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A Study of the Double Porosity Model of Fluid Flow in Fractured Rock and
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The present work is a study of three degenerate, linear parabolic
systems of equations, each of which represents a version of the so-called
double porosity model for underground fluid flows in natural fractured
rock.   These systems of equations together with initial and boundary
conditions describe single-phase flows in fluids, slightly compressible,
in large confined homogeneous reservoirs under general conditions
including those of typical well tests. Analytic solutions are given as
convolutions of initial and source data with fundamental solutions for
each system. We establish that the problems are well-posed for all
initial conditions likely to arise in practice. We obtain, using the
theory of generalized functions, classes of all functions in which
solutions exist, are unique and depend continuously on the initial data
for appropriate restrictions. We consider two of the models to be the
main versions relevant for flows in natural fractured rock, including
typical basalt formations. They apply to short-time flows on order of
the duration of typical pressure transient tests, and they are
potentially useful for obtaining accurate values of formation parameters
from well test data.   We also show that the inverse problems of

estimating parameters from well data are well-posed, i.e., that parameter
values obtained are unique and depend continuously on the data.

Solutions to Some Linear Evolutionary Systems of Equations:
A Study of the Double Porosity Model of Fluid Flow
in Fractured Rock and its Applications

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TABLE OF CONTENTS

LIST OF FIGURES

SOLUTIONS TO SOME LINEAR EVOLUTIONARY SYSTEMS OF EQUATIONS:
A STUDY OF THE DOUBLE POROSITY MODEL OF FLUID FLOW
IN FRACTURED ROCK AND ITS APPLICATIONS

INTRODUCTION

Many problems that arise in the physical sciences are modeled mathematically as partial differential equations of various types. Partial differential equations form a well-developed branch of classical applied mathematics and are an essential tool in many investigations in the applied sciences. The mathematical formulation of e.g. a physical problem often results in such an equation and its solution usually allows insights and yields important information about the physical process under consideration. In general, a differential equation expresses in mathematical terms a relationship between various quantities involved in a physical process, such as, for example, balances between forces acting upon moving objects and rates of change in velocities of motion, local mass density changes and variations in the fields of flow of fluids, etc. Many physical problems of practical interest can be represented by relatively simple equations which may be treated and solved by standard methods. In this work we make use of the theory and methods of linear partial differential equations to study systems of equations which include cases of the so-called parabolic types and which arise often in problems concerned with flow of ground fluids in natural rock formations.

A partial differential equation is (in contrast to an ordinary differential equation) an equation in an unknown function of several independent variables together with its partial derivatives. In physical problems the independent variables are usually the spacial coordinates

x and time t. The equation is said to be of order n if it contains at least one derivative of order n and none of order higher than n, and the equation is linear if it is linear in the unknowns and its derivatives. A complete formulation of a problem consists of one or more equations given with initial and boundary conditions which the equations must satisfy, and the system may then have a unique solution, many solutions or no solution at all. In most cases involving physical problems the equations together with the initial and boundary conditions have a unique solution.

We mention three simple, well-known linear partial differential equations which occur frequently in the physical sciences. These are the wave equation (1), the diffusion or heat equation (2) and Laplace's equation (3).

$$\frac{\partial^2 u}{\partial t^2} - a \nabla^2 u = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} - a \nabla^2 u = 0 \tag{2}$$

$$\nabla^2 u = 0 \tag{3}$$

We denote here by u=u (x,t) the unknown function of the spacial variable x = $(x_1, x_2, \ldots, x_m)$ and time t, $a > 0$ is a constant, and

$$\nabla^2 = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian operator in m-dimensions. Equations (1), (2), (3) are given here in their homogeneous form, i.e., with no terms ("sources" or "sinks") on the right hand sides. As the names suggest the wave equation

(1) describes the propagation of waves in space and it arises in fields such as electromagnetism, acoustics, elasticity and fluid dynamics. The diffusion/heat equation (2) describes diffusion of heat, fluids or other substances in conducting solids, porous media, and liquids or gases, respectively. Laplace's equation (3) characterizes many stationary processes in electrodynamics, hydrostatics, gravitation and steady-state diffusion. In contrast to Laplace's equation the other two describe time-dependent processes. The diffusion equation is a simple example of a parabolic equation and it will be of special interest to us here, for it is a basic equation from which other systems of equations treated here are derived, and it is itself a special and simplest case of these systems. The diffusion equation will therefore receive detailed attention again later.

A mathematical representation of a physical process is commonly referred to as a model of the process or problem. A mathematical model may be analytic or numerical and of varying degrees of complexity, but it always represents an idealized and simplified version of the actual problem. The appropriate choice of a model in a given case depends on both the nature and purpose of the investigation and on the physical problem. One is likely to employ a sophisticated model in a case when accurate quantitative information is being sought about a complex physical system, while in studies concerned primarily with qualitative features of a general process it is often useful and convenient to use simple models that represent only the essential characteristics of the physical system. Important considerations include also cost and computational effort involved.

Many problems in the applied sciences may be classified as of either the so-called <u>direct</u> or <u>inverse</u> type. A typical direct problem involving the use of a mathematical model consists in the simulation or prediction of the behavior of a physical system based on known properties of that system, while with an inverse problem one seeks to obtain quantitative information about the properties of the system from observations of its behavior. More specifically, the direct problem consists of computing the theoretical model output given known values of the model parameters and the initial conditions, while with an inverse problem the model parameters are unknowns to be estimated and functions representing model output are known and given. The values of the parameters are determined from the input functions by various methods of analysis. The input functions are usually measured data which represent the behavior of the physical system, and model parameters represent physical properties of the system.

In groundwater hydrology and petroleum engineering direct problems arising in practice are prediction and simulation of underground flows in natural formations, while inverse problems are typically estimations of numerical values of certain flow-related properties (parameters) of the rock masses from analysis of measured data on the subsurface flows. (The relevant parameters here are "permeability" and "storage capacity", which we define in Chapter 1.) This inverse problem is of particular importance in all practical work on reservoir flows, for values of these parameters are typically unknowns that would be needed for quantitative modeling of predicting of the behavior of fluids in a given natural formation. The standard and simplest mathematical model used for

treating problems of both types is the diffusion equation and its solutions. For the inverse problem in particular, a specific solution to the diffusion equation, known as the "Theis solution" (model), is widely used as a simple and convenient model for well test responses in natural reservoirs. This model together with a simple graphical data interpretation technique (type curve matching) has for decades been a standard tool for estimating formation parameters in groundwater hydrology and petroleum engineering. The Theis model is treated in Chapters 1 and 4.

The diffusion equation governs, as we discuss in Chapter 1, flows, single-phase, of slightly compressible fluids in "ordinary" homogeneous porous media, and it is applicable to flows in natural rock formations which are reasonably homogeneous and unfractured. Many natural rock formations are highly fractured as well as heterogeneous in general, and these conditions complicate the behavior of ground flows such that it cannot in many cases be adequately described by the simple homogeneous diffusion model. Fracturing of rock masses affects particularly short-term flow processes such as those associated with typical field experiments, or well tests, conducted to yield parameter estimates for rock formations, and the accuracy of such estimates for a fractured rock mass may therefore be affected when interpretation is based on the conventional diffusion or Theis model. Consequently special models are needed for flows in fractured formations and for enabling more accurate estimation of parameters. One such model, or type of a model, is a simple modified diffusion model known as the "double porosity model". This model, as we discuss in detail in Chapter 2, is represented by

degenerate parabolic systems derived from two coupled diffusion equations and it describes flows in idealized systems thought of as composed of two overlapping porous media; the rock matrix and the fractures. Conventional diffusion models, in contrast, view both fractures and rock matrix together as a single porous medium. The double porosity model typically assumes that virtually all lateral flow in the rock mass occurs along the fractures and that the fractures form a uniformly distributed system within the rock formation. The double porosity approach was introduced in the literature in 1960 by Barenblatt et al [5] and modified slightly by Warren and Root [50] in 1963. The main difference between these two versions of the model is neglect by Barenblatt et al of fluid storage capacity of the fractures. The double porosity model has been a subject of numerous investigations by workers mostly in groundwater hydrology and petroleum engineering. Most of these investigations, reviewed in more detail in Chapter 2, are studies of variants of the model version considered by Warren and Root, and a few studies compare model results with data from natural reservoirs. Equations governing the flow are given with initial and boundary conditions that represent conditions of typical well tests. Analytic solutions and numerical results are usually presented and compared with corresponding solutions to the Theis model. The solutions are usually obtained by Laplace transform methods with numerical inversion or by finite difference methods. Certain distinguishing characteristics of the models are observed from the solutions which we describe later (Chapters 2 and 4). The double porosity models are found to behave as distinct models that

differ from the behavior of the Theis model on a short-time scale of the flow, or during the so-called "early-time" phase of the response of a typical well test. Some publications report data sets from natural fractured formations and compare these with the theoretical model results (see Chapters 2 and 4). A few authors, e.g. [33], [50], consider methods for estimating parameters based on the double porosity models. Some studies in more recent years, [1], [2], [19], [20], treat double porosity models as finite element systems, and a recent paper [32] presents analytic treatment of special cases of the model. In all except the last five studies above the authors leave open questions of well-posedness of the models, (i.e., that in addition to existence the solutions are unique and depend continuously on the initial and boundary data). Moreover, well-posedness of the inverse problems of estimating parameters from data is not considered.

The present work is a study of three versions of the double porosity model, of which two are the forms due to Barenblatt et al [5] and Warren and Root [50] and a third case of the model which is not normally considered in the literature. We obtain analytic solutions for more general conditions including those of typical well tests. The solutions are given for each model as convolutions between fundamental solutions (Green's functions) and the initial and boundary data. We establish that each model is well-posed for all initial conditions likely to arise in practice; i.e., we obtain (larger) classes of functions in which solutions exist, are unique and depend continuously on the data for appropriate restrictions. To do this we treat the systems and the solutions as generalized functions and make use of the theory of

generalized functions which is reviewed in Chapter 3. We consider the inverse problems of estimating parameters from well test data and show for each model that the inverse problem is also well-posed, or that parameter estimates obtained for a given data function are unique and depend continuously on the data. We discuss validity and application of the models to natural formations in general and physical conditions and time scales for which the models apply. We review observational evidence in the literature for double porosity behavior in natural formations.

This work began in 1983 as part of a research program which was being conducted at the time by Rockwell-Hanford Operations (RHO) in Richland, Washington. The purpose of this program was to evaluate the Columbia River Basalt formation as a site for the storage of nuclear waste. Substantial amounts of toxic and potentially hazardous waste materials have accumulated over the years with the operation of nuclear power plants, and consequently there is a growing need for effective methods of disposal of this waste. The principal method considered is storage underground in sealed containers in areas of low permeability with respect to groundwater flow. The Columbia River Basalts were one of a number of such areas in this country considered for this purpose. The site evaluation program would require, among other things, investigation of the permeability and fluid storage capacity for this rock mass. These are the properties which determine the behavior of underground fluids and allow prediction of the groundwater flow over a long time scale. Permeability and storage capacity would be determined from analysis of well test data using suitable mathematical models which describe flow on the short time scale that characterizes the data. The

Columbia River Basalts as a typical basalt formation are likely to be highly fractured and we chose therefore to study and apply the double porosity model to estimating parameters for this formation. The site evaluation program was terminated before the completion of this study, expected field data were never obtained and this study remained thus limited to an investigation of the theoretical models only. We may note here that as a highly fractured rock formation the Columbia River Basalts are unlikely to possess the low fluid permeabilities desired for a nuclear waste storage site.

Based on the present work as well as the published work in the literature we find the Barenblatt and the Warren-Root models to be, for suitable physical conditions, the main potentially useful forms of the double porosity model for ground fluid flows in natural fractured rock masses in general, including typical basalt formations. That is, given reasonably uniform fracturing of the rock mass and temperature conditions such that the fluid remains in the liquid state, it is likely to be useful to model short-time flow in fractured rock such as the "early-time" response of a typical pressure transient test as a double porosity system of the Barenblatt or the Warren-Root type. The principal value of these models lies in allowing more accurate estimation of formation parameters from early-time pressure transient data than the conventional diffusion or Theis model, although for somewhat longer time scales of flow the Theis model is equally valid. These models (as indeed all the models herein) are applicable for most or all initial conditions likely to be encountered in practice. We find that the Barenblatt model requires reasonable smoothness of initial data but imposes no

restrictions on their behavior at infinity, while the other models (including the diffusion model) allow initial data having growth at infinity up to order two. Features of the Warren-Root model would be the most easily recognized in data from typical well tests.

CHAPTER 1

THE STANDARD MODEL OF GROUND FLUID FLOW

A. Basic Principles of Flow in Porous Media

In this section we review the basic physics of flow of a fluid in a typical porous solid. We derive from first principles the diffusion equation which, as mentioned, is a standard mathematical description for such flows. For a more complete treatment of the subject see e.g. [14] or [17].

A regular or ordinary porous medium, distinguished for example from a fractured medium, is a solid which contains numerous tiny and densely distributed regions of void spaces (pores) that allow storage and movement of fluids within the solid to occur. Familiar examples of such materials include rock, sand, soil, and cement. The pores are typically quite small, usually of the size as the mineral grains in rock, and they are sufficiently interconnected so that fluids may seep through these media. Fluid seepage occurs due to internal forces arising from the fluid pressure, the fluid mass distribution, temperature variations and gravity. A typical order for the velocity of flow in rock can be centimeters per day. The transport of fluids in a porous medium has the character of diffusion. Owing to the low velocities of flow inertial terms in the equations of motion are negligible and a virtual balance holds between the pressure gradient forces and friction. Under these conditions the flow in a porous medium may usually be described by Darcy's law:

$$\vec{q} = -\frac{k}{\mu}\left(\nabla p + \rho\vec{g}\right),$$

(1.1)

where $\vec{q}$ is the volumetric flow rate or flux having the units of velocity, p is the fluid pressure, k is the <u>permeability</u> of the medium, μ is the (dynamic) viscosity, ρ is the fluid density and $\vec{g}$ = (0,0,g) is the acceleration due to gravity. The permeability k, a property of the medium, is actually a tensor, but since an ordinary porous medium is typically isotropic k is usually a scalar and, in most cases, taken to be a constant if the properties of the medium do not vary significantly in space. The permeability depends strongly on the <u>porosity</u> (ϕ), or fractional void volume of the medium, and the value of k is higher for higher values of ϕ. In cases of uniform temperature μ is a constant. Both fluid and solid are usually slightly compressible and under isothermal conditions the fluid density ρ and medium porosity ϕ are functions of the pressure p only. For temperatures within appropriate limits the fluid can be assumed to remain in the liquid state only, or single-phase. In many cases the role of gravity is negligible and may be omitted from the equations.

To derive the diffusion equation we assume a single-phase fluid in an arbitrary, elementary volume V in the medium. We assume both the fluid and the solid to be slightly compressible and the temperatures to be constant. Let B be the boundary of V and ϕ, ρ and $\vec{q}$ = (q_1, q_2, q_3) are the porosity, fluid density and volumetric flow rate, respectively, at a point x = (x_1, x_2, x_3) in V. Then the following equation holds for the mass balance of fluid in V:

$$\frac{\partial}{\partial t}\int_V (\phi \rho)\, dx \;+\; \int_B (\rho \vec{q})\cdot d\vec{s} \;=\; \int_V F\, dx \;, \qquad (1.2)$$

where $dx = dx_1 dx_2 dx_3$, $\vec{ds}$ is the vector-element of surface on B, and F is a source or sink of fluid mass, or the rate at which mass is being produced per unit volume. Equation (1.2) states simply that the time rate of change of fluid mass inside V plus net outward flux across B is equal to the amount of fluid mass generated or lost in V per unit time due to the source term F. We may apply Gauss' Divergence Theorem to the second integral of (1.2), interchange integration and differentiation in the first integral and, combining all terms under one integral sign, we obtain

$$\int_V \left(\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\vec{q}) - F \right) dx = 0.$$

(1.3)

We assume the integrand to be continuous. Since the volume element V is arbitrary, we obtain the mass continuity equation

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\vec{q}) = F .$$

(1.4)

The flow in the porous medium obeys Darcy's law (1.1), so we get, after combining (1.4) and (1.1),

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot\left(-\rho\frac{k}{\mu}(\nabla p + \rho\vec{g}) \right) = F.$$

(1.5)

The fluid density ρ and the porosity ϕ are, as noted, assumed to be functions of the pressure p only, and these functions may be linearized

for "small" variations in ρ, ϕ, p, or if $|\partial p/p| \ll 1, |\partial \rho/\rho| \ll 1$ and $|\partial \rho/\rho| \ll 1$, where p, ∂p and $\partial \phi$ represent deviations from mean or equilibrium values p_o, ρ_o and ϕ_o. Let c_1 and c_s be compressibilities at constant temperatures (T) for the liquid and solid, respectively, defined by

$$ c_\ell = \frac{1}{\rho} \frac{d\rho}{dP} \bigg|_T, \qquad \text{and} \qquad (1- \phi_o) \, c_s = \frac{d\phi}{dP} \bigg|_T $$

(both c_ℓ and c_s are taken to be constant). The first term in equation (1.4) may then be written

$$ \frac{\partial}{\partial t}(\rho\phi) = \left(\phi \frac{\partial \rho}{\partial P} + \rho \frac{\partial \phi}{\partial P}\right) \frac{\partial P}{\partial t} \approx \rho_o c_o \frac{\partial P}{\partial t}, \tag{1.6} $$

where $c_o = \phi_o c_\ell + (1- \phi_o)c_s$ is the <u>storage capacity</u> of the system and we have approximated ρ and ϕ by $\rho_o = \rho(p_o)$ and $\phi_o = \phi(p_o)$ at the constant equilibrium pressure p_o. We assume that the medium is homogeneous and isotropic so k is taken to be a constant, and since the system is isothermal μ is also a constant; the effects on the flow of density variations and gravity are negligible and so the second term in equation (1.4) may be simplified and written

$$ \nabla \cdot (\rho \vec{q}) \approx - \rho_o \frac{k}{\mu} \nabla^2 P . \tag{1.7} $$

From (1.6) and (1.7) equation (1.4) takes the form

$$ \rho_o c_o \frac{\partial P}{\partial t} - \rho_o \frac{k}{\mu} \nabla^2 P = F, \tag{1.8} $$

which, after division by ρ_0^2 and c_o and assuming no sources or sinks (F≡0), yields the diffusion equation for the pressure p

$$\frac{\partial p}{\partial t} - a \nabla^2 p = 0, \tag{1.9}$$

where $a = k/\mu c_o$ is the diffusivity of the system. Equation (1.9) expresses a balance between the rate of change of local fluid density and the divergence of flow at any point and it governs time-dependent diffusive transport in a slightly compressible porous medium. For flow of an incompressible fluid or in case of steady-state diffusion the first term in equation (1.9) would vanish and the flow would be governed by Laplace's equation. We will return to equation (1.9) again and give detailed treatment to it in Chapter 4.

B. The Theis Model

Although we present solutions to the diffusion equations for general conditions in Chapter 4 it is useful to list here the particular solution which, as noted earlier, is a standard simple model of well test response for parameter estimation for natural reservoirs. This so-called Theis model is the solution to equation (1.9) for the infinite radially symmetric plane with a constant-strength fluid point source applied at the origin (r=0) at t≥0 and fluid pressures are assumed to be initially equal to zero. Specifically, the Theis model is the solution to the problem

$$\frac{\partial u}{\partial t} - a \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = \frac{f_o}{c_o} \delta(r) U_+(t),$$

$$u(r, 0) \equiv 0, \tag{2.1}$$

where we define u=u(r,t) as the deviation of fluid pressure from the (constant) mean or equilibrium pressure (p_o), f_o is the (constant) strength of the source, and $U_+(t)$ is the unit step function defined by

$$U_+(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} .$$

Physically, problem (2.1) represents mean fluid pressure behavior in a confined, infinite horizontal homogeneous layer of a constant thickness H, and a constant-strength line source/sink extending through the depth of the layer is applied at r=0, t≥0, to the system, assumed to be initially in equilibrium. The line source has the strength f_o per unit length. The following (inhomogeneous) diffusion equation may be written for this system in cylindrical coordinates,

$$\frac{\partial u}{\partial t} - a\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right) = \frac{f_o}{c_o}\delta(r)U_+(t) . \tag{2.2}$$

With the boundary conditions expressing no flow across the top and bottom surfaces of the layer, i.e.

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0 , \qquad \left.\frac{\partial u}{\partial z}\right|_{z=H} = 0 , \tag{2.3}$$

we find, after vertical integration of equation (2.2) from z=H to z=o, that the vertical derivatives vanish, since by (2.3),

$$\int_{z=H}^{z=0} \frac{\partial^2 u}{\partial z^2}dz = \left.\frac{\partial u}{\partial z}\right|_{z=H}^{z=0} = 0 ,$$

and equation (2.2) thus reduces to (2.1). The solution to problem (2.1) is

$$u(r,t) = \frac{\mu f_o}{4\pi k}Ei\left(-\frac{r^2}{4\pi t}\right) , \tag{2.4}$$

where the function Ei(-x) is given by

$$- Ei(-x) = \int_0^\infty exp(-y)/y \, dy \, .$$

(2.5)

Equation (2.4) was introduced into groundwater hydrology in 1935 by Theis [49] (see also [22]) as a simple model of a drawdown test response in a homogeneous reservoir having the form of an infinite horizontal layer. The Theis solution (2.4) is shown in log-log form in Figure 1.1. (We describe typical well tests, including drawdown tests, in the next section). If the volumetric flowrate out of a pumping well used in the drawdown test is q_o, then the line source strength is $f_o = -q_o/2\pi H$. The Theis solution (2.4) is used for determining values for the unknown parameters k and c_o ($a=k/\mu c_o$) given known values of the fluid pressure $u(r,t)$ as a function of time t with known values of f_o, μ, and r.

The fluid source may be implemented in two equivalent ways; it may be given as a source term on the right hand side of the diffusion equation as expressed by equation (2.1), or it may be given with the homogeneous diffusion equations as a point boundary condition of the form

$$\lim_{r \to 0} \int_\Omega \frac{k}{\mu} \vec{\nabla} u \cdot d\vec{s} = q_o U_+(t)/H \, ,$$

(2.6)

where Ω denotes a closed surface (curve) of vanishing radius r surrounding the source point r=0. To prove this consider the (small) elemental volume V with the surface Ω containing the source point r=0. Without loss of generality we may take the arbitrary sources f(t), q(t) to be the constant f_o and q_o, respectively, and the source point to be r=0. The space domain is also arbitrary and it may be 2- or 3-dimensional, but for convenience we choose the xy- plane which represents

$$u^*(r,t) = - \quad Ei(-r^2/4at)$$
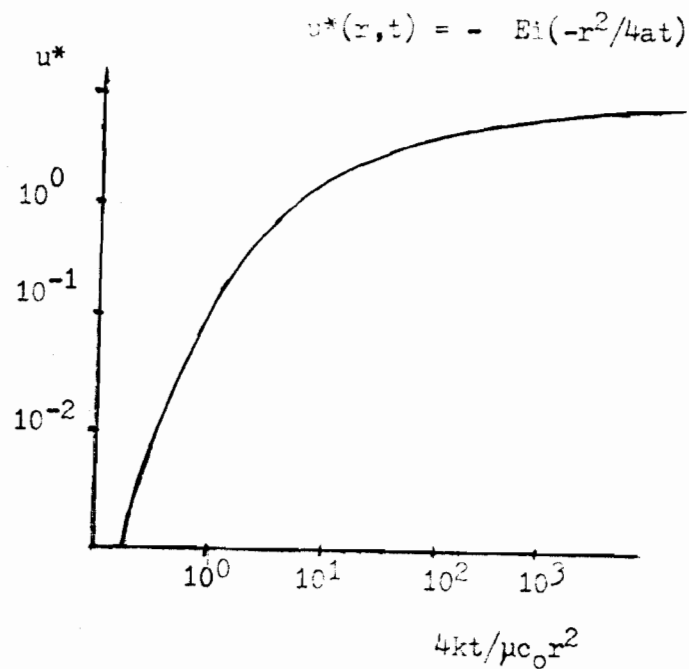
Fig. 1.1   Pressure drop $u^*(r,t) = 4\pi k/\mu f_o \, \delta \, p(r,t)$ in an infinite horizontal layer due to a point sink $f(t) = f_o U_+(t)$ applied at r=0.

the infinite layer of thickness H. The following equation for the mass balance of fluid in V may be written as

$$\frac{\partial}{\partial t} \int_V (\rho \phi) dV = - \int_\Omega \rho \vec{q} \cdot d\vec{s} + \int_V \rho F dV ,$$

(2.7)

where

$$F(r,t) = f_o \, \delta(r) \, U_+(t) .$$

(2.8)

The diffusion equation (2.1) multiplied by $\rho_o c_o$ ($\rho \approx \rho_o$) represents a simpler version of equation (2.7), of which the first term is given by

$$\frac{\partial}{\partial t} \int_V (\rho \phi) dV = \int_V c_o \rho_o \frac{\partial u}{\partial t} ,$$

(2.9)

the second term is

$$\int_\Omega \vec{q} \rho \cdot d\vec{s} = - \int_\Omega \frac{k}{\mu} \rho_o \vec{\nabla} u \cdot d\vec{s}$$

(2.10)

and the third term, after integration over V and using the property of the delta function, is

$$\int_V \rho f_o U_+(t) \delta(r) dV = \rho_o f_o U_+(t).$$

(2.11)

We let V-> 0 so that the integral (2.9) vanishes, $\frac{\partial u}{\partial t}$ is continuous and we obtain then a balance between the two remaining terms, (2.10) and

(2.11), of equation (2.7). This yields, as V -> 0, the boundary condition (2.6) after dividing through by ρ_0, or

$$\lim_{V \to 0} \int_\Omega \frac{k}{\mu} \nabla u \cdot d\vec{s} = \frac{q_0}{H} \, ,$$

where $q_0 = -2\pi f_0 H$.

C. Pressure Transient Tests

Pressure transient tests are standard types of field experiments conducted on natural rock formations for the purpose of observing the behavior of the subsurface flow and determining the permeabilities and storage capacities of the rock mass. These are well tests which involve perturbing the underground fluid and measuring the time-dependent response. The underground fluid is perturbed by withdrawal (pumping) or injection of fluid through wells or boreholes and measurements are made of the flow response by means of pressure-sensitive devices (piezometers) placed at suitable depths in wells. These instruments provide time series of fluid pressure from which the subsurface flow may be observed and which may be analyzed to yield information about the system and its properties.

There are two main types of pressure transient tests; the so-called drawdown tests and buildup tests. In case of a typical drawdown test the system is initially in equilibrium and both pumping and measurements are begun simultaneously at a time t=0 and continued for a suitable length of time as the pressure in the system drops (at a decreasing rate) due to the pumping. In case of a buildup test the system is initially in a perturbed state due to pumping performed prior to measurements, and at

t=0 the pumping is terminated and measurements begun and continued for a suitable length of time while the pressure in the system gradually rises (builds up) towards equilibrium. The typical duration of these tests is, for usual single-phase or "low-temperature" conditions, a few to several days.

Pressure transient tests may involve one or more wells for pumping and/or measurements. If only one well is used for both pumping and measurements the test is a single-well test, but if measurements are made at one or more separate "observation wells" located at some (known) distances from one or more "test wells" used for pumping the test is an interference test. Parameter values estimated from a given well tests is representative of the rock mass in an area surrounding the pump well on space scales of the drainage volume or the separation distance between test and observation wells. We note that all model solutions given here apply to interference test conditions only.

D. Parameter Estimation by Type Curve Matching

Type curve matching is a simple graphical method for deriving values of permeability and storage capacity from pressure transient data. The technique was introduced by Theis [49] along with the solution (2.4) above. The method consists of graphical matching of the data with the theoretical model in a manner which allows values of these two as well as other parameters to be derived quickly and easily. In principle any of a number of methods for analysis may be used, but curve fitting techniques are particularly convenient and practical because they require relatively little computational effort and time. When based on valid and

accurate interpretation models curve matching methods usually give good ballpark values of the parameters.

A <u>type curve</u> is a log-log graph of dimensionless fluid pressure (or some function of pressure) plotted versus a dimensionless time variable, usually $4kt/\mu c_o r^2$. The most commonly used type curve in practice is the Theis curve, shown in Figure 1.1. Another example of a curve frequently used for parameter estimation is the semilog curve of the Theis solution. The semilog plot of the Theis solution is characterized by an approximately linear shape over the most significant range of values of the time variable, and this linear part of the graph is the part used for estimating parameters. Log-log curves have thus a wider and more general applicability than semilog curves. Other examples of type curves which have been introduced in the literature in recent years are log-log plots of pressure derivatives versus time. These are considered useful for interpretation of data from double porosity systems in particular, because certain special characteristics of some double porosity systems (the Warren-Root types) are made more conspicuous by such plots of data. However, type curves based on pressure derivatives have the disadvantage of the unsmoothing effect of computing pressure derivatives, particularly with "noisy" data sets.

