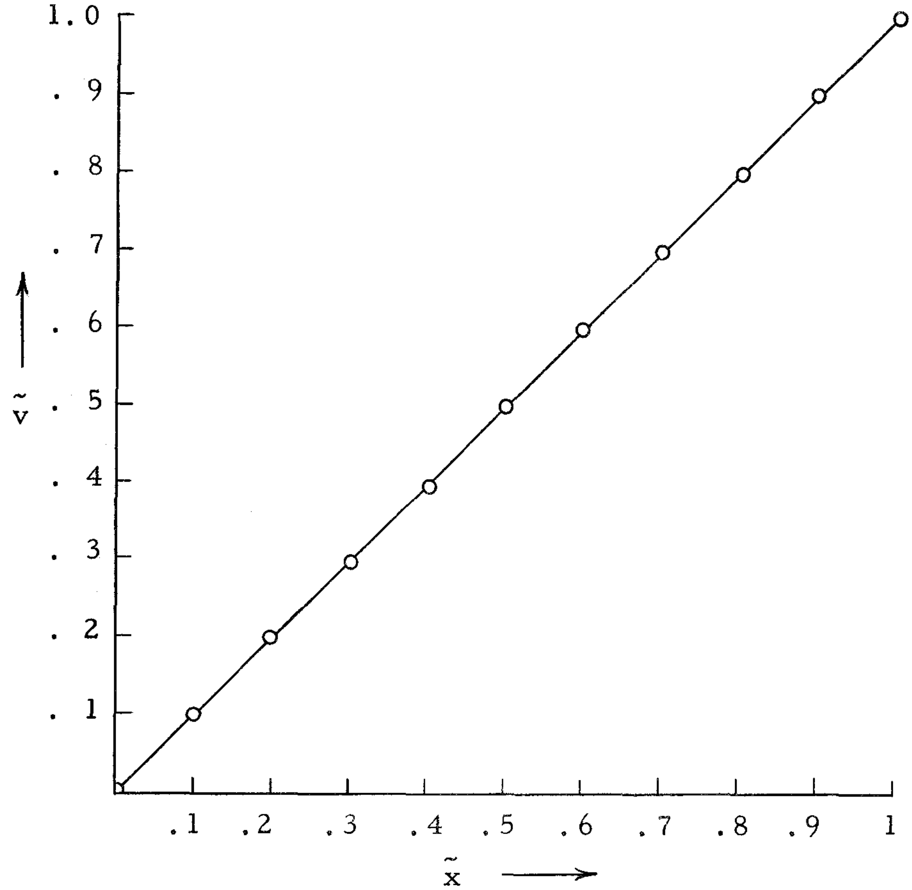


Figure 5.7. Velocity profile in simple shearing flow.

Equations (5.4.7) show that in order to maintain a shearing flow between two infinite parallel plates, shearing forces t_{zx} alone are not sufficient but we must, in addition, apply forces normal to the plates. These remarks also apply to the two previous cases of rectilinear flow. This normal stress phenomena does not occur in the rectilinear flow of a Newtonian fluid.

We further note that the normal stresses in all the three cases of rectilinear flow are even functions of the rate of shear, whereas shearing stress is an odd function. This is quite natural because reversing the rate of shear, that is, changing v' to $-v'$, would not affect the normal stresses whereas it would reverse the direction of flow and consequently also reverse the tangential force.

After having studied three types of rectilinear flows of a non-Newtonian fluid, we would like to make another important and interesting observation. We know from the classical theory (Landau and Lifshitz, 1959; Schlichting, 1960) that the velocity profiles in the rectilinear flows of a Newtonian fluid are given by

$$v = \frac{V}{h} x - \frac{\Delta p}{2\mu} (hx - x^2), \quad \text{(for generalized rectilinear flow)}$$

$$v = -\frac{\Delta p}{2\mu} (hx - x^2), \quad \text{(for channel flow)}$$

and $$v = \frac{V}{h} x . \quad \text{(for simple shearing flow)}$$

It is evident that in this case, the velocity profile for the generalized rectilinear flow can be obtained by a mere superposition of the velocity profiles for the channel flow and the simple shearing flow. But, from the corresponding velocity profiles in rectilinear flows of a non-Newtonian fluid, viz,

$$v = \frac{3}{4}\left(\frac{\Delta p}{\alpha\beta}\right)^{1/3}\left[x + \frac{k_1}{\Delta p}\right]^{4/3} + k_2 ,$$

$$v = -\frac{3}{4}\left(\frac{\Delta p}{\alpha\beta}\right)^{1/3}\left[\left(\frac{h}{2}\right)^{4/3} - \left(x-\frac{h}{2}\right)^{4/3}\right] ,$$

and

$$v = \frac{V}{h}x ,$$

we see that the velocity profile for the generalized rectilinear flow can no longer be obtained by an application of the superposition principle. Such a situation is quite characteristic of nonlinear problems of continuous media.

CHAPTER 6

HELICAL FLOWS

6. 1. Preliminary Remarks

This chapter is devoted to the discussion of the generalized helical flow and its two important special cases. Section 2 deals with the Poiseuille-Couette flow. Since helical flow with narrow annular gap is of some practical importance, such as in lubrication theory, we study it in detail. In Sections 3 and 4 a detailed treatment of Poiseuille and Couette flows respectively is presented. Besides other non-Newtonian effects, the swelling and thinning in the case of Poiseuille flow and climbing in Couette as also the pressure variations are discussed at sufficient length. The velocity profiles and pressure variations are also shown graphically.

6. 2. Poiseuille-Couette Flow

Formulation of the Problem

We now consider a steady combined axial and tangential flow of a mass of fluid in the annular region between two infinite coaxial cylinders (Figure 6. 1). We assume a nonvanishing pressure gradient Δp along the axis of the cylinders and also that the outer cylinder is rotating with constant angular velocity Ω. The velocity field in cylindrical coordinates $(r, \theta, z$ with z taken along the axis of the cylinders) is given by

$$v_r = 0$$

$$v_\theta = r\omega(r) , \qquad (6.2.1)$$

$$v_z = u(r) ,$$

where $\omega(r) \equiv \frac{d\theta}{dt}$ is the angular velocity at a radial distance r from the axis of rotation.

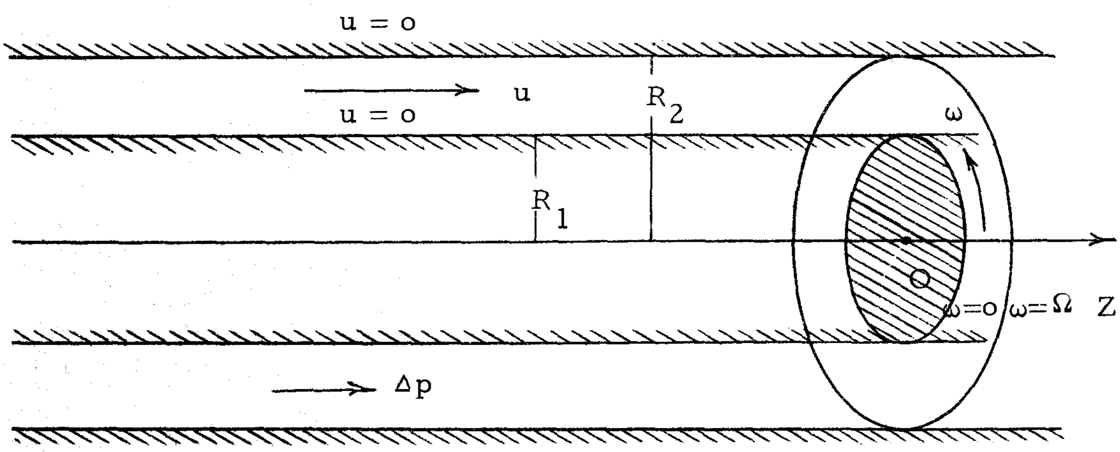


Figure 6.1. Poiseuille-Couette flow.

Equations of Motion

Using (6.2.1) we get the following expressions for the first and the second deformation-rate tensors d^i_j and b^i_j respectively:

$$\left\| d^i_j \right\| = \left\| \begin{matrix} 0 & d^1_2 & d^1_3 \\ d^2_1 & 0 & 0 \\ d^3_1 & 0 & 0 \end{matrix} \right\| ,$$

(6. 2. 2)

$$= \left\| \begin{matrix} 0 & \frac{r^2\omega'}{2} & \frac{u'}{2} \\ \frac{\omega'}{2} & 0 & 0 \\ \frac{u'}{2} & 0 & 0 \end{matrix} \right\| ,$$

$$\left\| b^i_j \right\| = \left\| \begin{matrix} b^1_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| ,$$

(6. 2. 3)

$$= \left\| \begin{matrix} (u'^2 + r^2\omega'^2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

where $u' \equiv \frac{\partial u}{\partial r}$ and $\omega' \equiv \frac{\partial \omega}{\partial r}$.

Substituting the values of II_D and I_B from $(6.2.2)_1$ and $(6.2.3)_1$

in the constitutive equation (4. 4. 10) we have

$$t^i_j = -p\,\delta^i_j + 8\alpha\beta\,(d^1_3 d^3_1 + d^1_2 d^2_1)\,d^i_j + 2\alpha[1 + 4\beta^2(d^1_3 d^3_1 + d^1_2 d^2_1)]\,d^i_\ell d^\ell_j$$

$$+ \gamma\, b^1_1\, b^i_\ell\, b^\ell_j\,. \qquad (6.\,2.\,4)$$

Using $(6.\,2.\,2)_2$ and $(6.\,2.\,3)_2$ in (6. 2. 4) we obtain the stress components:

$$t_{rr} = -p + \frac{\alpha}{2}[1+\beta^2(u'^2+r^2\omega'^2)]\,(u'^2+r^2\omega'^2) + \gamma\,(u'^2+r^2\omega'^2)^3,$$

$$t_{\theta\theta} = -p + \frac{\alpha}{2}[1+\beta^2(u'^2+r^2\omega'^2)]\,r^2\omega'^2\,,$$

$$t_{zz} = -p + \frac{\alpha}{2}[1+\beta^2(u'^2+r^2\omega'^2)]\,u'^2\,, \qquad (6.\,2.\,5)$$

$$t_{r\theta} = \alpha\beta\,(u'^2+r^2\omega'^2)\,r\omega'\,,$$

$$t_{\theta z} = \frac{\alpha}{2}[1+\beta^2(u'^2+r^2\omega'^2)]\,u'r\omega'\,,$$

$$t_{zr} = \alpha\beta\,(u'^2+r^2\omega'^2)\,u'\,.$$

The equations of motion (2. 4. 9) now become

$$\frac{\partial(t_{rr})}{\partial r} + \frac{t_{rr} - t_{\theta\theta}}{r} = -\rho\, r\omega^2 ,$$

$$\frac{\partial(t_{r\theta})}{\partial r} + 2\frac{t_{r\theta}}{r} = 0 , \tag{6.2.6}$$

$$\frac{\partial(t_{zr})}{\partial r} + \frac{t_{zr}}{r} - \frac{\partial p}{\partial z} = 0 .$$

Boundary Conditions

If R_1 and R_2 are the radii of the inner and the outer cylinders respectively, then the no slip condition on the cylindrical walls requires

$$u(R_1) = u(R_2) = 0 \tag{6.2.7}$$

and

$$\omega(R_1) = 0$$
$$\omega(R_2) = \Omega \neq 0 . \tag{6.2.8}$$

Solution of the Equations

Pressure Gradient. Differentiating $(6.2.6)_1$ with respect to z, we get

$$\frac{\partial^2 p}{\partial z \partial r} = 0 . \tag{6.2.9}$$

From equation (6.2.9) it follows that $\frac{\partial p}{\partial z}$ is not a function of r;

it may be a function of z. But $(6.2.6)_3$ shows that $\frac{\partial p}{\partial z}$ is a function of r and not of z. Since r and z are mutually independent, $\frac{\partial p}{\partial z}$ must be a constant. We denote this constant by Δp.

Velocity Field. Integrating $(6.2.6)_3$ we get

$$t_{zr} = \frac{\Delta p}{2} r + \frac{A}{r}, \tag{6.2.10}$$

where A is a constant of integration.

After integration $(6.2.6)_2$ yields

$$t_{r\theta} = \frac{B}{r^2}, \tag{6.2.11}$$

where B is a constant of integration.

