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Title : The Existence and Uniqueness of Solutions in a Weighted Sobolov Space for an Initial-Boundary Problem of a Degenerate Parabolic Equation with Principal Part in Divergence Form

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In this dissertation I consider degenerate, parabolic, quasilinear equations with principal part in divergence form: $u_t = \mathrm{div} a(x, t, u, u_x) + b(x, t, u, u_x)$. Degenerate equations cannot be treated by standard methods. Because of the degeneracy, they do not always have classical solutions and have different characteristics than the solutions of nondegenerate equations. I explicitly treat one dimensional porous medium equations of the form $u_t = (u^m)_{xx}$ and also compare them to an equation of the form $\varepsilon u_{tt} + u_t = (u^m)_{xx}$. This is obtained by using a more general form of Darcy's law. Recent, concrete results for the unique solvability of the problems contained in the class of the initial-boundary value problems of degenerate parabolic quasilinear equations with principal part in divergence form, are given. I also consider the unique solvability of the first initial-boundary value problems of the form : $(x^\alpha u_x)_x = u_t,\ 0 < \alpha \leq 2$ in a weighted Sobolov space. The treatment of the problem depends on whether $0 < \alpha < 1$ or $1 \leq \alpha \leq 2$. By giving an explicit solution to the problem when $0 < \alpha < 1$, I show that it serves as an example for the existence

of a unique solution to the initial, boundary problem with zero boundary conditions for $u_t = \operatorname{div}(a(x) \operatorname{grad} u(x,t)) + f(u(x,t))$ when $f = 0$. When $1 \leq \alpha \leq 2$, I show that we can not give boundary conditions for some parts of the boundary in order to have a unique solution in a weighted Sobolov space. Then as a generalization of this case, I give the main theorem of this dissertation: for the unique solvability in a weighted Sobolov space for a generalized solution of the problem $u_t = \operatorname{div}(a(x) \operatorname{grad} u(x,t)) + f(u(x,t))$ when $1 \leq \alpha \leq 2$ with no boundary conditions on some parts of the boundary and zero on the others. An explicit solution to satisfy the theorem is also given. Finally, I give an application of degenerate quasilinear parabolic equations with principal part in divergence form to a physical problem of unsaturated flows of liquids in a porous medium.

The Existence and Uniqueness of Solutions in a Weighted Sobolov Space for an Initial-Boundary Problem of a Degenerate Parabolic Equation with Principal Part in Divergence Form

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Hanku Lee

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CHAPTER 1

Introduction and Some Preliminary Mathematical Facts

In this dissertation we consider linear and quasilinear , degenerate parabolic equations with principal part in divergence form, that is, equations of the form

$$u_t = \frac{d}{dx_i} a_i(x, t, u, u_x) + b(x, t, u, u_x), \tag{1.1}$$

or in vector form,

$$u_t = \mathrm{div} a(x, t, u, u_x) + b(x, t, u, u_x). \tag{1.2}$$

Here the summation convention is used, that is, when the same index appears twice in a term, one sums over the range of the indices indicated. Here i runs from 1 to n unless the contrary is explicitly stated. u_x denotes the gradient of u with respect to the spatial variables and the divergence is calculated with respect to x as well.

1.1 Introduction

These equations occur in many physical problems such as nonlinear diffusion problems and ground water problems in a porous medium. Groundwater problems are governed by a transport equation such as Darcy's law:

$$\vec{q} = -\frac{k}{\mu}\left(\vec{\nabla} p + \rho \vec{g}\right). \tag{1.3}$$

Here p is the pressure, ρ the density of the fluid (or gas), $\vec{g} = (0, 0, g)$ the vector giving the acceleration due to gravity, here assumed to be constant and k is the permeability tensor which in most applications is assumed to be strictly positive definite. $\vec{q}$ is the volumetric flow rate , also called the seepage velocity. For the applications we deal with, it is only nonnegative definite. $\mu > 0$ is the dynamic viscosity of the fluid. Equation (1.3) is combined with the equation of continuity, or conservation of mass,

$$\phi \rho_t = -\text{div}(\rho \vec{q}), \tag{1.4}$$

where ϕ is the porosity of the medium. An equation for p is obtained by inserting (1.3) into (1.4) and making use of constitutive relations for the coefficients.

When the equations are nondegenerate parabolic, they can be treated by the standard methods found in [1] and [2]. Some known results for nondegenerate parabolic equations will be given in Chapter 4.

When the equations are degenerate, they cannot be treated by the standard methods because of the degeneracy and need not always have classical solutions. However, under certain assumptions it is possible to establish the unique solvability for the Cauchy problem and the initial-boundary value problem for these degenerate parabolic equations. The solutions often have properties different from those of solutions to nondegenerate equations. The most important property, which is distinct from that of nondegenerate equations, is that the solutions may have a finite speed of propagation.

Now I elaborate the content mentioned above using a concrete example. Consider the flow of an ideal gas in a homogeneous, porous medium. The flow is governed by three laws. The first is Darcy's law (1.3), neglecting gravity, which was originally established for the case of water but also holds for the flow of gasses underground. The second is the law of conservation of mass (1.4) where, for the case of gas, the porosity ϕ of the medium is the volume fraction available to the gas and we assume it to be constant. The third law is an equation of state, or constitutive relation, for the gas which I take to be

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{\lambda}}. \tag{1.5}$$

Here $p_0, \rho_0 \in \mathbb{R}^+$, and $\lambda \in [1, \infty)$ are constant. The constant $\lambda > 1$ occurs when one assumes the expansion of the gas to be adiabatic. Multiply both sides of (1.3) by ρ, neglect gravity, and insert $p = p_0(\rho/\rho_0)^{1/\lambda}$, to obtain

$$\rho\vec{q} = -\frac{kp_0\rho}{\mu\rho_0^{1/\lambda}}\nabla\rho^{\frac{1}{\lambda}} = -\frac{kp_0}{\mu\rho_0^{1/\lambda}(\lambda+1)}\nabla\rho^{\frac{\lambda+1}{\lambda}}$$

Take the divergence of both sides,

$$\operatorname{div}(\rho\vec{q}) = -\frac{kp_0}{\mu\rho_0^{1/\lambda}(\lambda+1)}\Delta\rho^{\frac{\lambda+1}{\lambda}}$$

Then, from (1.4),

$$\rho_t = \frac{kp_0}{\phi\mu\rho_0^{1/\lambda}(\lambda+1)}\Delta\rho^{\frac{\lambda+1}{\lambda}}$$

By rescaling we can take $kp_0/\phi\mu\rho_0^{1/\lambda}(\lambda+1)$ to be equal to one and we set $(\lambda+1)/\lambda = m$ to obtain $\rho_t = \Delta\rho^m$ $(m > 1)$. The density ρ of a gas flowing in a homogeneous, porous medium must obey the equation

$$\rho_t = \Delta\rho^m \quad (m > 1)$$

This equation is known as the porous media equation which models the density distribution of a gas in a porous medium. The equation is parabolic at any point (x, t) at which $\rho > 0$. However, at a point where $\rho = 0$, it is degenerate parabolic.

Now let us consider the following problem :

$$\begin{cases} \rho_t &= (\rho^m)_{xx} \\ \rho|_{t_0} &= \psi(x) \end{cases} \tag{1.6}$$

where $\psi(x) > 0$ in $|x| < a$ and vanishes outside $|x| = a$. Generally, solutions for the problem fail to be classical solutions at precisely those points where the solution passes from positive to zero values. The lateral boundary of suppρ is called the interface. Thus, we need a concept of weak solutions. Classes of weak solutions for equation (1.6) were introduced by Oleinik, Kalashnikov and Yui-Lin in the paper [3]. There they proved the existence and uniqueness of such solutions and showed that if at some constant t_0 a weak solution $\rho(x, t_0)$ has compact support, then $\rho(x, t)$ has compact support for any $t \geq t_0$. Moreover, they showed that in a neighborhood of any point (x, t) where $\rho > 0, \rho$ is a classical solution of the problem. Some explicit solutions of the porous media equation were found and these were all self similar solutions. Self similar solutions are solutions of the form $u(x, t) = g(\eta)$ with $\eta = x^m t^n$, where m and n are constants and $g(\eta)$ satisfies a boundary value problem such that the differential equation with respect to η is obtained by inserting $g(\eta)$ to the porous media equation instead of u(x,t) and a boundary condition is given such as g(0) =C constant and $g(\infty) = 0$. The most important one was found by G.I. Barenblatt in [4]. From Barenblatt's self similar solutions and the work of others , see e.g. [5], [6], [7], [8], [9], [10], we know that there exists a function $r(t) > 0$ which describes an interface so that $\rho(x, t) = 0$ for $x > r(t)$, where $r(0) = a$ and $r(t)$ is monotonically increasing.

This means that the solution has a finite speed of propagation. Chapter 2 covers this phenomenom explicitely in more detail and also contains a comparison with $\varepsilon\rho_{tt} + \rho_t = \Delta\rho^m$ which is obtained when using a more general form of Darcy's law:

$$\varepsilon\vec{q}_t + \vec{q} = -\frac{k}{\mu}\vec{\nabla}p \tag{1.7}$$

where ϵ is a positive constant.

In the case when $m = 1$, equation (1.6) reduces to the heat equation :

$$\rho_t = \rho_{xx}$$

This equation is a nondegenerate parabolic equation and a solution to the initial value problem is given by

$$\rho(x,t) = \frac{1}{(4\pi t)^{1/2}}\int_{-\infty}^{\infty}\psi(\xi)e^{-\frac{(\xi-x)^2}{4t}}\,d\xi\ , \tag{1.8}$$

where $\rho(x,0) = \psi(x)$ represents the initial values of ρ . We see from (1.8) that $\rho > 0$ for any $t > 0$. In other words, ρ becomes positive everywhere after an arbitrarily small increment in time provided it is nonzero initially. This means that the speed of propagation of the solution is infinite. This property remains valid for nonlinear nondegenerate parabolic equations.

Generally, it is known that the properties such as the unique solvability and finite speed of propagation hold for solutions to the Cauchy problem

$$\begin{cases} u_t &= \frac{d}{dx_i}a_i(x,t,u,u_x) + b(x,t,u,u_x) \\ u|_{t_0} &= u_0 \end{cases} \tag{1.9}$$

as well as for solutions to the initial-boundary value problem

$$\begin{cases} u_t &= \frac{d}{dx_i}a_i(x,t,u,u_x) + b(x,t,u,u_x) \\ u|_{t_0} &= u_0 \\ u|_{\partial\Omega\times[0,T]} &= \psi(x,t)|_{\partial\Omega\times[0,T]} \end{cases} \tag{1.10}$$

Allowing certain degeneracies for the functions a_i, b, u_0, ψ, we see the possibility of the unique solvability for (1.10) in the linear case in [11]. There Showalter considered a class of implicit , linear evolution equations of the form $\frac{d}{dt}\mathcal{U}u(t) + \mathcal{L}u(t) = f(t),\ t > 0$ in a Hilbert space where $\mathcal{U}$ and $\mathcal{L}$ are operators and their realization in function spaces as initial-boundary value problems for partial differential equations which may contain a degeneracy. He used the theory of analytic semigroups on a weighted Sobolev space and proved theorems in his paper for the classical diffusion equation $\frac{\partial}{\partial t}\left(c(x)u(x,t)\right) - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = F(x,t)$. However, the proof of the unique solvablity for problems (1.9) and (1.10) in the nonlinear case still remains open.

Recently, more concrete and direct proofs for the unique solvability of the problems which are contained in the class of problems (1.9) and (1.10) have been given in several papers [12], [13], [14], [15]. In Chapter 4 the known results from the papers will be mentioned briefly. Among the previously mentioned papers, I am mainly interested in the paper by Stahel [13]. This paper deals with the existence of a unique solution for the initial boundary value problem:

$$\begin{cases} u_t = \operatorname{div}\left(a(x) \operatorname{grad} u(x,t)\right) + f(u(x,t)) & \text{in } \Omega \times [0,T] \\ u(x,t) = 0 & \text{on } \quad \partial\Omega \times [0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{1.11}$$

where Ω is a smooth, bounded domain of $\mathbb{R}^N$ and T is an arbitrary, positive number. The coefficient matrix, $a(x)$, is assumed to be a positive definite, symmetric, $N \times N$ matrix but its smallest eigenvalue might converge to zero as x approaches the boundary of the domain.

The linear case of this problem is a special case of the results previously mentioned by Showalter [11]. However, the existing result applies to a wider class of functions and a does not have to degenerate with a given order but need only be bouneded from below (see equations (1.12), (1.13), and (1.14)). Nevertheless, the result is a natural extension of Showalter [11] to the semilinear case, but the methods of the proof are different. The main theorems of Stahel [13] are given as follows : (see the appendix for unfamiliar notation and basic spaces).

Theorem 1.1 Problem (1.11) has, for each $u_0 \in W_2^2(\Omega) \cap L_\infty(\Omega)$ whose support is compactly contained in Ω, a global weak solution which is Lipschitz continuous with respect to time and with values in $L_2(\Omega)$ under

Condition 1.1 Let $a(x) = [a_{ij}(x)]_{1 \leq i,\ j \leq N}$ be an $N \times N$ metrix where each $a_{ij}(x)$ is a smooth function on Ω which extends continuously to the closure of Ω . Let

$$\begin{aligned} \underline{a}(x) &= \min\{\sum_{i,j} a_{ij}(x)\xi_i\xi_j \mid \|\xi\| = 1\} \\ \overline{a}(x) &= \max\{\sum_{i,j} a_{ij}(x)\xi_i\xi_j \mid \|\xi\| = 1\} \end{aligned} \tag{1.12}$$

and suppose that there is a $c_1 > 0$ with $\underline{a}(x) \geq c_1\, d(x)^\alpha$ with $0 < \alpha < 1$ in a neighborhood of Γ where $d(x) = \operatorname{dist}(x, \Gamma)$.

Next we introduce the following condition.

Condition 1.2 With $0 < \alpha < 1$ for constants $c_1,\ c_2 > 0$,

$$c_2 d(x)^\alpha \geq \overline{a}(x) \geq \underline{a}(x) \geq c_1 d(x)^\alpha \tag{1.13}$$

Theorem 1.2 Suppose condition 1.2 holds , problem (1.11) has, for a given $u_0 \in L_2(\Omega)$, at most one weak solution which is Lipschitz continuous with respect to time and takes which values in $L_2(\Omega)$.

Theorem 1.3 If we assume the conditions from Theorem 1.1 and 1.2 at $f(0) = 0$, then the solution u of (1.11) satisfies

$$u \in C^1\big((0,T),\, L_2(\Omega)\big) \cap C^0\big((0,T),\, H_a^1(\Omega)\big). \tag{1.14}$$

In this dissertation all of these theorems are generalized. First, let us consider the following problem in one spatial dimension :

$$\begin{cases} u_t &= \frac{\partial}{\partial x}(x^\alpha u_x) \qquad \text{in } (0,1)\times[0,T] \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= \psi(x),\ \psi \in L_2\big((0,1)\big) \quad \text{where } 0<\alpha<2. \end{cases} \tag{1.15}$$

Then it is shown that (1.15) can be separated into two problems in order to have solutions in the space

$$C^0\big((0,T),\ H^1_{x^\alpha}(0,1)\big) \equiv C^0\big((0,T),\ \overset{0}{w}{}^1_{2,\,x^\alpha}(0,1)\big)$$

as follows:

$$\begin{cases} u_t &= \frac{\partial}{\partial x}(x^\alpha u_x) \quad \text{when } 0<\alpha<1 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= \psi(x) \in L_2\big((0,1)\big) \end{cases} \tag{1.16}$$

and

$$\begin{cases} u_t &= \frac{\partial}{\partial x}(x^\alpha u_x) \quad \text{when } 1\le\alpha\le 2 \\ u(1,t) &= 0 \\ |u(0,t)| &< \infty \quad \text{for finite solution or } u(0,t) \text{ has no condition} \\ u(x,0) &= \psi(x) \in L_2\big((0,1)\big) \end{cases} \tag{1.17}$$

Problem (1.16) serves as an example satisfying the theorems of the paper by Stahel [13] and the explicit solution is given in Chapter 3. One of the main result in this dissertation is a theorem for a generalized equation resulting from (1.17). The generalized equation of (1.17) is expressed as

$$\begin{cases} u_t &= \nabla\cdot(a\nabla u) + f(u(x,t)) \quad \text{in } \Omega\times I \\ u &= 0 \quad \text{on } \Gamma_1\times I \quad \text{no boundary condition on } \quad \Gamma_0\times I \\ u(0) &= u_0 \quad \text{in} \quad \Omega \end{cases} \tag{1.18}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with a smooth boundary Γ and lying in the half space $x_N > 0$ with the part Γ_0 of its boundary Γ adjoining the plane $x_N = 0$. We assume that $F(x, \overline{\xi}) = \sum_{i,j} a_{ij}(x)\xi_i\overline{\xi}_j$ of the equation above degenerates on the part Γ_0. The remaining part of the boundary Γ of Ω is denoted by Γ_1 so that $\Gamma_0 \cup \Gamma_1 = \Gamma$. Let $a(x) = [a_{ij}(x)]_{1 \le i, j \le N}$ be a matrix of smooth functions on Ω, which extend continuously to the closure of Ω, such that

$$c_2 d(x)^\alpha \geq \overline{a}(x) \geq \underline{a}(x) \geq c_1 d(x)^\alpha \text{ with } 1 \leq \alpha \leq 2 \tag{1.19}$$

for constants c_1, $c_2 > 0$. Here we assume that

(1) $f(u(x,t))$ is Hölder continuous with exponents $\beta/2$ in $(x,t) \in \bar{Q}_T$, $|u| \leq M$, and $|p| \leq M_1$. See Chapter 4 for β, M and M_1.

(2) $f(u(x,t))$ is Lipschitz continuous in t ; $f(u(x,t))$ is differentiable in u and p in $(x,t) \in \bar{Q}_T$, $|u| \leq M$, and $|p| \leq M_1$.

(3) The Lipschitz constants, $|\partial f/\partial u|$, and $|\partial f/\partial p_k|$ are bounded by a constant C.

(4) there is a number M_2 such that $f(s)s < 0$ for all $|s| \geq M_2$. This assumption implies an L_∞ a priori bound.

Then the main result in this dissertation is contained in the following theorem.

Theorem 1.4 Problem (1.18) has, for a given $u_0 \in L_2(\Omega) \cap C^2(\overline{\Omega})$, which is assumed to be a nonnegative function, a weak solution which is Lipschitz continuous with respect to time with values in $W_2^1(\Omega;\, d,\, \alpha) \subset L_2(\Omega;\, d,\, \alpha)$ and the solution is unique.

The proof of Theorem 1.4 and an explicit solution of (1.17) are given in Chapter 3.

Now let us consider the physical meaning for the problem (1.11). Generally, $a(x)$ represents a permability matrix in the case of porous media equations. We usually assume that the eigenvalues of a(x) are positive at each point in the given domain with the eigenvalues of $a(x)$ converging to zero at specified rates as x approaches on the boundary. The specified rates are given by the order of some power α of the distance x from the boundary of the given domain Ω. Generally, the power α is given as between zero and one. However, the Stahel paper [13] deals with the case that $a(x)$ does not have to degenerate with a given order but need only be bounded from below as x approaches on the boundary. See (1.12) and (1.13).

The situation of the Stahel paper [13] comes up in a porous medium with a tight boundary that is the permeability vanishes at parts of the boundary. We assume therefore that the porous medium consists of several components, among which some components have a tight boundary and the others do not , which means that the permeability is not zero on all of the boundary. Thus we cannot say that the eigenvalues of $a(x)$ approach zero as x approaches the boundary everywhere. In fact, on the loose boundary , the permeabilities will be nonzero and in general vary from point to point. Thus, the condition on the boundary cannot be given. Problem (1.18) deals with this situation under the assumptions on the rate of decay of $a(x)$ as x approaches the boundary. See (1.19).

Finally, applications of these theorems to other physical problems, such as unsaturated flows of liquids (incompressible fluids) in a porous medium, will be given in Chapter 5.

1.2 Some Preliminary Mathematical Notations and Theorems

This section provides some preliminary mathematical facts needed in this dissertation. However, a more detailed discussion and elaboration of the basic notations and definitions are given in the Appendices 1 and 3 . The basic function spaces used in this dissertation also are described in Appendix 2.

Definition 1. Let

$$Lu = a_0 D^n u + a_1 D^{n-1} u + \cdots + a_n u = f \tag{1.20}$$

where the a_i are smooth functions of their arguments and D stands for differentiation with respect to the spatial variables. Then

(1) any distribution satisfying (1.20) is called a generalized solution.

(2) a classical solution of (1.20) is an ordinary function which is n times continuously differentiable and satisfies it (and therefore generates a regular distribution which satisfies (1.20) in the generalized sense).

(3) a weak solution is an ordinary function which may not be n times differentiable, and therefore may not be a classical solution, but which generates a regular distribution which is a generalized solution.

(4) a distribution solution is a singular distribution satisfying (1.20).

(See [16])

Definition 2. Quasi-linear equations of parabolic type with principal part in divergence form have the form :

$$\mathcal{L}_u \equiv u_t - \frac{d}{dx_i}(a_i(x,t,u,u_x)) - b(x,t,u,u_x) = 0 \tag{1.21a}$$

or

$$\mathcal{L}_u \equiv u_t - \operatorname{div} a(x,t,u,u_x) - b(x,t,u,u_x) = 0 \tag{1.21b}$$

resulting in a quasilinear equation of the general form:

$$\mathcal{L}_u \equiv u_t - a_{ij}u_{x_i x_j} + b(x,t,u,u_x) = 0 \tag{1.22}$$

Definition3. If for an arbitrary nonzero real vector $\xi = (\xi_1, \ldots \xi_N)$,

$$\frac{\partial a_i(x,t,u,p)}{\partial p_j}\xi_i\xi_j > 0$$

for $(x,t) \in \overline{Q}_T$ and arbitrary u and p, then we say that the operator $\mathcal{L}$ is of parabolic type.

Definition 4. If there exist functions $\nu(\tau)$ and $\mu(\tau)$, where $\nu(\tau)$ is a positive, nonincreasing, continuous function defined for $\tau \geq 0$ and $\mu(\tau)$ is a positive, nondecreasing, continuous function defined for $\tau \geq 0$, such that for an arbitrarily real vector, $\xi = (\xi_1, \ldots, \xi_N) \neq 0$,

$$\nu(|u|)|\xi|^2 \leq \frac{\partial a_i(x,t,u,p)}{\partial p_j}\xi_i\xi_j \leq \mu(|u|)|\xi|^2$$

for arbitrary u and p and $(x,t) \in \overline{Q}_T$, then we say that $\mathcal{L}$ is uniformly parabolic.

Definition 5. If the coefficient functions of the principal part of equation (1.21a,b) satisfying the parabolic condition vanish for certain values of $(x,t) \in \overline{Q}_T$, u and u_x, then the equation is said to be degenerate parabolic.

Example. $u_t - \operatorname{div}(|Du|^{p-2}Du) = 0,\ p > 2$. When $|Du| = 0$, this equation is degenerate.

Definition 6. Let J be a nonnegative, real-valued function belonging to $C_0^\infty(\mathbb{R}^N)$ and having the properties

(1) $J(x) = 0 \quad \text{if} \quad |x| \geq 1 \quad \text{and}$

(2) $\int_{\mathbb{R}^N} J(x)\, dx = 1$

If $\epsilon > 0$, the function $J_\epsilon(x) = \epsilon^{-N} J(\frac{x}{\epsilon})$ is nonegative , belong to $C_0^\infty(\mathbb{R}^N)$ and satisfies

(1) $J_\epsilon(x) = 0 \quad \text{if} \quad |x| \geq \epsilon \quad \text{and}$

(2) $\int_{\mathbb{R}^N} J_\epsilon(x)\, dx = 1$

J_ϵ is called a mollifier, and the convolution

$$J_\epsilon * u(x) = \int_{\mathbb{R}^N} J_\epsilon(x-y)u(y)\, dy$$

is called a mollification of u.

For example , we may take

$$J(x) = \begin{cases} ke^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

where $k > 0$ is so chosen that condition (2) is satisfied.

(See [1] and [33])

Theorem 1.5 Let u be a function which is defined in $\mathbb{R}^N$ and vanishes identically outside the domain Ω.

(1) If $u \in L^1_{loc}(\overline{\Omega})$, then $J_\epsilon * u \in C^\infty$ $(\mathbb{R}^N)$.

(2) If also $\operatorname{supp} u \subset\subset \Omega$, then $J_\epsilon * u \in C_0^\infty(\Omega)$ provided $\varepsilon <$ dist ($\operatorname{supp} u, \partial\Omega$).

(3) If $u \in L_p(\Omega)$ where $1 \leq p \leq \infty$, then $J_\epsilon * u \in L_p(\Omega)$. Moreover, $||J_\epsilon * u||_p \leq ||u||_p$ and $\lim_{\varepsilon \to 0^+} ||J_\varepsilon * u - u||_p = 0$.

(4) If $u \in C(\Omega)$ and $G \subset\subset \Omega$, then $\lim_{\varepsilon \to 0^+} J_\varepsilon u(x) = u(x)$ uniformly on G.

(5) If $u \in C(\overline{\Omega})$, then $\lim_{\varepsilon \to 0^+} J_\varepsilon u(x) = u(x)$ uniformly on Ω

(See [18], [19] pp. 29-31)

Theorem 1.6 Let $-\infty \le a \le b \le \infty$. Let $\lambda = \lambda(t)$ be a continuously differentiable function defined on (a, b) and such that $\lambda'(t) > 0$ for $t \in (a, b)$.