For the purpose of illustration let us consider the Theis solution (2.4) as our interpretation model and we want to estimate the permeability (k) and the storage capacity (c_o) from a hypothetical data set by type curve matching. The model and a "data" curve are shown in Figure 1.2. Both curves are (ideally) of the same shape but they are shifted relative to one another along both axes by amounts which depend

$$u^*(r,t) = - Ei(-r^2/4(k/\mu c_o)t)$$

$$u^*(r,t) = (4\pi/uf_o)\delta p$$

$$= (1/k)(-Ei(-r^2/4(k/\mu c_o)t))$$
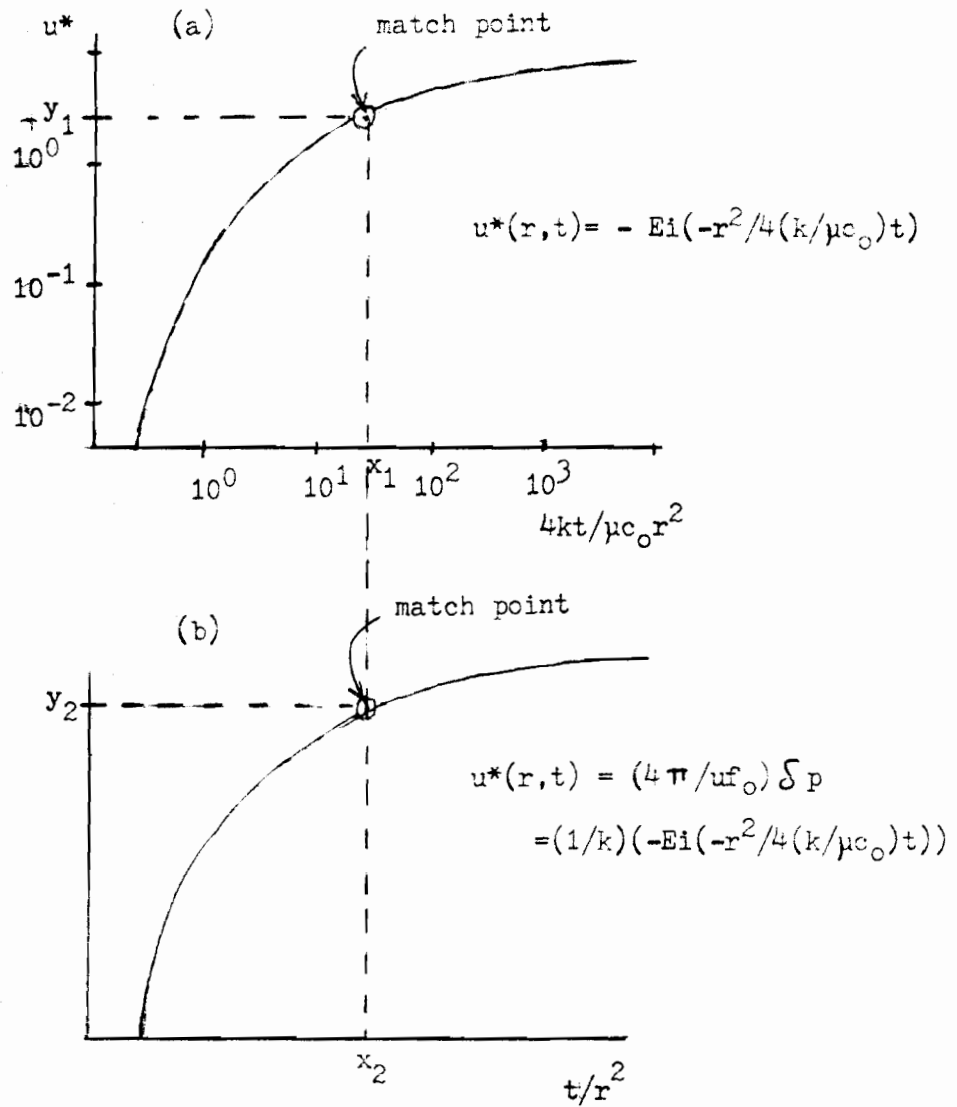
Figure 1.2   (a)   Model curve
              (b)   Data curve (hypothetical; coordinates
                        omitted).

on the values of the parameters we wish to find. The procedure for estimating the parameters is as follows: The data and model curves are matched against one another (one of the curves is overlaid on top of the other) such that they coincide, and an arbitrary "match point" (one point for each curve) is selected, as shown in Figure 1.2. The coordinates for the match point on the data curve are, as shown, x_1, y_1, and the corresponding coordinates for the point on the model curve are, respectively, x_2, y_2. All quantities other than k and c_o are assumed known. The values of y_1, y_2 differ by a factor k and thus we obtain the value of k from the ratio y_1/y_2. Similarly, the values of x_1, x_2 differ by the factor $4k/\mu c_o$ and thus, using the value of k just found, we obtain the value of c_o from the ratio $4kx_2/\mu x_1$.

Type curves analogous to the Theis curve above may be constructed for other models and similar parameter estimation procedures applied. Other models may contain independent parameters in addition to permeability and storage capacity. For type curve plots of pressure against a time variable such as $4kt/\mu c_o r^2$ permeability and storage capacity are estimated from match point coordinates in the manner just described, while values of additional unknown parameters are deduced from best fits with the model curve. That is, we infer values of additional parameters for the data to be the values associated with the best fitting model curve. Curve matching procedures involving unknown parameters in addition to permeability and storage capacity require in general sets of several model curves.

CHAPTER 2

MODELS OF FLOW IN FRACTURED ROCK

A. Naturally Fractured Rock

Many natural rock formations are highly fractured, i.e., they contain significant amounts of void space in the form of "cracks", "fissures", "vugs", etc., which occur as narrow, extended regions of void space superimposed as a "secondary porosity" upon the primary pore structure of the rock (see Figure 2.1a). (Up to as much as 41% of natural reservoirs of commercial interest in the oil industry are significantly fractured [48]). The fracturing of a rock mass is the result of thermal, mechanical and chemical effects which have acted generally after the formation of the primary rock itself. Fractures of natural rock are of variable size and shape, but they are of characteristically of very different geometric structure and space scales than the primary pores. Typically they are of a flattened, elongated or sheet-like shape and often also jagged and irregular, and their lengths and widths are usually considerably greater than the typically microscopic sizes of the pores. Fractures range in size from the small "microfractures" or millimeters length to large structures of hundreds of meters or more. Spacing or distance between fractures is generally in the order of the scales of the fracture lengths. Usually, fractures, except for the largest ones, are embedded within the rock mass and not directly visible except at surfaces and outcrops.

Because of their characteristic lengths and shapes fractures allow effective passage of fluid and thus profoundly affect character of ground

fluid flow and the overall permeability of the rock mass. The presence of fractures results, for example, in rapid fluid communication between widely separated wells during a well test, or a more rapid response at a well located farther from a test well than at a well nearer by [48]. The contribution to the overall formation permeability by fractures in a natural rock mass typically exceeds the contribution of the primary pores by a few to several orders of magnitude. Moreover, the permeability of natural fractured rock tends to be anisotropic due to preferential orientation of fractures in most rock masses. In general, the flow of fluids in fractued rock is more complex than flow in unfractured rock.

B. Models of Flow in Fractured Rock

The behavior of fluids in natural fractured rock masses differs in general from flows in homogeneous porous media and thus it cannot in most cases be adequately described by simple homogeneous diffusion models such as the Theis model. Moreover, due to considerable variability in the fracturing of natural rock the behavior of fluids varies considerably among individual fractured formations. Therefore the problem of modeling flow in natural fractured rock is relatively complicated and requires special models. A substantial amount of research on the behavior of fluids in natural fractured rock and fractured porous media has led to a number of theoretical models presented to date in the literature. It appears that two main approaches to the problem are taken [29]: one which treats the fractured formation as a discontinuous medium, and the other which views it as a statistically homogeneous continuum. More

specifically, the theoretical models may be broadly classified into three types or categories; (1) the "deterministic" models, (2) the "equivalent homogeneous reservoirs" and (3) the double porosity models. The deterministic approach is based on a detailed description of individual fractures which are of known location and geometry, and the flow in each fracture is modeled as a separate system of flow coupled to the flow in the surrounding porous matrix. Examples of deterministic of models are the "discrete fracture models", in which individual fractures as idealized as planes of known location, width and orientation. The equivalent homogeneous approach considers main trends of the flow in the fractured system and tries to describe the fluid behavior by means of models of lower complexity. Examples of this approach are diffusion models which attempt to incorporate effects of fractures by suitable choices of parameters. The double porosity models, which we describe in more detail in the next section, view the reservoir as composed of two overlapping (intermeshed) continua, the rock matrix and the collective system of fractures, which both contain separate but interacting systems of flow. In contrast to conventional diffusion models the double porosity and discrete fracture models take special account of the effect of fractures by modeling flow in the fractures as a separate system coupled to the flow in the rock matrix, and they describe therefore more accurately flow in natural fractured system than do conventional diffusion models. In practical applications, however, deterministic models such as the discrete fracture models have the disadvantage of being limited to systems with only a few fractures of known location and geometry. Moreover, these models are case-specific and complex versions

of these models tend to require considerable computational effort by numerical methods. Double porosity models, on the other hand, treat the reservoir as a continuum and do not require knowledge of the location and geometry of individual fractures but only that they be fairly uniformly distributed. The fractures should be of suitable scales of length, width and spacing relative to the scales of the flow. This type of a situation is, we believe, likely to be more often encountered in practice. Moreover, treating natural reservoirs and fracture systems as continuous media allows the models to be treated analytically and the model solutions to be applicable to larger numbers of natural systems.

C. The Double Porosity Model.
The Physical System and Basic Equations

The double porosity (dual- or two-porosity) model was as we noted before, introduced in 1960 by three Russian authors, Barenblatt, Zheltov and Kochina [5] as an analytic approach to describe more realistically the behavior of fluids in fractured rock. The basic idea of this model is, as mentioned, to treat the fractures as a second continuum intermeshed with the rock matrix and containing a separate system of flow interacting with the flow in the matrix. In other words, the rock mass is viewed as composed of two overlapping media; one being the rock matrix, an ordinary porous medium, and the other is the collective system of fractures which form an interconnected system of conduits. The fracture system is assumed to have lengths, widths and spacing such that with local averaging it may be treated as a continuous medium having distinct macroscopic properties (permeability, storage capacity). For convenience the fractures and matrix are treated as homogeneous and

isotropic media. For conceptual simplicity the fractured rock mass is usually thought of as a homogeneous porous medium intersected by a series of mutually perpendicular planes of narrow void space; these intersecting planes constitute the fracture system and subdivides the reservoir into equal rectangular or cube-shaped matrix "blocks" as shown in Figure 2.1. This simple and highly idealized "cube model" (unrealistic for any natural rock mass) was originally described by Barenblatt et al [5]; it guarantees homogeneity and isotropy of the fracture system without violating basic assumptions of the model. (Homogeneity and isotropy of the system simplifies mathematical treatment. Anisotropy of a natural system can be eliminated from the model by proper coordinate transformation.) In contrast to conventional diffusion models, a double porosity system is thus characterized by two sets of properties; a permeability k_1 and storage capacity c_1 for the rock matrix and corresponding values k_2, c_2 for the fracture system. Similarly, two fluid pressures rather than one are defined for any point in the system; p_1 for the matrix and p_2 for the fractures. Fluid seepage is assumed to occur according to Darcy's law in each medium and we may write, following Barenblatt et al, the two coupled diffusion equations governing the flow in the system (with no external sources or sinks given);

(2.1)

$$c_1 \frac{\partial P_1}{\partial t} - \frac{k_1}{\mu} \nabla^2 P_1 + q_c = 0,$$

$$c_2 \frac{\partial P_2}{\partial t} - \frac{k_2}{\mu} \nabla^2 P_2 - q_c = 0,$$

where
$$q_c = \frac{\alpha}{\mu}(p_1 - p_2)$$

(2.2)

is a transfer (crossflow) term, or an internal source or sink of flow between the matrix and fractures due to the pressure difference p_1-p_2, and α is a (dimensionless) transfer coefficient (or a "crossflow" or "interporosity flow coefficient"), assumed to be a constant. The value of α would be obtained empirically and it would be proportional to the matrix permeability k_1 and the fracture surface area per unit volume of the reservoir. The expression of q_c as being proportional to p_1-p_2 is based on the assumption that the interporosity flow is in a quasi-steady state and that divergence of flow within individual matrix "blocks" is negligible. The quantities p_1, p_2, q_c, etc., though representing functions of points (x) in space (as well as time t), are understood to

VUGS   MATRIX   FRACTURE   MATRIX   FRACTURES
(a) ACTUAL RESERVOIR   (b) MODEL RESERVOIR
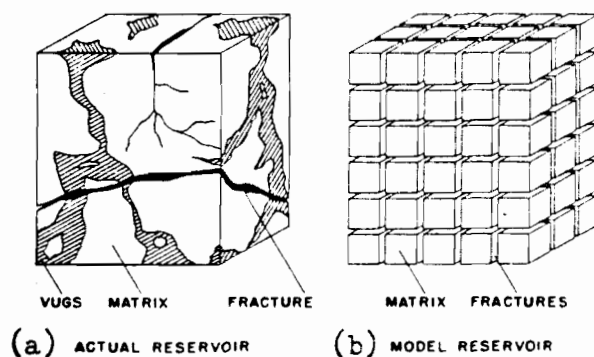
Figure 2.1   Idealization of the fractured reservoir [50].

be local spacial averages on scales exceeding characteristic fracture spacing or block length. The model thus requires that the space scales of the flow be considerably greater than the characteristic scales of the fractures. Equations (2.1) are reduced to simpler forms by invoking one or both of the following physically reasonable assumptions; (1) the matrix permeability k_1 is negligible compared with the fracture permeability k_2, and (2) the fracture storage capacity c_2 is small or negligible compared with the storage capacity c_1 of the matrix. In other words, it is reasonable to assume that essentially all lateral flow occurs within the fractures and that most of the fluid is stored in the matrix. Barenblatt et al [5] invoke both (1) and (2), thus eliminating two terms from equation (2.1), and hence they propose as their "double porosity model" the following system of equations

$$c_1 \frac{\partial P_1}{\partial t} \qquad + \frac{\alpha}{\mu}(P_1 - P_2) = 0,$$
$$- \frac{k_2}{\mu}\nabla^2 P_2 - \frac{\alpha}{\mu}(P_1 - P_2) = 0. \qquad (2.3)$$

Most investigators prefer to retain the fissure storage terms by allowing c_2 to be small but not negligible compared with c_1 and thus consider rather the system of equations

$$c_1 \frac{\partial P_1}{\partial t} \qquad + \frac{\alpha}{\mu}(P_1 - P_2) = 0,$$
$$c_2 \frac{\partial P_2}{\partial t} - \frac{k_2}{\mu}\nabla^2 P_2 - \frac{\alpha}{\mu}(P_1 - P_2) = 0. \qquad (2.4)$$

This version of the model was first considered by Warren and Root [50] and it is the form of the model most often treated in the literature.

Equations (2.3) and (2.4) are degenerate forms of the parabolic system of equations (2.1). An additional degenerate form of (2.1) may be derived by invoking assumption (2) only, or allowing for some (small) lateral flow the matrix also, and we then have the system

$$\varsigma_1 \frac{\partial P_1}{\partial t} - \frac{k_1}{\mu} \nabla^2 P_1 + \frac{\alpha}{\mu}(P_1 - P_2) = 0$$

$$- \frac{k_2}{\mu} \nabla^2 P_2 - \frac{\alpha}{\mu}(P_1 - P_2) = 0 \qquad (2.5)$$

This version of the model, which we call the "dual permeability model", does not usually appear in the literature. Systems (2.3), (2.4) and (2.5), being derived from (2.1) by omitting at least one of the above-mentioned two terms in the equations, are collectively considered here as fractured media models of the dual porosity/permeability type. The original (parabolic) system (2.1) represents simply two interacting diffusion systems, and parabolic systems in general are treated in e.g. [23].

The models above are completed by adjoining suitable initial and boundary conditions with the equations. Typically an external source or sink is applied to the fracture system and given by

$$\lim_{r \to 0} \int_{\Omega} \frac{k_2}{\mu} \nabla P_2 \cdot d\vec{s} = Q , \qquad (2.6)$$

where r is the radius of the closed surface or curve surrounding the source point (x_o), and Q, the source strength, is usually a constant ($q = q_o U_+(t)$). In most cases the reservoir is also assumed to be confined and the no-flow conditions applied at boundaries;

$$\nabla P_i \cdot \vec{n} \Big|_{\Sigma} = 0 \qquad (i = 1, 2) , \qquad (2.7)$$

where $\vec{n}$ is the unit vector normal to the boundary $\sum$. Many models in the literature assume reservoirs in the form of the infinite, confined horizontal layer of constant thickness (H); the space domain is thus the infinite horizontal plane, usually radially symmetric, and the boundary condition (2.7) does not appear explicitly. Solutions to these cases are, with the source condition (2.6) and Q constant, analogous to the Theis solution (2.4) discussed in Chapter 1.

Several variants of mostly the basic model (2.4) above have, as we noted, been treated in the literature. The differences among them are usually minor only and there are only minor differences among the solutions. In some of the cases there are minor modifications in the formation of the basic equations; for example, "transient" forms of interporosity flow are sometimes used instead of the usual quasi-steady form given by (2.2). Some studies consider multi-layered systems rather than the cube model forms with slight modifications of the basic equations. Other studies approach double porosity models as finite element systems composed of finite sets of matrix blocks separated by regular fracture space. The solutions have the same basic characteristics as the solutions given in Chapter 4 to the systems (2.3) and (2.4).

D. Scaling of the Equations

We may justify the neglect of terms and the simplification of equations (2.1) above by more rigorous arguments based on scaling as follows: Let P, L and T be some characteristic values of pressure,

length and time for a given problem of flow, and define the variables p_1, p_2, x, and t by

$$p_1 = P p_1^*, \quad p_2 = P p_2^*, \quad x = L x^*, \quad t = T t^*,$$

where the "starred" variables represent dimensionless quantities. We may then rewrite the system of equations (2.1) in terms of these variables as

$$\frac{\partial p_1^*}{\partial t^*} - \frac{k_1 T}{\mu c_1 L^2} \nabla^{*2} p_i^* + \frac{\alpha T}{\mu c_1}(p_1^* - p_2^*) = 0,$$

$$\frac{\partial p_2^*}{\partial t^*} - \frac{k_2 T}{\mu c_2 L^2} \nabla^{*2} p_2^* - \frac{\alpha T}{\mu c_2}(p_1^* - p_2^*) = 0, \quad (2.8)$$

where $\nabla^2 = \nabla^{*2}/L^2$ and we have canceled out P. The relative magnitudes of terms in equation (2.8) are determined by the magnitudes of the "consolidation coefficients", or the values of the various coefficients and characteristic values. It is usual to assume that dimensionless variables are of order one (although a wide range of values may occur; see e.g. Figure 2.1). It is common practice to consider terms as "negligible" and to omit them from equations if they are smaller than other terms by orders of three or more. Thus, with the first terms in (2.8) of order one, the order of the others is determined by the magnitudes of $k_i T/\mu c_i L^2$ and $T\alpha/\mu c_i$ (i=1,2).

Systems of equations (2.3) and (2.4) imply the following relative magnitudes:

(i) $\quad \dfrac{\alpha T}{\mu c_1} \sim 1,$

(ii) $\quad 1 \leq \dfrac{k_2 T}{\mu c_2 L^2} \sim \dfrac{\alpha T}{\mu c_2},$

(iii) $\quad \dfrac{k_1 T}{\mu c_1 L^2} \ll 1.$

From (i) and (ii) we have

(iv) $$\frac{k_2 T}{\mu c_1 L^2} \approx 1$$

and
$$c_2 \lesssim c_1.$$

Inequality (iii) is the basic assumption of both models (2.3) and (2.4) that $k_1 \ll k_2$. (Indeed, for rock masses such as basalts we should have $k_1 \ll k_2$ by at least three orders of magnitude.) If also $c_2 \ll c_1$ by three orders or more we have the system (2.3) (the Barenblatt model), but if $c_2 \leq c_1$ by less than three orders system (2.4) (the Warren-Root model) is obtained. Systems (2.3) and (2.4) both assume that the parameter α is of an order such that (i) holds.

For an impression of realistic orders of magnitudes of the various quantities and coefficients we may consider the following example. Choosing a length scale of, say, 5000 meters and the following parameter values for flow in a basalt formation (ref. [4]):

$$k_2 = 10 \text{ md} \approx 10^{-10} \text{ cm}^2,$$
$$\phi_1 = 10^{-3}$$
$$c_1 = 4.5*10^{-11}/\text{dyne}$$
$$\mu = 3*10^{-3} \text{ cm}^2/\text{sec*gm}$$

We approximate $c_1 \approx \phi_1 c_1$ and obtain from (iv) the value

$$T \approx \mu c_1 L^2/k_2 = 3.5*10^5 \text{ sec} \approx 4 \text{ days},$$

and (from (i))

$$\alpha \approx \mu c_1/T = 4.5*10^{-22}.$$

One may attempt to estimate (theoretically) the order of magnitude of the coefficient α on the basis of physical considerations as follows: Let σ be the fracture surface area per unit volume of the reservoir and let ∇P_m be characteristic value of small-scale pressure gradients across matrix-fracture interfaces. Then we may write for the interporosity flow a characteristic value given by

$$q_c = k_1/\mu \ \nabla P_m \ \sigma .$$

Taking, for simplicity, a reservoir as a cube model where ℓ is the length of the blocks we obtain, with

$$\sigma \approx \ell^{-1}$$

and

$$\nabla P_m \approx (P_1 - P_2)/\ell ,$$

the following estimate for q_c ,

$$q_c \approx \frac{k_1}{\mu}(P_1 - P_2)/\ell^2 . \tag{2.9}$$

From (2.2) and (2.9) we have

$$\alpha \approx k_1/\ell^2 .$$

If we assume a block length of 10 m and a matrix permeability of 10^{-3} md ($\approx 10^{-14}$ cm^2) we obtain an estimated value of

$$\alpha \approx 10^{-14}/10^6 = 10^{-20} .$$

The double porosity model does not apply if

$$\frac{\alpha T}{\mu c_1} \ll 1 , \qquad \frac{\alpha T}{\mu c_2} \ll 1 ,$$

for then no interporosity flow would occur; the system consists then of two uncoupled diffusion equations. If, moreover,

$$\frac{k_i \, T}{\mu \, c_i \, L^2} \gg 1 \qquad\qquad (i=1,2)$$

the system reduces to two uncoupled Laplace's equations which describe, for the boundary conditions given, stationary flow processes. Finally, if all terms except the first two are of orders significantly less than one so that

$$\frac{\partial P_i}{\partial t} \approx 0$$

the solutions describe time-independent fields of pressure, $P_i(x,t) = P_i(x,o)$.

E. Basalt Formations as Double Porosity Systems

A typical basalt formation such as the Columbia River Basalts is an extensive, deep and highly fractured rock formation composed of successive layers of basalt rock separated by thin layers of sediments (scoria, tuffs, rock fragments, etc.). The basalt layers are a dark, fine-grained igneous type of rock formed as cooled lava layers from volcanic eruptions in usually the recent geologic past. These formations are typically tens of kilometers across and up to a few kilometers deep, and the individual layers are usually from 10 to 30 meters thick. These rock formations are common in regions of active crustal (e.g. sea floor) spreading such as in Iceland and the Pacific Northwest. The basalt layers contain an abundance of small to moderate-size fracture systems, including microfractures, along layer surfaces as well as large fractures which extend through the layers. For example, tabular and columnar

fracture systems with diameters in tens of centimeters are often seen on surface outcrops of the Columbia River Basalts. In addition, the formations are often transected by numerous dikes, or vertical basalt layers. Flow of groundwater may occur readily within these formations; horizontally between the basalt layers via the sediments and the small to moderate fracture systems, and vertically along large fractures and dikes. Basalt rock has a relatively low porosity, or 10^{-4} to 10^{-3} compared with values of 10^{-3} to 10^{-2} for most rocks, and thus it has a rather low storage capacity, but a fair amount of fluid may be stored in microfractures along layer boundaries and in the interbed sediments. It appears likely that these formations meet conditions required for modeling as large and deep double porosity systems, where microfractures and possibly also the rock matrix itself constitute the "matrix" and all larger fracture systems and the interbed sediments represent the "fracture system". It is reasonable to assume that the fracture systems are fairly uniformly distributed and that characteristic scales of flow, being typically in orders of several hundreds of meters or more, would exceed space scales of most of the fractures.

F. Literature Review

The study of flows in porous media began more than a hundred years ago since Henry Darcy published his book (1854) which contains results of experiments leading to his aforementioned law (equation (1.1)). The theory developed since for flows in porous media has been applied by groundwater hydrologists and petroleum engineers to modeling flows in natural rock formations and oil reservoirs. Mathematical models of

ground fluid flows have become increasingly sophisticated over the years with advances in computer technology as well as in methods for testing and measuring subsurface flows. Techniques for pressure transient testing and data analysis began to develop in the 1930's at about the time of publication of Theis' classic paper (1935). A large body of literature exists on these subjects today. Interest in the behaviors of fluids in natural fractured formations began in the 1940's when it became known that fractured reservoirs were important oil producers. Quantitative study of flows in fractured formations began in the 1950's and it was recognized that the behavior of fluids in fractured formations differed in general from behavior in homogeneous porous media. We discussed briefly special theoretical models developed for flows in fractured media earlier. For reviews on main types of theoretical models for fractured reservoirs in the literature up to the time of their publication two articles [29] [48] may be consulted. One (Gringarten, 1982, [29]) gives a broad review of the various types of theoretical models with particular reference to well testing and analysis. The other article (Streltsova-Adams, 1978, [48]) reviews a few variants of the double porosity model with a brief discussion on fracturing in natural rock and some data sets.

We discussed the concept of the double porosity model as originally proposed in the paper by Barenblatt et al [5], 1960, in which they present equations of forms (2.1) and (2.3) for flow in the highly idealized cube model described earlier. They derive analytic solutions for two special cases of their model; one is a 1-dimensional initial value problem for the fissure pressure response to a sudden pressure drop

in an infinite channel, and the other is for a drawdown test response in an infinite, radially symmetric plane, analogous to the Theis model. Their solution for the second case is identical to a solution given here for the matrix pressure response for the same model (Chapter 4). (Their formulation of the fluid source differs from that given here). They compare their double porosity model solutions with corresponding diffusion model solutions and note that significant differences between the two occur over a finite time range in the early-time end of the response, but beyond that time range the two models are asymptotically the same. (The double porosity model is thus relevant as a special model only for short-time flows, or for a finite, early-time phase of the pressure transient response.)

A few years later, in 1963, the slightly modified form of the model represented by equations (2.4) was considered by Warren and Root [50]. This model version retains, as we noted, the fissure storage term in the equations by letting c_2 be smaller than c_1 but not negligible. Other minor modifications in their model include an anisotropic fracture permeability k_2 and a small but non-zero radius of the fluid source. They give analytic solutions for the fluid pressure response in the fracture systems during single-well drawdown and buildup tests and present these solutions graphically in semilog form (Figure 2.2). They note from the solutions distinctive characteristics of their model; namely the two parallel linear segments seen in Figure 2.2. The linear features characterize, as noted before, the Theis model, and these two linear segments represent short-term and long-term asymptotic diffusion model responses. The left-most line is the short-time response and it
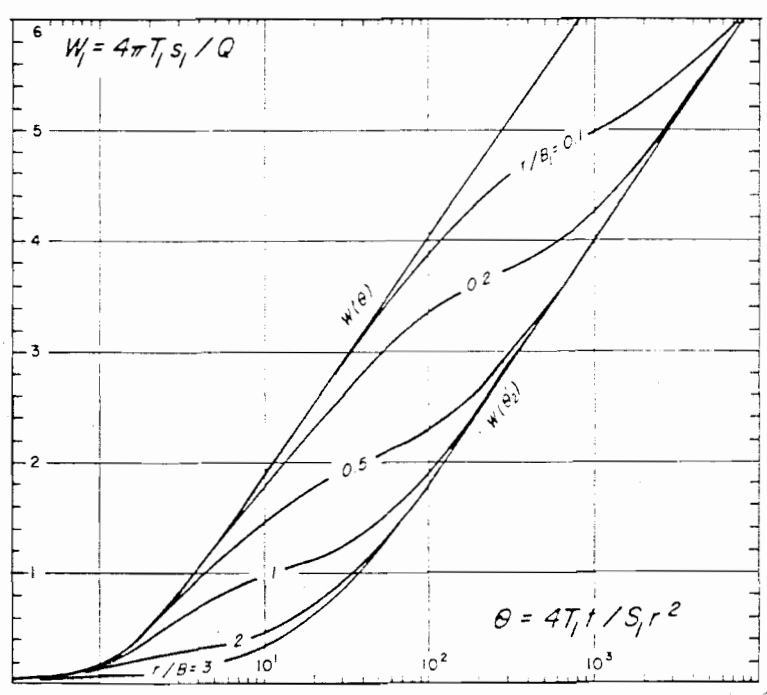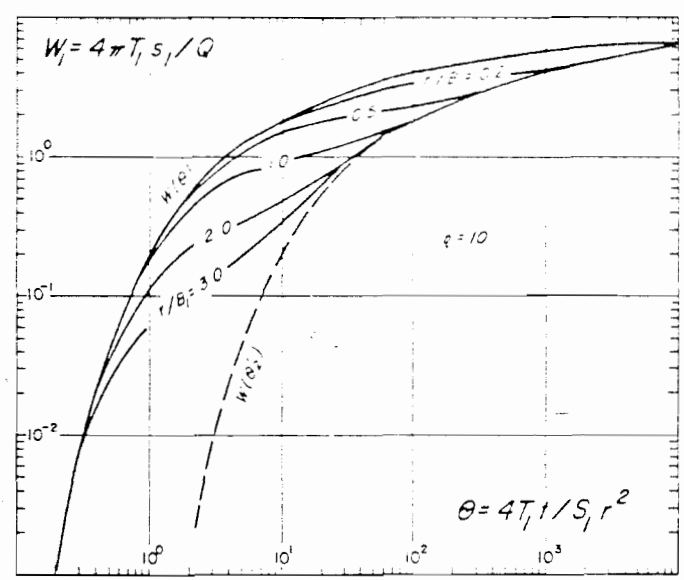
Figure 2.2   Dimensionless fracture pressure drawdown in an infinite horizontal layer with a constant-strength line source.   The multi-layered Warren-Root model after Boulton and Streltsova [10] with $c_2/(c_1+c_2)$ = .1 Top: Log-log curve; bottom: semilog curve. ($W(x)$ = $Ei(-x)$; "B_1" = k_2/α; r = distance from source) [48]

is (approximately) equal to the Theis model solution for a diffusivity of $k_2/\mu c_2$ while the right-most line is the Theis model solution for a diffusivity of $k_2/\mu(c_1+c_2)$. The short-term response represents thus a diffusion model response of the fracture system and the long-term solution describes the behavior of a composite diffusion system having properties of both the matrix and the fractures. The portion of the graph joining the two linear segments represents the effect of the interporosity flow and the exact position of this part of the graph is determined by the value of the crossflow coefficient. Warren and Root consider also a method for interpretation based on their model of data from single-well tests, and compare their model with data from a natural fractured reservoir [51].

Several studies followed on variants of this model by a number of investigators. Among these are studies by Kazemi [33] and Kazemi et al [34] on virtually the same model. The former [33] considers flow in a finite circular domain with a layered rather than a cube model structure, and solutions are obtained for both "transient" and quasi-steady forms of crossflow allowed. (Only slight differences are found in the solutions for the two different formulation of cross flow.) The second study [34] considers flow in the infinite horizontal layer with quasi-steady interporosity flow, and obtains analytic solutions, the Laplace transforms of which are identical to solutions given here (Chapter 4) for the same model. These solutions represent a flow response for interference tests. They observe the same features of the fissure pressure response as noted earlier by Warren and Root; i.e., the early- and late-time diffusion model behavior indicated by two parallel straight

lines on semilog graphs. Kazemi et al [34] discuss graphical interpre-
tation as suggested by Warren and Root extended to interference test
data. Additional studies of the same basic model include those of
Streltsova [45], De Swaan [18] and Odeh [40]. The third of these [40]
contradicts the results of Warren and Root [50] and claims that for
practical purposes no difference exists between the Warren-Root model and
the conventional diffusion model, but this conclusion was found to be due
to the manner in which the role of parameters was estimated [51].

Multi-layered forms of the double porosity model were presented by
Boulton and Streltsova [10], [11]. These are models of flow in
idealized, infinitely deep systems, composed of alternating layers of
matrix and fracture space. Most or all of the lateral flow is assumed
to occur within the fracture layers while vertical flow takes place as
interporosion flow between matrix and fractures. The flow between the
layers is modeled as (local) diffusion across the matrix-fracture
interface, proportional to the matrix permeability. These models produce
results numerically similar to those of the Warren-Root model, and they
are shown graphically in Chapter 4. A study by Duguid and Lee [21]
considers a relatively complete system of equations and obtain solutions
describing flow in a leaky aquifer under conditions including those of
typical drawdown tests. In summary, the various models above are
formulated as systems of linear equations similar to (2.2), (2.4), that
govern single-phase, usually radially symmetric flows in homogeneous
layer reservoirs, and initial and boundary conditions are given so that
the solutions describe flow responses under typical pressure transient
conditions. Analytic treatment is limited to obtaining solutions to

these systems, usually by methods of Laplace transforms together with numerical inversion or by finite difference methods. The results are usually presented graphically in either type curve or semilog forms and characteristic properties of the models are deduced from the results.