From $(6.2.5)_6$ and (6.2.10) we obtain

$$\alpha\beta\,(u'^2 + r^2\omega'^2)u' = \frac{\Delta p}{2} r + \frac{A}{r}. \tag{6.2.12}$$

Again from $(6.2.5)_4$ and (6.2.11) we get

$$\alpha\beta\,(u'^2 + r^2\omega'^2)r\omega' = \frac{B}{r^2}. \tag{6.2.13}$$

From (6.2.12) and (6.2.13), after squaring and adding, we get

$$\alpha^2\beta^2(u'^2 + r^2\omega'^2)^3 = \left(\frac{\Delta p}{2} r + \frac{A}{r}\right)^2 + \left(\frac{B}{r^2}\right)^2,$$

whence

$$u'^2 + r^2\omega'^2 = \frac{1}{\alpha^{2/3}\beta^{2/3}}\left[\left(\frac{\Delta p}{2}r + \frac{A}{r}\right)^2 + \left(\frac{B}{r^2}\right)^2\right]^{1/3} . \quad (6.2.14)$$

Using (6. 2. 14) in (6. 2. 12) and (6. 2. 13) we get

$$u' = \frac{\left(\frac{\Delta p}{2}r + \frac{A}{r}\right)}{\alpha^{1/3}\beta^{1/3}\left[\left(\frac{\Delta p}{2}r + \frac{A}{r}\right)^2 + \left(\frac{B}{r^2}\right)^2\right]^{1/3}} \quad (6.2.15)$$

and

$$\omega' = \frac{\frac{B}{r^3}}{\alpha^{1/3}\beta^{1/3}\left[\left(\frac{\Delta p}{2}r + \frac{A}{r}\right)^2 + \left(\frac{B}{r^2}\right)^2\right]^{1/3}} \quad (6.2.16)$$

Integration of (6. 2. 15) and (6. 2. 16) and application of the boundary conditions $(6.2.7)_1$ and $(6.2.8)_1$ yields

$$u = \int_{R_1}^{r} \frac{\left(\frac{\Delta p}{2}\xi + \frac{A}{\xi}\right)}{\alpha^{1/3}\beta^{1/3}\left[\left(\frac{\Delta p}{2}\xi + \frac{A}{\xi}\right)^2 + \left(\frac{B}{\xi^2}\right)^2\right]^{1/3}}\, d\xi \quad (6.2.17)$$

and

$$\omega = \int_{R_1}^{r} \frac{\frac{B}{\xi^3}}{\alpha^{1/3}\beta^{1/3}\left[\left(\frac{\Delta p}{2}\xi + \frac{A}{\xi}\right)^2 + \left(\frac{B}{\xi^2}\right)^2\right]^{1/3}}\, d\xi . \quad (6.2.18)$$

Application of the remaining boundary conditions gives

$$0 = \int_{R_1}^{R_2} \frac{\left(\frac{\Delta p}{2}\,\xi + \frac{A}{\xi}\right)}{\alpha^{1/3}\beta^{1/3}\left[\left(\frac{\Delta p}{2}\,\xi + \frac{A}{\xi}\right)^2 + \left(\frac{B}{\xi^2}\right)^2\right]^{1/3}}\, d\xi\,, \qquad (6.2.19)$$

$$\Omega = \int_{R_1}^{R_2} \frac{\frac{B}{\xi^3}}{\alpha^{1/3}\beta^{1/3}\left[\left(\frac{\Delta p}{2}\,\xi + \frac{A}{\xi}\right)^2 + \left(\frac{B}{\xi^2}\right)^2\right]^{1/3}}\, d\xi\,. \qquad (6.2.20)$$

Equations (6.2.19) and (6.2.20) can be solved numerically for A and B. One method would be by trial and error. We would assume reasonable values of A and B and perform numerical integration. We will have to repeat the trials until we obtain such values of A and B as satisfy (6.2.19) and (6.2.20). We can then use the values of A and B so obtained in (6.2.17) and (6.2.18) and carry out numerical integration to find u and ω. Since the number of trials may be quite large, the digital computer will have to be used in any practical problem.

Pressure Field and Stress Distribution. Substituting $(6.2.5)_{1,2}$ in $(6.2.6)_1$ we get

$$\frac{\partial p}{\partial r} = \frac{\alpha}{2}\,\frac{\partial}{\partial r}\left[\{1+\beta^2(u'^2+r^2\omega'^2)\}(u'^2+r^2\omega'^2)\right] + \gamma\,\frac{\partial}{\partial r}(u'^2+r^2\omega'^2)^3$$

$$+ \frac{\alpha}{2r}\left[\{1+\beta^2(u'^2+r^2\omega'^2)\}\,u'^2\right] + \frac{\gamma}{r}(u'^2+r^2\omega'^2)^3 + \rho r\omega^2.$$

Integrating the above equation between R_1 and r, $(R_1 \le r \le R_2)$, we obtain

$$p(r,z) = p(R_1,z) + \Big[\frac{\alpha}{2}\{(u'^2+r^2\omega'^2)+\beta^2(u'^2+r^2\omega'^2)^2\}+\gamma(u'^2+r^2\omega'^2)^3\Big]_{r=R_1}^{r=r}$$

$$+\int_{R_1}^{r} \frac{\alpha}{2\xi}[\{1+\beta^2(u'^2+\xi^2\omega'^2)\}u'^2]\, d\xi$$

$$+\int_{R_1}^{r} \frac{\gamma}{\xi}(u'^2+\xi^2\omega'^2)^3 d\xi + \int_{R_1}^{r} \rho\xi\omega^2 d\xi\,, \qquad (6.2.21)$$

where u' and ω' are given by (6. 2. 15) and (6. 2. 16) respectively and $p(R_1, z)$ is assumed to be known.

Substitution of this value of $p(r,z)$ into $(6.2.5)_{1,2,3}$ determines the normal stresses.

Torque. The torque M per unit height required to maintain a steady rotation of the fluid inside the cylinder of radius r, $(R_1 \le r \le R_2)$, is given by

$$\begin{aligned} M &= 2\pi r \cdot r \cdot t_{r\theta} \\ &= 2\pi B\,, \end{aligned} \qquad (6.2.22)$$

after using (6. 2. 11). Since M is independent of r, the torque per unit height on the outer cylinder is given by (6. 2. 22). Incidentally, the constant B is now assigned a physical meaning in terms of torque.

Volumetric Flow Rate. The volumetric flow rate Q through the annulus is given by

$$
\begin{aligned}
Q &= \int_{R_1}^{R_2}\int_0^{2\pi} u(r)\, r d\theta dr \\
&= 2\pi \int_{R_1}^{R_2} u(r)\, r\, dr \,.
\end{aligned}
\qquad (6.2.23)
$$

So far we have discussed the helical flow problem in a very general way. We would now like to discuss a special case of some physical interest.

Helical Flow with a Narrow Annular Gap

The case of helical flow when the gap between the two cylinders is very small is of special interest in the theory of lubrication because the flow pattern of a lubricant in the narrow clearance between the bearing and the journal is, for all practical purposes, helical. To discuss this flow, our natural assumption would be,

$$
\frac{R_2 - R_1}{R_1} \ll 1 \qquad (6.2.24)
$$

where R_1 and R_2 are the radii of the inner and outer cylinders respectively.

We first introduce the dimensionless variables $\tilde{r}$, $\tilde{u}$, $\tilde{\omega}$ defined by

$$\tilde{r} = \frac{r}{R_1} ,$$

$$\tilde{u} = \frac{u}{U} , \qquad (6.2.25)$$

$$\tilde{\omega} = \frac{\omega}{\Omega} ,$$

where U may be taken as the average axial velocity.

Then the dimensionless forms of (6.2.17), (6.2.18), (6.2.19) and (6.2.20) are respectively

$$\tilde{u} = \int_1^{\tilde{r}} \frac{\tilde{\gamma}\,(\tilde{\xi} + \frac{\tilde{A}}{\tilde{\xi}})}{[(\tilde{\xi} + \frac{\tilde{A}}{\tilde{\xi}})^2 + (\frac{\tilde{B}}{\tilde{\xi}^2})^2]^{1/3}}\, d\tilde{\xi} , \qquad (6.2.26)$$

$$\tilde{\omega} = \int_1^{\tilde{r}} \frac{\tilde{\delta}\,\frac{\tilde{B}}{\tilde{\xi}^3}}{[(\tilde{\xi} + \frac{\tilde{A}}{\tilde{\xi}})^2 + (\frac{\tilde{B}}{\tilde{\xi}^2})^2]^{1/3}}\, d\tilde{\xi} , \qquad (6.2.27)$$

$$0 = \int_1^{\tilde{R}} \frac{(\tilde{\xi} + \frac{\tilde{A}}{\tilde{\xi}})}{[(\tilde{\xi} + \frac{\tilde{A}}{\tilde{\xi}})^2 + (\frac{\tilde{B}}{\tilde{\xi}^2})^2]^{1/3}}\, d\tilde{\xi} , \qquad (6.2.28)$$

$$1 = \int_1^{\tilde{R}} \frac{\tilde{\delta}\,\frac{\tilde{B}}{\tilde{\xi}^3}}{[(\tilde{\xi} + \frac{\tilde{A}}{\tilde{\xi}})^2 + (\frac{\tilde{B}}{\tilde{\xi}^2})^2]^{1/3}}\, d\tilde{\xi} , \qquad (6.2.29)$$

where

$$\tilde{A} = \frac{2A}{R_1^2 \Delta p}$$

$$\tilde{B} = \frac{2B}{R_1^3 \Delta p}$$

$$\tilde{\gamma} = \left(\frac{R_1^4 \Delta p}{2\alpha\beta U^3}\right)^{1/3} \tag{6.2.30}$$

$$\tilde{\delta} = \left(\frac{R_1 \Delta p}{2\alpha\beta \Omega^3}\right)^{1/3}$$

$$\tilde{R} = \frac{R_2}{R_1} > 1$$

are the dimensionless quantities.

From (6. 2. 28) we see that $\tilde{A} < 0$, so that $\tilde{A} = -s^2$ (say). [A $\neq$ 0, since in that case the integrand would be positive throughout the interval $1 \leq \tilde{\xi} \leq \tilde{R}$. Also, A cannot be > 0, since in this case the integrand would become positive again throughout the same interval, and hence (6. 2. 28) will not be satisfied.] Again, $\tilde{\xi}^2 - s^2$ has to be negative for part of the interval $[1, \tilde{R}]$ and positive for the remaining part in order to satisfy (6. 2. 28).

Hence

$$1 < s^2 < \tilde{R}^2. \tag{6.2.31}$$

Also

$$1 \leq \tilde{\xi}^2 \leq \tilde{R}^2. \tag{6.2.32}$$

Since by (6. 2. 24), $\tilde{R} - 1 << 1$, that is, the interval $[1, \tilde{R}]$ is very small we have $|\tilde{\xi}^2 - s^2| << 1$ and consequently

$$|\tilde{\xi} + \frac{A}{\tilde{\xi}}| << 1 . \qquad (6.\,2.\,33).$$

From (6. 2. 29) we see that

$$1 \leq \tilde{R}^{4/3}(\tilde{R} - 1)|\tilde{\delta}|\;|\tilde{B}^{1/3}| . \qquad (6.\,2.\,34)$$

Now $\tilde{R} - 1 << 1$, by hypothesis. Further, in the flow of a lubricant between the bearing and the journal, it is reasonable to assume that the pressure gradient Δp is small whereas the angular velocity Ω is large. In view of the above situation and the inequality (6. 2. 34), δ can be chosen sufficiently small so that $|\tilde{B}|$ and consequently $(\frac{\tilde{B}}{\tilde{\xi}^2})^2$ becomes large enough as compared with $(\tilde{\xi} + \frac{\tilde{A}}{\tilde{\xi}})^2$ in the interval $[1, \tilde{R}]$.