[A] Let $\lim_{t \to b} \lambda(t) = \infty$; Further, let us denote by $\lambda(a) = \lim_{t \to a} \lambda(t)$ and define weight functions σ_1, σ_0 in the following way:

$$\sigma_1(t) = e^{(1-p)\lambda(t)}[\lambda'(t)]^{1-p}, \quad \sigma_0(t) = e^{\lambda(t)}\lambda'(t)[e^{\lambda(t)} - e^{\lambda(a)}]^{-p}.$$

[B] Let $\lim_{t \to a} \lambda(t) = -\infty$; Further, let us denote by $\lambda(b) = \lim_{t \to b} \lambda(t)$ and define weight functions σ_1, σ_0 in the following way:

$$\sigma_1(t) = e^{(p-1)\lambda(t)}[\lambda'(t)]^{1-p}, \quad \sigma_0(t) = e^{-\lambda(t)}\lambda'(t)[e^{-\lambda(t)} - e^{-\lambda(b)}]^{-p}.$$

Let $1 < p < \infty$ and let $u = u(t)$ be a function which is almost everywhere differentiable on (a, b) such that

$$\int_a^b |u'(t)|^p \sigma_1(t)\, dt < \infty.$$

Further let u satisfy the condition $u(a) = \lim_{t \to a} u(t) = 0$ in Case A and $u(b) = \lim_{t \to b} u(t) = 0$ in Case B. Then the inequality

$$\int_a^b |u(t)|^p \sigma_0(t)\, dt \le \left(\frac{p}{p-1}\right)^p \int_a^b |u'(t)|^p \sigma_1(t)\, dt.$$

(See Kufner [20] , pp. 30-34)

Theorem 1.7 POINCARÉ INEQUALITIES :

(i)

$$\int_\Omega \frac{u^2(x)}{|x-y|^2}\, dx \le \frac{4}{(N-2)^2} \int_\Omega u_x^2(x)\, dx, \qquad N > 2$$

where Ω is any domain in $\mathbb{R}^N$, $u(\cdot)$ is an element of $C_0^\infty(\Omega)$, and $y \in \mathbb{R}^N$. If $N = 2$, then the inequality

(ii)

$$\int_\Omega \frac{u^2(x)}{|x-y|^2(\ln|x-y|^2)}\,dx \le 4\int_\Omega u_x^2(x)\,dx$$

for any domain $\Omega \subset \mathbb{R}^2$, $y \notin \Omega$, and $u \in C_0^\infty(\Omega)$

(iii)

$$\int_\Omega \frac{|u(x)|^p}{|x-y|^l}\,dx \le \left|\frac{p}{N-l}\right|^p \int_\Omega \frac{|u_x(x)|^p}{|x-y|^{l-p}}\,dx$$

for all $p > 0, l \neq N$ and $u \in C_0^\infty(\Omega)$

(See [21], pp. 41-42)

Theorem 1.8 Let Ω be a bounded smooth domain. Then the set $C^\infty(\overline{\Omega})$ is dense in $W_p^k(\Omega; d, \varepsilon)$ for $\varepsilon \ge 0$. That is, $W_p^k(\Omega; d, \varepsilon) = \overline{C^\infty(\overline{\Omega})}$ holds, where the closure of the set $C^\infty(\overline{\Omega})$ refers to the norm $||u||_{k,p;d,\varepsilon}$

(See Kufner [20] , pp . 43-49)

Theorem 1.9 If $x(t)$ is a solution of the Gronwall Inequality

$$|x(t)| \le a(t) + \int_0^t |x(s)|\,b(s)\,ds\,, \quad t \ge 0,$$

where $a(\cdot)$ and $b(\cdot)$ are continuous, nonnegative functions, then

$$|x(t)| \le a_\infty(t)e^{B(t)},$$

where $a_\infty(t) = \max_{0 \le s \le t} a(s)$ and $B(t) = \int_0^t b(s)\,ds$.

(See [22], pp. 141)

CHAPTER 2

A NONLINEAR POROUS MEDIA EQUATION OF PARABOLIC TYPE

In this chapter, I treat a nonlinear porous media equation of parabolic type and a more general porous media equation in one spatial dimension of hyperbolic type. The hyperbolic porous media equation is obtained by applying the more general form of Darcy's law (1.7) instead of the usual form Darcy's law (1.3). First I will show that the solutions of the nonlinear porous media equation have a finite speed of propagation, which is the most important property of degenerate parabolic equations, by obtaining a similarity solution of the equation explicitly. Then, I will show, in an extended result, that a similarity solution for the general porous media equation, at large time scales, is actually one of the similarity solutions of the nonlinear porous media equation.

2.1 A Finite Speed of Solutions for a One Dimensional Porous Media Equation

Now consider the Cauchy problem for a one-dimensional porous media equation.

$$\begin{cases} u_t & = (u^m)_{xx} \quad \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) & = u_0(x) \quad \text{for } x \in \mathbb{R} \end{cases} \tag{2.1}$$

where $m > 1$ and u_0 is a bounded continuous nonnegative function and u represents the density of a gas where the units have been chosen so that the constants are one. (Here we use u as a generic notation for a sought after function instead of ρ which is used in the introduction.)

Then the porous media equation is degenerate in the neighborhood of any point where u vanishes. As mentioned in the introduction, the solution of the porous media equation fails to be a classical solution at precisely those points on the interface of the solution . We need a concept of weak solutions but some explicit solutions of the porous media equation were found. These were all self similar solutions and the most important one of them was found by G.I. Barenbatt in the paper " *On some unsteady motions of a liquid or a gas in a porous medium* " [4]. I seek the Barenblatt solution for (2.1) here. Roughly speaking, the Barenblatt solution is the solution of (2.1) whose initial datum is a mass M concentrated at the origin. That is, $u(x,0) = u_0(x) = M\delta(x)$, where δ is the Dirac delta function and $M = \int_{\mathbb{R}} u_0(x)\,dx$ is the total mass (initial data representing an instantaneous point source).

The flow of an ideal gas in a homogenous porous medium is governed by three laws:

(1) conservation of mass

(2) Darcy's law

(3) equation of state

Now let us write equation (2.1) as a conservation law for the mass:

$$u_t + (u \cdot v)_x = 0\,,$$

where u is the density of a gas and v is the velocity vector. (Here we change notation from q to v where q was used in the introduction.) Then

$$u_t + (u \cdot (-mu^{m-2}u_x))_x = 0.$$

Thus the local velocity of the gas is given by

$$v = -mu^{m-2}u_x = -\left(\tfrac{m}{m-1}u^{m-1}\right)_x$$

and by Darcy's law we get $v = -(k/\mu)\,\mathrm{grad}\,p$. Hence, we can define the pressure p as the potential of the vector field. That is, $v = -p_x$ so that $p = \frac{m}{m-1}u^{m-1}$. In general, we define the presure p and define a transformation $p = \psi(u)$ of a classical solution u of (2.1) as follows. The transformation ψ is given by

$$\psi(s) = \int_0^s m\xi^{m-2}\, d\xi \qquad \text{for } s.$$

Here we assume that $\int_0^s m\xi^{m-2}\, d\xi < \infty$.

In the general case of $u_t = (\phi(u))_{xx}$, the transformation ψ is given by

$$\psi(s) = \int_0^s \frac{\phi'(\xi)}{\xi}\, d\xi \qquad \text{for } s.$$

Thus we have

$$p(x,t) = \psi(u(x,t)) = \int_0^{u(x,t)} m\xi^{m-2}\, d\xi = \tfrac{m}{m-1}u^{m-1}$$

and since $\frac{\partial\psi(u)}{\partial t} = mu^{m-2}u_t$, then

$$u_t = \frac{1}{mu^{m-2}}\frac{\partial\psi(u)}{\partial t} = \frac{1}{mu^{m-2}}p_t$$

and

$$(u^m)_x = mu^{m-1}u_x = up_x, \quad (u^m)_{xx} = up_{xx} + u_xp_x$$

Since $u_t = (u^m)_{xx}$ and $u_t = p_t/mu^{m-2}$, we have

$$p_t = (m-1)p\cdot p_{xx} + p_x^2.$$

Hence, the evolution of p is governed by the pressure equation

$$p_t = (m-1)p\cdot p_{xx} + p_x^2.$$

Now let us seek a self similar solution of the following type in the domain $0 < x < \infty,\ 0 < t \leq T$ where T is some positive constant ;

$$u(x,t) = (t+\tau)^{\beta} f(\zeta), \quad \zeta = x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}}$$

where $\tau \in \mathbb{R}$ is arbitrarily , $t + \tau > 0$.

By substituting this solution into equation (2.1) we get

$$(f^m(\zeta))'' + \frac{1}{2}\{1+(m-1)\beta\}\zeta f'(\zeta) = \beta f(\zeta), \ 0 < \zeta < \infty.$$

At the boundary we impose the condition $f(0) = K, \ f(\infty) = 0$, where K is constant. Thus the solution $u(x,t)$ satisfies the lateral boundary condition $u(0,t) = (t+\tau)^{\alpha} K$ and $u(x,t) \to 0$ as $x \to \infty$ for fixed $t \in [0,T]$.
Let $\tilde{p} = (1+(m-1)\beta)/2, \ \tilde{q} = \beta$. Then the following is obtained:

$$\begin{cases} (f^m)'' + \tilde{p}\zeta f' = \tilde{q} f \\ f(0) = K, \quad f(\infty) = 0 \end{cases} \tag{2.2}$$

Now it is necessary to consider weak solutions to the problem (2.2) because I am looking for a weak solution of $u(x,t) = (t+\tau)^{\beta} f(\zeta)$.
Here a function f will be said to be a weak solution of (2.2) if

(i) f is bounded, continuous and nonnegative on $[0,\infty)$.

(ii) $(f^m)(\zeta)$ has a continuous derivative with respect to ζ on $[0,\infty)$.

(iii) f saisfies the identity

$$\int_0^{\infty} \phi' \{(f^m)' + \tilde{p}\zeta f\} \, d\zeta + (\tilde{p} + \tilde{q}) \int_0^{\infty} \phi f \, d\zeta = 0 \text{ for all } \phi \in C_0^1([0,\infty)).$$

Then the following results are known in [23] :

Theorem 2.1) when $K > 0$, the equation (2.2) has a weak solution with compact support if and only if $\tilde{p} \geq 0$ and $2\tilde{p} + \tilde{q} > 0$. This solution is unique. And, if we let $f(\zeta)$ be a weak solution of problem (2.2) with compact support, then the solution is of the form: $f(\zeta) > 0$ on $[0,a)$, $f(\zeta) = 0$ on $[a,\infty)$ for some $a > 0$.

We are considering the Berenblatt solution when $\beta = -\frac{1}{m+1}, K > 0$, so $2\tilde{p} + \tilde{q} = 1 + m\beta = \frac{1}{m+1} > 0$ and $\tilde{p} = \frac{1+(m-1)\beta}{2} = \frac{1}{m+1} > 0$. Thus, by the Theorem 2.1 , problem (2.2) has a weak solution with compact support. Hence, by integrating (2.2) , we have

$$f(\zeta) = \begin{cases} \{\frac{m-1}{2m(m+1)}(a^2 - \zeta^2)\}^{\frac{1}{m-1}} & 0 \le \zeta \le a \\ 0 & a < \zeta < \infty \end{cases}$$

Consequently ,

$$u(x,t) = \begin{cases} (t+\tau)^{-\frac{1}{m+1}} f(x(t+\tau)^{-\frac{1}{m+1}}) & 0 \le x \le a(t+\tau)^{\frac{1}{m+1}} \\ 0 & a(t+\tau)^{\frac{1}{m+1}} < x < \infty \end{cases} \tag{2.3}$$

Let $A = \frac{m-1}{2m(m+1)}a^2, \quad B = \frac{m-1}{2m(m+1)}, \quad \alpha = \frac{1}{m+1}$.

Then we can write (2.3) as

$$u(x,t) = (t+\tau)^{-\alpha}\left\{\left[A - Bx^2(t+\tau)^{-2\alpha}\right]_+\right\}^{1/(m-1)} \tag{2.4}$$

Now expand (2.4) where $t \in \mathbb{R}^+$ and take the initial mass to be concentrated at the origin. That is, $u(x,0) = M\delta(x)$ and M is the total mass for all $t \in \mathbb{R}^+$. Let $u(x,t,M)$ denote the solution of (2.1) when the total mass is M. Let $\tau = 0$ in (2.4). The Barenblatt solution of (2.1) is given by

$$u(x,t;M) = t^{-\alpha}\{[A - B\,|x|^2\, t^{-2\alpha}]_+\}^{1/(m-1)}$$

which satisfies $\int_{\mathbb{R}} u(x,t;M)\,dx = M$ for all $t \in \mathbb{R}^+$.

Let $F(\xi) = \{[A - B\xi^2]_+\}^{1/(m-1)}$ where $\xi = |x|t^{-\alpha}$. Then,

$$\int_{\mathbb{R}} u(x,t;M)\,dx = w_1 \int_0^\infty F(\xi)\,d\xi = M \quad \text{for all}\, t \in \mathbb{R}^+,$$

where w_1 denotes the volume of the unit ball in $\mathbb{R}^1$. Thus we get the relation

$$A^{(m+1)/2(m-1)}B^{-1/2}w_1 \int_0^{\frac{\pi}{2}} (\cos\theta)^{(m+1)/(m-1)}\,d\theta = M$$

In general , in n dimension , $w_n \int_0^\infty F(\xi)\xi^{n-1}d\xi = M$. Thus

$$w_n A^{(m+1)/2(m-1)} B^{-1/2} \int_0^{\frac{\pi}{2}} (\cos\theta)^{(m+1)/(m-1)} (\sin\theta)^{n-1}\, d\theta = M$$

Now, let $C = w_1 \int_0^{\frac{\pi}{2}} (\cos\theta)^{(m+1)/(m-1)}\, d\theta$. Then

$$M = A^{(m+1)/2(m-1)} B^{1/2} C.$$

Hence,

$$A = (M^2 B C^{-2})^{(m-1)/(m+1)}$$

It is obvious that the Barenblatt solution u is also a classical solution of (2.1) and is a C^∞ function on the set $\mathcal{P}[u] = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : u(x,t) > 0\}$.

But let us consider $p = mu^{m-1}/(m-1)$. Then

$$\begin{aligned} p &= \tfrac{m}{m-1} \left(t^{-\alpha} \left\{ \left[A - B|x|^2 t^{-2\alpha} \right]_+ \right\}^{\frac{1}{m-1}} \right)^{m-1} \\ &= \tfrac{m}{m-1} t^{-\alpha(m-1)} \tfrac{B}{t^{2\alpha}} \left[\left(\sqrt{\tfrac{A}{B}} t^\alpha \right)^2 - |x|^2 \right]_+ = \tfrac{\alpha}{2t} [r^2(t) - |x|^2]_+ \end{aligned}$$

from the fact that $mt^{-\alpha(m-1)} B/(m-1)t^{2\alpha} = \alpha/2t$ and where we have set $r(t) = \sqrt{\frac{A}{B}} t^\alpha$. Here

$$r(t) = \sqrt{\tfrac{A}{B}} t^\alpha = \left(M^{\frac{2(m-1)}{m+1}} B^{\frac{m-1}{m+1}} C^{-\frac{2(m-1)}{m+1}} B^{-1} \right)^{\frac{1}{2}} t^\alpha = c_m M^{\frac{m-1}{m+1}} t^{\frac{1}{m+1}}$$

where $c_m = B^{-1/(m+1)} C^{-(m-1)/(m+1)}$. Thus,

$$r(t) = c_m M^{(m-1)/(m+1)} t^{1/(m+1)}.$$

The pressure p is positive on the set

$$\mathcal{Q}[u] = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : |x| < r(t)\}$$

Hence , u(x,t) is positive for all (x,t) $\in \mathcal{Q}[u]$. Thus

$$\mathcal{Q}[u] = \mathcal{P}[u] = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : u(x,t) > 0\}.$$

Therefore,

$$I[u] = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : |x| = r(t)\}$$

is the interface, since it is the boundary of supp $u = \overline{\mathcal{P}[u]}$. where the bar indicates closure. So, u is actually a classical solution of (2.1) in $\mathbb{R} \times \mathbb{R}^+ \setminus I[u]$ but it is not a classical solution in all of $\mathbb{R} \times \mathbb{R}^+$ since $\nabla(u^{m-1})$ has jump discontinuities across $I[u]$.

By Darcy's law we expect velocity $v = -p_x$ and we also expect that the interface will move with the local velocity of the gas. Hence we can expect $\dot{r}(t) = -p_x(r(t)^-,t) = v(r(t),t)$ where $p_x(r(t)^-,t) = \lim_{x\uparrow r(t)} p_x(x,t)$. But $D^+r(t) = -p_x(r(t)^-,t)$ is in $\mathbb{R}^+$ where D^+ denotes the right-hand derivative. See Herrero and Vazquez [24].

Now the outer right interface of u is defined to be the curve $x = r(t)$, where $r(t) = \sup\{x : u(x,t) > 0\}$ if $0 < t < T^* < \infty$,

$$r(0) = \sup\{x : \int_{(x,\infty)} du_0 = \int_{(x,\infty)} dM\delta(x)\} > 0.$$

The interface does not necessarily begin to move at $t = 0$ although it must ultimately move at some point. There is a $t^* \in [0,\infty)$, called the waiting time such that (i) $r(t) \equiv 0$ on $[0,t^*]$; and (ii) $r(t)$ is strictly increasing on (t^*,∞) which means that once the interface begins to move it never stops (See Aronson , Caffarelli and Kamin [8] , and Vazquez [10] for work on the determination of the waiting times). The *speed of propogation* refers to the speed of the interface. Since the interface will move with the local velocity of the gas, from $r(t) = c_m M^{(m-1)/(m+1)} t^{1/(m+1)}$, we can compute $\dot{r}(t) = -p_x(r(t)^-, t)$. And since $p = \frac{\alpha}{2t}\left[r^2(t) - |x|^2\right]_+$, we get

$$p_x = -\alpha|x|/t.$$

Thus,

$$\dot{r}(t) = -p_x(r(t)^-, t) = -\lim_{x \uparrow r(t)} p_x(x,t) = \lim_{x \uparrow r(t)} \tfrac{\alpha}{t}|x| \approx \tfrac{\alpha}{t} r(t)$$

and by letting $r(t) = c_m M^{\frac{m-1}{m+1}} t^{\frac{1}{m+1}}$, we get

$$\dot{r}(t) \approx \tfrac{1}{m+1} c_m M^{\frac{m-1}{m+1}} t^{-\frac{m}{m+1}}.$$

Furthermore, as $t \to \infty$,

$$\dot{r}(t) = \tfrac{1}{m+1} c_m M^{\frac{m-1}{m+1}} t^{-\frac{m}{m+1}} + \circ\left(\tfrac{1}{t}\right).$$

The proof that $\dot{r}(t) t^{m/m+1}$ converges to $c_m M^{(m-1)/(m+1)}/(m+1)$ can be found in Aronson , Caffarelli , and Kamin paper [8] and [25].

From $\dot{r}(t) = c_m M^{(m-1)/(m+1)}/(m+1) t^{m/(m+1)}$, it is known that the velocity $\dot{r}(t)$ is bounded for all t such that $t^* > 0$ and $t \in [t^*, \infty)$, where t^* is the waiting time. In fact, for every solution of (2.1), the velocity is bounded in every strip $S_{\tau,\infty} = \mathbb{R} \times (\tau, \infty)$ where $\tau > 0$. Hence we know that the speed of propagation is finite. Moreover Vazquez[9] and, Herrero and Vazquez[24] prove that

$$p_x^2 = v^2(x,t) \leq \tfrac{2}{m+1} ||p(x,0)||_\infty t^{-1}.$$

2.2 An Extension to a General Porous Media Equation and a Comparison Solutions of Nonlinear Porous Media Equations

Let us consider the case when the more general form of Darcy's law, (1.7), instead of (1.3). Here we assume that $\rho/\rho_0 = 1+\theta$ where θ is small. Then $\rho = \rho_0(1+\theta)$. Thus, $\text{div}\ (\rho q) = \rho_0 \text{ div } q + \rho_0 \text{ div } \theta q$.
Here we neglect the term $\rho_0 \text{ div } \theta q$ so $\text{div}\ (\rho q) = \rho_0 \text{ div } q$. By conservation of mass, (1.4), we have

$$\phi \rho_t = -\rho_0 \text{ div } q. \tag{2.5}$$

By differentiating both sides of (2.5) with respect to t we have

$$\phi \rho_{tt} = -\rho_0 \text{ div } q_t. \tag{2.6}$$

By taking the divergence in (1.7), we have

$$\varepsilon \,\text{div}\, q_t + \text{div}\, q = -\tfrac{k}{\mu}\Delta p. \tag{2.7}$$

Therefore, by inserting (2.5) and (2.6) into (2.7), we obtain

$$-\tfrac{\phi\varepsilon}{\rho_0}\rho_{tt} - \tfrac{\phi}{\rho_0}\rho_t = -\tfrac{k}{\mu}\Delta p. \tag{2.8}$$

From the equation of state, (1.5), $p = p_0(\rho/\rho_0)^{1/\lambda}$. By inserting this into (2.8),

$$\tfrac{\phi\varepsilon}{\rho_0}\rho_{tt} + \tfrac{\phi}{\rho_0}\rho_t = \tfrac{k\,p_0}{\mu(\rho_0)^{1/2}}\Delta\rho^{1/\lambda}.$$

Hence since $\rho \approx \rho_0$,

$$\varepsilon\rho_{tt} + \rho_t = \tfrac{k\,p_0\rho_0}{\phi\mu(\rho_0)^{1/\lambda}}\Delta\rho^{\frac{1}{\lambda}} \approx \tfrac{k\,p_0}{\phi\mu\rho_0^{1/\lambda}(\lambda+1)}\Delta\rho^{(\lambda+1)/\lambda}$$

By rescaling we can take $k\,p_0/\phi\mu\rho_0^{1/\lambda}(\lambda+1)$ to be equal to 1 and we let $(\lambda+1)/\lambda = m$. Then we obtain

$$\varepsilon\rho_{tt} + \rho_t = (\rho^m)_{xx}.$$

Replace ρ by u so that the standard form of the equation is

$$\varepsilon u_{tt} + u_t = (u^m)_{xx}. \tag{2.9}$$

In order to find a solution to (2.9), let us try a similar solution of the following type :

$$u(x,t) = (t+\tau)^{\beta} f(\zeta), \qquad \zeta = x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}}$$

where $\tau \in \mathbb{R}$ is arbitrarily , $t+\tau > 0$. Then by differentiating the $u(x,t)$ term with respect to t, we have

$$u_t = \beta(t+\tau)^{\beta-1} f(\zeta) + (t+\tau)^{\beta} f'(\zeta)\tfrac{\partial\zeta}{\partial t}. \tag{2.10}$$

and

$$u_{tt} = \beta(\beta-1)(t+\tau)^{\beta-2} f(\zeta) + \beta(t+\tau)^{\beta-1}\tfrac{\partial\zeta}{\partial t} f'(\zeta)$$

$$+\left\{\beta(t+\tau)^{\beta-1}\tfrac{\partial\zeta}{\partial t} + (t+\tau)^{\beta}\tfrac{\partial^2\zeta}{\partial t}\right\} f'(\zeta) + (t+\tau)^{\beta}\left(\tfrac{\partial\zeta}{\partial t}\right)^2 f''(\zeta) \tag{2.11}$$

and

$$(u^m)_{xx} = (f^m)''(\zeta)(t+\tau)^{\beta-1}. \tag{2.12}$$

By inserting (2.10), (2.11), and (2.12) into (2.9), and dividing by $(t+\tau)^{\beta-1}$, we have

$$(f^m)'' - \varepsilon(t+\tau)\left(\tfrac{\partial\zeta}{\partial t}\right)^2 f'' - \left\{2\varepsilon\beta\tfrac{\partial\zeta}{\partial t} + \varepsilon(t+\tau)\tfrac{\partial^2\zeta}{\partial t} + (t+\tau)\tfrac{\partial\zeta}{\partial t}\right\} f'$$

$$-\{\varepsilon\beta(\beta-1)(t+\tau)^{-1} + \beta\} f(\zeta) = 0. \tag{2.13}$$

Also, from $\zeta = x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}}$, we obtain

$$\left\{\begin{array}{ll} \frac{\partial\zeta}{\partial t} & = -\frac{1}{2}\{1+(m-1)\beta\}\, x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}-1} \\ \frac{\partial^2\zeta}{\partial t^2} & = \{\frac{1}{4}\{1+(m-1)\beta\}^2 \\ & +\frac{1}{2}\{1+(m-1)\beta\}\} x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}-2} \end{array}\right. \tag{2.14}$$

Then by inserting (2.14) into (2.13), we have

$$
\begin{aligned}
(f^m)'' & -\tfrac{\varepsilon}{4}\{1+(m-1)\beta\}^2\zeta^2(t+\tau)^{-1}f'' \\
& -\zeta\left(-\varepsilon\beta\{1+(m-1)\beta\}(t+\tau)^{-1}-\tfrac{1}{2}\{1+(m-1)\beta\}\right. \\
& \left.+\varepsilon\left[\tfrac{1}{4}\{1+(m-1)\beta\}^2+\tfrac{1}{2}\{1+(m-1)\beta\}\right](t+\tau)^{-1}\right)f' \\
& -\left(\varepsilon\beta(\beta-1)(t+\tau)^{-1}+\beta\right)f=0. \qquad (2.15)
\end{aligned}
$$

Let T be sufficiently large so that for all $t \geq T$, $\frac{1}{t+\tau} \approx 0$. Thus, (2.15) becomes

$$
(f^m)''+\tfrac{1}{2}\{1+(m-1)\beta\}\zeta f'=\beta f(\zeta). \qquad (2.16)
$$

If we impose the boundary condition $f(0)=K$, $f(\infty)=0$, $K>0$ constant on (2.16), we obtain (2.2). If we take $\beta=-1/(m+1)$, then the solution is the Barenblatt solution. If we take $\beta=1/(m-1)>0$ for $m>1$, (2.16) becomes

$$
(f^m)''+\zeta f'=\tfrac{1}{m-1}f(\zeta). \qquad (2.17)
$$

then (2.17) has a weak solution with compact support by the Theorem 2.1.