Some studies investigate the validity of the assumption of quasi-steady interporosity flow by comparing results with corresponding solutions for "transient" crossflow, in which the interporosity flow is modeled as a more realistic local diffusion across the matrix-fracture interface of individual blocks (e.g. [46], [47]). Though differences in the solutions are found to be small, they may nevertheless lead to significant error in parameter estimation [46]. Another study [26] supports the validity of the usual quasi-steady assumption by considering that fracture walls of many natural reservoirs are likely to be lined by thin mineral deposits which act as impermeable barriers to the interporosity flow and thus produce a similar effect on fluid behavior in the system as the quasi-steady approximation.

Published work in recent years by Douglas et al [19], [20] and Arbogast [1], [2] present double porosity models as finite element systems and apply numerical finite element methods to their solution. These models view the reservoir as a finite collection of spatially disjoint, regular homogeneous matrix blocks separated by regular fracture space. The flow in each matrix block is a separate system coupled to the flow in the surrounding fracture space and thereby to the flow in the adjacent matrix blocks. These models are shown to be well-posed for appropriate conditions on initial and boundary data. We also mention a paper by Hornung and Showalter [32] who present simple special cases of

the double porosity model called the "compartment model" and the "microstructure model". These models, formulated as simple systems of equations governing flow in homogeneous reservoirs, are also shown to be well-posed for suitable restrictions on initial and boundary data. These studies find that unique solutions exist in the spaces L_2 and H^1.

Observational evidence for double porosity behavior in natural fractured formations is presented in a few publications including [48], [51] cited above. The most easily recognizable feature characteristic of a double porosity system are the previously noted "two parallel straight lines" seen on semilog plots of fluid pressure versus time, and this suggest systems of the Warren-Root type. (On log-log curves the transition segment between the two parallel linear segments appears as an inflection point, as seen in Figure 2.2.) These features are, as mentioned, observed in field recovery data reported by Warren and Root [51] and in a data set by Borevsky et al (1973) [48]. A study by Sauveplane [44] reports good fits between data from fractured coal aquifers and the multilayered model of Boulton and Streltsova [10] and also with a discrete fracture model of Gringarten and Witherspoon (1972). Other reports in the literature include two studies of well interference tests from Klamath Falls, Oregon, by Benson [6] and Benson and Lai [7]. In these studies good agreement is found between the data and the Warren-Root model and also with a numerically similar composite model [7]. (The "composite model" refers to a two-part piecewise homogeneous diffusion model.) We return to further discussion of the observational evidence for the models in Chapter 5.

Little work if any appears to have been done on the behavior of fluids in basalt formations which involves applications of the double porosity models. (We may note that Klamath Falls, Oregon, contains small amounts of basalt rock underground.) Other types of theoretical models have been devised for flows in basalt formations, and we mention the "lumped ladder model" developed by Axelsson [4]. This model represents systems composed of combinations of conductance and capacitance elements which take account of the layered and fractured structure of basalt formations. This model is suited for long-time flows in these formations in which compressibility effects are negligible but free surface effects are important. Another numerically similar but much simpler analytic model for a similar long-time flows is given by Bodvarsson [8], and it describes flow of an incompressible fluid in a homogeneous semi-infinite halfspace with a free fluid surface. The characteristic time scales for flows described by these models is in order of months to years. A good fit was obtained between these models and a set of well production data from "low-temperature" basalt sites in Iceland.

Some recent publications consider the general problem of data interpretation for fractured reservoirs, of choosing proper interpretation models for such formations and of recognizing natural double porosity systems in particular (e.g. [30], [31], [37]). As mentioned, earlier special type curves have been developed for (Warren-Root) double porosity systems based on pressure derivatives (see e.g. [28]). In general, the problem of modeling flow in natural fractured formations remains at present a complex one and an area of active research.

CHAPTER 3

SOLUTIONS AS GENERALIZED FUNCTIONS

The systems of equations and their solutions studied here are treated in the sense of generalized functions, and we use the theory of generalized functions to obtain existence, uniqueness and continuous dependence of the solutions on initial data for various restrictions which may be required on the data. The theory of generalized functions is useful for doing so, for it allows us to obtain so-called uniqueness and correctness classes for these systems, or function classes in which solutions exist, are unique and depend continuously in the initial data. Moreover, we encounter functions, or fundamental solutions to systems herein, which are not ordinary functions but are meaningful only as generalized functions. Uniqueness classes define, as as we will discuss, classes of functions in which only uniqueness of the solutions is guaranteed. Since we are interested also in solutions which exist as ordinary functions we derive also the so-called correctness classes for each system; these are subclasses of the uniqueness classes which contain solutions for which existence is also guaranteed as well as continuous dependence on the initial data.

We therefore devote this chapter to a review of the basic theory of generalized functions, and we present fundamental theorems by which we obtain the two function classes above for rather general systems of parabolic equations. These theorems are then applied to the specific systems of equations which we treat in the next Chapter. The material

presented here is adapted from the texts, <u>Generalized Functions</u>, vols. 2 (1967) and 3 (1968), by Gel'fand and Shilov [26], [27].

A. Generalized Functions and Function Spaces

1.   Some Definitions

A <u>generalized function</u> is a continuous linear functional (f, φ) in some <u>fundamental</u> or <u>test function space</u> Φ, $\varphi \in \overline{\Phi}$. Generalized functions therefore depend, in contrast to "ordinary" or "classical" functions, on the selected space Φ. Many generalized functions may be given by an integral

$$(f, \phi) = \int_R f(x) \phi(x) dx, \tag{1.1}$$

where f(x) is an ordinary function, "locally integrable" in the domain R.   (Linear functionals given by (1.1) are called "regular func-functionals".) A generalized solution to a system of equations is thus understood to be a linear functional or an integral in the form (1.1). The class of all generalized functions f defined on the fundamental space Φ is the <u>conjugate</u> or <u>dual space</u>, $\overline{\Phi}'$ of Φ. A well-known example of a generalized function which is not an ordinary function is the familiar delta function $\delta(x)$. In general, for a given class Φ to be suitable as a test function space it must be nontrivial (i.e., contain at least one function $\phi(x)$ not identically zero); in fact it must be "sufficiently rich" in functions, which means that f(x) = 0 almost everywhere if the integral (1.1) is zero for all ϕ in Φ.

2.   Fundamental (Test Function) Spaces

Classes of functions used as test function spaces are the so-called K, Z, S and W spaces. These spaces consist of functions with specific properties and restrictions on behavior at infinity such that the integrals (1.1) converge. Subspaces of these classes are defined by suitable numbers which are indicated below. Briefly, these spaces are characterized as follows:

Spaces of type K   Functions in these spaces are infinitely differentiable and of bounded support. Specifically, the space K(a) consists of functions of bounded support in the domain $|x| \le a$ ($|x_i| \le a_i$, i=1,...,n for n-dimensional cases), and the space K is the union of the spaces K(a) over all a.

Spaces of type Z   Functions $\phi(x)$ in these spaces are entire, analytic and increase in the complex z=x+iy plane and increase along the y=axis as

$$z^k \phi(z) \le C \exp(b|y|)$$

(1.2)

for some constants C, b > 0 and k (integer).

Spaces of type S   These spaces consist of functions $\phi(x)$ which are infinitely differentiable and decrease with $|x|$ at infinity faster than any power of $1/|x|$. More specifically, the behaviors of $\phi(x)$ in the various subspaces of S (α, β, A, B being nonnegative constants, and k,q = 0,1,2,...) are characterized as follows:

Functions in $S_{\alpha,A}$

$$|x^k \phi^{(q)}(x)| \le C_k A^k k^{k\alpha},$$

(1.3)

$$\left| x^k \phi^{(q)}(x) \right| \leq C \exp\left(- a |x|^{\frac{1}{\alpha}}\right) , \tag{1.4}$$

where

$$a \leq \frac{\alpha}{e A^{\frac{1}{\alpha}}} . \tag{1.5}$$

(if $\alpha = 0$ $\phi(x)$ have bounded support and the space coincides with spaces of K.)

Functions in $S^{\beta, B}$

$$\left| \phi^{(q)}(x) \right| \leq C_q B^{(q)} q^{q\beta} . \tag{1.6}$$

Functions in this space can be continued analytically into the $z = x + iy$ plane where the inequality holds;

$$\left| x^k \phi(z) \right| \leq C \exp\left(b |y|^{\frac{1}{1-\beta}}\right) , \tag{1.7}$$

where

$$b \geq \frac{1-\beta}{e}(B e)^{\frac{1}{1-\beta}} \qquad (\beta < 1) . \tag{1.8}$$

Functions in $S_{\alpha, A}^{\beta, B}$

$$\left| x^k \phi^{(q)}(x) \right| \leq C A^k B^{(q)} k^{k\alpha} q^{q\beta} , \tag{1.9}$$

and these functions may be continued analytically into the z-plane where they satisfy

$$\left| \phi(z) \right| \leq C \exp\left(- a |x|^{\frac{1}{\alpha}} + b |y|^{\frac{1}{1-\beta}}\right) \tag{1.10}$$

with a and b given by (1.5) and (1.8), respectively.

We note that the smaller the values of α, β, A, B the greater the restrictions on the growth or decrease of the functions at infinity and

the smaller the function classes above. (In general, the smaller the test function space, the larger the space of generalized functions defined on that test function space.)

The following imbeddings apply for the S spaces:

$$S_\alpha = S_\alpha^\infty = \bigcup_A S_{\alpha,A} \quad , \qquad S^\beta = S_\infty^\beta = \bigcup_B S^{\beta,B}$$

$$S_\alpha^\beta = \bigcup_{A,B} S_{\alpha,A}^{\beta,B} \quad , \qquad S = S_\infty^\infty = \bigcup_{\alpha,\beta,A,B} S_{\alpha,A}^{\beta,B} \quad .$$

For the spaces above to be nontrivial we have the following restrictions on the constants α, β, A, B :

$$S_{\alpha,A} \, , \qquad S^{\beta,B} \qquad \text{for any} \quad \alpha \geq 0, \quad \beta \geq 0 , \qquad A, B;$$

$$S_{\alpha,A}^{0,B} \, , \qquad S_{0,A}^{\beta,B} \qquad \text{for any} \quad \alpha > 1, \quad \beta > 1 , \qquad A, B;$$

$$S_{\alpha,A}^{\beta,B} \qquad \text{for } \alpha + \beta > 1 \qquad \text{and any} \qquad A,B;$$

$$S_{\alpha,A}^{\beta,B} \qquad \text{for } \alpha + \beta = 1, \quad AB = \gamma , \text{ where } \gamma \text{ is any positive number.}$$

The spaces $S_{0,A}^{\beta,\beta}$ and $S_{\alpha,A}^{0,B}$ are, though nontrivial, not "sufficiently rich in functions", but the unions of these spaces,

$$S_0^{\beta,B} = \bigcup_A S_{0,A}^{\beta,B} \qquad \text{and} \qquad S_{\alpha,A}^0 = \bigcup_B S_{\alpha,A}^{0,B} , \text{ are.}$$

<u>Spaces of type W</u>   Functions in these spaces are entire analytic functions with the following exponential behaviors along x and y:

<u>In W_M</u> : $$|\phi(x)| \leq C \exp(-M(x)) ,$$

where M(x) is a non-negative convex function of x;

<u>In W^Ω</u> : $$|\phi(x)| \leq C \exp(\Omega(y)) ,$$

where   is a non-negative convex function of y;

$\underline{\text{In } W_M^{\Omega}}$: $\qquad |\varphi(x)| \le C \exp(-M(x) + \Omega(y)),$

$\underline{\text{In } W_{M,a}^{\Omega,b}}$: $\qquad |\varphi(x)| \le C \exp(-M(ax) + \Omega(by)).$

3. Generalized Functions on Fundamental Spaces

The class of generalized functions Φ' defined on the given test function space Φ requires, as mentioned, that for each $f(x)$ in Φ' and $\phi(x)$ in Φ, the integral

$$(f, \phi) = \int f(x)\phi(x)dx$$

converges (absolutely). This condition is automatically met for all locally integrable functions $f(x)$ on spaces of type K, since these consist of functions of bounded support. For other types of spaces, for example spaces of type S, it is necessary that the functions $f(x)$ not increase too fast with $|x|$; in fact $f(x)$ should not increase faster than a power of x at infinity. Similar rules hold for spaces Z and W.

B. Operations on Test Function Spaces

A number of operations (linear, continuous) are defined on the fundamental spaces Φ that map the space into itself or into some other space Φ_1. That is, operations on a test function $\phi(x)$ results in a function $\psi(x)$ which belongs either to Φ or to some other space Φ_1. Rules of linearity hold, i.e., if A and B are operators which transform the space Φ into Φ_1 then for any $\phi(x)$ in Φ and number λ,

$$(A+B)\,\phi(x) = A\,\phi(x) + B\,\phi(x)$$

and $\qquad\qquad\qquad (\lambda A)\,\phi(x) = \lambda A\,\phi(x) .$

In particular, functions in the test function spaces may be added or multiplied and the result belongs again to the same space. Some of the

simplest and most common operations on test function spaces of importance in analysis are, however, multiplication by the independent variable x, (or by any finite power of x), multiplication by (other) functions of x, differentiation and Fourier transformation. We list briefly below the results of these operations on spaces of type S (K and Z are subsets of S) and spaces W.

1. Operations on Spaces S

Multiplication by x. This operation, or multiplication by any polynomial P(x) of finite degree, on a function $\phi(x)$ belonging to e.g. $S_{\alpha,A}^{\beta,B}$ results in a function again in $S_{\alpha,A}^{\beta,B}$. (The functions x and P(x) are thus multipliers in the space $S_{\alpha,A}^{\beta,\beta}$.)

Differentiation (with respect to x). The derivative $\frac{d}{dx}\phi(x)$ (or $\frac{d^n}{dx^n}\phi(x)$) up to any finite order n) belongs, for any $\phi(x)$ in $S_{\alpha,A}^{\beta,\beta}$, again in $S_{\alpha,A}^{\beta,\beta}$.

Multiplication by infinitely differentiable functions. Consider the space $S_{\alpha,A}$ and a function f(x) satisfying the inequality

$$|f(x)| \leq C \exp\left(a_1 |x|^{1/\alpha}\right) . \tag{2.1}$$

If $\alpha = 0$ then any f(x), infinitely differentiable, defines a bounded continuous operator in this space. Assume $\alpha > 0$. Functions $\phi(x)$ have the character of decrease at infinity given by (2.1) and so for f(x) to be defined on this space we must have $a_1 < a = \frac{\alpha}{e}A^{1/\alpha}$. The product $f(x)\phi(x)$ then belongs to the (broader) space $S_{\alpha,A'}$, where $a_1 - a = \frac{\alpha}{e}A'^{1/\alpha}$. If a_1 in (2.1) is replaced by ϵ for any $\epsilon > 0$ then $f(x)\phi(x)$ belongs to the same space $S_{\alpha,A}$.

Similarly f(x) satisfying (2.2) below is a bounded operation of a multiplication in the space S ,

$$|f^{(q)}(x)| \leq C B_o^q q^{q\beta} (1 + |x|^h) .$$ (2.2)

More specifically, if $\phi(x)$ belongs to the space $S^{\beta,B}$ then $f(x)\phi(x)$ belongs to $S^{\beta,B+B_o}$. If B_o, however, is replaced by ϵ for any $\epsilon > 0$ then $f(x)\phi(x)$ belongs to the same space $S^{\beta,B}$.

The function f(x) satisfying the inequality (2.3) is a bounded

$$|f^{(q)}(x)| \leq C B_o^q q^{q\beta} exp(a_c |x|^{1/\alpha})$$ (2.3)

operator of multiplication in the space S_α^β ($\alpha > 0$). Considering functions $\phi(x)$ in the space $S_{\alpha,A}^{\beta,B}$ ($\alpha > 0$) with $a_o < a = \frac{\alpha}{\ell} A^{1/\alpha}$ the multiplication by f(x) maps the space $S_{\alpha,A}^{\beta,B}$ into the space $S_{\alpha,A'}^{\beta,B'}$ where $a-a_o = \frac{\alpha}{\ell} A^{1/\alpha}$ and $B' = B+B_o$. If for any $\epsilon > 0$, however, f(x) satisfies

$$|f^{(q)}(x)| \leq C B_o^q q^{q\beta} exp (\epsilon |x|^{1/\alpha}) ,$$ (2.4)

then f(x) maps $S_{\alpha,A}^{\beta,B}$ into $S_{\alpha,A}^{\beta,B+B_o}$, and if f(x) satisfies

$$|f^{(q)}(x)| \leq C \epsilon^q q^{q\beta} exp (\epsilon|x|^{1/\alpha})$$ (2.5)

the f(x) transforms $S_{\alpha,A}^{\beta,B}$ into itself. If, for $|x| < A$, f(x) satisfies

$$|f^{(q)}(x)| \leq C B_o^q q^{q\beta}$$ (2.6)

then f(x) transforms the space $S_{c,A}^{\beta,B}$ into $S_{\varnothing,A}^{\beta,B+B_o}$, and if B_o is replaced by $\in$ in (1.10) for any $\epsilon > 0$ then f(x) maps $S_{c,A}^{\beta,B}$ into itself.

2.   Multiplication by Entire Analytic Functions (Multipliers)

Consider the spaces $S_{\alpha,A}^{\beta,B}$ of S_α^β which consist, as we recall, of entire analytic functions $\phi(z)$ which satisfy

$$|\phi(x+iy)| \leq C \exp\left(-a|x|^{1/\alpha} + b|y|^{\frac{1}{1-\beta}}\right),$$

where $\qquad a = \dfrac{\alpha}{e A^{1/\alpha}} \qquad$ and $\qquad b \geq \dfrac{1-\beta}{e}(B e)^{\frac{1}{1-\beta}}$

Let an entire analytic function $f(z)$ satisfy

$$|f(x+iy)| \leq C \exp\left(a_1|x|^h + b_1|y|^\gamma\right).$$

The function $f(x)$ is a multiplier in the space S_α^β for $\alpha = 1/h$ and $\beta = 1-1/\gamma$; for $a_1 < 0$ $f(x)$ is also an element in this space. Moreover, if

$$|f(x+iy)| \leq C(1+|x|^h)\exp\left(b|y|^\gamma\right)$$

$f(x)$ is a multiplier in the space $S^{1-1/\gamma}$.

3.   Operations on Spaces W

Multiplication by z   of functions $\phi(z)$ in W^Ω results in a function also in W^Ω.

Differentiation   with respect to z of any function in W_M^Ω results in a function again in W_M^Ω.

Multiplication by entire analytic functions. If an entire analytic function $f(z)$ satisfies

$$|f(z)| \leq C \exp\left(\Omega(b_0 y)\right)(1+|x|^h)$$

then it is a multiplier in the space W^{Ω} and takes the space $W^{\Omega,b}$, into the space $W^{\Omega,b+b_0}$. If f(z) satisfies

$$|f(z)| \leq C \exp\left(M(a_c x) + \Omega(b_o y)\right)$$

then it is a bounded multiplication operator on $W_{M,a}^{\Omega,b}$ for a > a_0 and takes the space $W_{M,a}^{\Omega,b}$ into the space $W_{M,a-a_0}^{\Omega,b+b_0}$. If, on the other hand, f(z) satisfies

$$|f(z)| \leq C \exp\left(\Omega(\epsilon y)\right)$$

or

$$|f(z)| \leq C \exp\left(M(\epsilon x) + \Omega(\epsilon y)\right)$$

for any $\epsilon > 0$, then f(z) is a multiplier in $W^{\Omega,b}$ or $W_{M,a}^{\Omega,b}$, respectively.

4.   Operations on S by Infinite Order Differential Operators

We list the result of this operation on spaces S by the theorem (3.1) below which we also prove. This theorem will be useful later.

Let f(s) be the entire analytic function (in the complex variable s = σ + i τ) given by

$$f(s) = \sum_{\nu=0}^{\infty} c_\nu s^\nu ,$$

and   consider the differential operator

$$f\left(\frac{d}{dx}\right) = \sum_{\nu=0}^{\infty} c_\nu \frac{d^\nu}{dx^\nu} .$$

This operator is defined on some fundamental space $\overline{\Phi}$ if for any $\phi(x)$ in Φ the series

$$f\left(\frac{d}{dx}\right)\phi(x) = \sum_{\nu=0}^{\infty} c_\nu \phi^{(\nu)}(x)$$

is again a function in Φ or some other space Ψ. For certain constraints on the growth of the function f(s) the operator f(d/dx) is defined and bounded on the space Φ, and for spaces Φ of type S the growth of f(s) is related to the numbers α, β which define the space.

We say that the function f(s) has an _order of growth_ $\leq \lambda$ and _type_ < b if the following inequality holds;

$$|f(s)| \leq C \, exp\left(b_1 |s|^{\lambda}\right) ,$$

(2.7)

where $b_1 < b$ is some constant. We state the following theorem: (This theorem will be used in later discussions.)

__Theorem 3.1__ If $f(s) = \sum\limits_0^\infty c_\nu s^\nu$ is an entire analytic function of order

of growth $\leq 1/\beta$ and type $< \beta/B^{1/\beta} e^2$, then the operator f(d/dx) is defined and bounded in the space $S^{\beta,B}$ and transforms this space into the space S^{β, Be^β}

__Proof__. We recall that functions $\phi(x)$ in $S^{\beta,B}$ satisfy

$$|x^k \phi^{(q)}(x)| \leq C_{\rho k} (B + \rho)^q q^{q\beta}$$

(for any arbitrary $\rho > 0$). Let be given

$$\psi(x) = f\left(\frac{d}{dx}\right)\phi(x) = \sum_o^\infty c_\nu \phi^{(\nu)}(x) .$$

We show that the function $\psi(x)$ belongs to S^{β, Be^β}. We have the inequality for the function $\psi(x)$:

$$|x^k \psi^{(q)}(x)| \leq \left| \sum_\iota^\infty c_\nu x^k \phi^{(\nu + q)}(x)\right|$$

$$\leq C_{k\rho}(\beta+\rho)^{q} \sum |c_{\nu}| (\beta+\rho)^{\nu} (\nu+q)^{(\nu+q)\beta}$$

$$\leq C_{k\rho}(\beta+\rho)^{q} q^{\beta q} \sum |c_{\nu}| (\beta+\rho)^{\nu} \frac{(\nu+q)^{(\nu+q)\beta}}{q^{\beta q}}$$

$$\leq C_{k\rho}(\beta+\rho)^{q} q^{q\beta} \sum |c_{\nu}| (\beta+\rho)^{\nu} (\nu+q)^{\nu\beta} (1+\nu/q)^{q\beta}$$

$$\leq C_{k\rho}(\beta+\rho)^{q} q^{q\beta} \sum |c_{\nu}| (\beta+\rho)^{\nu} \nu^{\nu\beta} (1+q/\nu)^{\nu\beta} (1+\nu/q)^{q\beta}$$

$$\leq C_{k\rho}(\beta+\rho)^{q} q^{q\beta} e^{q\beta} \sum |c_{\nu}| (\beta+\rho)^{\nu} \nu^{\nu\beta} e^{\nu\beta}. \tag{2.8}$$

We may obtain, for the function f(s) of the growth order and type given, the upper bound on the coefficients $|c_{\nu}|$ given by (for proof see below):

$$|c_{\nu}| \leq C \frac{\theta^{\nu}}{B^{\nu} e^{\nu\beta} \nu^{\nu\beta}}, \tag{2.9}$$

where $\theta < 1$ is some positive number, C a constant. If, moreover, $1 + \rho/B < 1/\theta$ (ρ sufficiently small) the series (2.8) will converge and we finally obtain

$$|x^{k} \psi^{(q)}(x)| \leq C'_{k\rho}(Be^{\beta}+\rho e^{\beta}) q^{q\beta},$$

where $C'_{k\rho} = C_{k\rho} C$. Thus the function $\psi(x) = f(\frac{d}{dx})\phi(x)$ belongs to the space $S^{\beta, Be^{\beta}}$, as we wanted to show. ∎

For the proof of inequality (2.9) we have by the Cauchy formula

$$c_{\nu} = \frac{1}{2\pi i} \int_{|s|=r} \frac{f(s)}{s^{\nu+1}} ds$$

and the inequality (2.7) that

$$|c_\nu| \le C \; \frac{\exp(b_1 r^\lambda)}{r^\nu} \; ,$$

(2.10)

To find the minimum on the right hand side with respect to r, we take logarithm, differentiate and equate to zero. This gives for a minimum $r = r_0$

$$b_1 \lambda \, r_0^{\lambda-1} - \nu/r_0 = 0 \; ,$$

from which

$$r_0 = \left(\frac{\nu}{b_1 \lambda}\right)^{1/\lambda} \; .$$

Substituting this value of r_0 into (2.10) we find

$$|c_\nu| \le C \left(\frac{b_1 e \lambda}{\nu}\right)^{\lambda/\nu} \; .$$

(2.11)

Using $\lambda = 1/\beta$, $b_1 = \beta/B^{1/\beta}e^2$ in (2.11) we obtain the inequality (2.9) above.

5. Fourier Transformations

The Fourier transform $\hat{\phi}(\sigma)$ of the test function $\phi(x)$ in Φ is given by the linear functional

$$\hat{\phi}(\sigma) = F(\phi(x)) = \frac{1}{(2\pi)^{n/2}} \int \phi(x) \exp(i\sigma x) dx$$

and belongs to the space $\Psi = F(\Phi)$. Similarly, the inverse (Fourier) transform

$$\phi(x) = F^{-1}(\hat{\phi}(\sigma)) = \frac{1}{(2\pi)^{n/2}} \int \hat{\phi}(\sigma) \exp(-i\sigma x) d\sigma \; .$$

of $\hat{\phi}(\sigma)$ in Ψ is again in $\Phi = F^{-1}(\Psi)$. Transforms of partial derivatives (or finite-degree polynomials of partial derivatives) of functions may be given by

$$P(D)\,\hat{\phi}(\sigma) = F\left(P(ix)\phi(x)\right),$$
$$F\left(P(D)\phi(x)\right) = P(-i\sigma)\,\hat{\phi}(\sigma),$$
$$F^{-1}\left(P(D)\hat{\phi}(\sigma)\right) = P(ix)\,\phi(x),$$
$$P(D)\phi(x) = F^{-1}\left(P(-i\sigma)\hat{\phi}(\sigma)\right).$$

Fourier transforms and inverse transforms are linear continuous one-to-one mappings (isomorphisms).

We list transforms of the fundamental spaces as follows:

$$F(K(a)) = Z(a), \qquad F(Z(a)) = K(a),$$
$$F(K) = Z, \qquad\qquad F(Z) = K,$$
$$F(S_{\alpha,A}^{\beta,B}) = F^{-1}(S_{\alpha,A}^{\beta,B}) = S_{\beta,B}^{\alpha,A},$$
$$F(S_{\alpha}^{\beta}) = F^{-1}(S_{\alpha}^{\beta}) = S_{\beta}^{\alpha},$$
$$F(S) = F^{-1}(S) = S.$$

For transforms of spaces of type W let $M(x)$ and $\Omega(y)$ be non-negative convex functions which are dual "in the sense of Young", i.e., they satisfy the inequality

$$xy \leq M(x) + \Omega(y).$$

Then the transforms of spaces of type W are

$$F(W_M) = F^{-1}(W_M) = W^{\Omega},$$
$$F(W^{\Omega}) = F^{-1}(W^{\Omega}) = W_M.$$

Given M(x) dual (in the sense of Young) to $\Omega_1(y)$ and $M_1(x)$ dual to $\Omega(y)$ we have

$$F\left(W_M^{\Omega}\right) = F^{-1}\left(W_M^{\Omega}\right) = W_{M_1}^{\Omega_1} .$$

C. Operations With Generalized Functions

Continuous linear operations may similarly be performed on generalized functions in the space Φ' defined on test function space Φ, and the results of these operations on Φ' are again elements in Φ'. For example, we may, as simple common operations, apply differentiation, multiplication by functions and Fourier transformation on generalized functions. In general, we may define an operation by an operator A on a generalized function f(x) in terms of the operation on $\phi(x)$ by its conjugate A* ;

$$\left(Af, \phi\right) = \left(f, A^* \phi\right).$$

$$(3.1)$$

In the case of the operation by the differential operator A = d/dx, we have

$$\left(\frac{df}{dx}, \phi\right) = \left(f, -\frac{df}{dx}\right),$$

and multiplication by a function g(x) which is a multiplier in the space Φ,

$$\left(gf, \phi\right) = \left(f, g\phi\right),$$

or, more generally, for differentiation by a finite order polynomial P(D), D = $\partial/\partial x$, $\left(P(D)f, \phi\right) = \left(f, P(-D)\phi\right).$

The following relations hold between (generalized) functions and their Fourier transforms (in n-dimensional space) ;

$$\left(g, \psi\right) = \left(F(f), F(\phi)\right) = \left(2\pi\right)^n \left(f, \phi\right),$$

$$\left(f, \phi\right) = \left(F^{-1}(g), F^{-1}(\psi)\right) = \left(2\pi\right)^{-n} \left(g, \psi\right).$$

Moreover, we have the symmetrical relations between differentiation and multiplication by independent variables $(g = F(f))$;

$$F(P(D)f) = P(-i\sigma)F(f),$$
$$F^{-1}(P(D)g) = P(ix)F^{-1}(g),$$
$$P(D)F(f) = F(P(ix)f),$$
$$P(D)F^{-1}(g) = F^{-1}(P(-i\sigma)g).$$

In particular,

$$F(\delta) = 1, \qquad F^{-1}(\delta) = 1/(2\pi)^n,$$
$$F(1) = \bar{\delta}(x)/(2\pi)^n,$$
$$F(x^k) = (-iD)^k \delta(\sigma),$$
$$F(P(x)) = P(-iD)\delta(\sigma).$$

D. Existence and Uniqueness of Generalized Solutions

We now consider solutions to systems of linear partial differential equations in the sense of generalized functions, and we treat, in particular, their existence and uniqueness. In the remainder of this chapter we develop theorems on uniqueness and correctness classes (defined below) which are applicable to solutions to certain general systems of parabolic equations. Special cases of these theorems are derived in Chapter 4 which are applicable to the specific systems of equations treated there. In Section 1 we develop a fundamental theorem

on uniqueness classes, and in Sections 4 and 5 we present fundamental theorems for two classes of systems separately; so-called hyperbolic and parabolic systems. A number of additional theorem used in the proofs of these theorems are also presented. In addition we consider also briefly applications of the fundamental theorems on uniqueness classes for ordinary solutions (Section 2).

The above-mentioned function classes are defined as follows:

A <u>uniqueness class</u> is a (linear topological) space of (ordinary or generalized) functions of the argument x, for which the uniqueness of the solution to the Cauchy problem is guaranteed for given initial conditions, provided the solution exists.

A <u>correctness class</u> is the totality of <u>ordinary</u> functions of the argument x, for which the existence of the solution to the Cauchy problem is guaranteed for arbitrary initial conditions (again within a given class of functions) as well as its uniqueness and continuous dependence of the solution on the initial conditions.

As mentioned before a function f(x) has <u>order of growth</u> p and <u>type</u> b if it satisfies the inequality

$$|f(x)| \leq C \exp(b|x|^p) \tag{4.1}$$

for some constant C.