By the foregoing reasoning, we can now ignore the second order terms in $(\tilde{\xi} + \frac{\tilde{A}}{\tilde{\xi}})$, occurring in (6. 2. 26) to (6. 2. 29). Thus we get from (6. 2. 28) and (6. 2. 29)

$$0 = \int_1^{\tilde{R}} (\tilde{\xi}^{\,7/3} \quad \tilde{A}\,\tilde{\xi}^{1/3})d\tilde{\xi} \qquad (6.\,2.\,35)$$

and

$$1 = \int_1^{\tilde{R}} \tilde{\delta}\,\tilde{B}^{\,1/3}\,\tilde{\xi}^{-5/3}\,d\tilde{\xi} . \qquad (6.\,2.\,36)$$

Integrating (6. 2. 35) and (6. 2. 36) and solving for $\tilde{A}$ and $\tilde{B}$ we get

$$\tilde{A} = \frac{2(1 - \tilde{R}^{10/3})}{5(\tilde{R}^{4/3} - 1)} , \qquad (6.2.37)$$

and

$$\tilde{B} = \frac{8}{27(\tilde{\delta})^3 [1 - \tilde{R}^{-2/3}]^3} . \qquad (6.2.38)$$

Again in a similar way, we obtain from (6.2.26) and (6.2.27)

$$\tilde{u} = \int_1^{\tilde{r}} \frac{\tilde{\gamma}}{\tilde{B}^{2/3}} (\tilde{\xi}^{7/3} + \tilde{A}\tilde{\xi}^{1/3}) \, d\tilde{\xi} , \qquad (6.2.39)$$

and

$$\tilde{\omega} = \int_1^{\tilde{r}} \tilde{\delta}\tilde{B}^{1/3} \tilde{\xi}^{-5/3} \, d\tilde{\xi} . \qquad (6.2.40)$$

Integrating we get

$$\tilde{u} = \frac{27}{40} \tilde{\gamma} \tilde{\delta}^2 (1 - \tilde{R}^{-2/3})^2 (\tilde{R}^{10/3} - 1) \left[\frac{\tilde{r}^{10/3} - 1}{\tilde{R}^{10/3} - 1} - \frac{\tilde{r}^{4/3} - 1}{\tilde{R}^{4/3} - 1} \right] , \qquad (6.2.41)$$

and

$$\tilde{\omega} = \frac{[1 - \tilde{r}^{-2/3}]}{[1 - \tilde{R}^{-2/3}]} . \qquad (6.2.42)$$

Equations (6.2.41) and (6.2.42) give the velocity profile for the flow.

Discussion of the Results

In the solution for the Poiseuille-Couette flow, we notice one important fact that the axial and the angular velocities are not independent of each other, so that the solution for this flow can not be obtained from the solutions for the Poiseuille and Couette flows separately. In the Newtonian case, however, the two velocities are independent and the superposition principle is applicable. This observation is in conformity with a similar conclusion holding in the case of other physical problems too.

The angular velocity as given by (6.2.42) for a helical flow of a lubricant in the narrow gap between the journal and the bearing is found to be the same as the velocity in the Couette flow. But the axial velocity of the helical flow under similar conditions as given by (6.2.41) is found to be different from that of the Poiseuille flow and thus is affected by its angular velocity.

6.3. Poiseuille Flow

Formulation of the Problem

We consider a steady axial flow in an infinite circular cylinder produced by the application of a nonvanishing pressure gradient Δp along the axis of the cylinder (Figure 6.2). The velocity field in cylindrical coordinates (r, θ, z) is given by

$$\begin{aligned} v_r &= 0, \\ v_\theta &= 0, \qquad (6.3.1) \\ v_z &= u(r) . \end{aligned}$$

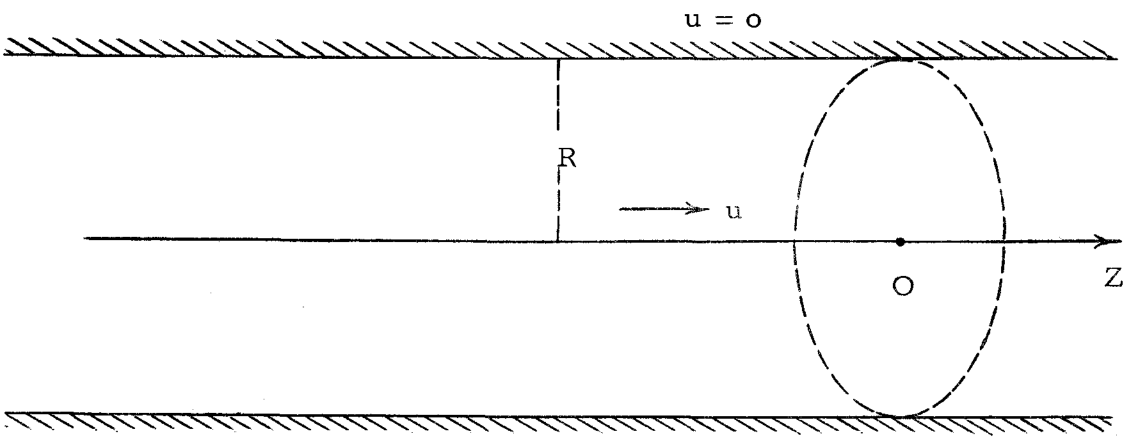


Figure 6.2. Poiseuille flow.

Equations of Motion

Since in this flow in addition to v_r, the tangential velocity v_θ also vanishes, the first and the second deformation-rate tensors $d^i_{.j}$ and $b^i_{.j}$ respectively take the following forms:

$$\left\|d^i_{.j}\right\| = \begin{Vmatrix} 0 & 0 & d^1_3 \\ 0 & 0 & 0 \\ d^3_1 & 0 & 0 \end{Vmatrix} = \begin{Vmatrix} 0 & 0 & \frac{u'}{2} \\ 0 & 0 & 0 \\ \frac{u'}{2} & 0 & 0 \end{Vmatrix}, \tag{6.3.2}$$

$$\left\| b^i_j \right\| = \left\| \begin{matrix} b^1_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| \tag{6.3.3}$$

$$= \left\| \begin{matrix} u'^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| ,$$

where $u' = \frac{\partial u}{\partial r}$.

By using $(6.3.2)_1$ and $(6.3.3)_1$ in (4.4.10) we have the constitutive equation in the form

$$t^i_j = -p\ \delta^i_j + 8\alpha\beta\ d^1_3 d^3_1 d^i_j + 2\alpha\left[1+4\beta^2 d^1_3 d^3_1\right] d^i_\ell d^\ell_j$$

$$+ \gamma\ b^1_1\ b^i_\ell\ b^\ell_j . \tag{6.3.4}$$

With the help of $(6.3.2)_2$ and $(6.3.3)_2$ we get from (6.3.4) the expressions for the stresses:

$$\begin{aligned} t_{rr} &= -p + \frac{\alpha}{2}(1 + \beta^2 u'^2)\, u'^2 + \gamma u'^6, \\ t_{\theta\theta} &= -p , \\ t_{zz} &= -p + \frac{\alpha}{2}(1 + \beta^2 u'^2)\, u'^2 , \end{aligned} \tag{6.3.5}$$

$$t_{zr} = \alpha \beta u'^3 ,$$

$$t_{\theta z} = 0 ,$$

$$t_{r\theta} = 0 .$$

The equations of motion (2.4.9) now take the form

$$\frac{\partial(t_{rr})}{\partial r} + \frac{t_{rr} - t_{\theta\theta}}{r} = 0 ,$$

$$\frac{\partial(t_{zr})}{\partial r} + \frac{t_{zr}}{r} - \frac{\partial p}{\partial z} = 0 . \tag{6.3.6}$$

Boundary Conditions

If R is the radius of the cylinder, then the no-slip condition on the cylindrical wall requires

$$u(R) = 0 . \tag{6.3.7}$$

Solution of the Equations

Pressure Gradient. By the same argument as given in the case of generalized helical flow, we conclude that the axial pressure gradient Δp is constant.

Velocity Field. Integrating $(6.3.6)_2$ we obtain

$$t_{zr} = \frac{\Delta p}{2} r + \frac{A}{r} , \tag{6.3.8}$$

where A is a constant of integration.

Since the stresses are finite at $r=0$, we must have $A=0$ in (6.3.8), so that after substituting the value of t_{zr} from $(6.3.5)_4$ in (6.3.8) we have

$$\alpha\beta\, u'^3 = \frac{\Delta p}{2} r . \qquad (6.3.9)$$

Integrating and using the boundary condition we have

$$u = \frac{3}{4}\left(\frac{\Delta p}{2\alpha\beta}\right)^{1/3}(r^{4/3}-R^{4/3}) . \qquad (6.3.10)$$

Pressure Field and Stress Distribution. Substituting the values of t_{rr} and $t_{\theta\theta}$ from $(6.3.5)_{1,2}$ in (6.3.6) and then replacing u' by its value from (6.3.9) we get

$$\frac{\partial p}{\partial r} = \frac{\alpha}{2}\frac{\partial}{\partial r}\left[\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3} r^{2/3} + \beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3} r^{4/3}\right]$$

$$+ \frac{\alpha}{2}\left[\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3} r^{-1/3} + \beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3} r^{1/3}\right]$$

$$+ \gamma\frac{\partial}{\partial r}\left(\frac{\Delta p r}{2\alpha\beta}\right)^2 + \gamma\left(\frac{\Delta p}{2\alpha\beta}\right)^2 r .$$

Integrating the above equation between 0 and r we obtain

$$p(r,z) = p(0,z) + \frac{\alpha}{4}\left[5\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3} r^{2/3} + \frac{7}{2}\beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3} r^{4/3}\right]$$

$$+ \frac{3\gamma}{2}\left(\frac{\Delta p}{2\alpha\beta}\right)^2 r^2 . \qquad (6.3.11)$$

Substituting for u' from (6. 3. 9) into (6. 3. 5) we get the stress components:

$$t_{rr} = -p(r, z) + \frac{\alpha}{2} [(\frac{\Delta p}{2\alpha\beta})^{2/3} r^{2/3} + \beta^2 (\frac{\Delta p}{2\alpha\beta})^{4/3} r^{4/3}] + \gamma (\frac{\Delta p}{2\alpha\beta})^2 r^2 ,$$

$$t_{\theta\theta} = -p(r, z) ,$$

$$t_{zz} = -p(r, z) + \frac{\alpha}{2} [(\frac{\Delta p}{2\alpha\beta})^{2/3} r^{2/3} + \beta^2 (\frac{\Delta p}{2\alpha\beta})^{4/3} r^{4/3}],$$

$$t_{zr} = \frac{\Delta p}{2} r , \qquad (6. 3. 12)$$

$$t_{\theta z} = 0 ,$$

$$t_{r\theta} = 0 ,$$

where $p(r, z)$ is given by (6. 3. 11).

Volumetric Flow Rate. The volumetric flow rate Q through the pipe is given by

$$Q = \int_0^R \int_0^{2\pi} u(r) r \, d\theta \, dr$$

$$= -\frac{3}{10} \pi \left(\frac{\Delta p}{2\alpha\beta}\right)^{1/3} R^{10/3} . \qquad (6. 3. 13)$$

Discussion of the Results

In the Poiseuille flow of a Newtonian fluid, the velocity profile is given by

$$u = -\frac{\Delta p}{4\mu} (R^2 - r^2) \qquad (6. 3. 14)$$

where μ is the Newtonian viscosity; whereas for a non-Newtonian fluid it is given by (6. 3. 10). The nondimensional forms of (6. 3. 14) and (6. 3. 10) are respectively

$$\tilde{u} = 1 - \tilde{r}^2 \quad , \tag{6.3.15}$$

$$\tilde{u} = 1 - \tilde{r}^{4/3} \quad , \tag{6.3.16}$$

where

$$\tilde{u} = \frac{u}{u_{max}}$$

and

$$\tilde{r} = \frac{r}{R} \quad . \tag{6.3.17}$$

These are exactly similar to the ones that we found in the case of channel flow [cf (6. 3. 18) and (6. 3. 19)] . Therefore,the velocity curves in the case of Poiseuille flow are similar to those of the channel flow. Thus the Poiseuille flow, like the channel flow, also exhibits a flattening of the velocity profile for the non-Newtonian fluids which accounts for the viscoelastic nature of these fluids.

Again, like the channel flow, the shearing stress for the Poiseuille flow is a linear function of r, in spite of the fact that the velocity is nonlinear in r. Further, we need only to know the axial pressure gradient Δp to determine the shearing stress at a point in the flow region.

The Poiseuille formula

$$Q = \frac{-\pi R^4 \Delta p}{8\mu} \tag{6.3.18}$$

for the volumetric flow rate holds good for every Newtonian fluid. According to this formula, the volumetric flow rate is always proportional to the fourth power of the radius of the pipe. But experiments have shown that the flow of non-Newtonian fluids is not governed by this formula. This is what our analysis also confirms. In the case of the non-Newtonian fluid that we have considered here, we see from (6.3.13) that the volumetric flow rate is proportional to $R^{10/3}$ instead of R^4, R being the radius of the pipe.