The function

$$
f(\zeta)=\begin{cases}\left\{\left[\frac{m-1}{m}\right]a(a-\zeta)\right\}^{\frac{1}{m-1}}, & 0\leq\zeta\leq a\\ 0, & a<\zeta<\infty\end{cases}
$$

satisfies (2.17) with boundary conditions $f(0)=K$, $f(\infty)=0$. Therefore,

$$
u(x,t)=\begin{cases}(t+\tau)^{\frac{1}{m-1}}f(\zeta), & 0\leq\zeta\leq a\\ 0, & a<\zeta<\infty\end{cases}
$$

so

$$
u(x,t)=\begin{cases}\left\{\left[\frac{m-1}{m}\right]a[a(t+\tau)-x]\right\}^{\frac{1}{m-1}}, & 0\leq x\leq a(t+\tau)\\ 0, & a(t+\tau)<x<\infty\end{cases}
$$

This is the wave solution found in *Oleinik* , *Kalashnikov and Yui-Lin* [3] and is one of the similarity solutions of the equation $u_t = (u^m)_{xx}$ (See Gilding and Peletier [23]). Thus for sufficiently large time scales, a similarity solution of $\varepsilon u_{tt} + u_t = (u^m)_{xx}$ reduces to the similarity solution of $u_t = (u^m)_{xx}$.

CHAPTER 3

THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR AN INITIAL BOUNDARY PROBLEM OF A DEGENERATE PARABOLIC EQUATION WITH PRINCIPAL PART IN DIVERGENCE FORM

In this chapter, I deal with the existence of a unique solution of initial-boundary value problems for the following equation

$$u_t - \operatorname{div}(a(x)\operatorname{grad} u(x,t)) = f(u(x,t)). \tag{3.1}$$

The existence of a unique solution of the initial-boundary problem (1.11) under the conditions 1.1 and 1.2 is shown in Theorems 1.1 , 1.2 and 1.3 in the introduction. A proof has also been given by Andreas Stahel [13]. In his paper he assumes the existence of a sequence of open sets Ω_n with smooth boundaries such that

$$\Omega_n \subset \Omega_{n+1} \subset \Omega,\ \cup_{n\in N}\Omega_n = \Omega,\ \lim_{n\to\infty} \sup\{\underline{a}(x) | x \in \Omega/\Omega_n\} = 0$$

and also assumes that there are functions a_n defined on Ω such that

(1) $a_n\big|_{\Omega_n} = a$;

(2) $\underline{a}_n(x) \geq \frac{1}{n}$;

(3) $a_n(x)\xi\cdot\xi \geq a(x)\xi\cdot\xi$, for all $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

(4) $\lim_{n\to\infty} \sup\{||a_n(x) - a(x)||\ |x \in \Omega\} = 0$.

The above assumptions are such that the coefficient matrix **a** can only degenerate on a part of Γ. Then Stahel [13] defines u to be a weak solution of (1.11) if the following conditions are satisfied:

$$u \in C^\circ(I, L_2(\Omega)) \cap C^\circ(I, H^1_a(\Omega'))$$

for any compact subset Ω' of Ω and

$$||u(t)||_{H^1_a(\Omega} + ||u(t)||_\infty \leq c, \; u(0) = u_0, \; u(t)|_\Gamma = 0\,,$$

and

$$\langle\psi(T), u(T)\rangle - \langle\psi(0), u_0\rangle - \int_0^T \langle\dot{\psi}(t), u(t)\rangle\, dt$$
$$= \int_0^T -\langle a\nabla\psi(t),\; \nabla u(T)\rangle + \langle\psi(t),\; f(u(t))\rangle\, dt$$

for all $\psi \in C^1(I, L_2(\Omega)) \cap C^0(I, \overset{0}{W}{}^1_2(\Omega))$, for all $T \in I$.

In section 3.1 , I consider (3.1) in one-dimensional spatial setting $\Omega = (0,1)$ when $a(x) = x^\alpha$, $0 < \alpha \leq 2$. Then equation (1.11) is the same as (1.15). It will be shown that (1.15) must be separated into either (1.16) or (1.17) depending on whether $0 < \alpha < 1$ or $1 \leq \alpha \leq 2$ in order to have solutions in the space $C^\circ\big((0,T),\; H^1_{x^\alpha}(0,1)\big)$. It will also be shown that (1.16) serves as an example satisfying Theorems 1.1, 1.2, and 1.3 . Then the explicit solutions for (1.16) and (1.17) are also given.

In section 3.2 , I consider equation (1.18), which is a generalized form of (1.17) in higher spatial dimensions. Theorem 1.4 will also be proved for the existence of a unique solution of (1.18).

3.1 Boundary Conditions for a Unique Solution and Explicit Solutions for Examples in One Dimensional Space

Let us consider (1.15). This equation degenerates at the boundary point $x = 0$. Let us try to solve this by the separation of variables method. For this purpose, set $u(x,t) = X(x)T(t)$ and insert it into (1.15). We get:

$$\begin{cases} \frac{\partial}{\partial x}\left(x^\alpha X_x\right) + kX & = 0 \quad \text{for } k > 0,\; 0 < \alpha \leq 2 \\ X(0) = X(1) & = 0 \end{cases} \tag{3.2}$$

and

$$T' + kT = 0 \tag{3.3}$$

First, let us consider (3.2). We can think of (3.2) as a problem for an elliptic equation degenerating at the point $x = 0$. Then, for investigation of the boundary value problem, it is reasonable to take as the Hilbert space the space $\overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$ with the inner product :

$$(u, v) = \int_\Omega (x^\alpha u_x v_x + uv)\, dx \tag{3.4}$$

and note that the set $C_0^\infty(\Omega)$ is dense in $\overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$.

In other words, $\overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in the norm corresponding to (3.4) and has such properties that $X(x) \in \overset{\circ}{W}{}^1_{2,x^\alpha}((0,1))$ is continuous on $(0, 1]$ and has generalized first derivatives on $(0, 1)$ and $X(1) = 0$. However, the space has different properties at the boundary point x $= 0$ depending on the size of α , as $0 < \alpha < 1$ or $1 \le \alpha \le 2$, though the inner products are defined in the same way. That is ,

[A] : for $\alpha \in (0, 1)$, all elements of $\overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$ are equal to zero at $x = 0$. But

[B] : for $\alpha \in [1, 2]$, this is not the case. Hence, we must remove the condition $X(0) = 0$ in order to solve (3.2) with $\alpha \in [1, 2]$.

Proof of [A]: When $0 < \alpha < 1$ and $X \in \overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$, $\Omega = (0, 1)$, then

$$\begin{aligned}[X(x)]^2 &= \left[\int_0^x \tau^{-\frac{\alpha}{2}} \tau^{\frac{\alpha}{2}} \frac{\partial X}{\partial \tau}\, d\tau\right]^2 \le \int_0^x \left(\tau^{-\frac{\alpha}{2}}\right)^2 d\tau \int_0^x \tau^\alpha \left(\frac{\partial X}{\partial \tau}\right)^2 d\tau \\ &= \frac{x^{1-\alpha}}{1-\alpha} \int_0^x \tau^\alpha \left(\frac{\partial X}{\partial \tau}\right)^2 d\tau \le \frac{x^{1-\alpha}}{1-\alpha} \int_0^1 \tau^\alpha \left(\frac{\partial X}{\partial \tau}\right)^2 d\tau\end{aligned}$$

since $\tau^\alpha \left(\frac{\partial X}{\partial \tau}\right)^2 \ge 0$ on $x \in (0, 1)$. So, let $x = h$. Then $[X(h)]^2 \le ch^{1-\alpha}(X_x, X_x)$ for a bounded constant c. Then if we let h go to zero, since $(X_x, X_x) < \infty$ and $0 < \alpha < 1$, then $\lim_{h\to 0}[X(h)]^2 \leq 0$. Thus, $[X(0)]^2 \leq 0$ and hence, $X(0) = 0$.

Proof of [B] : Suppose all elements of $\overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$ are zero at $x = 0$ when $1 < \alpha \leq 2$. Let $u \in \overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$. Then

(i) $\displaystyle\int_0^1 x^\alpha u_x^2(x)\, dx < \infty.$

By using Poincare inequalities (Theorem 1.3) when $p = 2$, $y = 0$, $l = 2-\alpha$, $n = 1$, and $\Omega = (0,1)$, we obtain the inequality

$$\int_0^1 x^{\alpha-2} u^2(x)\, dx \leq \frac{4}{(\alpha-1)^2} \int_0^1 x^\alpha u_x^2(x)\, dx$$

for all $u \in C_0^\infty(\Omega)$. Hence, $\int_0^1 x^{\alpha-2}u^2(x)\, dx$ must be finite by (i). Therefore, since $u \in \overset{\circ}{W}{}^1_{2,x^\alpha}$ and

(ii) $\int_0^1 \tau^{\alpha-2}\, d\tau$ converge for $2-\alpha < 1$ (that is, $1 < \alpha$).

These results imply that

(iii) $\int_0^1 \tau^{\alpha-2}|u'(\tau)|^2\, d\tau < \infty$ for $1 < \alpha < 2$.

Then using Hölder inequality, we find for any $h > 0$

$$\begin{aligned}
|u(x+h) - u(x)| &= \left| \int_x^{x+h} u'(\tau)\, d\tau \right| \\
&= \left| \int_x^{x+h} u'(\tau) \left(\tau^{\alpha-2}\right)^{1/2} \left(\tau^{\alpha-2}\right)^{-1/2}\, d\tau \right| \\
&\leq \left(\int_x^{x+h} |u'(\tau)|^2 \tau^{\alpha-2}\, d\tau \right)^{1/2} \left(\int_x^{x+h} \tau^{2-\alpha}\, d\tau \right)^{1/2}
\end{aligned}$$

From (ii) and (iii), we know that the last term converges to zero uniformly with respect to x when h tends to zero for $1 < \alpha \leq 2$. Thus u is uniformly continuous. Consequently, the limit $\lim_{x\to 0} u(x) = a_0$ exists. But, since $\int_0^1 x^{\alpha-2}u^2(x)\, dx < \infty$ and $\int_0^1 x^{\alpha-2}\, dx < \infty$ for $\alpha > 1$, then $\lim_{x\to 0} u(x) = a_0$ is not necessarily zero. Hence, $a_0 \neq 0$ leads to a contradiction.

The case $\alpha = 1$ requires some special considerations. Suppose all elements of $\overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$ are zero at $x = 0$. Let $u \in \overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$. Then , $\int_0^1 x u_x^2 \, dx < \infty$,

$$[u(x)]^2 = \left[\int_x^1 \tau^{-1/2}\tau^{1/2}\partial u/\partial\tau \, d\tau\right]^2 \leq \left|\int_x^1 \tau^{-1} \, d\tau\right| \left|\int_x^1 \tau \left(\partial u/\partial\tau\right)^2 \, d\tau\right|$$

$$= |\ln 1 - \ln x| \int_x^1 \tau \left(\partial u/\partial\tau\right)^2 \, d\tau$$

By multiplying both sides by $x^{-1}|\ln x|^{-2-\varepsilon}$, $\varepsilon > 0$ and integrating in $\Omega = (0,1)$, we obtain

(iv) $\int_0^1 u^2(x)\, x^{-1}|\ln x|^{-2-\varepsilon} \, dx \leq \int_0^1 \frac{1}{x|\ln x|^{1+\varepsilon}} \, dx \int_0^1 x \left(\partial u/\partial x\right)^2 \, dx$

Let $\theta(x) = x^{-1}|\ln x|^{-2-\varepsilon}$ and $\int_0^1 \frac{1}{x|\ln x|^{1+\varepsilon}} \, dx \leq c$ where $c < \infty$ is a constant. The inequality (iv) can also be obtained from the generalized Hardy inequality (Theorem 1.2) and thus, since $\int_0^1 x u_x^2 \, dx < \infty$, then $\int_0^1 \theta(x) u^2(x) \, dx < \infty$ from (iv) and $\int_0^1 \theta(x) u_x^2(x) \, dx < \infty$. Then apply the Hölder inequality ,

$$|u(x+h) - u(x)| = \left|\int_x^{x+h} u'(\tau) \, d\tau\right| = \left|\int_x^{x+h} u'(\tau)\theta(\tau)^{\frac{1}{2}}\theta(\tau)^{-\frac{1}{2}} \, d\tau\right|$$

$$\leq \left(\int_x^{x+h} |u'(\tau)|^2 \tau^{-1}|\ln\tau|^{-2-\varepsilon} \, d\tau\right)^{\frac{1}{2}} \left(\int_x^{x+h} \tau|\ln\tau|^{2+\varepsilon} \, d\tau\right)^{\frac{1}{2}}$$

We know that $\int_0^1 x^m (\ln x)^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is a positive integer and $m > -1$. Thus, the last term converges to zero uniformly with respect to x as h tends to zero. Hence, u is uniformly continuous and, consequently, $\lim_{x\to 0} u(x) = a_0$ exists. Since

$$\int_0^1 x^{-1}|\ln x|^{-2-\varepsilon} u^2(x) \, dx < \infty \text{ and } \int_0^1 x^{-1}|\ln x|^{-2-\varepsilon} \, dx < \infty,$$

$\lim_{x\to 0} u(x) = a_0$ is not necessarily zero. Therefore, $a_0 \neq 0$ leads to a contradiction. Thus , we must remove the condition $X(0) = 0$ to solve (3.2) with $\alpha \in [1,2]$.

Hence by [A] and [B], problem (3.2) must be separated into two problem depending on whether $0 < \alpha < 1$ or $1 \leq \alpha \leq 2$ as follows :
For $k > 0$ and $1 < \alpha < 1$,

$$\begin{cases} \frac{\partial}{\partial x}(x^\alpha X_x) + kX & = 0 \\ X(0) = X(1) & = 0 \end{cases} \tag{3.5}$$

and for $k > 0$ and $1 \leq \alpha \leq 2$,

$$\begin{cases} \frac{\partial}{\partial x}(x^\alpha X_x) + kX & = 0 \\ |X(0)| & < \infty \\ \text{or no boundary} & \text{conditions are given on } x = 0 \\ X(1) & = 0 \end{cases} \tag{3.6}$$

If the solution u of (1.15) is given by separation of variables, then the solution u is given by

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

where $X_n \in \overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$ for all n and $T_n(t) \in L_2((0,T))$ for all n. Actually $T_n(t) = e^{-kt}$ by (3.3)

If $X_n \in \overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$ when $0 < \alpha < 1$, then $X_n(0) = X_n(1)$ must be zero for all n by [A]. Thus $u(0,t) = \sum_{n=1}^{\infty} T_n(t)X_n(0)$ and $u(1,t) = \sum_{n=1}^{\infty} T_n(t)X_n(1)$ must be zero. Also, if $X_n \in \overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega)$, when $1 \leq \alpha \leq 2$, then $X_n(1) = 0$ and $X_n(0)$ is not given for any n by [B]. Hence, $u(1,t)$ must be zero and $u(0,t)$ is not specified by instead is calculated from the solution.

Therefore, in view of the above results, we can separate (1.15) into (1.16) and (1.17) . We see from the following discussion that (1.16) satisfies the conditions of the theorems in Stahel [13]. Hence, the problem (1.16) serves as an example satisfying the theorems in Stahel [13].

In (1.16) the $a(x)$, for conditions (1.13), is given as $a(x) = x^\beta$, $0 < \beta < 1$ satisfying $a(x) = \underline{a}(x) = \overline{a}(x) = x^\beta$ and $d(x) = x$ in a neighborhood of $\{x = 0\}$, since $d(x) = \text{dist}\,(x, \Gamma) = \text{dist}\,(x, 0) = x$. Then we take α and β to be equal and $c_2 = 2,\ c_1 = \frac{1}{2}$. Therefore, the inequality $2x^\alpha \geq x^\alpha \geq \frac{1}{2}x^\alpha$ for $0 < \alpha < 1$ is satisfied. Thus the conditions of (1.13) are satisfied. Hence, (1.16) satisfies all the conditions of Theorem 1.3 .

However, let us consider (1.17) . The $a(x)$ for the condition 1.1 and 1.2 are given as $a(x) = \underline{a}(x) = \overline{a}(x) = x^\beta$ with $1 \leq \beta \leq 2$ and $d(x) = x$ in a neighborhood of $\{x = 0\}$. Then let us assume that there exists a $c_1 > 0$ such that $x^\beta \geq c_1 x^\alpha$ for all x in a neighborhood of $\{x = 0\}$ with $0 < \alpha < 1$. Then $\frac{1}{c_1} \geq \frac{1}{x^{\beta-\alpha}}$, $0 < \beta - \alpha < 2$ for all x in a neighborhood of $\{x = 0\}$. Since $c_1 \frac{1}{\beta - \alpha} \neq 0$, we can choose x' in a neighborhood of $\{x = 0\}$ such that $0 < x' < c_1^{1/(\beta-\alpha)}$. Then $\frac{1}{(x')^{\beta-\alpha}} > \frac{1}{c1}$ since $0 < \beta - \alpha < 2$. Therefore, $c_1(x')^\alpha > (x')^\beta$. This is a contradiction to the assumption . Thus we conclude that Stahel [13] does not deal with the case of equation (1.17) (when boundary conditions are zero).

Example 3.1) Now let us seek an explicit solution to (1.16) by separation of variables. Let $u = X(x)T(t)$. Then, we have (3.3) and (3.5) . First let us solve (3.5). Let $y = \frac{1}{1-\alpha}x^{1-\alpha}$, with $x = ((1-\alpha)y)^{1/(1-\alpha)}$ and let $\tilde{X}(y) = X(\{(1-\alpha)y\}^{1/(1-\alpha)}) = X(x)$. Then by the chain rule for $\tilde{X}(y)$, when k is given by λ^2 , the (3.2) is converted to

$$\frac{d^2\tilde{X}}{dy^2} + \lambda^2((1-\alpha)y)^{\frac{\alpha}{1-\alpha}}\tilde{X} = 0$$

Now let $\tilde{\lambda} = \lambda^2(1-\alpha)^{\alpha/(1-\alpha)}$ and $\alpha/(1-\alpha) = \nu$. Then

$$\frac{d^2\tilde{X}}{dy^2} + \tilde{\lambda}y^{\nu}\tilde{X}(y) = 0. \tag{i}$$

Let $Z(y) = y^{-1/2}\tilde{X}(y)$. So , $\tilde{X}(y) = y^{1/2}Z(y)$. Then by the chain rule for $Z(y)$, (i) is converted to

$$-\tfrac{1}{4}y^{-\frac{3}{2}}Z(y) + y^{-\frac{1}{2}}Z'(y) + y^{\frac{1}{2}}Z''(y) + \tilde{\lambda}y^{\nu+\frac{1}{2}}Z(y) = 0. \tag{ii}$$

By multiplying $y^{2/3}$ in (ii), we have

$$y^2Z''(y) + yZ'(y) + (\tilde{\lambda}y^{\nu+2} - \tfrac{1}{4})Z(y) = 0. \tag{iii}$$

Let $t = \rho y^k, \tilde{\lambda} = \rho^{2k^2}, \nu = 2k-2$. Then $y = (t/\rho)^{1/k}$. Let

$$\tilde{Z}(t) = Z((t/\rho)^{1/k}) = Z(y).$$

So, by the chain rule, (iii) is converted to

$$t^2\frac{d^2\tilde{Z}}{dt^2} + t\frac{d\tilde{Z}}{dt} + (t^2 - (\frac{1}{2k})^2)\tilde{Z} = 0. \tag{iv}$$

This equation is known as Bessel's equation of order $1/2k$. The general solution of Bessel's equation or order $1/2k$ (non integer) is given by

$$\tilde{Z}(t) = AJ_{\frac{1}{2k}}(t) + BJ_{-\frac{1}{2k}}(t) \tag{v}$$

By inserting $t = \rho y^k$ and $\tilde{Z}(t) = Z(y)$ into (v), we obtain

$$Z(y) = AJ_{\frac{1}{2k}}(\rho y^k) + BJ_{-\frac{1}{2k}}(\rho y^k) \tag{vi}$$

Since $Z(y) = -y^{1/2}\tilde{X}(y)$, (vi) becomes

$$\tilde{X}(y) = Ay^{\frac{1}{2}}J_{\frac{1}{2k}}(\rho y^k) + By^{\frac{1}{2}}J_{-\frac{1}{2k}}(\rho y^k) \tag{vii}$$

Since $\tilde{X}(y) = X(x)$ and $y = \frac{1}{1-\alpha}x^{1-\alpha}$, (vii) becomes

$$X(x) = A(\tfrac{1}{1-\alpha}x^{1-\alpha})^{\frac{1}{2}}J_{\frac{1}{2k}}(\rho(\tfrac{1}{1-\alpha}x^{1-\alpha})^k) + B(\tfrac{1}{1-\alpha}x^{1-\alpha})^{\frac{1}{2}}J_{-\frac{1}{2k}}(\rho(\tfrac{1}{1-\alpha}x^{1-\alpha})^k) \tag{viii}$$

Here k is given by $\nu=2k-2$ and $\nu=\alpha/(1-\alpha)$. Thus $k=(2-\alpha)/2(1-\alpha)$. Then $1/2k=(1-\alpha)/(2-\alpha)$, $0<\alpha<1$. Since $0<(1-\alpha)/(2-\alpha)<1$, $1/2k$ is not integer. Also, $\tilde{\lambda}$ is given by $\tilde{\lambda}=\lambda^2(1-\alpha)^{\alpha/(1-\alpha)}$ and $\tilde{\lambda}=\rho^2k^2$. Then $\rho=\tilde{\lambda}^{1/2}/k=2\lambda(1-\alpha)^{(2-\alpha)/2(1-\alpha)}/(2-\alpha)$ and $\rho(1/(1-\alpha))^k=2\lambda/(2-\alpha)$.

Thus (viii) becomes

$$X(x) = A(\tfrac{1}{1-\alpha})^{\frac{1}{2}}x^{\frac{1-\alpha}{2}}J_{\frac{1-\alpha}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha}x^{\frac{2-\alpha}{2}}) + B(\tfrac{1}{1-\alpha})^{\frac{1}{2}}x^{\frac{1-\alpha}{2}}J_{-\frac{1-\alpha}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha}x^{\frac{2-\alpha}{2}}). \tag{ix}$$

Since $J_{-\frac{1}{2k}}(\frac{2\lambda}{2-\alpha}x^{\frac{2-\alpha}{2}})$ is unbounded on [0,1] and we are seeking only bounded solutions , we must take $B = 0$. From the boundary condition $X(1) = 0$, we have

$$X(1) = A(\tfrac{1}{1-\alpha})^{\frac{1}{2}}J_{\frac{1-\alpha}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha}) = 0.$$

Thus $J_{\frac{1-\alpha}{2-\alpha}}(\frac{2\lambda}{2-\alpha})$ should be zero. $J_{\frac{1-\alpha}{2-\alpha}}(x)$ has an infinite number of isolated zeroes. Let $x_i = \frac{2\lambda_i}{2-\alpha}$ for $i = 1, 2, \cdots$ where $J_{\frac{1-\alpha}{2-\alpha}}(x_i) = 0$. Then $\lambda_i = \frac{2-\alpha}{2}x_i$. Hence , the equation (1.16) has an infinite number of solutions

$$X_n = \{x^{\frac{1-\alpha}{2}}J_{\frac{1-\alpha}{2-\alpha}}(\tfrac{2\lambda_n}{2-\alpha}x^{\frac{2-\alpha}{2}})\}.$$

By (3.3) , $T_n = a_n e^{-\lambda_n^2 t}$. Thus ,

$$u = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} X_n$$

where $X_n = x^{\frac{1-\alpha}{2}} J_{\frac{1-\alpha}{2-\alpha}}(\frac{2\lambda_n}{2-\alpha} x^{\frac{2-\alpha}{2}})$.
From $u(x,0) = \psi(x)$, where $\psi(x) = \sum_{n=1}^{\infty} b_n X_n$,

$$(\psi(x), X_l) = b_l(X_l, X_l) \quad \text{for } \; l = 1, 2, \cdots .$$

Accordingly , b_l is given by

$$b_l = \frac{(\psi(x), X_l)}{(X_l, X_l)} \quad \text{for all } l = 1, 2, \cdots .$$

Thus

$$u = \sum_{n=1}^{\infty} \frac{(\psi(x), X_n)}{(X_n, X_n)} e^{-\lambda_n^2 t} x^{\frac{1-\alpha}{2}} J(\frac{2\lambda_n}{2-\alpha} x^{\frac{2-\alpha}{2}})$$

and the solution

$$u \in \overset{\circ}{W}{}^{1,0}_{2,x^\alpha}(Q_T) = L_2((0,T); \overset{\circ}{W}{}^{1}_{2,x^\alpha}(\Omega)).$$

$u \in \overset{\circ}{W}{}^{1,0}_{2,x^\alpha}(Q_T)$ means that

$$u \in C^1((0,T), L_2(\Omega)) \cap C^0((0,T), H^1_{x^\alpha}(\Omega)).$$

Since C^0 is dense in L_2, u is a weak solution of (1.16) because u satisfies the conditions of the weak solution in the Stahel paper [13] .