The <u>order</u> (p) of a given system of equations is (the same as) the order of the resolvent matrix/function for that system;

The <u>reduced order</u> ($p_0 \leq p$) of the smallest value of the order (p) of the system.

An initial value problem, or a system of equations together with initial conditions, is <u>always</u> <u>solvable</u> in a given space of functions if for some t_o in an interval $0 \le t_o \le T$ and each initial condition ϕ_o there exists a solution $\phi(t)$ defined for all $0 \le t \le T$ which for $t=t_o$ becomes equal to the initial function ϕ_o and is such that $\phi(t)$ depends linearly and continuously on ϕ_o.

1. Uniqueness Classes

Consider the following general system of parabolic equations and initial conditions, which we write in the form

$$\frac{\partial u(x,t)}{\partial t} = - A^*\left(i\frac{\partial}{\partial x}\right)u(x,t) ,$$

$$u(x,0) = u_o(x) .$$

(4.2)

In the general case $u(x,t)$, the dependent variable, is an m-component vector $(u_j(x,t), j=1,\ldots m)$, the operator $-A_t^*(i\partial/\partial x) = P(i\partial/\partial x)$ is an mxm matrix of polynomials in the differential operator $i\partial/\partial x$ and x is in general an n-component vector $(x_1,x_2,\ldots,x_n)$. The (generalized) function $u(x,t)$ is in the space Φ' of functions conjugate to some test function space Φ. Let us write also the corresponding initial value problem for functions $\phi(x,t)$ in the space Φ; i.e.,

$$\frac{\partial \phi(x,t)}{\partial t} = A\left(i\frac{\partial}{\partial x}\right)\phi(x,t),$$

$$\phi(x,t_o) = \phi_o(x),$$

(4.3)

where, similarly, $\phi(x,t)$ is an m-component vector and A_t $(i\partial/\partial x)$, the conjugate operator, is an mxm matrix of polynomials P $(i\partial/\partial x)$ in $i\partial/\partial x$. Both t_o and t are assumed to lie in the finite interval $0 \leq t \leq t_o \leq T$. The solution to system (4.2) in Φ', it turns out, is closely related to the solution to system (4.3). Let the solution to system (4.3) be given by

$$\phi(x,t) = Q\left(i\frac{\partial}{\partial x}, t_o, t\right)\phi_o(x),$$

$$(4.4)$$

where

$$Q\left(i\frac{\partial}{\partial x}, t_o, t\right) = exp\left(P\left(i\frac{\partial}{\partial x}\right)(t-t_o)\right).$$

$$(4.5)$$

The following theorem holds:

Theorem 3.2. If the Cauchy problem (4.3) is always solvable in the test function space Φ, then the corresponding Cauchy problem (4.2) in the conjugate space Φ' has a solution $u(x,t)$ for any initial functional $u_o(x)$; this solution is unique in Φ' and depends continuously on $u_o(x)$ in the sense of the topology of the space $\overset{\star}{\Phi}{}'$.

Proof. We show that the only solution $u(t)$ to system (4.2) given the initial condition $u_o \equiv 0$ is $u(t) \equiv 0$. (This implies uniqueness of the solution to system (4.2), for $u_o \equiv 0$ is the initial condition for the difference $u_1(t)-u_2(t) = u(t)$ between any two solutions $u_1(t)$ and $u_2(t)$ to system (4.2).) Differentiating the linear functional $(u(t), \phi(t))$ with respect to t and using equations (4.2, (4.3) and (3.1) we obtain

$$\frac{d}{dt}\left(u(t),\phi(t)\right) = \left(\frac{\partial u(t)}{\partial t},\phi(t)\right) + \left(u(t),\frac{\partial \phi(t)}{\partial t}\right)$$

$$= \left(-A_t^* \, u(t), \phi(t)\right) + \left(u(t), A_t \, \phi(t)\right)$$

$$= \left(-A_t^* \, u(t), \phi(t)\right) + \left(A_t^* \, u(t), \phi(t)\right) = 0.$$

Thus we have that $(u(t), \phi(t))$ is constant and equal to the functional $(u(t_o), \phi_o)$. By setting $u_o(0) = 0$ we obtain that $(u(t_o), \phi_o) = 0$ for all t_o. Since ϕ_o is arbitrary we have that $u_o(t_o) = 0$ for all t_o, and hence, by arbitrary choice of t_o and also of t in the interval $0 \le t \le t_o \le T$, we obtain that $u(t) = 0$ for all t in $0 \le t \le T$. The solution $u(t)$, which may be formally written as

$$u(t) = Q^*(0, t) u_o,$$

depends continuously on the initial function u_o due to the continuity of $Q^*(0,t)$. ∎

Let us also state the following theorem:

<u>Theorem 3.3</u>. Let for any t_o and t in $0 \le t \le t_o \le T$ a linear continuous operator $Q(t_o,t)$ be defined on the space Φ which maps this space into the space Φ_1 and, in addition, if applied to an arbitrary vector ϕ_o (in Φ), yields a solution to the Cauchy problem (4.3), where ϕ_o is in Φ and $\phi(t)$ is in Φ_1. Then the Cauchy problem (4.2) admits in the interval $0 \le t \le T$ the unique solution $u(t)$ in E', where E' is the space conjugate to the space E, and E contains Φ and Φ_1 as dense subsets.

<u>Proof</u>. We may obtain uniqueness of the solution to system (4.2) in the same manner as in the proof of the preceeding theorem. That is, with the

67

initial condition $u_o(0) = 0$ we obtain that $(u_o(t_o), \phi_o) = 0$ so $u_o(t_o)$

vanishes on the space Φ; t_o is arbitrary so $(u(t), \phi(t)) = 0$ and $u(t)$

vanishes on the space Φ_1. The functional $u_o(t_o)$ is by assumption defined

on all of E, $\Phi \subset \Phi_1$, $\subset E$ and Φ is dense in E, so $u_o(t_o) = u(t) = 0$ in E'. ∎

Let us return to the systems of equations (4.2) and (4.3). We note

that the elements Q_{ij} $(i\partial/\partial x, t_o, t)$ of the matrix/function (4.5) are

entire analytic functions in the arguments $\partial/\partial x_1, \cdots, \partial/\partial x_n$. Or,

alternatively, Q_{ij} (s, t_o, t) are entire analytic functions of the complex

variable $s = \sigma + i\tau = s_1, .., s_n$. The matrix/function

$$Q(s, t_o, t) = \exp\left((t - t_o) P(s)\right)$$

is the "resolvent" for system (4.3), obtained upon Fourier transforma-

tion of (4.3), in the complex variable $s = \sigma + i\tau$. That (Fourier

transformed) system has the solution

$$\hat{\phi}(s, t) = Q(s, t_o, t) \hat{\phi}_o(s) .$$

The norm of $Q(s, t_o, t)$ is determined by the norm of the matrix $P(s)$. We

may obtain the estimate for $\|P(s)\|$:

$$\|P(s)\|^2 \le \sum_{j=1}^{m} \sum_{k=1}^{m} |P_{ij}(s)|^2 \le C_1^2 |s|^{2p}$$

for some constant C_1, where p is the maximal order of each $P_{ij}(s)$ such

that $P_{ij}(s)| \le C|s|^p$. We then have the estimate

$$\|Q(s, t_o, t)\| = \|\exp\left((t - t_o)P(s)\right)\|$$
$$\le \exp\left((t - t_o)C_1 |s|^p\right) .$$

The elements of $Q(s,t_o,t)$ are entire analytic functions of order equal to or smaller than p. In fact the matrix/function $Q(s,t_o,t)$ may have a smaller (reduced) order $p_o \leq p$, and we may write the inequality for $Q(s,t_o,t)$ in the more general form in terms of p_o; i.e.

$$\|Q(s,t_o,t)\| \leq C_2(1+|s|)^{P(m-1)} exp(b|t-t_o||s|^{P_o}). \quad (4.6)$$

In general, the reduced order p_o characterizes the system more precisely than p (the highest order of the differential operators of $P(i\partial/\partial x)$). Indeed, the reduced order determines the uniqueness class for system (4.2). We state the following "fundamental" theorem:

Theorem 3.4 If a system of equations of the form (4.2) has a reduced order $p_o > 1$, then the totality of functions $f(x)$ satisfying the inequalities

$$|f(x)| \leq C exp(b_o|x|^{p_o'}) \quad (4.7)$$

with p_o' given by

$$\frac{1}{P_o} + \frac{1}{P_o'} = 1$$

and arbitrary but fixed C and b_o, forms a uniqueness class for the Cauchy problem (4.2). In other words, there exists at most one solution to that system, for which at $t=0$ equals the given initial (vector) function $u_o(x)$ and all components of which satisfy (4.7) for fixed t in the interval $0 \leq t \leq T$. If $p_o = 1$ then the uniqueness class consists of the totality of functions of class (4.7) with p_o' arbitrary but fixed. For $p_o < 1$ the solution to problem (4.2) is unique in the class of all functions $f(x)$ with no restrictions on their growth at infinity. (The admissible inter-

val of the time parameter $0 \le t \le T$ depends only on the constants C and b_o but not on the choice of the initial functions u_o.)

Proof: We assume that t varies in the interval $t_o \le t \le t_o + T$ and $bT < \theta$, where θ is arbitrary and b is the constant in the estimate (4.6) of the resolvent $Q(s, t_o, t)$ for the adjoint system (4.3), which is of the reduced order p_o and type $< \theta$. We want to choose a test function space Φ for the system (4.3) on which the operator $Q(i\partial/\partial x, t_o, t)$ is defined and meaningful. For this we make use of spaces of type S. and choose as the space $S_{\alpha,A}^{\beta,B}$. This space, as we recall, consists of functions which may be characterized by

$$\left| x^k \phi^{(q)}(x) \right| \le C \left(A + \delta \right)^k k^{k\alpha} (B + \rho)^q q^{q\beta}$$

$(\rho > 0, \delta > 0$; arbitrarily small). (For simplicity we may choose the 1-dimensional variable x varying on the line $-\infty < x < \infty$; the results are easily generalized to the n-dimensional case.) We apply Theorem 3.1 given earlier to the resolvent $Q(s, t_o, t)$ of system (4.3). As we recall, the theorem states that if an entire analytic function f(s) is of order $1/\beta$ and type smaller than $\beta/B^{1/\beta}e^2$ then the operator f(d/dx) is defined and bounded in the space $S_{\alpha,A}^{\beta,B}$ and maps this space into the space $S_{\alpha,A}^{\beta,Be^\beta}$. We are given the growth order $p_o = 1/\beta$ for $Q(s, t_o, t)$ and the type $\theta < \beta/B^{1/\beta}e^2$ and we deduce values of α, β, A, B such that the space $S_{\alpha,A}^{\beta,B}$ is nontrivial and sufficiently rich in functions, but as small as possible such that the conjugate space of generalized functions is as large as possible. With $\beta = 1/p_o$ and $B = (\beta/\theta e^2)^\beta$ given we choose the numbers α, A such that the above-stated conditions on $S_{\alpha,A}^{\beta,B}$ are met. We consider in turn the following three cases (values) of p_o; $p_o > 1$, $p_o = 1$ and $p_o < 1$.

(a). Assume $p_o > 1$. Then $\beta < 1$ and we may choose $\alpha = 1-\beta$, by which we have $A = \gamma/B$ for some positive number γ. This test function space has functions characterized, as we recall, by

$$|\varphi(x)| \le C \exp\left(-a|x|^{1/\alpha}\right), \tag{4.8}$$

with $a = \alpha/eA^{1/\alpha}$. We have $1/\alpha$ in terms of p_o ;

$$\frac{1}{\alpha} = \frac{1}{1-\beta} = \frac{1}{1-1/p_o} = p_o' .$$

The initial function $\phi_o(x)$ belongs to the space $\Phi = S_{\alpha,A}^{\beta,B}$ and the solution $\phi(x,t) = Q(i\partial/\partial x, t_o, t)\, \phi_o(x)$ of the system (4.3) belongs to the space $\Phi_1 = S_{\alpha,A}^{\beta,Be^\beta}$. The operator $\phi(i\partial/\partial x, t_o, t)$ maps the space $\Phi = S_{\alpha,A}^{\beta,B}$ into the space $\Phi_1 = S_{\alpha,A}^{\beta,Be^\beta}$. The functions $\phi(x,t)$ and $\Delta\phi(x)t)/\Delta t$ are bounded for t in $0 \le t \le T$ in the space $S_{\alpha,A}^{\beta,Be^\beta}$ and converge in $S_{\alpha,A}^{\beta,Be^\beta}$ to $\phi_o(x)$ and $\partial\phi(x,t)/\partial t$, respectively, i.e.,

$$\lim_{t \to t_o} \phi(x,t) = \lim_{t \to t_o} Q\left(i\frac{\partial}{\partial x}, t_o, t\right)\phi_o(x) = \phi_o(x) ,$$

and

$$\lim_{\Delta t \to 0} \frac{\Delta\phi(x,t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\left(Q\left(i\frac{\partial}{\partial x}, t_o, t\right)\phi_o(x)\right)}{\Delta t}$$

$$= P\left(i\frac{\partial}{\partial x}\right)Q\left(i\frac{\partial}{\partial x}, t_o, t\right) = \frac{\partial\phi(x,t)}{\partial t}$$

in the topology of $S_{\alpha,A}^{\beta,Be^\beta}$. We then have, by Theorem 3.3, that the adjoint problem (4.2) can have, for $0 \le t \le T$, only one solution u(x,t) in the space E', where E' is the conjugate space to E which contains Φ and Φ_1 as dense subsets. The functions in $\bar{\Phi} = S_{\alpha,A}^{\beta,B}$ and $\bar{\Phi}_1 = S_{\alpha,A}^{\beta,B\alpha^\beta}$ have the same character of decrease at infinity given by (4.8), and we obtain thus the

class of generalized functions in E' characterized (almost everywhere) by the inequality

$$|f(x)| \leq C \exp\left(\tfrac{1}{2} a |x|^{p'_o}\right). \tag{4.9}$$

Consequently, the Cauchy problem (4.2) has, for $0 \leq t \leq T$, $bT < \theta$, a unique solution in the class (4.7) for a given b_o. (The constant a in (4.9), which is related to $bT < \theta$, is arbitrary and can be chosen equal to a given b_o, for $a = \frac{\alpha}{\ell} A^{1/\alpha} = \frac{\alpha}{\ell}\left(\frac{\beta}{\gamma}\right)^{1/\alpha} = C\theta^{-\theta/\alpha}$ for any constant C greater than 0.)

(b) Let next $p_o = 1$. For this case we have $\beta = 1$. If we let $\alpha = 1 - \beta = 1$ we obtain the trivial space S_o^1 so we let therefore $\alpha > 0$, though α may be arbitrarily small. The resulting space $\Phi = S_{\alpha,A}^{1,B}$ is nontrivial and, by the same reasoning as above, we obtain the indicated uniqueness class (4.7) with arbitrary p_o'.

(c) Finally, let $p_o < 1$. For this case we have $\beta = 1/p_o > 1$ and we may choose, for a nontrivial space, $\alpha = 0$. Resulting spaces $S_{o,A}^{\beta,B}$, though nontrivial, are not sufficiently rich in functions; however, we may take the union with respect to A, $S_o^{\beta,B} = \bigcup_A S_{o,A}^{\beta,B}$, as the space Φ ; this space is sufficiently rich in functions. Recalling that spaces $S_{o,A}^{\beta,B}$ consist of functions of compact support, we have that generalized functions defined on these spaces are arbitrary, locally integrable functions and we thus obtain the uniqueness class for system (4.2), $p_o < 1$, all arbitrary functions, with no restrictions on growth with x as $|x| \to \infty$. ∎

The above discussion, based on 1-dimensional space, can be extended to n=dimensional space for $n > 1$ by replacing spaces $S_{\alpha,A}^{\beta,B}$ by the n-

dimensional counterparts, $S_{\alpha 1,\ldots \alpha n, A1,\ldots An}^{\beta 1,\ldots \beta n, B1,\ldots Bn}$, and $Q(s,t_o t)$ by $Q(s_1,\ldots,s_n,t_o,t)$ satisfying

$$|Q_{ij}(s_1,\ldots s_n, t_o, t)| \le C \exp(\theta_1|s_1|^{p_o} + \cdots + \theta_n|s_n|^{p_o}).$$

The numbers αj, βj, Aj, Bj are chosen according to p_o in the same manner as above. We arrive at a uniqueness class characterized, for $p_o > 1$, by

$$|f(x_1,\ldots x_n)| \le C \exp(a_1|x_1|^{p_o'} + \cdots + a_n|x_n|^{p_o'});$$

for $p_o = 1$ the same class (4.7) but with arbitrary p_o', and for $p_o < 1$, we have, as before, a uniqueness class of all arbitrary functions.

2.   Ordinary Solutions as Generalized Solutions

The foregoing considered the solutions to the systems of equations (4.2) as generalized functions, with the operators $\partial/\partial t$ and $P(i\partial/\partial x)$ also understood in the sense of generalized function. This means that for the solution $u(x,t)$ to the system we have, for any test function $\phi(x)$, that

$$\frac{\partial}{\partial t}\big(u(x,t),\phi(x)\big) = \big(u(x,t), P^*(i\frac{\partial}{\partial x})\phi(x)\big)$$

and

$$\lim_{t \to 0}\big(u(x,t),\phi(x)\big) = \big(u_c(x),\phi(x)\big).$$

(We were dealing with the problem "weakly".) We can apply Theorem 3.4 to ordinary solution also, when we can verify that the corresponding generalized solution $(\mu(x,t), \phi(x))$ is a solution to problem (4.2) in the sense of the theory of generalized functions. The following theorem allows us to do that under quite general conditions, and we state it as follows ([27], p. 49):

Theorem 3.5.   Let $u(x,t) = [u_j(x,t), j=1,..,m]$ be ordinary functions which are differentiable with respect to t and which admit application

of the differential operators $P(i\partial/\partial x)$ (i.e., admit derivatives with respect to x up to order p). Let these functions $u(x,t) = [u_j(x,t)]$ transform the system (4.2) into a system of identities and satisfy the inequalities

$$|u_j(x,t)| \leq C \, exp\left(\tfrac{1}{2}a|x|^{1/\alpha}\right).$$

Then the system of functionals $\left(u_j(x,t), \phi(x)\right) = \int u_j(x,t)\phi(x)dx,$

defined on the space $\Phi = S_{\alpha,A}^{\beta,B}$, $a = \alpha/eA^{\alpha}$, is a solution to the Cauchy problem (4.2) in the sense of generalized functions. The initial functionals are determined by the functions $u(x,0) = [u_j(x,0)]$.

3.   Correctness Classes

We now consider the classes of unique solutions to the systems which exist as ordinary functions and which depend continuously on the initial data. In other words, we want to determine correctness classes for the solutions to the systems (4.2), since, as we have noted, we are interested especially in solutions which are also ordinary functions. As before, we treat the systems of equations and the solutions initially as generalized functions and clarify under what conditions the solutions to the problems are also ordinary functions. As noted earlier, we discuss in particular the two classes of systems; hyperbolic and parabolic, and we formulate fundamental theorems on the correctness classes for these systems. To do this we make extensive use of Fourier transforms.

Let us return to the general system of equations (4.2), apply Fourier transformation to it and thus obtain the (transformed) system

$$\frac{\partial \hat{u}}{\partial t}(s,t) = P(s)\hat{u}(s,t) ,$$

$$\hat{u}(s,o) = \hat{u}_o(s) , \tag{4.10}$$

where again s = σ + iτ is complex. This system has the formal solution

$$\hat{u}(s,t) = Q(s,t)\hat{u}_o(s) , \tag{4.11}$$

where

$$Q(s,t) = \exp(tP(s)) \tag{4.12}$$

is the resolvent matrix/function. The function $\hat{u}(s,t)$ is a linear functional on the (Fourier transformed) test function space $\Psi = F(\Phi)$, where Φ is the space on which u(x,t) is defined as a generalized function. Φ corresponds to the appropriate uniqueness class for the problem with the solution u(x,t) and therefore $\hat{u}(s,t)$ are unique within the given class. The inverse transform of the solution (4.11) yields the convolution

$$u(x,t) = G(x,t) * u_o(x) , \tag{4.13}$$

where G(x,t), the Green's matrix, is the inverse Fourier transform of Q(s,t),

$$G(x,t) = F^{-1}(Q(s,t)) . \tag{4.14}$$

The convolution (4.13) in general defines a generalized function, but it may also be an ordinary function, depending on properties of the function G(x,t) and the initial function $u_o(x)$. Various smoothness conditions as well as growth restrictions at infinity may need to be imposed on the

functions $u_o(x)$ in order to obtain an ordinary function within the uniqueness class for the problem from the convolution (4.13), depending on how well "behaved" the function $G(x,t)$ is. If $G(x,t)$ is (consists of) an ordinary function, then no smoothness requirements need to be imposed on $u_o(x)$; only that $u_o(x)$ not grow too rapidly with x such that we have convergence of the integral which defines the convolution (4.13). If, however, $G(x,t)$ consists of generalized functions which are derivatives of some order h of ordinary functions, then the convolution (4.13) will exist as an ordinary function if $u_o(x)$ admit derivatives up to order h. If $G(x,t)$ consists of generalized functions of "infinite order", i.e., generalized functions which are not derivatives of a finite order of ordinary functions, then the initial functions $u_o(x)$ need to be infinitely often differentiable or even analytic. The various possibilities depend on the structure of the resolvent $Q(s,t)$, particularly its behavior in the complex plane.

We may distinguish three classes of systems which correspond to the three general characteristics of $G(x,t)$ and requirements on $u_o(x)$ above: (a) parabolic systems; (b) Petrovskii-correct systems, and (c) incorrect systems. The first of these (a) (which we will discuss in more detail below) are characterized by exponential decrease of the resolvent $Q(\sigma,t)$ to some order h > 1 along the real axis, i.e., the resolvent $Q(s,t)$ may be characterized by the inequality

$$\| Q(\sigma,t) \| \leq C_1 \left(1 + |\sigma| \right)^{p\,(m-i)} \exp\left(-a\,|\sigma|^h\right), \qquad (4.15)$$

where h > 1. For these (parabolic) systems the functions $G(x,t)$ turn out to be ordinary functions which decrease exponentially with $|x| \to \infty$ and

yield solutions for any (locally integrable) initial data $u_o(x)$ with no requirements on smoothness. Petrovskii-correct systems (b) (which include "hyperbolic" systems) are characterized by behavior of $Q(s,t)$ which increase at most as a power of $|\sigma|$ along the real axis, or by an inequality

$$\| Q(\sigma,t) \| \le C_1 (1 + |\sigma|)^{P(m-1)} \, .$$

$$(4.16)$$

For these cases $G(x,t)$ are generalized functions of "finite fixed order", i.e., contain derivatives of finite order of ordinary functions, and so the initial functions $u_o(x)$ must admit derivaties up to that finite order for the solution to exist as an ordinary function. Incorrect systems are characterized by exponential increase of the resolvent $Q(s,t)$ by some order h along the real axis, i.e., by an inequality

$$\| Q(\sigma,t) \| \le C_i (1 + |\sigma|)^{P(m-1)} \, \mathcal{e} x p(a |\sigma|^h) \, ,$$

$$(4.17)$$

and the convolution (4.13) will be an ordinary function only if the initial functions $u_o(x)$ are infinitely differentiable or analytic.

Returning the Fourier transformed problem (4.10) ($\hat{u}(s,t)$ is the generalized function defined on $\Psi = F(\Phi)$), we formulate the following theorem:

<u>Theorem 3.6</u> The Cauchy problem (4.10) admits the solution (4.11) with the resolvent (matrix-function) given by (4.12).

<u>Proof</u>. Let, as before, the corresponding problem in the test function space Ψ_1, be written

$$\frac{\partial \hat{\phi}(s,t)}{\partial t} = P^*(s) \, \hat{\phi}(s,t),$$

$$\hat{\phi}(s,o) = \hat{p}_o(s) \, ,$$

with the solution given by

$$\hat{\phi}(s,t) = Q^*(s,t)\hat{\phi}_o(s) = exp\left(t P^*(s)\right)\hat{\phi}_o(s) .$$

The operations of multiplication by the functions $Q*(s,t)$ and $P*(s)Q*(s,t)$ $(\equiv \partial/\partial t \; Q^*(s,t))$ are defined and bounded in the space Ψ and map this space into the (wider) space Ψ_1. Therefore the operators $Q(s,t)$ and $P(s)Q(s,t)$ are defined and bounded in Ψ_1 and map that space into Ψ'. We also have convergence of the operator in Ψ, i.e.,

$$\lim_{\Delta t \to o} \frac{\Delta Q^*(s,t)}{\Delta t} = \lim_{\Delta t \to o} \frac{Q^*(s,t+\Delta t) - Q^*(s,t)}{\Delta t} = P^*(s)Q^*(s,t),$$

and therefore also of the adjoint operator in Ψ_1',

$$\lim_{\Delta t \to o} \frac{\Delta Q(s,t)}{\Delta t} = \lim_{\Delta t \to o} \frac{Q(s,t+\Delta t) - Q(s,t)}{\Delta t} = P(s)Q(s,t)$$

for each generalized function $\hat{u}_o(s)$. Therefore $\hat{u}(s,t) = Q(s,t)\hat{u}_o(s)$ defines a solution to equation (4.10). We have also the convergence

$$Q(s,t)\hat{u}_o(s) \longrightarrow \hat{u}_o(s) \qquad \text{as} \quad t \longrightarrow 0$$

in Ψ', since, in Ψ

$$Q^*(s,t)\hat{\phi}_o(s) \longrightarrow \hat{\phi}_o(s) \qquad \text{as } t \longrightarrow 0. \; \blacksquare$$

We then have the generalized solution (4.13) to the original system (4.2) with the initial functions $u_o(x)$ given, and this solution depends continuously on $u_o(x)$. The continuous dependence of $u(x,t)$ on $u_o(x)$ is

easily shown by Fourier transformation. By the continuity of the Fourier transforms $u_o(x) \rightarrow 0$ implies that $u_o(s) \rightarrow 0$; from this we also have that $u(s,t) = Q^*(s,t)u_o(s) \rightarrow 0$ as $u_o(s) \rightarrow 0$ (by the boundedness of the operator $Q^*(s,t)$) and hence, via the continuity of the Fourier transform, that $u(x,t) \rightarrow 0$. The solution (4.13), even if an ordinary function, is a solution in the conventional sense if $u(x,t)$ is sufficiently smooth so that it admits derivatives with respect to x and t to satisfy the system of equations (4.2), and this can be guaranteed by imposing sufficient smoothness requirements on the initial functions $u_o(x)$.

4. Hyperbolic Systems

These systems are characterized by reduced growth order of less than or equal to 1 and, as stated before, by inequalities of the resolvents $Q(s,t)$ given in a general form by

$$\| Q(s,t) \| \leq C (1 + |s|)^{p(m-1)} exp \left(bt|s| \right), \tag{4.18}$$

$$\| Q(\sigma,t) \| \leq C (1 + |\sigma|)^h, \quad h \leq p(m-1), \tag{4.19}$$

where h is the "correctness exponent" for the given system. Hyperbolic systems, and only such systems, admit for the Cauchy problem (4.2) a solution for any sufficiently smooth initial data without restrictions on their growth at infinity.

We formulate a "fundamental" theorem below for correctness classes for solutions to systems of the general form (4.2) of hyperbolic type; but before we do so we list (with proofs) the following two theorems (3.7, 3.8) which are used in the proof of the fundamental theorem (3.9).

Theorem 3.7 (Paley-Winer-Schwartz). If an entire function $f(z) = f(z_1,..,z_n)$ of first order growth and type $\leq b$ does not grow more rapidly than $|x|^q$ for some q, thereby defining a functional in the space Z,

$$(f, \phi) = \int \overline{f(x)} \, \phi(x) dx,$$

then the Fourier transform $F(f)$ in K' of the functional f has its support in the domain $G_b = \{|\sigma_j| \leq b_j\}$. Furthermore, for any $\epsilon > 0$, the functional $F(f)$ may be represented as the sum of the finite number (dependent only on q and n) of components, each of which is the result of applying a differential operator of order $\leq q+(1) = (q_1,\ldots,q_n+1)$ to some function $e(\sigma)$, integrable in the domain G_{b+} but vanishing outside this domain.

Proof. Let $\psi_\epsilon(\sigma) = F(\phi_\epsilon(x))$ be an infinitely differentiable function of support in the domain $G_\epsilon = \{|\sigma| < \epsilon\}$. Its inverse Fourier transform $\phi_\epsilon(x)$ is an entire analytic function of first order growth and type $\leq \epsilon$ which tends to zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$. The product $\phi_\epsilon(x)f(x)$ is again an entire function of first order growth and type $\leq b + \epsilon$ which tends to zero more rapidly than any power of $1/|x|$. The Fourier transform of this function (product) is the convolution

$$F(\phi_\epsilon(x)f(x)) = F(\phi_\epsilon) * F(f) = \gamma_\epsilon(\sigma)F(f),$$

and it vanishes outside the domain $G_{b+\epsilon}$. One may form a sequence of functions $\psi_{\epsilon\gamma}(\sigma) = F(\phi_{\epsilon\gamma})$ from the family $\psi_\epsilon(\sigma)$ in the space K' converging to the delta function $\delta(\sigma)$, and we then have

$$\gamma_{\epsilon\nu}(\sigma) * F(f) \longrightarrow \delta(\sigma) * F(f) = F(f) .$$

Since the functions $F(\phi_{\epsilon\gamma}(x)f(x))$ have supports in the domain $G_{b+\epsilon}$, their limit $F(f)$ also has support in this domain, or, in fact, in G_b, since $\epsilon > 0$ is arbitrarily small. We introduce the function

$$f_c(x) = \frac{f(x)}{(x_1-i)^{q_1+1} \cdot \ldots \cdot (x_n-i)^{q_n+1}} ,$$

(4.20)

This function belongs to the space $L_2(R)$ (by the condition on growth of $f(x)$ for real x). Let also be given $g_o(\sigma) = F(f_o(x))$, a function also in $L_2(R)$. We obtain from (4.20)

$$g(\sigma) = F(f(x)) = P(iD)g_o(\sigma),$$

where $P(x) = (x_1-1)^{q_1+1} \cdot \ldots \cdot (x_n-1)^{q_n+1}$. Hence $F(f)$ is the result of applying a differential operator $P(iD)$ of order $q+(1) = (q_1+1,..,q_n+1)$ to the square-integrable function $g_o(\sigma)$, which has its support in the domain G_b . ∎

Theorem 3.8 (Phragmen-Lindelof) If an analytic function $f(z)$, defined within and on the sides of an angle G_θ with aperture $\theta < \pi/p$, satisfies the inequality

$$|f(z)| \le C\exp(b|z|^p)$$

within the angle G_θ and is bounded on the sides of this angle by a constant C_1 , then it is bounded by the same constant C_1 within the angle G_θ also .