To study the pressure variation in a Poiseuille flow, we first write (6.3.11) in the nondimensional form

$$\tilde{p} = \tilde{r}^{2/3} + \tilde{m}\tilde{r}^{4/3} + \tilde{n}\tilde{r}^2, \tag{6.3.19}$$

where

$$\begin{aligned} \tilde{p} &= \frac{p(r,z) - p(0,z)}{\frac{5a}{4}\left(\frac{R\Delta p}{2a\beta}\right)^{2/3}}, \\ \tilde{m} &= \frac{7\beta^2}{10}\left(\frac{R\Delta p}{2a\beta}\right)^{2/3}, \\ \tilde{n} &= \frac{6}{5}\frac{\gamma}{a}\left(\frac{R\Delta p}{2a\beta}\right)^{4/3}, \\ \tilde{r} &= \frac{r}{R} \end{aligned} \tag{6.3.20}$$

are the dimensionless quantities.

We notice that (6.3.19) is exactly similar to (5.3.17) of (rectilinear) channel flow. Therefore the graph in this case must be similar to that of the pressure difference in channel flow. Consequently the behavior of radial pressure in Poiseuille flow is similar to that of pressure across the width in the channel flow.

Swelling and Thinning in Poiseuille Flow

So far in our discussion of the Poiseuille flow, we have assumed the pipe to be of infinite length, thus ignoring the end effects altogether. Actually, however, the pipes are finite in length. Consequently, the results obtained in this section can be only approximations to what we observe in the experiments or other real situations. But even from these approximate results, we can derive interesting conclusions. Let us for example, assume that the fluid is emerging from an end of the pipe with a velocity given by (6.3.10). We assume further that the stresses at this exit are given by (6.3.12). If the exit section is assumed to be the plane $z=0$, then according to $(6.3.12)_3$

$$t_{zz} = -\frac{\alpha}{2}\left[\frac{3}{2}\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3} r^{2/3} + \frac{3}{4}\beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3} r^{4/3}\right] - \frac{3}{2}\gamma\left(\frac{\Delta p}{2\alpha\beta}\right)^2 r^2 - p(0,0)\,. \qquad (6.3.21)$$

Therefore the total normal stress at the exit section is

$$2\pi \int_0^R t_{zz} r dr = -\pi \alpha \left[\frac{9}{16}\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3} R^{8/3} + \frac{9}{40}\beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3} R^{10/3}\right]$$

$$- \frac{3}{4}\pi\gamma\left(\frac{\Delta p}{2\alpha\beta}\right)^2 R^4 - \pi R^2 p(0,0). \quad (6.3.22)$$

If the atmospheric pressure at the exit section is p_a, then the total atmospheric pressure is given by $\pi p_a R^2$. Balancing the forces at the exit section, we get

$$- \pi \alpha \left[\frac{9}{16}\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3} R^{8/3} + \frac{9}{40}\beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3} R^{10/3}\right]$$

$$- \frac{3}{4}\pi\gamma\left(\frac{\Delta p}{2\alpha\beta}\right)^2 R^4 - \pi R^2 p(0,0) = -\pi p_a R^2 \quad (6.3.23)$$

After simplification we have

$$p(0,0) = p_a - \alpha\left[\frac{9}{16}\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3} R^{2/3} + \frac{9}{40}\beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3} R^{4/3}\right]$$

$$- \frac{3}{4}\gamma\left(\frac{\Delta p}{2\alpha\beta}\right)^2 R^2. \quad (6.3.24)$$

Substituting the value of $p(0,0)$ from (6.3.24) into $(6.3.12)_1$ we get the radial normal stress (at $z=0$)

$$t_{rr} = -p_a + \frac{\alpha}{2}\left[\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3}\left(\frac{9}{8}R^{2/3} - \frac{3}{2}r^{2/3}\right) + \beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3}\left(\frac{9}{20}R^{4/3} - \frac{3}{4}r^{4/3}\right)\right]$$

$$+ \frac{\gamma}{2}\left(\frac{\Delta p}{2\alpha\beta}\right)^2\left(\frac{3}{2}R^2 - r^2\right). \quad (6.3.25)$$

The value of the normal stress t_{rr} at $r=R$ is then

$$t_{rr}(R) = -p_a - \frac{3\alpha}{4}\left[\frac{1}{4}\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3}R^{2/3} + \frac{1}{5}\beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3}R^{4/3}\right]$$

$$+ \frac{\gamma}{4}\left(\frac{\Delta p}{2\alpha\beta}\right)^2 R^2 . \qquad (6.3.26)$$

Setting $E = p_a + t_{rr}(R)$ we obtain from (6.3.26), at the exit section,

$$E = -\frac{3\alpha}{4}\left[\frac{1}{4}\left(\frac{\Delta p}{2\alpha\beta}\right)^{2/3}R^{2/3} + \frac{1}{5}\beta^2\left(\frac{\Delta p}{2\alpha\beta}\right)^{4/3}R^{4/3}\right] + \frac{\gamma}{4}\left(\frac{\Delta p}{2\alpha\beta}\right)^2 R^2$$

$$(6.3.27)$$

$$= -\frac{3\alpha^{1/3}}{16\beta^{2/3}}\left(\frac{R\Delta p}{2}\right)^{2/3} - \frac{3\beta^{2/3}}{20\alpha^{1/3}}\left(\frac{R\Delta p}{2}\right)^{4/3} + \frac{\gamma}{4\alpha^2\beta^2}\left(\frac{R\Delta p}{2}\right)^2$$

$$= -c\chi - b\chi^2 + a\chi^3, \qquad (6.3.28)$$

where

$$c = \frac{3\alpha^{1/3}}{16\beta^{2/3}} ,$$

$$b = \frac{3}{20}\frac{\beta^{2/3}}{\alpha^{1/3}} ,$$

$$(6.3.29)$$

$$a = \frac{\gamma}{4\alpha^2\beta^2} ,$$

$$\chi = \left(\frac{R\Delta p}{2}\right)^{2/3} .$$

Now if $E < 0$, then $-t_{rr}(R) > p_a$, which means that the normal stress exerted by the fluid on the wall of the pipe exceeds the atmospheric pressure. In other words, the fluid stream emerging at the pipe exit swells (Figure 6.3, p. 99). Thus we can expect swelling when

$$a\chi^3 - b\chi^2 - c\chi < 0,$$

that is, when

$$\frac{b-\sqrt{b^2+4ca}}{2a} < \chi < \frac{b+\sqrt{b^2+4ca}}{2a}. \qquad (6.3.30)$$

Similarly if $E > 0$, then we will have thinning instead of swelling at the exit (Figure 6.3, p. 99). In other words, thinning will occur when

$$a\chi^3 - b\chi^2 - c\chi > 0,$$

that is,

$$\chi > \frac{b+\sqrt{b^2+4ca}}{2a}$$

or

$$\chi < \frac{b-\sqrt{b^2+4ca}}{2a}. \qquad (6.3.31)$$

Relations (6.3.30) and (6.3.31) can be expressed in terms of α, β, γ, Δp and R by using (6.3.29). This shows that swelling and thinning of the fluid at the exit depends not only on the fluid properties α, β, γ but also on the pressure gradient and radius of the pipe.

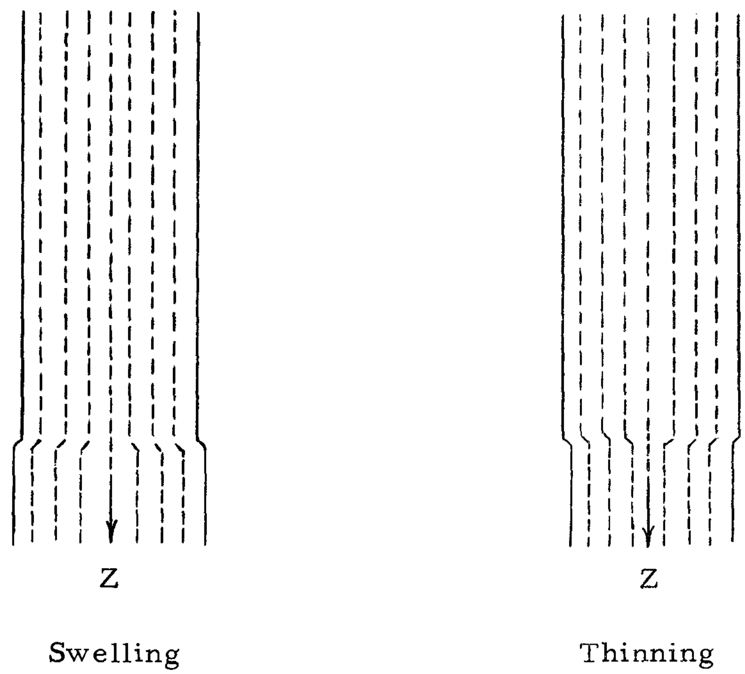


Figure 6.3. Swelling and thinning in Poiseuille flow.

We further note from (6.3.28) that but for the second deformation-rate term $a\chi^3$, there would only be swelling phenomena. The role of this term is thus very significant in controlling the behavior of the fluid.

In the foregoing discussion we have assumed that the rheological constant γ is positive. If, however, we allow the possibility of γ being negative, then we see from (6.3.27) that E is always negative. Consequently we can expect only swelling.

In the case of a Newtonian fluid

$$E = 0$$

Hence, there is neither swelling nor thinning of a Newtonian fluid at the exit of the pipe.

This swelling or thinning phenomenon occurring at the exit of a pipe is due to the normal stress t_{rr} and is for this reason called the normal stress effect. Since the swelling was first noticed experimentally by Merrington (1943), it is named after him as the Merrington effect.

6.4. Couette Flow

Formulation of the Problem

We now consider a steady rotational flow between two infinite coaxial circular cylinders (Figure 6.4). The inner cylinder is assumed to be at rest and the outer one to be moving with constant angular velocity. The axial pressure gradient is assumed to vanish. The velocity field in cylindrical coordinates (r, θ, z) is given by

$$\begin{aligned} v_r &= 0 , \\ v_\theta &= r\,\omega(r), \\ v_z &= 0 , \end{aligned} \tag{6.4.1}$$

where $\omega(r) \equiv \frac{d\theta}{dt}$ is the angular velocity of a cylindrical layer of fluid at a radial distance r from the axis of rotation.

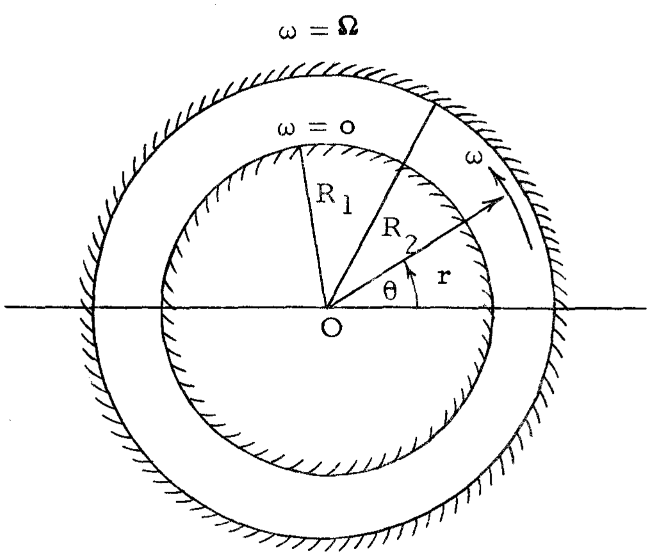


Figure 6.4. Couette flow

Equations of Motion

By virtue of (6.4.1), the first and the second deformation-rate tensors $d^i_{.j}$ and $b^i_{.j}$ respectively take the forms:

$$\left\| d^i_{.j} \right\| = \left\| \begin{matrix} 0 & d^1_2 & 0 \\ d^2_1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| = \left\| \begin{matrix} 0 & \frac{r^2\omega'}{2} & 0 \\ \frac{\omega'}{2} & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| , \tag{6.4.2}$$

$$\left\| b^i_{.j} \right\| = \left\| \begin{matrix} b^1_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| = \left\| \begin{matrix} r^2\omega'^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| , \tag{6.4.3}$$

where $\omega' \equiv \frac{\partial \omega}{\partial r}$.