Next, let us solve the equation (1.17) explicitly , which will serve as an example of the Theorem 1.4 by separation of variables. Let $u = X(x)T(t)$. Then (3.3) and (3.6) are obtained. Now let us solve (3.6) and do the same process going through the step (i) through (ix) as above. Then we obtain that when $(1-\alpha)/(2-\alpha)$ is not an integer,

$$X(x) = A(\tfrac{1}{1-\alpha})^{\frac{1}{2}} x^{\frac{1-\alpha}{2}} J_{\frac{1-\alpha}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}}) + B(\tfrac{1}{1-\alpha})^{\frac{1}{2}} x^{\frac{1-\alpha}{2}} J_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}}), \quad \text{(x)}$$

in the case when $1 < \alpha < 2$, $(\alpha-1)/(2-\alpha) > 0$ and $(1-\alpha)/(2-\alpha) < 0$. Thus since $J_{\frac{1-\alpha}{2-\alpha}}(\frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}})$ is unbounded on $[\,0,1\,]$, we take

$$X(x) = B(\tfrac{1}{1-\alpha})^{1/2} x^{\frac{1-\alpha}{2}} J_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}}) \quad \text{(xi)}$$

Recall that

$$J_\mu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\mu+n+1)} \left[\frac{x}{2}\right]^{\mu+2n} \quad \text{(xii)}$$

Thus when $\mu = (1-\alpha)/(\alpha-2)$,

$$J_{\frac{1-\alpha}{\alpha-2}}(\tfrac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\mu+n+1)} (\tfrac{\lambda}{2-\alpha})^{\frac{1-\alpha+2n(\alpha-2)}{\alpha-2}} x^{\frac{\alpha-1-2n(\alpha-2)}{2}}$$

Hence by (xii), we obtain from (xi)

$$X(x) = B(\tfrac{1}{1-\alpha})^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\mu+n+1)} (\tfrac{\lambda}{2-\alpha})^{\frac{1-\alpha+2n(\alpha-2)}{\alpha-2}} x^{-n(\alpha-2)},$$

for $1 < \alpha < 2$, $-n(\alpha-2) \geq 0$ and for all nonnegative integers n. Thus

$$X(0) = B\frac{(\frac{1}{1-\alpha})^{1/2}}{\Gamma(\mu+1)} (\tfrac{\lambda}{2-\alpha})^{\frac{1-\alpha}{\alpha-2}}.$$

Therefore it is possible that we take $B < \infty$ so that $|X(0)| < \infty$ since $(\frac{1}{1-\alpha})^{\frac{1}{2}}(\frac{\lambda}{2-\alpha})^{\frac{1-\alpha}{\alpha-2}}/\Gamma(\mu+1)$ is finite. Now from the boundary condition : $X(1) = 0$,

$$X(1) = B(\tfrac{1}{1-\alpha})^{\frac{1}{2}} J_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha}) = 0.$$

Thus $J_{\frac{\alpha-1}{2-\alpha}}(\frac{2\lambda}{2-\alpha}) = 0$. Because $J_{\frac{\alpha-1}{2-\alpha}}(x)$ has an infinite number of isolated zeroes, let $x_i = 2\lambda/(2-\alpha)$ for $i = 1, 2, \cdots$ where $J_{\frac{\alpha-1}{2-\alpha}}(x_i) = 0$. Then $\lambda = (2-\alpha)/2x_i$. Thus let $\lambda_n = (2-\alpha)/2x_n$. Hence the problem (3.6) has an infinite number of solutions

$$\{x^{\frac{1-\alpha}{2}} J_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda_n}{2-\alpha} x^{\frac{2-\alpha}{2}})\}$$

when $(1-\alpha)/(2-\alpha)$ is not integer and $1 < \alpha < 2$. In the case when $(1-\alpha)/(2-\alpha)$ is an integer, (in fact, a negative integer for $1 < \alpha < 2$ since $\frac{1-\alpha}{2-\alpha} = -1$ when $\alpha = \frac{3}{2}$) then (iv) in the previous example is equivalent to

$$t^2 \frac{d^2 \tilde{Z}}{dt^2} + t\frac{d\tilde{Z}}{dt} + (t^2 - (-\tfrac{1}{2k})^2)\tilde{Z} = 0$$

This equation is known as Bessel's equation of order $(-\frac{1}{2k}) = \frac{\alpha-1}{2-\alpha}$, where $1 < \alpha < 2$ (the order is a positive integer for $1 < \alpha < 2$).
The general solution is given by

$$X(x) = A((\tfrac{1}{1-\alpha})^{\frac{1}{2}} x^{\frac{1-\alpha}{2}}) J_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}})$$
$$+ B((\tfrac{1}{1-\alpha})^{\frac{1}{2}} x^{\frac{1-\alpha}{2}}) Y_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}})$$

Since $Y_{\frac{\alpha-1}{2-\alpha}}(\frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}})$ is unbounded on [0,1] , we take

$$X(x) = A(\tfrac{1}{1-\alpha})^{\frac{1}{2}} x^{\frac{1-\alpha}{2}} J_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}})$$

We proceed as before. We know that the problem has an infinite number of solutions

$$\{x^{\frac{1-\alpha}{2}} J_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda_n}{2-\alpha} x^{\frac{2-\alpha}{2}})\}$$

for both cases when $1 < \alpha < 2$.

When $\alpha = 1$, (3.6) becomes

$$\begin{cases} xX''(x) + X'(x) + \lambda^2 X(x) = 0 \\ X(1) = 0 \\ |X(0)| < \infty \\ \textit{or no boundary condition at } x = 0 \end{cases} \tag{3.7}$$

Since $x = 0$ is a regular singular point, by the Frobenius method , we can assume a series solution of the form

$$X(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}.$$

Then

$$X' = \sum_{n=0}^{\infty} c_n x^{n+r-1} \quad \text{and} \quad X'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}.$$

By inserting these into the equation of (3.7) , we obtain

$$\{r(r-1)c_0 + rc_0\} + \sum_{n=1}^{\infty} \{(n+r)(n+r-1)c_n + (n+r)c_n + \lambda^2 c_{n-1}\} x^{n+r-1} = 0$$

Thus

$$r(r-1)c_0 + rc_0 = 0 \quad \text{and} \quad (n+r)(n+r+1)c_n + (n+r)c_n + \lambda^2 c_{n-1} = 0$$

for all $n = 1, 2, \cdots$. Let $c_0 \neq 0$. Thus $r = 0$. From the recursion relationship

$$(n+r)(n+r-1)c_n + (n+r)c_n + \lambda^2 c_{n-1} = 0,$$

we have

$$c_n = \frac{-\lambda^2}{(n+r)^2} c_{n-1} \quad \text{for } n = 1, 2, \cdots .$$

Since $r = 0\,, c_n = \frac{-\lambda^2}{n^2} c_{n-1}$. Thus

$$c_n = \frac{(-1)^n(\lambda^2)^n}{(n!)^2} c_0.$$

Hence

$$X(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda^2 x)^n}{(n!)^2}.$$

Since $J_0(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} (\frac{x}{2})^{2n}$,

$$X(x) = c_0 J_0(2\sqrt{\lambda^2 x}).$$

Therefore, let $X_1(x) = c_0 J_0(2\sqrt{\lambda^2 x})$ be a solution for (3.7).Then the second linearly independent solution $X_2(x)$ is given in integral form by

$$X_2(x) = X_1(x) \int_0^1 \frac{e^{-\int \frac{1}{x} dx}}{X_1^2(x)} \, dx = \frac{1}{c_0} J_0(2\sqrt{\lambda^2 x}) \int_0^1 \frac{1}{x J_0^2(2\sqrt{\lambda^2 x})} \, dx$$

and the general solution has such a form that

$$X(x) = A J_0(2\sqrt{\lambda^2 x}) + B J_0(2\sqrt{\lambda^2 x}) \int_0^1 \frac{1}{x J_0^2(2\sqrt{\lambda^2 x})} \, dx$$

But since $B J_0(2\sqrt{\lambda^2 x}) \int_0^1 \frac{1}{x J_0^2(2\sqrt{\lambda^2 x})} \, dx$ is unbounded on [0,1] and we are only seeking bounded solutions, we must take $B = 0$. Thus the solution has the form $X(x) = A J_0(2\sqrt{\lambda^2 x})$. In order to satisfy the boundary condition $X(1) = 0$,

$$J_0(2\lambda^2) = 0.$$

Since $J_0(x)$ has an infinite number of isolated zeroes, x_i, let $2\lambda_i = x_i, i = 1, 2, \cdots$, Then $\lambda_i = x_i/2$. Hence the problem (3.7) has an infinite number of solutions $\{J_0(2\lambda_n\sqrt{x})\}$ which also have the property of being orthogonal sets.

When $\alpha = 2$, (3.6) becomes

$$\begin{cases} x^2 X''(x) + 2xX'(x) + \lambda^2 X = 0 \\ X(1) = 0 \\ |X(0)| < \infty \quad or \\ No \quad bondary \quad condition \quad at \quad x = 0 \end{cases} \tag{3.8}$$

This is an Euler eqution. For $x > 0$, take $X(x) = x^r$. Then

$$X'(x) = rx^{r-1} \quad \text{and} \quad X''(x) = r(r-1)x^{r-2}.$$

By inserting these into (3.8) , we obtain

$$r(r-1)x^r + 2rx^r + \lambda^2 x^r = 0$$

and so $r^2 + r + \lambda^2 = 0$. Thus

$$r_1 = \tfrac{-1+\sqrt{1-4\lambda^2}}{2} \quad \text{and} \quad r_2 = \tfrac{-1+\sqrt{1-4\lambda^2}}{2}.$$

We consider three separate cases

<u>Case 1)</u> When $1 - 4\lambda^2 > 0$, the general solution is given by

$$X(x) = c_1 x^{r_1} + c_2 x^{r_2}, x > 0.$$

If we want a bounded solution, we can take

$$X(x) = c_1 x^{\frac{-1+\sqrt{1-4\lambda^2}}{2}} \quad \text{when} \quad \tfrac{-1+\sqrt{1-4\lambda^2}}{2} > 0.$$

But since $X(1) = 0$, $c_1 = 0$. So $X(x) = 0$. This is meaningless. Hence, we cannot insist on a bounded solution. However, in view of the boundary condition that no boundary condition is given at $x = 0$, we take

$$X(x) = c_1 x^{\frac{-1+\sqrt{1-4\lambda^2}}{2}} + c_2 x^{\frac{-1-\sqrt{1-4\lambda^2}}{2}}.$$

From $X(1)=0$,

$$X(x)=c_1(x^{\frac{-1+\sqrt{1-4\lambda^2}}{2}}-x^{\frac{-1-\sqrt{1-4\lambda^2}}{2}}).$$

Case 2) When $1-4\lambda^2=0$, the general solution is given by

$$X(x)=(c_1+c_2\ln x)x^{-1/2} \quad , \quad x>0.$$

From $X(1)=0$,

$$X(x)=c_1(1-\ln x)x^{-1/2} \quad , \quad x>0.$$

At x=0 though , this solution is unbounded.

Case 3) When $1-4\lambda^2<0$, the general solution is given by

$$X(x)=c_1x^{-1/2}\cos(\tfrac{\sqrt{4\lambda^2-1}}{2}\ln x)+c_2x^{-1/2}\sin(\tfrac{\sqrt{4\lambda^2-1}}{2}\ln x).$$

Since $X(1)=0$ implies $c_1=0$, thus

$$X(x)=c_2x^{-1/2}\sin(\tfrac{\sqrt{4\lambda^2-1}}{2}\ln x).$$

Now let us solve (3.3). Then $T=ae^{-\lambda^2 t}$. So let $T_n(t)=a_ne^{-\lambda_n^2 t}$. Hence when $1\le\alpha<2$, the solution of (1.17) is given as

$$u=\sum_{n=1}^{\infty}T_n(t)X_n(x)=\sum_{n=1}^{\infty}b_ne^{-\lambda_n^2 t}X_n(x)$$

where X_n is given by

$$x^{\frac{1-\alpha}{2}}J_{\frac{\alpha-1}{2-\alpha}}(\tfrac{2\lambda_n}{2-\alpha}x^{\frac{2-\alpha}{2}})$$

and

$$b_l=\frac{(\psi(x),X_l)}{(X_l,X_l)} \quad \text{for } l=1,2,\cdots.$$

And when $\alpha = 2$, the solution of (1.17) is given as

$$u(x,t) = T(t)X(x) = be^{-\lambda^2 t}X(x)$$

where $X(x)$ is given by

(i) $x^{\frac{-1+\sqrt{1-4\lambda^2}}{2}} - x^{\frac{-1-\sqrt{1-4\lambda^2}}{2}}$ if $1 - 4\lambda^2 > 0$

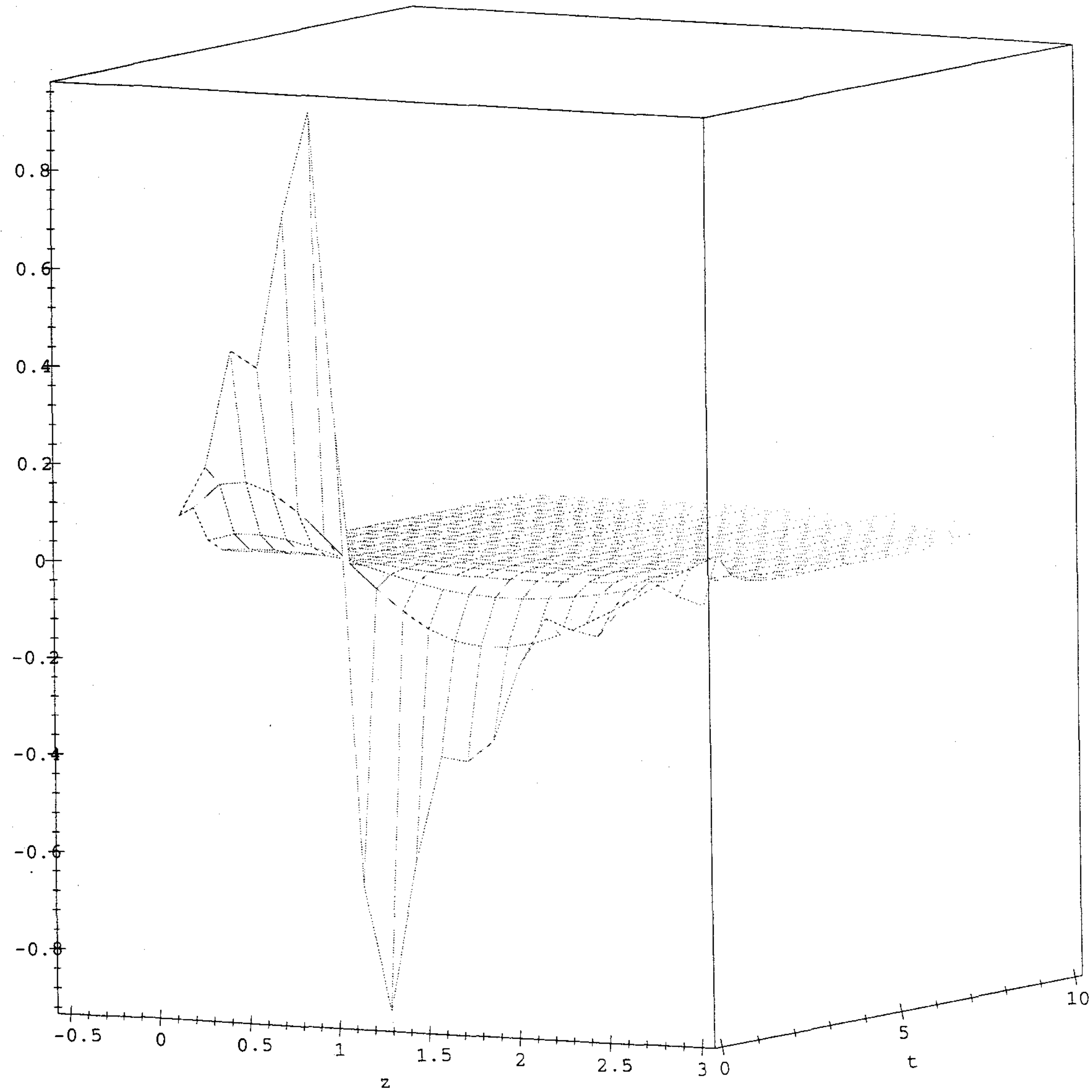
(ii) $(1 - \ln x)x^{-\frac{1}{2}}$ if $1 - 4\lambda^2 = 0$

(iii) $-1/2\sin(\frac{\sqrt{4\lambda^2-1}}{2}\ln x)$ if $1 - 4\lambda^2 < 0$

and

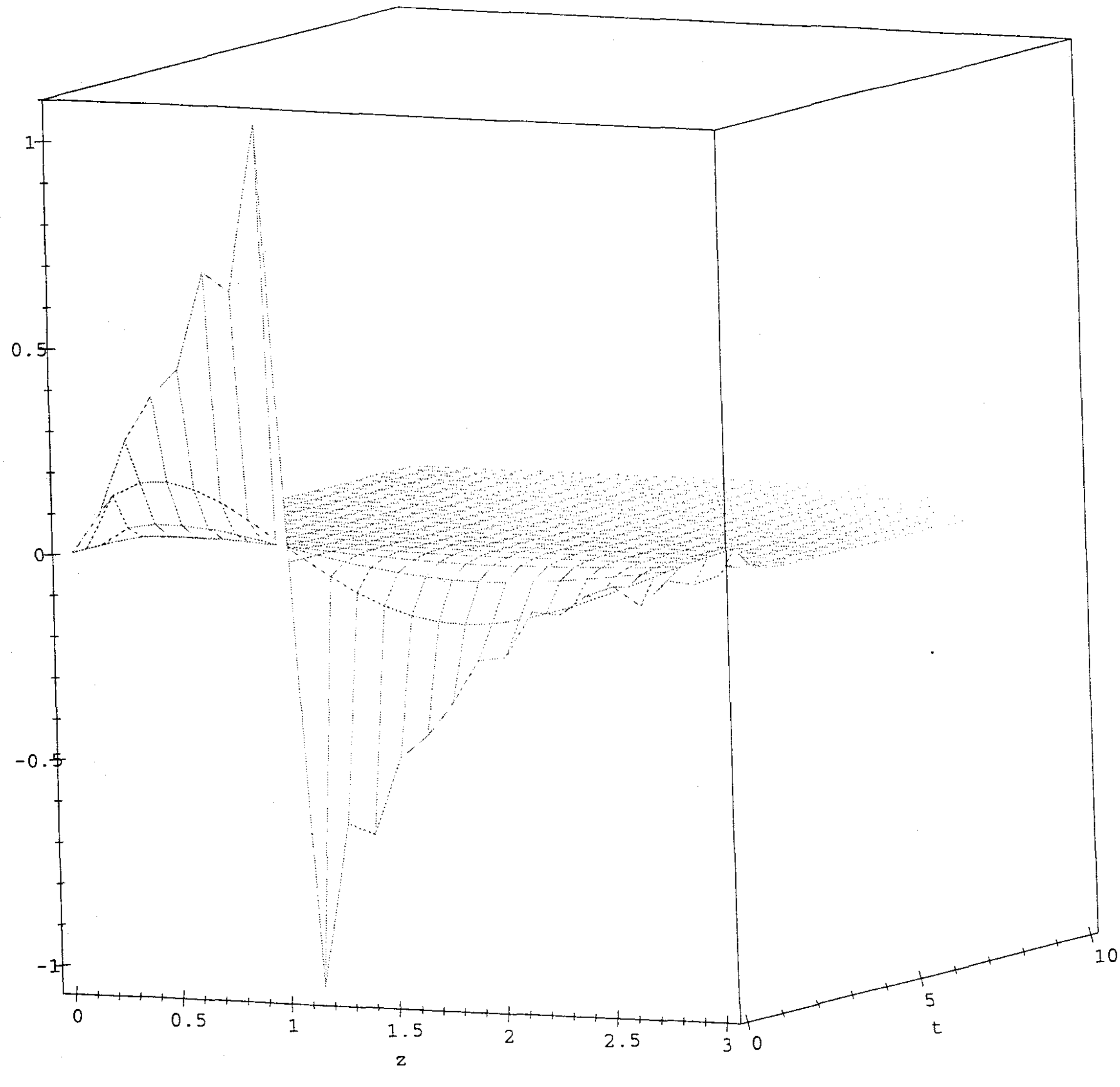
$$b = \frac{(\psi(x), X(x))}{(X(x), X(x))}.$$

1. The Graph of a Solution of Example 3.1
when $\alpha = \frac{1}{2}$, $k = 5$ and $\psi(x) = x$



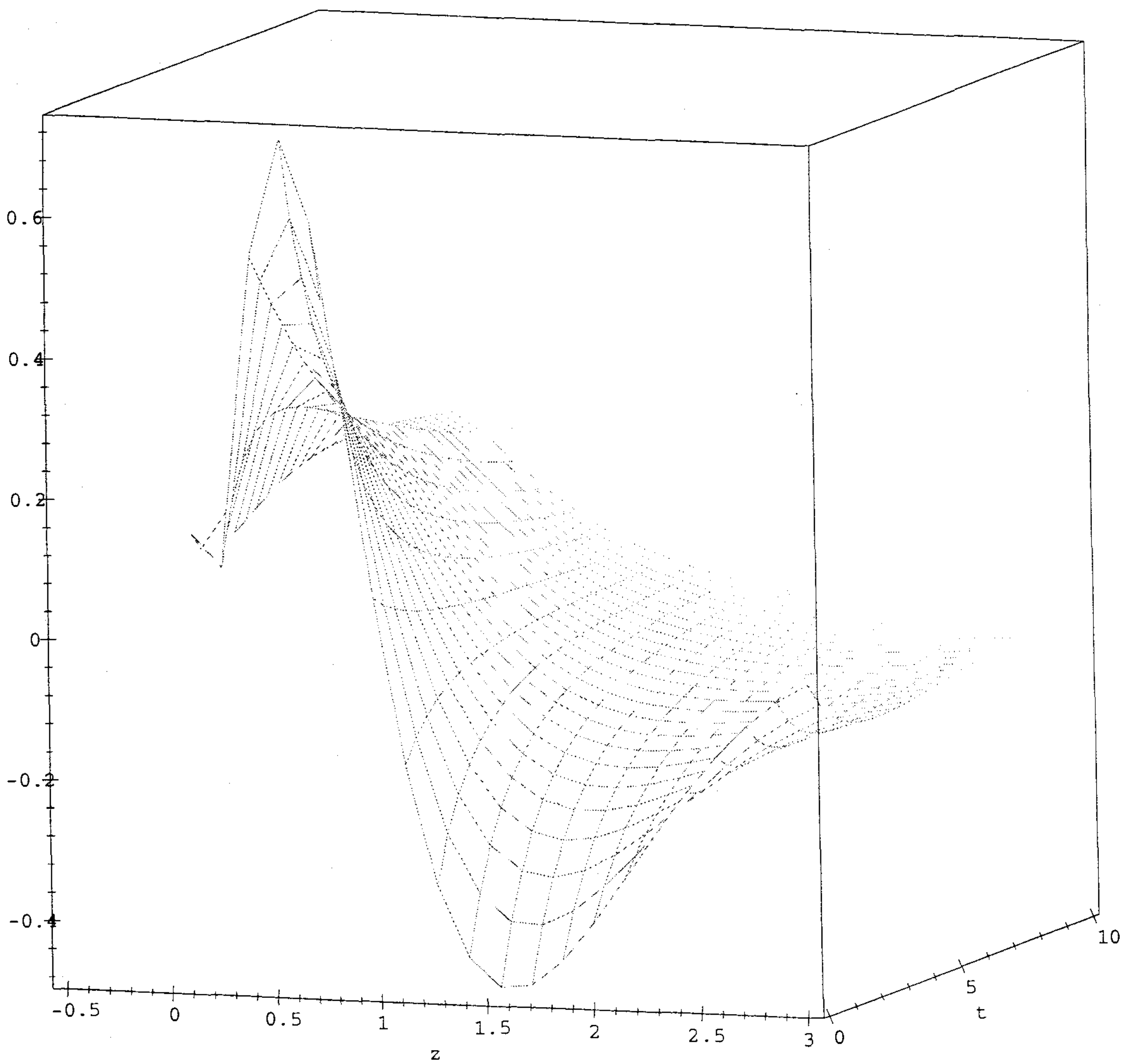
Look at the values at boundary points x=0 and x=1 , where the values at x=0 and x=1 must be zeroes but here the value at x=0 dose not appear obviously.

2. The Graph of a Solution of Example 3.1
when $\alpha = \frac{1}{2}$, $k = 10$ and $\psi(x) = x$



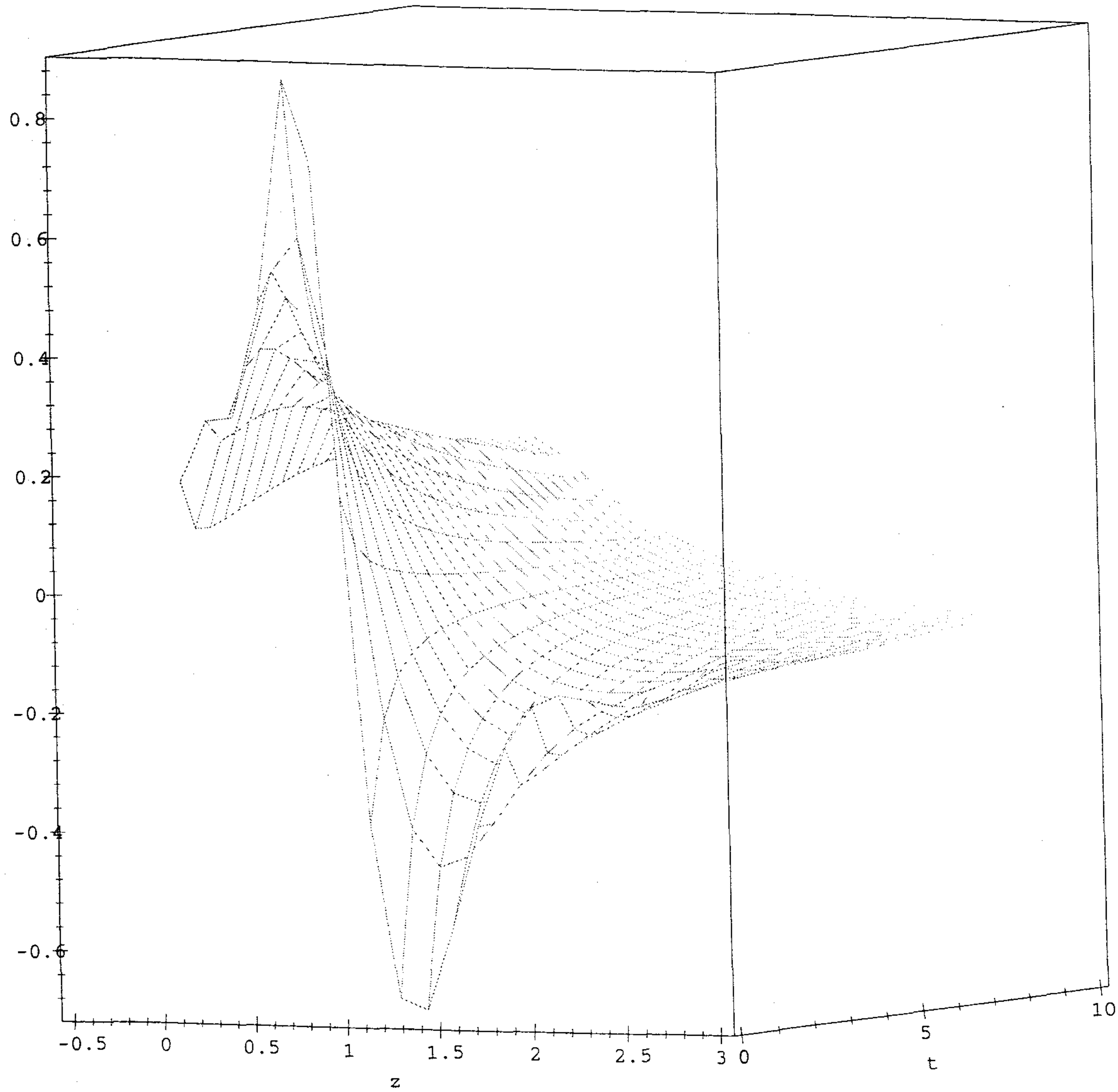
Look at the values at boundary points x=0 and x=1 , where the values at x=0 and x=1 must be zeroes.