Proof. Without loss of generality we may consider the angle G_θ to be bounded by the rays $\arg z = \pm \theta/2$. We may find a number p_1 such that $\theta < \pi/p_1 < \pi/p$. Consider the branch of the function

$$F_\epsilon(z) = \exp(-\epsilon z^{p_1}), \qquad (|\arg z| \le \theta/2) ,$$

which takes positive values on the real axis. Let us also construct the function

$$f_\epsilon(z) = f(z)F_\epsilon(z) ,$$

and we show that this function $f_\epsilon(z)$ is bounded within the limits of the angle G_θ. We have on the sides of the angle G_θ,

$$|f_\epsilon(z)| = |f(r\exp(\pm i\theta/2))F_\epsilon(r\exp(\pm i\theta/2))| \le C_1\exp(-r^{p_1}\cos(p_1\theta/2)) \le C_1 ,$$

since, by assumption, $p_1\theta/2 < \pi/2$, so $\cos(p_1\theta/2) > 0$. On the arc of a circle $z = r\exp(iw)$, $|w| < \theta/2$, we have

$$|f_\epsilon(r\exp(iw))| = |f(r\exp(iw))\cdot F_\epsilon(r\exp(iw))| \le C\exp(br^p - \epsilon r^{p_1}\cos(p_1 w)) \le$$

$$C\exp(br^p - \epsilon r^{p_1}\cos(p_1\theta/2)) \dashrightarrow 0$$

as $r \to \infty$, since $p_1 > p$. Hence, for sufficient large r we have

$$f_\epsilon(r\exp(iw)) \le C_1 \qquad \text{for some } r_0 .$$

Thus on the boundary of the region formed by the two rays $w = \pm\, \theta/2$ and on the arc of the circle of radius r_0 the function $f_\epsilon(z)$ is bounded by the constant C_1. By the maximum principle the function $f_\epsilon(z)$ is bounded by C_1 also within this region. We therefore obtain that within the angle G_θ (since $r \ge r_0$ is arbitrarily large) that the function $f(z)$ satisfies the inequality

$$|f(z)| = |f_\epsilon(z)\cdot F_\epsilon^{-1}(z)| \le C_1 \, |\exp(\epsilon z^{p_1})|,$$

and, since $\epsilon > 0$ is arbitrary, we have

$$|f(z)| \le C_1$$

for all z in G_ϵ. ∎

The following two generalized versions of Theorem 3.8 may be given (for proofs see [26], p. 212, p. 241):

<u>Theorem 3.8[I]</u> If an analytic function $f(z)$, defined within and on the sides of the angle G_ϵ with aperture $\epsilon \leq \pi/p$, satisfies

$$|f(z)| \leq (\exp \ (b|z|^p) \qquad\qquad \text{within } G_e$$

and

$$|f(z)| \leq C_1 \ (1 + |z|^h) \qquad\qquad \text{on the sides of } G_e \ ,$$

then

$$|f(z)| \leq C_1' \ (1 + |z|^h) \qquad\qquad \text{for all } z \text{ in } G_e \ .$$

<u>Theorem 3.8[II]</u> (generalized to n variables). Let   be given an analytic function $f(z) = f(z_1,\ldots,z_n)$, defined for values of the variables $z_1,z_2,\ldots,z_n$, each of which runs through an angle G_j of aperture $w_j < \pi/p_j$ in its plane, independently of the values of the remaining variables. Let the boundary of each G_j be denoted by Γ_j . Furthermore, let the function $f(z)$ satisfy the inequalities

$$|f(z_1, \ z_n)| \leq (\exp(b_1|z_1|^{p1} + \ldots + b_n \ |z|^{pn})$$

for z_j in G_j, and

$$|f(z_1,\ldots z_n)| \leq C_1 \qquad \text{for } z_1 \text{ in } \Gamma_1,\ldots, \ z_n \text{ in } \Gamma_n \ .$$

Then the inequality

$$|f(z_1,\ldots z_n)| \leq C_1 \qquad \text{for } z_1 \text{ in } G_1,\ldots, \ z_n \text{ in } G_n$$

is valid.

Let us now state the following (fundamental) theorem for the correctness classes of solutions to hyperbolic systems of form (4.2):

<u>Theorem 3.9</u> If the initial functions $u_o(x) = [u_j(x,o), \ j=1,\ldots m]$ of a hyperbolic system (4.2) with correctness exponent h admit continuous derivatives with respect to x up to order h+n+k (n is the number of independent space variables, k is a nonnegative integer), then the system

admits a unique continuous solution u(x.t) which is k times differentiable in x. The solution depends continuously on the initial functions $u_j(x,0)$ in the following sense: if the sequence of functions $u_{jy}(x,0)$ converge for $v \to \infty$, together with their derivatives up to order h+n+k, uniformly in each ball $|x| \le r$ to the functions $u_j(x,0)$ and their derivatives, respectively, then the corresponding solutions $u_{jy}(x,t)$ converge to the solution $u_j(x,t)$, together with their derivatives in x up to order k, uniformly in each ball $|x| \le r$. (No restrictions are imposed on the growth of the initial functions $u_j(x,0)$ and their derivatives as $|x| \to \infty$.)

<u>Proof</u>. We know from previous discussions that the Cauchy problem (4.2) admits for hyperbolic systems a unique solution within the class of generalized functions on the space K. We want to show that the convolusion (4.13) maps the initial function $u_o(x)$ which is h+n+k times differentiable into a function u(x,t) admitting derivatives up to order k with respect to x. Assume first a growth order of the system of $p_o = 1$. The elements of the resolvent matrix-function Q(s,t) are entire analytic functions of order (at most) one and type $\le \theta = bT$, which for real s = σ increase not faster than a polynomial of degree h. According to Theorem 3.7 the (inverse) Fourier transform of an entire function (Q(s,t)) of order one and type $\le \theta=bT$, which increases with s = σ not faster than a polynomial of degree h, is a generalized function over K with support in the region $|x| \le bT$ and admits a representation of the form

$$F^{-1}(Q(s,t)) = G(x,t) = R\left(\frac{\partial}{\partial x}\right)f(x,t), \tag{4.21}$$

where $R(\partial/\partial x)$ is a fixed polynomial of degree not higher than $n+h$, and $f(x)$ is a continuous function which vanishes outside the region $|x| \leq bT + \epsilon$ (for arbitrary small $\epsilon > 0$). Hence the solution is

$$u(x,t) = R\left(\frac{\partial}{\partial x}\right)f(x,t) * u_o(x) = f(x,t) * R\left(\frac{\partial}{\partial x}\right)u_o(x). \quad (4.22)$$

If $u_o(x)$ admits derivatives up to order $h+n+k$, then the function $R(\partial/\partial x)u_o(x)$ admits derivatives up to order k, and the integral (4.22) converges since it ranges only over a bounded region.

Next consider the case $p_o < 1$. The elements of the matrix $Q(s,t)$ are entire functions of order smaller than one and for real $s = \sigma$ increase at most as polynomials of degree h. Theorems 3.8-3.8^{11} imply that the elements of $Q(s,t)$ are polynomials of degree not higher than h in s (or the variables $s_1, \ldots, s_n$). We may obtain that the Green's matrix consists of elements of the form $P_{ij}(D,t)\delta(x)$, where P_{ij} are differential operators of order at most h. The solution (convolution) (4.13) becomes of the form

$$|u(x,t)| \leq |P(D,t)\delta(x) * u_o(x)| = P(D,t)u_o(x). \quad (4.23)$$

If, (by Theorems 3.8, 3.8^{11}) $Q(s,t)$ is bounded by a constant, then we obtain ($h=0$)

$$|u(x,t)| \leq |f(t)\delta(x) * u_o(x)| = |f(t)u_o(x)|, \quad (4.24)$$

where $f(t)$ is some continuous function. ∎

5.   Parabolic Systems

We now consider solutions to systems (4.2) which are parabolic. As mentioned, these systems have resolvents which are functions characterized by inequalities

$$\| Q(s,t) \| \leq C (1 + |s|)^{P(m-1)} \exp(ht|s|^{P_0}), \qquad (4.25)$$

and

$$\| Q(\sigma,t) \| \leq C_1 (1 + |\sigma|)^{P(m-1)} \exp(-at|\sigma|^{h}), \qquad (4.26)$$

with $p_0 \geq h > 1$. That is, in particular, the elements of $Q(s,t)$ decrease exponentially along the real line by an order h, the so-called "parabolicity exponent" of the system, which (like p_0) is strictly greater than one.   Another important number which characterizes a parabolic system is the "genus" (μ) of that system; this number determined the correctness classes for the (parabolic) Cauchy problem.

Let us use the following theorem:

<u>Theorem 3.10</u>   If an entire function $f(z)$ has order of growth $\leq p$ with a finite type, i.e., satisfies the inequality

$$|f(z)| \leq C_1 \exp(b|z|^p)$$

for all z, and, in addition, satisfies for real $z=x$ the inequality

$$|f(x)| \leq C_2 \exp(a|x|^h)$$

($a \neq 0$, $0 < h \leq p$),   then there exists a domain G_μ defined by the inequality

$$|y| \leq k_1 (1 + |x|)^\mu$$

with $\mu \geq 1 - (p-h)$, in which

$$|f(z)| \leq C_3 \exp (a'|x|^h) \ ,$$

where $C_3 = \max (C_1, C_1)$, and the constant a' differs arbitrarily little from a.

<u>Proof</u>. We may construct the analytic function

$$f_1(z) = f(z) \exp (-az^h) \ .$$

This function is bounded (by C_2) on the half-axis $x > 0$. In the right half-plane we have the inequality

$$|f_1(z)| \leq C_1 \exp (b_1 |z|^p) \quad ,$$

where we may take $b_1 = b + |a|$. Let, moreover, be given the function

$$f_2(z) = f_1(z) \exp (ib_2 z^p)$$

with $b_2 = b_1 + \epsilon$ (for arbitrary $\epsilon > 0$). This function is analytic in the first quadrant and bounded by the constant C_2 on the half-axis $x > 0$ also. It is, moreover, bounded on the ray $z = r \exp (i\pi/2p)$, since on this ray

$$\left| f_2 \left(r \exp \left(i \pi /2p \right) \right) \right| \leq C_1 \exp \left(b_1 r^p - b_2 r^p \right) \leq C_1 \ ,$$

and on the limits of the angle $0 \leq \arg z \leq \pi/2p$ the function is bounded by

$$|f_2(z)| \leq C_1 \exp \left(b_3 r^p \right) \ ,$$

where $b_3 = 2b_1 + \epsilon$. By Theorem 3.8 (Phragmen-Lindelof) the function $f_2(z)$ is also bounded within the angle above. It follows that within the angle $0 \leq w \leq \pi/2p$ the function $f(z)$ satisfies

$$\left| f \left(r \exp \left(i w \right) \right) \right| = |f(z)| = \left| f_1(z) \exp \left(a z^h \right) \right|$$

$$= \left| f_2(z) \exp \left(a z^h - i b_2 z^p \right) \right| \leq C_3 \exp \left(a r^h \cos h w + b_2 r^p \sin p w \right) \ . \tag{4.27}$$

Considering points $z = r \exp(iw)$ such that

$$a r^h \cos(hw) + b_2 r^p \sin(pw) \leq a_1 r^h \cos^h(w) \qquad (4.28)$$

($a_1 > a$ and of the same sign), the function $f(z)$ also satisfies

$$|f(z)| \leq C_3 \exp(a_1 r^h \cos^h(w)) = C_3 \exp(a_1 x^h). \qquad (4.29)$$

We may obtain that the inequality (4.28) is satisfied in the domain

$$r^{p-h} = \frac{a_1 \cos^h(w) - a \cos(hw)}{b_2 \sin(pw)} .$$

If $h = p$ the curve is the ray

$$a_1 \cos^h(w) - a \cos(hw) = b_2 \sin(pw),$$

and if $h < p$ we have that (since the numerator is bounded) $w \to 0$ as $r \to \infty$, and hence it may be assumed that

$$\sin(pw) = p \frac{y}{x} E_1, \quad \cos^h(w) = E_2, \quad \cos(hw) = E_3,$$

and $\quad r = x E_4,$

where the E's are variables approaching one. We obtain by (4.27) and (4.29) that

$$y = K E x^{1-(p-h)},$$

where $K = \dfrac{a_1 - a}{b_1}$, and $E(x) \to 1$ as $x \to \infty$. It turns out that the function $f(z)$ satisfies the inequality (4.29) in a domain defined by

$$y \leq \min(x \, tg(\pi/2p), K_1 x^{1-(p-h)}) \qquad (4.30)$$

for some K_1 which depends only on a, a_1, b_2. (a_1 may be taken arbitrarily close to a.) Obtaining similar inequalities (domains) as (4.30) in the remaining quadrants of the z-plane with x, y replaced by $|x|$, $|y|$, we will have, by continuity considerations, a domain defined by

$$|y| \le K_1'(1+|x|)^{1-(p-h)} ,$$

where the inequality

$$|f(z)| \le C' exp(a'|x|^h)$$

holds. ∎

The least upper bound of the number μ in Theorem 3.10 where the inequalities of the theorem hold is the genus of the parabolic system. We will obtain that the correctness class for the system will consist, for positive genus, of functions f(x) which for $|x| \to \infty$ have an exponential growth order $\le p_1 = p_o/(p_o - \mu)$, and for non-positive genus the correctness class will consist of functions f(x) of growth order $\le p_2 = h/(h - \mu)$.

Let us consider first systems of positive genus ($\mu > 0$). We state the following theorem (with proof) which we use in the proof of the fundamental theorem (3.12) below.

Theorem 3.11 Consider the entire analytic function f(z) of growth order $\le p$ and finite type which satisfies the inequality

$$|f(z)| \le C exp(b|z|^p),$$

Furthermore, let us assume that in some domain G, defined by

$$|y| \leq K_1 \left(1 + |x|\right)^{\mu} \qquad\qquad (0 < \mu \leq 1)$$

the function $f(z)$ satisfies the inequality

$$|f(z)| \leq C_1 \, exp\left(a|x|^h\right) \qquad\qquad (4.31)$$

($h < p$; for $a < 1$ $h = p$ is admissible also.)   Then the function $f(z)$ satisfies the inequality

$$|f(z)| \leq C_2 \, exp\left(a|x|^h + b'|y|^{p/\mu}\right) \qquad\qquad (4.32)$$

for all $z = x+iy$, where the constant b' only depends on a, b, K_1, and C_2 = max (C, C_1) .

<u>Proof</u>.   Let us put

$$M(y) = \sup_{x} |f(z)| \, exp\left(-a|x|^h\right). \qquad\qquad (4.33)$$

This expression is bounded, since every horizontal line belongs, except for some finite segment $\triangle y$, to the domain G, where the expression (4.33) is bounded by the constant C_1 .   We consider two possibilities: 1) either the upper bound (4.33) is reached for any z on the segment y, or 2) it is not reached on this segment.   We consider the first case, and let the upper bound (4.32) be reached outside the domain G for a given $y = \bar{y}$, at which point $x + iy$ we have the inequality

$$|\bar{y}| > K_1 \left(1 + |x|\right)^{\mu} > K_1' |x|^{\mu} .$$

We obtain (using the inequality $0 < \mu \leq 1$)

$$|x| < K_2 |y|^{1/\mu} .$$

We have

$$|\bar{z}| = \left(|\bar{x}|^2 + |\bar{y}|^2\right)^{1/2} \le \left(K_2^2 |\bar{y}|^{2/\mu} + |\bar{y}|^2\right)^{1/2} \le K_3 |\bar{y}|^{1/\mu}.$$

We may replace $-a|x|^h$ for $a < 0$ by the greater quantity $|a|K_4|\bar{y}|^{h/\mu}$ (or zero for $a \ge 0$) and obtain

$$M(\bar{y}) = |f(z)| \exp(-a|\bar{x}|^h) \le C \exp\left(b|\bar{z}|^p + |a|K_4|\bar{y}|^{h/\mu}\right),$$

and, moreover,

$$M(y) \le C \exp\left(b_1|\bar{y}|^{p/\mu} + b_2|\bar{y}|^{h/\mu}\right)$$
$$\le C \exp\left(b_3|\bar{y}|^{p/\mu}\right)$$

(since $h \le p$). Thus, for any x,

$$|f(z)| \exp(-a|x|^h) \le M(\bar{y}) \le C \exp\left(b_3|\bar{y}|^{p/\mu}\right),$$

and, therefore,

$$|f(z)| \le C \exp\left(a|x|^h + b_3|\bar{y}|^{p/\mu}\right).$$

Considering the other case in which the point z lies in the domain G, it follows from (4.31) that the quantity $|f(z)| \exp(-a|x|^h)$ is bounded by the constant C_1, from which we have

$$|f(z)| \le C_1 \exp\left(a|x|^h\right) \le C_1 \exp\left(a|x|^h + b_3|y|^{p/\mu}\right);$$

$b_3 \le b_1 + b_2 \le (|a| + b) K_5$, where K_5 depends only on K_1. ∎

We formulate the following theorem:

Theorem 3.12 Consider the parabolic system

$$\frac{\partial u(x,t)}{\partial t} = P\left(i\frac{\partial}{\partial x}\right)u(x,t) \, ;$$
$$u(x,0) = u_c(x) \tag{4.34}$$

$(u(x,t) = (u_j(x,t), j=1,..,m)$; $P(i\partial/\partial x)$ is an mxm matrix), and let this system have a positive genus $(\mu > 0)$. If the initial functions $u_o(x)$ of this system belong to the class $K_{p1,bo}$, which contains functions satisfying

$$|f(x)| \leq C \, exp\left(b_o|x|^{p_1}\right) \tag{4.35}$$

with

$$p_1 = p_o/(p_o - \mu) \, ,$$

then, for sufficiently small $t > 0$ and arbitrarily given $b_1 > b_o$, the solution of the system belongs to the class $K_{p1,b1}$.

Proof. Assume first that n=1 (n = number of independent variables). The class $K_{p1,b1}$ is contained, for sufficiently small $t \leq T$, in the uniqueness class for the Cauchy problem, for $p_1 = p_o/(p_o-\mu) \leq p_o/(p_o - 1)$ and we have that $0 \leq \mu \leq 1$. In general, the solution, given by the convolution

$$u(x,t) = G(x,t) * u_o(x) \tag{4.36}$$

transforms ordinary functions $u_o(x)$ into generalized functions $u(x,t)$, and we want to show that this convolution maps the class $K_{p1,bo}$ of ordinary functions into the class $K_{p1,b1}$ of ordinary functions. We investigate the properties of the resolvent $Q(s,t)$, the Fourier transform of the Green's matrix/function $G(x,t)$. Recalling the estimates for $Q(s,t)$,

$$\|Q(s,t)\| \leq C\left(1+|s|\right)^{P(m-1)} exp\left(bt|s|^{p_o}\right) \tag{4.37}$$

and

$$\|Q(\sigma,t)\| \leq C_1 (1 + |\sigma|)^{p(m-1)} \, exp\,(-a|\sigma|^h),$$

(4.38)

we use Theorem 3.11 and obtain that

$$\|Q(s,t)\| \leq C' \, exp\,(-at|\sigma|^h + b't|\tau|^{P_0/\mu}),$$

(4.39)

where b' is not larger than $B_1(a+b_0)$ and B_1 depends only on the domain G in Theorem 3.11. Assuming that $0 \leq t \leq T$ the inequality (4.39) implies that $Q(s,t)$ belongs to the space $W^{P_0/\mu,\theta}$, which consists of entire functions characterized by

$$|s^k \phi(s)| \leq C_k \, exp\,\left(\frac{\mu}{P_0}|\bar{\theta}_\tau|^{P_0/\mu}\right)$$

for arbitrary $\bar{\theta} > \theta$ and $(\mu/p_0)\theta^{p_0/\mu} = b'T$. The Fourier transform of this space is given by

$$F\left(W^{P_0/\mu,\theta}\right) = W_{(P_0/\mu)',\,1/\theta} \, ,$$

where

$$\frac{1}{P_0/\mu} + \frac{1}{(P_0/\mu)'} = 1 .$$

Consequently,

$$\left(P_0/\mu\right)' = P_0/(P_0-\mu) = P_1 .$$

It follows that the functions G(x,t) belong to

$$\|G(x,t)\| \leq C \, exp\,\left(-\frac{1}{P_1}\left|\frac{x}{\bar{\theta}}\right|^{P_1}\right), \qquad (\bar{\theta} > \theta),$$

with this estimate being valid in $0 \leq t \leq T$, where $(\mu/p_o)e^{po/\mu} = b'T$.
$G(x,t)$ is thus an ordinary function with exponential decrease, and we may
show that the convolution (4.36) is an ordinary function belonging to the
class $K_{p1,b1}$. We use the following lemma (for proof, see [27], p. 119):
Lemma 1:   Let $\lambda > 0$.   For arbitrary $\gamma > \beta > 0$ there exists a number α
> 0 such that for all x and ξ we have the inequality

$$- \alpha |\xi|^{\lambda} + \beta |x - \xi|^{\lambda} \leq \gamma |x|^{\lambda} \; .$$

The integrand in

$$G(x,t) \times u_o(x) = \int G(\xi,t) u_o(x-\xi) d\xi$$

admits the majorant

$$exp \left(- \frac{1}{p_1} \left| \tilde{\theta}^{-p_1} \xi \right|^{p_1} + b_o |x - \xi|^{p_1} \right)$$

(from   the assumptions about $u_o(x)$ and the class of functions containing
$G(x,t)$).   We can choose $(1/2p_1)\theta^{-p1}$ large enough so that we have the
inequality

$$- \frac{1}{2p_1} \tilde{\theta}^{-p_1} |\xi|^{p_1} + b_o |x-\xi|^{p_1} \leq b_1 |x|^{p_1} \; .$$

We then obtain for the integral

$$\left| \int G(\xi,t) u_o(x-\xi) d\xi \right| \leq exp (b_1 |x|^{p_1} \int exp \left(- \frac{1}{2p_1} \tilde{\theta}^{-p_1} |\xi|^{p_1} d\xi \right.$$

$$= C_2 exp \left(b_1 |x|^{p_1} \right).$$

Thus the convolution (4.36) exists in the class $K_{p1,b1}$ of ordinary functions for sufficiently small T. The choice of $(\frac{1}{2}p_1)\theta^{-p1}$ fixes the interval $0 \leq t \leq T$, since $\mu/p_0\theta^{p_0/\mu} = b'T$ and $\theta > e$. For $b_1 \to b_0$ we have $\theta^{-p1} \to \infty$ and the interval $0 \leq t \leq T$ contracts to zero. ∎

We next consider the case of a nonpositive genus ($\mu \leq 0$) and quote the following theorem (see [27], p. 123):

Theorem 3.13 Let the initial functions $u_0(x) = \{u_j(x,0), j=1,..,m\}$ of the parabolic system (4.34) with $\mu \leq 0$ belong to the class $K_{p1,0}$ of functions $f(x)$, which for every $\epsilon > 0$ satisfy the inequality

$$\left| f(x) \right| \leq C_\epsilon \, exp\left(\epsilon |x|^{P_1} \right)$$

with

$$p_1 = h/(h-\mu) \; .$$

Then for sufficiently small $T > 0$ and $t \leq T$, the solution of the system belongs to the same class.

CHAPTER 4

THE MODELS. EQUATIONS AND SOLUTIONS

In this chapter we treat the three systems of equations derived from the coupled diffusion equations (2.1) in Chapter 2. Two of these, (2.3) and (2.4), represent, as noted, the double porosity models of Barenblatt et al [5] and Warren and Root [50], respectively, and the third case, equations (2.5), is included as a remaining case of a set of degenerate parabolic systems obtained from (2.1). (Parabolic systems in general are treated in e.g. [23].) For purposes of comparison we also treat the diffusion equation as a standard model of flow in a homogeneous porous medium. We refer to each of the systems as Models 1 to 4, which together with initial and boundary conditions govern flow of fluids, single-phase and slightly compressible, in confined homogeneous isotropic reservoirs under general conditions including those of typical drawdown tests. We take as the space domain the infinite horizontal plane, which represents the infinite horizontal layer with a constant thickness (H), and place a fluid point source (physically a line source) at the origin. Analytic solutions, including fundamental solutions, are obtained by methods of Fourier and Laplace transforms, and they are, as mentioned, treated as generalized solutions for which uniqueness and correctness classes are obtained. Solutions are compared with corresponding solutions to the diffusion equations (Model 1) and we investigate properties of each model by comparing with those of the diffusion model. We discuss the inverse problem of estimating model parameters when the solutions are given as numerical input, and we establish uniqueness and

continuous dependence of the parameter estimates with respect to the input functions.

We denote fluid pressures by u=u(x,t) for Model 1; subscripted as u_1 and u_2 for matrix and fractures, respectively, for Models 2 to 4. The fluid pressures represent deviations from a mean or equilibrium pressure such that $u \equiv 0$, $u_1 = u_2 \equiv 0$ at equilibrium. We consider in turn two sets of conditions for each model; (a) arbitrary initial conditions and no fluid sources, and (b) a fluid source given, nonzero and arbitrary, and usually zero initial pressures. The second case (b) represents conditions of an ideal drawdown test. Fluid sources are given as source terms in the right hand sides of the basic equations. In cases involving actual systems fluid pressures may usually be assumed to vanish at infinity.

Before we proceed to treat the systems in the next section we list here the following (Fourier, Hankel and Laplace) transforms and inverse transforms:

Fourier transforms (in the general n-dimensional space)

$$\hat{f}(\sigma) = \frac{1}{(2\pi)^{n/2}} \int f(x) \, exp(-i\sigma x)dx , \qquad (1)$$

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int \hat{f}(\sigma) \, exp(i\sigma x)d\sigma. \qquad (2)$$

Haenkel transforms (for the radially symmetric plane)

$$\hat{\tilde{f}}(\rho) = {}_o\!\int^\infty f(r) \, r \, J_o(r\rho)dr , \qquad (3)$$

$$f(r) = {}_o\!\int^\infty \hat{f}(\rho) \, \rho J_o(r\rho)d\rho . \qquad (4)$$

(Jo (x) is the Bessel function of zero order, first kind.)

Laplace transforms

$$\bar{f}(s) = \int_0^\infty f(t)\, exp\,(-st)\,dt, \tag{5}$$

$$f(t) = \int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, exp\,(st)\,ds . \tag{6}$$

We also list the following two theorems which will be useful shortly:

Theorem 4.1 (Plancherel) (ref. [43], p. 200)

One can associate to each function $f(x)$ in L_2 a function $\hat{f}(\sigma)$ in L_2 such that the following properties hold:

(a) If $f(x)$ is in $L_1 \cap L_2$, then $\hat{f}(\sigma)$ is the Fourier transform given by (1);

(b) For every $f(x)$ in L_2, $\|f\|_2 = \|\hat{f}\|_2$;

(c) The mapping $f \to \hat{f}$ is a Hilbert space isomorphism of L_2 onto L_2 ;

(d) The following symmetry relations exist between $f(x)$ and $\hat{f}(\sigma)$:

If
$$\phi_D = \frac{1}{(2\pi)^{n/2}} \int_D f(x)\, exp\,(-i\sigma x)\,dx$$

and
$$\psi_D = \frac{1}{(2\pi)^{n/2}} \int_D \hat{f}(\sigma)\, exp\,(i\sigma x)\,d\sigma$$

then $$\|\phi_D - \hat{f}\| \to 0 \qquad and \qquad \|\psi_D - f\| \to 0$$

as $D \to \infty$.

Theorem 4.2 (Riemann-Lebesgue) (ref [42], p. 90). If $f(x)$ is integrable

function on $(-\infty, \infty)$, then

$$\lim_{\nu \to \infty} \int_{-\infty}^{\infty} f(x) \cos(\nu x) dx = 0.$$

$(\cos(\nu x)$ may be replaced by $\exp(i\nu x)$, and the result of Theorem 4.2 may be generalized to n-space.)

A. The Standard Model

1. Model 1. The Diffusion Equation

$$\frac{\partial u}{\partial t} - a \nabla^2 u = f(t)/c_0 \, \delta(x - x_c), \tag{1.1}$$

$$u(x, 0) = u_0(x).$$

Solution (a): The case with no source.

We set $f(t) \equiv 0$ in (1.1), take the Fourier transform of (1.1) and $u_0(x)$ and we obtain

$$\frac{\partial \hat{u}}{\partial t} + a|\sigma|^2 \hat{u} = 0, \tag{1.2}$$

$$\hat{u}(\sigma, 0) = \hat{u}_0(\sigma).$$

From this we have

$$\hat{u}(\sigma, t) = Q(\sigma, t)\hat{u}_0(\sigma),$$

$$Q(\sigma, t) = \exp(-a|\sigma|^2 t), \tag{1.3}$$

and the solution is thus, after inverse transformation of (1.3)

$$u(x, t) = G(x, t) * u_0(x)$$

$$= \int G(x, x', t) u_0(x') dx', \tag{1.4}$$

where $G(x, x', t)$ is the fundamental solution (Green's function) in n-

dimensional space, given by

$$G(x,x',t) = F^{-1}(Q(\sigma;t)) = \frac{1}{(2\pi)^n}\int \exp(-\partial|\sigma|^2 t - i\sigma(x-x'))d\sigma$$

$$= \frac{1}{(2\pi \partial t)^{n/2}}\exp\left(\frac{-|x-x'|^2}{4\partial t}\right).$$

(1.5)

The function $G(x,x',t)$ has the properties,

(1.6)

$$\lim_{t \to 0} G(x,x',t) = \delta(x-x'),$$

$$\lim_{|x-x'| \to \infty} G(x,x',t) = \lim_{t \to \infty} G(x,x',t) = 0,$$

(1.7)

and

$$\int G(x,x',t)dx = 1.$$

(1.8)

The delta function $\delta(x)$ has the property such that for any function $f(x)$ we have

$$\int \delta(y-x)f(y)dy = f(x).$$

(1.9)

(The delta function is as noted before, meaningful only as a generalized function.) By property (1.9) we obtain from (1.4) the initial conditions $u_o(x)$ at t=0.

The solution (1.4) to the system describes a process of diffusion of fluid from the initial state $u_o(x)$ outward in the reservoir towards asymptotic equilibrium at $t \to \infty$. The diffusion is characterized by movement of fluid from regions of relatively high fluid density towards regions of low density until the fluid mass is uniformly distributed throughout the reservoir. The fundamental solution $G(x,x',t)$ describes the response of the system at a point x, time t, to a delta input at x',

t=0.

Existence and Uniqueness

We observe that the system (1.1) has the reduced order $p_o = 2$ and we may obtain the following uniqueness class:

Theorem 4.3.   All functions f(x) satisfying the inequality

$$\left| f(x) \right| \leq C \exp\left(b_o |x|^2 \right) \qquad (1.10)$$

(C and b_o arbitrary but fixed) form a uniqueness class for system (1.1). That is, there exists at most one solution u(x,t) to the system which for t=0, t fixed in the interval $0 \leq t \leq T$, equals the initial function $u_o(x)$, which also belongs to class (1.10).

Proof. With $p_o = 2$ given we apply Theorem 3.4 in Chapter 3 and obtain, with $p_o = p_o/(p_o-1) = 2$, the above uniqueness class (1.10). By Theorem 3.4 growth restrictions need to be imposed on $u_o(x)$ such that the inequality (1.10) holds for $u_o(x)$. ∎

The system (1.1) is, as mentioned, parabolic with a parabolicity exponent $h = p_o = 2$, by which the genus of the system is $\mu \geq 1 - (p_o-h) = 1 > 0$. Hence we may obtain the following correctness class:

Theorem 4.4.   Given that the initial function $u_o(x)$ satisfies the inequality (1.10) (b_o arbitrary but fixed), the system (1.1) has, for sufficiently small t, a unique solution in the class $K_{2,b1}$ of functions f(x), which are characterized by the inequality

$$\left| f(x) \right| \leq C \exp\left(b_i |x|^2 \right) \qquad (1.11)$$

with arbitrary $b_1 > b_o$.