Using $(6.4.2)_1$ and $(6.4.3)_1$ we reduce the constitutive equation (4.4.10) to the following form:

$$t^i_j = -p\delta^i_j + 8\alpha\beta\, d^1_2 d^2_1 d^i_j + 2\alpha(1+4\beta^2 d^1_2 d^2_1) d^i_\ell d^\ell_j + \gamma b^1_1 b^i_\ell b^\ell_j \;. \tag{6.4.4}$$

Using $(6.2.2)_2$ and $(6.4.3)_2$ we obtain from (6.4.4) the following stress components:

$$\begin{aligned}
t_{rr} &= -p + \frac{\alpha}{2}(1+\beta^2 r^2 \omega'^2) r^2 \omega'^2 + \gamma r^6 \omega'^6 ,\\
t_{\theta\theta} &= -p + \frac{\alpha}{2}(1+\beta^2 r^2 \omega'^2) r^2 \omega'^2 ,\\
t_{zz} &= -p ,\\
t_{r\theta} &= \alpha\beta\, r^3 \omega'^3 ,\\
t_{\theta z} &= 0 ,\\
t_{zr} &= 0 .
\end{aligned} \tag{6.4.5}$$

The equations of motion (2.4.9) then become

$$\begin{aligned}
\frac{\partial(t_{rr})}{\partial r} + \frac{t_{rr} - t_{\theta\theta}}{r} &= -\rho r \omega^2 ,\\
\frac{\partial(t_{r\theta})}{\partial r} + 2\frac{t_{r\theta}}{r} &= 0 .
\end{aligned} \tag{6.4.6}$$

Boundary Conditions

If R_1 and R_2 are the radii of the inner and the outer cylinders respectively, then assuming that the fluid adheres to the cylindrical walls, we have

$$\omega(R_1) = 0 \ , \qquad \omega(R_2) = \Omega \neq 0 \ . \tag{6.4.7}$$

Solution of the Equations

Velocity Field. Integrating $(6.4.6)_2$ and substituting the value of $t_{r\theta}$ from $(6.4.5)_4$ we obtain

$$\omega' = \left(\frac{B}{\alpha\beta}\right)^{1/3} r^{-5/3} \ , \tag{6.4.8}$$

where B is a constant of integration.

Integration of (6.4.8) gives

$$\omega = -\frac{3}{2}\left(\frac{B}{\alpha\beta}\right)^{1/3} r^{-2/3} + C \ , \tag{6.4.9}$$

where C is a constant of integration.

Applying the boundary conditions (6.4.7) we have

$$0 = -\frac{3}{2}\left(\frac{B}{\alpha\beta}\right)^{1/3} R_1^{-2/3} + C \ ,$$

$$\Omega = -\frac{3}{2}\left(\frac{B}{\alpha\beta}\right)^{1/3} R_2^{-2/3} + C \ ,$$

so that

$$\frac{3}{2}\left(\frac{B}{\alpha\beta}\right)^{1/3} = \frac{\Omega R_1^{2/3} R_2^{2/3}}{R_2^{2/3} - R_1^{2/3}}$$

and (6.4.10)

$$C = \frac{\Omega R_2}{R_2^{2/3} - R_1^{2/3}} .$$

Substituting the values of the constants B and C from (6.4.10) into (6.4.9) we obtain the velocity profile

$$v = \frac{\Omega r \left[1 - \left(\frac{R_1}{r}\right)^{2/3}\right]}{1 - \left(\frac{R_1}{R_2}\right)^{2/3}} . \qquad (6.4.11)$$

Pressure Field and Stress Distribution. Substituting the values of t_{rr} and $t_{\theta\theta}$ from $(6.4.5)_{1,2}$ into $(6.4.6)_1$ we get

$$\frac{\partial p}{\partial r} = \frac{\alpha}{2} \frac{\partial}{\partial r}(r^2\omega'^2 + \beta^2 r^4 \omega'^4) + \gamma \frac{\partial}{\partial r}(r^6\omega'^6)$$

$$+ \gamma r^5 \omega'^6 + \rho r \omega^2 .$$

Integrating the above equation from R_1 to r we get

$$p(r) = p(R_1) + \frac{1}{\left[1 - \left(\frac{R_1}{R_2}\right)^{2/3}\right]^2} \left[\frac{2\alpha\Omega^2}{9} \left\{ \left(\frac{R_1}{r}\right)^{4/3} - 1 \right\} \right.$$

$$+ \frac{8\alpha\beta^2\Omega^4 \left\{ \left(\frac{R_1}{r}\right)^{8/3} - 1 \right\}}{81 \left\{ 1 - \left(\frac{R_1}{R_2}\right)^{2/3} \right\}^2} + \frac{16\gamma\Omega^6 \left\{ \left(\frac{R_1}{r}\right)^4 - 1 \right\}}{243 \left\{ 1 - \left(\frac{R_1}{R_2}\right)^{2/3} \right\}^4}$$

$$\left. + \frac{\rho R_1^2 \Omega^2}{2} \left\{ \left(\frac{r}{R_1}\right)^{2/3} - 1 \right\}^3 \right]. \qquad (6.4.12)$$

Substituting the value of ω' from (6.4.8) into (6.4.5) we get the stress components:

$$t_{rr} = -p(r) + \frac{2\Omega^2}{9\left[1 - \left(\frac{R_1}{R_2}\right)^{2/3}\right]^2} \left[\alpha \left(\frac{R_1}{r}\right)^{4/3} \right.$$

$$\left. + \frac{4\alpha\beta^2\Omega^2 \left(\frac{R_1}{r}\right)^{8/3}}{9 \left\{ 1 - \left(\frac{R_1}{R_2}\right)^{2/3} \right\}^2} + \frac{32\gamma\Omega^4 \left(\frac{R_1}{r}\right)^4}{81 \left\{ 1 - \left(\frac{R_1}{R_2}\right)^{2/3} \right\}^4} \right],$$

$$t_{\theta\theta} = -p(r) + \frac{2\alpha\,\Omega^2}{9\left[1-\left(\frac{R_1}{R_2}\right)^{2/3}\right]^2}\left[\left(\frac{R_1}{r}\right)^{4/3}\right.$$

$$\left. + \frac{4\,\beta^2\Omega^2\left(\frac{R_1}{r}\right)^{8/3}}{9\left\{1-\left(\frac{R_1}{R_2}\right)^{2/3}\right\}^2}\right] , \qquad (6.4.13)$$

$$t_{zz} = -p(r)$$

$$t_{r\theta} = \frac{8\,\alpha\beta\Omega^3\left(\frac{R_1}{r}\right)^2}{27\left[1-\left(\frac{R_1}{R_2}\right)^{2/3}\right]^3} ,$$

$$t_{\theta z} = 0 ,$$

$$t_{zr} = 0 ,$$

where $p(r)$ is given by (6.4.12) .

Torque. The torque M per unit height required to maintain a steady flow of the fluid inside a cylinder of radius r $(R_1 \leq r \leq R_2)$ is given by

$$M = 2\pi\, r \cdot r \cdot t_{r\theta} \qquad (6.4.14)$$

$$= \frac{16\,\pi\,\alpha\beta\,\Omega^3 R_1^2}{27\left[1-\left(\frac{R_1}{R_2}\right)^{2/3}\right]^3} .$$

Discussion of the Results

For the Newtonian fluid, the velocity profile is given by

$$v = \frac{r\,\Omega\,[1-(\frac{R_1}{r})^2]}{1-(\frac{R_1}{R_2})^2} ,$$

or in the nondimensional form by :

$$\tilde{v} = \frac{\tilde{r}\,[1-(\frac{1}{\tilde{r}})^2]}{\tilde{R}[1-(\frac{1}{\tilde{R}})^2]} , \qquad (6.4.15)$$

where

$$\tilde{v} = \frac{v}{R_2 \Omega} ,$$

$$\tilde{r} = \frac{r}{R_1} , \qquad (6.4.16)$$

$$\tilde{R} = \frac{R_2}{R_1} ,$$

are the nondimensional quantities.

From (6.4.11) we get the nondimensional velocity for the non-Newtonian case, viz,

$$\tilde{v} = \frac{\tilde{r}[1-(\frac{1}{\tilde{r}})^{2/3}]}{\tilde{R}\,[1-(\frac{1}{\tilde{R}})^{2/3}]} . \qquad (6.4.17)$$

The graphs of the velocities in the two cases (Figure 6. 5, p. 110) reveal that there is damping of velocity in the non-Newtonian case as compared to the Newtonian one. This phenomena may be attributed to the viscoelastic nature of the non-Newtonian fluid.

We know that in the Newtonian case, there are no extra normal stresses. But in the non-Newtonian case, it is interesting to note that there exists a normal thrust t_{rr} per unit area on the cylindrical layers of the rotating fluid as well as on the walls of the cylinders. Besides, the fluid exerts an axial thrust t_{zz} per unit area which as we shall show presently, is responsible for the climbing phenomena occurring at the surface of the inner cylinder.

For the Newtonian fluid, the torque M per unit height that must be applied to the cylinders to maintain the flow is given by

$$M = \frac{4\pi \mu \Omega R_1^2}{1 - \left(\frac{R_1}{R_2}\right)^2} . \qquad (6.4.18)$$

M is thus always proportional to the angular velocity Ω whereas (6. 4. 14) shows that for a non-Newtonian fluid it could be proportional to the cube of the angular velocity.

To study the radial pressure difference across the annular gap, we first write (6. 4. 12) in the nondimensional form:

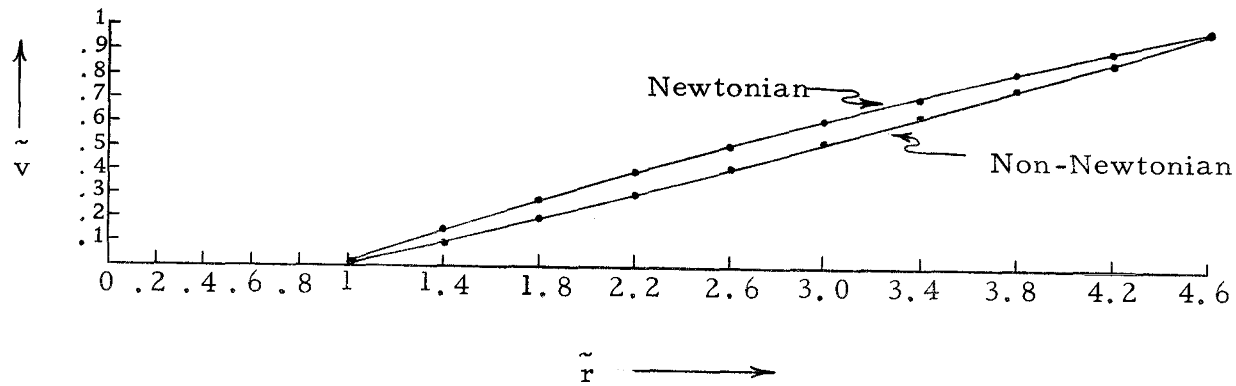


Figure 6.5. Velocity profiles of the Newtonian and the non-Newtonian fluids in Couette flow ($\tilde{R}$ = 4.6).

$$\tilde{p} = \tilde{\ell}\left\{\left(\frac{1}{\tilde{r}}\right)^{4/3}-1\right\} + \tilde{m}\left\{\left(\frac{1}{\tilde{r}}\right)^{8/3}-1\right\} + \tilde{n}\left\{\left(\frac{1}{\tilde{r}}\right)^{4}-1\right\}$$

$$+\left\{(\tilde{r})^{2/3}-1\right\}^{3}, \qquad (6.4.19)$$

where

$$\tilde{p} = \frac{[p(r)-p(R_1)]\left[1-\left(\frac{R_1}{R_2}\right)^{2/3}\right]^2}{\left(\frac{\rho R_1^2 \Omega^2}{2}\right)},$$

$$\tilde{\ell} = \frac{4}{9}\frac{\alpha}{\rho R_1^2},$$

$$\tilde{m} = \frac{16}{81}\frac{\alpha\beta^2\Omega^2}{\rho R_1^2\left[1-\left(\frac{R_1}{R_2}\right)^{2/3}\right]^2}, \qquad (6.4.20)$$

$$\tilde{n} = \frac{32}{243}\frac{\gamma\,\Omega^4}{\rho R_1^2\left[1-\left(\frac{R_1}{R_2}\right)^{2/3}\right]^4},$$

$$\tilde{r} = \frac{r}{R_1},$$

are the dimensionless quantities.

We plot the graph of the pressure difference for various values of the dimensionless parameters $\tilde{\ell}$, $\tilde{m}$ and $\tilde{n}$ (Figure 6.6, p.112). For $\tilde{\ell} = \tilde{m} = \tilde{n} = 1$, we see from the graph that the pressure drops very rapidly near the inner cylinder until it reaches its lowest limit.