3. The Graph of a Solution of Example 3.1
when $\alpha = \frac{3}{2}$, $k = 5$ and $\psi(x) = x$



Look at the values at boundary points x=0 and x=1, where no boundary condition at x = 0 is given and the value at x=1 must be zero.

4. The Graph of a Solution of Example 3.1
when $\alpha = \frac{3}{2}$, $k = 10$ and $\psi(x) = x$



Look at the values at boundary points x=0 and x=1, where no boundary condition at x = 0 is given and the value at x=1 must be zero.

3.2 The Existence and Uniqueness of Solutions in a Weighted Sobolov Space for an Initial-Boundary Problem of a Degenerate Parabolic Equation with Principal Part in Divergence Form

In this dissertation I allow the ellipticity function to go to zero on the boundary of Ω. In fact , I assume that the " degeneration of the ellipticity " is of the order of some power α of the distance of x from the boundary of Ω . Hence, we are led to study the Banach space $W_p^k(\Omega; d, \alpha)$: A power type weighted Sobolev space. (See Appendix 2 (12) for the space). According to Remark 4.9 in [36], if $2k - \alpha - 1 \leq 0$, then in general, there are no reasonable boundary values that can be assigned to $u \in W^k_{2,\alpha}(\Omega)$. This situation, for the case k = 1 (this is , $\alpha \geq 1$) and with a more general weight function than d^α(x) , is discussed in Section 3 of [37]. However, when certain additional conditions are satisfied and a condition is given for the solution to vanish on a part of the boundary, the existence of solutions for intial boundary problems like (1.17) are possible and even with uniqueness is possible. We see that possibility from boundary value problems of elliptic type equations in [38]. However, it is not reasonable to take $\overset{\circ}{W}{}^k_{p,\alpha}(\Omega)$ as the solution space for the initial boundary problems like (1.17) because the solutions do not have compact support in Ω and even though we can extract a subsequence of functions in $C_0^\infty(\Omega)$ converging to some $u \in W^k_{p,\alpha}(\Omega)$, we cannot reasonably assign a boundary values to u , that is, in (1.17), u can have another value different from zero as x goes to the boundary $\{x = 0\}$. However , since the function $u|_{\partial\Omega}$ makes sense for $u \in C^\infty(\overline{\Omega})$, and the fact $C^\infty(\overline{\Omega})$ is dense in $W_p^k(\Omega; d, \alpha)$ for $\alpha \geq 0$ (see theorem 7.2 in [4]),it is reasonable to seek approximate solutions in $C^\infty(\overline{\Omega})$ converging to some $u \in W_p^k(\Omega; d, \alpha)$ in order to solve an initial-boundary problem like (1.17).

Remark 3.1) When k=1 and p=2 , $W_p^k(\Omega; d, \alpha) \equiv W_{p,\alpha}^k(\Omega)$.
If $d(x, \Gamma) = x$, $\overset{\circ}{W}{}_{2,x^\alpha}^1(\Omega) \equiv \overset{\circ}{W}{}_2^1(\Omega; x, \alpha) \equiv \overset{\circ}{W}{}_{2,\alpha}^1(\Omega) \equiv H_{x^\alpha}^1(\Omega)$, where $W_{p,\alpha}^k$ is a Sobolov space used in Meyer [36] and $H_{x^\alpha}^1$ is also a Sobolov space used in Stahel paper [13]. (See the appendix 2)

Now let us consider the problem of finding generalized solutions to (1.18) satisfing (1.19). Then we have the following result which is the main theorem in this dissertation. Here we make the same assumptions as in [13] , which are given at the beginning of this chapter. Here u is a weak solution of (1.18) if all the following conditions are satisfied :
$u \in C^0(I, L^2(\Omega; d, \alpha)) \cap C^0(I, \overset{\circ}{W}{}_2^1(\Omega'; d, \alpha)$ for any compact subset Ω' of Ω and $\|u(t)\|_{k,p;d,\alpha} \le c, \quad u(0) = u_0, \quad u(t)|_{\Gamma_1} = 0.$

$$< \psi(T), u(T) > - < \psi(0), u_0 > - \int_0^T < \psi'(t), u(t) > dt$$
$$= - \int_0^T < a\nabla\psi(t), \nabla u(t) > dt + \int_0^T < \psi(t), f(u(t)) > dt$$

for all $\psi \in C^1(I, L_2(\Omega)) \cap C^0(I, \overset{\circ}{W}{}_2^1(\Omega; d, \alpha))$ and all $T \in [0, T^+]$.

Theorem 1.4 **The problem (1.18) has , for a given $u_0 \in L_2(\Omega) \cap C^2(\bar{\Omega})$ which is assumed to be a nonnegative function , a weak solution which is Lipschitz continuous with respect to time with values in $W_2^1(\Omega; d, \alpha) \subset L_2(\Omega; d, \alpha)$. Moreover the solution is unique.**

<u>**Idea of the proof**</u> First let us extend the given domain Ω so that the condition (1.19) and all assumptions are satisfied in the extended domain $\tilde{\Omega}$ and simultaneously extend $u_0, (a_{ij})$, and coefficient functions of f(u) continuously to $\tilde{\Omega}$ such that the extended $\tilde{u_0}$ to $\tilde{\Omega}$ has compact support in the extended boundary $\partial\tilde{\Omega}$ and the extended $(\tilde{a}_{ij})$ to $\tilde{\Omega}$ also satisfies

the condition (1.19) and the extended coefficient functions also satisfies the assumptions given in problem (1.18) for the coefficient functions of f(u). Here the extended domain is translated so that the translated domain is situated in half space $x_N > 0$. We do this since we expect that the extended and translated function of (a_{ij}) is also positive definite in the extended and translated domain except the part on which the problem is degenerating. Then the compatibility condition at the boundary $\partial\tilde{\Omega}$ of the extended and translated initial condition $\tilde{u}_0$ for the solution of an extended problem to the extended and translated domain is forced for the solution to have zero at the whole extended and translated boundary $\partial\tilde{\Omega}$. Thus, the problem (1.18) is transformed to a problem on the extended and translated domain $\tilde{\Omega}$ with zero at the whole extended and translated boundary $\partial\tilde{\Omega}$. However, the transformed problem is still degenerate, parabolic. Hence replacing the domain and initial-boundary data (it is not necessarily for the boundary data since boundary conditions are zero.) with their smooth approximations, we obtain approximating problems which are nondegenerate and we know the existence of solutions for these problems (See Chapter 4). We can also extract a subsequence of the sequence of approximate solutions which converges to a solution uniformly on each compact subset of $\tilde{\Omega}$ for each $t \in I$. Then I will show that we can also extract a subsequence from the sequence of the mollifers of the approximate solutions , which converges to some solution in the norm of $W_2^1(\tilde{\Omega}; d, \alpha)$ for each $t \in I$. The fact that $C^\infty(\bar{\tilde{\Omega}})$ is dense in $W_2^1(\tilde{\Omega}; d, \alpha)$ means the existence of a solution in $W_2^1(\tilde{\Omega}; d, \alpha)$ for each t. Then the solution which is restricted and translated to the original domain Ω is forced to be the one for the original problem.

The proof of the existence.

Notation I) Let $\tilde{\Omega}$ denote an extended domain of Ω which is translated so as to be situated in half space $x_N > 0$. $\partial\tilde{\Omega}$ denotes the boundary of $\tilde{\Omega}$.

Notation II) $\tilde{a} = (\tilde{a}_{ij})$ denotes an extension of (a_{ij}) to $\tilde{\Omega}$ so that $(\tilde{a}_{ij})$, $\frac{\partial \tilde{a}_{ij}}{\partial x_i}$ are continuous in $\tilde{\Omega}$ and $c_2 d(x)^\alpha \geq \bar{\tilde{a}}(x) \geq \underline{\tilde{a}}(x) \geq c_1 d(x)^\alpha$ with $1 \leq \alpha \leq 2$, $c_1, c_2 > 0$ in a neighborhood of $\partial\tilde{\Omega}$ is satisfied. Here the extension of (a_{ij}) to $\tilde{\Omega}$ is done by a translation of the function after an extension of (a_{ij}) to an extended domain of Ω.

Notation III) $\tilde{u}_0$ denotes an extension (also translated) of u_0 so as to be defined continuously and uniformly with a compact support in $\tilde{\Omega}$ so that outside of Γ_1 become zero.

Notation IV) $\tilde{f}(\tilde{u})$ denotes an extension of $f(u)$ to $\tilde{\Omega}$ such that

(1) $\tilde{f}(\tilde{u})$ is Hölder continuous with exponents $\beta/2$ in $(x,t) \in \bar{\tilde{Q}}_T$, $|\tilde{u}| \leq M$, and $|\tilde{p}| \leq M_1$ See Chapter 4 for β , M ,and M_1.

(2) $\tilde{f}(\tilde{u})$ is Lipschitz continuous in t ; $\tilde{f}(\tilde{u})$ is differentiable in $\tilde{u}$ and $\tilde{p}$ in $(x,t) \in \bar{\tilde{Q}}_T$, $|\tilde{u}| \leq M$ and $|\tilde{p}| \leq M_1$.

(3) The Lipschitz constants, $|\partial\tilde{f}/\partial\tilde{u}|$, $|\partial\tilde{f}/\partial\tilde{p}_k|$ are bounded by a constant C.

(4) There is a number M_2 such that $\tilde{f}(s)s < 0$ for all $|s| \geq M_2$. This implies an L_∞ a priori bound.

Step 1) We extend Ω to a domain $\tilde{\Omega}$ with smooth boundary denoted as the Notation I) and simunenously (a_{ij}) , u_0 , and coefficient functions of $f(u)$ are extended to $\tilde{\Omega}$ so that the conditions of Notation II) , III) and IV) are satisfied , which are denoted by $(\tilde{a}_{ij})$, $\tilde{u}_0$ and $\tilde{f}(\tilde{u})$ respectly , and a sequence of open sets $\tilde{\Omega}_i$ with smooth boundary such that $\Omega \equiv \tilde{\Omega}_0 \subset \tilde{\Omega}_1 \subset$

$\tilde{\Omega}_2 \cdots \subset \tilde{\Omega}$, $\bigcup_{n=0} \tilde{\Omega}_n = \tilde{\Omega}$ and $\lim_{n\to\infty} \sup\{\underline{\tilde{a}}(x) \mid x \in \tilde{\Omega}\backslash\tilde{\Omega}_n\} = 0$ exist. Then the compatibility condition at the boundary of the extended and translated initial condition $\tilde{u}_0$ with the solution of an extended problem to the domain $\tilde{\Omega}$ is force for the solution to have zero at the whole boundary $\partial\tilde{\Omega}$ since $\tilde{u}_0$ is zero at the whole boundary $\partial\tilde{\Omega}$. Thus the problem (1.18) is transformed to the extended domain $\tilde{\Omega}$ as follows;

$$\begin{cases} \tilde{u}_t & = \nabla \cdot (\tilde{a}\nabla\tilde{u}) + \tilde{f}(\tilde{u}) & \text{in } \tilde{\Omega} \times I \\ \tilde{u} & = 0 & \text{on } \partial\tilde{\Omega} \times I \\ \tilde{u}(0) & = \tilde{u}_0 & \text{in } \tilde{\Omega} \end{cases} \tag{3.9}$$

Step 2)

$$\text{Let} \quad \tilde{u}_{0k} = \begin{cases} \tilde{u}_0(x) & \text{if} \quad \tilde{u}_0(x) > \frac{1}{k} \\ \frac{1}{k} & \text{if} \quad 0 \le \tilde{u}_0(x) \le \frac{1}{k} \end{cases}$$

Since $\tilde{a}$ is still degenerate, we replace $\tilde{a}$ by a nondegenerate approximating smooth function $\tilde{a}_{n\epsilon}$ for each $t \in I$ where $\tilde{a}_n$ is defined on $\tilde{\Omega}$ such that

$$\tilde{a}_n|_{\tilde{\Omega}_n} = \tilde{a}, \quad \underline{\tilde{a}}_n(x) \ge \tfrac{1}{n}, \tilde{a}_n(x)\ \xi \cdot \xi \ge \tilde{a}(x)\ \xi \cdot \xi$$

for all $x \in \tilde{\Omega}$ and $\xi \in R^N$, $\lim_{n\to\infty} \sup\{\ \|\tilde{a}_n(x) - \tilde{a}(x)\| \ \mid x \in \tilde{\Omega}\} = 0$ and $\tilde{a}_{n\epsilon}$ is defined on $\tilde{\Omega}$ such that

$$\sup_{x\in\tilde{\Omega}}\{\ \|\tilde{a}_{n\epsilon}(x) - \tilde{a}_n(x)\| \mid x \in \tilde{\Omega}\} < \epsilon \quad \text{for} \quad \epsilon > 0,$$

and $\tilde{a}_{n\epsilon}(x)\ \xi \cdot \xi \ge \tilde{a}_n(x)\ \xi \cdot \xi$ for all $x \in \tilde{\Omega}$ and $\xi \in R^N$. Here $\tilde{f}_{n\epsilon}$ is a smooth approximation of $\tilde{f}_n$ such that

$$\sup_{x\in\tilde{\Omega}}\{\left\|\tilde{f}_{n\epsilon} - \tilde{f}_n\right\| \mid x \in \tilde{\Omega}\} < \epsilon \quad \text{for} \quad \epsilon > 0,$$

where $\tilde{f}_n$ is defined such that

$$\tilde{f}_n\big|_{\tilde{\Omega}_n} = \tilde{f} \quad , \quad \text{the coefficient functions of } \tilde{f}_n(\tilde{u}) \text{ are greater than } \tfrac{1}{n},$$

and $\sup_{n\to\infty}\{\|\tilde{f}_n - \tilde{f}\| \mid x \in \tilde{\Omega}\} = 0$.

Then the problem (3.9) is replaced by the problem solving the nondegenerate approximating problems such that

$$\begin{cases} (\tilde{u}_{n\epsilon_i})_t &= \nabla \cdot (\tilde{a}_{n\epsilon_i} \nabla \tilde{u}_{n\epsilon_i}) + \tilde{f}_{n\epsilon}(\tilde{u}_{n\epsilon_i}) & \text{in } \tilde{\Omega} \times I \\ \tilde{u}_{n\epsilon_i} &= 0 & \text{on } \partial\tilde{\Omega} \times I \\ \tilde{u}_{n\epsilon_i}(0) &= \tilde{u}_{0,n\epsilon_i} & \text{in } \tilde{\Omega} \end{cases} \tag{3.10}$$

The condition $c_2 d(x)^\alpha \geq \bar{\tilde{a}}_{n\epsilon_i} \geq \underline{\tilde{a}}_{n\epsilon_i} \geq c_1 d(x)^\alpha$ with $1 \leq \alpha \leq 2$ and $c_1, c_2 > 0$, and the condition of Notation IV for $\tilde{f}$ are sufficient to satisfy the conditions solving the nondegenerate problems (see Chapter 4). Then its solution , $\tilde{u}_{n\epsilon_i}$, exits and belongs to $C^{2,1}(\tilde{\Omega}) \cap C(\bar{\tilde{\Omega}})$ for all $t \in I$.

Step 3) Let $J_\delta \tilde{u}_{n\epsilon_i}$ be a mollifier of $\tilde{u}_{n\epsilon_i}$ having a compact support in $\tilde{\Omega}$. Then $J_\delta \tilde{u}_{n\epsilon_i} \in C^\infty(\bar{\tilde{\Omega}})$. Let us show that we can extract a subsequence $\{ J_\delta \tilde{u}_{n\epsilon_n} \}$ such that

$$\left\| J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u} \right\|_{H^1_{\tilde{a}}(\tilde{\Omega})} < \epsilon \quad \text{for} \quad \epsilon > 0.$$

Consider $\left\| J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u} \right\|^2_{H^1_{\tilde{a}}(\tilde{\Omega})}$. Then

$$\left\| J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u} \right\|^2_{H^1_{\tilde{a}}(\tilde{\Omega})} \leq \left\| J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u} \right\|^2_{H^1_{\bar{\tilde{a}}_n}(\tilde{\Omega})} \leq \int_{\tilde{\Omega}} \bar{\tilde{a}}_n \left|\nabla(J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u})\right|^2 dx$$

$$\leq \int_{\tilde{\Omega}\setminus\tilde{\Omega}_m} \bar{\tilde{a}}_{n\epsilon_n} \left|\nabla(J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u})\right|^2 dx + \int_{\tilde{\Omega}_m} \bar{\tilde{a}}_{n\epsilon_n} \left|\nabla(J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u})\right|^2 dx \tag{3.11}$$

Lemma 3.1 If $c_2 d(x)^2 \geq \bar{\tilde{a}}_{n\epsilon_n}(x) \geq \underline{\tilde{a}}_{n\epsilon_n}(x) \geq c_1 d(x)^\alpha$ is satisfied for $1 \geq \alpha \geq 2$ and $c_1, c_2 > 0$, then we have a relation such that

$\bar{\tilde{a}}_n(x) \le \bar{\tilde{a}}_{n\epsilon_n}(x) \le c\underline{\tilde{a}}_{n\epsilon_n}^{1+\tilde{\alpha}}(x)$ for all x in a neighborhood of $\partial\tilde{\Omega}$.

proof We obtain $c\underline{\tilde{a}}_{n\epsilon_n}(x)\underline{\tilde{a}}_{n\epsilon_n}^{\tilde{\alpha}}(x) \ge cc_1\underline{\tilde{a}}_{n\epsilon_n}^{\tilde{\alpha}}(x)d(x)^{\alpha}$ by multifying $c\underline{\tilde{a}}_{n\epsilon_n}^{\tilde{\alpha}}(x)$ into $\underline{\tilde{a}}_{n\epsilon_n}(x) \ge c_1 d(x)^{\alpha}$,where $\tilde{\alpha}\,, c > 0$ are chosen so that $cc_1\underline{\tilde{a}}_{n\epsilon_n}^{\tilde{\alpha}}(x) \ge c_2$ for all x in the a neighborhood of $\partial\tilde{\Omega}$. Thus we have $c\underline{\tilde{a}}_{n\epsilon_n}^{1+\tilde{\alpha}}(x) \ge c_2 d(x)^{\alpha} \ge \bar{\tilde{a}}_{n\epsilon_n}(x)$ for all x in a neighborhood of $\partial\tilde{\Omega}$. Hence $c\underline{\tilde{a}}_{n\epsilon_n}^{1+\tilde{\alpha}}(x) \ge \bar{\tilde{a}}_{n\epsilon_n}(x)$ for all x in a neighborhood of $\partial\tilde{\Omega}$.

Therefore, By Lemma 3.1 we obtain

$$
\begin{aligned}
(3.11) \le \int_{\tilde{\Omega}\setminus\tilde{\Omega}_m} & c\underline{\tilde{a}}_{n\epsilon_n}^{1+\tilde{\alpha}} \left|\nabla(J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u})\right|^2 dx \\
&+ \int_{\tilde{\Omega}_m} \bar{\tilde{a}}_{n\epsilon_n}\left|\nabla(J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u})\right|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&\le c\sup\{\underline{\tilde{a}}_{n\epsilon_n}^{\tilde{\alpha}} | x \in \tilde{\Omega}\setminus\tilde{\Omega}_m\}\left\| J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}\right\|_{H^1_{\bar{\tilde{a}}_{n\epsilon_n}}(\tilde{\Omega}\setminus\tilde{\Omega}_m)} \\
&\qquad + \left\| J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}\right\|_{H^1_{\bar{\tilde{a}}_{n\epsilon_n}}(\tilde{\Omega}_m)} \qquad (3.12)
\end{aligned}
$$

For a given $\epsilon > 0$, we can make the first term smaller than ϵ by choosing m sufficiently large. Now let us consider in (3.12)

$$\| J_\delta\tilde{u}_{n\epsilon_n}(t) - \tilde{u}(t)\|^2_{H^1_{\tilde{a}_{n\epsilon_n}}(\tilde{\Omega}_m)} \quad \text{for all}\, t \in I.$$

Then

$$
\begin{aligned}
&\| J_\delta\tilde{u}_{n\epsilon_n}(t) - \tilde{u}(t)\|^2_{H^1_{\tilde{a}_{n\epsilon_n}}(\tilde{\Omega}_m)} \\
&\qquad = < \tilde{a}_{n\epsilon_n}\nabla(J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u})\,,\,\nabla(J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}) > \\
&\qquad = \int_{\tilde{\Omega}_m} \tilde{a}_{n\epsilon_n}\nabla(J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u})\cdot\nabla(J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u})\,dx \quad (3.13)
\end{aligned}
$$

Remark 3.2) $a\nabla w\cdot\nabla w = \nabla\cdot(wa\nabla w) - w\nabla\cdot(a\nabla w)$

$$(3.13) = \int_{\tilde{\Omega}_m} \nabla \cdot ((J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u})\tilde{a}_{n\epsilon_n} \nabla (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}))\, dx$$

$$- \int_{\tilde{\Omega}_m} (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}) \nabla \cdot (\tilde{a}_{n\epsilon_n} \nabla (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}))\, dx \quad (3.14)$$

Lemma 3.2 $\int_{\tilde{\Omega}_m} \nabla \cdot ((J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u})\tilde{a}_{n\epsilon_n} \nabla (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}))\, dx = 0$
for a sufficiently large m.

Proof $\int_{\tilde{\Omega}_m} \nabla \cdot (aw \nabla w)\, dx = \int_{\partial\tilde{\Omega}_m} (aw \nabla w) \cdot n\, d\sigma = \int_{\partial\tilde{\Omega}_m} aw \frac{\partial w}{\partial \eta}\, d\sigma$
by green's formula . Let $a = \tilde{a}_{n\epsilon_n}$ and $w = J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}$. Then since $J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u} = 0$ for sufficiently large m ,

$$\int_{\partial\tilde{\Omega}_m} \tilde{a}_{n\epsilon_n} (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}) \frac{\partial (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u})}{\partial \eta}\, d\sigma \equiv 0 \quad \square$$

By applying Lemma 3.2 and (3.10) to (3.14), we obtain

$$(3.14) = - \int_{\tilde{\Omega}_m} (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}) \nabla \cdot (\tilde{a}_{n\epsilon_n} \nabla (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}))\, dx$$

$$= - \int_{\tilde{\Omega}_m} (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}) \{ (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u})_t - (\tilde{f}_{n\epsilon_n}(J_\delta \tilde{u}_{n\epsilon_n}) - \tilde{f}_{n\epsilon_n}(\tilde{u})) \}\, dx$$

$$\leq \left\| J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u} \right\|_2 \left(\left\| (J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u})_t \right\|_2 + \left\| \tilde{f}_{n\epsilon_n}(J_\delta \tilde{u}_{n\epsilon_n}) - \tilde{f}_{n\epsilon_n}(\tilde{u}) \right\|_2 \right) \quad (3.15)$$

Lemma 3.3 $\dot{\tilde{u}}_{n\epsilon_n}$ is equibounded in $L_2(\tilde{\Omega})$.