<u>Proof</u>. Given $p_o = h = 2$, the genus $\mu \geq 1 > 0$ and initial functions $u_o(x)$ belonging to the class K_2, b_o, where K_2, b_o consists of functions satisfying (1.10), we apply Theorem 3.12 in Chapter 3 and obtain, with $p_1 = p_o/(p_o - \mu)$ = 2, the correctness class (1.11) above. ∎

<u>Solution (b)</u>. <u>The case with a source</u>

We next include the source term, nonzero, in equation (1.1), and we let x_o be the origin. Fourier transformation of equation (1.1) gives

$$\frac{\partial \hat{u}(\sigma,t)}{\partial t} + a|\sigma|^2 \hat{u}(\sigma,t) = f(t)/c_o , \tag{1.12}$$

$$\hat{u}(\sigma,0) = \hat{u}_o(\sigma) .$$

We may write the solution to (1.12)

$$\hat{u}(\sigma,t) = \hat{u}_o(\sigma) \exp(-a|\sigma|^2 t)$$

$$+ \frac{1}{c_o} \int_o^t \exp(-a|\sigma|^2(t-\tau)) f(\tau) d\tau , \tag{1.13}$$

which, upon inverse transformation, yields the solution to system (1.13),

$$u(x,t) = G(x,t) * u_o(x)$$

$$+ \frac{1}{c_o} \int_o^t G(x,x_o,t-\tau) f(\tau) d\tau . \tag{1.14}$$

We recognize the first integral on the right hand side of (1.14) as the solution obtained earlier to the initial value problem (a), while the second integral is due to the source term. This second integral converges for t in the finite interval $0 \leq t \leq T$, $r > 0$.

Let us consider the special case of a constant-strength source

given by

$$f(t) = f_0 \, U_+ \, (t),$$

and let $u_o(x) = 0$ at $t=0$. The solution (1.5) then consists of the second term on the right hand side of (1.5), which we may integrate explicitly with respect to time τ. The solution is radially symmetric and we obtain for the plane case the Theis solution (3.4), Chapter 1, i.e.,

$$u(r,t) = \frac{f_0}{c_0} \int_0^t G(r, t-\tau) d\tau = \frac{-\mu f_0}{4\pi k} E_i \left(\frac{-r^2}{4 \partial t} \right). \tag{1.15}$$

We write (1.15) also in terms of its Hankel transform, for it will be useful again in the next section:

$$u(r,t) = \frac{\mu f_0}{2\pi k} \int_0^\infty \frac{J_o(r\rho)}{\rho} \left(1 - exp(-\partial \rho^2 t) \right) d\rho. \tag{1.16}$$

The solution (1.15), (1.16) converges for t in the finite interval $0 \le t \le T$ and $r > 0$, but diverges for $t \to \infty$ and/or $r = 0$.

We may list, for comparison, the corresponding solution for the case of a confined semi-infinite halfspace, in which a fluid source is applied at a point at depth z_0 below the origin. The boundary condition

$$\left. \frac{\partial u}{\partial z} \right|_{z=0} = 0$$

applies at the z=0 plane, the surface of the halfspace. The solution for this case is given by

$$u(r,t) = \frac{\mu f_0}{4\pi k} \left[\frac{erfc \left(r_+ / \sqrt{4\partial t} \right)}{r_+} + \frac{erfc \left(r_- / \sqrt{4\partial t} \right)}{r_-} \right], \tag{1.7}$$

where
$$r = (x^2 + y^2 + z^2)^{1/2} \,,$$

$$r_{\pm} = (x^2 + y^2 + (z \mp z_0)^2)^{1/2} \qquad (r_{\pm} > 0) \,,$$

and
$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-y^2)\,dy \,.$$

We note that in the limit $t \to \infty$ we have for (1.17)

$$\lim_{t \to \infty} u(r,t) = \frac{\mu f_0}{4\pi k}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) \,,$$

i.e., (1.17) is bounded for all $t \geq 0$, $r > 0$, in contrast to the unbounded behavior of (1.16) for the plane case.

One may explain the unbounded increase of $u(r,t)$ for the plane case roughly as follows: The physical system in 3-dimensional space represented by the 2-dimensional solution is a (porous) medium with an infinite vertical line source, while the source for the halfspace solution (1.17) is confined to a point only. The line source is assumed to have some source strength per unit length of the line. An observation point x is located in the medium at some minimum distance r_0 from the line. The line source is activated at $t = 0$ and signals are emitted from all points of the line outward into the medium. After a short time lapse signals are received at x from all points along short segments of the line only which are located near by x, but as time increases signals reach x from all points along increasingly longer segments of the line, and thus the total intensity of signals arriving at x continues to increase with time without limit. In contrast, the signal intensity arriving at x from one source point only approaches a finite limit at $t \to \infty$ equal to the intensity leaving x.

The Inverse Problem

The method of estimating the parameters k and c_o from input functions u(r,t) for the Theis model was discussed earlier in Chapter 1. We want to show also that the values obtained for k,c_o are unique and depend continuously on u(r,t). All parameters are assumed to be positive constants. To show uniqueness we may recall the Fourier/Hankel transform of u(r,t), i.e.,

$$\hat{u}(\rho,t) = \frac{\mu f_o}{2\pi k} \cdot \frac{1}{\rho^2} \left(1 - exp\left(-a\rho^2 t\right)\right).$$

(We ignore the factor, $\mu f_o/2\pi$, for it is assumed to be known and a constant.) It is easy to see that $\hat{u}(\rho,t)$ is a unique function (numerically) for each unique pair of values of k and a, and therefore of k and c_o. For the function $exp(-a\rho^2 t)$ is unique for each value of a , and thus also the function $\frac{1}{\rho^2}(1-exp(-a\rho^2 t))$. The function $\frac{1}{k_2\rho^2}(1-exp(-a\rho^2 t))$, and hence $\hat{u}(\rho,t)$, is unique for every a and k, or c_o and k.

Moreover, $\hat{u}(\rho,t)$ is in L_2 and so, by (Plancherel's) Theorem 4.1(c) u(r,t) is a unique function for each pair k and c_o. The continuity of $\hat{u}(\rho,t)$, and thus of u(r,t) with respect to k and c_o is easily seen also, or k and c_o depend continuously on the input function u(r,t). We summarize the above as follows:

Theorem 4.5. To each function u(r,t) given by (1.15) there corresponds a unique pair of values of k and c_o (positive) which depend continuously on u(r,t).

B. The Double Porosity/Permeability Models

2. Model 2. The Barenblatt Model

$$c_1 \frac{\partial u_1}{\partial t} \qquad + \frac{\alpha}{\mu}(u_1 - u_2) = 0,$$

$$- \frac{k_2}{\mu} \nabla^2 u_2 - \frac{\alpha}{\mu}(u_1 - u_2) = f(t)\delta(x - x_0), \qquad (2.1)$$

$$u_1(x, 0) = u_{10}(x).$$

Solution (a). The case with no source

We may combine the two equation (2.1) into one by eliminating either u_1 or u_2 and we obtain, with $f(t) \equiv 0$, the equation

$$\frac{\partial u_i}{\partial t} - b \nabla^2 u_i - c \nabla^2 \frac{\partial u_i}{\partial t} = 0 \qquad (i = 1, 2),$$

where

$$b = \frac{k_2}{\mu c_1} \qquad \text{and} \qquad c = \frac{k_2}{\alpha}.$$

We seek again solutions to this system by Fourier transforms. After transforming (2.2) and the initial condition u_{10} we have

$$\frac{\partial \hat{u}_i}{\partial t} + b|\sigma|^2 \hat{u}_i + c|\sigma|^2 \frac{\partial \hat{u}_i}{\partial t} = 0,$$

$$\hat{u}_1(\sigma, 0) = \hat{u}_{10}(\sigma), \qquad (2.3)$$

from which we have $\hat{u}_1$ and $\hat{u}_2$ given by

$$\hat{u}_1(\sigma, t) = Q_1(\sigma, t)\hat{u}_{10}(\sigma), \qquad (2.4)$$

$$\hat{u}_2(\sigma, t) = Q_2(\sigma, t)\hat{u}_{10}(\sigma) = \frac{\hat{u}_1(\sigma, t)}{1 + c|\sigma|^2}, \qquad (2.5)$$

with the resolvents $Q_1(\sigma, t)$, $Q_2(\sigma, t)$ given by

$$Q_1(\sigma, t) = \exp\left(\frac{-b|\sigma|^2 t}{1 + c|\sigma|^2}\right), \tag{2.6}$$

$$Q_2(\sigma, t) = \frac{1}{1 + c|\sigma|^2} \exp\left(\frac{-b|\sigma|^2 t}{1 + c|\sigma|^2}\right). \tag{2.7}$$

Inverse transformation of (2.4) and (2.5) yields the solution u_1 and u_2;

$$u_1(x, t) = G_1(x, t) \ast u_{1o}(x) = \int G_1(x - x', t) u_{1o}(x') dx', \tag{2.8}$$

$$u_2(x, t) = G_2(x, t) \ast u_{1o}(x) = \int G_2(x - x', t) u_{1o}(x') dx', \tag{2.9}$$

where the fundamental solutions $G_1(x, t)$ and $G_2(x, t)$ are given, in n-dimensions, by

$$G_1(x, t) = F^{-1}\left(Q_1(\sigma, t)\right)$$
$$= \frac{1}{(2\pi)^n} \int \exp\left(-i\sigma x\right) \exp\left(\frac{-b|\sigma|^2 t}{1 + c|\sigma|^2}\right) d\sigma, \tag{2.10}$$

and

$$G_2(x, t) = F^{-1}\left(Q_2(\sigma, t)\right)$$
$$= \frac{1}{(2\pi)^n} \int \frac{\exp\left(-i\sigma x\right)}{1 + c|\sigma|^2} \exp\left(\frac{-b|\sigma|^2 t}{1 + c|\sigma|^2}\right) d\sigma. \tag{2.11}$$

The integrals $G_1(x, t)$ and $G_2(x, t)$ diverge at $x = 0$ for all $t \geq 0$, and they are meaningful only as generalized functions or as integrals on test function spaces. In fact, $G_1(x, t)$ has properties of the delta function

for all $t \geq 0$. Both G_1 and G_2 are integrable in space. The following limits hold for G_1 and G_2:

$$\lim_{t \to 0} G_1(x, t) = \delta(x) , \tag{2.12}$$

$$\lim_{t \to 0} G_2(x, t) = \frac{1}{(2\pi)^n} \int \frac{\exp(-i\sigma x)}{1 + c|\sigma|^2} d\sigma , \tag{2.13}$$

$$\lim_{t \to \infty} G_1(x, t) = \lim_{t \to \infty} G_2(x, t) = 0 \quad (|x| > 0), \tag{2.14}$$

and, by the (Riemann-Lebesgue) Theorem 4.2

$$\lim_{|x| \to \infty} G_1(x, t) = \lim_{|x| \to \infty} G_2(x, t) = 0 . \tag{2.15}$$

For the radially symmetric plane G_2 has properties of the modified Bessel function K_o , and we have for (2.13)

$$\lim_{t \to 0} G_2(r, t) = \frac{1}{2\pi} \int_0^\infty \frac{\rho J_c(r\rho) d\rho}{1 + c\rho^2} = \frac{1}{2\pi c} K_o\left(\frac{r}{\sqrt{c}}\right). \tag{2.16}$$

Properties of the functions G_1 and G_2 may be seen more clearly by partitioning them into singular and nonsingular parts as follows. We note that the exponential factor below may be expanded into the series

$$\exp\left(\frac{-b|\sigma|^2 t}{1 + c|\sigma|^2}\right) = \exp(-bt/c)\exp\left(\frac{bt/c}{1 + c|\sigma|^2}\right)$$

$$= \exp(-bt/c)\left[1 + \frac{bt/c}{1 + c|\sigma|^2} + \cdots + \frac{1}{n!}\left(\frac{bt/c}{1 + c|\sigma|^2}\right)^n + \cdots\right], \tag{2.17}$$

and we may find that the integrals (2.10), (2.11) for G_1 and G_2 may be written

$$G_1(x,t) = \exp(-bt/c)\left[\delta(x) + bt\,G_2(x,0) + \Delta_1(x,t)\right], \quad (2.18)$$

$$G_2(x,t) = \exp(-bt/c)\left[G_2(x,0) + \Delta_2(x,t)\right]. \quad (2.19)$$

G_1 and G_2 contain the terms in the brackets, singular and nonsingular, multiplied by $\exp(-bt/c)$; $\delta(x)$ and $G_2(x,0)$ are singular at $x=0$ while Δ_1 and Δ_2 converge for all $x.t \geq 0$. The functions Δ_1 and Δ_2 are due to all the terms in the series (2.17) of order higher than 2 or 3. Both Δ_1 and Δ_2 tend to zero as $t \to 0$, $t \to \infty$ and $x \to \infty$. By (2.12) we obtain for u_1 the initial condition u_{10} at $t = 0$. The properties of G_1 and G_2 as shown by (2.18) and (2.19) indicate that the solutions u_1 and u_2 given by (2.8) and (2.9) undergo roughly a uniform exponential decay in magnitude with time with relatively little change in shape from the initial states u_{10}, $u_2(x,0)$ compared with the diffusion model solution (1.4); that is, they exhibit less diffusion of the fluid than the solution for the preceeding Model 1.

Existence and Uniqueness

We apply the theory in the preceeding Chapter and obtain for the system (2.1) the following uniqueness class:

Theorem 4.6: The Cauchy problem (2.1) has, for $0 \leq t \leq T$, at most one solution $u_1(x,t)$, $u_2(x,t)$ in the class of all arbitrary functions with no restrictions on their growth at infinity. (That is, there is at most one solution $u_1(x,t)$, $u_2(x,t)$ to system (2.1), which for $t = 0$, with t fixed in $0 \leq t \leq T$, equals the initial functions $u_{10}(x)$ and $u_{20}(x)$, where u_{10} is

arbitrary anc u_{2_0} is given by (2.9).)

Proof. We observe that the resolvents $Q_1(s,t)$ and $Q_2(s,t)$ for this system ($s = \sigma + i\tau$) have a growth order of $p_0 = 0$ at infinity and thus obey the inequality

$$|Q_i (s,t)| \leq C_i \qquad (i = 1, 2) \qquad (2.20)$$

for some C_i . We note also that $Q_1(s,t)$ and $Q_2(s,t)$ have singular points $s = \pm i/\sqrt{c}$ on the τ-axis and are therefore not entire analytic functions in the variable s. However, since these singular points do not lie on the real axis and $Q_1(\sigma,t)$, $Q_2(\sigma,t)$ are defined for all σ the solutions and their Fourier transforms are not affected. We may use Theorem 3.4 in Chapter 3 with the order $p_0 = 0$ (< 1) and obtain the above uniqueness class for this system. ∎

We may also obtain the following correctness class:

Theorem 4.7   If the initial function $u_{1_0}(x)$ of system (2.1) admits continuous derivatives up to order n+k (n is the number of independent space variables and k is a nonnegative integer), then system (2.1) admits a (unique) continuous solution $u_1(x,t)$, $u_2(x,t)$ which is k times differentiable in x. The solution depends continuously on $u_{1_0}(x)$ in the following sense: if a sequence of functions $u_{0\nu}(x)$ converges for $\nu \to \infty$ together with its derivatives in x up to order n+k uniformly in each ball $|x| \leq$ r to the function $u_{1_0}(x)$ and its derivatives, respectively, up to order n+k, then the corresponding solutions $u_{1\nu}(x,t)$, $u_{2\nu}(x,t)$ converge to the solution $u_1(x,t)$, $u_2(x,t)$, together with their derivatives in x up to order k uniformly in each ball $|x| \leq r$. (There are no restrictions on the growth of $u_{1_0}(x)$ and its derivatives as $|x| \to \infty$.)

Proof.   Since the resolvents $Q_1(s,t)$, $Q_2(s,t)$ satisfy the inequality

(2.20) at infinity the system (2.1) is hyperbolic with a correctness exponent h = 0. (Again, the singular points of Q_1 and Q_2, lying off the real axis, do not affect the solutions or their Fourier transforms.) We may thus apply Theorem 3.9 in Chapter 3 with h = 0 and obtain the above-mentioned correctness class of solutions for this system with the smoothness properties indicated. ▌

The proof of Theorem 3.9, Chapter 3, indicates that the fundamental solution G_1 has properties of a delta function, as we have seen already, and that the solution u_1, u_2 can be characterized by an inequality

$$\left| u_i(x,t) \right| \leq \left| f(t) u_{i0}(x) \right| \tag{2.21}$$

for some function f(t).

Solution (b): The case with a source

We now include the source term (non-zero) in the basic equations. Equation (2.2) then becomes in u_1

$$\frac{\partial u_1}{\partial t} - b \nabla^2 u_1 - c \nabla^2 \frac{\partial u_1}{\partial t} = f(t)/c_1 \, \delta(x - x_a). \tag{2.22}$$

Fourier transformation of equation (2.22) yields

$$\frac{\partial \hat{u}_1}{\partial t} + \frac{b|\sigma|^2}{1 + c|\sigma|^2} \hat{u}_1 = \frac{f(t)/c_1}{1 + c|\sigma|^2}, \tag{2.23}$$

from which we obtain the solution for $\hat{u}_1(\sigma, t)$,

$$\hat{u}_1(\sigma, t) = \hat{u}_{10}(\sigma) Q_1(\sigma, t) + \frac{1}{c_1} \int_0^t Q_2(\sigma, t - \tau) f(\tau) d\tau. \tag{2.24}$$

Inverse transformation of (2.24) gives the solution $u_1(x, t)$,

$$u_1(x,t) = G_1(x,t) * u_{10}(x) + \frac{1}{c_1} \int_0^t G_2(x-x_0, t-\tau) f(\tau) d\tau, \quad (2.25)$$

We recognize the first integral on the right hand side as the solution due to the initial condition u_{10}, and the second integral is the contribution of the source term. We may obtain similarly for the fissure pressure u_2 the equation (using the basic equations (2.1), Fourier transformed):

$$\hat{u}_2(\sigma, t) = \frac{\ddot{u}_1(\sigma, t) + \mu/\alpha\, f(t)}{1 + c|\sigma|^2}, \quad (2.26)$$

Substituting for $\hat{u}_1$ from (2.24) we have for (2.26) the equation

$$\hat{u}_2(\sigma, t) = \hat{u}_{10}(\sigma)\, Q_2^1(\sigma, t)$$
$$+ \frac{1}{c_1} \int_0^t Q_3(\sigma, t-\tau) f(\tau) d\tau + \frac{\mu}{\alpha}\, \frac{f(t)}{1+c|\sigma|^2}, \quad (2.27)$$

where

$$Q_3(\sigma, t) = \frac{Q_2(\sigma, t)}{1+c|\sigma|^2} = \frac{1}{1+c|\sigma|^2} \exp\left(\frac{-b|\sigma|^2 t}{1+c|\sigma|^2}\right). \quad (2.28)$$

Inverse Fourier transformation of (2.27) yields

$$u_2(x,t) = G_2(x,t) * u_{10}(x) + \frac{\mu}{\alpha} f(t)\, G_2(x-x_0, 0)$$
$$+ \frac{1}{c_1} \int_0^t G_3(x-x_0, t-\tau) f(\tau) d\tau, \quad (2.29)$$

where

$$G_3(x,t) = F^{-1}(Q_3(\sigma, t))$$
$$= \frac{1}{(2\pi)^n} \int \frac{\exp(-i\sigma x)}{(1+c|\sigma|^2)^2} \exp\left(\frac{-b|\sigma|^2 t}{1+c|\sigma|^2}\right) d\sigma. \quad (2.30)$$

Again the first integral on the right hand side of (2.29) is the solution

found earlier due to the initial condition while the other two terms are due to the source.

We consider again the special case of a constant-strength source given by $f(t) = f_o U_+(t)$ and let $u_{1o}(x) \equiv 0$. Then we may integrate the terms in (2.27) and (2.29) due to the source explicitly with respect to time and we obtain, after some rearrangement, the solutions

$$u_1(x,t) = \frac{\mu f_o}{k_2 (2\pi)^n} \int \frac{\exp(-i\sigma(x-x_o))}{|\sigma|^2} \left[1 - \exp\left(\frac{-b|\sigma|^2 t}{1+c|\sigma|^2}\right)\right] d\sigma, \quad (2.31)$$

$$u_2(x,t) = u_1(x,t) + \frac{\mu}{\alpha} f_o G_2(x-x_o, t). \quad (2.32)$$

For the radially symmetric plane, with $x_o=0$, (2.31) and (2.32) are of the form

$$u_1(r,t) = \frac{\mu f_o}{2\pi k_2} \int_0^\infty \frac{J_o(r\rho)}{\rho}\left[1 - \exp\left(\frac{-b\rho^2 t}{1+c\rho^2}\right)\right] d\rho, \quad (2.33)$$

$$u_2(r,t) = u_1(r,t) + \frac{\mu f_o}{2\pi\alpha} \int_0^\infty \frac{\rho J_o(r\rho)}{1+c\rho^2}\exp\left(\frac{-b\rho^2 t}{1+c\rho^2}\right) d\rho. \quad (2.34)$$

(The solution (2.33) was, as mentioned before, given by Barenblatt et al [5] for the "fissure pressure" response.) Solutions to a multi-layer form of this model are shown in Figure 4.1.

The integral (u_1) converges for t in $0 \le t \le T$ and r > 0, but it diverges if r=0 or t → ∞. The second integral of (2.34), the function G_2 multiplied by $\mu f_o/\alpha$, vanishes for t → ∞, r > 0. We take u_1 and u_2 in the sense of generalized functions, or defined as linear functionals on test function spaces. We may obtain the following limits for u_1 and

u_2:

$$\lim_{t \to 0} u_1(r, t) = 0,$$

$$\lim_{t \to 0} u_2(r, t) = \lim_{t \to 0} \frac{\mu f_o}{\alpha} G_2(r, t) = \frac{\mu f_o}{2 \pi k_2} K_o\left(\frac{r}{\sqrt{c}}\right),$$

$$\lim_{t \to \infty} u_1(r, t) = \lim_{t \to \infty} u_2(r, t) = 0,$$

and

$$\lim_{r \to \infty} u_1(r, t) = \lim_{r \to \infty} u_2(r, t) = 0.$$

The solution (2.33) converges, as noted before [5], to the Theis solution (1.14) as $t \to \infty$. To prove this, recall the Theis solution in the form (1.15), with "a" equal to b;

$$u(r, t) = \frac{\mu f_o}{2 \pi k_2} \int_0^\infty \frac{J_o(r\rho)}{\rho} \left[1 - exp\left(- b \rho^2 t\right)\right] d\rho. \quad (2.35)$$

Define the difference

$$\Delta u(r, t) = u_1(r, t) - u_2(r, t)$$

$$= \frac{\mu f_o}{2 \pi k_2} \int_0^\infty \frac{J_o(r\rho)}{\rho} \left[exp\left(\frac{-b \rho^2 t}{1 + c \rho^2}\right) - exp\left(- b \rho^2 t\right)\right] d\rho. \quad (2.36)$$

Let $g(\rho, t_o)$ be a function, integrable on $(0, \infty)$, defined by

$$g(\rho, t_o) = \frac{\mu f_o}{2 \pi k_2} \frac{J_o(r\rho)}{\rho} \left[exp\left(\frac{-b \rho^2 t_o}{1 + c \rho^2}\right) + exp\left(- b \rho^2 t_o\right)\right], \quad (2.37)$$

with $t \geq t_o > 0$ for some $t_o > 0$. We use the following theorem:

<u>Theorem 4.8</u>   (Lebesque Convergence Theorem; ref. [41], p. 88).   Let g be an integrable function over a domain E and let $f\nu$ be a sequence of measurable functions such that $|f\nu| \leq g$ on E and for almost all   x   in E   we have $f(x) = \lim\limits_{\nu \to \infty} f\nu(x)$.   Then

$$\int_{E} f(x)dx = \lim_{\nu \to \infty} \int_{E} f_{\nu}(x)dx .$$

Taking the function $g(\rho, t_o)$ defined above as the (integrable) function g in Theorem 4.8, E is the domain ($0 \leq r \leq \infty$) and let the sequence f_v be given by

$$f_{\nu}(\rho) = f(\rho, t_{\nu})$$

$$= \frac{M f_o}{2 \pi k_2} \frac{J_o(r\rho)}{\rho} \left[exp\left(\frac{-b\rho^2 t_{\nu}}{1+c\rho^2}\right) - exp\left(-b\rho^2 t_{\nu}\right)\right] \qquad (2.38)$$

with $t_{\nu} \geq t_o$, and $t_{\nu} \to \infty$ as $\nu \to \infty$.   We have

$$|f_{\nu}(\rho)| \leq g(\rho) = g(\rho, t_o) ,$$

and

$$\lim_{\nu \to \infty} f_{\nu}(\rho) = 0 .$$

Thus, by Theorem 4.6, we have that

$$\lim_{\nu \to \infty} \int_{0}^{\infty} f_{\nu}(\rho)d\rho = \lim_{\nu \to \infty} \Delta u(r_1 t_{\nu}) = 0 , \qquad (2.39)$$

and so

$$u_1(r,t) \to u(r,t) \qquad \text{and } t \to \infty.$$

(We note here that since both u_1 and u tend to infinity as $t \to \infty$ we mean by "$t \to \infty$" that "t becomes large" though still finite in the above

discussion.)

The physical process described by the solutions (2.33), (2.34) is roughly the following: When the external fluid source (sink) is applied to the fracture system at x_o at t=0, and with the system initially in equilibrium, the pressure u_2 in the fracture system immediately jumps (drops) to a finite nonzero value $(\alpha K_o(r/\sqrt{c}))$, radially symmetric about x_o and decreasing in magnitude with distance from x_o. This abrupt pressure drop at t=0 is due to the fluid depletion by the activation of the source/sink at t≥0 and the negligible storage capacity of the fracture system. At the same time flow begins to occur from the matrix into the fractures due to the pressure difference $u_1 - u_2$ immediately induced at t=0, and the matrix pressure u_1 begins to drop as well. The crossflow and drop rate of u_1 are greatest at first, but diminish towards zero as t → ∞. Both u_1 and u_2 approach with time the behavior of the standard diffusion model having the parameters k_2 and c_1 of both the fractures and the matrix.

The Inverse Problem

This model has the three unknown parameters, k_2, c_1 and α to be estimated. Because of the third parameter α more than one type curve for different values of α are required in the curve matching process. Solutions are plotted against $4k_2/\mu c_1 r^2$ and we obtain several curves for different values of α (see Figure 4.1). We select the model curve which best fits the data and estimate k_2 and c_1 in the same manner as described earlier. The value of α characterizing the data is the value associated with the best fitting model curve.
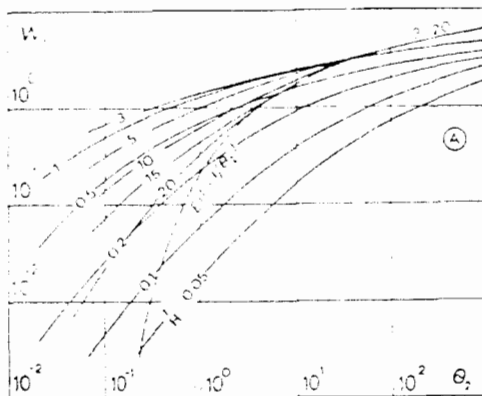
Fracture pressure
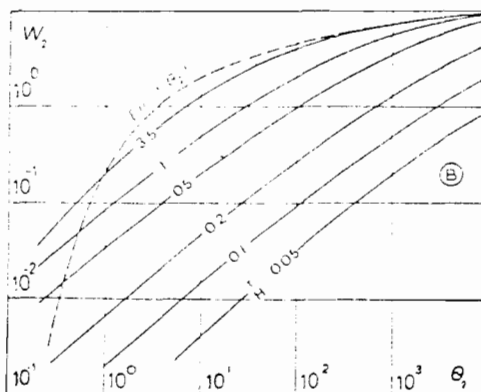
Matrix pressure

$$\Theta = 4k_2t/\mu c_1 r^2$$

Figure 4.1   Dimensionless pressure drawdown in an infinite horizontal layer with a constant-strength line source. The multi-layered Barenblatt model after Boulton and Streltsova [10].   (r = radial distance from source: 2H = matrix layer thickness)

We next show that the values of the parameters k_2, c_1, α are unique for each input function u_1 or u_2. Although u_2 is the quantity more likely to be measured than u_1 we choose for convenience u_1 as the input function. The same reasoning below applies to u_2. We recall the Hankel transform of $u_1(r,t)$, i.e.,

$$\hat{u}_1(\rho,t) = \frac{\mu f_o}{k_2} \frac{1}{\rho^2} \left[1 - exp\left(\frac{-b\rho^2 t}{1+c\rho^2}\right) \right], \tag{2.40}$$

Again we ignore the factor μf_o, for it is assumed known. As before, the parameters k_2, c_1, α are positive constants. The function $\hat{u}_1(\rho,t)$, we find, is unique for each set of values of k_1, b and c, and therefore for k_2, c_1 and α. For the function $\exp(-b\rho^2 t/(1+c\rho^2))$ is unique for each b and c; thus the function

$$\frac{1}{\rho^2} \left[1 - exp\left(\frac{-b\rho^2 t}{1+c\rho^2}\right) \right]$$

is unique for each b and c, and hence the function

$$\frac{1}{k_2 \rho^2} \left[1 - exp\left(\frac{-b\rho^2 t}{1+c\rho^2}\right) \right]$$

is unique for each set of values of k_2, b and c, and therefore we have uniqueness of $\hat{u}_1(\rho,t)$ for each set of values of k_2, c_1 and α. Moreover, $\hat{u}_1(\rho,t)$ is in L_2 for t in the finite interval $0 \le t \le T$, and so we have by (Plancherel's) Theorem 4.1 that $u_1(r,t)$ is unique for each set of k_2, c_1 and α. Continuity of $\hat{u}_1(\rho,t)$ and thus of $u_1(r,t)$, with respect to values of k_2, c_1 and α is obvious. We summarize the above as follows:

Theorem 4.9. To each function $u_1(r,t)$ or $u_2(r,t)$ given by (2.33) or (2.34), respectively, we may associate a unique set of (positive) values

of k_2, c_1 and α, and these values depend continuously on $u_1(r,t)$ and $u_2(r,t)$.