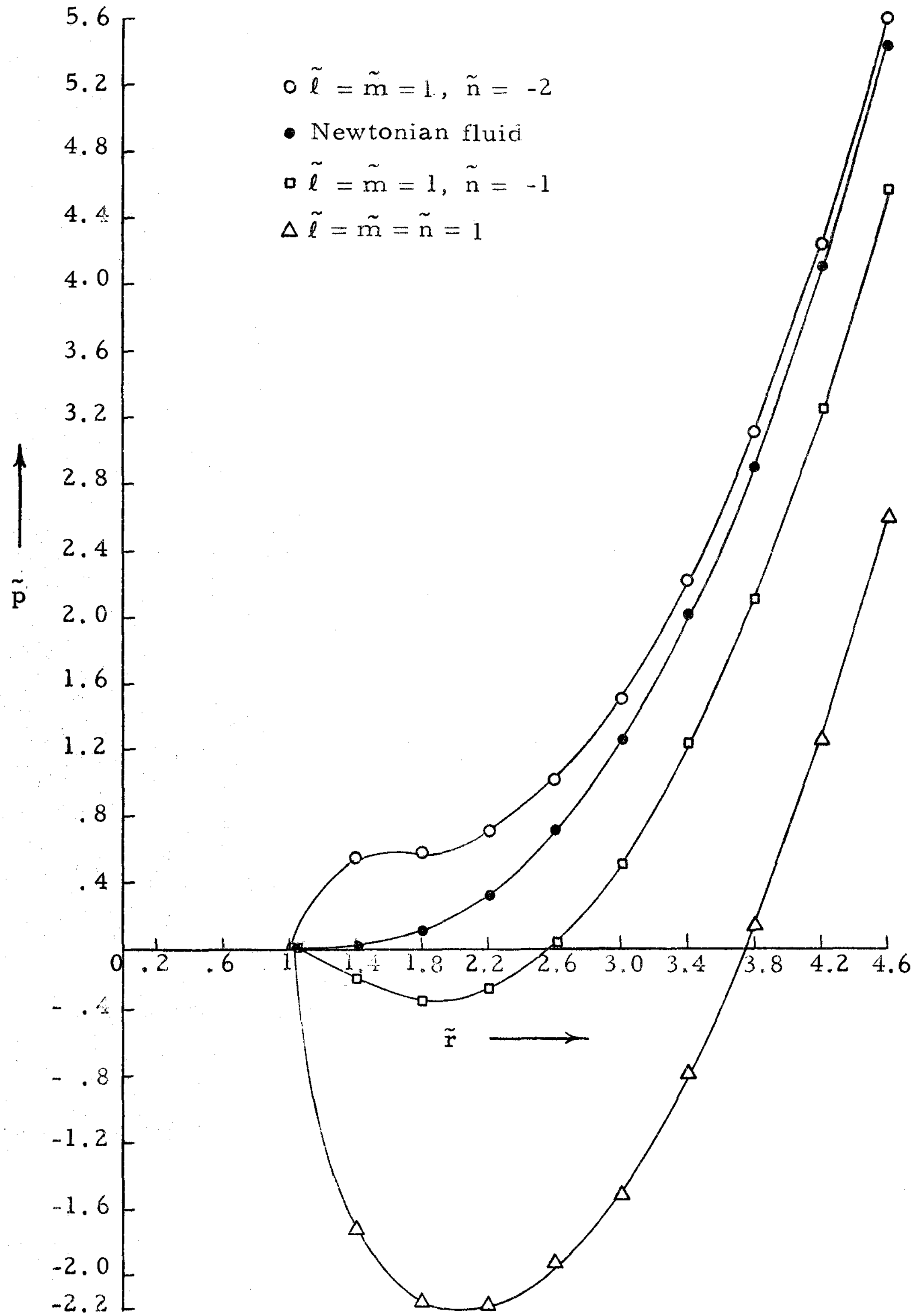


Figure 6.6. Pressure variation in Couette flow.

After this limit the pressure starts increasing until it becomes equal to that at the inner cylinder. From this point on, there is a sharp increase in pressure. We further notice that as the value of $\tilde{n}$ decreases the pressure drop also undergoes a corresponding decrease. In other words, we can say from $(6.4.20)_3$ that the decrease in the value of the rheological constant γ results in a corresponding decrease in the pressure drop. We see, however, from the figure that depending on the values of the dimensionless parameters $\tilde{\ell}$, $\tilde{m}$, and $\tilde{n}$ and consequently the rheological constants of α, β and γ, there could even be no pressure drop at all across the annular gap. Surely the non-Newtonian nature of the fluid is responsible for this 'anomalous' variation in pressure in a Couette flow.

In the case of a Newtonian fluid, only the last term in (6.4.19) will survive. Consequently the pressure difference $\tilde{p}$ is always nonnegative, and the pressure is always increasing as we go from the inner to the outer cylinder.

The preceding analysis brings out very clearly the visco-elastic effect on the fluid pressure.

Climbing in Couette Flow

In the investigation of Couette flow carried out in this section, we assumed the two cylinders to be infinitely long. Since the cylinders used in the laboratory or in some other real situation are of finite length, the predictions of our theory can be expected to tally with experimental results only approximately. If therefore, we consider a Couette flow apparatus in which the revolving fluid is in contact with an atmosphere of pressure p_a, we may assume that the normal thrust in the axial direction at the exposed end is given by t_{zz}.

Now let

$$\pi = p_a + t_{zz} . \tag{6.4.21}$$

After differentiation with respect to r we get from (6.4.21)

$$\frac{\partial \pi}{\partial r} = - \frac{\partial p}{\partial r}$$

$$= \frac{\Omega^2}{\left[1 - \left(\frac{R_1}{R_2}\right)^{2/3}\right]^2} \left[\frac{8\alpha}{27r}\left(\frac{R_1}{r}\right)^{4/3} + \frac{64\,\alpha\,\beta^2\Omega^2\left(\frac{R_1}{r}\right)^{8/3}}{243\, r\left\{1 - \left(\frac{R_1}{R_2}\right)^{2/3}\right\}^2} \right.$$

$$\left. + \frac{64\,\gamma\,\Omega^4\left(\frac{R_1}{r}\right)^4}{243 r\left\{1 - \left(\frac{R_1}{R_2}\right)^{2/3}\right\}^4} \right] - \rho\, r\, \omega^2 . \tag{6.4.22}$$

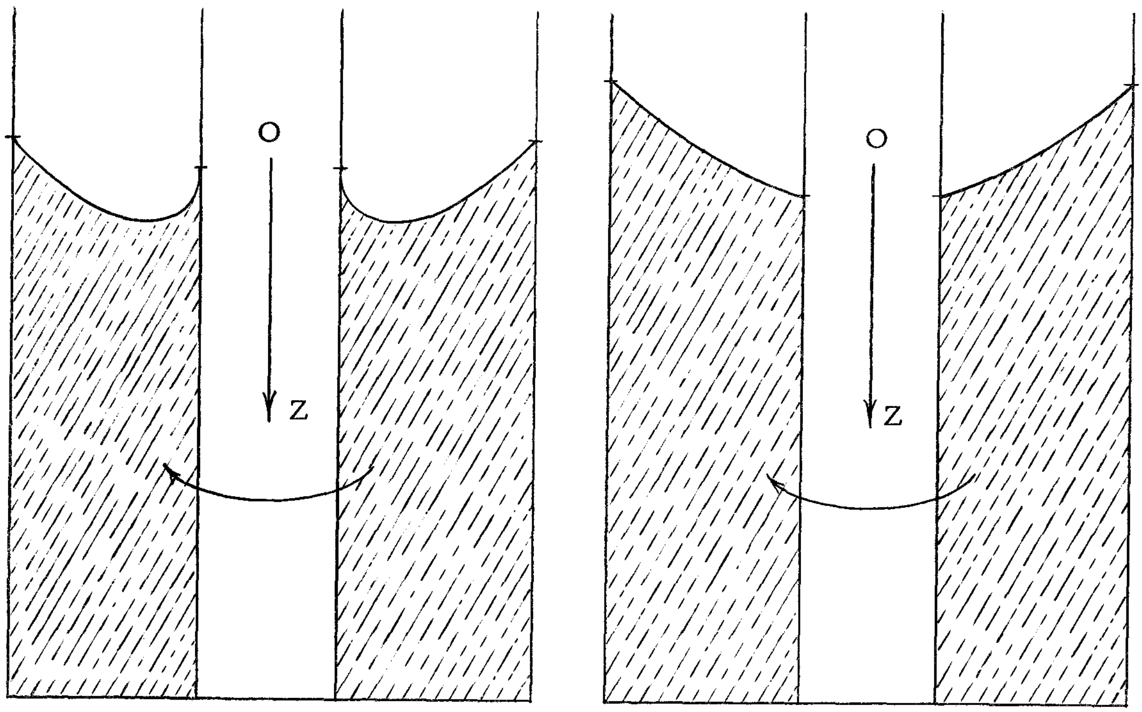
Assuming γ to be positive we observe from (6.4.22) that the only negative term in the expression for $\frac{\partial \pi}{\partial r}$ is $-\rho\, r\, \omega^2$. Close to the inner cylinder, which is at rest, r and ω are small. Consequently the term $\rho r \omega^2$ is also small. Whereas the positive part in the expression for $\frac{\partial \pi}{\partial r}$, which contains r in the denominator is large near the inner cylinder. We can therefore expect that unless the density ρ is not too large $\frac{\partial \pi}{\partial r}$ is positive near the inner cylinder. Physically, this means that near the inner cylinder the fluid surface slopes downwards from the inner cylinder to the outer one (Figure 6.7, p. 116). In other words, the fluid will show a tendency to climb up the inner cylinder. This is known as the Weissenberg effect and has been noticed in experiments with non-Newtonian fluids. From our analysis we can say something more also. Since with the increase of r the positive part in $\frac{\partial \pi}{\partial r}$ is decreasing, whereas the negative part is increasing, it may happen that $\frac{\partial \pi}{\partial r}$ vanishes at some point within the annular gap. Then, after that point $\frac{\partial \pi}{\partial r}$ would be negative. Physically, this means that near the outer cylinder the fluid surface slopes upwards (Figure 6.7, p. 116). It must, however, be emphasized that this latter phenomena will occur provided $\frac{\partial \pi}{\partial r}$ vanishes at some point within the annulus.

For a Newtonian fluid we have from $(6.4.5)_{1,2}$ and $(6.4.6)_1$

$$\frac{\partial p}{\partial r} = \rho r \omega^2 .$$

Therefore

$$\frac{\partial \pi}{\partial r} = -\rho r \omega^2 . \qquad (6.4.23)$$



Non-Newtonian fluid

Newtonian fluid

Figure 6.7. Climbing in Couette flow.

Equation (6.4.23) shows that $\frac{\partial \pi}{\partial r}$ is always negative for a Newtonian fluid. This means that the fluid surface always slopes upwards from the inner cylinder to the outer one (Figure 6.7., p. 116). In other words, there will be a depression near the inner cylinder.

CHAPTER 7

TORSIONAL FLOW

7.1. Preliminary Remarks

In Section 2 we study the torsional flow of a cylindrical mass of fluid. The graphs of the velocity field and pressure drop are also drawn.

7.2. Parallel Plate Viscometer

Formulation of the Problem

We assume that the flow takes place in a right cylindrical region (Figure 7.1). We use cylindrical coordinates (r, θ, z), where z is measured along the axis of the cylindrical mass with radius R. We suppose that, of the two rigid parallel discs which bound this region, the one at $z=0$ is at rest and the other at $z=h$ has angular velocity Ω. We assume axial symmetry in the problem and accordingly the velocity and the pressure are independent of θ. The velocity field is given by

$$\begin{aligned} v_r &= 0 , \\ v_\theta &= v(r, z) = r\,\omega(z) , \qquad (7.2.1) \\ v_z &= 0 . \end{aligned}$$

where $\omega(z)$ is the angular velocity of the concentric layer of the cylindrical mass at a distance z from the disc at $z=0$.

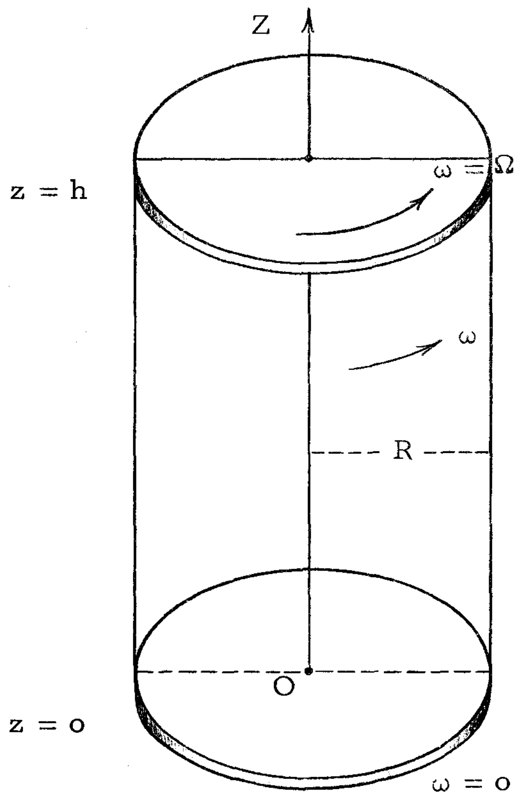


Figure 7. 1. Parallel plate viscometer.