Proof Let $A_{n\epsilon_n} \tilde{u}_{n\epsilon_n} = \nabla \cdot (\tilde{a}_{n\epsilon_n} \nabla \tilde{u}_{n\epsilon_n})$ and $v_n(t) = \dot{\tilde{u}}_{n\epsilon_n}(t)$.
Thus the equation of (3.10) implies

$$v_n(t) = A_{n\epsilon_n} \tilde{u}_{n\epsilon_n}(t) + \tilde{f}_{n\epsilon_n}(\tilde{u}_{n\epsilon_n}(t))$$

and we set

$$v_{n,0} = \dot{\tilde{u}}_{0,n} = A_{n\epsilon_n} \tilde{u}_{0,n} + \tilde{f}_{n\epsilon_n}(\tilde{u}_{0,n}).$$

Then from variation of constants formula , we obtain

$$v_n(t) = e^{tA_{n\epsilon_n}} v_{n,0} + \int_0^t e^{(t-\tau)A_{n\epsilon_n}} \tilde{f}_{n\epsilon_n}{}'(\tilde{u}_n(\tau)) v_n(\tau)\, d\tau.$$

The Notation IV (3) implies that $\tilde{f}_{n\epsilon_n}{}'(\tilde{u}_n(\tau))$ is bounded by a constant. Thus

$$\|v_n(t)\|_2 \leq \|v_{n,0}\| + \int_0^t C\,\|v_n(\tau)\|_2\, d\tau.$$

and Theorem 1.8 (A theorem for the Gronwall Inequality) implies

$$\|v_n(t)\|_2 \leq \|v_{n,0}\|_2\, e^{Ct}.$$

Since $\|v_{n,0}\|_2 \leq \|A_{n\epsilon_n}\tilde{u}_{0,n}\|_2 + \left\|\tilde{f}_{n\epsilon_n}(\tilde{u}_{0,n})\right\|_2$, we have

$$\|v_n(t)\|_2 \leq \|v_{n,0}\|_2 \leq C.$$

Thus $\left\|\dot{\tilde{u}}_{n\epsilon_n}\right\|_2 \leq C$ where the constant C does not depend on n and t $\square$

Therefore , in (3.15) by Lemma 3.3 we know

$$\left\|\,(J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u})_t\,\right\|_2 + \left\|\tilde{f}_{n\epsilon_n}(J_\delta\tilde{u}_{n\epsilon_n}) - \tilde{f}_{n\epsilon_n}(\tilde{u})\right\|_2 \leq C.$$

From (V) of Theorem 1.1 , it is known that if $\tilde{u}_{n\epsilon_n} \in C(\bar{\tilde{\Omega}})$, then $\lim_{\delta\to 0+} J_\delta\tilde{u}_{n\epsilon_n}(x) = \tilde{u}_{n\epsilon_n}(x)$ and $\lim_{n\to\infty}\tilde{u}_{n\epsilon_n} = \tilde{u}$ uniformly on $\tilde{\Omega}$. Hence, we can make $\|J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}\|_2$ in (3.15) sufficiently small. Therefore , we can make

$$\left\|\,J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}\,\right\|^2_{H^1_{\tilde{a}_{n\epsilon_n}}(\tilde{\Omega}_m)} < \epsilon$$

for sufficiently large m,n, given $\epsilon > 0$. Thus, finally , we have

$$\left\|\,J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}\,\right\|_{H^1_{\tilde{a}_n}(\tilde{\Omega})} < \epsilon.$$

That is , a sequence $\{\,J_\delta\tilde{u}_{n\epsilon_n}\,\}$ converges to $\tilde{u}$ in the seminorm $\|\cdot\|^2_{H^1_{\tilde{a}}(\tilde{\Omega})}$. We also can make

$$\left\|\,J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}\,\right\|_{L_2(\tilde{\Omega};d,\alpha)} < \epsilon_1 \quad \text{for a given}\ \epsilon_1 > 0.$$

Hence from such a relation that

$$\left|J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}\right| \leq \left|J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u}_{n\epsilon_n}\right| + \left|\tilde{u}_{n\epsilon_n} - \tilde{u}\right|,$$

we have

$$\left\| J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u} \right\|_{H^1_{\tilde{a}}(\tilde{\Omega})} + \left\| J_\delta \tilde{u}_{n\epsilon_n} - \tilde{u} \right\|_{L_2(\tilde{\Omega};d,\alpha)} < \epsilon$$

for sufficiently large n , $\epsilon > 0$. Thus we can extract a subsequence $\{J_\delta \tilde{u}_{n\epsilon_n}\}$ converging to $\tilde{u}$ in the norm of the weighted Sobolev space $W_2^1(\tilde{\Omega}; d, \alpha)$. Since $C^\infty(\bar{\tilde{\Omega}})$ is dense in $W_2^1(\tilde{\Omega}; d, \alpha)$ and $J_\delta \tilde{u}_{n\epsilon_n} \in C^\infty(\bar{\tilde{\Omega}})$, we have $\tilde{u} \in W_2^1(\tilde{\Omega}; d, \alpha) \subset L_2(\tilde{\Omega}; d, \alpha)$. Now let us translate $\tilde{u}$ back to original position and restrict it to the original domain Ω and let this be denoted by $(\tilde{u}_b)\big|_\Omega = u_b$. Then u_b is given as a solution of the original problem (1.18) because it satisfies the boundary condition of the original problem. That is , zero at Γ_1 , no boundary condition at Γ_0 , and $u_b \in W_2^1(\Omega; d, \alpha) \subset L_2(\Omega; d, \alpha)$.

Now from Lemma 3.3 , we know that

$$\left\|\dot{\tilde{u}}_{n\epsilon_n}\right\|_2 \leq C,$$

where the constant C does not depend on n and t. We can also obtain

$$\left\|\dot{\tilde{u}}_{n\epsilon_n}\right\|_{L_2(\tilde{\Omega};d,\alpha)} \leq C \tag{3.16}$$

by using the $L_2(\tilde{\Omega}; d, \alpha)$-norm instead of the $\|\cdot\|_2$ -norm given in Lemma 3.3. Then (3.16) implies

$$\| \tilde{u}_{n\epsilon_n}(t+h) - \tilde{u}_{n\epsilon_n}(t) \|_{L_2(\tilde{\Omega};d,\alpha)} \leq Ch \quad \text{for all}\, t \in I \,\text{and}\, n \in N \tag{3.17}$$

where C does not depend on n and t . Since $\tilde{u}_{n\epsilon_n}$ converges uniformly to $\tilde{u}$ in $L_2(\tilde{\Omega}; d, \alpha)$, it is clear that (3.17) implies the Lipschitz continuity of $\tilde{u}$ in $L_2(\tilde{\Omega}; d, \alpha)$. Thus (3.17) also implies the Lipschitz continuity of u_b since the property of Lipschitz continuity is invariant under translation and restriction of the domain .

It remains to show that u_b is a weak solution of the problem (1,18). The proof is similar to that of [13] . Let us take a fixed test function

$$\psi \in C^1(I, L_2(\tilde{\Omega}; d, \alpha)) \cap C^0(I, \overset{\circ}{W}{}_2^1(\tilde{\Omega}; d, \alpha))$$

Since $\tilde{u}_{n\epsilon_n}$ is a classical solution of (3.10), we have for $T \in I$,

$$< \psi(T), \tilde{u}_{n\epsilon_n}(T) > - < \psi(0), \tilde{u}_{0,n} > - \int_0^T < \dot{\psi}(t), \tilde{u}_{n\epsilon_n}(t) > dt$$
$$= - \int_0^T < \tilde{a}_{n\epsilon_n} \nabla \psi(t), \nabla \tilde{u}_{n\epsilon_n}(t) > dt + \int_0^T < \psi(t), \tilde{f}_{n\epsilon_n}(\tilde{u}_{n\epsilon_n}(t)) > dt \tag{3.18}$$

Remark 3.3 By maximum principle,

$$-\frac{e^{-2c_2T}}{n} \leq \tilde{u}_{n\epsilon_n} \leq M = e^{2c_2T} \max\{\sup_{\tilde{\Omega}} \tilde{u}_{0,n}, 0\}$$

$\tilde{u}_{n\epsilon_n} \in C^{2,1}(\bar{\tilde{Q}})$ and $||\tilde{u}_{n\epsilon_n}||_{C^{1,2}}$ is independent of n.
Therefore $\{\tilde{u}_{n\epsilon_n}\}$ is compact in $C(\bar{\tilde{Q}})$ for all $t \in I$. (See Dong [12])

By Remark 3.3 , it is clear that the first line in (3.18) converges to the same expression , where $\tilde{u}_{n\epsilon_n}$ is replaced by $\tilde{u}$ if n tends to infinity. Similarly

$$< \psi(t), \tilde{f}_{n\epsilon_n}(\tilde{u}_{n\epsilon_n}(t)) > \to < \psi(t), \tilde{f}(\tilde{u}(t)) >$$

uniformly with respect to $t \in [0, T]$ by Remark 3.3 and the assumptions of Notation IV. Let us show that $< \tilde{a}_{n\epsilon_n} \nabla \psi(t), \nabla \tilde{u}_{n\epsilon_n}(t) >$ converges to $< \tilde{a} \nabla \psi, \nabla \tilde{u}(t) >$ as n goes to infinity. For let $\epsilon > 0$ be given,

$$
\begin{aligned}
&|< \tilde{a}_{n\epsilon_n}\nabla\tilde{u}_{n\epsilon_n} - \tilde{a}\nabla\tilde{u}, \nabla\psi >| \\
&\leq \int_{\tilde{\Omega}} |\tilde{a}_{n\epsilon_n} - \tilde{a}|\ |\nabla\tilde{u}_{n\epsilon_n}|\ |\nabla\psi|\ dx + |< \tilde{a}\nabla(\tilde{u}_{n\epsilon_n} - \tilde{u}), \nabla\psi >| \\
&\leq c\|\tilde{u}_{n\epsilon_n}\|_{H^1_{\tilde{a}_{n\epsilon_n}}(\tilde{\Omega})} \sup\left\{|\tilde{a}_{n\epsilon_n}(x) - \tilde{a}(x)|\ |\ x \in \tilde{\Omega}\right\}^{\frac{1}{2}} \left\|\psi\right\|_{1,2} \\
&+\left|\int_{\tilde{\Omega}_m} \tilde{a}\nabla(\tilde{u}_{n\epsilon_n} - \tilde{u}) \cdot \nabla\psi\, dx\right| + \left|\int_{\tilde{\Omega}\setminus\tilde{\Omega}_m} \tilde{a}\nabla(\tilde{u}_{n\epsilon_n} - \tilde{u}) \cdot \nabla\psi\, dx\right| \\
&\leq c\|\tilde{u}_{n\epsilon_n}\|_{H^1_{\tilde{a}_{n\epsilon_n}}(\tilde{\Omega})} \sup\{|\tilde{a}_{n\epsilon_n}(x) - \tilde{a}(x)|\,|x \in \tilde{\Omega} \setminus \tilde{\Omega}_n\} \left\|\psi\right\|_{1,2} \\
&+c\|\tilde{u}_{n\epsilon_n} - \tilde{u}\|_{H^1_{\tilde{a}}(\tilde{\Omega}_m)} \left\|\psi\right\|_{1,2} \\
&+(\|\tilde{u}_{n\epsilon_n}\|_{H^1_{\tilde{a}}(\tilde{\Omega})} + \|\tilde{u}\|_{H^1_{\tilde{a}}(\tilde{\Omega})}) \sup\left\{|\tilde{a}(x)|\ |x \in \tilde{\Omega} \setminus \tilde{\Omega}_m\right\}^{\frac{1}{2}} \left\|\psi\right\|_{1,2} \\
&\leq \epsilon \left\|\psi\right\|_{1,2},
\end{aligned}
$$

for sufficiently large n.

Lemma 3.4 $\|\tilde{u}_{n\epsilon_n} - \tilde{u}\|_{H^1_{\tilde{a}}(\tilde{\Omega}_m)} \leq \epsilon$ for sufficiently large m , n and $\|\tilde{u}_{n\epsilon_n}\|_{H^1_{\tilde{a}}(\tilde{\Omega})} \leq c_1$, $\|\tilde{u}\|_{H^1_{\tilde{a}}(\tilde{\Omega})} \leq c_2$, where c_1 and c_2 does not depend on n.

Proof $\|\tilde{u}_{n\epsilon_n} - \tilde{u}\|_{H^1_{\tilde{a}}(\tilde{\Omega}_m)} \leq \|J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}_{n\epsilon_n}\| + \|J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}\|$.
The right side can be made sufficiently small since $\|J_\delta\tilde{u}_{n\epsilon_n} - \tilde{u}\| \leq \epsilon$ for sufficiently large n , m from Step 3. $\|\tilde{u}_{n\epsilon_n}\|_{H^1_{\tilde{a}}(\tilde{\Omega}_m)} \leq c_1$, $\|\tilde{u}\|_{H^1_{\tilde{a}}(\tilde{\Omega}_m)} \leq c_2$ are shown by (3.17) , because $\tilde{u} \in W^1_2(\tilde{\Omega}; d, \alpha)$.

To prove the last inequality one has to use Lemma 3.4 and the assumption of Step 1 . First we can choose m such that the third term is as small as desired and then make n so great that the first two terms are small. This implies that $\tilde{u}$ is a weak solution of (3.9). Thus u_b is also a weak solution of (1.18). □

The Proof of the Uniqueness

It is sufficient to show the uniqueness of a solution of the problem (3.9) in order to show the uniqueness of a solution of the problem (1.18). Thus suppose we have two solutions $\tilde{v}$ and $\tilde{w}$ of the problem (3.9) with the same initial value. Then $\tilde{u} = \tilde{v} - \tilde{w}$ would be a weak solution with $\tilde{u}_0 = 0$ and $\tilde{f}(\tilde{u})$ replaced by $F(t) := \tilde{f}(\tilde{v}(t)) - \tilde{f}(\tilde{w}(t))$. We must show that $\tilde{u}$ is identically zero. Let us introduce the function

$$\psi_{r,n}(x) = \psi_{r,n}(x_N) = \begin{cases} 0 & \text{for} \quad 0 \leq x_N \leq r \\ (\ln|\ln r|)^{\epsilon} - (\ln|\ln x_N|)^{\epsilon} & \text{for} \quad r \leq x_N \leq \frac{1}{n} \\ 1 & \text{for} \quad x_N \geq \frac{1}{n} \end{cases}$$

where $(\ln|\ln r|)^{\epsilon} - (\ln\left|\ln \frac{1}{n}\right|)^{\epsilon} = 1\,,\ 0 < \epsilon < \frac{1}{2}$.

Let $S_n = \{x = (\tilde{x}, x') \in \tilde{\Omega}\,\text{with}\, x' > \frac{1}{n}\,,\ \tilde{x} \in R^{N-1}\}$. Let us also define $R_{r,n}\tilde{u} = \tilde{u}(x)\cdot\psi_{r,n}(x)$. Then it is clear that $R_{r,n}\tilde{u}(x) = \tilde{u}(x)$ for all $x \in S_n$.

Lemma 3.5 $lim_{r\to 0} < \tilde{a}\nabla R_{r,n}\tilde{u}, \nabla\tilde{u} > = < \tilde{a}\nabla\tilde{u}, \nabla\tilde{u} >$ and

$\lim_{n\to\infty} < \tilde{a}\nabla R_{r,n}\tilde{u}, \nabla\tilde{u} > = < \tilde{a}\nabla\tilde{u}, \nabla\tilde{u} >$.

Proof)

$$\begin{aligned} &\lim_{r\to 0} < \tilde{a}\nabla R_{r,n}\tilde{u}, \nabla\tilde{u} > = \lim_{r\to 0} < \tilde{a}\nabla(\tilde{u}(x)\cdot\psi_{r,n}(x)), \nabla\tilde{u} > \\ &= \lim_{r\to 0} < \tilde{a}\psi_{r,n}(x)\nabla\tilde{u}(x), \nabla\tilde{u}(x) > + \lim_{r\to 0} < \tilde{a}\tilde{u}(x)\nabla\psi_{r,n}(x), \nabla\tilde{u}(x) > \end{aligned}$$

The first limit on the right-hand side is equal to $< \tilde{a}\nabla\tilde{u}(x), \nabla\tilde{u}(x) >$ since $\psi_{r,n}(x)$ goes to one as r tends to zero. We show first that the second limit does not exceed $\lim_{r\to 0} < \tilde{a}\nabla\tilde{u}, \nabla\tilde{u} >^{\frac{1}{2}} \cdot < \tilde{u}\nabla\psi_{r,n}, \tilde{u}\nabla\psi_{r,n} >^{\frac{1}{2}}$ in modulus and is equal to zero. By virtue of the fact that $\frac{\partial\psi_{r,n}}{\partial x_j} = 0$ for $0 < j \leq N-1$, we have

$$< \tilde{u}\nabla\psi_{r,n}, \tilde{u}\nabla\psi_{r,n} > = \int_{\tilde{\Omega}} a_{NN}(x)\Big(\frac{\partial\psi_{r,n}}{\partial x_N}\Big)^2 \tilde{u}^2\, dx$$
$$\leq c_2 \int_{\tilde{\Omega}} x_N^{\alpha}\Big(\frac{\partial\psi_{r,n}}{\partial x_N}\Big)^2 \tilde{u}^2\, dx \qquad (3.19)$$

since $c_2 x_N^{\alpha} \geq a_{NN}(x)$ from condition (1.19).

Lemma 3.6 We will always consider that the function u(x) for $x_N > 0$ is extended to zero outside the domain $\tilde{\Omega}$. If $u \in W_2^1(\tilde{\Omega}; d, \alpha)$ and the condition (1.19) is satisfied , then

$$\begin{cases} |\tilde{u}| \leq c x_N^{\frac{1-\alpha}{2}} & \text{when} \quad \alpha > 1 \\ |\tilde{u}| \leq c|\ln x_N|^{\frac{1}{2}} & \text{when} \quad \alpha = 1 \end{cases}$$

where $c > 0$ is a constant.

<u>**Proof**</u> In the case $\alpha > 1$,

$$\tilde{u}^2(x) = \Big[\int_{x_N}^{A} x_N^{-\frac{\alpha}{2}} x_N^{\frac{\alpha}{2}} \frac{\partial\tilde{u}}{\partial x_N}\, dx_N\Big]^2$$
$$= |1-\alpha|^{-1}\, \big|x_N^{1-\alpha} - A^{1-\alpha}\big| \int_{x_N}^{A} x_N^{\alpha}\Big(\frac{\partial\tilde{u}}{\partial x_N}\Big)^2 dx_N$$

and in the case $\alpha = 1$,

$$\tilde{u}^2(x) = \Big[\int_{x_N}^{A} x_N^{-\frac{1}{2}} x_N^{\frac{1}{2}} \frac{\partial\tilde{u}}{\partial x_N}\, dx_N\Big]^2$$
$$\leq \left|\int_{x_N}^{A} x_N^{-1}\, dx_N\right| \left|\int_{x_N}^{A} x_N\Big(\frac{\partial\tilde{u}}{\partial x_N}\Big)^2 dx_N\right|$$
$$= |\ln A - \ln x_N| \int_{x_N}^{A} x_N\Big(\frac{\partial\tilde{u}}{\partial x_N}\Big)^2 dx_N$$

where A is so large a number that all of $\tilde{\Omega}$ is situated in $x_N < A$. Then $u \in W_2^1(\tilde{\Omega}; d, \alpha)$ and the condition (1.19) implies

$$\begin{cases} |\tilde{u}| \le c x_N^{\frac{1-\alpha}{2}} & \text{when} \quad \alpha > 1 \\ |\tilde{u}| \le c|\ln x_N|^{\frac{1}{2}} & \text{when} \quad \alpha = 1 \end{cases}$$

where $c > 0$ is a constant $\square$

Thus in the case $\alpha > 1$ by Lemma 3.6 ,

$$\begin{aligned} (3.19) &\le c_2 c^2 \int_{\tilde{\Omega}} x_N^{\alpha} (x_N^{\frac{1-\alpha}{2}})^2 (\tfrac{\partial \psi_{r,n}}{\partial x_N})^2 \, dx \\ &= c_3 \int_{\tilde{\Omega}} x_N (\tfrac{\partial \psi_{r,n}}{\partial x_N})^2 \, dx \end{aligned} \tag{3.20}$$

where $c_3 = c_2 c^2$.

Remark 3.4

$$\frac{\partial \psi_{r,n}}{\partial x_N} = \begin{cases} 0 & \text{for} \quad x_N > r, \\ \frac{\epsilon}{x_N |\ln x_N| (\ln|\ln x_N|)^{1-\epsilon}} & \text{for} \quad r \le x_N \le \frac{1}{n}, \\ 0 & \text{for} \quad x_N > \frac{1}{n} \end{cases}$$

$\frac{\partial \psi_{r,n}}{\partial x_N}$ is obtained by direct differential of $\psi_{r,n}$ function.

Thus by Remark 3.4 ,

$$(3.20) = c_4 \int_{\tilde{\Omega} \cap \{r \le x_N \le \frac{1}{n}\}} \frac{1}{x_N |\ln x_N|^2 (\ln|\ln x_N|)^{2-2\epsilon}} \, dx$$

where $c_4 = c_3 \epsilon^2$ and the last term comverges to zero as r converges to zero. In the case $\alpha = 1$, by Lemma 3.6 and 3.7, (3.19) becomes

$$c_2 \int_{\tilde{\Omega}} x_N (\tfrac{\partial \psi_{r,n}}{\partial x_N})^2 \tilde{u}^2 \, dx \le c_4 \int_{\tilde{\Omega} \cap \{r \le x_N \le \frac{1}{n}\}} \frac{1}{x_N |\ln x_N| (\ln|\ln x_N|)^{2-2\epsilon}} \, dx$$

The last term converges to zero as r converges to zero. Thus we have that since $1/n$ goes to zero as $n \to \infty$ and r goes to zero , for $\alpha \ge 1$,

$$\lim_{r \to 0} < \tilde{u} \nabla \psi_{r,n}, \tilde{u} \nabla \psi_{r,n} >= 0 \text{ and } \lim_{n \to \infty} < \tilde{u} \nabla \psi_{r,n}, \tilde{u} \nabla \psi_{r,n} >= 0.$$

Since $\tilde{u}$ is only defined for $t \in I$, we extend it continuously to $\mathbb{R}$ by 0 for negative arguments and by $u(T)$ for $t > T$. To obtain uniqueness of the solution , we use a test function such that

$$\varphi_{n,\delta}(t) = \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} R_{r,n}\tilde{u}(\tau)\, d\tau \quad \text{where}\, \delta > 0.$$

Now we have

$$\varphi_{n,\delta} \in C^1(I, L_2(\tilde{\Omega})) \cap C^0(I, \overset{\circ}{W}{}_2^1(\tilde{\Omega}; d, \alpha))$$

and

$$\dot{\varphi}_{n,\delta}(t) = \frac{1}{2\delta} R_{r,n}(u(t+\delta) - u(t-\delta)).$$

We first want to let δ converge to zero.

$$\begin{aligned}\int_0^T < \dot{\varphi}_{n,\delta}(t), \tilde{u}(t) > dt &= \int_0^T \frac{1}{2\delta} < \tilde{u}(t+\delta) - \tilde{u}(t-\delta), \tilde{u}(t) >_{S_n} dt \\ &+ \int_0^T \int_{S_n^c} \frac{1}{2\delta} R_{r,n}(\tilde{u}(t+\delta) - \tilde{u}(t-\delta))\tilde{u}(t)\, dxdt \\ &\to \frac{1}{2}(\|\tilde{u}(T)\|^2_{L_2(S_n)} - \|\tilde{u}(0)\|^2_{L_2(S_n)}) + K_n\end{aligned}$$

where $K_n := \lim_{\delta\to 0+} \int_0^T \int_{S_n^c} \frac{1}{2\delta} R_{r,n}(\tilde{u}(t+\delta) - \tilde{u}(t-\delta))\tilde{u}(t)\, dxdt$, $|K_n| \le \int_0^T cM\mathrm{vol}(S_n^c)^{\frac{1}{2}}\, dt$. To verify the above inequality one uses the L_∞-bound and the Lipschitz condition on $\tilde{u}$.

Since $\tilde{u} \in C^0(I, L_2(\tilde{\Omega})) \cap C^0(I, W_2^1(\tilde{\Omega}'; d, \alpha))$, we deduce

$$-\int_0^T < \tilde{a}\nabla\varphi_{n,\delta}(t), \nabla\tilde{u}(t) > dt \to -\int_0^T < \tilde{a}\nabla R_{r,n}\tilde{u}(t), \nabla\tilde{u}(t) > dt$$

and

$$\int_0^T < \varphi_{n,\delta}(t), F(t) > dt \to \int_0^T < R_{r,n}\tilde{u}(t), F(t) > dt.$$

as δ approaches zero . Since $\tilde{u}$ is a weak solution , we obtain

$$\|\tilde{u}(T)\|^2_{L_2(\tilde{\Omega})} - \|\tilde{u}(0)\|^2_{L_2(\tilde{\Omega})} - \frac{1}{2}(\|\tilde{u}(T)\|^2_{L_2(S_n)} - \|\tilde{u}(0)\|^2_{L_2(S_n)}) + K_n$$

$$= - \int_0^T < \tilde{a}\nabla R_{r,n}\tilde{u}(t), \nabla\tilde{u}(t) > dt + \int_0^T < R_{r,n}\tilde{u}(t), F(t) > dt$$

Now we let n tend to infinity and use the above lemma and $\tilde{u}(0) = 0$. Then we obtain

$$\|\tilde{u}(T)\|^2_{L_2(\tilde{\Omega})} = -2\int_0^T < \tilde{a}\nabla\tilde{u}(t), \nabla\tilde{u}(t) > dt + \int_0^T < \tilde{u}(t), F(t) > dt \tag{3.21}$$

Now from assumptions of Notation IV for $\tilde{f}$,

$$\|F(t)\|_2 = \|\tilde{f}(\tilde{v}(t)) - \tilde{f}(\tilde{w}(t))\|_2 \leq c\|\tilde{v}(t) - \tilde{w}(t)\|_2 = \|\tilde{u}(t)\|_2 \tag{3.22}$$

Thus applying (3.22) into (3.21) , we obtain

$$\|\tilde{u}(T)\|^2_{L_2(\tilde{\Omega})} + 2\int_0^T \|\tilde{u}(t)\|^2_{H^1_{\tilde{a}}(\tilde{\Omega})} dt \leq 2c\int_0^T \|\tilde{u}(t)\|^2_{L_2(\tilde{\Omega})} dt.$$

Thus

$$\|\tilde{u}(T)\|^2_{L_2(\tilde{\Omega})} \leq 2c\int_0^T \|\tilde{u}(t)\|^2_{L_2(\tilde{\Omega})} dt.$$

From Theorem 1.8 (A theorem for the Gronwall Inequality),

$$\|\tilde{u}(T)\|_{L_2(\tilde{\Omega})} \leq 0.$$

This implies that $\tilde{u}$ has vanished identically. $\Box$

CHAPTER 4

THEOREMS FOR NONDEGENERATE PARABOLIC EQUATIONS AND DEGENERATE PARABOLIC EQUATIONS

This chapter provides some known results for nondegenerate parabolic equations and degenerate parabolic equations with principal part in divergence form : (1.1) or (1.2). We deal mainly with the solvability of the first boundary value problem for them.