3. Model 3. The Warren-Root Model

$$c_1 \frac{\partial u_1}{\partial t} \qquad\qquad + \frac{\alpha}{\mu}(u_1 - u_2) = 0 \,,$$

$$c_2 \frac{\partial u_2}{\partial t} - \frac{k_2}{\mu}\nabla^2 u_2 - \frac{\alpha}{\mu}(u_1 - u_2) = f(t)\delta(x - x_0) \,, \qquad (3.1)$$

$$u_1(x,0) = u_{10}(x) \,, \qquad u_2(x,0) = u_{20}(x) \,.$$

Solution (a) The case with no sources

We may combine the two equations (3.1) into one (by eliminating either u_1 or u_2) and obtain, with $f(t) \equiv 0$,

$$\frac{\partial^2 u_i}{\partial t^2} + a \frac{\partial u_i}{\partial t} - b \nabla^2 u_i - c \nabla^2 \frac{\partial u_i}{\partial t} = 0 \quad (i=1,2) \,, \qquad (3.2)$$

where

$$a = \frac{\alpha(c_1 + c_2)}{\mu c_1 c_2} \,, \qquad b = \frac{\alpha k_2}{\mu^2 c_1 c_2} \,, \qquad c = \frac{k_2}{\mu c_2} \,.$$

As before we apply Fourier transformation to obtain a solution to this problem. Fourier transformation of (3.2) results in the equation

$$\frac{\partial^2 \hat{u}_i}{\partial t^2} + a_0(\sigma) \frac{\partial \hat{u}_i}{\partial t} + b_0(\sigma) \hat{u}_i = 0 \,, \qquad (3.3)$$

where

$$a_0(\sigma) = a + c|\sigma|^2 = \frac{\alpha}{\mu c_1} + \frac{k_2|\sigma|^2 + \alpha}{\mu c_2} \,,$$

$$b_0(\sigma) = b|\sigma|^2 = \frac{\alpha k_2 |\sigma|^2}{\mu^2 c_1 c_2} \,.$$

Equation (3.3) has the general solution

$$\hat{u}_i(\sigma, t) = A(\sigma) \exp(r_1(\sigma)t) + B(\sigma) \exp(r_2(\sigma)t), \tag{3.4}$$

where $r_1(\sigma)$ and $r_2(\sigma)$ are roots of the equation (3.3) given by

$$r_1(\sigma) = \frac{1}{2}\left(-a_c(\sigma) + D(\sigma)\right)$$

and

$$r_2(\sigma) = \frac{1}{2}\left(-a_c(\sigma) - D(\sigma)\right).$$

The "discriminant" $D(\sigma)$ is given by

$$D(\sigma) = \left[\left(a_c(\sigma)\right)^2 - 4b_c(\sigma)\right]^{1/2}$$

$$= \left[\left(\frac{k_2|\sigma|^2 + \alpha}{\mu c_2} - \frac{\alpha}{\mu c_1}\right)^2 + \frac{4\alpha^2}{\mu^2 c_1 c_2}\right]^{1/2},$$

and the coefficients $A(\sigma)$ and $B(\sigma)$ are determined by the initial conditions. We obtain $A(\sigma)$ and $B(\sigma)$ from equation (3.4) in $\hat{u}_1$ using the initial conditions. We write the matrix-vector equation,

$$\begin{bmatrix} 1 & 1 \\ r_1(\sigma) & r_2(\sigma) \end{bmatrix} \begin{bmatrix} A(\sigma) \\ B(\sigma) \end{bmatrix} = \begin{bmatrix} \hat{u}_{1o} \\ \partial\hat{u}_{1o}/\partial t \end{bmatrix}, \tag{3.5}$$

where $\partial\hat{u}_{1o}/\partial t$ is obtained from the basic equations (3.1) with u_{1o} and u_{2o}. Solving for $A(\sigma)$ and $B(\sigma)$ in (3.5) gives

$$A(\sigma) = \frac{1}{D(\sigma)}\left(\frac{\partial\hat{u}_{1o}}{\partial t} - r_2(\sigma)\hat{u}_{1o}\right), \tag{3.6}$$

$$B(\sigma) = \frac{1}{D(\sigma)}\left(\Gamma_1(\sigma)\hat{u}_{1o} - \frac{\partial\hat{u}_{1o}}{\partial t}\right). \tag{3.7}$$

By inserting $A(\sigma)$ and $B(\sigma)$ given by (3.6) and (3.7) into (3.4) and rewriting $\partial\hat{u}_{1o}/\partial t$ in terms of $\hat{u}_{1o}$ and $\hat{u}_{2o}$ we obtain, after inverse transformation, the solution

$$\begin{bmatrix} u_1(x,t) \\ u_2(x,t) \end{bmatrix} = \int G(x-x',t)\begin{bmatrix} u_{1o}(x') \\ u_{2o}(x') \end{bmatrix}dx', \tag{3.8}$$

where $G(x,t)$ is the 2x2 Green's matrix for which the elements are given by (in n-dimensions)

$$G_{11}(x,t) = \frac{1}{(2\pi)^n}\int\frac{\exp(-i\sigma x)}{D(\sigma)}\left[-\left(\Gamma_2(\sigma)+\frac{\alpha}{\mu c_1}\right)\exp(\Gamma_1(\sigma)t)\right. \tag{3.9}$$
$$\left.+\left(\Gamma_1(\sigma)+\frac{\alpha}{\mu c_1}\right)\exp(\Gamma_2(\sigma)t)\right]d\sigma,$$

$$G_{12}(x,t) = \frac{1}{(2\pi)^n}\int\frac{\exp(-i\sigma x)}{D(\sigma)}\frac{\alpha}{\mu c_1}\left[\exp(\Gamma_1(\sigma)t)-\exp(\Gamma_2(\sigma)t)\right]d\sigma, \tag{3.10}$$

$$G_{21}(x,t) = \frac{1}{(2\pi)^n}\int\frac{\exp(-i\sigma x)}{D(\sigma)}\frac{\alpha}{\mu c_2}\left[\exp(\Gamma_1(\sigma)t)-\exp(\Gamma_2(\sigma)t)\right]d\sigma, \tag{3.11}$$

$$G_{22}(x,t) = \frac{1}{(2\pi)^n}\int\frac{\exp(-i\sigma x)}{D(\sigma)}\left[\left(\Gamma_1(\sigma)+\frac{\alpha}{\mu c_1}\right)\exp(\Gamma_1(\sigma)t)\right. \tag{3.12}$$
$$\left.-\left(\Gamma_2(\sigma)+\frac{\alpha}{\mu c_1}\right)\exp(\Gamma_2(\sigma)t)\right]d\sigma.$$

The roots $r_1(\sigma)$ and $r_2(\sigma)$ are zero or negative for all real σ, and they are of order ≤ 2 in σ so that the system (or the Green's matrix) exhibits damping with time in a manner similar to the diffusion model (Model 1). The integrals (3.9) - (3.12) converge for all $t \geq 0$, $x \neq 0$, and G_{ij} are integrable in space for all $t \geq 0$. We have the following limits;

$$\lim_{t \to 0} G(x,t) = \begin{bmatrix} \delta(x) & 0 \\ 0 & \delta(x) \end{bmatrix} ,$$

(3.13)

$$\lim_{t \to \infty} G_{ij}(x,t) = 0 \qquad (x \neq 0) ,$$

(3.14)

and (by Theorem 4.2),

$$\lim_{|x| \to \infty} G_{ij}(x,t) = 0 .$$

(3.15)

By (3.13) we obtain the initial conditions u_{1o}, u_{2o} at $t = 0$, and by (3.14) and (3.15) u_1 and u_2 vanish for $t \to \infty$ and $|x| \to \infty$.

Existence and Uniqueness

As before we treat the solution (3.8) as generalized functions and we may obtain the following uniqueness class for the system:

Theorem 4.10. The Cauchy problem (3.1) has a uniqueness class of all functions f(x) satisfying the inequality

$$|f(x)| \leq C \exp(\theta |x|^2)$$

(3.16)

(with arbitrary C and Θ, fixed). That is, there is at most one solution $u_1(x,t)$, $u_2(x,t)$ in (3.16) to the system (3.1), which for t=0, t fixed in

$0 \le t \le T$, equals the initial functions $u_{1o}(x)$, $u_{2o}(x)$, both also belonging to the class (3.16).

<u>Proof</u>. We have that the order of the system (3.1) is $p_o = 2$. For, as we noted, $r_1(\sigma)$ and $r_2(\sigma)$ are of order 2 in σ, so the resolvents $Q_{ij}(s,t)$ are of growth order 2 at infinity. Specifically, $Q_{ij}(s,t)$ obey the inequality for s, complex, at infinity

$$\| Q_{ij}(s,t) \| \le C_{ij} \, exp(\theta |s|^2) \qquad (\theta \le cT) . \qquad (3.17)$$

$Q_{ij}(s,t)$ for the complex variable $s = \sigma + i\tau$ have singularities $D(s) = 0$ at points which do not lie on the real σ-axis and therefore do not affect the solutions or their Fourier transforms. We then apply the fundamental Theorem 3.4 in Chapter 3 with $p_o = 2$, $p_1 = p_o/(p_o-1) = 2$, and obtain the uniqueness class (3.16) above. ∎

Similarly, we may derive the following correctness class for the system (3.1):

<u>Theorem 4.11</u>. Assume the initial functions $u_{1o}(x)$, $u_{2o}(x)$ belong to the class K_{2,θ_0} of functions satisfying the inequality

$$|f(x)| \le C \, exp(\theta_0 |x|^2) \qquad (3.18)$$

for some θ_0. Then for sufficiently small $t > 0$ and arbitrary $\theta_1 > \theta_0$ the system (3.1) has a unique solution in the class K_{2,θ_1} of ordinary functions satisfying

$$|f(x)| \le C_i \, exp(\theta_i |x|^2) . \qquad (3.19)$$

<u>Proof</u>. We observe that the system (3.1) is parabolic and of (reduced) order $p_o = 2$, has a parabolicity exponent $h = p_o = 2$, and therefore has

a genus $\mu \geq 1 - (p_o - h) = 1 > 0$. We then apply the (fundamental Theorem 3.12 for parabolic systems of positive genus, with $p_1 = p_o/(p_o - \mu) = 2$, type $b_o = \theta_o$, and obtain the correctness class (3.19). (As before, singular points of the resolvents $Q_{ij}(s,t)$, since lying off the real σ-axis, do not affect the solutions of the system.) ∎

Solution (b). The case with a nonzero source

We next include the source term, nonzero, in the basic equations (3.1). Equation (3.2) in u_1 then takes the form

$$\frac{\partial^2 u_1}{\partial t^2} + a\frac{\partial u_1}{\partial t} - b\nabla^2 u_1 - c\nabla^2 \frac{\partial u_1}{\partial t}$$

$$= \frac{b}{cc_2} f(t)\delta(x - x_o) \, . \tag{3.20}$$

Let us set the initial conditions $u_{1o}(x) = u_{2o}(x) \equiv 0$ and seek a solution for this case by method of Laplace transforms. Laplace transformation of equation (3.20) results in the following equation,

$$\nabla^2 \bar{u}_1 - \frac{s^2 + as}{b + sc}\bar{u}_1 = -\frac{b/cc_2}{b + sc}\bar{f}(s)\delta(x - x_c), \tag{3.21}$$

where here we denote by s the independent variable of the transformed function, $\bar{u}_1 = \bar{u}_1(x,s)$. A similar equation is obtained for $\bar{u}_2$; i.e.,

$$\nabla^2 \bar{u}_2 = \frac{s^2 + as}{b + sc}\bar{u}_2 = -\frac{\mu}{k_2}\bar{f}(s)\delta(x - x_o) \, . \tag{3.22}$$

We may use the following integral method to obtain a solution to (3.21) and (3.22). Define the operators

$$L = L* = \nabla^2 - \frac{s^2 + \partial s}{b + sc}$$

and a (Green's) function $\overline{G}(x,x',s)$ which satisfies

$$L^*\overline{G}(x,x',s) = \delta(x-x'),$$ (3.23)

where the gradient $\nabla = \nabla x'$ is taken with respect to x'. Equations (3.21) and (3.22) may then be written in terms of the operator L

$$L\overline{u}_i(x',s) = \overline{F}_i(x',s) \qquad (i = 1, 2),$$ (3.24)

where

$$\overline{F}_1(x',s) = - \frac{b/cc_2}{b+sc}\overline{f}(s)\delta(x'-x_o)$$ (3.25)

and

$$\overline{F}_2(x',s) = - \frac{\mu}{k_2}\overline{f}(s)\delta(x'-x_o).$$ (3.26)

We multiply $L\overline{u}_i$ by $\overline{G}$, $L*\overline{G}$ by $\overline{u}_i$, subtract one from the other and we obtain

$$\overline{G}L\overline{u}_i - \overline{u}_i L^*\overline{G} = \overline{G}(x,x',s)\overline{F}_i(x',s) - \overline{u}_i(x',s)\delta(x-x').$$ (3.27)

Pressures u_1 and u_2 may, as noted before, be assumed to vanish at infinity or outside some bounded volume, and thus there should be no flow at infinity or across the boundary of some finite volume. (This is physically reasonable for a system containing no sources at infinity.) Consequently, we assume that a no-flow condition holds for $\overline{G}$ at the boundary of a finite or an infinite volume. We integrate equation (3.27) with respect to x' over space and then apply the above-mentioned boundary condition on $\overline{G}$ as well as the property (1.9) of the delta function; the left hand side of (3.27) vanishes and we obtain from the right hand side the result,

$$\bar{u}_1(x,s) = \frac{\mu}{k_2} \frac{b}{b+sc} \bar{f}(s) \bar{G}(x,x_o,s), \tag{3.28}$$

$$\bar{u}_2(x,s) = \frac{\mu}{k_2} \bar{f}(s) \bar{G}(x,x_o,s). \tag{3.29}$$

Equations (3.28) and (3.29) show that u_1 and u_2 are convolutions with respect to time of the source function $f(t)$ with the functions $\mu/k_2 K_1(x,x_o,t)$ and $\mu/k_2 K_2(x,x_o,t)$ given by

$$K_1(x,x_o,t) = L_p^{-1}\left(\frac{b}{b+sc} \bar{G}(x,x_o,s)\right), \tag{3.30}$$

$$K_2(x,x_o,t) = L_p^{-1}\left(\bar{G}(x,x_o,s)\right). \tag{3.31}$$

The solutions u_1 and u_2 then take the form:

$$u_1(x,t) = \frac{\mu}{k_2}\int_0^t K_1(x,x_o,t-\tau)f(\tau)d\tau, \tag{3.32}$$

$$u_2(x,t) = \frac{\mu}{k_2}\int_0^t K_2(x,x_o,t-\tau)f(\tau)d\tau. \tag{3.33}$$

The function $\bar{G}$ is, for the radially symmetric plane with the source point x_o at the origin,

$$\bar{G}(r,s) = \frac{1}{2\pi} K_o\left(\sqrt{\frac{s^2+as}{b+sc}}\, r\right), \tag{3.34}$$

We may obtain expressions for the (inverted) functions $K_1(r,t)$, $K_2(r,t)$ and $G(r,t)$ for this case by means of tables and by using the following identity;

$$K_o(\gamma r) = \int_c^\infty \frac{\rho J_o(\rho r)}{\gamma^2 + \rho^2} \, d\rho \quad , \tag{3.35}$$

where γ is here a positive number. With K_o given by (3.35) we obtain for $\overline{K}_1(r,s)$, and $\overline{K}_2(r,s)$

$$\overline{K}_1(r,s) = \frac{b}{2\pi} \int_0^\infty \frac{\rho J_o(r\rho) \, d\rho}{s^2 + (a + c\rho^2)s + b\rho^2} \quad , \tag{3.36}$$

$$\overline{K}_2(r,s) = \frac{1}{2\pi} \int_0^\infty \frac{(b+sc) \, \rho J_o(r\rho) \, d\rho}{s^2 + (a + c\rho^2)s + b\rho^2} \quad . \tag{3.37}$$

We may invert the integrands of (3.36) and (3.37) using tables ([39], pp. 217, 220) and obtain for K_1 and K_2 the following expressions,

$$K_1(r,t) = \frac{b}{2\pi} \int_0^\infty \frac{\rho J_o(r\rho)}{D(\rho)} \Big[exp\,(\tau_1(\rho)t) - exp(\tau_2(\rho)t) \Big] d\rho , \tag{3.38}$$

$$K_2(r,t) = \frac{1}{2\pi} \int_0^\infty \frac{\rho J_o(r\rho)}{D(\rho)} \Big[(b + c\tau_1(\rho))exp(\tau_1(\rho)t) - (b + c\tau_2(\rho))exp(\tau_2(\rho)t) \Big] d\rho . \tag{3.39}$$

It is useful to consider the special case of a constant-strength source, $f(t) = f_o U_+(t)$. This allows explicit integration of solutions (3.32), (3.33) with respect to time, and we obtain for the case of the radially symmetric plane, with (3.38) and (3.39),

$$u_1(r,t) = \frac{\mu f_o}{2\pi k_2} \int_0^\infty \frac{\rho J_o(r\rho)}{D(\rho)} \left[\frac{\exp(r_1(\rho)t) - 1}{r_1(\rho)} \right.$$

$$\left. - \frac{\exp(r_2(\rho)t) - 1}{r_2(\rho)} \right] d\rho, \qquad (3.40)$$

$$u_2(r,t) = u_1(r,t)$$

$$\qquad (3.41)$$

$$+ \frac{\mu f_o c}{2\pi k_2} \int_0^\infty \frac{\rho J_o(r\rho)}{D(\rho)} \left[\exp(r_1(\rho)t) - \exp(r_2(\rho)t) \right] d\rho.$$

The integrals (3.40) and (3.41) converge for $0 \leq t \leq T$, $r > 0$; however, (3.40) diverges for $t \to \infty$ or $r = 0$. We show solutions for the multi-layered version of this model due to Boulton and Streltsova [10] in Figure 4.2.

Characteristic features of the solution are easily seen from the Laplace transforms in the limits as $s \to \infty$ and $s \to 0$. (These correspond to the limits as $t \to 0$ and $t \to \infty$, respectively.) We have from (3.28) and (3.29)

$$\bar{u}_1(r,t) = \frac{\mu f_o}{2\pi k_2} \frac{b}{(b+sc)s} K_o(\gamma(s)r), \qquad (3.42)$$
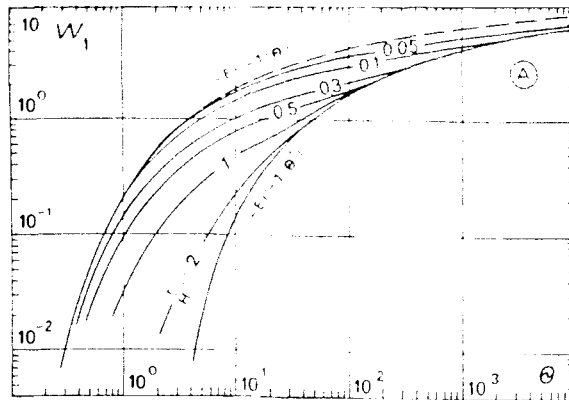
$$\bar{u}_2(r,t) = \frac{\mu f_o}{2\pi k_2 s} K_o(\gamma(s)r), \qquad (3.43)$$
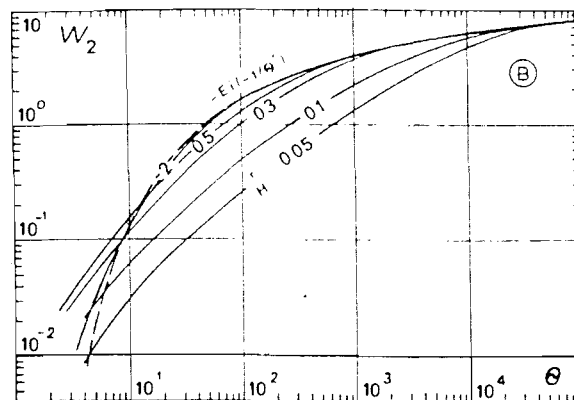
where

$$\gamma(s) = \sqrt{\frac{s^2 + as}{b + sc}} \qquad \text{and} \qquad \bar{f}(s) = f_o/s.$$

(Results identical to (3.42) and (3.43) are given by Kazemi et al [34].) We obtain the limits

Fracture pressure

Matrix pressure

$$\varepsilon = 4k_2 t/\mu c_2 r^2$$

Figure 4.2   Dimensionless pressure drawdown in an infinite
horizontal layer with a constant-strength source. The
multi-layered Warren-Root model after Boulton and
Streltsova [10] with $c_2/(c_1+c_2) = .1$.
(r = distance from source;
2H = thickness of matrix layer)

$$\lim_{s \to \infty} \bar{u}_1(r,s) = \frac{\mu f_o}{2\pi k_2} \frac{b}{cs^2} K_o\left(\sqrt{\frac{s}{c}}\, r\right), \tag{3.44}$$

$$\lim_{s \to \infty} \bar{u}_2(r,s) = \frac{\mu f_o}{2\pi k_2 s} K_o\left(\sqrt{\frac{s}{c}}\, r\right), \tag{3.45}$$

$$\lim_{s \to 0} \bar{u}_1(r,s) = \lim_{s \to 0} \bar{u}_2(r,s) = \frac{\mu f_o}{2\pi k_2 s} K_o\left(\sqrt{\frac{a}{b}s}\, r\right). \tag{3.46}$$

From tables (e.g. [39]) we have, with a > 0 any constant,

$$L_p^{-1}\left(\frac{1}{s} K_o\left(\sqrt{\frac{s}{a}}\, r\right)\right) = -\frac{1}{2} Ei\left(\frac{-r^2}{4at}\right). \tag{3.47}$$

Also, for an arbitrary function g(t),

$$L_p^{-1}\left(\frac{1}{s}\bar{g}(s)\right) = \int_0^t g(\tau)d\tau. \tag{3.48}$$

We obtain, after inverse transformation of (3.44) - (3.46),

$$\lim_{t \to 0} u_1(r,t) = \frac{-\alpha f_o}{4\pi k_2 c_1} \int_0^t Ei\left(\frac{-r^2}{4c\tau}\right)d\tau, \tag{3.49}$$

$$\lim_{t \to 0} u_2(r,t) = \frac{-\mu f_o}{4\pi k_2} Ei\left(\frac{-r^2}{4ct}\right), \tag{3.50}$$

and

$$\lim_{t \to \infty} u_1(r,t) = \lim_{t \to \infty} u_2(r,t) = \frac{-\mu f_o}{4\pi k_2} Ei\left(\frac{-r^2}{4(b/a)t}\right). \tag{3.51}$$

Equations (3.49), (3.50) and (3.51) indicate that u_1 and u_2 behave as the Theis model in the short- and long-term limits with the diffusivities $c = k_2/\mu c_2$ and $b/a = k_2/\mu(c_1 + c_2)$. respectively, and this accounts for the two parallel straight line segments on semilog graphs of the solution (Figure 2.2 , Chapter 2). In other words, the pressure in the fracture system responds initially as a diffusion model with properties of the fracture system, and in the late-time limit both fracture and matrix pressures behave as a diffusion model with properties of both the matrix and the fractures. The transition between these two asymptotic forms of the solution represents the effect of the interporosity flow and depends on the value of the transfer coefficient α.

We may describe the physical process roughly as follows: When the fluid source/sink is applied at t=0 to the fracture system at $x_o = r = 0$ the fracture system responds initially as an ordinary porous medium with properties k_2, c_2 of the fractures. The fluid pressure u_2 in the fractures drops in the manner described by the Theis model, and the drop u_2 leads to a pressure difference u_1-u_2 and flow between the matrix and fracture system to compensate the fluid loss in the fracture system. The fluid pressure u_1 in the matrix begins to drop as well. The pressure difference u_1-u_2 or the crossflow rises from zero to some maximum value at some finite time and diminishes thereafter towards zero as u_1 approaches u_2 and the system thereafter behaves as a diffusion model (an ordinary porous medium) with properties of both matrix and fractures.

The Inverse Problem

With this model there are the four unknown parameters, k_2, c_1, c_2

and α, to estimate, and the curve matching requires several model curves
for a good fit with the data. Specifically, the dimensionless pressure
u_1 or u_2 may be plotted against the variable $4k_2t/\mu(c_1+c_2)r^2$, and different
values (curves) result from different values of α and/or the ratio c_2/c_1
(Figure 4.2). Let us assume that u_2 is the quantity measured and we
select the model curve for u_2 which best fits the data. We recall that
the u_2 curve has the two Theis curve asymptotes for the short- and long-
time limits. By identifying, for example, the short-time Theis curve (in
the data plot) we may obtain the values of k_2 and c_2 by the usual matching
process, and, from the long-time Theis asymptote we obtain the value of
$c_1 + c_2$. The value of α is that associated with the (best-fitting) model
curve.

We now show that the set of values of the parameters, k_2, c_1, c_2,
α, is unique for each input function $u_2(r,t)$ and depends continuously on
$u_2(r,t)$. To do this it is convenient to recall the Laplace transform of
$u_2(r,t)$, i.e.

$$\bar{u}_2(r,s) = \frac{\mu f_o}{2\pi k_2 s} K_o(\gamma(s)r),$$

(3.52)

where
$$\gamma(s) = \sqrt{\frac{s^2 + as}{b + cs}}.$$

As before we ignore the known factor $\mu f_o/2\pi$. We may find from
inspection of (3.52) that $\bar{u}_2(r,s)$ is a unique function for each set of
values of k_2, a, b, and c, and therefore for each set of k_2, c_1, c_2 and
α. For the argument $\gamma(s)$ is a unique function for each set of values of
a, b, and c, thus the function $1/sK_o(\gamma(s)r)$ is unique for each set a, b
and c, and so the function $1/(k_2s)K_o(\gamma(s)r)$, or $\bar{u}_2(r,s)$, is unique for

each set of k_2, a, b, c, or k_2, c_1, c_2, α. We also have that $u_2(r,t)$ and $\bar{u}_2(r,s)$ are in L_2 for $0 \leq t \leq T$, and thus, by (Plancherel's) Theorem 4.1(c), the uniqueness of $\bar{u}_2(r,s)$ for each set k_2, c_1, c_2, α implies the uniqueness of $u_2(r,t)$ for each set k_2, c_1, c_2, α. The continuity of $\bar{u}_2(r,s)$ and therefore $u_2(r,t)$ with respect to k_2, c_1, c_2, α is obvious. We restate the above as follows:

<u>Theorem 4.12</u>. To each function $u_2(r,t)$ given by (3.41) we may associate a unique set of (positive) values of k_2, c_1, c_2 and α, and these parameters depend continuously on $u_2(r,t)$.

4. <u>Model 4. The Dual Permeability Model</u>

$$c_i \frac{\partial u_i}{\partial t} - \frac{k_1}{\mu} \nabla^2 u_1 + \frac{\alpha}{\mu}(u_1 - u_2) = f_1(t)\delta(x - x_0), \qquad (4.1)$$

$$- \frac{k_2}{\mu} \nabla^2 u_2 - \frac{\alpha}{\mu}(u_1 - u_2) = f_2(t)\delta(x - x_0),$$

$$u_i(x,0) = u_{i,0}(x).$$

<u>Solution (a): The case with no sources</u>

We take $f_1(t) = f_2(t) \equiv 0$ and we may combine equations (4.1) into one in u_1 or u_2 and obtain

$$\frac{\partial u_i}{\partial t} + a \nabla^4 u_i - b \nabla^2 u_i - c \nabla^2 \frac{\partial u_i}{\partial t} = 0 \qquad (i=1,2), \qquad (4.2)$$

where

$$a = \frac{k_1 k_2}{\mu \alpha c_1}, \quad b = \frac{k_1 + k_2}{\mu c_1}, \quad c = \frac{k_2}{\alpha}.$$

As before we apply Fourier transformation to the solution to this

problem. The transform of equation (4.1) may be written

$$\frac{\partial \hat{u_i}}{\partial t} + \frac{a|\sigma|^4 + b|\sigma|^2}{1 + c|\sigma|^2}\hat{u_i} = 0, \tag{4.3}$$

from which we obtain

$$\hat{u_1}(\sigma, t) = \hat{u}_{10}(\sigma)\, exp\,(-\gamma(\sigma)t), \tag{4.4}$$

$$\hat{u_2}(\sigma, t) = \frac{\hat{u}_{10}}{1 + c|\sigma|^2}\, exp\,(-\gamma(\sigma)t), \tag{4.5}$$

where

$$\gamma(\sigma) = \frac{a|\sigma|^4 + b|\sigma|^2}{1 + c|\sigma|^2}.$$

Inverse transformation of (4.4) and (4.5) yields the solution

$$u_i(x, t) = G_i(x, t) * u_{10}(x)$$

$$= \int G_i(x - x', t)\, u_{10}(x')\, dx', \tag{4.6}$$

where the fundamental solution G_1 and G_2 are given by, in n-dimensions,

$$G_1(x, t) = \frac{1}{(2\pi)^n} \int exp\,(-i\sigma x)\, exp\,(-\gamma(\sigma)t)\, d\sigma, \tag{4.7}$$

$$G_2(x, t) = \frac{1}{(2\pi)^n} \int \frac{exp\,(-i\sigma x)}{1 + c|\sigma|^2}\, exp\,(-\gamma(\sigma)t)\, d\sigma. \tag{4.8}$$

The functions (integrals) $G_1(x, t)$ and $G_2(x, t)$ converge for $t > 0$, $x = 0$, but diverge for $t = 0$ and $x = 0$. We treat G_1 and G_2 therefore as

generalized functions, and they are integrable in space for all $t \geq 0$. We have, specifically, the following limits of G_1 and G_2 ;

$$\lim_{t \to 0} G_1(x,t) = \delta(x) , \tag{4.9}$$

$$\lim_{t \to 0} G_2(x,t) = \frac{1}{(2\pi)^n} \int \frac{\exp(-i\sigma x) d\sigma}{1 + c|\sigma|^2} , \tag{4.10}$$

$$\lim_{t \to \infty} G_1(x,t) = \lim_{t \to \infty} G_2(x,t) = 0 , \tag{4.11}$$

and (by the Riemann-Lebesgue Theorem 4.2)

$$\lim_{|x| \to \infty} G_1(x,t) = \lim_{|x| \to \infty} G_2(x,t) = 0 . \tag{4.12}$$

For the radially symmetric plane, (4.10) is equal to

$$\lim_{t \to 0} G_2(r,t) = \frac{1}{2\pi c} K_0\left(\frac{r}{\sqrt{c}}\right) . \tag{4.13}$$

We may observe that G_1 and G_2 have characteristics of both the diffusion Model 1 and the Barenblatt Model 2. The argument $\gamma(\alpha)$ is of order 2 in α, the system is parabolic and the limits (4.9) to (4.13) are identical to the corresponding limits for the fundamental solutions G_1 and G_2 for Model 2. Indeed, in the limit as $k_1 \to 0$ we obtain the Barenblatt Model 2.

Existence and Uniqueness

The system (4.1) has, as noted, a reduced order $p_0 = 2$, and we thus

have, by Theorem 3.4 in Chapter 3, the following uniqueness class of all
functions $f(x)$ satisfying the inequality

$$|f(x)| = C \exp(\theta |x|^2) \tag{4.14}$$

(for arbitrary but fixed C and θ). That is, there is at most one
solution $u_1(x,t)$, $u_2(x,t)$ to the system (4.1) in the class of functions
(4.14), which for $t = 0$, t fixed in $0 \leq t \leq T$, equals the initial func-
tions $u_{1c}(x)$, $u_2(x,0)$, which both also belong to the class (4.14). (This
uniqueness class is the same as that obtained for Models 1 and 3.) More-
over, since the resolvents $Q_1 = F(G_1)$, $Q_2 = F(G_2)$ satisfy inequalities of
the form

$$\| Q_i(s,t) \| \leq C \exp(\theta |s|^2) \qquad (i = 1, 2),$$

$$\| Q_i(\sigma, t) \| \leq C_1 \exp(-\theta |\sigma|^2) \qquad (\theta \leq \frac{a}{c} T),$$

the system (4.1) is, as noted, parabolic with parabolicity exponent $h =$
$p_0 = 2$, and it has thus a (positive) genus $\mu \geq 1 - (p_0-h) = 1$. Hence,
given that the initial functions $u_{1o}(x)$, $u_2(x,0)$ satisfy the inequality
(4.14) for some θ, we obtain, by Theorem 3.12, Chapter 3, the correctness
class of solutions satisfying the inequality

$$|f(x)| \leq C_2 \exp(\theta_1 |x|^2) \tag{4.15}$$

for arbitrary $\theta_1 > \theta$ and sufficiently small $t > 0$. (This is also the
same correctness class as we obtained for Models 1 and 3.) The resol-
vents $Q_i(s,t)$ have singular points $s = \pm i/\sqrt{c}$ which do not lie on the
real axis and therefore do not affect the solutions.