Equations of Motion

With the velocity field given by (7. 2. 1) we get for the first and the second deformation-rate tensors:

$$\left\| d^i_j \right\| = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & d^2_3 \\ 0 & d^3_2 & 0 \end{matrix} \right\|$$

$$= \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\omega' \\ 0 & \frac{1}{2}r^2\omega' & 0 \end{Vmatrix}, \tag{7.2.2}$$

$$\| b^i_j \| = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b^3_3 \end{Vmatrix} \tag{7.2.3}$$

$$\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r^2\omega'^2 \end{Vmatrix},$$

where $\omega' = \frac{\partial \omega}{\partial z}$.

Now, with the help of $(7.2.2)_1$ and $(7.2.3)_1$, the constitutive equation (4.4.10) takes the following form:

$$t^i_j = -p\,\delta^i_j + 8\alpha\beta\, d^2_3 d^3_2 d^i_j + 2\alpha(1+4\beta^2 d^2_3 d^3_2) d^i_\ell d^\ell_j + \gamma\, b^3_3 b^i_\ell b^\ell_j \,. \tag{7.2.4}$$

Using $(7.2.2)_2$ and $(7.2.3)_2$ in (7.2.4), we get the following expressions for the physical components of the stress tensor:

$$t_{rr} = -p\,,$$

$$t_{\theta\theta} = -p + \frac{\alpha}{2}(1 + \beta^2 r^2 \omega'^2)\, r^2\omega'^2,$$

$$t_{zz} = -p + \frac{\alpha}{2}(1 + \beta^2 r^2 \omega'^2)\, r^2\omega'^2 + \gamma\, r^6 \omega'^6 \,,$$

$$t_{\theta z} = \alpha\beta r^3 \omega'^3 , \qquad (7.2.5)$$

$$t_{rz} = 0 ,$$

$$t_{r\theta} = 0 .$$

From (2.4.9) and (7.2.5), we get the following equations of motion:

$$\frac{\partial(t_{rr})}{\partial r} + \frac{t_{rr} - t_{\theta\theta}}{r} = - \rho r \omega^2 ,$$

$$\frac{\partial(t_{\theta z})}{\partial z} = 0 , \qquad (7.2.6)$$

$$\frac{\partial(t_{zz})}{\partial z} = 0 .$$

Boundary Conditions

Since the fluid is assumed to adhere to the plates we have

$$\omega(0) = 0$$

and (7.2.7)

$$\omega(h) = \Omega \neq 0 .$$

Solutions of the Equations

Velocity Field. From $(7.2.5)_4$ and $(7.2.6)_2$, we have

$$\frac{\partial(\omega'^3)}{\partial z} = 0$$

or

$$\omega' = \text{constant} = C(\text{say}). \qquad (7.2.8)$$

Therefore $$\omega = cz + d\,, \tag{7.2.9}$$

where d is an arbitrary constant.

Applying the boundary conditions (7.2.7) we get

$$d = 0$$

and

$$c = \frac{\Omega}{h}\,. \tag{7.2.10}$$

Hence the angular velocity becomes

$$\omega = \frac{\Omega}{h} z\,. \tag{7.2.11}$$

<u>Pressure Field and Stress Distribution.</u> From $(7.2.5)_3$, $(7.2.6)_3$ and (7.2.8), we obtain

$$\frac{\partial p}{\partial z} = 0\,. \tag{7.2.12}$$

Hence p depends on r alone.

Substituting the values of t_{rr}, $t_{\theta\theta}$, ω' and ω from $(7.2.5)_1$, $(7.2.5)_2$, (7.2.8) and (7.2.11) respectively into $(7.2.6)_1$, we get

$$\frac{1}{r}\frac{\partial p}{\partial r} + \frac{\alpha\Omega^2}{2h^2}\left(1 + \beta^2 \frac{\Omega^2}{h^2} r^2\right) = \rho \frac{\Omega^2}{h^2} z^2\,. \tag{7.2.13}$$

In (7.2.13) the left side is a function of r only while the right side is a function of z only. Therefore a solution of our problem consistent with the boundary conditions is obtained by setting each side of (7.2.13) equal to a constant K (say). In particular K can be taken equal to zero which is justifiable on the basis of experiments. Equation (7. 2. 13), then becomes

$$\frac{\partial p}{\partial r} = -\frac{\alpha\Omega^2}{2h^2}\left(r + \frac{\beta^2\Omega^2}{h^2} r^3\right). \tag{7. 2. 14}$$

Integrating (7. 2. 14), between the limits o and r, we have

$$p(r) = p(o) - \frac{\alpha\Omega^2}{4h^2}\left[r^2 + \frac{\beta^2\Omega^2}{2h^2} r^4\right]. \tag{7. 2. 15}$$

Equation (7. 2. 15) gives the pressure field in the cylindrical mass of fluid.

Substituting the value of ω' from (7. 2. 11) into (7. 2. 5) we get the following stress distribution:

$$
\begin{aligned}
t_{rr} &= -p(r) , \\
t_{\theta\theta} &= -p(r) + \frac{\alpha\Omega^2}{2h^2}\left(1 + \frac{\beta^2\Omega^2}{h^2} r^2\right) r^2 , \\
t_{zz} &= -p(r) + \frac{\alpha\Omega^2}{2h^2}\left(1 + \frac{\beta^2\Omega^2}{h^2} r^2\right) r^2 + \frac{\gamma\Omega^6}{h^6} r^6 , \\
t_{\theta z} &= \alpha\beta \frac{\Omega^3}{h^3} r^3 , \\
t_{rz} &= 0 , \\
t_{r\theta} &= 0 ,
\end{aligned}
\tag{7. 2. 16}
$$

where $p(r)$ is given by (7. 2. 15) .

The total normal force N that must be applied to hold the revolving disc in place is given by

$$N = \int_0^R \int_0^{2\pi} t_{zz} \, r d\theta dr$$

$$= -\pi R^2 p(o) + \frac{\pi \alpha \Omega^2 R^4}{4h^2}\left(\frac{3}{2} + \frac{5\beta^2\Omega^2 R^2}{6h^2}\right) + \frac{\pi\gamma\Omega^6 R^8}{4h^6} . \tag{7.2.17}$$

Torque. The torque M that must be applied to maintain the angular velocity Ω is given by

$$M = \int_0^R \int_0^{2\pi} t_{\theta z} \, r^2 \, d\theta dr$$

$$= \frac{\pi \alpha \beta \Omega^3 R^6}{3h^3} . \tag{7.2.18}$$

Discussion of the Results

We see that in order to maintain in a cylindrical mass of a non-Newtonian fluid, the motion described by (7.2.1) and (7.2.7), we have to apply to the plane ends of the fluid mass, normal stresses of the magnitude t_{zz} per unit area given by $(7.2.16)_3$ or equivalently a total normal force N given by (7.2.17) and azimuthal surface tractions of magnitude $t_{\theta z}$ per unit area given by $(7.2.16)_4$ or equivalently a couple of magnitude M given by (7.2.18). In the Newtonian case, the pressure p and consequently the normal stresses $-p$ would be constant throughout the fluid mass. In this case we do not need any extra normal stress to maintain the motion.

We next observe that in the non-Newtonian case under consideration, the shearing forces $t_{\theta z}$ are proportional to the cube of the radial distance r whereas in the Newtonian case these are always proportional to the radial distance only. This difference may be attributed to the viscoelastic nature of the non-Newtonian fluid.

It is interesting to note that the velocity profile in this particular case of the non-Newtonian fluid is the same as in the Newtonian case. From (7.2.11) we get the azimuthal velocity

$$v = \frac{\Omega}{h} zr , \tag{7.2.19}$$

Its nondimensional form is

$$\tilde{v} = \tilde{z}\tilde{r}, \tag{7.2.20}$$

where

$$\tilde{v} = \frac{v}{R\Omega}, \quad \tilde{z} = \frac{z}{h}, \quad \tilde{r} = \frac{r}{R}, \tag{7.2.21}$$

are the nondimensional variables.

From (7.2.20) we plot the graphs of the velocity profiles (Figure 7.2a, p. 127 and Figure 7.2b, p. 128).

To study the radial pressure drop we write from (7.2.15)

$$\tilde{p} = -\tilde{r}^2 - \tilde{m}\,\tilde{r}^4 , \qquad (7.2.22)$$

where

$$\tilde{p} = \frac{p(r) - p(o)}{\dfrac{\alpha\Omega^2 R^2}{4h^2}} ,$$

$$\tilde{m} = \frac{\beta^2\Omega^2 R^2}{2h^2} , \qquad (7.2.23)$$

$$\tilde{r} = \frac{r}{R}$$

are the nondimensional quantities.

We plot the graph of pressure drop given by (7.2.22) for $\tilde{m} = \frac{1}{4}, \frac{1}{2}$ and 1 (Figure 7.3, p. 129). We notice from the graph that the pressure drop is very slow near the axis of the viscometer whereas it is very rapid near its cylindrical surface. In the case of a Newtonian fluid, the radial pressure is constant everywhere. Consequently there is no radial pressure drop in the torsional flow of a Newtonian fluid while such a drop is found to occur in the non-Newtonian case.

We may note further that since the expression for radial pressure gradient does not involve the second deformation-rate term, the pressure drop in the torsional flow does not depend on the

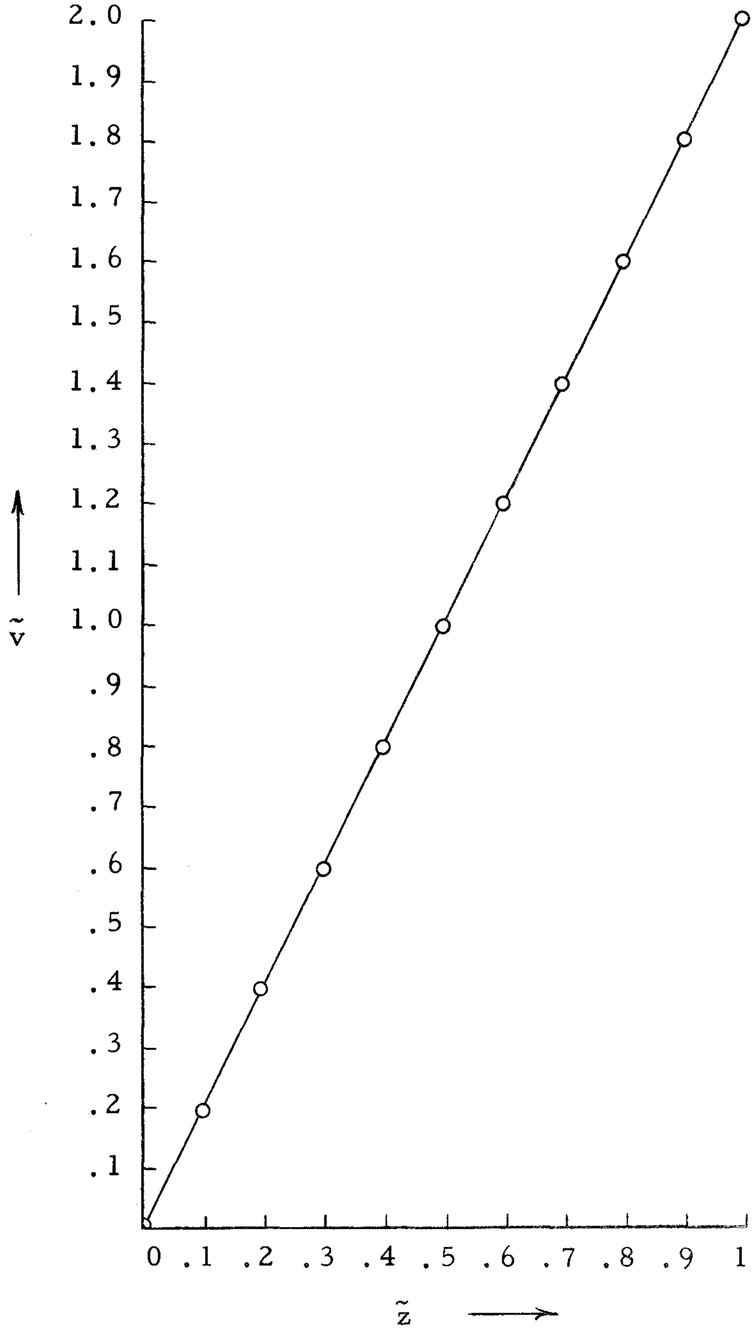


Figure 7.2a. Velocity profile in parallel plate viscometer ($\tilde{r} = 2$).