4.1 Nondegenerate Parabolic Equations

We begin with the case of nondegenerate parabolic equations. Let us consider the problem of finding a function u such that

$$\begin{cases} u_t & = \frac{d}{dx_i} a_i(x,t,u,u_x) + b(x,t,u,u_x) \\ u|_{\Gamma_T} & = \psi|_{\Gamma_T} \quad \text{for some given function } \psi. \end{cases} \tag{4.1}$$

We make the following assumptions :
(a) For $(x,t) \in \bar{Q}_T$ and arbitrary u , $\frac{\partial a_i}{\partial p_j}\xi_i\xi_j\Big|_{p=0} \geq 0$,
$A(x,t,u,0)u \geq -b_1u^2 - b_2$ are satisfied where b_1 and b_2 are nonnegative constants and

$$A(x,t,u,u_x) = b(x,t,u,u_x) - (\partial a_i/\partial u)u_{x_i} - \partial a_i/\partial x_i$$

Here for $(x,t) \in \bar{Q}_T$, $|u| \leq M$, and arbitrary p , M is taken to be $\max_{Q_T}|u^\tau| \leq M, \tau \in [0,1]$. $\{u^\tau\}$ is the solution to the family of the following problems. For each τ and for w(x,t) given, consider the linear problem :

$$\begin{cases} v_t - [\tau\frac{\partial a_i(x,t,w,w_x)}{\partial w_{x_j}} + (1-\tau)\delta_i^j]v_{x_ix_j} \\ \qquad +\tau A(x,t,w,w_x) - (1-\tau)[\psi_t - \Delta\psi] = 0 \\ v|_{\Gamma_T} = \psi|_{\Gamma_T}, \quad 0 \leq \tau \leq 1 \end{cases} \tag{4.2}$$

for determining the function v. We introduce a linear Banach space, B_δ, of functions, $w(x,t)$, that are continuous together with their derivatives with respect to x in $\bar{Q}_T$ and have the finite norm

$$|w|_{B_\delta} \equiv |w|_{Q_T}^{(\delta)} + |w_x|_{Q_T}^{(\delta)}$$

under certain restrictions on the functions a_i, a, ψ and S . The problem (4.2) defines an operator Ψ in B_δ which associates with each function w of B_δ a solution v of the linear problem (4.2) : $v = \Psi(w;\tau)$. This operator is nonlinear and depends on τ . Its fixed points for $\tau = 1$ are solutions to the problem (4.1). Let u^τ be one of the fixed points of the transformation $\Psi(w;\tau)$. That is, let $u^\tau = \Psi(u^\tau;\tau)$. This means that u^τ is a solution to the nonlinear problem

$$\begin{cases} u_t & -\frac{d}{dx_i}[\tau a_i(x,t,u,u_x) + (1-\tau)u_{x_i}] \\ & \qquad +\tau b(x,t,u,u_x) - (1-\tau)(\psi_t - \Delta\psi) = 0 \\ u|_{\Gamma_T} & = \psi|_{\Gamma_T}, \qquad 0 \le \tau \le 1 \end{cases} \tag{4.3}$$

where a_i and b are continuous, the $a_i(x,t,u,p)$ are differential with respect to x, u, and p , and a_i and b satisfy the inequalities

$$\nu\xi^2 \le \frac{\partial a_i}{\partial p_j}\xi_i\xi_j \le \mu\xi^2, \quad \nu > 0 \quad \text{and}$$

$\sum_{i=1}^{n}(|a_i| + \left|\frac{\partial a_i}{\partial u}\right|)(1+|p|) + \sum_{i,j=1}^{n}\left|\frac{\partial a_i}{\partial x_j}\right| + |b| \le \mu(1+|p|)^2$

(c) For $(x,t) \in \bar{Q}_T, |u| \le M$ and $|p| \le M_1$, Here M_1 is taken to be $\max_{Q_T}|u_x^\tau| \le M_1, \tau \in [0,1]$. The function $a_i, b, \frac{\partial a_i}{\partial u}, \frac{\partial a_i}{\partial p_j}$ and $\frac{\partial a_i}{\partial x_j}$ are continuous functions satisfing a Hölder condition in x,t,u and p with exponents $\beta, \frac{\beta}{2}, \beta$ and β respectively .

(d) a_i and b are Lipschitz continuous in t ; b is differentiable in u and p in $(x,t) \in \bar{Q}_T, |u| \le M, |p| \le M_1$. The Lipschitz constants , $\left|\frac{\partial a}{\partial u}\right|$, and $\left|\frac{\partial a}{\partial p_k}\right|$ are bounded.

(e) $\psi(x,t) \in H^{2+\beta,1+\frac{\beta}{2}}(\bar{Q}_T)$ (See Appendix 2 (18)) and satisfies compatability conditions on $S_0 \equiv \{x \in \partial\Omega, t=0\}$. Thus the given ψ satisfies $\psi_t - \frac{d}{dx_i}a_i(x,t,\psi,\psi_x) + b(x,t,\psi,\psi_x) = 0$ on $S_0 \equiv \{x \in \partial\Omega, t=0\}$.

(f) $\partial\Omega \in H^{2+\beta}$ $(\beta > 0)$. (See Appendix 2 (19) for $H^{2+\beta}$)

(g) $\psi|_{S_T} \in O^{2,1}(S_T)$; $\max_{x\in\Omega}|\psi_x(x,0)| < \infty$; $\psi \in H^{r,\frac{r}{2}}(\bar{Q}_T)$

(h) $S \in O^2$. (See Appendix 2 (22),(23) for $O^{2,1}(S_T)$, O^2 respectively)

Theorem 4.1. Suppose that the conditions (a) - (f) hold , then there exists a unique solution to problem (4.1) in the class $H^{2+\beta,1+\frac{\beta}{2}}(\bar{Q}_T)$. Moreover , this solution has derivatives u_{xt} in $L_2(Q_T)$.

Theorem 4.2. Suppose that the conditions (a) - (c), (g) and (h) hold. Then there exits at least one solution u(x,t) of the problem (4.1) belonging to $H^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ and having u_x bounded in $\bar{Q}_T$ and derivatives u_t, u_{xx} that belong to $H^{\beta,\frac{\beta}{2}}(Q_T)$. For the uniqueness of such a solution, it is sufficient that the function $b(x,t,u,p)$ satisfies a Lipschitz condition in u and p uniformly on any compactum of the form $\{(x,t) \in \bar{Q}_T, |u| \le c, |p| \le c\}$.

Theorem 4.3. Suppose that the conditions (a),(b),(g) and (h) hold. Then the problem (4.1) has a solution u(x,t) in $H^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T) \cap W_2^{2,1}(Q_T)$ with u_x in $H^{r,\frac{r}{2}}(Q_T)$ and with finite $\max_{Q_T}|u_x|$. If, furthermore , $b(x,t,u,p)$ satisfies a Lipschitz condition in u and p (uniformly on any compactum), then the solutiion is unique in the indicated class.

(See Appendix 2 (7) for $W_2^{2,1}(Q_T)$)

Actually it is possible to go further in weakening the conditions of the theorem and arriving at generalized solutions of equations of problem (4.1) having only derivatives of first order.

Theorem 4.4. Suppose the following conditions hold.

(i) For $(x,t) \in \bar{Q}_T$ and arbitrary u and p , inequality

$$a_i(x,t,u,p)p_i + b(x,t,u,p)u \le \nu p^2 - \phi_2(x,t)|u|^\alpha - \phi_1(x,t),$$

where $\nu > 0, \phi_1(x,t) \in L_1(Q_T), \alpha \in (0,2), \phi_2(x,t) \in L_{q_2,r_2}(Q_T)$ and the number q_2 and r_2 are subject to the conditions $\frac{1}{r_2} + \frac{n}{2q_2} = 1 + \frac{n}{4}(2-\alpha)$, $r_2 , q_2 \ge 1$ is valid and $|a_i(x,t,u,p)|(1+|p|) \le \mu(1+|p|)^2 + (1+|u|^2)\phi_3(x,t)$ and $|b(x,t,u,p)| \le \mu(1+|p|)^2 + (1+|u|^\alpha)\phi_3(x,t)$ with $\alpha < 2$ and $\phi_3 \in L_{q_3,r_3}$, $\frac{1}{r_3} + \frac{n}{2q_3} < 1$; $r_3, q_3 \ge 1$.

(j) For $(x,t) \in \bar{Q}_T$, arbitrary p , and $|u|$ exceeding some constant K $a_i(x,t,u,p)p_i \ge \nu|p|^2 - \mu|u|^\beta - u^2\phi_3(x,t)$ and $-b(x,t,u,p)u \le \nu|p|^2 + \mu|u|^\beta + u^2\phi_3(x,t)$ with $\beta < 2 + \frac{4}{n}$.

(k) For (x,t) and $(x',t') \in \bar{Q}_T$ and arbitrary u , v , p , and q ,

$$(p_i - q_i)[a_i(x,t,u,p) - a_i(x,t,u,q)] \ge \nu(|u|)|p-q|^2$$

and $|a_i(x,t,u,p) - a_i(x,t,u,q)| \le \mu(|u|)|p-q|$

$$|a_i(x,t,u,p) - a_i(x',t',u,p)| \le$$
$$\epsilon(|x-x'| + |t-t'| + |u-v|)[|p| + \mu(|u|+|v|) + \phi_4(x,t) + \phi_4(x',t')]$$

$$|b(x,t,u,p) - b(x',t',v,q)| \le \epsilon(|x-x'| + |t-t'|$$
$$+ |u-v| + |p-q|)[\mu(|u|+|v|+|p|+|q|) + \phi_1(x,t) + \phi(x',t')]$$

where $\nu(\tau)$ and $\mu(\tau)$ are continuous, positive functions of $\tau \ge 0$, $\epsilon(\tau)$ is a continuous function of $\tau \ge 0$ that is equal to zero for $\tau = 0$, $\phi_4 \in L_2(Q_T)$, and $\phi_1 \in L_1(Q_T)$. Then the problem (4.1) with $\psi|_{S_T} = 0$ has at least one solution in $H^{r,\frac{r}{2}}(\bar{Q}_T) \cap W_2^{1,0}(Q_T)$ with some $r > 0$ for any function $\psi \in H^{\delta,\frac{\delta}{2}}(\Gamma_T)$.

Actually the existence of generalized solutions u of the boundary value problems for equations of the problem (4.1) in the class of functions having only derivatives u_x can be proved in another way without using theorem (4.1) - (4.3) on their classical solvability . Namely , such solutions can be obtained as limits of approximate solutions , u^N, computed by Galerkin's method. Let us consider such a problem as follows ;

$$\begin{cases} u_t & = \frac{d}{dx_i} a_i(x,t,u,u_x) + b(x,t,u,u_x) \\ u|_{S_T} & = 0 \\ u|_{t=0} & = 0 \quad \text{for any} \quad \psi_0(x) \in L_2(\Omega) \end{cases} \tag{4.4}$$

We assume such conditions as follow ;

(l) For $(x,t,u,p) \in \{\bar{\Omega} \times [0,T] \times E_1 \times E_n\}$, the function a_i and b are measurable in (x,t,u,p) and continuous in (u,p) for almost all $(x,t) \in Q_T$; a_i satisfy the inequality

$$|a_i| \leq \phi_1(x,t) + c|u|^{\frac{q^*}{m'}} + c|p|^{m-1},$$

$\phi_1 \in L_{m'}(Q_T)$ where $q^* < q = \frac{m(n+2)}{m}, m' = \frac{m}{m-1}$ with $m > \frac{2n}{n+2}$ for $n \geq 2$ and $m > 1$ for n = 1 : and b satisfies the inequality

$$|b| \leq \phi_2(x,t) + c|u|^{\frac{q^*}{q}} + c|p|^{\frac{m^*}{q'}}, \phi_2 \in L_{q'}(Q_T)$$

where $q' = \frac{q}{q-1}$ and $m^* < m$.

(m) For any function $u(x) \in \overset{\circ}{W}{}^1_m(\Omega)$, (See Appendix 2 (6) for $\overset{\circ}{W}{}^1_m(\Omega)$)

$$\int_\Omega [a_i u_{x_i} + bu]\, dx \geq \nu \int_\Omega |u_x|^m dx - c(t) \int_\Omega (1+u^2)\, dx, \nu > 0, \int_0^T c(t)dt \leq c$$

(n) A monotonicity condition of the form

$$\int_\Omega [a_i(x,t,v,v_x) - a_i(x,t,v,u_x)](v_{x_i} - u_{x_i})\, dx \geq \int_\Omega \nu(|v|\,,|u_x|)\, |v_x - u_x|\, dx$$

holds, where $\nu(\tau_1,\tau_2)$ is a continuous positive function for $\tau_1 \geq 0$ and $\tau_2 \geq 0$, and u and v are arbitrary elements of $\overset{\circ}{W}{}^1_m(\Omega)$.

Theorem 4.5. The problem (4.4) , for any $\psi_0 \in L_2(\Omega)$, has at least one generalized solution u from $\overset{\circ}{V}{}^{1,0}_{m,2}(Q_T)$ where $\int_0^T \frac{\|u(x,t+h)-u(x,t)\|^2_{2,Q_{T-h}}}{h^2} < \infty$ if the conditions (l),(m) and (n) hold.

(See Appendix 2 (21) for $\overset{\circ}{V}{}^{1,0}_{m,2}(Q_T)$)

The proofs of Theorems (4.1)-(4.5) are found in Ladyzhenskaya [2].

4.2 Degenerate Parabolic Equations

Let us consider the case of degenerate parabolic equations . We deal with degenerate parabolic equations which are contained in the class of the form (1.1).

[**I**] Let us consider the first boundary value problem

$$\begin{cases} u_t & = (a_{ij}(x,t,u)u_{x_j})_{x_i} + b_i(x,t,u)u_{x_i} + c(x,t,u)u \\ u|_{t=0} & = u_0(x) \\ u|_{\partial\Omega\times[0,T]} & = \psi(s,t) \end{cases} \tag{4.5}$$

where u_0, ψ are nonnegative continuous functions satisfying the compatibility condition $u_0|_{\partial\Omega} = \psi|_{t=0}$. We assume the coefficients of the equation and the domain to satisfy ;

(o) a_{ij} , b_i , c, $\frac{\partial a_{ij}}{\partial x_i}$, and $\frac{\partial b_i}{\partial x_i}$ are in $C(\bar{Q} \times \mathbb{R})$.

(p) There exists a constant Λ and a function ν such that

$$\frac{1}{\Lambda}\nu(|r|)|\xi|^2 \leq a_{ij}(x,t,r)\xi_i\xi_i \leq \Lambda\nu(|r|)|\xi|^2, \text{ for all } \xi \in \mathbb{R}^n$$

and $(x,t,r) \in C(\bar{Q} \times \mathbb{R})$. Here the function $\nu(r)\,(0 \le r \le \infty)$ satisfies $\nu \in C[0,\infty), \nu(0) = 0, \nu(r) > 0\,(r > 0)$ and $\exists\,\delta > 0, m > 1$ such that $1 \le r\nu(r)/\int_0^r \nu(s)\,ds \le m$ for $0 < r \le \delta$.

(q) $\partial\Omega \in H^{1+\beta_0}, \quad \beta_0 > 0.$

Theorem 4.6. Under assumptions (o),(p),(q) for $u_0 \in H^{\beta}(\bar{\Omega})$ and $\psi \in H^{\beta,\frac{\beta}{2}}(\partial\Omega \times [0,T]), \beta > 0$, the first boundary value problem (4.5) has a generalized solution u . Moreover , u is Hölder continuous in $\bar{Q}$. Here a generalized solution for the first boundary value problem (4.5) is defined to be a nonnegative continuous function u in $\bar{Q}$, if it satisfies the initial and boundary values pointwise and satisfies the equation in the following sense : For all $\varphi \in H^{2,1}(Q) \cap H^1(\bar{Q})$, $\varphi|_{t=T} = 0 \quad (x \in \Omega)$, $\varphi|_{\partial\Omega\times[0,T]} = 0$

$$\begin{aligned}
&\int_Q [u\varphi_t + A_{ij}(x,t,u)\varphi_{x_i x_j} - (A_i(x,t,u) + B_i(x,t,u))\varphi_{x_i} \\
&\qquad + (c(x,t,u) + B(x,t,u))\varphi]\,dxdt + \int_\Omega u_0(x)\varphi(x,0)\,dx \\
&\quad - \int_{\partial\Omega\times[0,T]} A_{ij}(s,t,\psi(s,t))\frac{\partial\varphi(s,t)}{\partial N}\cos(N,x_i)\cos(N,x_j)\,dsdt = 0
\end{aligned}$$

where N is the unit outer normal at $\partial\Omega$ and

$$\begin{aligned}
A_{ij}(x,t,r) &= \int_0^r a_{ij}(x,t,s)\,ds, \qquad & A_j(x,t,r) &= \int_0^r \tfrac{\partial a_{ij}}{\partial x_i}(x,t,s)\,ds \\
B_i(x,t,r) &= \int_0^r b_i(x,t,s)\,ds, \qquad & B(x,t,r) &= \int_0^r \tfrac{\partial b_i}{\partial x_i}(x,t,s)\,ds
\end{aligned}$$

(See Dong [12] for the proof of the Theorem 4.6).

[II] We consider a problem in a class of degenerate parabolic equations on a bounded domain with mixed boundary conditions such that

$$\begin{cases}
u_t = \mathrm{div}(\nabla\phi(x,t,u) + \vec{f}(x,t,u)) + h(x,t,u) \quad \text{on} \quad Q_T \\
u = u_0 \quad \text{on} \quad (\partial\Omega \setminus \Sigma)_T \cup \tilde{\Omega} \times \{0\} \\
(\nabla\phi(x,t,u) + \vec{f}(x,t,u)) \cdot n = g(x,t,u) \quad \text{on} \quad \Sigma_T
\end{cases} \tag{4.6}$$

where Ω is a bounded domain in $\mathbb{R}^N$ which satisfies the uniform C^1-regularity property . Σ is a relatively open subset of $\partial\Omega$, and $\bar{\Sigma}$ and $\partial\Omega \setminus \Sigma$ are C^2 surfaces with boundary which meet in a C^2 manifold of dimension N-2 . n denotes the outerward unit normal to Σ at x . Then we assume that the coefficient functions and the domain satisfy the following conditions ;

(i) $\phi, \Delta_x\phi \in C(\bar{Q}_T \times \mathbb{R}), \nabla_x\phi \in \prod_{i=1}^N C(\bar{Q}_T \times \mathbb{R})$ and $\phi_u \in C(\bar{Q}_T \times \mathbb{R} \setminus \{0\})$, such that $\phi(x,t,0) = 0$ and $\phi_u(x,t,u) > 0$ for all $u \neq 0$.

(ii) $\vec{f} \in \prod_{i=1}^N C(\bar{Q}_T \times \mathbb{R}), \nabla_x \cdot \vec{f} \in C(\bar{Q}_T \times \mathbb{R})$, and
$\vec{f}_u \in \prod_{i=1}^N C(\bar{Q}_T \times \mathbb{R} \setminus \{0\})$, such that $\vec{f}(x,t,0) = 0$.

(iii) $h \in C(\bar{Q}_T \times \mathbb{R})$ and $h_u \in C(\bar{Q}_T \times \mathbb{R} \setminus \{0\})$, with h(x,t,0) = 0 .

(iv) $g \in C(\bar{\Sigma}_T \times \mathbb{R})$ and $g_u \in C(\bar{\Sigma}_T \times \mathbb{R} \setminus \{0\})$, with $g(x,t,0) = 0$.

(v) $u_0 \in L^\infty((\partial\Omega \setminus \Sigma)_T \cup \tilde{\Omega} \times \{0\})$, with $u_0 \geq 0$.

(vi) $\phi_u, f_u, h_u \in L^\infty(Q_T \times [-M, M])$ for every $M > 0$.

(vii) Given $M > 0$, there exists a constant C = C(M) such that $|g_u(x,t,u)| \leq C\phi_u(x,t,u)$ for all $(x,t,u) \in \Sigma_T \times (0, M]$.

(viii) Either $\partial\Omega = \Sigma$ or N = 1 .

Then since it is possible that $\phi_u(x,t,u) = 0$, the differential equation of the problem (4.6) is degenerate parabolic .

Theorem 4.7. (LOCAL EXISTENCE AND CONTINUATION)
If (i) - (v) are satisfied , for some $T_1 = T_1(u_0) \in (0, T)$, the problem (4.6) has a solution , $u(x,t) = u(x,t;u_0)$, on Q_T . Furthermore , if $T = \infty$ and T_1 is redefined to be the maximum possible value of t such that u is a solution of the problem (4.6) on Q_s for each $s \in [0,t)$, then $\lim_{t \to T_1} \sup\{t + \|u(\cdot,t)\|_\infty\} = \infty$.

Theorem 4.8. (COMPARISION AND CONTINUOUS DEPENDENCE)
Suppose (i) - (viii) hold , and let $u(x,t;u_0)$ and $v(x,t;u_0)$ be nonnegative solutions of the problem (4.6) on Q_T . Here $T < \min\{T_1(u_0), T_1(v_0)\}$. If $u_0 \leq v_0$, then $u \leq v$ on Q_T . Furthermore there exist a constant C such that

$$\int_\Omega |u(x,t) - v(x,t)|\, dx \leq C \int_\Omega |u_0(x,0) - v_0(x,0)|\, dx$$
$$+ C \int_0^t \int_{\partial\Omega\setminus\Sigma} |u_0 - v_0|\, ds$$

for all $t \in [0,T]$.

See Anderson [15] for the proofs of the Theorems (4.7) and (4.8).

[III] Let us consider the following boundary value problem

$$\begin{cases} u_t & = \operatorname{div} a(x,t,u,Du) + b(x,t,u,Du) \quad \text{in } Q_T \\ u(\cdot,t)|_{\partial\Omega} & = g(\cdot,t) \quad \text{for almost every} \quad t \in (0,T) \\ u(\cdot,0) & = u_0 \end{cases} \tag{4.7}$$

Here the functions $a : Q_T \times R^{N+1} \to R^N$ and $b : Q_T \times R^{N+1} \to R$ are measurable and satisfy

$$(a) \quad a(x,t,u,Du)\cdot Du \geq c_0|Du|^p - c_0'|u|^\delta - \varphi_0(x,t)$$
$$(b) \quad |a(x,t,u,Du)| \leq c_1|Du|^{p-1} + c_1'|u|^{\delta\frac{\delta-1}{\delta}} + \varphi_1(x,t)$$
$$(c) \quad |b(x,t,u,Du)| \leq c_2|Du|^{p\frac{\delta-1}{\delta}} + c_2'|u|^{\delta-1} + \varphi_2(x,t)$$

for $p > 1$ and a.e $(x,t) \in \Omega \times [0,T]$. c_i, c_i' $i = 0,1,2$ are positive constants and δ is in $p \leq \delta \leq p\frac{N+2}{N}$. The nonnegative functions φ_i , i = 0,1,2 are defined in Q_T and satisfy $\varphi_0, \varphi_1^{\frac{p}{p-1}}, \varphi_2^{\frac{\delta}{\delta-1}} \in L_{\tilde{q}}(Q_T)$ where $\frac{1}{\tilde{q}} = (1-k_0)\frac{p}{N+p}$ $\quad k_0 \in (0,1]$.

Theorem 4.9. Assume that u is a nonnegative weak subsolution of the problem (4.7) and $g \in L_\infty(S_T)$, then u is bounded in $\Omega\times(\epsilon, T], \forall\epsilon \in (0, T)$.

Theorem 4.10. Let u be a nonnegative weak subsolution of the problem such that

$$\begin{cases} u_t & = \operatorname{div} a(x,t,u,Du) \quad \text{in } Q_T, p > 1 \\ u(\cdot,t)|_{\partial\Omega} & = g(\cdot,t) \quad \text{for almost every } t \in (0,T) \\ u(\cdot,0) & = u_0 \end{cases}$$

Here the function $a : Q_T \times R^{N+1} \to R^N$ is measurable and satisfy

$$\begin{cases} a(x,t,u,Du)\cdot Du \geq c_0|Du|^p \\ |a(x,t,u,Du)| \quad\quad \leq c_1|Du|^{p-1} \end{cases}$$

for two given constants , $0 < c_0 \leq c_1$. Then

$$\sup_{Q_T} u \leq \max\{\operatorname{ess\,sup}_{S_T} g; \operatorname{ess\,sup}_{\Omega} u_0\}.$$

This is called the weak maximum priciple.

Let us consider the inequality above. Then

$$u(x,t) \leq \max\{\operatorname{ess\,sup}_{S_T} g; \operatorname{ess\,sup}_{\Omega} u_0\} \quad \text{for all} \quad (x,t) \in Q_T.$$

If $g \equiv 0$ on S_T and intially $u_0 > 0$ is given , then

$$u(x,t) \leq \operatorname{ess\,sup}_{\Omega} u_0 = \operatorname{ess\,sup}_{\Omega} u(x,0) \quad \text{for all} \quad (x,t) \in Q_T.$$

Thus for each t , $0 \leq t \leq T$, there exists a constant R_t such that $u(x,t) = 0$, $|x| \leq R_t$ provided that u(x,0) has compact support . This means that the speed of propagation of the solution is finite.

See [41] for the proofs of the Theorems (4.9) and (4.10)

[IV] Theorem 4.11. (GLOBAL MAXIMUM PRINCIPLE)
Let u be a weak solution of (1.2) in Q_T such that $u \leq M$ on Γ . Then $u(x,t) \leq M + ck$ for almost every $(x,t) \in Q_T$. Here M , c depends only on T , $|\Omega|$ and the structure of (1.2) while $k = (\|b\| + \|d\|)\,|M| + (\|f\| + \|g\|)$.

Theorem 4.12. (LOCAL BOUNDEDNESS)
Let u be a weak solution of (1.2) in Q_T . Suppose that the set $Q_T(3\rho)$ is contained in Q_T . Then for almost every (x,t) in $Q_T(\rho)$,

$$|u(x,t)| \leq c(\rho^{-\frac{n+2}{2}} \|u\|_{2,2,3\rho} + \rho^{\theta} k),$$

where c is a constant depending only on ρ and the structure of (1.2) , and $k = \|f\| + \|g\| + \|h\|$. In particular , weak solutions of (1.2) must be locally essentially bounded . Here we denote by $R(\rho)$ the open cube in E^n of edge length ρ centered at $\bar{x}$, and define $Q_T(\rho) = R(\rho) \times (\bar{t} - \rho^2, \bar{t})$ and let $(\bar{x}, \bar{t})$ be a fixed point in Q_T.