Solution (b). The case with sources

We next include the source term (nonzero) in the equation (4.1) and we obtain for the Fourier transforms $\hat{u}_1$ and $\hat{u}_2$

$$\hat{u}_1(\sigma, t) = \hat{u}_{1o}(\sigma)\, exp\left(-\gamma(\sigma)\, t\right)$$
$$+ \int_o^t \hat{F}_1(\sigma, \tau)\, exp\left(-\gamma(\sigma)(t-\tau)\right) d\tau , \tag{4.16}$$

and

$$\hat{u}_2(\sigma, t) = \frac{\hat{u}_1(\sigma, t) + \frac{\mu}{\alpha} f_2(t)}{1 + c|\sigma|^2} , \tag{4.17}$$

where

$$\hat{F}_1 = \frac{1}{c_1}\left(f_1(t) + \frac{f_2(t)}{1 + c|\sigma|^2}\right) .$$

Inverse transformation of (4.16) and (4.17) yields the solution

$$u_1(x, t) = G_1(x, t) * u_{1o}(x) + \frac{1}{c_1}\int_o^t G_1(x-x_o, t-\tau) f_1(\tau)\, d\tau$$
$$+ \frac{1}{c_1}\int_o^t G_2(x-x_o, t-\tau) f_2(\tau)\, d\tau , \tag{4.18}$$

$$u_2(x, t) = G_2(x, t) * u_{1o}(x) + \frac{1}{c_1}\int_o^t G_2(x-x_o, t-\tau) f_1(\tau)\, d\tau$$
$$+ \frac{1}{c_1}\int_o^t G_3(x-x_o, t-\tau) f_2(\tau)\, d\tau + \frac{\mu}{\alpha} f_2(t) G_2(x-x_o, o) , \tag{4.19}$$

where we define

$$G_3(x, t) = \frac{1}{(2\pi)^n}\int \frac{exp(-i\sigma x)}{(1 + c|\sigma|^2)^2} exp\left(-\gamma(\sigma) t\right) d\sigma.$$

We consider again the special case of constant-strength sources, i.e., let $f_1(t) = f_{1o} U_+(t)$, $f_2(t) = f_{2o} U_+(t)$, and take, for convenience, $u_{1o}(x) \equiv 0$. We then obtain from (4.18) and (4.19)

$$u_1(x,t) = \frac{f_{10}}{c_1} I_1(x-x_0,t) + \frac{f_{20}}{c_1} I_2(x-x_0,t), \qquad (4.20)$$

$$u_2(x,t) = \frac{f_{10}}{c_1} I_2(x-x_0,t) + \frac{f_{20}}{c_1} I_3(x-x_0,t)$$

$$+ \frac{\mu}{\alpha} f_{20} U_+(t) G_2(x-x_0,0), \qquad (4.21)$$

where

$$I_i(x,t) = \int_0^t G_i(x,t-\tau)d\tau \qquad (i=1,2,3).$$

We have the following limits of I_i (with $x \neq 0$):

$$\lim_{t\to 0} I_1(x,t) \sim \delta(x) t = 0,$$

$$\lim_{t\to 0} I_2(x,t) \sim G_2(x,0) t = 0,$$

$$\lim_{t\to 0} I_3(x,t) \sim G_3(x,0) t = 0,$$

and

$$\lim_{t\to\infty} I_1(x,t) = \lim_{t\to\infty} I_2(x,t) = \lim_{t\to\infty} I_3(x,t) = \infty.$$

Consequently,

$$\lim_{t\to 0} u_1(x,t) = 0, \qquad (x \neq 0),$$

$$\lim_{t\to 0^+} u_2(x,t) = \frac{\mu}{\alpha} f_{20} G_2(x-x_0,0),$$

and

$$\lim_{t\to\infty} u_1(x,t) = \lim_{t\to\infty} u_2(x,t) = \infty.$$

Moreover, we note that

$$0 \leq \lim_{t \to \infty} |u_1 - u_2| = \lim_{t \to \infty} \left[\frac{f_{10}}{c_1}(I_1 - I_2) + \frac{f_{20}}{c_1}(I_2 - I_3) + \frac{\mu}{\alpha} f_{20} U_+(t) G_2(x - x_0, 0) \right]$$

$$\leq \frac{c}{c_2} \frac{1}{(2\pi)^n} \int \frac{\exp(-i\sigma(x - x_0))}{a|\sigma|^2 + b} \left| f_{10} + \frac{f_{20}}{1 + c|\sigma|^2} \right| d\sigma$$

$$+ \left| \frac{\mu}{\alpha} f_{20} G_2(x - x_0, 0) \right| < \infty \qquad (x \neq x_0), \qquad (4.22)$$

The solutions indicate, as noted before, that this model behaves similarly to the Barenblatt Model 2 and has also properties of the diffusion Model 1. Comparison of (4.18), (4.19) and (2.25), (2.29) shows, for example, that the contribution of the fracture source f_2 to u_1 and u_2 is of analogous form and (numerically) similar in the short-time region to the solutions u_1, u_2 for Model 2. In fact, in the limit $t \to 0$ the solutions are identical for these two Models; at $t = 0^+$, like for Model 2, u_1 rises gradually from zero and u_2 jumps abruptly to the value $\mu/\alpha f_2 G_2(x - x_0, 0)$. The contribution of the matrix source f_1 to u_1 is numerically similar to the diffusion Model 1 solution. Both u_1 and u_2 grow without limit as time increases (due to both f_1 and f_2). However, as (4.22) shows, u_1 and u_2 do not in general approach the same value in the limit as $t \to \infty$; i.e., a finite pressure difference remains between the matrix and the fracture system as $t \to \infty$.

The Inverse Problem

For this model there are the four unknown parameters to estimate, k_1, k_2, c_1 and α, and therefore several model curves are needed in the matching process. Moreover, since the matrix permeability is assumed to be nonzero so that lateral flow in the matrix is not neglected, we should expect that some part (f_1) of the total source (f) involves flow to or from the matrix, and the remaining part (f_2) represents flow to or from the fractures. Similarly, the measured fluid pressure would consist of some linear combination of the pressures u_1 and u_2. These combinations are unknowns, but it would be reasonable to assume, for example, that u = $k_1u_1/(k_1+k_2)$ + $k_2u_2/(k_1+k_2)$, f_1 = $k_1f/(k_1+k_2)$ and f_2 = $k_2f/(k_1+k_2)$. We obtain u as a linear combination of the integrals I_1, I_2, I_3 and $G_2(x-x_o,0)$ multiplied by the fractional values of k_1, k_2 above. One may, in principle, construct a set of type curves for different values of k_1/k_2 and α, plotted against $4k_2t/\mu c_1r^2$, select a model curve which best fits the data and obtain values of the parameters k_1, k_2, c_1 and α in a similar manner as before. We note, however, that curve matching and parameter estimation is the least practical for this model.

We show that the set of values of the parameters k_1, k_2, c_1 and α are unique for each input function $u(r,t)$ and that these parameters depend continuously on $u(r,t)$. For this it is sufficient to establish that the function $1/k_1G_2(r,t)$ (or $1/k_iG_i$ (r,t) for any i=1,2,3) is unique for each set k_1, a, b, c, which implies that it is unique for each set of k_1, k_2, c_1, α. Recall the Fourier/Hankel transform of G_2, i.e.,

$$\hat{G}_2(\sigma,t) = \frac{1}{1+c|\sigma|^2} \, exp\big(-\gamma(\sigma)t\big), \qquad (4.23)$$

with
$$\gamma(\sigma) = \frac{a|\sigma|^4 + b|\sigma|^2}{1 + c|\sigma|^2} \ .$$

We have that $\gamma(\sigma)$ is a unique function for each set of a, b, c, $\hat{G}_2(\sigma,t)$ by (4.23) is unique for each a, b, c, and thus $1/k_1\hat{G}_2(\sigma,t)$ is unique for each k_1, a, b, c. Also, $\hat{G}_2(\sigma,t)$ and $G_2(r,t)$ are both in L_2 for $0 \le t \le$ T, and so by (Plancherel's) Theorem 4.1(c) $1/k_1G_2(r,t)$ is a unique function for each set of k_1, a, b and c. In the same manner we could obtain uniqueness of $1/k_1G_1(r,t)$ and $1/k_1G_3(r,t)$ for each set k_1, a, b, c, and therefore of $1/k_1I_1$, $1/k_1I_2$, $1/k_1I_3$, since I_1, I_2 and I_3 are time integrals of G_1, G_2 and G_3. Moreover, the functions G_1 and I_i (i=1,2,3) are in L_2. Since $u(r,t)$ is a sum of I_1, I_2, I_3 and G_2 we obtain that $u(r,t)$ is unique for each set of values of k_1, k_2, c_1 and α. Continuity of $u(r,t)$ with respect to k_1, k_2, c_1 and α is easily seen. We restate the above by the following Theorem:

<u>Theorem 4.13</u>. To each function $u_1(r,t)$ or $u_2(r,t)$ given by (4.20) or (4.21), respectively (or a given linear combination of u_1 and u_2) there corresponds a unique set of (positive) values of k_1, k_2, c_1 and α, and these are continuous with respect to $u_1(r,t)$ and $u_2(r,t)$.

CHAPTER 5

APPLICATIONS TO NATURAL FORMATIONS

A. Structure, Space and Time Scales

We have seen that in order for the double porosity/permeability models to be valid certain conditions and constraints apply to the structure of the system and the space and time scales of the flow. For example, as we discussed earlier in Chapter 2, the fractures must consist of a fairly uniform and interconnected system of conduits distributed throughout the formation, and they should have lengths, widths and spacing (between fractures) which are well below the characteristic space scales of the flow. This means that if we take a length scale of the flow to be 1000 meters, and consider a uniformly fractured elemental volume of the rock mass to be, say, 50 meters across, then the fracture systems should have scales of length and spacing within tens of meters. These length scales would be likely to include all small to moderate size fracture systems in natural rock. Secondly, given that these as well as other assumptions of the models apply, we found from the solutions in Chapter 4 that, if we compare with solutions to the diffusion model, the double porosity models exhibit their distinctive behavior over a finite time range only, but beyond that time range these models become, as has been noted before, indistinguishable from the standard diffusion model. This time range is of obvious importance when considering modeling of a natural reservoir as a double porosity system, and it is determined by the values of the model parameters as well as by distance between source and observation points. The value of the interporosity flow parameter

(α) has decisive influence of this time scale. This time scale lies within the order of the so-called transient or early-time phase of a typical pressure transient test, which for the conventional Theis or diffusion model refers (vaguely) to the time range when most of the time change of the flow takes place.

To obtain an indication of the order of this time scale of relevance to double porosity modeling we may calculate the following example based on the solution to the Warren-Root model as shown in Figure 4.2, Chapter 4. We select two curves; one (a) for a value of the parameter "r/H" = 1, and (b) for "r/H" = .05. We determine the approximate position of the points along the abscissas where each of these curves approaches the right-most Theis curve to within, say, 5% of the ordinate values. For curve (a) we have for this point the value $4at/r^2 \geq$ 150, and for curve (b) the value is $4at/r^2 \geq 2000$, where $a = k_2/\mu(c_1+c_2)$. We choose a length scale of 1000 meters and assumed the following values for rock and water:

$$k_2 = 10^{-10} \text{ cm}^2 \quad (10 \text{ md}),$$

$$\phi_1 = 10^{-3},$$

$$c_\ell = 4.5*10^{-11}/\text{dyne},$$

$$\mu = 3*10^{-3} \text{ gm*cm}^2/\text{sec},$$

We approximate $c_1+c_2 \approx c_1 \approx \phi_1 c_\ell$; which gives $a = k_2/\mu c_1 \approx k_2/\mu\phi_1 c_\ell$, and we obtain thus for curve (a) the value

$$t \geq 105 \; r^2/4a = 5*10^5 \text{ sec} \approx 6 \text{ days},$$

and for curve (b)

$$t \geq 1000 \; r^2/4a = 6.7*10^6 \text{ sec} \approx 78 \text{ days}.$$

Although these values obviously depend on the values of r/H we may conclude from these solutions that the order of the time scale we are seeking is days to possibly weeks. (r/H is the interporosity flow parameter for the multi-layered model in Figure 4.2.) For longer time scales of flow, such as months or years, all four models in Chapter 4 are no longer appropriate, for compressibility effects which are predominant in short-time flows become negligible for longer-time flows, and instead effects of a free fluid surface become important and should be taken into account.

B. On Parameter Estimation With the Double Porosity Models

Parameter estimation and type curve matching appears to be feasible with the Barenblatt and Warren-Root models and should yield more accurate values for natural double porosity systems than the Theis model. As noted before the model solutions apply to well interference tests and data sets obtained should cover an adequate time range of the flow such that both the short- and long-time Theis curves can be identified. This time range would be of the order calculated above. The distinctive characteristics of the drawdown response predicted by the Warren-Root model, i.e., the two short- and long-time Theis model behavior, is easily recognized. Distinctive characteristics of the fracture response for the Barenblatt model is the initial abrupt jump (drop) in the pressure and slow rise with time towards the long-time Theis model behavior (Figure 4.1 , Chapter 4). Different curves correspond, as Figure 4.1 shows, to different values of the parameter α ($c=k_2/\alpha$), with curves for lower values of α lying farther to the right. Type curves for the matrix response (Figure 4.1, Chapter 4) show an initial roughly linear rise

with time towards the long-time Theis solution; curves for different values of α are of roughly similar shape and approximately parallel with those for lower values of α which lie farther to the right. The position of these curves suggest that parameter values obtained by curve matching with the Barenblatt model, particularly storage capacity (c_1), are sensitive to the value of α and the time range of the data, and that fitting with the Theis model may produce significant errors in the estimates. Parameter estimation using Model 4 (the dual permeability model) would be impractical due to number of independent parameters involved, absence of distinctive features in the solutions and unknown relative contribution of fractures and matrix to measured pressure and source functions.

C. Double Porosity Behavior in Natural Formations

In general fluid flows in natural fractured formations is, as we have noted, relatively complex and variable among individual natural formations, and so the problem of modeling flows in fracture formations is not a simple one. Since also different types of theoretical models sometimes produce similar output, some difficulty exists therefore in identifying a given natural formation as a double porosity system. However, as discussed before, some evidence for double porosity behavior in natural formations has been observed and reported in the literature. Such evidence is typically obtained by comparing well test data from fractured reservoirs with output from theoretical models and observing characteristics of these models in the data. Most notably, features of the Warren-Root model are easily recognized. Indeed, Warren and Root [51] first reported, as mentioned before, data from a fractured formation

showing the characteristics two parallel straight lines on semilog plots and infer this as field evidence for their model (Figure 5.1). Similar features appear in other data sets; as, for example, shown in Figure 5.2. We noted earlier two publications which report good fits of well data from Klamath Falls, Oregon, with the Warren-Root model as well as also a composite model [6], [7] (Figure 5.3). Another study, we also noted, reports good fits between data from fractured aquifers and the multi-layered double porosity model of Boulton and Streltsova [10] (Figure 5.5) and a discrete fracture model (Figure 5.6). Other examples of well test data from fractured formations are shown in Figure 5.7. For comparison we show a data set from [7] matched against the Theis model in Figure 5.4. We may consider the   suggestions of double porosity behavior in the various reports above, particularly the evidence for the Warren-Root model, as strongly suggestive, if not necessarily conclusive.

D. Discussion on the Validity of the Double Porosity Models

The double porosity models attempt to describe more adequately by simple systems of equations the more complex process of flow in natural fracture formations than does the conventional homogeneous diffusion model. Like most fractured media models this type of a model takes effects of fractures into account by considering separate processes of flow in fractures and matrix, whereas, in contrast, the conventional diffusion models view matrix and fractures together as a single porous medium with a single system of flow. The double porosity models apply to short-time single-phase flows in homogeneous reservoirs with the same simplifications and assumptions as outlined in Chapter 1 for the diffusion equation. The double porosity approach appears valid and
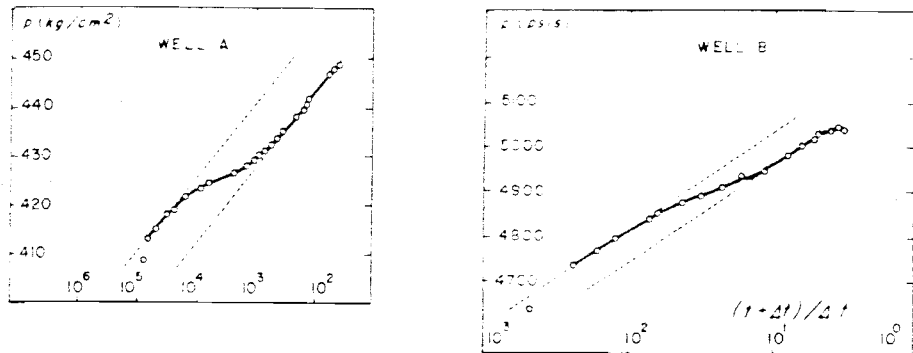
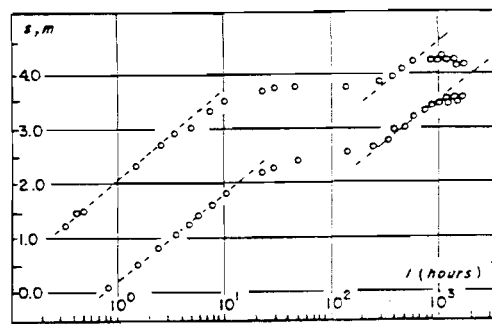Figure 5.1. Field recovery data for a fractured reservoir, after Warren and Root, 1965 [48]

Figure 5.2. Field drawdown data for flow to a well in a fractured limestone, after Borevsky et al, 1973. [48]
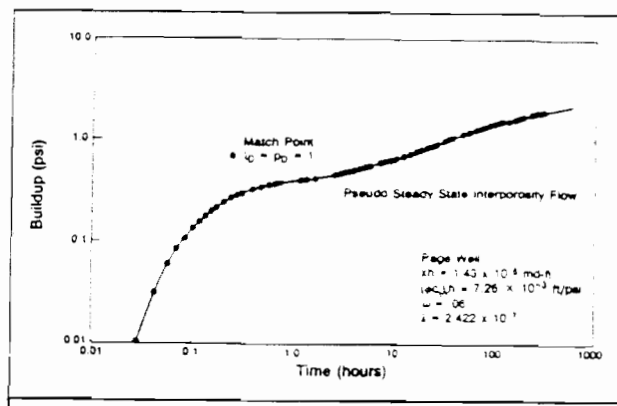
Figure 5.3. Log-log plot and double porosity type curve match for well buildup data, after Benson and Lai, 1986 [7]. (The Page well, Klamath Falls, Oregon)
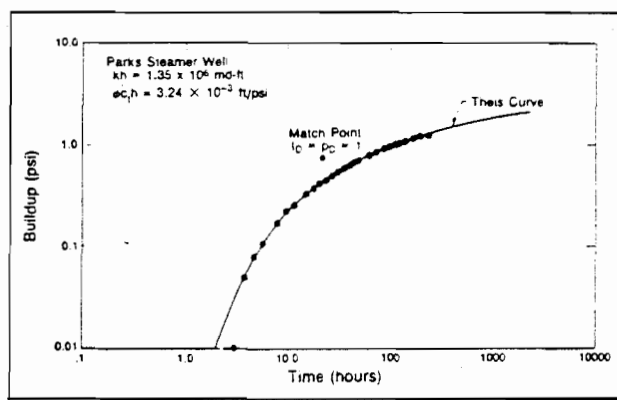

Figure 5.4. Log-log plot and Theis curve match for well buildup data, after Benson and Lai, 1986 [7]. (The Steamer well, Klamath Falls, Oregon.)
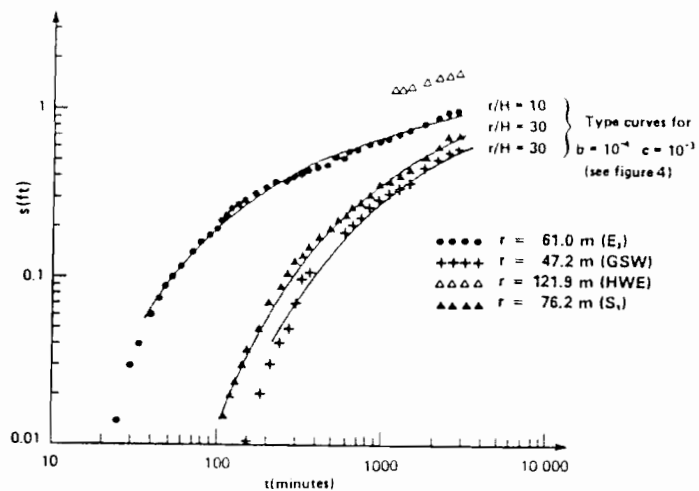
Figure 5.5. Well test data from a coal aquifier interpreted with the Boulton-Streltsova model. Exact solution. (After Sauveplane. [44])
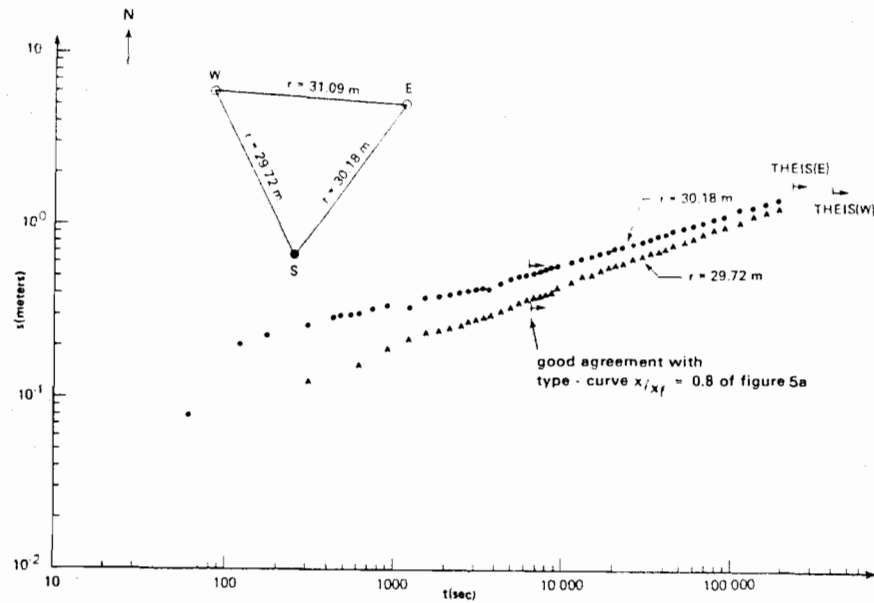
Figure 5.6. Barrhead coal aquifier interpretation with the discrete (vertical) fracture model of Gringarten and Witherspoon, 1972. [44]
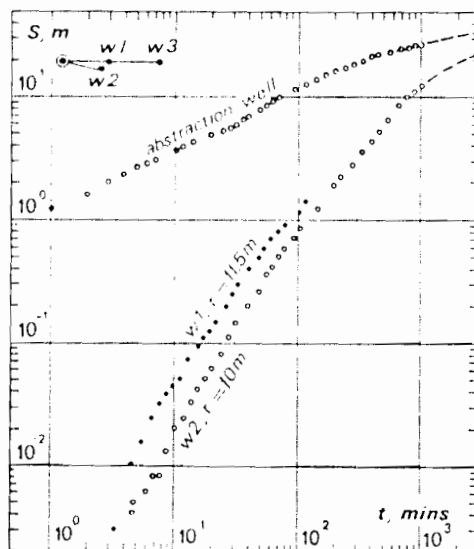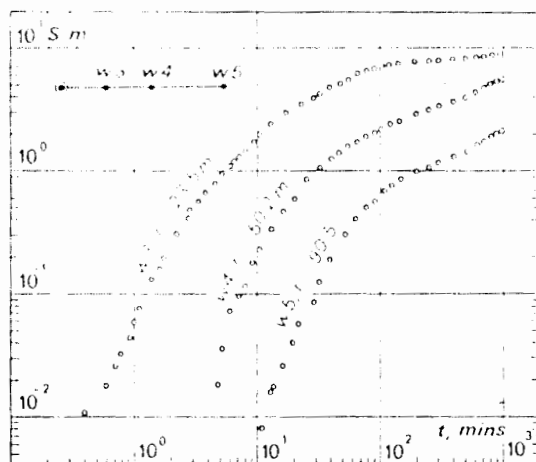
Figure 5.7. Field measurements of (top) porous flow and (bottom) fissure flow in a fractured formation. (After Herbert, 1976. [45])

realistic at a first order level given that the diffusion equation is a valid model for flow in natural (unfractured) formations in general. Indeed we may view the double porosity models as analogs of the homogeneous diffusion model extended to fractured porous media. Of the three models studied here we consider the two due to Barenblatt and Warren and Root (Models 2 and 3) to be the main ones relevant for flows in natural fractured formations based on the consideration that fracture permeability is likely to be considerably greater than matrix permeability. Structural requirements such that the double porosity models apply are, as we discussed, a reasonably uniformly distributed fracture system on space scales well below the scales of the flow such that the fracture system may be treated as a continuum. Moreover, the fracture system should be characterized by some predominant effective space scale such that we may associate with it distinct macroscopic properties which differ from corresponding properties of the matrix. Double porosity models have, as we noted, the advantage over e.g. discrete fracture models in not requiring knowledge of exact size, shape, location, of individual fractures, but only that they form a fairly uniformly distributed system in the rock masses. These conditions involving natural fractured rock are likely to arise most often in practice. Moreover, this type of model, being simple, allows analytic solutions to be obtained easily and made applicable to a large number of cases.

Limitations of the double porosity model lies, on the other hand, in its simplicity, particularly with regard to the assumptions made about the regular and uniform structure of the fracture system in a natural

rock mass and a predominant and uniform space scale characterizing it. Fractures in most natural rock masses typically comprise a range of sizes, shapes, lengths and spacing [37], and this space scale variability would invite effects of e.g. "multiple interacting continua". Secondly, since most natural formations are heterogeneous in structure and composition some spacial variability is likely to exist in fracture permeability, interporosity flow coefficient, etc. Heterogeneity and multiple fracture scales of the structure both tend to obscure distinctive, recognizable double porosity characteristics. Refinements of the models increases number of unknown independent parameters involved, thus increasing computational effort required and hence generally reducing the practical applicability.

As we have noted, different types of models may sometimes produce similar results. For example, as noted in the study [7], two-part composite diffusion model is found to be numerically similar to the Warren-Root model, and on longer time scales the double porosity models behave similarly to the homogeneous diffusion (e.g. Theis) model. Therefore the choice of a suitable model in a given case is based on the simplicity and convenience in the use of a given model as well as on the physical system.

It appears likely, as we have discussed, that the double porosity models of the Barenblatt or the Warren-Root types (Models 2 and 3), could be used for describing flows in basalt formations. Typical basalt formations appear to possess properties required for modeling as large and deep double porosity system, where the rock matrix and microfractures along rock interfaces forms the matrix and other larger fractures and

interbed sediments constitute the fracture system. It is reasonable also
to expect the fracture system to be fairly uniformly distributed in
the formation, some anisotropy of the fracture permeability (k_2) is
likely to exist (e.g. vertical). but this, as mentioned, may be
eliminated from the model. Based on previous studies (e.g. [4])
formation permeabilities for basalts are in orders of 1 to 10 md (10^{-11}
to 10^{-10} cm^2), and porosisites are usually from 10^{-4} to 10^{-2}. The
formation may be modeled as a thick infinite horizontal layer or a semi-
infinite halfspace. If the line source extends the depth of the layer
the system may be modeled as 2-dimensional, but otherwise as a 3-
dimensional system (e.g. a halfspace), for which vertical variability may
need to be included in the solutions.

E. Summary and Conclusions

In this work we study three degenerate systems of linear parabolic
equations which collectively represent forms of the "double porosity/per-
meability model" and which is used for describing fluid flows in natural
fractured rock. These systems are modified diffusion models derived from
two coupled diffusion equations. Two of these systems represent the so-
called double porosity model often treated in the literature and are
considered to be the main forms of the model relevant for flows in
natural fractured rock. These models describe short-term flows in highly
idealized homogeneous systems. Like the diffusion equation each system
of equations governs flows, single-phase, of slightly compressible fluids
in homogeneous and uniformly fractured reservoirs. Initial and boundary
conditions are given such that the solutions describe flows in confined
reservoirs under general conditions including those of typical well

tests. We obtain analytic solutions by Fourier and Laplace transforms and give these as convolutions of fundamental solutions with initial conditions and source functions. We established that the models are well-posed, i.e., that the solutions exist, are unique and depend continuously on the initial/source data for all initial conditions likely to arise in practice. In fact, by treating the systems as generalized functions we obtain, by the theory of generalized functions, uniqueness and correctness classes for each model for certain restrictions required on the intial data. It turns out that the uniqueness and the correcness classes for the Barenblatt model, a hyperbolic system, consist of all functions with no restrictions on growth at infinity, although the correctness class requires sufficient smoothness of the initial data. For the other models, including the standard diffusion model, all of which are parabolic, the uniqueness and correctness classes consist of functions of growth at infinity restricted to order 2. These classes of functions would include all initial conditions likely to be given in practical applications with these models, for initial conditions for actual physical systems would be expected to consist of functions, smooth and bounded, that vanish at infinity. We compare the various model solutions with corresponding solutions to the diffusion equation, the standard model for flow in unfractured porous media. We observe that the fractured media models exhibit behaviors which are distinguishable from that of the diffusion model over a finite time scale only, but on a somewhat longer time scale the solutions are approximately the same. We treat also the inverse problem of estimating parameters with each model (by e.g. the conventional method of type curve matching), and we

establish uniqueness and continuous dependence of each set of parameter values with respect to the input data. We consider physical conditions required for the models to apply in practice as well as their general validity and usefulness.

As we find, under appropriate physical conditions the models treated here are relevant for short-time, or transient flows such as during typical drawdown and buildup test. Based on calculations with typical values for rock and water this time scale is of order up to days to weeks. For flows on time scales beyond this the standard diffusion models are equally valid and may, because of its greater simplicity, be more practical. We conclude that the double porosity models due to Barenblatt et al [5] and Warren and Root [50] are the most valid forms of the double porosity model for fractured rock in general, including typical basalt formations, and their main value lies in their potential for providing more accurate estimates of formation parameters from pressure transient data. These two models are likely to give more accurate values for permeability and storage capacity for fractured rock than the conventional Theis model, particularly for data in the short-time end of the well test response. A number of reports in the literature of well test data from natural fractured formations support the validity of the double porosity approach in modeling such systems.

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