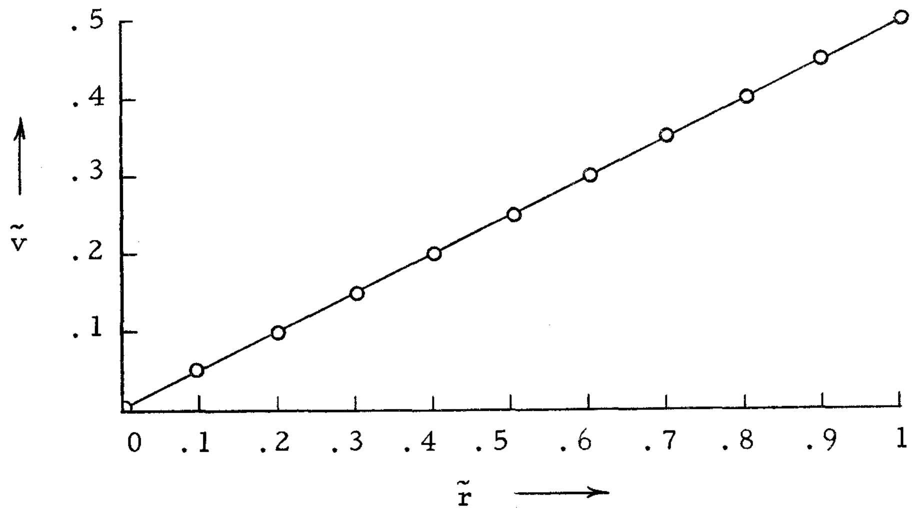


Figure 7.2b. Velocity profile in parallel plate viscometer ($\tilde{z} = 1/2$).

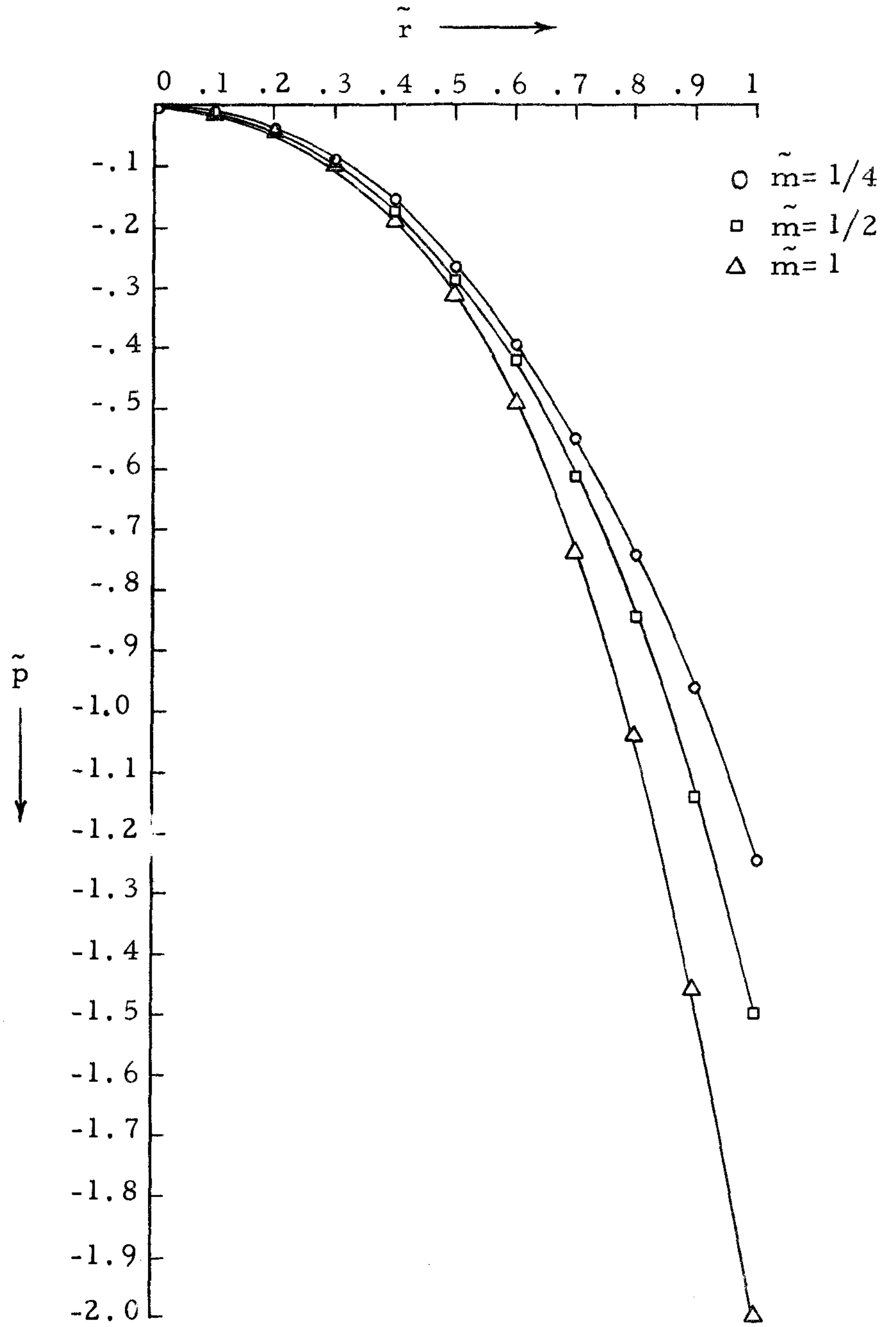


Figure 7.3. Pressure variation in parallel plate viscometer.

rheological constant γ. Also from Figure 7.3 it is found that the rate of pressure drop decreases as the value of $\tilde{m}$ and consequently that of the rheological constant β decreases. Similarly with the increase of β, the rate of the pressure drop also increases.

CHAPTER 8

SUMMARY, DISCUSSION AND SCOPE OF FURTHER WORK

8.1. Summary and Discussion

It is now being recognized more and more that the response of real materials to external forces is, in general, nonlinear in character. The classical theories which were designed to explain the behavior of materials subjected to deforming forces do not, however, take cognizance of this nonlinearity in Nature. For example, it is the classical theory of elasticity, in which the displacement is proportional to the applied load. Again, it is the classical theory of fluid dynamics in which the rate of deformation is proportional to the viscous forces. These theories which lead to linear expressions relating the load to the response fail, as one should expect, to interpret even remotely the physical phenomena which is essentially nonlinear.

The failure of the classical theories to explain the nonlinear response of materials has lately stimulated widespread interest in the search for more general theories. In our present work, we have given a brief outline of the various theories governing the behavior of real materials which have appeared since 1945. It is seen from our description that the pioneering work in this field has been done by Reiner, Rivlin, Ericksen, Green, Oldroyd, Noll, Seth, etc. We noticed that all these workers except Seth have used in their theories, ordinary measures of deformation or deformation-rate. The result is, as we have seen, that they have obtained very complicated constitutive equations involving terms in powers and products of ordinary measures and also a number of unknown response functions. These theories are, no doubt, very general and do explain much of the nonlinear phenomena but the investigation of such phenomena with the help of these sophisticated theories is not an easy matter either for the theorist or for the experimentalist. As we have already pointed out, the main source of this difficulty is due to the generalized measures not being used in the constitutive equations of nonlinear materials.

We have seen how Seth has attempted to resolve this difficulty by the introduction of generalized measures of deformation or deformation-rate in continuum mechanics. We have extended the generalized measures of Seth in such a way as to explain adequately the rheological behavior of materials. The constitutive equations that we have set up using the extended generalized measures, contain essentially two terms and at the most four rheological constants. Thus we have achieved the maximum possible simplicity without losing any generality as to the prediction of nonlinear effects. Unlike some previous theories, the constitutive equations of our theory do not contain any unknown functions of the invariants of kinematic matrices, etc. This has been accomplished by fixing the order of the generalized measure before writing the constitutive equation in its explicit form. Because we can always fix the order of the generalized measure and also adjust the values of the four rheological constants (which reduce to three in number for many problems) appropriately, our theory furnishes unlimited flexibility and generality.

To illustrate the simplicity and power of our theory, we have solved rectilinear, helical and torsional flow problems. For this purpose we arbitrarily fixed certain orders for the measures D^* and B^*. In actual practice, however, the choice of order of measures will have to be determined by experiments. Besides the freedom to choose the order of the measures, we have seen that the rheological parameters entering the constitutive equation can also be suitably varied so as to correlate the theory with experiments. We have seen that the use of generalized measures of any given order gives rise to response coefficients which are known functions of the invariants of the kinematic matrices D and B, containing only a finite number of terms. Consequently, we have been able to obtain very precise expressions for velocity, pressure, stresses, volumetric flow rates, etc. On the other hand, a reference to the literature on the solution of similar problems employing the constitutive equations of Reiner-Rivlin, Rivlin-Ericksen, Green-Rivlin, Noll, etc. would reveal that the corresponding expressions obtained for the kinematic and dynamic variables involve a number of unknown functions of the invariants. As a result, this would need a lot of experimental work to be done before any information of practical interest can be obtained from such solutions. Since the constitutive equations using generalized measures are much simpler than their counterparts in the existing theories, it has enabled us to discuss with greater ease and more clarity the well-known normal stress effects like the Merrington and the Weissenberg effects. Moreover, we have carried out detailed discussions on the phenomena of back flow, the helical flow of a lubricant in a narrow annular gap, 'anomalous' pressure variations, damping of velocity profiles, etc. We must state that the above advantages which are claimed for our theory are not merely of academic or theoretical nature. They are of far-reaching practical importance. After the order of the generalized measure has been fixed, we need to know only the values of the four rheological constants at the most, to obtain from our analysis , concrete information on the behavior of any fluid. For example, the stresses, the velocity profiles, the torques, the volumetric flow rates and the pressure variations are completely determined in any flow, once we have found suitable values for the rheological constants characterizing a fluid. Moreover, we have seen that our theory has explained swelling and thinning in Poiseuille flow and climbing in Couette flow, not only qualitatively but a glance through the expressions controlling these phenomena would at once reveal that we can also compute the difference of atmospheric pressure and the normal stresses. Also, it is evident from (6.4.22) that one can determine the point in the Couette flow at which the fluid surface at the free end of the annulus changes from sloping downwards to upwards as we move from the inner to the outer cylinder. Similarly our theory has not merely given a qualitative picture of backflow, but it also enables us to determine in a particular problem of rectilinear flow, the actual point of separation where the back flow occurs. It is this theory which provides in a definitive manner the qualitative as well as the quantitative information on the behavior of fluids.

Before we close the discussion, we may recall that we arbitrarily fixed the order of the generalized measures as follows:

D^*:

$$n = 4$$

$$q = 2$$

B^*:

$$n' = 2$$

$$q' = 3$$

From the expressions for the shear stresses that we consequently obtained, we note that the apparent viscosity of the fluid characterized by the above choice of the order of the measure, increases with the rate of shear. [A fluid with such a behavior is known as a dilatant fluid.]

8.2 Scope of Further Work

From what has been said in Section 8.1, it is reasonable to believe that a nonlinear theory based on generalized measures has a clear advantage over other nonlinear theories based on ordinary measures. There is, therefore, a natural desire to exploit the idea of generalized measures in the investigation of other nonlinear phenomena as well. In the present work, we have restricted the use of generalized measures to isotropic materials. The extension of this idea to the anisotropic case would be very rewarding since the occurrence of anisotropic materials is quite common in Nature. We have explained a number of non-Newtonian phenomena with the help of generalized measures but the time-dependent effects, stress-relaxation phenomena, etc., still need to be investigated. Moreover, the construction of any further models for continuous media based on generalized measures, must take into consideration the effects of temperature, electromagnetic effects, phase and chemical transformations, and in certain cases quantum effects at low temperatures, etc. These are not the only possible avenues of application of generalized measures. These measures, because of their simplicity and effectiveness are bound to find still wider applications.

Our work has thus opened a new but vast area of a very rewarding research in nonlinear physical phenomena.

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