Theorem 4.13. (HARNACK INEQUALITY)
Let u be a nonnegative weak solution of (1.2) in Q_T . Suppose that the set $Q_T(3\rho)$ is contained in Q_T . Then $\max_{Q_T^*}(\rho) u \leq C \min_{Q_T(\rho)}(u + \rho^{\theta} k)$ where C is a constant depending only on ρ and the structure of (1.2) and $k = \|f\| + \|g\| + \|h\|$, $Q_T^*(\rho) \times (\bar{t} - 8\rho^2, \bar{t} - 7\rho^2)$

See Aroson and Serrin [14] for the proofs of (4.11),(4.12), and (4.13)

CHAPTER 5

AN APPLICATION FOR DEGENERATE PARABOLIC EQUATIONS WITH PRINCIPAL PART IN DIVERGENCE FORM

This chapter provides an application of theorems for degenerate parabolic equations . We consider unsaturated flows of liquids (incompressible fluids) in a porous medium . Then in a given domain a certain amount of liquid is concentrated at a relatively high pressure . As time progresses, the liquid will flow toward areas of lower pressure . The flow will continue as long as a sufficiently high saturation is maintained. When the saturation falls below a certain residual value , the flow will cease . Thus the problem is to determine the amount of liquid in any given point of the domain at any given time . Underground, the flow is usually quite slow, and temperature considerations play no role , so it is governed by two laws, the first being Darcy's law :

$$q = -A(\operatorname{grad} p + f) \tag{5.1}$$

which relates the mass flux , q , to the gradient of the pressure , p , and to the external body forces , f , and the second being the continuity equation:

$$\frac{\partial}{\partial t}(\rho\varphi S) + \operatorname{div}(\rho q) = g, \tag{5.2}$$

where ρ denotes the density of the liquid and φ denotes the porosity of the medium which is a measure of the pore volume avariable to the fluid and S denotes the saturation which gives the fraction of the pore space actually occupied by the fluid. $A = (a_{ij})$ denotes a positive (or nonnegative) definite , symmetric matrix which represents the resistence of medium to the flow of the particular fluid in question and g denotes a function which arises , in the event that sources or sinks are present, e.g. absorption or pumping of the liquid out of or into the domain. The known functions $\rho, \varphi, S, g, f, a_{ij}$ are , in general , functions of pressure , temperature , position , time , etc . and their dependence on these quantities is complicated and difficult to measure. Consequently , it is necessary to make assumptions of both a physical and a mathematical nature to make the system (5.1) and (5.2) amenable to a mathematical treatment. Thus we make the following physical assumptions :

PA 1) Temperature dependence will be neglected.

PA 2) $A = (a_{ij})$ is a positive (or nonnegative) definite , symmetric matrix which depends only on x .

PA 3) ρ is a positive constant so the fluid is incompressible.

PA 4) f and g are functions of x , t , and p .

PA 5) The porosity φ depends upon x and p and it satisfies the inequality $0 < \varphi \leq 1$. As a function of p , φ is nondecreasing and for p sufficiently small , φ is independent of p .

PA 6) S is a function of x and p and satisfies the inequality $0 \leq S \leq 1$. As a function of p , S(x,t) is nondecreasing and for p sufficiently small , S is independent of p .

By combining equation (5.1) with (5.2) , we obtain

$$\rho \frac{\partial}{\partial t}(\varphi S) - \frac{\partial}{\partial x_i}(a_i(x,t,p,Dp)) = g \tag{5.3}$$

where $a_i(x,t,p,Dp) = a_{ij}\frac{\partial p}{\partial x_j} + (a_{ij}f_j)$. In view of the assumptions PA 1) - PA 6) , this equation is a nonlinear , partial differential equation in p alone with principal part in divergence form which is of parabolic type but which for certain values of (x,t) , p , Dp , degenerates. Let the role of p in (5.3) to be played by u and the role of $\rho\varphi S$ to be played by b(x,u) . Then the equation (5.3) becomes

$$\frac{\partial}{\partial t}b(x,u) - \frac{\partial}{\partial x_i}a_i(x,t,u,Du) = g(x,t,u) \qquad (5.4)$$

Let us consider a problem where the equation (5.4) is combined with the initial-boundary condition

$$u(x,0) = u_0(x), x \in \Omega \quad \text{ and } \quad u(x,t) = 0, \quad (x,t) \in S_T. \qquad (5.5)$$

We now make the functions in (5.4) the following mathematical assumptions on which will allow us to treat the initial-boundary value problem (5.4) , (5.5) :

MA 1) b(x,u) is defined and continuous for all $x \in \bar{\Omega}, -\infty < u < \infty$ and $0 \leq b(x,u) \leq 1$. As a function of u , b is nondecreasing and satisfies a uniform Lipchitz condition . Finally, for any $x \in \bar{\Omega}$, b(x,u) = 0 if $u \leq 0$ and b(x,u) is strictly increasing in u for $u > 0$.

MA 2) The a_{ij} , with $a_{ij} = a_{ji}$ are defined as only a function of x and continuous on $\bar{\Omega}$. Furthermore, there exist constants a_0, a_1, $a_1 \geq a_0 > 0$, such that for all $x \in \bar{\Omega}$, $\xi \in \mathbb{R}^n$,

$$a_0 \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq a_1 \sum_{i=1}^{n} \xi_i^2.$$

MA 3) C(x,t) and f(x,t) are defined and continuous on $\bar{Q}_T$. Further, $C(x,t) \geq 0$ and as a function of t , C(x,t) is continuously differentiable in $\bar{Q}_T$ and $\frac{\partial C}{\partial t} \geq 0$. Finally, f(x,t)=0 for $(x,t) \in S_T$.

MA 4) $u_0(x)$ is defined and continuously differentiable for $x \in \bar{\Omega}$. Further, $u_0(x) > 0$ for $x \in \Omega$ and $u_0(x) = 0$ for $x \in \partial\Omega$.

Now we write the equation (5.4) as

$$\sum_{i,j=1}^{\infty} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) - \frac{\partial}{\partial t} b(x,u) = C(x,t)u + f(x,t), \quad (x,t) \in Q_T, \quad (5.6)$$

where div (A f) + g is given by C(x,t) u + f(x,t) .

Then we define a generalized solution for such a problem that the equation (5.6) is combined with an initial boundary condition (5.5) as following ;

A function $u \in \overset{\circ}{W}{}_2^{1,0}(Q_T)$ is said to be a generalized solution of the problem (5.6) from $W_2^{1,0}(Q_T)$ if for all functions $\varphi \in \overset{\circ}{W}{}_1^1(Q_T)$ with $\varphi(x,T) = 0$, it satisfies the integral identity

$$\int_{Q_T} [\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial t} b(x,u)]\, dxdt = \int_{\Omega} \varphi(x,0) b(x, u_0(x))\, dx$$
$$- \int_{Q_T} [C(x,t)u\varphi + f(x,t)\varphi]\, dxdt \quad (5.7)$$

Then the problem (5.6) with an initial boundary condition (5.5) under the assumptions MA 1) - MA 4) has at most one solution in sense of the definition (5.7) for a generalized solution . A special case of this result appears in the paper [42]. The existence theory in the Guenther paper [42] actually is carried by means of finite differences so that a numerical method for solving the relevant equations is obtained as well .

Let us seek a generalized solution for the equation (5.4) with conditions $u(x,0) = u_0(x)$ and $u(x,t) = \psi(x,t)$, $(x,t) \in S_T$ where u_0, ψ are nonnegative continuous functions satisfying the compatibility condition $u_0|_{\partial\Omega} = \psi|_{t=0}$. In order to apply Theorem (4.6) for the existence of a generalized solution, we need some mathematical assumptions. By the assumptions PA 1) - PA 6) , we know that $0 \leq b(x,t) \leq 1$ and b(x,t) is a nondecreasing function of u. Thus, under conditions of slow flow , we can assume that b(x,u) is almost linear as a function of u. Thus , let us replace b(x,t) by

$$cu + d_k = \begin{cases} cu + d_1, & \text{if}\, u > 0\,, \\ cu + d_2, & \text{if}\, u \leq 0\,, \end{cases} \tag{5.8}$$

where c and b are constants satisfying conditions so that $c > 0\,,\, 0 \leq d_k \leq 1\,,\; k = 1{,}2$ and $d_1 \geq d_2$ and $\sup_{\{-\infty<u<\infty, x\in\Omega\}} |b(x,u) - (cu + d_k)| < \epsilon$, almost every (x,u) for arbitrarily small ϵ. Hence by inserting $cu + d_k$ into (5.4) instead of b(x,u) , we obtain

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(\frac{1}{c} a_{ij} \frac{\partial u}{\partial x_j}\right) = \frac{1}{c} g(x,t,u)$$

If div $(a_{ij} f_j) + g(x,t,u)$ can be expressed as $b_i(x,t,u)u_{x_i} + C(t,u)u$ and the assumptions (0),(p),(q) are satisfied in the problem (4.5) with $u_0 \in H^{\beta}(\bar{\Omega})$ and $\psi \in H^{\beta,\frac{\beta}{2}}(\partial\Omega \times [0,T])\,,\, \beta > 0\,,$ then Theorem (4.6) can be applied. Thus the equation (5.4) with $u(x,0) = u_0(x)$ and $u(x,t) = \psi(x,t)\,,\,(x,t) \in S_T$ has such a generalized solution u as defined in the Theorem (4.6) and the u is Hölder continuous in $\bar{Q}_T$.

Next let us consider a problem with the equation (5.4) under such boundary conditions that the value on a certain boundary is zero and the other boundary has no condition. This problem occurs in a porous medium consisting of several components and allows certain degeneracies in $A = (a_{ij})$. That is , (a_{ij}) is a nonnegative definite , symmetric matrix which depends only on x. If the conditions of Notation I - IV and assumptions in Chapter 3 are satisfied , then Theorem 1.4 can be applied. Thus, there is a weak solution which is Lipschitz continuous with respect to time with values in $L_2(\Omega; d, \alpha)$.

My Theorem 1.4 is a generalization of a Theorem in [42] though the methods are completely different and it is really a better theorem since it allows certain degeneracies in $A = (a_{ij})$ and is more useful to investigate certain degenerate systems , in particular, those that arise in modeling fractured media consisting of several components , among which some components have a tight boundary and the others do not.

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APPENDICES

APPENDIX 1

Basic Notation

Let us introduce a number of the symbols and notation used in this dissertation .

$\mathbb{R}^n$ is the n - dimensional Euclidean space .

$x = (x_1, \cdots, x_n)$ is a point in $\mathbb{R}^n$.

Ω is a domain in $\mathbb{R}^n$.

$S = \partial\Omega$ is the boundary of the domain $\Omega : \bar{\Omega} = \Omega \cup S$.

$Q_T = \{(x,t) : x \in \Omega, t \in (0,T)\}$ is a space-time cylinder in $\mathbb{R}^{n+1}$.

$S_T = \{(x,t) : x \in \partial\Omega, t \in [0,T]\}$ is the lateral surface of Q_T .

$\Gamma_T = S_T \cup \{(x,t) : x \in \Omega, t = 0\}$.

n is the outward unit normal to $\partial\Omega$.

Symbols $u_{xi} = \frac{\partial u}{\partial x_i}$ or $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ denote classical and generalized derivatives

$|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$, $x^2 = |x|^2$, $p = (p_1, p_2, \cdots, p_n)$, $u_x = (u_{x_1}, \cdots, u_{x_n})$,

$|p| = (\sum_{i=1}^n p_i^2)^{\frac{1}{2}}$, $\quad p^2 = |p|^2$, $\quad |u_x| = (\sum_{i=1}^n u_{x_i x_j}^2)^{\frac{1}{2}}$, $\quad u_x^2 = |u_x|^2$,

$u_{x_i}^2 = (u_{x_i})^2$, $|u_{xx}| = (\sum_{i,j=1}^n u_{x_i x_j}^2)^{\frac{1}{2}}$,

$a(x,t,u,p) = a(x_1, \cdots, x_n, t, u, p_1, \cdots, p_n)$

$\frac{d}{dx_i}[a(x,t,u(x,t),u_x(x,t))] = \frac{\partial a}{\partial x_i} + \frac{\partial a}{\partial u} u_{x_i} + \frac{\partial a}{\partial u_{x_k}} u_{x_k x_i}$

The summation convention is assumed to hold , that is when the same index occurs twice in a term, one sums

$$\frac{\partial a}{\partial u_{x_k}} u_{x_k x_i} = \sum_{k=1}^n \frac{\partial a}{\partial u_{x_k}} u_{x_k x_i}$$

$\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, that is an n-tuple of nonnegative integers α_i .

$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ denotes a differential operator of order $|\alpha| = \sum_{j=1}^n \alpha_j$ if $D_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq n$.

$Au = \nabla \cdot (a \nabla u), \quad A_n u = \nabla \cdot (a_n \nabla u)$

$\|u\|_p = |u\|_{L_p(\Omega)}, \quad \|u\|_{k,p} = \|u\|_{W_p^k(\Omega)}$, $\|u\|^2_{H_a^1(\Omega)} =< a\nabla u, \nabla u >$

$< u, v >= \int_\Omega u(x)v(x)\, dx$,

$< a\nabla u, \nabla v >= \int_\Omega \sum_{i,j=1}^N a_{i,j}(x) \partial_i u(x) \partial_i v(x)\, dx.$

APPENDIX 2

Definitions of the Basic Function Spaces

(1) $L_p(\Omega)$, $p \geq 1$ is the Banach space consisting of all measurable functions on Ω having the finite norm :

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}} \; 1 \leq p < \infty \quad \text{and} \quad \|u\|_{\infty,\Omega} = \operatorname*{ess\,sup}_{\Omega} |u|$$

(2) $L_{q,r}(Q_T)$ is the Banach space consisting of all measurable functions on Q_T with a finte norm :

$$\|u\|_{q,r,Q_T} = \left(\int_0^T \left(\int_\Omega |u(x,t)|^q \, dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}} \quad \text{where} \quad q \geq 1 \quad \text{and} \quad r \geq 1$$

$L_{q,q}(Q_T)$ will be denoted by $L_q(Q_T)$ and the norm $\|\cdot\|_{q,q,Q_T}$ by $\|\cdot\|_{q,Q_T}$. $L_{q,r} \equiv L^r((0,T); L^q(\Omega))$.

(3) $C^l(\Omega)$: For any nonnegative integer l , $C^l(\Omega)$ is the vector space consisting of all functions ϕ which , together with all their partial derivatives $D^\alpha \phi$ of order $|\alpha| \leq l$, are continuous on Ω . The subspaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consist of all those functions in $C(\Omega)$ and C^∞ , respectively , which have compact support in Ω .

(4) $C^{2,1}(\Omega)$ is the set of all continuous functions in Ω having continuous derivatives u_x, u_{xx}, u_t in Ω .

(5) $W_p^k(\Omega)$ for k nonnegative integer is the Banach space consisting of all elements of $L_p(\Omega)$ having generalized derivatives of order up to k, inclusively , that are pth-power summable on Ω . The norm in $W_p^k(\Omega)$ is defined by the equality

$$\|u\|_{k,p,\Omega} = \sum_{j=0}^{k} \sum_{(j)} \|D_x^j u\|_{p,\Omega} = \int_{|\alpha| \le k} \|D_x^\alpha u\|_{p,\Omega}$$

The symbol D_x^j denotes any derivative of u(x) with respect to x of order $|j|$ while $\sum_{(j)}$ denotes summation over all possible derivatives of u of orderj.

(6) $\underline{\overset{\circ}{W}{}_p^k(\Omega)}$ is the subspace of $W_p^k(\Omega)$ in which the set of all functions that are infinitely differentiable and finite in Ω is dense .

$W_p^1(\Omega)$ space is the comletion of $C^\infty(\Omega)$ under the norm ;

$$\|u\|_{1,p,\Omega} \equiv \|u\|_{p,\Omega} + \|Du\|_{p,\Omega} \quad \text{for} \quad u \in C^\infty(\Omega) \cap L_P(\Omega)$$

$\overset{\circ}{W}{}_p^1(\Omega)$ space is the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{1,p,\Omega}^{(0)} \equiv \|Du\|_{p,\Omega}\, , u \in C_0^\infty(\Omega)$$

.

(7) $\underline{W_q^{2l,l}(Q_T)}$ for l integer $(q \ge 1)$ is the banach space consisting of the elements of $L_q(Q_T)$ having generalized derivatives of the form $D_t^r D_x^s$ with any r and s satisfying the inequality $2r + s \le 2l$. The norm is defined by the equality

$$\|u\|_{q,Q_T}^{(2l)} = \sum_{j=0}^{2l} \sum_{(2r+s=j)} \|D_t^r D_x^s u\|_{q,Q_T}$$

(8) $\underline{W_2^{1,0}(Q_T)} \equiv L_2((0,T); W_2^1(\Omega))$ is the Hilbert space consisting of the elements u(x,t) of the space $L_2(Q_T)$ having generalized derivatives $\partial u/\partial x_i$, $i = 1, \cdots, n$ square summable on Q_T . The scalar product and the norm are defined by the equalities

$$(u,v)_{1,2,Q_T} = \int_{Q_T} (uv + u_x v_x)\, dxdt, \quad \|u\|_{1,2,Q_T} = \sqrt{(u,u)_{1,2,Q_T}}$$

(9) $\underline{\overset{\circ}{W}{}_2^{1,0}(Q_T)}$ $\equiv L_2((0,T);\overset{\circ}{W}{}_2^1(\Omega))$ is a subspace of $W_2^{1,0}(Q_T)$ in which the set of smooth functions equal to zero near S_T is dense .

(10) $\underline{W_2^1(Q_T)}$ is defined on every cross-section Ω_t of the cylinder Q_T by the plane $t = t_1 \in [0,T]$ as functions from $L_2(\Omega_{t_1})$ and they change continuously with t in the norm $L_2(\Omega)$ with a change $t \in [0,T]$.

(11) $\underline{W_p^k(\Omega;\sigma)}$ is defined as the set of all functions u(x) which are defined almost everywhere on Ω and whose generalized derivatives $D^\alpha u$ for order $|\alpha| \leq k$ satisfy

$$\int_\Omega |D^\alpha u(x)|^p \sigma_\alpha(x)\, dx < \infty.$$

It is a normed linear space when it is equipped with the norm

$$|u|_{k,p,\sigma} = (\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u(x)|^p \sigma_\alpha(x)\, dx)^{\frac{1}{p}}$$

Let ϵ be a real number and let us denote $\sigma(x) = [d_M(x)]^\epsilon$ where $M \subset \partial\Omega$ and $d_M = d(x) = \operatorname{dist}(x,M)$. Then the weight $\sigma(x)$ is called a power type weight .

(12) $\underline{W_p^k(\Omega;d,\epsilon)\ \textbf{or}\ W_p^k(\Omega;d_M,\epsilon)}$ is the power type weighted space corresponding to the Sobolev weighted space $W_p^k(\Omega;\sigma)$.

$$W_p^k(\Omega;d_M,\epsilon) = \{u = u(x) \mid \int_\Omega |D^\alpha u(x)|^p d_M^\alpha(x)\, dx < \infty\}$$

for all $\alpha, |\alpha| \leq k$. $L_p(\Omega;d_M,\epsilon)$ is the set of all functions u = u(x) satisfying

$$\|u\|_{p;d_M,\epsilon} = (\int_\Omega |u(x)|^p\, d_M^\epsilon(x)\, dx)^{\frac{1}{p}} < \infty$$

The norm of the $W_p^k(\Omega; d_M, \epsilon)$ is given by the formula

$$\|u\|_{k,p;d_M,\epsilon} = \Big(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{p;d_M,\epsilon} \Big)^{\frac{1}{p}}$$

(13) $\underline{\overset{\circ}{W}{}_p^k(\Omega; d, \epsilon)}$ is the closure of the space $C_0^\infty(\Omega)$ in the norm $\|u\|_{k,p;d,\epsilon}$

(14) $\underline{H_a^1(\Omega)}$ space is the closure of the smooth functions $C_0^\infty(\Omega)$ with support in Ω with respect to the norm

$$\left(\|u\|^2_{H_a^1(\Omega)} + \|u\|^2_{L_2}\right)^{\frac{1}{2}} = (< a\nabla u, \nabla u > + < u, u >)^{\frac{1}{2}}$$
$$= \left\{ \int_\Omega a\,|\nabla u|^2\, dx + \int_\Omega u^2\, dx \right\}^{\frac{1}{2}}$$

(15) $\underline{W^k_{p,\alpha}}$ is the Banach space defined to be the space of functions $u \in L_p(\Omega)$ such that

$$\|u\|^p_{k,p,\alpha} = \int_\Omega |u|^p\, dx + \int_\Omega \rho^\alpha(x) \sum_{|s|=k} |D^s u|^p\, dx < \infty,$$

where $\rho(x) = \text{dist}\,(x, \partial\Omega)$

(16) $\underline{W^1_{2,x^\alpha}}$ is the corresponding to $W_2^1(\Omega; d, \alpha)$ when $d(x, \Gamma) = x$.

$$\overset{\circ}{W}{}^1_{2,x^\alpha}(\Omega) \equiv \overset{\circ}{W}{}^1_2(\Omega; x, \alpha) \equiv \overset{\circ}{W}{}^1_{2,\alpha}(\Omega) \equiv H^1_{x^\alpha}(\Omega)$$

(17) $\underline{H^l(\bar{\Omega})}$ is the Banach space whose elements are continuous functions u(x) in Ω having in $\bar{\Omega}$ continuous derivatives up to order $[l]$ inclusively and a finite valve for the quantity $|u|_\Omega^{(l)} = < u >_\Omega^{(l)} = \sum_{j=0}^{[l]} < u >_\Omega^{(j)}$ where $< u >_\Omega^{(0)} = |u|_\Omega^{(0)} = \max_\Omega |u|$, $< u >_\Omega^{(j)} = \sum_{(j)} \left|D_x^j u\right|_\Omega^{(0)}$, $< u >_\Omega^{(l)} = \sum_{([l])} < D_x^{[l]} u >_\Omega^{(l-[l])}$ where l is always a nonintegral positive number.

(18) $\underline{H^{l,\frac{l}{2}}(\bar{Q}_T)}$ is the banach space of functions u(x,t) that are continuous in $\bar{Q}_T$, together with all derivatives of the form $D_t^r D_x^s$, for $2r+s<l$,and have a finite norm . (See [2] for the norm) . Roughly speaking , if $u \in H^{2+\beta,1+\frac{1}{\beta}}$, then $u_{ij}(1 \leq i,j \leq n)$ and u_t are Hölder continuous in t with exponent $\frac{\beta}{2}$.

(19) $\underline{H^{l+\alpha}(\bar{\Omega})}$ $(\alpha \in (0,1), l=0,1,\cdots)$ is the Banach space consisting of the elements $C^l(\bar{\Omega})$ for which the derivatives of order l satisfy the Hölder condition in $\bar{\Omega}$ with the power α .

(20) $\underline{\overset{\circ}{V}_{m,2}(Q_T)}$, $m \geq 1$ consists of all measurable functions u(x,t) that are equal to zero on S_T and have the finite norm defined as

$$\|u\|_{\overset{\circ}{V}_{m,2}(Q_T)} \equiv \operatorname{ess}\sup_{0 \leq t \leq T} \|u\|_{2,\Omega} + \|u_x\|_{m,Q_T}$$

(21) $\underline{\overset{\circ}{V}{}^{1,0}_{m,2}(Q_T)}$ is a completion in the norm defined in (20) of all smooth functios that are equal to zero on S_T .

(22) $\underline{O^l(\bar{\Omega})}$ $(l=1,2)$ is the set of all continuous functions in $\bar{\Omega}$ having continuous derivatives in $\bar{\Omega}$ up to order $l-1$, with the derivatives of order $l-1$ having a first differential at each point of $\bar{\Omega}$ and the derivatives of order l being bounded in $\bar{\Omega}$.

(23) $\underline{O^{2,1}(S_T)}$ is the set of all continuous functions in S_T having at each point of S_T derivatives u_x and u_t with the u_x being continuous in x and having a first differential with respect to x at each point of S_T and the functions u_x, u_t, u_{xx} being bounded in S_T .

APPENDIX 3

Definition 1) If $G \subset \mathbb{R}^n$, we denote by $\bar{G}$ the closure of G in $\mathbb{R}^n$. We shall write $G \subset\subset \Omega$ and $\bar{G}$ is a compact subset of $\mathbb{R}^n$. If u is a function defined on G , we define the support of u as supp u $= \overline{\{x \in G : u(x) \neq 0\}}$. We say that u has a compact support in Ω if supp u $\subset\subset \Omega$.

Definition 2) A sequence $\{\phi\}$ of functions belongings to $C_0^\infty(\Omega)$ is said to converge in the sense of the space $D(\Omega)$ to the function $\phi \in C_0^\infty(\Omega)$ provided the following conditions are satisfied :

(1) There exists $K \subset\subset \Omega$ such that supp $(\phi_n - \phi) \subset K$ for every n

(2) $\lim_{n\to\infty} D^\alpha \phi_n(x) = D^\alpha \phi(x)$ uniformly on K for each multi-index α.

There exists a locally convex topology on the vector space C_0^∞ with respect to which a linear functional T is continuous if and if $T(\phi_n) \to T(\phi)$ in C whenever $\phi_n \to \phi$ in the sense of the space $D(\Omega)$. This topological vector space is called $D(\Omega)$ and its elements are testing functions .

Difinition 3) The dual space $D'(\Omega)$ of $D(\Omega)$ is calles the space of distributions :

If S, T , $T_n \in D'(\Omega)$ and $c \in C$,

(1) (S + T) (ϕ) = S(ϕ) + T(ϕ) for $\phi \in D(\Omega)$.

(2) (c T) (ϕ) = c T (ϕ) for $\phi \in D(\Omega)$.

$T_n \to T$ in $D'(\Omega)$ if and only if $T_n(\phi) \to T(\phi)$ in C for $\phi \in D(\Omega)$.

Definition 4) The derivative $D^\alpha T$ of a distribution $T \in D'(\Omega)$ is defined as $(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$ for all $\phi \in D(\Omega)$.

Definition 5) A function u defined almost everywhere on Ω is said to be locally integrable on Ω provided $u \in L(A)$ for every measurable $A \subset\subset \Omega$. We write $u \in L^1_{\text{loc}}(\Omega)$. Corresponding to every $u \in L^1_{\text{loc}}(\Omega)$ there is a distribution $T_u \in D'(\Omega)$ defined by $T_u(\phi) = \int_\Omega u(x)\phi(x)\,dx, \quad \phi \in D(\Omega)$. This distribution T_u is said to be generated by the function u .

Definition 6) A distribution which is generated by a locally integrable function is called regular. All other distribution are called singular.